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WEYL INVARIANCE AND THE COSMOLOGICAL CONSTANT*

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ABSTRACT

A systematic study of a gravitational action possessing local conformal invariance is undertaken. The spontaneous breaking of the conformal symmetry induces general relativity as an effective long distance limit through the vacuum expectation value of an unphysical scalar field. The Ward identities of the broken theory guarantee the stability of Minkowski spacetime or, equivalently, the vanishing of the cosmological constant to all orders in any consistent perturbation expansion. This result persists when the theory is coupled to the standard $U(1) \times SU(2) \times SU(3)$ model with its electroweak symmetry broken by radiative corrections. A particularly natural small parameter is the ratio of the gravitational degrees of freedom to the matter ones, $1/N$. In this perturbation expansion the theory is asymptotically free and renormalizable in a simple way. We show that this expansion predicts the spontaneous breaking of local conformal symmetry. Furthermore, all ghost degrees of freedom acquire gauge fixing dependent masses.

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1. Motivation

There is no experimental evidence for the quantization of the gravitational field but one expects quantization should apply to all the fundamental fields of physics. They all interact with one another, and it is difficult to see how some could be quantized while others not. Of the four fundamental forces determining low energy physics, three have been adequately described as quantum field theories. All experimental results indicate these quantum theories possess gauge symmetries based on the group $U(1) \times SU(2)_L \times SU(3)_C$. The remaining force, gravity, has a similar gauge symmetry, the coordinate invariance in a spacetime manifold, but resists quantization. This prevents us from constructing a quantum theory of all known interactions based on the gauge principle.

Classically, the gravitational force at large distances is very well explained by Einstein's general theory of relativity which relates it to the curvature of spacetime. The underlying spacetime manifold is Riemannian and the associated curvature tensor can be obtained by noting that under parallel transport around a closed loop the final direction of a vector differs from its initial direction. Weyl, in his attempt to unify the gravitational and electromagnetic forces [1], generalized the Riemannian space by allowing the final vector to have a different length as well as a different direction, which is a very natural generalization of the manifold's gauge invariance.

Although there exists a range of classical distances in which Einstein's theory has no experimental confirmation, there is no overwhelming reason to modify it on the classical level. It is quantum physics which reveals the serious problems of general relativity. The experimental upper bound on the value of the cosmological constant is [2] approximately 10^{-48} $(\text{GeV})^4$, a very small number indeed. However, the presence of a single fundamental particle in the Einstein universe induces an infinite contribution to the vacuum expectation value of the energy momentum tensor or, equivalently, to the cosmological constant. Unless severe unnatural fine tuning is employed, general relativity contradicts the observed world.

There are two equivalent ways to quantize a classical theory; the first involves the canonical formalism while the second the functional integral formalism. Using generalized canonical methods, Einstein's theory can be put into Hamiltonian form [3] whose consistency cannot be established on the quantum level due to the presence of singular order ambiguous operator products [4]. This problem is intrinsic to the canonical formulation of any coordinate invariant metric theory [5]. Nevertheless, the naive expression for quantum inner products as a functional integral weighted by the classical action does define some quantum theory of gravity. It is this definition of quantum gravity which is employed to analyze its properties. The same methods developed for the renormalization of non-abelian gauge theories can be applied to Einstein's gravity and demonstrate its perturbative nonrenormalizability [6]. This phenomenon, which becomes more transparent when general relativity is coupled to renormalizable matter theories [7], can be traced to the dimensionful coupling constant entering Einstein's Lagrangian and defining the perturbation series. Consequently, the perturbation series of quantum general relativity leads to uncontrollable ultraviolet divergences. It could be that by solving exactly the quantum theory sensible finite results emerge. However, the intricate nonlinearity of gravity makes such hopes infinitesimal. Another possibility, within perturbation theory, is to reorganize the terms of the perturbation series consistently and hope to obtain answers that are finite or renormalizable order by order. One such effort considers expectation values of invariant Green's functions which correspond to physically meaningful experiments in curved spacetime but calculational difficulties prevent definitive conclusions [8].

It is reasonable to assume that quantum gravity is not derived by quantizing general relativity but some other renormalizable gravitational Lagrangian which leads to Einstein's theory at large distances in the same way that the Glashow-Weinberg-Salam $U(1) \times SU(2)_L$ gauge model leads to Fermi's four-fermion Lagrangian. Although this analogy is not exact since gravitation, unlike weak interactions, is a long-range force, it provides physical insight.

Consider the four-fermion interaction Lagrangian:

$$\bar{\psi}\, \not{\partial}\, \psi - \frac{1}{2}\, g^2(\bar{\psi}\Gamma_\mu \psi)^2$$

which in terms of an auxiliary field A_μ takes the form:

$$\bar{\psi}\, \not{\partial}\, \psi + g(\bar{\psi}\Gamma_\mu \psi)A^\mu + \frac{1}{2}\, A_\mu^2 \quad .$$

This Lagrangian is not renromalizable since it has one-loop higher derivative counterterms of the form $\alpha F_{\mu\nu}^2 + \beta(\partial_\mu A^\mu)^2$ which give kinetic energy to A_μ. By adding kinetic terms to the classical Lagrangian, one obtains a new theory which at low energies reproduces the four-fermion interaction and is renormalizable but not unitary. The introduction of gauge invariance restores unitarity but, at the same time, eliminates the A_μ mass term and, therefore, the desired low energy limit is lost. This limit is recovered by supplying a mass to A_μ through the vacuum expectation value of a scalar field which spontaneously breaks the gauge invariance.

Starting from the Einstein Lagrangian and calculating quantum corrections, we obtain higher derivative coordinate invariant counterterms [6]. By analogy, we are led to consider theories containing additional terms quadratic in the curvature tensor. Dimensional analysis concludes that the linear Einstein term dominates the long distance behavior of the theory while the quadratic terms dominate at short distances. Furthermore, it can be shown that such Lagrangians define perturbatively renormalizable quantum theories [9]. Nevertheless, these theories are not unitary in the ordinary loop expansion although nonperturbative techniques seem to suggest that no ghosts are present [10-12]. Independent of their unitarity properties, higher derivative gravitational actions still suffer from the cosmological constant problem.

A new gauge symmetry is needed. It is an old idea in particle physics that, in some sense, at sufficiently high energies the masses of the elementary particles should become unimportant. On the other hand, Weyl's generalized Riemannian space naturally incorporates this idea by possessing local conformal invariance.

In the resulting Lagrangian, which is quadratic in the Weyl curvature tensor, a cosmological constant term is forbidden by the extra symmetry. At the same time, in analogy with the previous example of the weak interactions, Einstein's theory at large distances is not reproduced as long as the symmetry remains unbroken. By analogy with the Higgs mechanism, general relativity can be induced [13,14] if a scalar field acquires a vacuum expectation value which spontaneously breaks the conformal invariance [15].

In this work, we construct a spontaneously broken conformally invariant quantum theory, Weyl's gravity, and study its general properties. There are two problems which have to be faced immediately. On one hand, the unresolved question of unitarity can be powerfully attacked using an expansion in a naturally small parameter, $1/N$, where N is the number of fundamental matter fields [10]. In this expansion higher derivative gravity theories are unitary to leading order but require the Lee-Wick prescription [16] from there on. On the other hand, one expects that conformal invariance is explicitly broken through the renormalization scale [17]. However, we directly connect this scale with the vacuum expectation value of the scalar field which induces the Einstein term. At the same time, the scalar is the dilaton which in turn is the Goldstone boson of the spontaneously broken dilatation invariance, the global counterpart of the conformal gauge symmetry.

In the context of Weyl's gravity [30]:

(a) We review the classical Weyl theory and present the properties of its $1/N$ expansion. The theory has one coupling constant, in the simplest case, and is asymptotically free [10] (section 2).

(b) Independent of the expansion parameter, we demonstrate that the Ward identities of any spontaneously broken conformally invariant theory imply a zero cosmological constant to all orders in a perturbation series around flat spacetime and an extremum of the dilaton potential (section 3).

(c) There exists an extension of dimensional regularization using the dilaton field which preserves the Ward identities of the spontaneously broken conformal invariance without anomalies [15] (section 4).

(d) The theory is renormalizable in a simple way (section 5).

(e) To realize the spontaneous breakdown of the conformal symmetry, the vacuum expectation value v_0 of the dilaton field has to be determined in terms of the renormalization group invariant scale Λ where the gravitational coupling constant becomes strong. We achieve this by using the equations of motion of the dilaton to determine the ratio v_0/Λ. Thus, the theory has a single scale parameter of the order of the Planck mass and the dilaton becomes the Goldstone mode of the spontaneously broken dilatation invariance (section 6).

(f) Any matter theory can be extended to a spontaneously broken conformally invariant theory coupled to Weyl's quantum gravity such that the cosmological constant vanishes. As a simple example, we demonstrate this explicitly in scalar quantum electrodynamics, the generalization to the standard $U(1) \times SU(2)_L \times SU(3)_C$ model being straightforward (section 7).

(g) We show that all ghost degrees of freedom receive a gauge dependent mass suggesting that the theory may be unitary in the $1/N$ expansion without using the Lee-Wick prescription (section 8).

2. Weyl's Gravity and the 1/N Expansion

We wish to study Weyl's gravity coupled to N matter fields. The classical action of the theory consists of two parts:

$$S \equiv S_G + S_M \tag{2.1}$$

and possesses coordinate and conformal invariance, the two elements of Weyl's local symmetry. The fundamental classical gravitational field variable is the real, symmetric tensor $g_{\mu\nu}(x)$ with signature $(+\,-\,-\,-)$ and x a point in some four-dimensional spacetime manifold (greek suffixes take on spacetime values). Under local coordinate transformations with infinitesimal parameter $\omega^{\mu}(x)$, the metric changes by:

$$\Delta g_{\mu\nu}(x) = -g_{\mu\rho}(x)\,\partial_\nu\,\omega^\rho(x) - g_{\nu\rho}(x)\,\partial_\mu\,\omega^\rho(x) - \omega^\rho(x)\,\partial_\rho\,g_{\mu\nu}(x) \qquad (2.2a)$$

while the result of a local conformal transformation parametrized by $\Omega(x)$ is:

$$\delta g_{\mu\nu}(x) = 2\Omega(x)\,g_{\mu\nu}(x) \quad . \qquad (2.2b)$$

The curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}(x)$ can be expressed in terms of the connection $\Gamma^{\alpha}{}_{\beta\gamma}(x)$ which is a well known function of $g_{\mu\nu}(x)$ and its first derivatives:

$$R^{\alpha}{}_{\beta\gamma\delta}(x) \equiv \partial_\gamma\,\Gamma^{\alpha}{}_{\delta\beta}(x) - \partial_\delta\,\Gamma^{\alpha}{}_{\gamma\beta}(x) + \Gamma^{\alpha}{}_{\gamma\rho}(x)\Gamma^{\rho}{}_{\delta\beta}(x) - \Gamma^{\alpha}{}_{\delta\rho}\Gamma^{\rho}{}_{\gamma\beta}(x) \quad . \qquad (2.3)$$

Weyl's tensor $C^{\alpha}{}_{\beta\gamma\delta}(x)$ can be constructed out of the curvature tensor and its contractions. It vanishes for all contractions of its indices, is conformally invariant since:

$$\delta\, C^{\alpha}{}_{\beta\gamma\delta}(x) = 0 \qquad (2.4)$$

and its square, which equals:

$$\begin{aligned} C^2(x) &\equiv C_{\alpha\beta\gamma\delta}(x)\,C^{\alpha\beta\gamma\delta}(x) \\ &= R_{\alpha\beta\gamma\delta}(x)\,R^{\alpha\beta\gamma\delta}(x) - 2R_{\alpha\gamma}(x)\,R^{\alpha\gamma}(x) + \frac{1}{3}\,R^2(x) \end{aligned} \qquad (2.5)$$

provides the simplest Weyl invariant action term depending only on $g_{\mu\nu}(x)$:

$$-\frac{1}{G^2}\int d^4x\,\sqrt{-g(x)}\;C^2(x) \qquad (2.6)$$

where G is some dimensionless coupling constant while $g(x)$ is the determinant of $g_{\mu\nu}(x)$.

However, Weyl's theory needs a scalar field $\phi_0(x)$ as part of the gravitational piece of the action. The transformation properties:

$$\Delta\phi_0(x) = -\omega^\rho(x)\,\partial_\rho\,\phi_0(x) \tag{2.7a}$$

and

$$\delta\phi_0(x) = -\Omega(x)\,\phi_0(x) \tag{2.7b}$$

of the scalar field imply that, up to an overall sign, after normalizing its kinetic energy term to have the standard coefficient, only one combination containing derivatives of $\phi_0(x)$ is invariant under both symmetries:

$$\pm\int d^4x\,\sqrt{-g(x)}\left[\frac{1}{2}\,g^{\mu\nu}(x)\,\partial_\mu\,\phi_0(x)\,\partial_\nu\,\phi_0(x) + \frac{1}{12}\,R(x)\,\phi_0^2(x)\right] \quad . \tag{2.8}$$

The requirement on our theory to induce the Einstein action, after $\phi_0(x)$ acquires a vacuum expectation value (VEV), fixes the overall sign of (2.8) and we infer:

$$S_G \equiv \int d^4x\,\sqrt{-g(x)}\left[-\frac{1}{G^2}\,C^2(x) - \frac{1}{2}\,g^{\mu\nu}(x)\,\partial_\mu\,\phi_0(x)\,\partial_\nu\,\phi_0(x) - \frac{1}{12}\,R(x)\,\phi_0^2(x)\right] . \tag{2.9}$$

Thus, as a direct consequence of the conformal part of the full invariance, the gravitational action contains only one coupling constant. The addition of the Weyl invariant self-interaction term:

$$\int d^4x\,\sqrt{-g(x)}\;\lambda_0\,\phi_0^4(x) \tag{2.10}$$

would introduce another coupling constant but, for simplicity, we do not consider it now as our conclusions are not affected (the most general case is treated in sections 4 and 6). Furthermore, the scalar $\phi_0(x)$ has to enter with a negative kinetic energy term; the correct Einstein term forces it to become a ghost. For spacetime manifolds topologically equivalent to Minkowski spacetime, the Gauss-Bonnett theorem relates the various quadratic terms in the curvature:

$$\int d^4x\,\sqrt{-g(x)}\left[R_{\alpha\beta\gamma\delta}(x)\,R^{\alpha\beta\gamma\delta}(x) - 4R_{\alpha\beta}(x)\,R^{\alpha\beta}(x) + R^2(x)\right] = 0 \quad . \tag{2.11}$$

The equations of motion following from (2.9) are:

$$\frac{\delta S_G}{\delta g_{\mu\nu}(x)} = -\frac{1}{G^2}\sqrt{-g(x)}\Big[\frac{1}{2}\,g^{\mu\nu}(x)\,C_{\alpha\beta\gamma\delta}(x)\,C^{\alpha\beta\gamma\delta}(x) + 2R^{\mu\alpha\nu\beta}(x)_{;\alpha\beta}$$

$$+ 2R^{\mu\alpha\nu\beta}(x)_{;\beta\alpha} - 2R^{\mu}_{\ \alpha\beta\gamma}(x)\,R^{\nu\alpha\beta\gamma}(x) - 2g^{\mu\nu}\,R^{\alpha\beta}(x)_{;\alpha\beta} - 2R^{\mu\nu}(x)_{;}{}^{\alpha}{}_{\alpha}$$

$$+ 2R^{\mu\alpha}(x)_{;}{}^{\nu}{}_{\alpha} + 2R^{\nu\alpha}(x)_{;}{}^{\mu}{}_{\alpha} + 4R^{\nu\alpha}(x)\,R^{\nu}{}_{\alpha}(x)$$

$$+\frac{2}{3}\,g^{\mu\nu}(x)\,R(x)_{;}{}^{\alpha}{}_{\alpha} - \frac{2}{3}\,R(x)_{;}{}^{\mu\nu} - \frac{2}{3}\,R(x)\,R^{\mu\nu}(x)\Big]$$

$$+\frac{1}{2}\sqrt{-g(x)}\Big(g^{\mu\alpha}(x)\,g^{\nu\beta}(x) - \frac{1}{2}\,g^{\mu\nu}(x)\,g^{\alpha\beta}(x)\Big)\Big(\partial_\alpha\phi_0(x)\Big)\Big(\partial_\beta\phi_0(x)\Big)$$

$$+\frac{1}{12}\sqrt{-g(x)}\Big[\Big(R^{\mu\nu}(x) - \frac{1}{2}\,g^{\mu\nu}\,R(x)\Big)\,\phi_0^2(x)$$

$$+ g^{\mu\nu}(x)\,\phi_0^2(x)_{;}{}^{\alpha}{}_{\alpha} - \phi_0^2(x)_{;}{}^{\mu\nu}\Big] = 0 \tag{2.12}$$

$$\frac{\delta S_G}{\delta \phi_0(x)} = -\frac{1}{6}\sqrt{-g(x)}\;R(x)\,\phi_0(x) + \partial_\alpha\Big(\sqrt{-g(x)}\;g^{\alpha\beta}(x)\,\partial_\beta\,\phi_0(x)\Big)$$

$$= 0 \tag{2.13}$$

where the semicolon denotes covariant differentiation. The scalar field variational equation contains no additional dynamics being the trace of the gravitational equation:

$$g^{\mu\nu}(x)\;\frac{\delta S_G}{\delta g_{\mu\nu}(x)} = \frac{1}{2}\;\phi_0(x)\;\frac{\delta S_G}{\delta \phi_0(x)} \tag{2.14}$$

This result is due to the conformal invariance present and is most clearly exhibited by redefining [18]:

$$g'_{\mu\nu}(x) \equiv v_0^{-2}\,\phi_0^2(x)\,g_{\mu\nu}(x) \quad ; \quad v_0 \neq 0 \tag{2.15}$$

in the gravitational action (2.9). The latter becomes:

$$S'_G = \int d^4x\;\sqrt{-g'(x)}\Big[-\frac{1}{G^2}\,C'^2(x) - \frac{v_0^2}{12}\,R'(x)\Big] \tag{2.16}$$

and the scalar field is entirely removed; it is a "gauge" degree of freedom. Weyl's gravity, based on S_G, needs conformal gauge fixing and, provided $\phi_0(x)$ has a nontrivial VEV, the action S'_G is the form the theory takes in the gauge $\phi_0(x) = v_0$ which can be thought as the "unitarity gauge" for the conformal symmetry. Notice that the general theory of relativity is induced by defining $v_0^2 \equiv 3(4\pi G_N)^{-1}$, where G_N is Newton's constant.

The equations of motion (2.12) and (2.13) have a one-parameter family of solutions given by:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad , \quad \phi_0(x) = v_0 \tag{2.17}$$

which classically spontaneously break the conformal invariance; $\eta_{\mu\nu}$ denotes the Minkowski spacetime metric.

The matter part S_M of the action will in general contain N_S scalars ϕ, N_F Dirac-fermions ψ and N_V vector bosons A_μ all of which, in a realistic theory, are associated with a Lie group H. There are many such fundamental matter field in nature and consequently a small expansion parameter is the ratio of the gravitational degrees of freedom to the matter ones. Thus, the perturbation theory of the action:

$$\begin{aligned} S = \int d^4x \, \sqrt{-g(x)} \Big[-\frac{1}{G^2} \, C^2(x) - \frac{1}{2} \, g^{\mu\nu}(x) \, \partial_\mu \, \phi_0(x) \, \partial_\nu \, \phi_0(x) - \frac{1}{12} \, R(x) \, \phi_0^2(x) \Big] \\ + S_M \, [\phi, \, \psi, \, A_\mu] \end{aligned} \tag{2.18}$$

will be organized in powers of $1/N$, where $N \equiv N_S + 6N_F + 12N_V$ [see below eq. (2.27)], such that the product G^2N is fixed. The background fields corresponding to the vacuum of (2.18) determine the expansion points of our perturbation:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + Gh_{\mu\nu}(x) \tag{2.19a}$$

$$\phi_0(x) = v_0 + \sigma(x) \quad . \tag{2.19b}$$

The matter fields, with the possible exception of some scalars, have trivial backgrounds. Finally, a simple set of gauge fixing conditions for all local invariances is:

$$\partial_\mu h^{\mu\nu}(x) = 0 \quad , \quad h^\mu_\mu(x) = 0 \quad ; \quad \partial_\mu A^\mu(x) = 0 \quad . \tag{2.20}$$

The first two conditions correspond to the coordinate and conformal symmetries respectively while the third to the gauge symmetry which $S_M[\phi, \psi, A_\mu]$ may have.

In the $1/N$ expansion, the quantities v_0^2/N and $g_i^2 N$, where g_i denote the matter coupling constants, are fixed as well. As a result, the free graviton propagator is given by (see Appendix A):

$$D_{\alpha\beta,\gamma\delta}(p) = \frac{i}{-p^4 + \frac{G^2 v_0^2}{24}\, p^2}\; P^{(2)}_{\alpha\beta,\gamma\delta}(p) \tag{2.21}$$

and can be decomposed into two terms:

$$D_{\alpha\beta,\gamma\delta}(p) = \frac{24}{G^2 v_0^2}\left[\frac{i}{p^2} - \frac{i}{p^2 - \frac{G^2 v_0^2}{24}}\right] P^{(2)}_{\alpha\beta,\gamma\delta}(p) \tag{2.22}$$

the first of which represents the massless spin-two graviton state while the second a massive spin-two state appearing in (2.22) with a negative sign. Since all physical states of a quantum theory must have positive-definite norm and energy, an appropriate causal prescription for shifting the poles of the propagator is needed. The massless part of $D_{\alpha\beta,\gamma\delta}(p)$ when replaced by:

$$\frac{i}{p^2 + i\epsilon} \tag{2.23}$$

leads to a good quantum state. However, when the massive part is changed to:

$$-\frac{i}{p^2 - \frac{G^2 v_0^2}{24} - i\epsilon} \tag{2.24a}$$

or

$$-\frac{i}{p^2 - \frac{G^2 v_0^2}{24} + i\epsilon} \tag{2.24b}$$

the corresponding physical state has either negative energy (2.24a), resulting in a perturbatively unstable theory, or negative norm (2.24b), resulting in a perturbatively nonunitary theory. Unless the massive pole is shown to be unphysical, Weyl's gravity will be perturbatively meaningless.

Our perturbation expansion is in powers of $1/N$ and, therefore, matter loop corrections to the free graviton propagator should be included as part of the leading order graviton propagator. Consider the renormalized contribution of N_S scalars to the one loop graviton self energy (fig. 2.1):

$$\Pi^{(S)}_{\alpha\beta,\gamma\delta}(p^2) = -\frac{i}{240}\,\frac{G^2 N_S}{(4\pi)^2}\; p^4\,\ell n\left(-\frac{p^2}{\mu^2}\right) P^{(2)}_{\alpha\beta,\gamma\delta}(p) \tag{2.25}$$

where dimensional regularization has been used with μ^2 as the subtraction point; the nonlogarithmic p^4-terms have been absorbed in the definition of μ^2. Analogous diagrams for fermions and vectors bosons are simple multiples of (2.25):

$$\Pi^{(V)}_{\alpha\beta,\gamma\delta}(p^2) = 2\Pi^{(F)}_{\alpha\beta,\gamma\delta}(p^2) = 12\Pi^{(S)}_{\alpha\beta,\gamma\delta}(p^2) \tag{2.26}$$

and the total contribution:

$$\Pi_{\alpha\beta,\gamma\delta}(p^2) = -\frac{i}{240}\,\frac{G^2}{(4\pi)^2}\,(N_S + 6N_F + 12N_V)\,p^4\,\ell n\left(-\frac{p^2}{\mu^2}\right) P^{(2)}_{\alpha\beta,\gamma\delta}(p) \tag{2.27}$$

is gauge invariant and, moreover, justifies the choice $N = N_S + 6N_F + 12N_V$ as the overall expansion parameter. The complete leading order $1/N$ graviton propagator $\bar{D}_{\alpha\beta,\gamma\delta}(p)$ is given by the sum of all graviton diagrams with an arbitrary number of one loop matter corrections (fig. 2.2). The sum of the resulting geometric series is:

$$\bar{D}_{\alpha\beta,\gamma\delta}(p) = \frac{iP^{(2)}_{\alpha\beta,\gamma\delta}(p)}{p^4\left(-1+\frac{G^2 v_0^2}{24p^2}\right)}\cdot\frac{1}{1-\frac{1}{240}\frac{G^2 N}{(4\pi)^2}\left(-1+\frac{G^2 v_0^2}{24p^2}\right)^{-1}\ell n\left(-\frac{p^2}{\mu^2}\right)} \tag{2.28}$$

and simplifies to:

$$\bar{D}_{\alpha\beta,\gamma\delta}(p) = \frac{iP^{(2)}_{\alpha\beta,\gamma\delta}(p)}{p^2\left[-p^2+\frac{G^2 v_0^2}{24}-\frac{1}{240}\frac{G^2 N}{(4\pi)^2}\,p^2\,\ell n\left(-\frac{p^2}{\mu^2}\right)\right]} \;. \tag{2.29}$$

Furthermore, the beta-function of the coupling constant G can be calculated to leading order in $1/N$:

$$\beta(G) = -\frac{1}{240}\frac{N}{(4\pi)^2}G^3 \tag{2.30}$$

and the theory is asymptotically free [10]. Each matter field gives a negative contribution to the beta-function in contrast to gauge theories where the opposite happens. Therefore, there exists a "QCD-like" renormalization group invariant scale Λ at which the coupling constant becomes strong:

$$\Lambda = \mu \exp\left\{-\frac{120(4\pi)^2}{G^2N}\right\} \quad . \tag{2.31}$$

In term of Λ, the propagator takes the form:

$$\bar{D}_{\alpha\beta,\gamma\delta}(p) = \frac{iP^{(2)}_{\alpha\beta,\gamma\delta}(p)}{p^2\left[\frac{G^2v_0^2}{24} - \frac{1}{240}\frac{G^2N}{(4\pi)^2}p^2\,\ell n\left(-\frac{p^2}{\Lambda^2}\right)\right]} \quad . \tag{2.32}$$

The benefits of the $1/N$ expansion are apparent. Weyl's theory, due to its asymptotic freedom, defines a well behaved nontrivial continuum quantum theory. This theory is studied by using a completely invariant expansion parameter so that the resulting perturbation series is insensitive to the strength of the coupling constant at all scales. The graviton propagator (2.32) possesses better ultraviolet convergence properties due to the presence of the logarithm. More importantly, the troublesome massive propagator pole may be avoided, again due to the logarithm. In Euclidean space, with signature (++++), the condition for the existence of such a pole is:

$$\frac{160\pi^2v_0^2}{\Lambda^2N} + x\,\ell n\,x = 0 \tag{2.33}$$

where the variable $x = \Lambda^{-2}p^2$ is always positive. Equation (2.33) has no real solutions provided (fig. 2.3):

$$\frac{160\pi^2v_0^2}{\Lambda^2N} > \frac{1}{e} \quad . \tag{2.34}$$

However, as Tomboulis showed [10], in the Minkowski complex p^2 plane a pair of complex conjugate poles exists on the physical sheet. Thus, these poles do not contribute to the absorptive part of the graviton propagator (2.32) and the theory is unitary to this order. In higher orders of the $1/N$ expansion unitarity violations may be avoided by using the Lee-Wick prescription [16] which is a well-defined but potentially problematic diagrammatic procedure [19].

Finally, we note that the presence of a scale Λ breaks the conformal symmetry. A true conformally invariant quantum theory would have a zero beta-function for its dimensionless coupling constant. Upon quantization of Weyl's gravity, however, infinities are encountered and a momentum subtraction is needed to absorb them. As a result, the theory acquires a scale Λ and a nonzero beta-function; conformal invariance has been broken explicitly. Suppose we could break the symmetry spontaneously and not explicitly by directly associating the scale Λ with the VEV of some scalar field. Then, the existence of a conformally invariant regularization scheme would imply that all the information extracted from the spontaneously broken theory can be preserved order by order in the perturbation.

3. Ward Identities and the Vanishing of the Cosmological Constant

Let us assume that a well-defined spontaneously broken Weyl invariant quantum theory exists. Then, the regularization and renormalization scheme will preserve its Ward identities (W-I's) a subset of which implies the vanishing of the induced cosmological constant term to all orders in the perturbation expansion. There is such a scheme [15] (see section 4) based on writing the theory in a Weyl invariant way in n dimensions where the conformal transformations on our fields take the form:

$$\delta g_{\mu\nu} = 2\Omega g_{\mu\nu} \tag{3.1a}$$

$$\delta\phi = -\frac{n-2}{2}\,\Omega(v_\phi + \phi) \tag{3.1b}$$

$$\delta\psi = -\frac{n-1}{2}\,\Omega\psi \tag{3.1c}$$

$$\delta A_\mu = 0 \tag{3.1d}$$

for the graviton and any spin 0, 1/2, 1 field. Moreover, assume that the presence of a nonzero scalar field VEV v_ϕ drives the spontaneous breaking of the conformal invariance.

Consider the n-dimensional Weyl invariant Lagrangian:

$$\mathcal{L}_{INV} \equiv \mathcal{L}_G(\eta_{\mu\nu} + Gh_{\mu\nu}, v_0 + \sigma) + \mathcal{L}_M(v_\phi + \phi, \psi, A_\mu) \tag{3.2}$$

with general linear gauge conditions (ς, Φ_μ) for the coordinate symmetry, (ξ, Φ) for the conformal and (α, Φ_A) for any gauge symmetry of the matter Lagrangian. The effective Lagrangian $\mathcal{L}$ consists of two parts:

$$\mathcal{L} = \mathcal{L}_{INV} + \mathcal{L}_{GF} \tag{3.3}$$

and possesses the following Becchi-Rouet-Stora (BRS) [20] symmetry:

$$\begin{aligned} sh_{\mu\nu} =& -[\partial_\mu c_\nu + \partial_\nu c_\mu + G(h_{\mu\rho}\partial_\nu c^\rho + h_{\nu\rho}\partial_\mu c^\rho + c^\rho\partial_\rho h_{\mu\nu})] \\ & + 2c(\eta_{\mu\nu} + Gh_{\mu\nu}) \end{aligned} \tag{3.4a}$$

$$s\phi = -Gc^\rho\partial_\rho\phi - \frac{n-2}{2}\,Gc(v_\phi + \phi) + s_A\phi \tag{3.4b}$$

$$s\psi = -Gc^\rho\partial_\rho\psi - \frac{n-1}{2}\,Gc\psi + s_A\psi \tag{3.4c}$$

$$sA_\mu = -G(A_\rho\partial_\mu c^\rho + c^\rho\partial_\rho A_\mu) + s_A A_\mu \tag{3.4d}$$

$$s\,\bar{c}_\mu = \frac{1}{\varsigma}\,\Phi_\mu \tag{3.4e}$$

$$sc_\mu = -Gc^\rho\partial_\rho c_\mu \tag{3.4f}$$

$$s\,\bar{c} = \frac{1}{\xi}\,\Phi \tag{3.4g}$$

$$sc = -\,Gc^{\rho}\partial_{\rho}c \tag{3.4h}$$

$$s\,\bar{c}_A = \frac{1}{\alpha}\,\Phi_A \tag{3.4i}$$

$$sc_A = -\,Gc^{\rho}\partial_{\rho}c_A + s_A c_A \tag{3.4j}$$

where (c_μ, c, c_A) represent the coordinate, conformal and gauge ghosts respectively, s_A is the internal gauge piece of the BRS invariance and all indices are raised and lowered with the Minkowski metric. Elementary inspection of (3.4) shows the BRS transformation acting on $h_{\mu\nu}$, ϕ, ψ, A_μ as the sum of a coordinate, conformal and internal transformation with infinitesimal parameter the respective ghost fields (c_μ, c, c_A). Furthermore, BRS tranformations leave the functional integral measure invariant, have unit functional Jacobian [9] and are nilpotent ($s^2 = 0$ except when acting on antighost fields). The part $\mathcal{L}_{GF}$ of the effective Lagrangian containing the gauge fixing and ghost interaction terms is given by:

$$\mathcal{L}_{GF} = \frac{1}{2\varsigma}\,\Phi_\mu\Phi^\mu + \frac{1}{2\xi}\,\Phi^2 + \frac{1}{2\alpha}\,\Phi_A^2 - \bar{c}^\mu\,s\Phi_\mu - \bar{c}\,s\Phi - \bar{c}_A\,s\Phi_A \quad . \tag{3.5}$$

in the presence of fermions, the vierbein must be introduced as the fundamental gravitational field since it transforms spinor into coordinate indices. Consequently, local Lorentz gauge fixing and ghost terms must be added leading to a straightforward generalization with no effect on our conclusions.

The generating functional $W[J_F, J_F^s]$ for connected Green's functions has a functional integral representation:

$$\exp\{iW[J_F, J_F^s]\} \equiv \int \left[\prod_F dF\right] \exp\left\{i\int d^n x\left(\mathcal{L} + \sum_F J_F\cdot F + \sum_{F\neq\bar{c}_a} J_F^s\cdot sF\right)\right\} \tag{3.6}$$

where J_F, J_F^s are the sources associated with the field F and its BRS transformation sF respectively. A Legendre transformation leads to the generating functional $\Gamma[\hat{F}, J_F^s]$ of one particle irreducible (1-PI) Green's functions:

$$\Gamma[\hat{F}, J_F^s] \equiv W[J_F, J_F^s] - \int d^n x \left\{ \sum_F J_F \cdot \hat{F} + \frac{1}{2\varsigma_a} \hat{\Phi}_a^2 \right\} \tag{3.7}$$

where the classical field $\hat{F}$ is defined by:

$$\hat{F} \equiv \frac{\delta}{\delta J_F} W[J_F, J_F^s] \tag{3.8}$$

and the subscript "a" is a generic index for all gauge fixing conditions. The equations of motion are:

$$\frac{\delta W}{\delta J_F^s} = \frac{\delta \Gamma}{\delta J_F^s} \tag{3.9a}$$

$$J_F = -\frac{\delta \Gamma}{\delta \hat{F}} - \frac{1}{\varsigma_a} \frac{\delta \hat{\Phi}_a}{\delta \hat{F}} \hat{\Phi}_a \quad ; \quad F \neq \bar{c}_a, c_a \tag{3.9b}$$

$$J_{c_a} = \frac{\delta \Gamma}{\delta \hat{\bar{c}}_a} \tag{3.9c}$$

$$J_{\bar{c}_a} = \frac{\delta \Gamma}{\delta \hat{c}_a} \quad . \tag{3.9d}$$

The Ward identities of a theory express its invariance under the BRS tranformations; only the source terms break the BRS invariance in the functional integral (3.6) and as a result the W-I's emerge:

$$\left\langle \sum_F J_F \cdot sF \right\rangle = 0 \tag{3.10}$$

where

$$\langle O \rangle \equiv \int \left[\prod_F dF \right] O \exp\left\{ i \int d^n x \left(\mathcal{L} + \sum_F J_F \cdot F + \sum_{F \neq \bar{c}_a} J_F^s \cdot sF \right) \right\} \tag{3.11}$$

for any operator O. Combining (3.10) with (3.9) and the antighost equation:

$$\left\langle \frac{\delta}{\delta \bar{c}_a} \right\rangle = 0 \tag{3.12}$$

the familiar form of the W-I's is reached:

$$\sum_{F\neq \bar{c}_a} \frac{\delta\Gamma}{\delta\hat{F}}\frac{\delta\Gamma}{\delta J^s_F}=0 \tag{3.13a}$$

$$\sum_{F\neq \bar{c}_b,c_b} \frac{\delta\hat{\Phi}_a}{\delta\hat{F}}\frac{\delta\Gamma}{\delta J^s_F}+\frac{\delta\Gamma}{\delta\hat{\bar{c}}_a}=0 \tag{3.13b}$$

Finally, we note the existence of a conserved quantity, the ghost number N_c:

$$N_c[\hat{c}_a]=1 \quad , \quad N_c[\hat{\bar{c}}_a]=-1 \quad , \quad N_c[J^s_{c_a}]=-2 \tag{3.14a}$$

$$N_c[\hat{F}]=0 \quad , \quad N_c[J^s_F]=-1 \quad \text{for} \quad F\neq \bar{c}_a\,,c_a \quad . \tag{3.14b}$$

Implications for the Cosmological Constant

Let us functionally differentiate (3.13a) with respect to the classical ghost field $\hat{c}$ of the conformal symmetry and then set all $\hat{F}$ and J^s_F equal to zero. By using ghost and fermion number conservation together with Lorentz invariance, we obtain:

$$\frac{\delta\Gamma}{\delta\hat{h}_{\mu\nu}}\frac{\delta^2\Gamma}{\delta J^s_{\mu\nu}\delta\hat{c}}+\sum_\phi \frac{\delta\Gamma}{\delta\hat{\phi}}\frac{\delta^2\Gamma}{\delta J^s_\phi\delta\hat{c}}=0 \tag{3.15}$$

evaluated at zero momentum. Furthermore, the BRS transformations (3.4a) and (3.4b) imply:

$$\frac{\delta^2\Gamma}{\delta J^s_{\mu\nu}\delta\hat{c}}=2(\eta_{\mu\nu}+X_{\mu\nu}) \ , \quad \text{with} \quad X_{\mu\nu}=\eta_{\mu\nu}X \tag{3.16a}$$

$$\frac{\delta^2\Gamma}{\delta J^s_\phi\delta\hat{c}}=-\frac{n-2}{2}\,G(v_\phi+Y) \tag{3.16b}$$

where $X_{\mu\nu}$ and Y are the one ghost matrix elements (1-PI) of the operators $Gch_{\mu\nu}$ and $c\phi$ respectively (see fig. 3.1). Thus, eq. (3.15) becomes:

$$(1+X)\eta_{\mu\nu}\frac{\delta\Gamma}{\delta\hat{h}_{\mu\nu}}=\frac{n-2}{4}\,G\sum_\phi(v_\phi+Y)\frac{\delta\Gamma}{\delta\hat{\phi}} \quad . \tag{3.17}$$

The action principle implies that, for all ϕ, the constant background v_ϕ should take the value minimizing the effective potential $V(\phi)$ of the theory. At this value, the scalar tadpoles $\delta\Gamma/\delta\,\hat{\phi}$ vanish and (3.17) automatically guarantees the simultaneous elimination of the graviton tadpoles $\delta\Gamma/\delta\,\hat{h}_{\mu\nu}$. Therefore, the cosmological constant is zero or, equivalently, Minkowski spacetime is stable to all orders in the perturbation expansion around the minimum of the scalar potential.

Any coordinate invariant metric theory coupled to matter contains graviton tadpoles $\delta\Gamma/\delta\,\hat{h}_{\mu\nu}$ which in general will diverge. As a result, a cosmological constant term $\Lambda_c \int d^4x\,\sqrt{-g}$ has to be introduced to eliminate these infinities and Minkowski spacetime will not be a solution of the equations of motion any more. To retain flat space as a solution, the bare cosmological constant must be fine tuned order by order to make its renormalized value equal to zero. One would expect the spontaneous breaking of the conformal invariance to induce both the Einstein and the undesirable cosmological term. However, the W-I (3.17) of the broken theory relates the graviton to the scalar tadpoles. The latter vanish at an extremum of the scalar effective potential order by order. Then, the W-I implies a zero cosmological constant without fine tuning.

In the Landau type gauges (2.20) the W-I (3.17) assumes a particularly simple form (see fig. 3.2):

$$\eta_{\mu\nu}\,\frac{\delta\Gamma}{\delta\,\hat{h}_{\mu\nu}} = \frac{n-2}{4}\,G\sum_{\phi} v_\phi\,\frac{\delta\Gamma}{\delta\,\hat{\phi}} \tag{3.18}$$

since the graviton propagator becomes transverse and traceless so that the relevant (see fig. 3.3) ghost-ghost-graviton vertex vanishes when contracted with propagators leading to $X = Y = 0$.

We conclude by noting the validity of the above results for both perturbative expansions: ordinary loop and $1/N$. The corresponding effective actions are related to each other by a proportionality factor N [10].

4. Regularization of the Theory Without Anomalies

Any renormalizable classically Weyl invariant local quantum field theory needs a renormalization scale which explicitly breaks the scale invariance and is the source of the conformal, or trace, anomalies [21]. In the context of dimensional regularization this scale μ is naturally introduced once any dimensionless coupling constant e acquires dimensions as the theory is continued to $n < 4$ spacetime dimensions:

$$e = e_0\, \mu^{(4-n)/2} \tag{4.1}$$

where e_0 is dimensionless. By replacing the scale μ with the unphysical dilaton scalar field ϕ_0 raised to an appropriate number of dimensions, a conformally invariant theory in n dimensions can be written when [15]:

$$e \;\rightarrow\; e_0\, \phi_0^{(4-n)/(n-2)} \quad . \tag{4.2}$$

However, the theory at the background solution $\phi_0 = 0$ is not analytic and a perturbation expansion exists only around $\phi_0 = v_0 \neq 0$ or, equivalently, when the conformal invariance is spontaneously broken [see eq. (2.19b)]:

$$e \;\rightarrow\; e_0\, (v_0 + \sigma)^{(4-n)/(n-2)} \quad . \tag{4.3}$$

As a result, the spontaneously broken theory appears to have two scales, v_0 and the subtraction point μ. This is clearly exhibited by substituting (4.3) with:

$$e \;\rightarrow\; e\left(1+\frac{\sigma}{v_0}\right)^{(4-n)/(n-2)} = e_0\, \mu^{(4-n)/2}\left(1+\frac{\sigma}{v_0}\right)^{(4-n)/(n-2)} \tag{4.4}$$

which we use from here on.[1] Nevertheless, the requirement of a stable perturbation expansion around $\sigma = 0$ will determine the ratio v_0/μ and prove the existence of a single scale parameter for the theory directly associated with the VEV v_0 of the scalar field ϕ_0 (see section 6).

[1]Since the Lagrangian (2.9) has a reflection symmetry $\phi_0 \rightarrow -\phi_0$, we can always choose ϕ_0 and, therefore, v_0 to be positive. Then, $1 + \frac{\sigma}{v_0} \geq 0$.

The general rule for constructing n-dimensional Lagrangian terms in a spontaneously broken Weyl invariant theory is quite simple. Start with any coordinate invariant theory of gravitons and matter fields. Introduce an unphysical scalar field σ, the dilaton, and perform conformal transformations [see eqs. (3.1)] with a σ-dependent parameter to all the dynamical variables:

$$g_{\mu\nu} \rightarrow g_{\mu\nu}\left(1+\frac{\sigma}{v_0}\right)^{4/(n-2)} \tag{4.5a}$$

$$\phi \rightarrow \phi\left(1+\frac{\sigma}{v_0}\right)^{-1} \tag{4.5b}$$

$$\psi \rightarrow \psi\left(1+\frac{\sigma}{v_0}\right)^{-(n-1)/(n-2)} \tag{4.5c}$$

$$A_\mu \rightarrow A_\mu \quad . \tag{4.5d}$$

The parameter has been chosen such that the transformed fields are conformally invariant when the transformation properties of the dilaton itself are taken into account. When the replacements (4.5) are effected in the coordinate invariant terms of the theory, they trivially become conformally invariant as well. Then, by making the dilaton a dynamical field we obtain a Weyl invariant theory of the metric, dilaton and matter fields. The "unitarity gauge" $\sigma = 0$ of the conformal symmetry naively reproduces the original coordinate invariant theory. This corresponds to performing the inverse transformations (4.5) to the fields as we have already seen in section 2 [see eq. (2.15)]. A list of all Weyl invariant Lagrangian terms containing up to four derivatives of the metric is the following:

(i) Pure gravity terms,

$$-\frac{1}{G^2}\sqrt{-g}\,C^2 \rightarrow -\frac{1}{G^2}\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{2\frac{n-4}{n-2}}\frac{n-2}{2(n-3)}\,C^2 \tag{4.6a}$$

$$\gamma\sqrt{-g}\,R^2 \rightarrow \gamma(\sqrt{-g}\,R^2)' \equiv \gamma\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{2\frac{n-4}{n-2}}$$
$$\times\left\{R-\frac{4(n-1)}{n-2}\left[\left(\frac{\partial_\mu\sigma}{v_0+\sigma}\right)^{;\mu}+g^{\mu\nu}\,\frac{(\partial_\mu\sigma)(\partial_\nu\sigma)}{(v_0+\sigma)^2}\right]\right\}^2 \tag{4.6b}$$

$$-v_0^2\,\sqrt{-g}\,R \;\rightarrow\; -\sqrt{-g}\left\{\frac{1}{2}\,g^{\mu\nu}\,(\partial_\mu\sigma)(\partial_\nu\sigma)+\frac{n-2}{8(n-1)}\,R(v_0+\sigma)^2\right\} \tag{4.6c}$$

$$\Lambda_c\sqrt{-g} \;\rightarrow \lambda_0\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{-2\frac{n-4}{n-2}}(v_0+\sigma)^4 \quad ; \quad \lambda_0\equiv\frac{\Lambda_c}{v_0^4} \quad . \tag{4.6d}$$

(ii) Scalar field terms,

$$\sqrt{-g}\left\{\frac{1}{2}\,g^{\mu\nu}\,(\partial_\mu\phi)(\partial_\nu\phi)+\frac{n-2}{8(n-1)}\,R\phi^2\right\}$$
$$\rightarrow\;\sqrt{-g}\left\{\frac{1}{2}\,g^{\mu\nu}\,(\partial_\mu\phi)(\partial_\nu\phi)+\frac{n-2}{8(n-1)}\,R\phi^2\right\} \tag{4.6e}$$

$$\beta\sqrt{-g}\,R\phi^2 \;\rightarrow \beta\,(\sqrt{-g}\,R\phi^2)'\equiv\beta\sqrt{-g}\,\phi^2$$
$$\times\left\{R-\frac{4(n-1)}{n-2}\left[\left(\frac{\partial_\mu\sigma}{v_0+\sigma}\right)^{;\mu}+g^{\mu\nu}\,\frac{(\partial_\mu\sigma)(\partial_\nu\sigma)}{(v_0+\sigma)^2}\right]\right\} \tag{4.6f}$$

$$-\lambda\sqrt{-g}\,\phi^4 \;\rightarrow\; -\lambda\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{-2\frac{n-4}{n-2}}\phi^4 \tag{4.6g}$$

$$-\frac{1}{2}\,m_\phi^2\sqrt{-g}\,\phi^2 \;\rightarrow\; -\frac{1}{2}\,\lambda'\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{-2\frac{n-4}{n-2}}(v_0+\sigma)^2\phi^2 \quad ;$$
$$\lambda'\equiv\frac{m_\phi^2}{v_0^2} \quad . \tag{4.6h}$$

(iii) Fermion terms,

$$iV\,\bar\psi\,\gamma_a\,V_\mu^a\,\nabla^\mu\psi \;\rightarrow iV\,\bar\psi\,\gamma_a\,V_\mu^a\,\nabla^\mu\psi \tag{4.6i}$$

$$-f\sqrt{-g}\,\bar\psi\,\phi\psi \;\rightarrow\; -f\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{-\frac{n-4}{n-2}}\bar\psi\,\phi\psi \tag{4.6j}$$

$$-m_\psi\sqrt{-g}\,\bar\psi\,\psi \;\rightarrow\; -f'\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{-\frac{n-4}{n-2}}\bar\psi\,(v_0+\sigma)\psi \;;\; f'\equiv\frac{m}{v_0} \;. \tag{4.6k}$$

(iv) Gauge boson terms,

$$-\frac{\sqrt{-g}}{4e^2}\,g^{\mu\nu}\,g^{\lambda\rho}\,Tr\,F_{\mu\lambda}\,F_{\nu\rho} \;\rightarrow\; -\frac{1}{4e^2}\sqrt{-g}\left(1+\frac{\sigma}{v_0}\right)^{2\frac{n-4}{n-2}}g^{\mu\nu}\,g^{\lambda\rho}\,Tr\,F_{\mu\lambda}\,F_{\nu\rho} \quad . \tag{4.6l}$$

In eq. (4.6i), $V_\mu^a(x)$ is the vierbein field with coordinate index μ, Lorentz index a and determinant $V(x)$, while ∇^μ is the covariant derivative formed out of the spin connection. Furthermore, $F_{\mu\nu}(x)$ is the nonabelian field strength with the trace acting on the gauge Lie algebra and G, $\gamma > 0$, β, λ, f, e are the various coupling constants. Finally, the n-dimensional conformally invariant Weyl tensor $C^\mu{}_{\alpha\beta\gamma}(x)$ is given by:

$$C^\mu{}_{\alpha\beta\gamma} \equiv R^\mu{}_{\alpha\beta\gamma} - \frac{1}{n-2}\left(\delta^\mu{}_\beta R_{\alpha\gamma} - \delta^\mu{}_\gamma R_{\alpha\beta} - g_{\alpha\beta} R^\mu{}_\gamma + g_{\alpha\gamma} R^\mu{}_\beta\right) - \frac{1}{(n-1)(n-2)} R\left(\delta^\mu{}_\gamma g_{\alpha\beta} - \delta^\mu{}_\beta g_{\alpha\gamma}\right) \tag{4.7a}$$

while its square:

$$C^2 = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - \frac{4}{n-2} R_{\mu\nu} R^{\mu\nu} + \frac{2}{(n-1)(n-2)} R^2 \quad . \tag{4.7b}$$

The passage from coordinate invariance to Weyl invariance is not equivalent to the inverse process. In a Weyl symmetric theory, the spontaneous breakdown of the conformal component provides a relation between the parameters of the theory and enables us to naturally achieve a zero cosmological constant. Thus, the "conformal unitarity gauge" of the Weyl theory is a coordinate invariant action containing one less parameter, namely no cosmological constant term. We shall see how this happens in section 6.

It is easy to verify the invariance of (4.6) under the set of the spontaneously broken conformal transformations (3.1). Therefore, the Ward identities of the regularized theory are preserved and no anomalies appear in the regularized quantities.[2] The renormalization should be done with a similar spirit. In contrast

[2]Since in two dimensions any scalar field becomes dimensionless it looks as if the theory could develop a singularity at $n = 2$ from the regularization factors [see eq. (4.6)]. However, by normalizing the coefficient of the $R\phi_0^2$ term to unity, the kinetic energy term for ϕ_0 is multiplied by $4(n-1)/(n-2)$ resulting in a dilaton propagator which vanishes at $n = 2$. Therefore, a smooth $n = 2$ limit exists for the dimensional continuation of the theory.

with gauge theories whose n-dimensional continuation has no explicit n dependence in the Lagrangian, Weyl invariant theories need such dependence. Thus, the naive minimal subtraction, which provides gauge invariant finite answers, does not give traceless finite results for the conformal symmetry. The correct algorithm consists of subtracting n-dimensional conformally invariant counterterms as an elementary example shows.

Consider the result of a regularized diagram in the theory:

$$\frac{1}{\epsilon}\, f(\epsilon)\,(n_{\mu\nu}\,p^2 - n p_\mu p_\nu) \tag{4.8}$$

which is traceless in n dimensions; the function $f(\epsilon)$ has a power series expansion $f(\epsilon) = f(0) + \epsilon f'(0) + \ldots$, where $\epsilon \equiv (4-n)/2$. If minimal subtraction is used, the pole part:

$$\frac{1}{\epsilon}\, f(0)\,(n_{\mu\nu}\,p^2 - 4 p_\mu p_\nu) \tag{4.9}$$

is traceless in 4 dimensions but the resulting finite part is not:

$$f'(0)\,(n_{\mu\nu}\,p^2 - 4p_\mu p_\nu) + 2f(0)\,p_\mu p_\nu \quad . \tag{4.10}$$

On the other hand, by subtracting the n-dimensional traceless counterterm:

$$\frac{1}{\epsilon}\, f(0)\,(n_{\mu\nu}\,p^2 - n p_\mu p_\nu) \tag{4.11}$$

the renormalized diagram:

$$f'(0)\,(n_{\mu\nu}\,p^2 - 4p_\mu p_\nu) \tag{4.12}$$

is traceless in 4 dimensions.

The above method of regularization and renormalization preserves the W-I's of the spontaneously broken Weyl invariant theory and trace anomalies do not arise as they cancel against the new vertices the regularization introduces. This has been shown in explicit examples in ref. [15]. In this context, it is interesting to study the relation between internal conformal transformations and dilatations, since the latter are the ultimate source of the anomalies.

In a Weyl invariant theory, a dilatation D with infinitesimal constant parameter ω is the sum of a coordinate transformation Δ with infinitesimal parameter $\omega^\mu(x) = \omega x^\mu$ (Killing vector) and a conformal transformation δ with infinitesimal parameter $\Omega(x) = \omega$: [3]

$$Dh_{\mu\nu} = -\omega\, x^\rho\, \partial_\rho\, h_{\mu\nu} \tag{4.13a}$$

$$D\sigma = -\omega \left[\frac{n-2}{2}\, v_0 + \left(\frac{n-2}{2} + x^\rho\, \partial_\rho\right) \sigma\right] \tag{4.13b}$$

$$D\psi = -\omega \left(\frac{n-1}{2} + x^\rho\, \partial_\rho\right) \psi \tag{4.13c}$$

$$DA_\mu = -\omega\, (1 + x^\rho\, \partial_\rho)\, A_\mu \tag{4.13d}$$

and is spontaneously broken since $v_0 \neq 0$. For simplicity, we assume that only ϕ_0 develops a VEV.

Under a dilatation, the effective Lagrangaian (3.3) is invariant up to a total derivative:

$$D\mathcal{L} = (n + x^\rho\, \partial_\rho)\, \mathcal{L} = \partial_\rho\, (x^\rho\, \mathcal{L}) \tag{4.14}$$

provided the gauge fixing terms Φ_a have dimensionality two. The W-I's obtained from dilatations take the form:

$$\left\langle \sum_F J_F \cdot DF \right\rangle = 0 \tag{4.15a}$$

or

$$\int d^n x \left\{ \sum_{F \neq \sigma} \frac{\delta\Gamma}{\delta \hat{F}} \left(d_F + x^\rho\, \partial_\rho\right) \hat{F} + \frac{\delta\Gamma}{\delta\, \hat{\sigma}} \left[d_\sigma\, v_0 + (d_\sigma + x^\rho\, \partial_\rho)\, \hat{\sigma}\right]\right\} = 0 \tag{4.15b}$$

where d_F is the canonical dimension of the field F and equals $\frac{n-2}{2}$, $\frac{n-1}{2}$, 1, 0 for spin 0, 1/2, 1, 2 fields respectively. When eq. (4.15b) is applied to a 1-PI Green's function $\Gamma^{(m)}$ with m external legs we get in momentum space:

$$\left[\sum_{i=1}^{m-1} p_i\, \frac{\partial}{\partial p_i} - n + \sum_F m_F\, d_F\right] \Gamma^{(m)}(p_j) + d_\sigma\, v_0\, \Gamma_\sigma^{(m+1)}(p_j) = 0 \tag{4.16}$$

---

[3] We thank S. Coleman for pointing this out.

where m_F is the number of external F fields ($m = \sum_F m_F$) and $\Gamma_\sigma^{(m+1)}$ is the same Green's function with $\Gamma^{(m)}$ except for the extra zero momentum external σ field it contains.

The renormalized Green's functions $\Gamma_R^{(m)}(p_j)$ in four dimensions satisfy:

$$\left(\sum_{i=1}^{m-1} p_i \frac{\partial}{\partial p_i} + v_0 \frac{\partial}{\partial v_0} + \mu \frac{\partial}{\partial \mu}\right) \Gamma_R^{(m)}(p_j) = d_m \Gamma_R^{(m)}(p_j) \tag{4.17}$$

where d_m is the dimension of $\Gamma_R^{(m)}$. Combining (4.16) with (4.17) we find:

$$\left(\mu \frac{\partial}{\partial \mu} + v_0 \frac{\partial}{\partial v_0}\right) \Gamma_R^{(m)} = v_0 \Gamma_{\sigma,R}^{(m+1)} \quad . \tag{4.18}$$

Recall that the scale parameters μ and v_0 are not independent (see section 6); eq. (4.18) becomes (see fig. 4.1):

$$\mu \frac{\partial}{\partial \mu} \Gamma_R^{(m)} = v_0 \Gamma_{\sigma,R}^{(m+1)} \tag{4.18$'$}$$

and corresponds to the low energy theorem for the dilaton field σ which is the Goldstone boson of the spontaneously broken dilatation invariance. Equation (4.18) does not contradict the renormalization group equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i \frac{\partial}{\partial g_i} + \sum_F m_F \gamma_F\right) \Gamma_R^{(m)} = 0 \tag{4.19}$$

where β_i is the beta function of the coupling constant g_i and γ_F the anomalous dimension of the field F. Let us verify the above claim with a simple example.

Example

Consider a non-abelian gauge field $A_\mu(x)$ interacting with fermions. We will check the validity of (4.18) for the self-energy $\Pi^{\mu\nu}$ of the gauge boson up to one loop. The relevant Lagrangian terms for the calculation are:

$$-\frac{1}{4} Z_A \left(1 + \frac{\sigma}{v_0}\right)^{2\frac{n-4}{n-2}} Tr\, F_{\mu\nu} F^{\mu\nu} + i Z_\psi \bar{\psi} \not{D} \psi \tag{4.20}$$

where D_μ is the gauge covariant derivative. There are five diagrams contributing to $\Pi^{\mu\nu}$ (see fig. 4.2) and the sum of the first four is given, in the Landau gauge, by:

$$\Pi_1^{\mu\nu} = ip^2\,\theta^{\mu\nu}\left[-1 + e^2\,\frac{1}{\epsilon}\,f(\epsilon)\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\right] \tag{4.21}$$

where $\theta^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$ and

$$f(\epsilon) = \frac{1}{(4\pi)^{2-\epsilon}}\,\Gamma(1+\epsilon)\,B(1-\epsilon,1-\epsilon)\left[\frac{26-33\epsilon+14\epsilon^2}{4(3-2\epsilon)}\,C_A - \frac{4(1-\epsilon)}{3-2\epsilon}\,T_\psi\right] \tag{4.22}$$

with C_A and T_ψ the group Casimirs for the gauge bosons and fermions. The fifth diagram (4.1e) is the one loop counterterm:

$$\Pi_2^{\mu\nu} = -ie^2\,\frac{1}{\epsilon}\,z(\epsilon)\,p^2\,\theta^{\mu\nu} \tag{4.23}$$

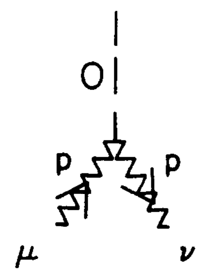
where

$$Z_A = 1 + e^2\,\frac{1}{\epsilon}\,z(\epsilon) \tag{4.24}$$

such that $z(0) = f(0)$. Therefore, the renormalized self-energy $\Pi_R^{\mu\nu}$ is:

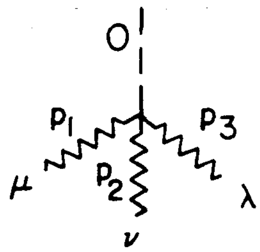
$$\begin{aligned}\Pi_R^{\mu\nu} &= \Pi_1^{\mu\nu} + \Pi_2^{\mu\nu} \\ &= ip^2\,\theta^{\mu\nu}\left[-1 + e^2\left(-f(0)\,\ell n\left(-\frac{p^2}{\mu^2}\right) + f'(0) - z'(0)\right)\right] \quad .\end{aligned} \tag{4.25}$$

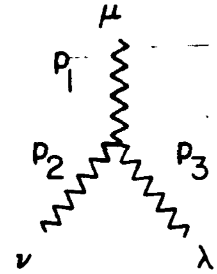
There are three diagrams contributing to $\Pi_\sigma^{\mu\nu}$ (see fig. 4.3). The two new vertices are:



$$= -2\,\frac{i}{v_0}\,\frac{n-4}{n-2}\,Z_A\,p^2\,\theta^{\mu\nu} \tag{4.26a}$$

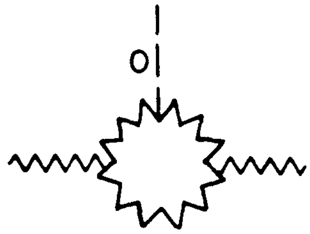
and

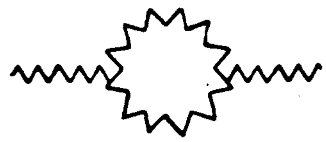


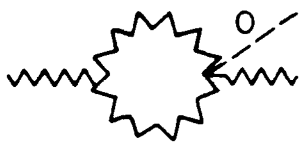
$$= \frac{2}{v_0}\,\frac{n-4}{n-2}\,Z_A$$


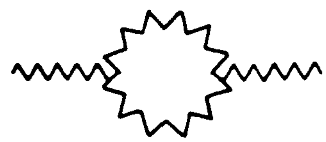
(4.26b)

It is easy to check, up to order e^2, the identities:



$$= -\frac{4}{v_0}\,\frac{n-4}{n-2}$$

(4.27a)



$$= \frac{4}{v_0}\,\frac{n-4}{n-2}$$

(4.27b)

so that the sum of the last two contributions to $\Pi_\sigma^{\mu\nu}$ is zero. Thus, the first diagram gives the full answer:

$$\Pi_{\sigma,R}^{\mu\nu} = \frac{2i}{v_0}\, p^2\, \theta^{\mu\nu}\, e^2\, z(0) \tag{4.28}$$

and (4.18) is trivially verified when we use $f(0) = z(0) = \frac{1}{(4\pi)^2}\left(\frac{13}{6}\, C_A - \frac{4}{3}\, T_\psi\right)$.

This simple example shows the kind of inconsistency arising when the above regularization scheme is not used: there are no new vertices (4.26) and $\Pi_\sigma^{\mu\nu} = 0$ which does not verify the W-I (4.18).

The $v_0 = 0$ limit of the theory does not exist in perturbation theory reflecting the nonanalyticity at $\phi_0 = 0$. We shall see in section 6 that this is due to the existence of a nonvanishing beta-function which forces v_0 to be nonzero and the conformal symmetry to be broken spontaneously. The symmetric limit can be approached at extremely high energies only for an asymptotically free theory. However, the limit could be reached exactly if some unified theory was finite.

Finally, in our theory the dilatation transformations $\bar{D} = \bar{\Delta} + \delta$, given by (4.13), define a conserved current $J^{\mu}(x)$. The entire contribution to the current comes from the coordinate part Δ and J^{μ} has the form:

$$J^{\mu}(x) = x_{\nu}\Theta^{\mu\nu}(x) \tag{4.29}$$

where $\Theta^{\mu\nu}$ is the energy momentum tensor. The conservation of the dilatation current implies that:

$$\Theta^{\mu}{}_{\mu}(x) = 0 \quad . \tag{4.30}$$

In the Minkowski spacetime limit, $\Theta^{\mu\nu}$ becomes the improved energy momentum tensor [22].

5. Renormalization

The perturbative renormalizability of gravitational actions including terms quadratic in the curvature tensor has been demonstrated by Stelle [9]. More precisely, it was shown that in a class of linear gauges coordinate invariant as well as coordinate noninvariant divergences appear. The invariant infinities can be absorbed by redefining the parameters and fields of the bare action while the noninvariant infinities require nonlinear renormalizations of the gravitational and ghost fields (and of their BRS transformations). However, in the Landau-type gauge or a class of linear gauges containing more than two derivatives, the coordinate variant divergences disappear and the renormalization procedure is considerably simplified.

The above results are not directly applicable to a Weyl invariant theory due to the presence of local conformal invariance. The spontaneous breakdown of the latter and the particular regularization scheme required present additional departures from the analysis of Stelle. Therefore, we carry the renormalization program for a Weyl symmetric gravitational action along the same general lines with ref. [9] and prove the renormalizability of the theory.

It is adequate to consider a matter Lagrangian with N scalar fields ϕ; any renormalizable Weyl invariant matter coupling gives the same results. The classes of coordinate and conformal gauges we use to study the renormalization are:

$$\Phi^{\mu} = \partial_{\nu} h^{\mu\nu} - \frac{1}{n} \partial^{\mu} h^{\nu}_{\nu} \tag{5.1a}$$

with parameter ς, and

$$\Phi = \frac{1}{2(n-1)} \left(\Box h^{\mu}_{\mu} - \partial_{\mu} \partial_{\nu} h^{\mu\nu}\right) \tag{5.1b}$$

with parameter ξ, respectively. These gauge conditions force the ghost propagator matrix of the theory to be diagonal to lowest order (see Appendix A). Further restrictions on (5.1) lead to a set of gauge fixing conditions in which one expects, in analogy with ref. [9], that the renormalization becomes simple. The restrictions consist of taking $\varsigma = 0$ and introducing extra derivatives in the conformal gauge fixing term:

$$\mathcal{L}_{GF} = \lim_{\varsigma \to 0} \frac{1}{2\varsigma} \Phi_{\mu} \Phi^{\mu} + \frac{1}{2\xi} \Phi \frac{\Box^2}{m^4} \Phi - \bar{c}^{\mu} s\Phi_{\mu} - \bar{c}\, s\Phi \tag{5.2}$$

where the dimensionful constant m ensures that ξ stays dimensionless. Notice that when $\xi = 0$ as well, we recover the simple covariant gauges (2.20) which provide a transverse traceless graviton propagator.

The effective Lagrangian (3.3) with (5.2) as its gauge fixing part, is still BRS invariant provided (3.4g) changes to:

$$s\bar{c} = \frac{1}{\xi} \Box^2 \Phi \quad . \tag{5.3}$$

The coordinate ghost term contributes:

$$\begin{aligned} -\bar{c}^{\mu} s\Phi_{\mu} = {} & \bar{c}^{\mu} \left(\Box \eta_{\mu\nu} + \frac{n-2}{n} \partial_{\mu}\partial_{\nu}\right) c^{\nu} + G\bar{c}^{\mu}\Big[\partial_{\nu}\Big(h^{\nu}_{\rho}\, \partial_{\mu} c^{\rho} + h_{\mu\rho} \partial^{\nu} c^{\rho} + c^{\rho} \partial_{\rho} h^{\nu}_{\mu}\Big) \\ & - \frac{1}{n} \partial_{\mu}\Big(2h_{\nu\rho} \partial^{\nu} c^{\rho} + c^{\rho} \partial_{\rho} h^{\nu}_{\nu}\Big)\Big] + 2G\bar{c}^{\mu}\Big[-\partial^{\nu}\Big(h_{\mu\nu} c\Big) + \frac{1}{n} \partial_{\mu}\Big(h^{\nu}_{\nu} c\Big)\Big] \end{aligned} \tag{5.4a}$$

and the conformal:

$$-\bar{c}\, s\Phi = -\bar{c}\Box c + \frac{1}{2(n-1)}\,\bar{c}\Big[\Box\Big(2h_{\mu\nu}\,\partial^\mu c^\nu + c^\mu\,\partial_\mu h^\nu_\nu\Big) -$$

$$-\,\partial_\mu\partial_\nu\Big(2h^\nu_\rho\,\partial^\mu c^\rho + c^\rho\,\partial_\rho h^{\mu\nu}\Big)\Big] + \frac{1}{n-1}\,\bar{c}\Big[\partial_\mu\partial_\nu\Big(h^{\mu\nu}c\Big) - \Box\Big(h^\mu_\mu c\Big)\Big] \quad . \tag{5.4b}$$

Before analyzing the ultraviolet behavior of the various propagators, which are derived in Appendix A for the gauges (5.1), we note the presence of a nondiagonal graviton-dilaton free propagator. The ξ-dependent part of all propagators behaves like $1/k^8$, as $k \to \infty$, while the ξ-independent pieces of the graviton and dilaton propagators behave like $1/(k^4\,\ell n\,k^2)$ and $1/k^4$ respectively.

The renormalization of the theory is most easily proved by using its BRS invariance which is preserved by the regularization scheme (see section 4). Then, the Ward identities (3.13) are satisfied by the regularized 1-PI Green's functions. We wish to show, by induction, that the renormalized 1-PI functions satisfy the W-I's as well with the appropriate redefinitions of the bare parameters and fields only. It is obvious that the W-I's are verified to lowest order in perturbation theory. Assume this to be true to k^{th} order and try to prove the assertion for the $(k+1)$ order. Thus, we consider the superficial degree of divergence D of a general 1-PI diagram.

Let E_F be the number of external lines of the field F; $I_{F_1F_2}$ the number of internal propagators F_1F_2; $V_{F_1F_2F_3}$ the number of vertices with fields $F_1F_2F_3$; V_h^4 and V_h^2 the number of graviton vertices with four and two derivatives; V_σ^4, V_σ^2 and V_σ the number of dilaton vertices with four, two and zero derivatives and an arbitrary number of gravitons; $V^2_{2\phi}$, $V_{2\phi}$, $V_{4\phi}$ the number of vertices possessing the indicated number of scalar fields and derivatives as well as arbitrary numbers of gravitons and dilatons; O_{Fc_a} the number of insertions of the operator which is the nonlinear part of sF [see eqs. (3.4)]; L the number of loops. In terms of the above quantities:

$$D = 4L + (4+\ell)V_h^4 + (2+\ell)V_h^2 + 4V_\sigma^4 + 2V_\sigma^2 + 2V_{2\phi}^2 + 2V_{\bar{c}_\mu h c_\nu} + V_{\bar{c}_\mu h c}$$
$$+ 3V_{\bar{c}hc_\mu} + 2V_{\bar{c}hc} - (4+\ell)I_{hh} - 8I_{h\sigma} - 4I_{\sigma\sigma} - 2I_{\phi\phi} - 2I_{\bar{c}_a c_a} \tag{5.5}$$
$$+ O_{c_\mu c_\nu} + O_{cc_\mu} + O_{hc_\mu} + O_{\phi c_\mu} + O_{\sigma c_\mu} - E_{\bar{c}_\mu} - 2E_{\bar{c}}$$

where ℓ represents the extra convergence of the $\ell n\, k^2$ factor in the $1/N$ graviton propagator and the last two terms reflect the action of at least one derivative on $\bar{c}_\mu$ and two on $\bar{c}$ [see eqs. (5.4)]. The well-known formula:

$$L = \sum_i I_i - \sum_i V_i - \sum_i O_i + 1 \tag{5.6}$$

implies:

$$D = 4 - \ell(I_{hh} - V_h^4) - (2-\ell)V_h^2 - 2V_\sigma^2 - 4V_\sigma - 2V_{2\phi}^2 - 4V_{2\phi} - 4V_{4\phi}$$
$$- 2V_{\bar{c}_\mu h c_\nu} - 3V_{\bar{c}_\mu h c} - V_{\bar{c}hc_\mu} - 2V_{\bar{c}hc} - 4I_{h\sigma} + 2I_{\phi\phi} + 2I_{\bar{c}_a c_a} - 3O_{c_\mu c_\nu}$$
$$- 3O_{cc_\mu} - 3O_{hc_\mu} - 4O_{hc} - 3O_{\phi c_\mu} - 3O_{\sigma c_\mu} - 4O_{\phi c} - 4O_{\sigma c} - E_{\bar{c}_\mu} - 2E_{\bar{c}} \tag{5.7}$$

Finally, using the topological relations for the conservation of σ, ϕ, $\bar{c}_\mu$, $\bar{c}+c$ field lines:

$$2V_{2\phi}^2 + 2V_{2\phi} + 4V_{4\phi} + O_{\phi c_\mu} + O_{\phi c} = 2I_{\phi\phi} + E_\phi \tag{5.8a}$$

$$V_{\bar{c}_\mu h c_\mu} + V_{\bar{c}_\mu h c} = I_{\bar{c}_\mu c_\nu} + E_{\bar{c}_\mu} \tag{5.8b}$$

$$2V_{\bar{c}hc} + V_{\bar{c}_\mu hc} + V_{\bar{c}hc_\mu} + O_{cc_\mu} + O_{hc} + O_{\phi c} + O_{\sigma c} = 2I_{\bar{c}c} + E_{\bar{c}} + E_c \tag{5.8c}$$

the superficial degree of divergence D becomes:

$$D = 4 - \ell(I_{hh} - V_h^4) - (2-\ell)V_h^2 - 2V_\sigma^2 - 4V_\sigma - 2V_{2\phi} - 4I_{h\sigma} - E_\phi$$
$$- 3E_{\bar{c}_\mu} - 3E_{\bar{c}} - E_c - 3O_{c_\mu c_\nu} - 2O_{cc_\mu} - 3O_{hc_a} - 2O_{\phi c_a} - 3O_{\sigma c_a} \quad . \tag{5.9}$$

The divergent part of any diagram has a ξ-independent piece, which is given by calculating in the $\xi = 0$ gauge, and a ξ-dependent one. But when the $\varsigma =$

$\xi = 0$ gauge conditions are used, inspection of (5.4) shows that at least one derivative acts on the ghost fields c_a and the degree of divergence (5.9) becomes $D - E_{c_\mu} - E_c$. Also, the ξ-dependent part of the propagators, which is responsible for the ξ-dependent divergences, behaves like $1/k^8$ in the ultraviolet and either lowers D by four, when at least one graviton or dilaton propagator is used, or in (5.9) $I_{h\sigma} \geq 1$ when at least one graviton-dilaton propagator is used.

As a result, all divergent diagrams ($D \geq 0$) involving external ghost lines are given in fig. 5.1 and have zero degree of divergence. Certain diagrams with an odd number of external dilaton lines are excluded since either V_σ^2 or $I_{h\sigma}$ is different than zero in (5.9). Moreover, all these diagrams of fig. 5.1 have ξ-independent divergences. Adding an arbitrary number of external gravitons and dilatons does not change D, leads to an infinity of divergent graphs and causes nonlinear renormalizations of the fields and the operators. The conclusions of this paragraph are valid in the ordinary loop expansion [$\ell = 0$ in (5.9)].

In the $1/N$ expansion the ξ-independent piece of the graviton propagator, which is the only relevant part for the divergences of fig. 5.1, has better ultraviolet behavior than the ordinary one. Because of (5.4) all divergent diagrams involving external ghost lines satisfy $I_{hh} > V_h^4$ and become finite using the $1/N$ graviton propagator when two or more internal graviton lines are present (see Appendix C). The absence of ghosts-dilaton couplings forces these diagrams to contain at least one graviton propagator except for the diagram of the operator $O_{\sigma c_a}$ which has at least two such propagators and hence always converges. Furthermore, the degree of divergence D in fig. 5.1 remains unchanged by adding arbitrary numbers of gravitons but becomes negative when even one dilaton line is attached. As a result, nonlinear redefinitions of $h_{\mu\nu}$, c_a, $sh_{\mu\nu}$, sc_a, $s\phi$ are needed to absorb the infinities. The only remaining divergences are those of L_{INV} which can be shown [9] to be independent of the gauge parameter. In fig. 5.2, the divergent diagrams involving scalar field external lines are shown with their degree of divergence which is unaffected by the addition of arbitrary numbers of gravitons and dilaton lines. In the $1/N$ expansion, the infinities of the scalar sector come from dilaton, scalar and one graviton internal lines.

If the connection with the simple covariant gauges (2.20) and the resulting convenience of a transverse traceless graviton propagator is sacrificed, there exists a class of gauges where the renormalization program becomes very simple even in the ordinary loop expansion. In $\mathcal{L}_{GF}$, given by (5.2), instead of (5.1) use:

$$\Phi^{\mu} = -\partial_{\nu} H^{\mu\nu} - \frac{2}{Gv_0}\, \partial^{\mu}\sigma \tag{5.1a$'$}$$

$$\Phi = \frac{2}{n-2}\, \frac{1}{Gv_0}\, \Box\sigma \tag{5.1b$'$}$$

where the perturbation expansion is in terms of the fluctuating field $H^{\mu\nu}(x)$ defined by:

$$\sqrt{-g(x)}\; g^{\mu\nu}(x) \equiv \eta^{\mu\nu} + GH^{\mu\nu}(x) \quad . \tag{2.19a$'$}$$

In these gauges, all graphs involving external ghost lines become convergent since, in the $\varsigma = \xi = 0$ limit, inspection of (5.2) shows that two derivatives always act on c_{μ} [9] and one on c modifying the degree of divergence (5.9) to $\mathcal{D} - 2E_{c_{\mu}} - E_c$. Thus, the theory is renormalized by simple ξ-independent redefinitions of the parameters and fields of the Weyl invariant part, $\mathcal{L}_{INV}$, of the effective Lagrangian $\mathcal{L}$. The renormalized Lagrangian is:

$$\mathcal{L}_R = \mathcal{L}_G + \mathcal{L}_M + \mathcal{L}_{GF} \tag{5.10a}$$

where

$$\begin{aligned} \mathcal{L}_G = &- \frac{Z_G}{G^2}\, \frac{n-2}{2(n-3)} \left(1 + \frac{\sigma}{v_0}\right)^{2\frac{n-4}{n-2}} \sqrt{-g}\; C^2 + \frac{(n-2)^2}{32(n-1)^2}\, Z_{\gamma}\, \gamma \left(\sqrt{-g}\; R^2\right)' \\ &- Z_{\sigma} \sqrt{-g} \left[\frac{1}{2}\, g^{\mu\nu}(\partial_{\mu}\sigma)(\partial_{\nu}\sigma) + \frac{n-2}{8(n-1)}\, R(v_0 + \sigma)^2\right] \\ &+ Z_{\lambda_0} \lambda_0 \left(1 + \frac{\sigma}{v_0}\right)^{-2\frac{n-4}{n-2}} \sqrt{-g}\; (v_0 + \sigma)^4 \end{aligned} \tag{5.10b}$$

and $\mathcal{L}_{GF}$ remains unchanged [see eq. (5.2)] while $\mathcal{L}_M$ acquires its usual counterterms.

In the "conformal unitarity gauge" of (5.10) we obtain the most general higher derivative gravitational action with zero cosmological constant. Since the latter is renormalizable as well the infinite parts of the various counterterms should be the same in both gauges. Furthermore, the metric field $g_{\mu\nu}$ does not get renormalized. The fluctuating field $H_{\mu\nu}$, defined in $(2.19a)'$, acquires a Z_H such that $GH_{\mu\nu}$ stays unrenormalized. However, by using different definitions for $h_{\mu\nu}$, for example $\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} + \frac{1}{\sqrt{N}} H^{\mu\nu\prime}$, the fluctuating field remains unrenormalized by itself.

We conclude by observing that although the conformal gauge fixing term (5.2) makes the renormalization program simpler, it introduces ξ-dependent artificial infrared divergences which can be avoided by an appropriate change of the gauge fixing term $\frac{1}{2\xi}\, \Phi \frac{\Box^2}{m^4} \Phi$ to $\frac{1}{2\xi}\, \Phi \frac{(\Box + m^2)^3}{\Box m^4} \Phi$. The only effect of the modified term is to replace the ξ-dependent part of the propagators (see Appendix A):

$$\xi\, \frac{m^4}{k^8} \to \xi\, \frac{m^4}{k^2(k^2 - m^2)^3} \quad . \tag{5.11}$$

All invariant quantities are ξ and m independent.

6. Realization of the Spontaneous Breakdown of Conformal Invariance

We consider the simplest framework for studying the spontaneous breaking of the conformal symmetry; the most general one will be discussed later in this section. This is achieved when only the field ϕ_0 can potentially receive a VEV. Thus, the matter Lagrangian should contain no scalar fields since they mix with ϕ_0 and could develop VEV's. The only possible remaining ϕ_0-matter mixing term is the Yukawa coupling $\bar{\psi}\, \phi_0 \psi$ which for simplicity, is excluded by using chiral invariance; as we shall see its presence does not affect any results.

The nontrivial beta function (2.30) of the asymptotically free theory (5.10) implies the existence of a renormalization group invariant scale Λ and signals the breakdown of the conformal invariance. By using an appropriate regularization scheme (see section 4) the W-I's of the theory are preserved and the breakdown is spontaneous and due to the VEV v_0 of the dilaton field. By perturbing the theory around a solution of the equations of motion the ratio v_0/Λ is determined, order by order in the perturbation, and the theory has only one scale as well as vanishing cosmological constant. We show this by explicitly calculating the effective potential of the dilaton to leading order in $1/N$. For the calculation we will use Landau type gauges (2.20) in $\mathcal{L}_{GF}$.

To leading order in $1/N$, where $G^2N = \gamma/N = \lambda_0 N =$ fixed, in the Lagrangian (5.10) only the gravitational coupling constant G is renormalized and $Z_\gamma = Z_{\lambda_0} = Z_\sigma = 1$; this is exhibited in Appendix B where the relevant Feynman rules are given. Therefore, the calculations presented in section 2 are valid to this order. Moreover, to study the breaking of the conformal invariance to first order in $1/N$, it is consistent to assume $\lambda_0 N = \gamma/N = 0$. Although the effective potential $V_{eff}[\sigma]$ is zero to tree order, the leading $1/N$ corrections are calculable, finite and demonstrate the spontaneous breakdown of the conformal symmetry.

The diagrams contributing to $V_{eff}[\sigma]$ are given in fig. 6.1. Let V_ν represent a vertex with ν σ-fields at zero momentum and two gravitons. Then, a general such diagram $\Gamma_{n;n_j}$ with $n = n_1 + \ldots + n_j + \ldots + n_\nu$ external σ legs, where n_j is the number of V_j vertices present, equals:

$$\Gamma_{n;n_j} = \int \frac{d^n k}{(2\pi)^n} \frac{1}{n_1!} \cdots \frac{1}{n_\nu!}\, 2^{n-1}(n-1)!\, (\sigma V_1)^{n_1} \ldots (\sigma^\nu V_\nu)^{n_\nu}\; tr\, \bar{D}^n \qquad (6.1)$$

where the symmetry factors of the diagram have been included and the trace is over the indices of the graviton propagator $\bar{D}_{\mu\nu,\lambda\rho}$ [see eq. (2.32)]. The effective potential is:

$$V_{eff}[\sigma] = i \sum_{n=1}^{\infty} \sum_{\substack{n_j \\ n_1+\ldots+n_\nu = n}} \Gamma_{n;n_j} \qquad (6.2a)$$

$$= -\frac{5i}{2} \int \frac{d^n k}{(2\pi)^n}\, \ell n \left[1 - 2 \sum_{j=1}^{\infty} (\sigma^j V_j) \bar{D}\right] \qquad (6.2b)$$

where:

$$\bar{D} = \frac{i}{k^2\left[\frac{G^2 v_0^2}{24} - \frac{1}{240}\frac{G^2 N}{(4\pi)^2} k^2 \ell n\left(-\frac{k^2}{\Lambda^2}\right)\right]} \tag{6.3a}$$

and:

$$\sum_{j=1}^{\infty} (\sigma^j V_j) = \frac{iG^2}{48}\left[\frac{1}{5}\frac{N}{(4\pi)^2} k^4 \ell n\left(1+\frac{\sigma}{v_0}\right) + k^2(\sigma^2 + 2v_0\sigma)\right] \quad . \tag{6.3b}$$

The logarithmic term in (6.3b) comes from the cancellation of the $\frac{1}{n-4}$ pole of Z_G and the vertices proportional to $(n-4)$; the Feynman rules and the counterterm Z_G are given in Appendix B. By substituting (6.3) in (6.2b), we obtain:

$$V_{eff}[\sigma] = -\frac{5i}{2}\int \frac{d^n k}{(2\pi)^n}\, \ell n\left[1 + 2\,\frac{k^2 \ell n(1+\frac{\sigma}{v_0}) + \frac{80\pi^2}{N}(\sigma^2 + 2v_0\sigma)}{\frac{160\pi^2}{N} v_o^2 - k^2 \ell n\left(-\frac{k^2}{\Lambda^2}\right)}\right] \quad . \tag{6.4}$$

Define the variables:

$$\alpha \equiv \frac{160\pi^2}{N}\frac{v_0^2}{\Lambda^2} \quad , \quad \beta \equiv \left(1+\frac{\sigma}{v_0}\right)^2 \quad , \quad x \equiv \frac{k^2}{\Lambda^2} \tag{6.5}$$

and perform the integration in Euclidean space:

$$V_{eff}[\sigma] = \frac{5\Lambda^4}{32\pi^2}\int_o^\infty dx\; x^{1-\epsilon}\, \ell n\left[\beta\,\frac{\alpha + \frac{x}{\beta}\,\ell n\,\frac{x}{\beta}}{\alpha + x\,\ell n\, x}\right] \tag{6.6}$$

where the angular integrations have been done.

We are interested in finding the value of v_0 at which the effective potential has a minimum for $\sigma = 0$, since σ is the translated field. The necessary and sufficient condition for which $V_{eff}[\sigma]$ has no imaginary part is $\alpha > \frac{1}{e}$ and this inequality, which is identical to the unitarity condition (2.34), guarantees a stable minimum, provided one exists. It is amusing to observe that had we used a positive kinetic energy for ϕ_0 in our action, the variable α in equation (6.6) would have to be replaced by $-\alpha$ and the condition for lack of imaginary part would become $\alpha < -\frac{1}{e}$ forcing v_0 and, consequently, σ to be imaginary and us to change its kinetic energy sign. In the region $\alpha > \frac{1}{e}$, where (6.6) is real, the remaining integration can be done using methods of complex analysis and dimensional continuation and gives a finite answer (see Appendix C). The latter justifies why, except for the radial integration measure, we can use four dimensional quantities in calculating the effective potential.

The final answer is:

$$V_{eff}[\sigma] = \frac{5}{32\pi}\Lambda^4\left[\left(1+\frac{\sigma}{v_0}\right)^4 - 1\right]\alpha^2 \sum_{j=0}^{\infty} \frac{\sin 2\vartheta_j \sin^2\vartheta_j}{(\vartheta_j + 2\pi j)^2} \quad ; \quad 0 \le \vartheta_j \le \pi \qquad (6.7a)$$

$$\alpha = \frac{\vartheta_j + 2\pi j}{\sin\vartheta_j}\, e^{-\frac{\vartheta_j + 2\pi j}{\tan\vartheta_j}} \quad . \qquad (6.7b)$$

At the minimum for $\sigma = 0$ the first derivative vanishes (see Appendix C):

$$\left.\frac{d\,V_{eff}[\sigma]}{d\sigma}\right|_{\sigma=0} = 0 \;\Rightarrow\; \alpha \simeq 1.62 \qquad (6.8a)$$

$$v_0^2 \simeq \frac{N}{975}\Lambda^2 \quad . \qquad (6.8b)$$

Therefore, as claimed, eq. (6.8b) shows that the theory (5.10) has a single scale parameter v_0 such that $v_0/\sqrt{N}$, as required by a consistent $1/N$ expansion, is fixed. In terms of the Planck mass $M_{P\ell}$:

$$v_0 = \sqrt{\frac{3}{4\pi}}\, M_{P\ell} \sim\ .49\, M_{P\ell} \quad , \qquad (6.9a)$$

$$\Lambda^2 \simeq \frac{233}{N}\, M_{P\ell}^2 \quad . \qquad (6.9b)$$

At extremely high energies we have an asymptotically free Weyl invariant theory. Using the presently known number of fundamental matter particles, $233/N$ is almost unity so that close to the Planck mass, as eqs. (6.9) reveal, the gravitational coupling constant G becomes strong, the spontaneous breaking of the conformal symmetry is driven by v_0 and Einstein's term is induced; Planck's mass is the natural mass scale of the theory. As N increases, however, the mass $M_{P\ell}$ becomes larger than the scale Λ.

Having found v_0/Λ, we can check to one loop the vanishing of the cosmological constant as predicted by the W-I (3.18). The graviton tadpole $t_{\mu\nu}$ (see fig. 6.2 and Appendix B for Feynman rules) is given by:

$$t_{\mu\nu} = -\frac{5i}{64\pi} G\Lambda^4 \alpha^2 \sum_{j=0}^{\infty} \frac{\sin 2\vartheta_j \sin^2\vartheta_j}{(\vartheta_j + 2\pi j)^2} \eta_{\mu\nu} \tag{6.10}$$

and verifies (3.18):

$$\eta^{\mu\nu} t_{\mu\nu} = \frac{1}{2} v_0 G \left(-i \left. \frac{dV_{eff}[\sigma]}{d\sigma} \right|_{\sigma=0} \right) \quad . \tag{6.11}$$

For the solution (6.8), $t_{\mu\nu}$ vanishes and so does the cosmological constant to this order.

The most general renormalizable Weyl invariant action requires the introduction of a four derivative term for the scalar field σ [see eq. (4.6b)]. Therefore, one expects a doubling of the degrees of freedom and, by inspecting the pole structure of the σ-propagator [see Appendix A, eq. (A.9c)], infers the presence of a massless ghost, the dilaton, and a physical scalar particle with mass v_0^2/γ.

Notice that for the solution (6.8) the complete effective potential (6.7) is zero. This realizes the field σ as the "Goldstone mode" of the spontaneously broken dilatation invariance. Now we can interpret the W-I (4.18′) as the low energy theorem for the massless Goldstone boson σ. The above properties can be shown to all orders with the W-I's.

Consider the action of the following functional operations on (3.13a):

$$\left. \frac{\delta}{\delta \hat{c}} \frac{\delta}{\delta \hat{h}_{\mu\nu}} \right|_{\hat{F}, J^s_{\hat{F}}=0} \tag{6.12a}$$

$$\left. \frac{\delta}{\delta \hat{c}} \frac{\delta}{\delta \hat{\sigma}} \right|_{\hat{F}, J^s_{\hat{F}}=0} \quad . \tag{6.12b}$$

Using v_0 such that the dilaton (and, therefore, the graviton) tadpole vanishes, one finds:

$$\frac{\delta^2\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{h}_{\lambda\rho}}\,\frac{\delta^2\Gamma}{\delta J^s_{\lambda\rho}\delta\,\hat{c}}+\frac{\delta^2\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{\sigma}}\,\frac{\delta^2\Gamma}{\delta J^s_{\sigma}\delta\,\hat{c}}=0 \tag{6.13a}$$

$$\frac{\delta^2\Gamma}{\delta\,\hat{h}_{\lambda\rho}\,\delta\,\hat{\sigma}}\,\frac{\delta^2\Gamma}{\delta J^s_{\lambda\rho}\delta\,\hat{c}}+\frac{\delta^2\Gamma}{\delta\,\hat{\sigma}\,\delta\,\hat{\sigma}}\,\frac{\delta^2\Gamma}{\delta J^s_{\sigma}\delta\,\hat{c}}=0\ . \tag{6.13b}$$

When the graviton tadpole is zero, coordinate invariance guarantees the vanishing of the zero momentum piece of the graviton self energy. Equations (3.16) show that $\delta^2\Gamma/\delta J^s_\sigma\delta\,\hat{\sigma}$ is not zero and, therefore, at zero momentum:

$$\frac{\delta^2\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{h}_{\lambda\rho}}=\frac{\delta^2\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{\sigma}}=\frac{\delta^2\Gamma}{\delta\,\hat{\sigma}\,\delta\,\hat{\sigma}}=0 \tag{6.14}$$

so that to all orders in the perturbation the dilaton field σ is massless. By taking successive derivatives $\delta/\delta\,\hat{h}_{\alpha\beta}$ and $\delta/\delta\,\hat{\sigma}$ on (3.13a) and using the same procedure, it is elementary to prove, by induction, the vanishing of an arbitrary 1-PI function involving only dilaton and graviton external legs at zero momentum.[4]

Moreover, the dilaton field, like a Nambu-Goldstone boson, is unphysical. Its most striking property is its dual role as a Higgs scalar, acquiring a VEV which breaks a local symmetry, and a massless Goldstone boson, depositing its degree of freedom to the metric field and disappearing from the physical sector.

Besides using the conformal unitarity gauge (see section 2), we can see that the dilaton is unphysical by attempting to give it a ξ-dependent mass. In spontaneously broken gauge theories the t'Hooft class of gauges gives a gauge dependent mass to the Goldstone bosons and eliminates all Goldstone boson-gauge field mixings. The analogous condition for the conformal symmetry is obtained by requiring the cancellation of the dilaton-graviton mixing term coming from the Lagrangian (5.10). The resulting gauge fixing condition is:

$$\Phi'=\frac{1}{2(n-1)}\left(\Box h^\mu_\mu-\partial_\mu\partial_\nu\,h^{\mu\nu}\right)-\xi\,\frac{n-2}{2}\,G\!\left(v_0\,\sigma+\frac{\gamma}{v_0}\Box\sigma\right) \tag{6.15}$$

---

[4]This is not true if matter fields are added as external legs at zero momentum.

and becomes $\frac{1}{2\xi}\,\Phi'^2$ in $\mathcal{L}_{GF}$ providing a ξ-dependent mass to the dilaton:

$$\langle\sigma\sigma\rangle = i\,\frac{1}{p^2-\frac{v_0^2}{\gamma}}\cdot\frac{1}{p^2\left(\frac{\gamma}{v_0^2}+\xi\,(n-2)^2\,\frac{G^2\gamma^2}{v_0^2}\right)-\xi\,(n-2)^2G^2\gamma}\quad. \tag{6.16}$$

As $\xi\to\infty$ the σ-propagator vanishes and the dilaton decouples.

In the most general case, we allow all coupling constants of the theory to be nonzero. According to the analog of the Coleman-Weinberg [23] mechanism a value v_0 extremizing the dilaton effective potential exists provided λ_0 is of the order of G^4. Since, to leading order in $1/N$, $\beta_{\lambda_0}=0$ while $\beta_G<0$, there is a region in which $\lambda_0\sim G^4$ and a solution v_0 is always available; it is consistent to assume $G^2N=\gamma/N=\lambda_0N^2=$ fixed. Then, the coupling constant λ_0 gets renormalized since the divergent one-loop four-point function constructed only out of σ-vertices proportional to γ becomes of order $1/N^2$. Notice that for $\lambda_0\neq 0$ the classical background solution about which we should expand possesses curvature. However, to leading order in $1/N$ one can expand around flat space (using the $\lambda_0=0$ classical solution). Indeed, the one-loop corrections in general would give terms of the form G^4, γ^{-4} as well as terms proportional to λ_0 which are necessarily suppressed [they are $o(1/N^6)$ at least]. This algorithm persists for higher orders.

There are three independent parameters in the general theory and the solutions have a more complicated form: $v_0=\mu f(G,\gamma,\lambda_0)$. To get an insight, consider the extra contributions to the dilaton tadpole t_0 given in fig. 6.3(b):

$$t_0^\lambda = i\,\frac{2n}{n-2}\,\lambda_0 Z_{\lambda_0}\,v_0^3 \tag{6.17a}$$

$$t_0^\gamma = \frac{2}{n-2}\,\frac{v_0}{\gamma}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2-\frac{v_0^2}{\gamma}} \tag{6.17b}$$

where the counterterm $\lambda_0Z_{\lambda_0}$ is given by the pole part of the σ field four-point function:

$$\lambda_0Z_{\lambda_0}=\lambda_0-\frac{1}{n-4}\,\frac{1}{32\pi^2}\,\frac{1}{\gamma^2}\quad. \tag{6.18}$$

In four dimensions the sum is:

$$t_0^\lambda + t_0^\gamma = 4i\lambda_0 v_0^3 + \frac{i}{(4\pi)^2}\frac{v_0^3}{\gamma^2}\left[\ell n\frac{v_0^2}{4\pi\mu^2\gamma} + \gamma_E - \frac{3}{2}\right] \tag{6.19}$$

where γ_E is Euler's constant. The sum of the corresponding graviton tadpoles is:

$$t_{\mu\nu}^\lambda + t_{\mu\nu}^\gamma = \frac{i}{2}\lambda_0 G v_o^4 \eta_{\mu\nu} + \frac{i}{(4\pi)^2}\frac{G v_0^4}{8\gamma^2}\left[\ell n\frac{v_0^2}{4\pi\mu^2\gamma} + \gamma_E - \frac{3}{2}\right]\eta_{\mu\nu} \tag{6.20}$$

and the W-I (3.18) is satisfied (see fig. 6.3). To obtain the solution v_0 extremizing the total dilaton effective potential to leading order in $1/N$, we require:

$$t_0^G + t_0^\gamma + t_0^\lambda = 0 \tag{6.21}$$

where $t_0^G \equiv -i \left.\frac{dV_{eff}[\sigma]}{d\sigma}\right|_{\sigma=0}$ [see eq. (6.7)]. Then, a consequence of eqs. (6.11), (6.19) and (6.20) is the vanishing of the cosmological constant.

The generalization of the above analysis when the matter Lagrangian contains scalar fields which can acquire VEV's is straightforward. In the simplest case, there is the field ϕ_0 with VEV v_0, such that $\phi_0 = v_0 + \sigma$, and a scalar field $\bar{\phi}_1$ with VEV $v_1 < v_0$ (to preserve the sign of the induced Einstein term), such that $\bar{\phi}_1 = v_1 + \phi_1$. Then, $\sigma - \phi_1$ mixing terms will appear and we define:

$$\begin{pmatrix}\sigma' \\ \phi_1'\end{pmatrix} = \begin{pmatrix} ch\chi & -sh\chi \\ sh\chi & -ch\chi \end{pmatrix}\begin{pmatrix}\sigma \\ \phi_1\end{pmatrix} \tag{6.22}$$

where σ' is the dilaton field, ϕ_1' is the Higgs field and the Lorentz rotation is dictated by the relative sign difference of the kinetic energies of σ and ϕ_1. The angle χ is:

$$sh\chi = \frac{v_1}{\sqrt{v_0^2 - v_1^2}} \quad . \tag{6.23}$$

At zero momentum, in the Landan-type gauges (2.20) $X = Y = 0$ [see eq. (3.16)] and the W-I's analogous to (6.13) give:

$$\eta_{\lambda\rho} \frac{\delta^2 \Gamma}{\delta \hat{h}_{\mu\nu} \delta \hat{h}_{\lambda\rho}} = \frac{n-2}{4} G \sqrt{v_0^2 - v_1^2} \frac{\delta^2 \Gamma}{\delta \hat{h}_{\mu\nu} \delta \hat{\sigma}'} \tag{6.24a}$$

$$\eta_{\lambda\rho} \frac{\delta^2 \Gamma}{\delta \hat{h}_{\lambda\rho} \hat{\sigma}'} = \frac{n-2}{4} G \sqrt{v_0^2 - v_1^2} \frac{\delta^2 \Gamma}{\delta \hat{\sigma}' \delta \hat{\sigma}'} \tag{6.24b}$$

$$\eta_{\lambda\rho} \frac{\delta^2 \Gamma}{\delta \hat{h}_{\lambda\rho} \delta \hat{\phi}_1'} = \frac{n-2}{4} G \sqrt{v_0^2 - v_1^2} \frac{\delta^2 \Gamma}{\delta \hat{\sigma}' \delta \hat{\phi}_1'} \tag{6.24c}$$

and with the same arguments as before:

$$\frac{\delta^2 \Gamma}{\delta \hat{h}_{\mu\nu} \delta \hat{h}_{\lambda\rho}} = \frac{\delta^2 \Gamma}{\delta \hat{h}_{\mu\nu} \delta \hat{\sigma}'} = \frac{\delta^2 \Gamma}{\delta \hat{\sigma}' \delta \hat{\sigma}'} = 0 \tag{6.25}$$

at zero momentum. Thus, all the properties of the σ field in the case without other scalars are inherited by σ' which is the new dilaton field. At the same time, the Higgs field ϕ_1' is not prevented from getting a mass. In terms of the rotated fields, the W-I (3.18) takes the form:

$$\eta_{\mu\nu} \frac{\delta \Gamma}{\delta \hat{h}_{\mu\nu}} = \frac{n-2}{4} G \sqrt{v_0^2 - v_1^2} \frac{\delta \Gamma}{\delta \hat{\sigma}'} \tag{6.26}$$

and the necessary and sufficient condition for the vanishing of the cosmological constant is the vanishing of the tadpole of the dilaton field in the theory.

In concluding, we note that when physical scalar fields are present there is no a priori symmetry preventing them from acquiring masses of order $M_{p\ell}$, the natural mass scale of the theory, even if their VEV is zero. This is the usual hierarchy problem which can be unnaturally solved by fine tuning without affecting the vanishing of the cosmological constant.

7. Theories with Zero Cosmological Constant

As an outcome of all the previous results, the following is true:

Theorem: Any matter theory can be extended to a spontaneously broken conformally invariant quantum theory coupled to Weyl's quantum gravity. The Ward identities of the total theory imply a zero cosmological constant to all orders in a perturbation around an extremum of the dilaton field potential – if one exists – and around Minkowski spacetime for the metric field.

Low energy physics is adequately described by the standard model, a gauge theory based on a $U(1) \times SU(2)_L \times SU(3)_C$ local symmetry. Its electroweak part can be studied in the ordinary perturbation expansion but, for simplicity, we consider scalar QED which is perfectly adequate for demonstrating the validity of our theorem with explicit calculations. The complete Lagrangian is given by:

$$
\begin{aligned}
\mathcal{L} = \sqrt{-g}\Big\{ & -\frac{Z_G}{G^2}\,\frac{(n-2)}{2(n-3)}\left(1+\frac{\sigma'}{v_0'}\right)^{2\frac{(n-4)}{(n-2)}} C^2 \\
& - Z_\sigma\left[\frac{1}{2}\, g^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) - \frac{n-2}{8(n-1)}\, R(v_0+\sigma)^2\right] \\
& - \frac{Z_A}{4}\left(1+\frac{\sigma'}{v_0'}\right)^{2\frac{(n-4)}{(n-2)}} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} + Z_\phi\, g^{\mu\nu}\Big[\frac{1}{2}\sum_{i=1}^{2} (\partial_\mu\phi_i)(\partial_\nu\phi_i) \\
& + eA_\mu\Big((\partial_\nu\phi_1)\phi_2 - (\partial_\nu\phi_2)(v_1+\phi_1)\Big) + \frac{1}{2}\, e^2 A_\mu A_\nu\Big((v_1+\phi_1)^2+\phi_2^2\Big)\Big] \\
& + Z_\phi\, \frac{n-2}{8(n-1)}\, R\Big((v_1+\phi_1)^2+\phi_2^2\Big) \\
& - Z_\lambda\, \frac{\lambda}{8}\left(1+\frac{\sigma'}{v_0'}\right)^{-2\frac{(n-4)}{(n-2)}}\Big[(v_1+\phi_1)^2+\phi_2^2\Big]^2\Big\} \\
& + \mathcal{L}_{GF}\ (\text{Landau-type gauges}) + \tilde{\mathcal{L}}
\end{aligned}
\qquad (7.1)
$$

where A_μ and e are the photon field and coupling constant and ϕ_1, ϕ_2 are the real components of a complex scalar field. The Lagrangian $\tilde{L}$ contains all possible remaining Weyl invariant terms [see eqs. (4.6)] which we do not use for simplicity. The field σ' is used in the regularization factors and the higher derivative scalar term (4.6b) since it is the unphysical dilaton [see eq. (6.22)] with VEV $v_0' \equiv \sqrt{v_0^2 - v_1^2}$. Moreover, $\tilde{L}$ may contain additional matter fields; we only consider their contribution to the effective number N [see eq. (2.27)].[1]

The minimization of the scalar effective potential to determine v_0 and v_1 is equivalent to requiring the respective scalar tadpoles to be zero and organizing the perturbation theory such that $\lambda \sim e^4$ [24]. Let $t_{\mu\nu}$, t_0 and t_1 denote the gravitational, dilaton and Higgs tadpoles respectively (the relevant Feynman rules are derived in Appendix B). The tree level tadpoles $t^{(0)}$ [see fig. 7.1(a)] contribute:

$$t^{(0)}_{\mu\nu} = -\frac{i}{16} G\eta_{\mu\nu}\lambda Z_\lambda v_1^4 \tag{7.2a}$$

$$t^{(0)}_0 = -\frac{i}{4}\frac{n}{n-2}\lambda Z_\lambda \frac{v_1^4}{v_0'} \tag{7.2b}$$

$$t^{(0)}_1 = \frac{i}{2}\lambda Z_\lambda v_1^3 \frac{v_0}{v_0'} \tag{7.2c}$$

and verify the W-I (6.26):

$$\eta^{\mu\nu} t^{(0)}_{\mu\nu} = \frac{n-2}{4} G v_0' t^{(0)}_0 \quad . \tag{7.3}$$

[1]The general validity of the $1/N$ expansion is based on the fact that the graviton is the only field which couples universally to all matter. This expansion reduces to the ordinary loop expansion when the matter Lagrangian is a gauge theory; notice that in (2.27) for a gauge group $SU(J)$, $N_V = J^2 - 1$. For a consistent $1/N$ expansion, attention must be given to the dilaton field which through the regularization of the theory, could couple to all fields [see eqs. (4.6)].

There are one-loop tadpoles[2] of similar order [see fig. 7.1(b)] coming from A_μ:

$$t_{\mu\nu}^{(A)} = \frac{n-1}{n(n-2)}\, G\eta_{\mu\nu}\, e^4\, v_1^4\, I_{(A)} \tag{7.4a}$$

$$t_0^{(A)} = 4\,\frac{n-1}{(n-2)^2}\, e^4\, \frac{v_1^4}{v_0'}\, I_{(A)} \tag{7.4b}$$

$$t_1^{(A)} = -2\,\frac{n-1}{n-2}\, e^4\, v_1^3\, \frac{v_0}{v_0'}\, I_{(A)} \tag{7.4c}$$

where

$$I_{(A)} \equiv \int \frac{d^n k}{(2\pi)^n}\, \frac{1}{(k^2 - e^2 v_1^2)^2} = \frac{i}{(4\pi)^2} \left(\frac{e^2 v_1^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2} \Gamma\left(2-\frac{n}{2}\right) \quad . \tag{7.5}$$

The identity (6.26) still holds:

$$\eta^{\mu\nu}\, t_{\mu\nu}^{(A)} = \frac{n-2}{4}\, G v_0'\, t_0^{(A)} \quad . \tag{7.6}$$

Finally, the one-loop graviton tadpoles $t^{(h)}$ [see fig. 7.1(c)] give:

$$t_{\mu\nu}^{(h)} = -\frac{5i}{192(4\pi)^2}\, G\eta_{\mu\nu}\, \Lambda^4\, I_{(h)} \tag{7.7a}$$

$$t_0^{(h)} = -\frac{5i}{24(4\pi)^2}\, \frac{\Lambda^4}{v_0'}\, I_{(h)} \tag{7.7b}$$

$$t_1^{(h)} = 0 \tag{7.7c}$$

where

$$I_{(h)} \equiv \int_0^\infty dx\, x^{1-\epsilon}\, \frac{\alpha - x}{\alpha + x\,\ell n\, x} = 48\pi\,\alpha^2 \sum_{j=0}^{\infty} \frac{\sin 2\theta_j\, \sin^2\theta_j}{(\theta_j + 2\pi j)^2} \tag{7.8}$$

with $\alpha = \frac{160\pi^2}{N}\,\frac{v_0'^2}{\Lambda^2}$ and θ_j provided by (6.7b); see Appendix C. As expected:

$$\eta^{\mu\nu}\, t_{\mu\nu}^{(h)} = \frac{1}{2}\, G v_0'\, t_0^{(h)} \quad . \tag{7.9}$$

---

[2]In dimensional regularization, tadpoles with massless loops vanish.

Combining (7.2), (7.4) and (7.7), we find to one-loop:

$$t_{\mu\nu} = -\frac{i}{16}\, G\eta_{\mu\nu}\,\lambda v_1^4 - \frac{3i}{128\pi^2}\, G\eta_{\mu\nu}\, e^4 v_1^4 \left[\ell n\, \frac{e^2 v_1^2}{4\pi\mu^2} + \gamma - \frac{5}{6}\right] + t_{\mu\nu}^{(h)} \qquad (7.10a)$$

$$t_0 = -\frac{i}{2}\,\lambda\,\frac{v_1^4}{v_0'} - \frac{3i}{16\pi^2}\, e^4\,\frac{v_1^4}{v_0'}\left[\ell n\,\frac{e^2 v_1^2}{4\pi\mu^2} + \gamma - \frac{5}{6}\right] + t_0^{(h)} \qquad (7.10b)$$

$$t_1 = \frac{i}{2}\,\lambda v_1^3\,\frac{v_0}{v_0'} + \frac{3i}{16\pi^2}\, e^4 v_1^3\,\frac{v_0}{v_0'}\left[\ell n\,\frac{e^2 v_1^2}{4\pi\mu^2} + \gamma - \frac{1}{3}\right] \quad . \qquad (7.10c)$$

The counterterm Z_λ has been evaluated in the symmetric matter theory ($v_1 = 0$) and is the pole part of the four-point scalar 1-PI Green's function:

$$\lambda Z_\lambda = \lambda - \frac{12}{n-4}\,\frac{e^4}{(4\pi)^2} + \ldots \quad . \qquad (7.11)$$

The full one-loop results (7.10) satisfy:

$$\eta^{\mu\nu}\, t_{\mu\nu} = \frac{1}{2}\, G v_0'\, t_0 \qquad (7.12)$$

and, to achieve a stable perturbation theory, we determine v_0 and v_1 such that the dilaton and Higgs tadpoles are zero:

$$t_0 = t_1 = 0 \qquad (7.13a)$$

which, through (7.12), implies the vanishing of the cosmological constant to this order:

$$t_{\mu\nu} = 0 \qquad (7.13b)$$

Relation (7.10c) does not contain the graviton loop and, thus, the Higgs tadpole vanishes for a value v_1 independent of the presence of gravity; there is no hierarchy problem to this order but in principle it will appear in higher loops. The dilaton tadpole t_0, which controls the value of the graviton tadpole $t_{\mu\nu}$, vanishes for v_0'; this value differs from its original value v_0 given by (6.8) by terms of order v_1/v_0 as eq. (7.10b) indicates. When the ratio v_1/v_0 is very small, as it would be in the standard model, the value of v_0' is essentially equal to v_0 and, therefore, gravitation supplies the dominant contribution.

The addition of fermions with Yukawa couplings:

$$-f\sqrt{-g}\left(1+\frac{\sigma'}{v_0'}\right)^{-\frac{n-4}{n-2}}\bar{\psi}\,(v_1+\phi_1)\,\psi \tag{7.14}$$

and the requirement $\lambda \sim e^2 f^2$ for a consistent perturbation expansion, introduces three tadpoles $t^{(\psi)}$ to one-loop [see fig. 7.1(d)]:

$$t^{(\psi)}_{\mu\nu} = -\frac{1}{n-2}\,G\eta_{\mu\nu}\,e^2 f^2\,v_1^4\,I_{(F)} \tag{7.15a}$$

$$t^{(\psi)}_0 = -\frac{4n}{(n-2)^2}\,e^2 f^2\,\frac{v_1^4}{v_0'}\,I_{(F)} \tag{7.15b}$$

$$t^{(\psi)}_1 = 2\,\frac{n}{n-2}\,e^2 f^2\,v_1^3\,\frac{v_0}{v_0'}\,I_{(F)} \tag{7.15c}$$

where:

$$I_{(F)} \equiv \int \frac{d^n k}{(2\pi)^n}\,\frac{1}{(k^2-f^2v_1^2)^2} = \frac{i}{(4\pi)^2}\left(\frac{f^2v_1^2}{4\pi\mu^2}\right)^{\frac{n}{2}-2}\Gamma\left(2-\frac{n}{2}\right) \quad . \tag{7.16}$$

Equations (7.15) verify the W-I (6.26):

$$\eta^{\mu\nu}\,t^{(F)}_{\mu\nu} = \frac{n-2}{4}\,Gv_0'\,t^{(F)}_0 \quad . \tag{7.17}$$

By coordinate invariance, since $t_{\mu\nu}=0$, an arbitrary 1-PI function involving only graviton and dilaton external legs at zero momentum vanishes. To gain an insight, we show this to first order for the graviton self-energy $\Pi_{\mu\nu,\lambda\rho}$, as well as the graviton-dilaton mixing $\Pi_{\mu\nu,0}$ and the graviton-Higgs mixing $\Pi_{\mu\nu,1}$, taking into account only matter corrections. The relevant tree order diagrams [see fig. 7.2(a)] are:

$$\Pi^{(0)}_{\mu\nu,\lambda\rho} = -\frac{i}{32}\,\lambda Z_\lambda\,G^2 v_1^4\,(\eta_{\mu\nu}\,\eta_{\lambda\rho}-\eta_{\mu\lambda}\,\eta_{\nu\rho}-\eta_{\mu\rho}\,\eta_{\nu\lambda}) \tag{7.18a}$$

$$\Pi^{(0)}_{\mu\nu,0} = -\frac{i}{8}\,\frac{n}{n-2}\,\lambda Z_\lambda\,G\,\frac{v_1^4}{v_0'}\,\eta_{\mu\nu} \tag{7.18b}$$

$$\Pi^{(0)}_{\mu\nu,1} = \frac{i}{4}\,\lambda Z_\lambda\,Gv_1^3\,\frac{v_0}{v_0'}\,\eta_{\mu\nu} \quad . \tag{7.18c}$$

There are one-loop diagrams proportional to e^4, which also contribute:

$$\Pi^{(b)}_{\mu\nu,\lambda\rho} = \frac{2}{n-2}\, G^2 e^4 v_1^4\, I_{(A)} \left\{ \left[-\frac{n-3}{2n} - \frac{1}{n(n+2)} + \frac{n-1}{4n} \right] \eta_{\mu\nu}\,\eta_{\lambda\rho} + \left[-\frac{1}{n(n+2)} + \frac{n-1}{4n} \right] \Big(\eta_{\mu\nu}\,\eta_{\nu\rho} + \eta_{\mu\rho}\,\eta_{\nu\lambda} \Big) \right\} \tag{7.19a}$$

$$\Pi^{(b)}_{\mu\nu,0} = -\frac{4}{(n-2)^2}\, G e^4\, \frac{v_1^4}{v_0^l}\, I_{(A)} \left(\frac{n-3}{2} + \frac{1}{n} - 2\,\frac{n-1}{n} \right) \eta_{\mu\nu} \tag{7.19b}$$

$$\Pi^{(b)}_{\mu\nu,1} = -\frac{2}{n-2}\, G e^4\, v_1^3\, \frac{v_0}{v_0^l}\, I_{(A)} \left(-\frac{n-3}{2} - \frac{1}{n} + \frac{n-1}{2} \right) \eta_{\mu\nu} \tag{7.19c}$$

[see fig. 7.2(b)]

$$\Pi^{(c)}_{\mu\nu,\lambda\rho} = \frac{2}{n-2}\, G^2 e^4 v_1^4\, I_{(A)} \left[\frac{n-3}{2n}\, \eta_{\mu\nu}\,\eta_{\lambda\rho} - \frac{n-2}{2n} \Big(\eta_{\mu\lambda}\,\eta_{\nu\rho} + \eta_{\mu\rho}\,\eta_{\nu\lambda} \Big) \right] \tag{7.20a}$$

$$\Pi^{(c)}_{\mu\nu,0} = -\frac{4}{(n-2)^2}\, G e^4\, \frac{v_1^4}{v_0^l}\, I_{(A)} \left(-\frac{n-3}{2} - \frac{1}{n} + 2\,\frac{n-1}{n} - \frac{n-1}{2} \right) \eta_{\mu\nu} \tag{7.20b}$$

$$\Pi^{(c)}_{\mu\nu,1} = -\frac{2}{n-2}\, G e^4\, v_1^3\, \frac{v_0}{v_0^l}\, I_{(A)} \left(\frac{n-3}{2} + \frac{1}{n} \right) \eta_{\mu\nu} \tag{7.20c}$$

[see fig. 7.2(c)]

$$\Pi^{(d)}_{\mu\nu,\lambda\rho} = \frac{2}{n-2}\, G^2 e^4 v_1^4\, I_{(A)} \left[\frac{1}{n(n+2)}\, \eta_{\mu\nu}\,\eta_{\lambda\rho} + \left(\frac{1}{n(n+2)} - \frac{1}{2n} \right) \Big(\eta_{\mu\lambda}\,\eta_{\nu\rho} + \eta_{\mu\rho}\,\eta_{\nu\lambda} \Big) \right] \tag{7.21a}$$

$$\Pi^{(d)}_{\mu\nu,0} = 0 \tag{7.21b}$$

$$\Pi^{(d)}_{\mu\nu,1} = 0 \tag{7.21c}$$

[see fig. 7.2(d)].

Summing the one-loop diagrams, we find:

$$\Pi^{(A)}_{\mu\nu,\lambda\rho} = \frac{n-1}{2n(n-2)} G^2 e^4 v_1^4 I_{(A)} \Big(\eta_{\mu\nu}\,\eta_{\lambda\rho} - \eta_{\mu\lambda}\,\eta_{\nu\rho} - \eta_{\mu\rho}\,\eta_{\nu\lambda}\Big) \qquad (7.22a)$$

$$\Pi^{(A)}_{\mu\nu,0} = \frac{2(n-1)}{(n-2)^2} G e^4 \frac{v_1^4}{v_0^l} I_{(A)}\,\eta_{\mu\nu} \qquad (7.22b)$$

$$\Pi^{(A)}_{\mu\nu,1} = -\frac{n-1}{n-2} G e^4 v_1^3 \frac{v_0}{v_0^l} I_{(A)}\,\eta_{\mu\nu} \quad . \qquad (7.22c)$$

Combining (7.18) and (7.22) with (7.2) and (7.4), we get:

$$\Pi^{(0)}_{\mu\nu,\lambda\rho} + \Pi^{(A)}_{\mu\nu,\lambda\rho} = \frac{1}{2n} G\eta^{\alpha\beta}\Big(t^{(0)}_{\alpha\beta} + t^{(A)}_{\alpha\beta}\Big)\Big(\eta_{\mu\nu}\,\eta_{\lambda\rho} - \eta_{\mu\lambda}\,\eta_{\nu\rho} - \eta_{\mu\rho}\,\eta_{\nu\lambda}\Big) \quad (7.23a)$$

$$\Pi^{(0)}_{\mu\nu,0} + \Pi^{(A)}_{\mu\nu,0} = \frac{1}{2} G\Big(t^{(0)}_0 + t^{(A)}_0\Big)\eta_{\mu\nu} \qquad (7.23b)$$

$$\Pi^{(0)}_{\mu\nu,1} + \Pi^{(A)}_{\mu\nu,1} = \frac{1}{2} G\Big(t^{(0)}_1 + t^{(A)}_1\Big)\eta_{\mu\nu} \qquad (7.23c)$$

and by setting the tadpoles equal to zero, the constant parts of all the self-energies vanish as well. The graviton corrections would be included in a similar way to both sides of (7.23).

8. Gauge Invariance and Unitarity

Einstein's gravity can be written in Lagrangian form with the metric tensor as the dynamical field variable and the curvature scalar, reflecting the coordinate invariance of the theory, as the Lagrangian up to a dimensionful constant. The metric tensor has ten independent components only two of which are the physical degrees of freedom: there exist four constraint equations of motion and four coordinate gauge conditions. In the gravitational theory of Brans and Dicke [13] a scalar field is coupled in a Weyl invariant way to the curvature scalar: in addition to the four coordinate gauge conditions we have to impose one conformal gauge condition and the metric tensor, which is left with one physical degree of freedom, recovers its missing degree of freedom from the scalar field. In Weyl's gravity (5.10), the addition of the higher derivative terms implies, naively, the following spectrum: a massless spin-two graviton, a massive spin-two ghost and a massive scalar. The presence of the massive ghost state is a characteristic of all gravitational actions quadratic in the curvature and is the source of their non-unitarity in ordinary perturbation theory [see eqs. (2.21)-(2.24)].

However, non-perturbative techniques, like the strong coupling expansion [12] and lattice reqularization [11], seem to suggest that no ghosts are present. The latter methods are more relevant to the Weyl theory since, as shown in the $1/N$ expansion, it may strongly interact at its natural mass scale [see eq. (6.9)] and the loop expansion breaks down.

The $1/N$ perturbation theory shifts the massive spin-two pole of the tree propagator to the complex plane and makes the theory unitary to leading order in the expansion [10]. This becomes transparent by writing the "Lehmann representation" of the full graviton propagator $\tilde{D}_{\mu\nu,\lambda\rho}$ on the physical sheet:

$$\begin{aligned}\tilde{D}_{\mu\nu,\lambda\rho}(p) = \frac{24i}{v_0^2 G^2}\left\{\frac{1}{p^2} + \frac{r}{p^2 - M^2} + \frac{r^*}{p^2 - M^{*2}} + (\text{analytic terms})\right\} \\ \times P^{(2)}_{\mu\nu,\lambda\rho}(p) + \begin{pmatrix}\text{longitudinal} \\ \text{and trace terms}\end{pmatrix}\end{aligned} \tag{8.1}$$

where M, M^* and r, r^* denote the positions and residues of the complex poles. The quantities appearing (8.1) are calculated order by order in the $1/N$ expansion:

$$v_0 = \sqrt{N}\; v_{0(0)} + v_{0(1)} + \cdots \tag{8.2a}$$

$$M^2 = M^2_{(0)} + \frac{1}{N}\, M^2_{(1)} + \cdots \tag{8.2b}$$

and to lowest order:

$$r^{(0)} = -\frac{1}{1 + \dfrac{M^2_{(0)}}{160\pi^2 v^2_{0(0)}}} \quad . \tag{8.3}$$

Due to the presence of complex poles in the propagator, the proper way to distribute the contours of integration in a Feynman integral is not obvious.

The correct prescription is to place the contour in the complex p_0-plane above all singularities in the right half-plane and below all singularities in the left half-plane [19].

In this section we argue that our theory may be unitary order by order in the $1/N$ expansion without use of the Lee-Wick prescription: all physical quantities, like S-matrix elements, involving only helicity two massless gravitons and physical matter particles as external lines never need ghosts or other unphysical degrees of freedom as intermediate states. It is straightforward to extend the standard demonstration of the gauge invariance of physical quantities to this theory and guarantee the ξ-independence of the S-matrix. This is proved by relating a change of the gauge fixing condition in the functional integral with a local invertible change in the sources by nonlinear terms which do not affect the S-matrix [25]. Consequently, if a particle has a gauge dependent mass it should not contribute to any physical process.

The decoupling of the dilaton was proved section 6; we now establish the ξ-independence of its VEV v_0 and the ξ-dependence of the spin-two mass M. Notice that in gauge theories exactly the opposite happens: the VEV of the Higgs field is gauge dependent while the mass of the vector boson gauge independent. But in gravity the physical quantity is the Planck mass, which is associated with the VEV of the dilaton, while the spin-two ghost is an unphysical particle.

The derivation of the Ward identities for the ξ-dependence of 1-PI Green's functions follows that of ref. [9]. Define a new total Lagrangian $\mathcal{L}_\eta$:

$$\mathcal{L}_\eta \equiv \mathcal{L} - \eta\, \bar{c}\, \Phi \tag{8.4}$$

where $\mathcal{L}$ is given in (3.3) and η is an anticommuting constant ($\eta^2 = 0$). Using the extra term in (8.4) we can conveniently take derivatives with respect to the gauge parameter ξ. The generating functional of 1-PI Green's functions Γ [see eq. (3.7)] is defined by:

$$\Gamma \equiv W - \int dx^n \left\{ \sum_F J_F \cdot \hat{F} + \frac{1}{2\varsigma_a} \hat{\Phi}_a^2 - \eta\, \hat{\bar{c}}\, \hat{\Phi} \right\} \quad . \tag{8.5}$$

The equation of motion (3.9b) changes to:

$$J_F = -\frac{\delta\Gamma}{\delta\hat{F}} - \frac{1}{\varsigma_a}\frac{\delta\hat{\Phi}_a}{\delta\hat{F}}\hat{\Phi}_a + \eta\frac{\delta\hat{\Phi}}{\delta\hat{F}}\hat{c} \qquad ; \quad F \neq \bar{c}_a, c_a \tag{8.6a}$$

while (3.9d) becomes:

$$J_{\bar{c}_a} = \frac{\delta\Gamma}{\delta_{\hat{\bar{c}}_a}} - \eta\hat{\Phi} \quad . \tag{8.6b}$$

Equations (3.9a) and (3.9c) remain the same but there is a new one:

$$\xi\frac{dW}{d\xi} = \xi\frac{d\Gamma}{d\xi} - \frac{1}{2\xi}\hat{\Phi}^2 \quad . \tag{8.6c}$$

The W-I's replacing (3.13a) are:

$$2\eta\xi\frac{d\Gamma}{d\xi} = \sum_{F\neq\bar{c}_a}\frac{\delta\Gamma}{\delta\hat{F}}\frac{\delta\Gamma}{\delta J^s_F} \quad . \tag{8.7}$$

Notice that for $\eta = 0$ we recover the old effective action Γ and W-I's. Furthermore, since the counterterms in our theory are ξ-independent (see ref. [9] and section 5), there is only explicit ξ-dependence in the Green's functions and the $d/d\xi$ appearing in (8.7) gives the total gauge parameter dependence of the 1-PI functions.

Initially, all Green's functions depend on the various coupling constants g_i, the subtraction point μ, the parameter v_0 and ξ as independent variables. By requiring the vanishing of the dilaton tadpole we obtain v_0 as a function of the remaining parameters: $v_0 = \mathcal{V}_0(g_i, \mu, \xi)$. To calculate the ξ-dependence of v_0, consider the action of the following functional operation on (8.7):

$$\frac{d}{d\eta}\frac{\delta}{\delta\hat{\sigma}}\bigg|_{\hat{F}, J^s_F, \eta=0} \quad . \tag{8.8}$$

One finds:

$$\begin{aligned} 2\xi\frac{d}{d\xi}\frac{\delta\Gamma}{\delta\hat{\sigma}} = &\frac{\delta\Gamma}{\delta\hat{h}_{\mu\nu}}\frac{d}{d\eta}\frac{\delta^2\Gamma}{\delta J^s_{\mu\nu}\delta\hat{\sigma}} + \frac{\delta\Gamma}{\delta\hat{\sigma}}\frac{d}{d\eta}\frac{\delta^2\Gamma}{\delta J^s_\sigma\delta\hat{\sigma}} \\ &+ \frac{\delta^2\Gamma}{\delta\hat{h}_{\mu\nu}\delta\hat{\sigma}}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_{\mu\nu}} + \frac{\delta^2\Gamma}{\delta\hat{\sigma}\delta\hat{\sigma}}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_\sigma} \quad . \end{aligned} \tag{8.9}$$

By using $v_0 = \mathcal{V}_0(g_i, \mu, \xi)$ so that the dilaton (and, therefore, the graviton) tadpole is zero and eq. (6.14), the right hand side of (8.9) vanishes:

$$\frac{d}{d\xi}\frac{\delta\Gamma}{\delta\,\hat\sigma}\bigg|_{v_0=\mathcal{V}_0} = 0 \Rightarrow \frac{d}{d\xi}\mathcal{V}_0 = 0 \tag{8.10}$$

establishing the gauge independence of v_0 and, consequently, the Planck mass. In (8.10) the following identity was employed:

$$\frac{d}{d\xi}\frac{\delta\Gamma}{\delta\,\hat\sigma}\bigg|_{v_0=\mathcal{V}_0} = -\frac{d\mathcal{V}_0}{d\xi}\left(\frac{\partial}{\partial v_0}\frac{\delta\Gamma}{\delta\,\hat\sigma}\right)\bigg|_{v_0=\mathcal{V}_0} \quad . \tag{8.11}$$

The action of the functional operation:

$$\frac{d}{d\eta}\frac{\delta}{\delta\,\hat h_{\mu\nu}}\frac{\delta}{\delta\,\hat h_{\lambda\rho}}\bigg|_{\hat F=J^s_F=\eta=0;\ v_0=\mathcal{V}_0} \tag{8.12}$$

on the W-I (8.7) gives:

$$\begin{aligned} 2\xi\frac{d}{d\xi}\frac{\delta^2\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}} = &\frac{\delta^3\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}\,\delta\,\hat h_{\alpha\beta}}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_{\alpha\beta}} + 2\,\frac{\delta^2\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\alpha\beta}}\frac{d}{d\eta}\frac{\delta^2\Gamma}{\delta\,\hat h_{\lambda\rho}\,\delta J^s_{\alpha\beta}} \\ &+\frac{\delta^3\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}\,\delta\,\hat\sigma}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_\sigma} + 2\,\frac{\delta^2\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat\sigma}\frac{d}{d\eta}\frac{\delta^2\Gamma}{\delta\,\hat h_{\lambda\rho}\,\delta J^s_\sigma} \quad . \end{aligned} \tag{8.13}$$

Since the transverse traceless part of the graviton self-energy contains the complex poles [see eq. (8.1)], we multiply both sides of (8.13) with $P^{(2)}_{\mu\nu,\lambda\rho}$ and set the external momentum $p^2 = M^2$:

$$\begin{aligned} 2\xi\frac{d}{d\xi}P^{(2)}_{\mu\nu,\lambda\rho}\frac{\delta^2\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}} = &P^{(2)}_{\mu\nu,\lambda\rho}\frac{\delta^3\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}\,\delta\,\hat h_{\alpha\beta}}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_{\alpha\beta}} \\ &+ P^{(2)}_{\mu\nu,\lambda\rho}\frac{\delta^3\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}\,\delta\,\hat\sigma}\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_\sigma} \end{aligned} \tag{8.14}$$

where the field lines $\hat h_{\alpha\beta}$ and $\hat\sigma$ carry zero momentum. The relation:

$$-\frac{d}{d\xi}P^{(2)}_{\mu\nu,\lambda\rho}\frac{\delta^2\Gamma}{\delta\,\hat h_{\mu\nu}\,\delta\,\hat h_{\lambda\rho}} = -\frac{5i}{r}\frac{v_0^2G^2}{24}\frac{\partial}{\partial\xi}M^2 \quad ; \quad \text{at } p^2 = M^2 \tag{8.15}$$

which is obtained from (8.1), when combined with (8.14), the identity:

$$\frac{d}{d\xi}\, M^2 = \frac{\partial}{\partial \xi}\, M^2 + \frac{d\mathcal{V}_0}{d\xi}\left(\frac{\partial}{\partial v_0}\, M^2\right)\Big|_{v_0=\mathcal{V}_0} \qquad (8.16)$$

and (8.10) shows that $\frac{d}{d\xi}\, M^2 \neq 0$ unless miraculous cancellations happen in the right hand side of (8.14). However, with an explicit calculation to leading non-vanishing order in $1/N$ we exhibit this is not the case.

The three graviton and two graviton-dilaton vertices behave like $1/\sqrt{N}$ to lowest order (see Appendix B):

$$P^{(2)}_{\mu\nu,\lambda\rho}\,\frac{\delta^3\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{h}_{\lambda\rho}\,\delta\,\hat{h}_{\alpha\beta}} = \frac{5i}{24}\, NG^3\left(v^2_{0(0)} + \frac{1}{160\pi^2}\, M^2_{(0)}\right)p_\alpha p_\beta + o\left(\frac{1}{N^{3/2}}\right) \qquad (8.17a)$$

$$P^{(2)}_{\mu\nu,\lambda\rho}\,\frac{\delta^3\Gamma}{\delta\,\hat{h}_{\mu\nu}\,\delta\,\hat{h}_{\lambda\rho}\,\delta\,\hat{\sigma}} = \frac{5i}{12}\,\frac{\sqrt{N}\,G^2}{v_{0(0)}}\left(v^2_{0(0)} + \frac{1}{160\pi^2}\, M^2_{(0)}\right)M^2_{(0)} + o\left(\frac{1}{N^{3/2}}\right) \qquad (8.17b)$$

where $v_{0(0)}$ and $M^2_{(0)}$ are given by the expansions (8.2) and eqs. (8.17) are evaluated at $p^2 = M^2$. The lowest order contributions of $\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_{\alpha\beta}}$ and $\frac{d}{d\eta}\frac{\delta\Gamma}{\delta J^s_\sigma}$ are the one-loop vacuum diagrams of the operators forming the nonlinear part of $sh_{\alpha\beta}$ and $s\sigma$ respectively [see eqs. (3.4)]. These diagrams, shown in fig. 8.1, contain the additional vertex of the Lagrangian (8.4) once and when the gauge fixing and ghost terms (5.1), (5.2), (5.11) are employed, produce results proportional to $1/\sqrt{N}$:

$$\frac{d}{d\eta}\,\frac{\delta\Gamma}{\delta J^s_{\mu\nu}} = \eta_{\mu\nu}\,\frac{\xi G}{8\pi^2} + o\left(\frac{1}{N^{3/2}}\right) \qquad (8.18a)$$

$$\frac{d}{d\eta}\,\frac{\delta\Gamma}{\delta J^s_\sigma} = \frac{\xi v_{0(0)} G^2}{32\pi^2}\,\sqrt{N} + o\left(\frac{1}{N^{3/2}}\right) \qquad (8.18b)$$

where the various propagators used can be found in Appendix A. Substituting (8.17) together with (8.18) in (8.14) and using (8.3), (8.15) as well as (8.16), we find:

$$\frac{d}{d\xi}\, M^2 = \frac{1}{N}\left(\frac{3}{32\pi^2}\, G^2 N\, M_{(0)}\right) + o\left(\frac{1}{N^2}\right)\,. \qquad (8.19)$$

Thus, as $\xi \to \infty$, the pole M^2 goes to infinity along a ray in the complex plane; both its real and imaginary parts become gauge parameter dependent. Physical quantities, being gauge invariant, should not depend on the gauge variant pole M. Consequently, the spin-two ghost associated with the pole should be unphysical.

The same analysis can be applied to gauge theories. In particular consider the scalar QED Lagrangian (7.1) without any gravitational interactions. We choose covariant gauges (α, Φ_A) and a $\mathcal{L}_{GF}$ such that the ultraviolet behavior of the α-dependent piece of the photon propagator is sufficient to ensure only α-independent counterterms. Similar reasoning gives for the VEV v_1 of the Higgs field ϕ_1:

$$2\alpha \, \frac{d}{d\alpha} \, \frac{\delta \Gamma}{\delta \, \hat{\phi}_1} = \frac{\delta^2 \Gamma}{\delta \, \hat{\phi}_1 \, \delta \, \hat{\phi}_1} \, \frac{d}{d\eta} \, \frac{\delta \Gamma}{\delta J^s_{\phi_1}} \tag{8.20}$$

at zero external momenta and for $v_1 = \mathcal{V}_1(e, \lambda, \mu, \alpha)$ so that the Higgs tadpole is zero. Using (8.11) we obtain to lowest order:

$$\alpha \, \frac{d}{d\alpha} \, v_1 = -\frac{1}{2} \, \frac{d}{d\eta} \, \frac{\delta \Gamma^{(0)}}{\delta J^s_{\phi_1}} \neq 0 \tag{8.21}$$

and, in contrast to (8.10), v_1 is gauge parameter dependent.

However, this is not the case for the physical photon mass. The W-I, corresponding to (8.14), which provides the α-dependence of the photon propagator pole M_A is:

$$2\alpha \, \frac{d}{d\alpha} \, \theta_{\mu\nu} \, \frac{\delta^2 \Gamma}{\delta \, \hat{A}_\mu \, \hat{A}_\nu} = \theta_{\mu\nu} \, \frac{\delta^3 \Gamma}{\delta \, \hat{\phi}_1 \, \delta \, \hat{A}_\mu \, \delta \, \hat{A}_\nu} \, \frac{d}{d\eta} \, \frac{\delta \Gamma}{\delta J^s_{\phi_1}} \tag{8.22}$$

at $p^2 = M_A^2$ and $v_1 = \mathcal{V}_1$. To lowest order we infer from (8.22):

$$\alpha \, \frac{\partial}{\partial \alpha} \, M_A^2 = e^2 v_1 \, \frac{d}{d\eta} \, \frac{\delta \Gamma^{(0)}}{\delta J^s_{\phi_1}} \quad . \tag{8.23}$$

Since v_1 is α-dependent, when the identity (8.16) is applied in this case both terms contribute and cancel each other [see eqs. (8.21) and (8.22)]:

$$\frac{d}{d\alpha} \, M_A^2 = 0 \quad . \tag{8.24}$$

In Weyl's theory the gauge invariance of v_0 eliminated the second term of (8.16) and left a ξ-dependence to the unphysical pole M. The full complex pole M has a ξ-invariant lowest order term $M_{(0)}$ and the ξ-dependence starts from the $M_{(1)}$ piece. However, when the position of a propagator pole changes order by order in the expansion, perturbation theory breaks down at values of the momenta close to $M^2_{(0)}$ and the full propagator should be used in cutting diagrams [27,19]. But the pole of the full propagator is ξ-variant and can be made arbitrarily large.

It is important to note that the question of unitarity in the context of a higher derivative quantum theory of gravity is very complex. Thus, the interpretation and consequences of the gauge variance of the troublesome pole M may not be as clear as one would like. We feel that a more careful analysis is needed.

9. Concluding Remarks

Weyl's gravity has a more complicated dynamical structure than general relativity and satisfies a very powerful theorem which, under very general assumptions, explains the stability of flat spacetime. The theory contains an unphysical degree of freedom, the dilaton field, that expresses the unavoidable spontaneous breakdown of the conformal component of the full symmetry. The theorem states that the ability to set equal to zero the vacuum expectation value of the translated dilaton field of a Weyl invariant physical system, is the necessary and sufficient condition for the vanishing of the cosmological constant. The class of physical systems fullfilling this condition includes gauge theories. Consequently, the standard model $U(1) \times SU(2)_L \times SU(3)_C$ of low energy physics when coupled to Weyl's gravity provides a theory describing all four known interactions with zero cosmological constant. A natural small parameter in this theory is the inverse of the effective number of matter degrees of freedom N which equals 1/292 [see eq. (2.27)] and gives a natural mass scale Λ, where the gravitational coupling constant G becomes strong, practically equal to the Planck mass [$\Lambda = .9\, M_{p\ell}$; see eq. (6.9b)]. At these high energies the three gauge coupling constants g_1, g_2, g_3 have the same value ($g_1 \sim g_2 \sim g_3 \sim .5$). At low energies, the spontaneous breakdown of the electroweak symmetry, in its simplest form involving elementary scalars, occurs through radiative corrections as dictated by the Weyl invariance. The lack of an appropriate symmetry will eventually drive the electroweak breaking scale up to the Planck energies unless severe fine tuning is performed. There exist alternative Weyl invariant methods to break $U(1) \times SU(2)_L$ and avoid the hierarchy problem: technicolor [27] and possibly conformal supergravity [28] are good candidates.

The vacuum expectation value of any physical scalar field should never equal or exceed the Planck mass since the highly desirable induced Einstein term would vanish or acquire the wrong sign. This is an additional problem scalars create, in particular those of various grand unified models. A description of nature in terms of Weyl's gravity and a gauge theory containing only fermions and vector bosons seems to be preferred.

Besides being naturally small, $1/N$ is insensitive to the scale dependence of the coupling constant G. Thus, we could penetrate the potentially strong coupling regime of the theory and extract information demonstrating the spontaneous breakdown of conformal invariance by determining the vacuum expectation value of the dilaton field in terms of the intrinsic scale parameter Λ. However, the cosmological constant theorem is valid for any perturbation expansion parameter. If the dilaton tadpole can be consistently set equal to zero order by order in some perturbation expansion, the Ward identity will force the cosmological constant to vanish to all orders in that expansion. Finally, the strong arguments presented for the unitarity of the theory relied heavily on the $1/N$ expansion.

The quantum theory of gravity analyzed in this work:

(i) Has a stable Minkowski solution and the correct long distance gravitational limit.

(ii) Easily incorporates the standard low energy model.

(iii) Is renormalizable and probably unitary.

Therefore, it provides a well-defined quantum theory in which cosmological and other gravitational implications can be calculated.

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APPENDIX A: PROPAGATORS

The complete set of n-dimensional projectors for symmetric rank-two tensors is (see ref. [29]):

$$P^{(2)}_{\mu\nu,\lambda\rho} = \frac{1}{2}(\theta_{\mu\lambda}\theta_{\nu\rho} + \theta_{\mu\rho}\theta_{\nu\lambda}) - \frac{1}{n-1}\theta_{\mu\nu}\theta_{\lambda\rho} \tag{A.1a}$$

$$P^{(1)}_{\mu\nu,\lambda\rho} = \frac{1}{2}(\theta_{\mu\lambda}\omega_{\nu\rho} + \theta_{\mu\rho}\omega_{\nu\lambda} + \theta_{\nu\rho}\omega_{\mu\lambda} + \theta_{\nu\lambda}\omega_{\mu\rho}) \tag{A.1b}$$

$$P^{(0-s)}_{\mu\nu,\lambda\rho} = \frac{1}{n-1}\theta_{\mu\nu}\theta_{\lambda\rho} \tag{A.1c}$$

$$P^{(0-w)}_{\mu\nu,\lambda\rho} = \omega_{\mu\nu}\omega_{\lambda\rho} \tag{A.1d}$$

$$P^{(0-sw)}_{\mu\nu,\lambda\rho} = \frac{1}{\sqrt{n-1}}\theta_{\mu\nu}\omega_{\lambda\rho} \tag{A.1e}$$

$$P^{(0-ws)}_{\mu\nu,\lambda\rho} = \frac{1}{\sqrt{n-1}}\omega_{\mu\nu}\theta_{\lambda\rho} \tag{A.1f}$$

where the quantities $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are the transverse and longitudinal vector projection operators:

$$\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \tag{A.2a}$$

$$\omega_{\mu\nu} \equiv \frac{p_\mu p_\nu}{p^2} \quad . \tag{A.2b}$$

In equation (A.1), the projectors (A.1a)-(A.1d) select out the spin-two, spin-one and two spin-zero parts of a massive tensor field at rest while (A.1e) and (A.1f) are the two spin-zero transfer operators. The completeness is expressed by:

$$\mathbf{1} = P^{(2)} + P^{(1)} + P^{(0-s)} + P^{(0-w)} \tag{A.3}$$

while the orthogonality by:

$$P^{(i-a)}\,P^{(j-b)} = \delta^{ij}\,\delta^{ab}\,P^{(i-a)} \quad ; \quad i,j=0,1,2 \quad \text{and} \quad a,b=s,w \qquad (A.4a)$$

$$P^{(i-ab)}\,P^{(j-cd)} = \delta^{ij}\,\delta^{bc}\,P^{(i-a)} \qquad (A.4b)$$

$$P^{(i-a)}\,P^{(j-bc)} = \delta^{ij}\,\delta^{ab}\,P^{(i-ac)} \qquad (A.4c)$$

$$P^{(i-ab)}\,P^{(j-c)} = \delta^{ij}\,\delta^{bc}\,P^{(i-ac)} \quad . \qquad (A.4d)$$

Some useful relations are:

$$\omega P^{(i)} = 0, \quad \omega P^{(0-s)} = 0, \quad \omega P^{(0-w)} = \omega,$$

$$\omega\Big(P^{(0-sw)} + P^{(0-ws)}\Big) = \frac{1}{\sqrt{n-1}}\,\theta \qquad (A.5a)$$

$$\theta P^{(i)} = 0, \quad \theta P^{(0-s)} = \theta, \quad \theta P^{(0-w)} = 0,$$

$$\theta\Big(P^{(0-sw)} + P^{(0-ws)}\Big) = \sqrt{n-1}\,\omega \quad . \qquad (A.5b)$$

The free propagators for the various fields appearing in the theory can be derived from the general formula:

$$\Big[AP^{(2)} + BP^{(1)} + CP^{(0-s)} + D\,(P^{(0-sw)} + P^{(0-ws)}) + EP^{(0-w)}\Big]^{-1}$$

$$= \frac{1}{A}\,P^{(2)} + \frac{1}{B}\,P^{(1)} - \frac{E}{D^2-CE}\,P^{(0-s)} + \frac{D}{D^2-CE}\,(P^{(0-sw)} + P^{(0-ws)})$$

$$- \frac{C}{D^2-CE}\,P^{(0-w)} \qquad (A.6a)$$

and its simple matrix extension:

$$\begin{pmatrix} AP^{(2)} + BP^{(1)} + CP^{(0-s)} + D(P^{(0-sw)} + P^{(0-ws)}) + EP^{(0-w)} & -F\theta \\ F\theta & G \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} \frac{1}{A}P^{(2)} + \frac{1}{B}P^{(1)} + \frac{-GE\,P^{(0-s)}+GD(P^{(0-sw)}+P^{(0-ws)})+[(n-1)F^2-GC]\,P^{(0-w)}}{G(D^2-CE)+(n-1)EF^2} & \frac{EF\theta-\sqrt{n-1}\,DF\omega}{G(D^2-CE)+(n-1)EF^2} \\ \frac{EF\theta-\sqrt{n-1}\,DF\omega}{G(D^2-CE)+(n-1)EF^2} & \frac{D^2-CE}{G(D^2-CE)+(n-1)EF^2} \end{pmatrix}. \quad (A.6b)$$

Using the gauge fixing conditions (5.1):

$$(\varsigma,\,\Phi^\mu) \quad \text{such that} \quad \Phi^\mu = \partial_\nu h^{\mu\nu} - \frac{1}{n}\partial^\mu h^\nu_\nu \qquad (A.7a)$$

$$(\xi,\,\Phi) \quad \text{such that} \quad \Phi = \frac{1}{2(n-1)}\left(\Box h^\mu_\mu - \partial_\mu\partial_\nu h^{\mu\nu}\right) \qquad (A.7b)$$

in the Lagrangian (5.10) with L_{GF} given by (3.5), we obtain:

$$A = -p^4 + L^2p^2, \quad B = -\frac{1}{2\varsigma}p^2,$$

$$C = -(n-2)L^2p^2 + \frac{1}{4(n-1)\xi}p^4 - \frac{n-1}{n^2\varsigma}p^2 + \frac{(n-2)^2}{16(n-1)}G^2\gamma p^4,$$

$$D = \frac{-(n-1)^{3/2}}{n^2\varsigma}p^2, \quad E = -\frac{(n-1)^2}{n^2\varsigma}p^2, \quad G = -p^2 + \frac{\gamma}{v_0^2}p^4, \qquad (A.8)$$

$$F = -\frac{n-2}{4(n-1)}v_0G\,p^2 + \frac{n-2}{4(n-1)}\frac{G}{v_0}\gamma p^4,$$

$$F^2 = \frac{n-2}{n-1}L^2p^4, \quad G(D^2-CE) + (n-1)EF^2 = \frac{n-1}{4n^2\varsigma\xi}p^8\left(\frac{\gamma}{v_0^2}p^2 - 1\right)$$

where $L^2 = \frac{n-2}{16(n-1)} v_0^2 G^2$. Specializing to the $\varsigma \to 0$ limit, the propagators take the form:

$$\langle h_{\mu\nu}\, h_{\lambda\rho}\rangle = i\Big[-\frac{1}{p^2(p^2-L^2)}P^{(2)} + \frac{4\xi(n-1)}{p^4}P^{(0-s)} + \frac{4\xi\sqrt{n-1}}{p^4}\left(P^{(0-sw)} + P^{(0-ws)}\right) + \frac{4\xi}{p^4}P^{(0-w)}\Big] \qquad (A.9a)$$

$$\langle h_{\mu\nu}\,\sigma\rangle = -i(n-2)\xi\,\frac{Gv_0}{p^4}\,\eta_{\mu\nu} \qquad (A.9b)$$

$$\langle \sigma\,\sigma\rangle = i\Big[\frac{1}{p^2(\frac{\gamma}{v_o^2}p^2-1)} + 4(n-2)(n-1)\xi\frac{L^2}{p^4}\Big] \qquad (A.9c)$$

$$\langle \bar{c}_\mu\, c_\nu\rangle = -\frac{i}{p^2}\left[\theta_{\mu\nu} + \frac{n}{2(n-1)}\omega_{\mu\nu}\right] \qquad (A.9d)$$

$$\langle \bar{c}\, c\rangle = \frac{i}{p^2} \qquad (A.9e)$$

$$\langle \bar{c}_\mu\, c\rangle = \langle \bar{c}\, c_\nu\rangle = 0 \quad . \qquad (A.9f)$$

When explicit calculations are done with equations (5.2) and (5.11) as $\mathcal{L}_{GF}$, in the propagators (A.9) the substitution:

$$\frac{\xi}{p^4} \to \xi\,\frac{m^4}{p^2(p^2-m^2)^3}$$

should be carried.

APPENDIX B: FEYNMAN RULES

The definition of the gravitational fluctuating field $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + G h_{\mu\nu} \tag{B.1}$$

results in the following expansions of useful quantities to cubic order in $h_{\mu\nu}$:

$$g^{\mu\nu} = \eta^{\mu\nu} - G h^{\mu\nu} + G^2 h^{\mu}_{\alpha} h^{\alpha\nu} - G^3 h^{\mu}_{\alpha} h^{\alpha}_{\beta} h^{\beta\nu} \tag{B.2}$$

$$\begin{aligned}\sqrt{-g} =& 1 + \frac{1}{2} G h^{\alpha}_{\alpha} + \frac{1}{8} G^2 (h^{\alpha}_{\alpha} h^{\beta}_{\beta} - 2 h^{\alpha}_{\beta} h^{\beta}_{\alpha}) \\ &+ \frac{1}{48} G^3 (h^{\alpha}_{\alpha} h^{\beta}_{\beta} h^{\gamma}_{\gamma} - 6 h^{\alpha}_{\beta} h^{\beta}_{\alpha} h^{\gamma}_{\gamma} + 8 h^{\alpha}_{\beta} h^{\beta}_{\gamma} h^{\gamma}_{\alpha})\end{aligned} \tag{B.3}$$

$$\begin{aligned}\sqrt{-g}\; g^{\mu\nu} =& \eta^{\mu\nu} + G\left(\frac{1}{2}\eta^{\mu\nu} h^{\alpha}_{\alpha} - h^{\mu\nu}\right) \\ &+ G^2\left[h^{\mu}_{\alpha} h^{\alpha\nu} - \frac{1}{2} h^{\mu\nu} h^{\alpha}_{\alpha} + \frac{1}{8}\eta^{\mu\nu}(h^{\alpha}_{\alpha} h^{\beta}_{\beta} - 2 h^{\alpha}_{\beta} h^{\beta}_{\alpha})\right]\end{aligned} \tag{B.4}$$

$$\begin{aligned}\sqrt{-g}\; g^{\mu\nu}\, g^{\lambda\rho} =& \eta^{\mu\nu}\eta^{\lambda\rho} + G\left(\frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho} h^{\alpha}_{\alpha} - \eta^{\mu\nu} h^{\lambda\rho} - \eta^{\lambda\rho} h^{\mu\nu}\right) \\ &+ G^2\left[\frac{1}{8}\eta^{\mu\nu}\eta^{\lambda\rho}(h^{\alpha}_{\alpha} h^{\beta}_{\beta} - 2 h^{\alpha}_{\beta} h^{\beta}_{\alpha}) + \eta^{\mu\nu} h^{\lambda}_{\alpha} h^{\alpha\rho} \right. \\ &\left. + \eta^{\lambda\rho} h^{\mu}_{\alpha} h^{\alpha\nu} + h^{\mu\nu} h^{\lambda\rho} - \frac{1}{2}\eta^{\mu\nu} h^{\lambda\rho} h^{\alpha}_{\alpha} - \frac{1}{2}\eta^{\lambda\rho} h^{\mu\nu} h^{\alpha}_{\alpha}\right]\end{aligned} \tag{B.5}$$

$$\begin{aligned}\sqrt{-g}\, R^2_{\alpha\beta\gamma\delta} =& \frac{1}{4} G^2 (\partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\beta\partial_\delta h_{\alpha\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta})^2 \\ &+ G^3 (\partial_\beta\partial_\gamma h_{\alpha\delta} + \partial_\alpha\partial_\delta h_{\beta\gamma} - \partial_\beta\partial_\delta h_{\alpha\gamma} - \partial_\alpha\partial_\gamma h_{\beta\delta}) \\ &\times \left[-h^{\alpha}_{\lambda}(\partial^\beta\partial^\gamma h^{\lambda\delta} + \partial^\lambda\partial^\delta h^{\beta\gamma} - \partial^\beta\partial^\delta h^{\lambda\gamma} - \partial^\lambda\partial^\gamma h^{\beta\delta}) \right. \\ &+ \frac{1}{8} h^{\lambda}_{\lambda}(\partial^\beta\partial^\gamma h^{\alpha\delta} + \partial^\alpha\partial^\delta h^{\beta\gamma} - \partial^\beta\partial^\delta h^{\alpha\gamma} - \partial^\alpha\partial^\gamma h^{\beta\delta}) \\ &\left. + \frac{1}{2}(\partial^\alpha h^{\delta\lambda} + \partial^\delta h^{\lambda\alpha} - \partial^\lambda h^{\delta\alpha})(\partial^\beta h^{\gamma}_{\lambda} + \partial^\gamma h^{\beta}_{\lambda} - \partial_\lambda h^{\beta\gamma})\right]\end{aligned} \tag{B.6}$$

$$\sqrt{-g}\ R^2_{\alpha\beta} = \frac{1}{4}G^2(\partial_\alpha\partial_\gamma h^\gamma_\beta + \partial_\beta\partial_\gamma h^\gamma_\alpha - \partial_\alpha\partial_\beta h^\gamma_\gamma - \Box h_{\alpha\beta})^2$$

$$+ G^3(\partial_\alpha\partial_\gamma h^\gamma_\beta + \partial_\beta\partial_\gamma h^\gamma_\alpha - \partial_\alpha\partial_\beta h^\gamma_\gamma - \Box h_{\alpha\beta})$$

$$\times \Big[\frac{1}{8}h^\lambda_\lambda(\partial^\alpha\partial^\delta h^\beta_\delta + \partial^\beta\partial^\delta h^\alpha_\delta - \partial^\alpha\partial^\beta h^\delta_\delta - \Box h^{\alpha\beta})$$

$$-\frac{1}{2}h_{\lambda_\rho}(\partial^\alpha\partial^\rho h^{\beta\lambda} + \partial^\beta\partial^\lambda h^{\alpha\rho} - \partial^\alpha\partial^\beta h^{\lambda\rho} - \partial^\lambda\partial^\rho h^{\alpha\beta})$$

$$-\frac{1}{2}h^\alpha_\lambda(\partial^\lambda\partial^\delta h^\beta_\delta + \partial^\beta\partial^\delta h^\lambda_\delta - \partial^\lambda\partial^\beta h^\delta_\delta - \Box h^{\lambda\beta}) + \frac{1}{2}(\partial_\rho h^\alpha_\delta)(\partial^\rho h^{\beta\delta})$$

$$-\frac{1}{2}(\partial_\delta h^\alpha_\rho)(\partial^\rho h^{\delta\beta}) + \frac{1}{4}(\partial^\alpha h_{\delta\rho})(\partial^\beta h^{\delta\rho}) - (\partial^\alpha h^\beta_\rho)(\partial^\delta h^\rho_\delta)$$

$$+\frac{1}{2}(\partial^\alpha h^\beta_\rho)(\partial^\rho h^\delta_\delta) + \frac{1}{2}(\partial_\rho h^{\alpha\beta})(\partial^\delta h^\rho_\delta) - \frac{1}{4}(\partial_\rho h^{\alpha\beta})(\partial^\rho h^\delta_\delta)\Big] \qquad (B.7)$$

$$\sqrt{-g}\ R^2 = G^2(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h^\alpha_\alpha)^2$$

$$+ G^3(\partial_\lambda\partial_\rho h^{\lambda\rho} - \Box h^\lambda_\lambda)\Big[\frac{1}{2}h^\alpha_\alpha(\partial_\beta\partial_\gamma h_{\beta\gamma} - \Box h_{\beta\beta})$$

$$-2h_{\alpha\gamma}(2\partial^\alpha\partial^\beta h^\gamma_\beta - \partial^\alpha\partial^\gamma h^\beta_\beta - \Box h^{\alpha\gamma}) + \frac{3}{2}(\partial_\alpha h_{\beta\gamma})(\partial^\alpha h^{\beta\gamma})$$

$$-(\partial_\alpha h_{\beta\gamma})(\partial^\beta h^{\alpha\gamma}) - 2(\partial_\alpha h^\alpha_\beta)(\partial_\gamma h^{\beta\gamma})$$

$$+2(\partial_\alpha h^\alpha_\beta)(\partial^\beta h^\gamma_\gamma) - \frac{1}{2}(\partial_\alpha h^\beta_\beta)(\partial^\alpha h^\gamma_\gamma)\Big] \qquad (B.8)$$

$$
\begin{aligned}
\sqrt{-g}\;\; R =& G(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h^\alpha_\alpha) \\
&+ G^2\left\{\frac{1}{2}h^\gamma_\gamma(\partial_\alpha\partial_\beta\, h^{\alpha\beta} - \Box h^\alpha_\alpha) - h_{\alpha\gamma}(2\partial^\alpha\partial^\beta h^\gamma_\beta - \partial^\alpha\partial^\gamma h^\beta_\beta - \Box h^{\alpha\gamma})\right. \\
&+ \frac{1}{4}\Big[3(\partial_\alpha h_{\beta\gamma})(\partial^\alpha h^{\beta\gamma}) - 2(\partial_\alpha h_{\beta\gamma})(\partial^\beta h^{\alpha\gamma}) - 4(\partial_\alpha h^\alpha_\beta)(\partial^\gamma h^\beta_\gamma) \\
&\left. + 4(\partial_\alpha h^\alpha_\beta)(\partial^\beta h^\gamma_\gamma) - (\partial_\alpha h^\beta_\beta)(\partial^\alpha h^\gamma_\gamma)\Big]\right\} \\
&+ G^3\left\{\frac{1}{8}(\partial_\alpha\partial_\beta\, h^{\alpha\beta} - \Box h^\alpha_\alpha)(3h^\gamma_\gamma h^\delta_\delta - 2h^\gamma_\delta h^\delta_\gamma)\right. \\
&- \frac{1}{2}h^\delta_\delta h_{\alpha\gamma}(2\partial^\alpha\partial^\beta h^\gamma_\beta - \partial^\alpha\partial^\gamma h^\beta_\beta - \Box h^{\alpha\gamma}) \\
&+ \frac{1}{8}h^\delta_\delta\Big[3(\partial_\alpha h_{\beta\gamma})(\partial^\alpha h^{\beta\gamma}) - 2(\partial_\alpha h_{\beta\gamma})(\partial^\beta h^{\alpha\gamma}) - 4(\partial_\alpha h^\alpha_\beta)(\partial_\gamma h^{\beta\gamma}) \\
&+ 4(\partial_\alpha h^\alpha_\beta)(\partial^\beta h^\gamma_\gamma) - (\partial_\alpha h^\beta_\beta)(\partial^\alpha h^\gamma_\gamma)\Big] \\
&+ h_{\alpha\gamma}h_{\beta\delta}(\partial^\alpha\partial^\delta h^{\beta\gamma} - \partial^\alpha\partial^\gamma h^{\beta\delta}) + h_{\alpha\delta}h^\delta_\gamma(2\partial^\alpha\partial^\beta h^\gamma_\beta - \partial^\alpha\partial^\gamma h^\beta_\beta - \Box h^{\alpha\gamma}) \\
&- \frac{1}{4}h_{\gamma\delta}\Big[2(\partial_\alpha h^\gamma_\beta)(\partial^\beta h^{\delta\alpha}) + 2(\partial_\alpha h^\gamma_\beta)(\partial^\alpha h^{\delta\beta}) \\
&- 4(\partial_\alpha h^\gamma_\beta)(\partial^\delta h^{\alpha\beta}) + (\partial^\gamma h^\alpha_\beta)(\partial^\delta h^\beta_\alpha) - 4(\partial_\alpha h^{\gamma\alpha})(\partial_\beta h^{\delta\beta}) \\
&+ 4(\partial_\alpha h^{\gamma\alpha})(\partial^\delta h^\beta_\beta) - (\partial^\gamma h^\alpha_\alpha)(\partial^\delta h^\beta_\beta)\Big] \\
&- \frac{1}{2}h_{\alpha\gamma}\Big[-2(\partial_\beta h^\alpha_\delta)(\partial^\delta h^{\beta\gamma}) + 2(\partial_\delta h^\alpha_\beta)(\partial^\delta h^{\beta\gamma}) \\
&+ (\partial^\alpha h^\delta_\beta)(\partial^\gamma h^\beta_\delta) - 4(\partial^\alpha h^\gamma_\beta)(\partial^\delta h^\beta_\delta) + 2(\partial^\alpha h^\gamma_\beta)(\partial^\beta h^\delta_\delta) \\
&\left. + 2(\partial_\delta h^{\alpha\gamma})(\partial_\beta h^{\beta\delta}) - (\partial_\beta h^{\alpha\gamma})(\partial^\beta h^\delta_\delta)\Big]\right\}
\end{aligned}
\qquad (B.9)
$$

The Feynman rules presented below use the following definitions:

(a) Graviton line ($h_{\mu\nu}$):

Dilaton line (σ):

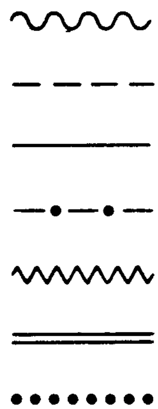
Scalar line (ϕ_1):

Goldstone boson line (ϕ_2):

Vector boson line (A_μ):

Fermion line :

Ghost line :



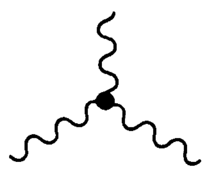
(B.10)

(b) The indices of diagram lines which will be contracted with transverse traceless projectors are encircled:  or 

Note that most of the calculations are performed in Landau-type gauges (2.20).

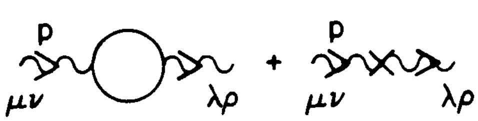
(c) $1/N$ corrected graviton propagator: 

(B.11)

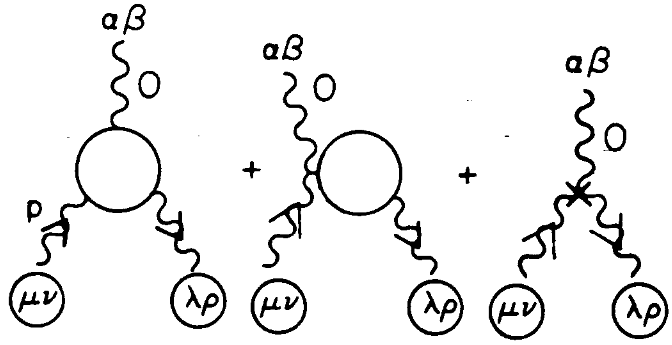
$1/N$ corrected graviton vertex: 

(d) Pair symmetrization in (α,β), (γ,δ), (μ,ν), (λ,ρ) is indicated with parentheses around the indices.

In Lagrangian (5.10), the one-loop matter contribution to the graviton self-energy and three point vertex is:



$$= -\frac{iG^2N}{(240)(4\pi)^2}\, p^4 \ln\left(-\frac{p^2}{\mu^2}\right) P^{(2)}_{\mu\nu,\lambda\rho} \qquad (B.12a)$$



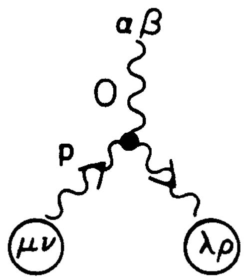
$$= -\frac{iG^3N}{(240)(4\pi)^2}\, p^2\Big\{\ell n\Big(-\frac{p^2}{\mu^2}\Big)\Big[\Big(\frac{1}{4}\,\eta_{\alpha\beta}p^2 - p_\alpha p_\beta\Big)\eta_{(\mu\lambda}\eta_{\nu\rho)} - \frac{1}{4}\,\eta_{(\alpha\mu}\eta_{\beta\lambda}\eta_{\nu\rho)}p^2\Big] - \frac{1}{2}\, p_\alpha p_\beta\, \eta_{(\mu\lambda}\eta_{\nu\rho)}\Big\} \tag{B.12b}$$

where the counterterm Z_G used is:

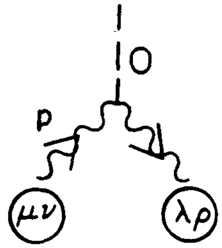
$$Z_G = 1 + \frac{G^2N}{240(4\pi)^2}\,\frac{1}{\epsilon} + \text{finite parts} \tag{B.12c}$$

with $n \equiv 4-2\epsilon$ and $N = N_S + 6N_F + 12N_V$ for N_S scalars, N_F Dirac-fermions and N_V vector bosons. The relevant Feynman rules in Landau-type gauges to leading order in $1/N$ are:

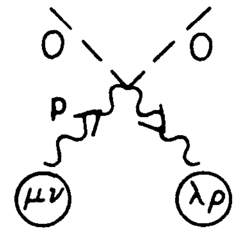
$$\frac{iP^{(2)}_{\mu\nu,\lambda\rho}}{p^2\Big[L^2 - \tau\, G^2N\, p^2\, \ell n\Big(-\frac{p^2}{\Lambda^2}\Big)\Big]} \tag{B.13}$$



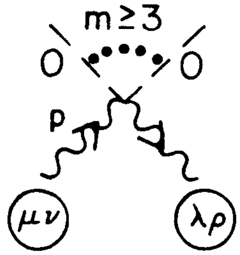
$$iGp^2\Big[L^2 - \tau\, G^2N\, p^2\, \ell n\Big(-\frac{p^2}{\Lambda^2}\Big)\Big] \times \Big\{\Big(\frac{1}{4}\,\eta_{\alpha\beta}p^2 - p_\alpha p_\beta\Big)\,\eta_{(\mu\lambda}\eta_{\nu\rho)} - \frac{1}{4}\,\eta_{(\alpha\mu}\eta_{\beta\lambda}\eta_{\nu\rho)}p^2\Big\} + \frac{i}{2}\,G\Big[L^2 + \tau\, G^2N\, p^2\Big]\, p_\alpha p_\beta\, \eta_{(\mu\lambda}\eta_{\nu\rho)} \tag{B.14}$$



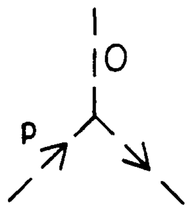
$$\frac{i}{v_0}\, p^2\left[L^2 + \tau\, G^2 N\, p^2\right]\eta_{(\mu\lambda}\eta_{\nu\rho)} \tag{B.15}$$



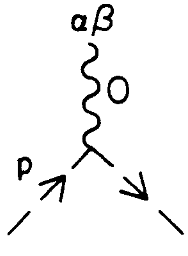
$$\frac{i}{v_0^2}\, p^2\left[L^2 - \tau\, G^2 N\, p^2\right]\eta_{(\mu\lambda}\eta_{\nu\rho)} \tag{B.16}$$



$$\frac{i}{v_0^m}\,(-)^{m-1}(m-1)!\,\tau\, G^2 N\, p^4\, \eta_{(\mu\lambda}\eta_{\nu\rho)} \tag{B.17}$$



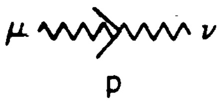
$$-\frac{4i}{n-2}\,\frac{\gamma}{v_0^3}\, p^4 \tag{B.18}$$



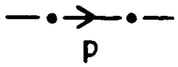
$$iG\left(-\frac{1}{2}\,\eta_{\alpha\beta}p^2 + p_\alpha p_\beta\right) + i\,\frac{G\gamma}{v_0^2}\left(\frac{1}{2}\,\eta_{\alpha\beta}p^4 - 2p_\alpha p_\beta p^2\right) \tag{B.19}$$

where $L^2 \equiv \dfrac{v_0^2 G^2}{24}$ and $\tau \equiv \dfrac{1}{240(4\pi)^2}$.

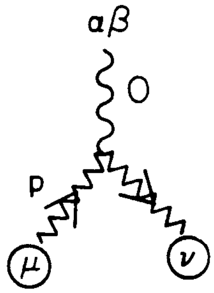
Additional Feynman rules for the scalar QED Lagrangian (7.1):



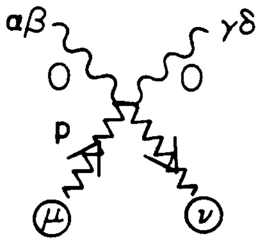
$$-\frac{i}{p^2 - e^2 v_1^2}\,\theta_{\mu\nu} \qquad (B.20)$$



$$\frac{i}{p^2} \qquad (B.21)$$

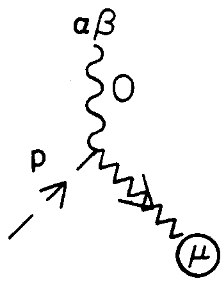


$$-\frac{i}{2}\,G\Big(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{(\mu\alpha}\eta_{\nu\beta)}\Big)(p^2 - e^2 v_1^2) + iG\,\eta_{\mu\nu}\,p_\alpha p_\beta \qquad (B.22)$$

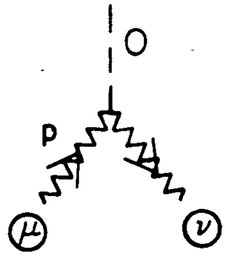


$$-\frac{i}{4}\,G^2\Big[\Big(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{(\alpha\gamma}\eta_{\beta\delta)}\Big)\eta_{\mu\nu} + \eta_{(\alpha\gamma}\eta_{\beta\mu}\eta_{\delta\nu)} - 2\eta_{\alpha\beta}\eta_{(\mu\gamma}\eta_{\nu\delta)}\Big](p^2 - e^2 v_1^2)$$

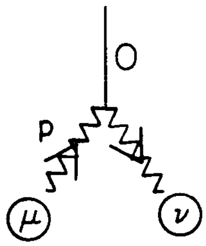
$$-\frac{i}{2}\,G^2\Big[\eta_{(\alpha\gamma}p_\beta p_{\delta)} - \eta_{\mu\nu}\Big(\eta_{\alpha\beta}p_\gamma p_\delta + \eta_{\gamma\delta}p_\alpha p_\beta\Big) + p_\alpha p_\beta\,\eta_{(\gamma\mu}\eta_{\delta\nu)} + p_\gamma p_\delta\,\eta_{(\alpha\mu}\eta_{\beta\nu)} - \frac{1}{2}\,\eta_{(\alpha\mu}\eta_{\gamma\nu}p_\beta p_{\delta)}\Big] \qquad (B.23)$$



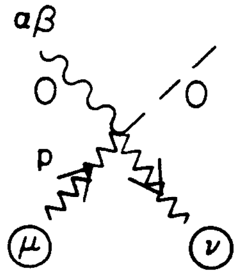
$$-\frac{1}{2}\,eGv_1\Big(\eta_{\alpha\mu}p_\beta + \eta_{\beta\mu}p_\alpha\Big) \qquad (B.24)$$



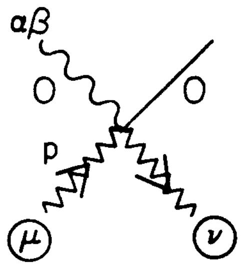
$$\frac{2i}{v_0'}\left(-\frac{n-4}{n-2}\,p^2+e^2v_1^2\right)\eta_{\mu\nu} \tag{B.25}$$



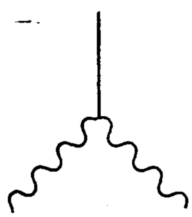
$$-2i\,e^2v_1\,\frac{v_0}{v_0'}\,\eta_{\mu\nu} \tag{B.26}$$



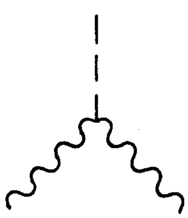
$$-i\,\frac{n-4}{n-2}\,\frac{G}{v_0'}\left(\eta_{\alpha\beta}\eta_{\mu\nu}p^2-\eta_{(\mu\alpha}\eta_{\nu\beta)}p^2-2\eta_{\mu\nu}p_\alpha p_\beta\right)$$
$$+\,iGe^2\,\frac{v_1^2}{v_0'}\left(\eta_{\mu\nu}\eta_{\alpha\beta}-\eta_{(\alpha\mu}\eta_{\beta\nu)}\right) \tag{B.27}$$



$$-\,iGe^2v_1\,\frac{v_0}{v_0'}\left(\eta_{\mu\nu}\eta_{\alpha\beta}-\eta_{(\alpha\mu}\eta_{\beta\nu)}\right) \tag{B.28}$$

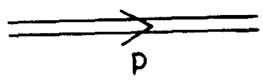


$$=0 \tag{B.29}$$

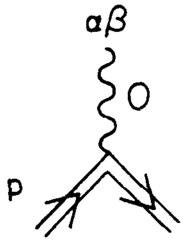


$$=(B.15) \tag{B.30}$$

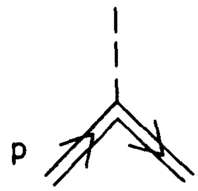
In the case of fermions [see eqs. (4.6j) and (7.14)]:



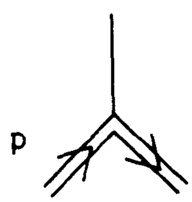
$$\frac{i}{\not p} \qquad (B.31)$$



$$\frac{i}{2} G\left[\eta_{\alpha\beta}(\not p - f v_1) - \frac{1}{2}\gamma_{(\alpha}p_{\beta)}\right] \qquad (B.32)$$



$$-\frac{2i}{n-2} f \frac{v_1}{v_0'} \qquad (B.33)$$



$$if \frac{v_0}{v_0'} \qquad (B.34)$$

APPENDIX C: INTEGRALS

At zero momentum, a general one-loop diagram contribution can be reduced to the following Euclidean space integral:

$$I_m^\nu(\alpha) \equiv \int_0^\infty dx \, \frac{x^\nu}{(\alpha + x\,\ell n\, x)^m} \tag{C.1}$$

where α and x are given in (6.5) and as shown in section 6, $\alpha > \frac{1}{e}$. Changing variables $x = e^u$:

$$\begin{aligned} I_m^\nu(\alpha) &= \int_{-\infty}^\infty du \, \frac{e^{(\nu+1)u}}{(\alpha + ue^u)^m} \\ &= \frac{1}{2}\int_{-\infty}^\infty du \left\{ \frac{e^{(\nu+1)u}}{(\alpha + ue^u)^m} + \frac{e^{-(\nu+1)u}}{(\alpha - ue^{-u})^m} \right\} \quad . \end{aligned} \tag{C.2}$$

There are pairs of complex conjugate solutions (u_j, u_j^*) of the equation $\alpha + ue^u = 0$, with:

$$u_j = \rho_j \, e^{i(\pi - \vartheta_j)} \quad ; \quad 0 \le \vartheta_j \le \pi \tag{C.3a}$$

where:

$$\rho_j = \frac{\vartheta_j + 2\pi j}{\sin \vartheta_j} \quad \text{and} \quad \alpha = \rho_j \, e^{-\rho_j \cos \vartheta_j} \quad . \tag{C.3b}$$

Thus, the integration can be performed using the residue theorem when $\nu + 1 < m$:

$$I_m^\nu(\alpha) = -2\pi \, Im \sum_{j=0}^\infty Res\left[\frac{e^{(\nu+1)u}}{(\alpha + ue^u)^m} \, , \, u_j \right] \quad ; \quad m = 2, 3, 4, \dots \quad . \tag{C.4}$$

Notice that:

$$\lim_{j \to \infty} \vartheta_j = \frac{\pi}{2} \tag{C.5a}$$

and:

$$\lim_{j \to \infty} |u_j| = \lim_{j \to \infty} \rho_j \sim j \quad . \tag{C.5b}$$

Then, the series (C.4) converges at least like $\sum j^{-2}$. If a Feynman graph in 4-dimensions is divergent, $\nu+1$ is an integer bigger than m. But using dimensional continuation ν is replaced by $\nu-\epsilon$, where $\epsilon \equiv \frac{4-n}{2}$, and the integral can be made convergent. Furthermore, in the final answer (C.4) no pole is recovered in the limit $\epsilon = 0$ [see eqs. (C.5)]; when $m \geq 2$ the integral (C.1) contains no logarithmic infinities and dimensional regularization is insensitive to higher divergences: $\int dx\, x^{\nu}\, (\ell n\, x)^{-m} = 0$.

Equation (C.4) is valid for $m = 1$ as well, although the integrand does not satisfy the asymptotic condition needed to apply directly the residue theorem. Consider the integral:

$$
\begin{aligned}
I_1^{\nu}(\alpha) &= -\int d\alpha \int_0^{\infty} dx\, \frac{x^{\nu}}{(\alpha + x\,\ell n\, x)^2} \\
&= 2\pi\, Im \sum_{j=0}^{\infty} \int d\alpha\, \frac{e^{(\nu-1)u_j}}{(u_j+1)^2}\left(\nu - \frac{1}{u_j+1}\right) \\
&= -2\pi\, Im \sum_{j=0}^{\infty} \int du_j\, \frac{d}{du_j}\, \frac{e^{\nu u_j}}{u_j+1} \\
&= -2\pi\, Im \sum_{j=0}^{\infty} \frac{e^{\nu u_j}}{u_j+1}
\end{aligned}
\tag{C.6}
$$

and the result agrees with the $m = 1$ value of (C.4). The final answer for $I_1^{\nu}(\alpha)$:

$$
I_1^{\nu}(\alpha) = -2\pi\, (-)^{\nu}\, \alpha^{\nu} \sum_{j=0}^{\infty} Im\, \frac{1}{u_j^{\nu}(u_j+1)} \tag{C.7}
$$

can be evaluated using the formulae:

$$
Im\, \frac{1}{u_j^{\nu}} = \frac{(-)^{\nu}}{\rho_j^{\nu}} \sin \nu\vartheta_j \tag{C.8a}
$$

$$
Im\, \frac{1}{u_j+1} = -\frac{\rho_j \sin\vartheta_j}{1+\rho_j^2 - 2\rho_j \cos\vartheta_j} \quad . \tag{C.8b}
$$

Using eqs. (C.5), we observe that the series (C.7) converges for $\nu > 0$ but diverges like $\sum j^{-1}$ when $\nu = 0$; the integral $I_1^{0}(\alpha)$ has ultraviolet logarithmic divergences, since $\int dx\, x^{-1} (\ell n\, x)^{-1} \sim \int (d\ell n\, x)(\ell n\, x)^{-1}$, and acquires an $1/\epsilon$ pole in dimensional regularization. The diagrams of fig. 5.1 have degree of divergence $D = 0$ and when only one graviton propagator is present correspond to $I_1^{-\epsilon}(\alpha)$ which diverges. However, if these diagrams contain two or more internal gravitons they are convergent since they behave like $\int dx\, x^{-1} (\ell n\, x)^{-m}$, $m \geq 2$.

The effective potential integration (6.6) is proportional to $[I_1^{2-\epsilon}(\alpha) - \alpha I_1^{1-\epsilon}(\alpha)]$:

$$I_{V_{eff}} \equiv \int_0^\infty dx\; x^{1-\epsilon}\, \ell n \left[\beta\, \frac{\alpha + \frac{x}{\beta}\, \ell n\, \frac{x}{\beta}}{\alpha + x\, \ell n\, x}\right] = \int_0^\infty dx\; x^{1-\epsilon} (\beta^{2-\epsilon} - 1)\, \ell n\, (\alpha + x\, \ell n\, x)$$

$$= -\frac{1}{2-\epsilon} (\beta^{2-\epsilon} - 1) \int_0^\infty dx\; x^{1-\epsilon}\, \frac{x - \alpha}{\alpha + x\, \ell n\, x} \qquad (C.9)$$

where we integrated by parts and ignored integrals of the form $\int_0^\infty dx\, x^{\nu-\epsilon}$ which vanish in dimensional regularization. Using (C.7) and (C.8) we find:

$$I_{V_{eff}} = \pi\, (\beta^2 - 1)\, \alpha^2\, Im \sum_{j=0}^{\infty} \frac{1}{u_j^2} \quad . \qquad (C.10)$$

Therefore [see (C.3)]:

$$I_{V_{eff}} = \pi(\beta^2 - 1)\, \alpha^2 \sum_{j=0}^{\infty} \frac{\sin\, 2\vartheta_j\, \sin^2 \vartheta_j}{(\vartheta_j + 2\pi j)^2} \qquad (C.11a)$$

$$= \pi(\beta^2 - 1) \sum_{j=0}^{\infty} \sin\, 2\vartheta_j\, e^{-2\frac{\vartheta_j + 2\pi j}{\tan \vartheta_j}} \qquad (C.11b)$$

and by substituting (C.11) in (6.6) we recover (6.7). The series (C.11) can be evaluated numerically and converges very fast. In fact, the contribution of the first pair of complex conjugate poles ($j = 0$) almost provides the full answer. At the minimum of $V_{eff}[\sigma]$ for $\sigma = 0$ (see section 6), the infinite sum in (C.11) has to vanish. The resulting numerical solution is $\alpha \simeq 1.62$ and all the poles lie very close to the imaginary axis. Notice that the solution coming from the $j = 0$ approximation is $\alpha = \frac{\pi}{2} \simeq 1.57$.

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FIGURE CAPTIONS

The correspondence between fields and diagram lines is given in Appendix B, eq. (B.10).

Figure 2.1. Matter corrections to the one-loop graviton self-energy.

Figure 2.2. Leading $1/N$ graviton propagator.

Figure 2.3. Graph of $x \ell n\, x$ versus x.

Figure 3.1. One ghost matrix elements of the operator $Gch_{\mu\nu}$ (a) and $c\phi$ (b) at zero momentum.

Figure 3.2. Ward identity for the cosmological constant in Landau-type gauges.

Figure 3.3. Ghost-ghost-graviton vertex.

Figure 4.1. Low energy dilaton theorem.

Figure 4.2. Vector boson self-energy diagrams up to one-loop.

Figure 4.3. Zero momentum dilaton-vector boson vertex diagrams up to one-loop.

Figure 5.1. Divergent diagrams involving external ghost lines with their degree of divergence D.

Figure 5.2. Divergent diagrams involving external scalar and dilaton lines with their degree of divergence D.

Figure 6.1. Dilaton effective potential to leading order in $1/N$.

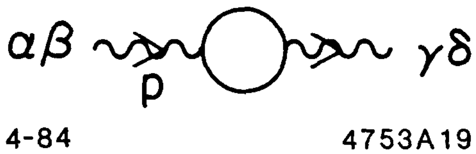
Figure 6.2. Leading $1/N$ graviton tadpole.

Figure 6.3. Corrections to the graviton tadpole (a) and the first derivative of the σ-potential coming from λ and γ contributions.

Figure 7.1. Graviton, dilaton, Higgs tadpoles up to one-loop in scalar QED.

Figure 7.2. Matter contributions to graviton-graviton, graviton-dilaton, graviton-Higgs self-energies up to one-loop.

Figure 8.1. One-loop vacuum diagrams of the operators forming the nonlinear part $\tilde{s}\, h_{\mu\nu}$ of $sh_{\mu\nu}$ (a) and $\tilde{s}\,\sigma$ of $s\sigma$ (b).



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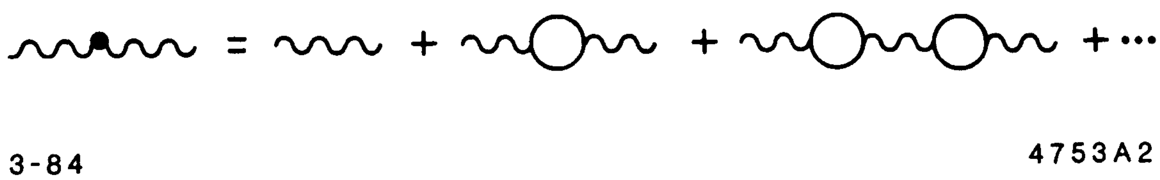

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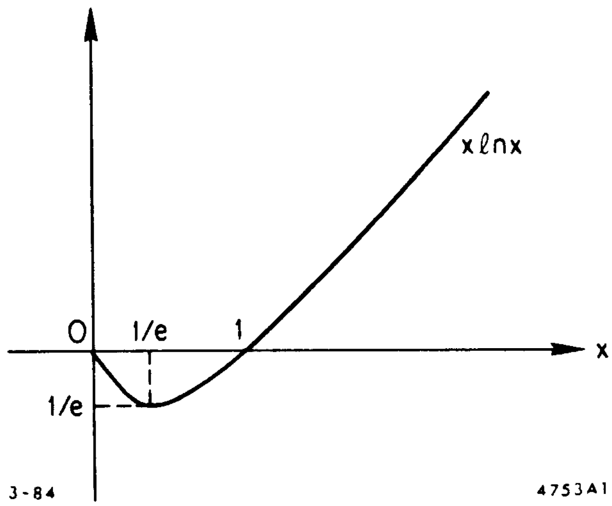

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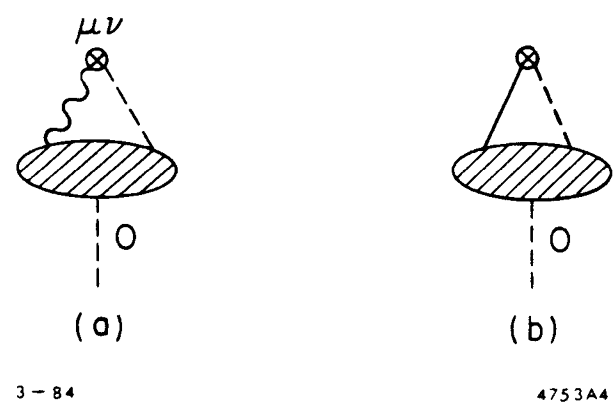

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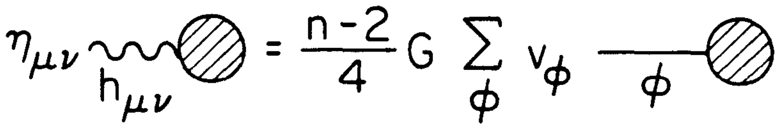

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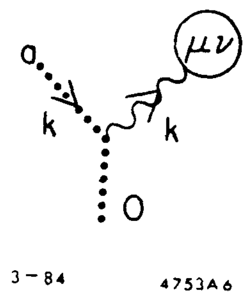

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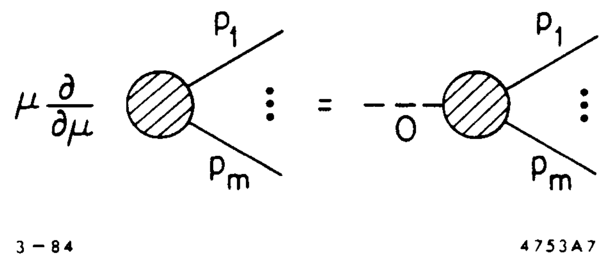

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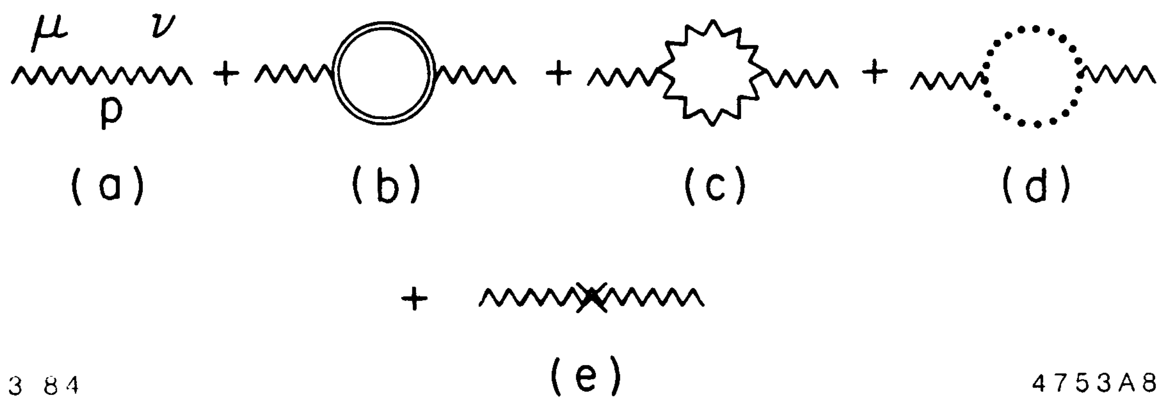

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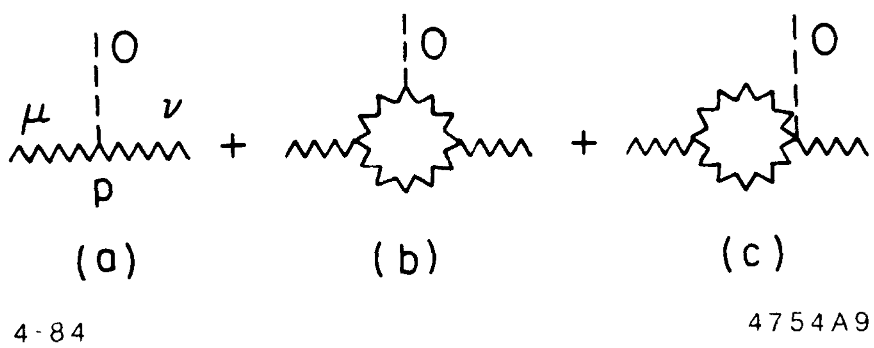

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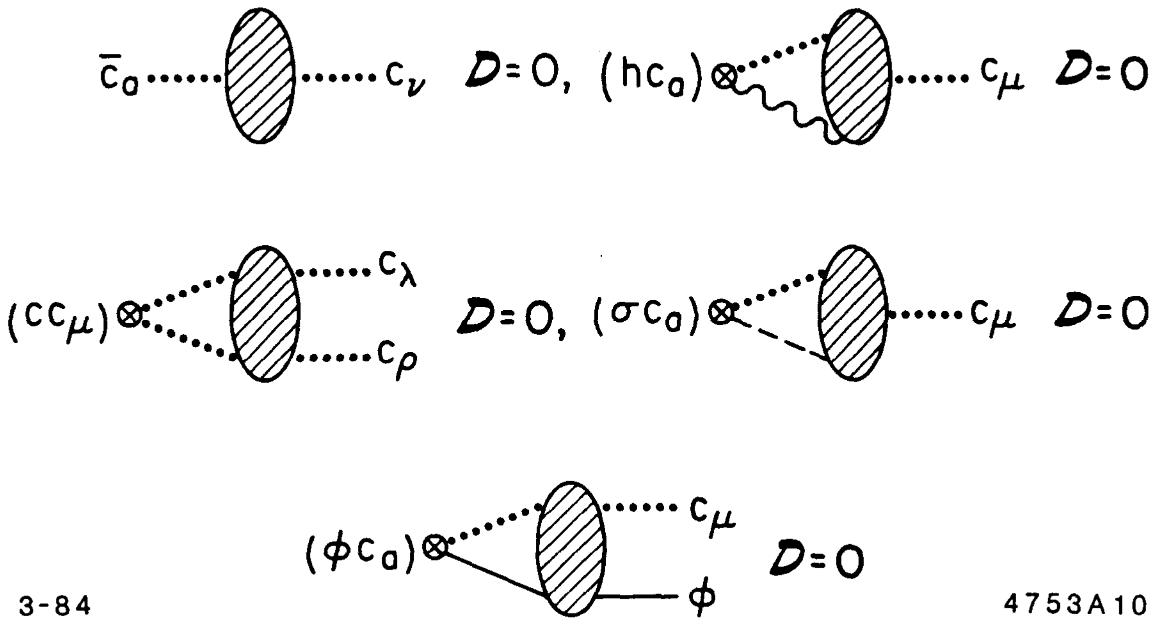

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c_ν
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c_μ
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(cc_μ)
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(σc_a)
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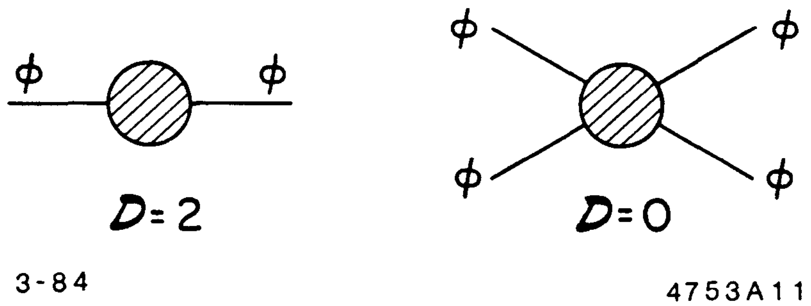
Fig. 5.1



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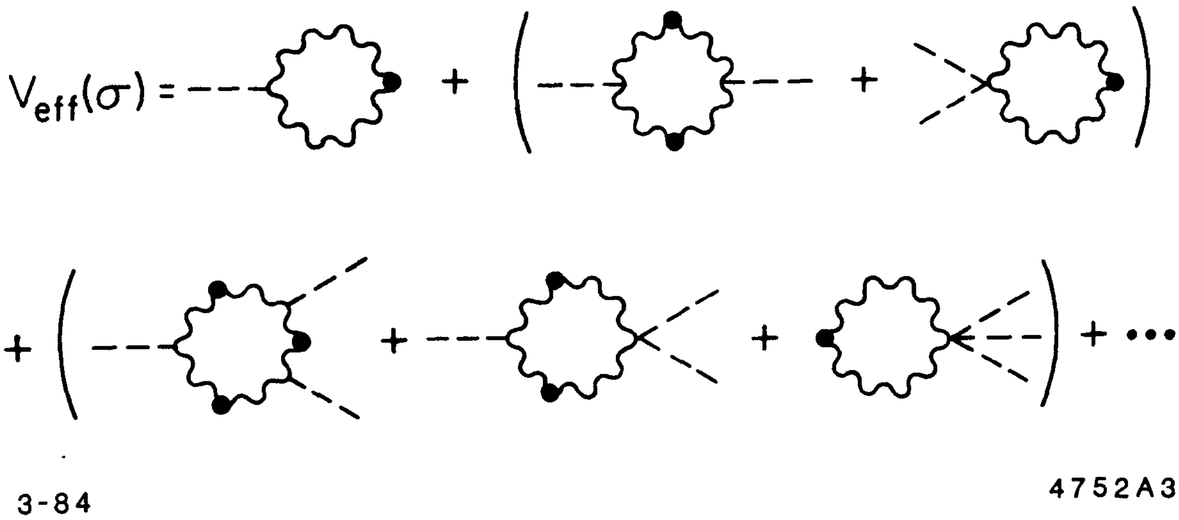

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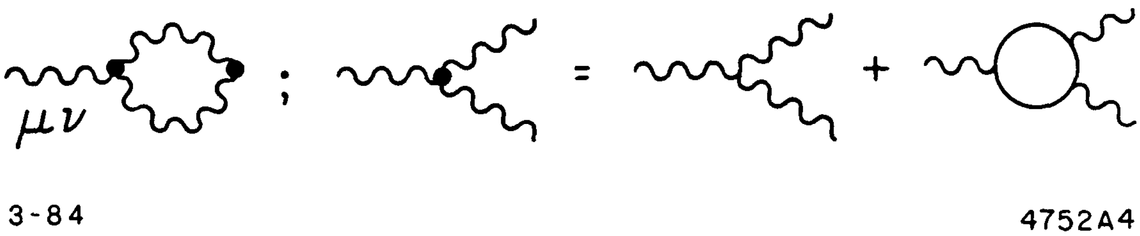

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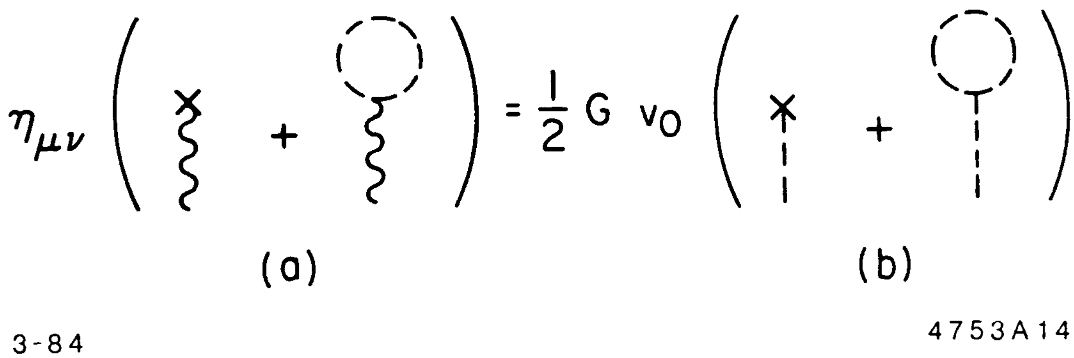

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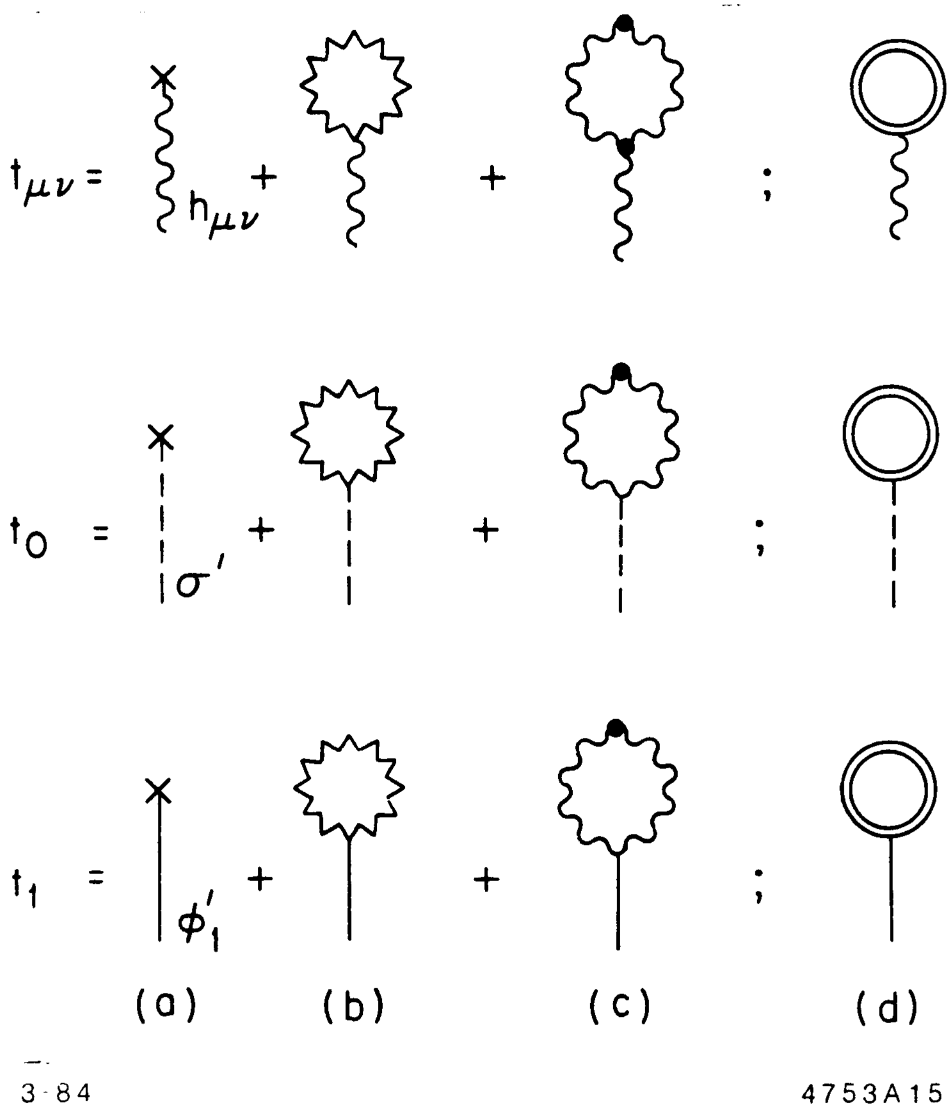

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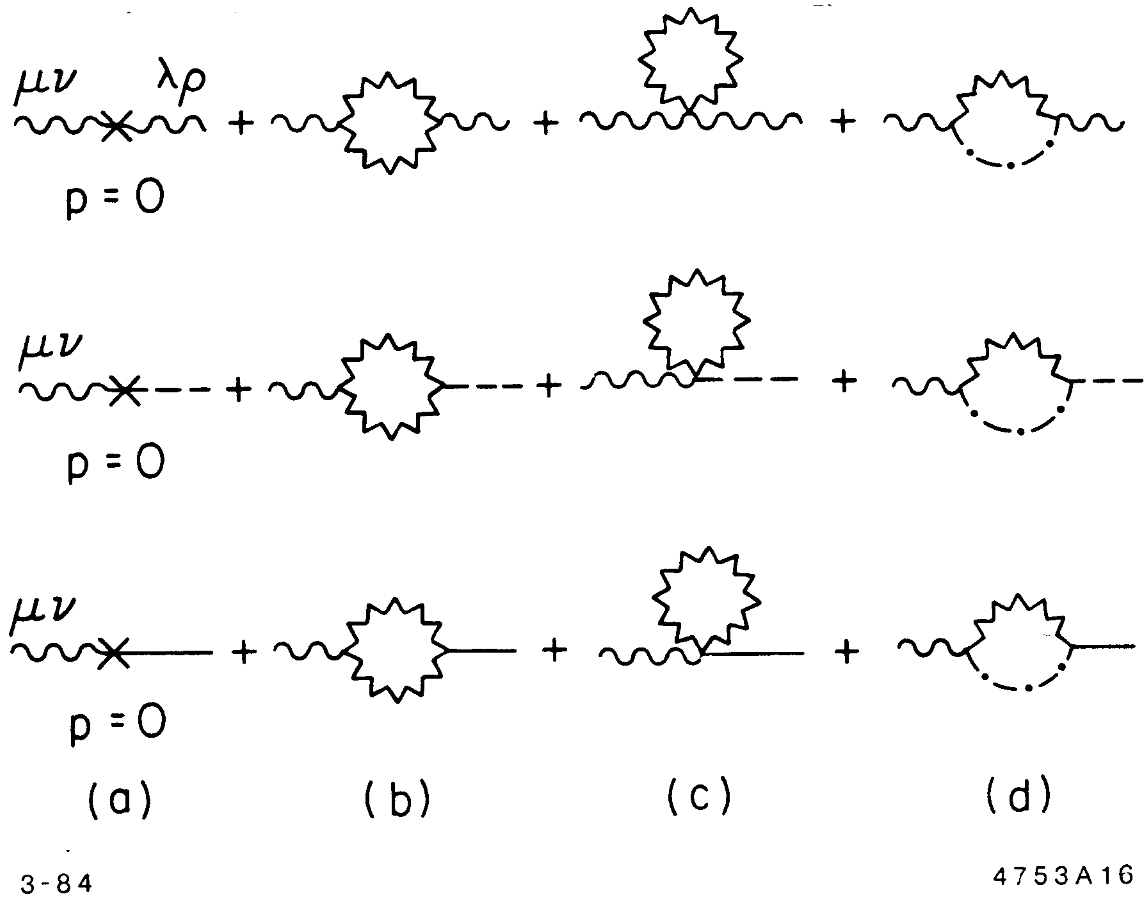

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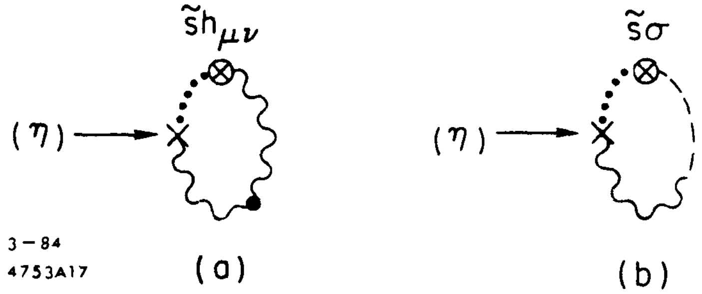

Fig. 8.1