Thermodynamics of Specific $f(R, T^{\phi})$ Gravity with Entropy Corrections

By

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I, **Zoya Khan** with registration number **CIIT/FA17-RMT-055/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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DEDICATED TO

My Family

Table of Contents

Table of Contents viii

List of Figures x

Acknowledgements xiii

Abstract xv

1 Introduction 1

2 Preliminaries 7
2.1 Cosmology . . . 8
2.2 Cosmic Acceleration . . . 8
2.3 Dark Energy . . . 9
2.4 ΛCDM Model . . . 9
2.5 Quintessence . . . 11
2.6 Laws of Thermodynamics . . . 12
2.6.1 Zeroth Law of Thermodynamics . . . 13
2.6.2 First Law of Thermodynamics . . . 13
2.6.3 Second Law of Thermodynamics . . . 16
2.6.4 Entropy . . . 16
2.6.5 Third Law of Thermodynamics . . . 19
2.7 Thermodynamics in Black Hole . . . 19
2.7.1 Zero Law of Black Hole Mechanics . . . 19
2.7.2 First Law of Black Hole Mechanics . . . 21
2.7.3 Second Law of Black Hole Mechanics . . . 21
2.7.4 Generalized Second Law of Thermodynamics . . . 21
2.7.5 Third Law of Black Hole Mechanics . . . 21
2.8 Thermal Equilibrium Condition . . . 22
2.8.1 Bekenstein-Hawking Entropy . . . 22
2.8.2 Rényi Entropy . . . 22
2.8.3 Logarithmic Corrected Entropy . . . 24

2.8.4 Sharma-Mittal Entropy 24
2.9 Modified Theories 24
2.10 $f(R, T)$ Theory 25
2.11 $f(R, T^{\phi})$ Theory 26

3 Thermodynamic Consequences of Specific Modified Gravity on the Apparent Horizon 28
3.1 Basics of $f(R, T^{\phi})$ theory 29
3.2 Thermodynamics Laws 32
3.2.1 First law of thermodynamics 32
3.2.2 Generalized second law of thermodynamics 34
3.3 GSLT for Flat Spacetime: $k = 0$ 36
3.4 GSLT for Closed Spacetime: $k = 1$ 38
3.5 GSLT for Open Spacetime: $k = -1$ 43

4 Thermodynamic Implications of Multiquintessence Scenario 48
4.1 First-order formalism of multiquintessence scenario 49
4.1.1 **Model 1** 50
4.1.2 **Model 2** 51
4.1.3 **Model 3** 52
4.2 GSLT and Thermal Equilibrium Condition 52
4.2.1 Sharma-Mittal Entropy 56
4.2.2 Logarithmic Corrected Entropy 59
4.2.3 Rényi Entropy 61

5 Summary 66

List of Figures

2.1 Expansion of the universe 10

2.2 Example of Zero Law of Thermodynamics 14

2.3 Example of First Law of Thermodynamics 15

2.4 Example of Second Law of Thermodynamics 17

2.5 Example of Entropy 18

2.6 Example of Third Law of Thermodynamics 20

2.7 The Bekenstein-Hawking entropy is the entropy to be attributed to any BH: one fourth of its HA expressed in units of the Planck area 23

3.1 Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 0$. 36

3.2 Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 0$. 36

3.3 Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 0$. 37

3.4 Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 0$. 37

3.5 Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 0$. 37

3.6 Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 0$. 37

3.7 Plot of S' with z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 38

3.8 Plot of S'' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 39
3.9 Plot of S' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 41
3.10 Plot of S'' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 41
3.11 Plot of S' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 42
3.12 Plot of S'' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right. 43
3.13 Plot of S' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. 44
3.14 Plot of S'' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. 44
3.15 Plot of S' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. . . . 45
3.16 Plot of S'' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. . . . 46
3.17 Plot of S' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. 47
3.18 Plot of S'' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right. 47

4.1 Plot of $\dot{S}_{tot}$ versus t for model 1. 55
4.2 Plot of $\ddot{S}_{tot}$ versus t for model 1. 55
4.3 Plot of $\dot{S}_{tot}$ versus t for model 2. 56
4.4 Plot of $\ddot{S}_{tot}$ versus t for model 2. 56
4.5 Plot of $\dot{S}_{tot}$ versus t for model 3. 57
4.6 Plot of $\ddot{S}_{tot}$ versus t for model 3. 57
4.7 Plot of $\dot{S}_{tot}$ versus t for model 1. 58
4.8 Plot of $\ddot{S}_{tot}$ versus t for model 1. 58
4.9 Plot of $\dot{S}_{tot}$ versus t for model 2. 59

4.10 Plot of $\ddot{S}_{tot}$ versus t for model 2. 59
4.11 Plot of $\dot{S}_{tot}$ versus t for model 3. 59
4.12 Plot of $\ddot{S}_{tot}$ versus t for model 3. 59
4.13 Plot of $\dot{S}_{tot}$ versus t for model 1. 60
4.14 Plot of $\ddot{S}_{tot}$ versus t for model 1. 60
4.15 Plot of $\dot{S}_{tot}$ versus t for model 2. 61
4.16 Plot of $\ddot{S}_{tot}$ versus t for model 2. 61
4.17 Plot of $\dot{S}_{tot}$ versus t for model 3. 62
4.18 Plot of $\ddot{S}_{tot}$ versus t for model 3. 62
4.19 Plot of $\dot{S}_{tot}$ versus t for model 1. 63
4.20 Plot of $\ddot{S}_{tot}$ versus t for model 1. 63
4.21 Plot of $\dot{S}_{tot}$ versus t for model 2. 64
4.22 Plot of $\ddot{S}_{tot}$ versus t for model 2. 64
4.23 Plot of $\dot{S}_{tot}$ versus t for model 3. 65
4.24 Plot of $\ddot{S}_{tot}$ versus t for model 3. 65

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Abstract

In this thesis, we examine the thermodynamical behavior of homogeneous and isotropic universe (flat and non-flat) in the framework of $f(R, T^{\phi})$ gravity, where R stands for Ricci scalar and T^{ϕ} represents the trace of the energy-momentum tensor of a scalar field ϕ. Throughout we follow the first-order formalism, that specifies the scalar field to the Hubble parameter which becomes $H = W(\phi)$. By using Bekenstein-Hawking entropy, we analyze the validity of the generalized second law of thermodynamics at apparent horizon for three different models of $W(\phi)$ and discuss the thermal equilibrium condition for these cases as well. We observe that, this generalized law gives better results only for one model and as well as thermal equilibrium condition satisfies for each value of $W(\phi)$ at apparent horizon with Bekenstein-Hawking entropy.

Also, we investigate the generalized second law of thermodynamics and thermal equilibrium condition in multi-component scalar field for flat Friedmann-Robertson-Walker universe. We are following the first-order formalism and we choose three superpotential models of the Hubble parameter, and by using these models we observe the validity of the generalized law as well as thermal equilibrium condition for Bekenstein entropy. Also, we take three different entropies which are logarithmic corrected, Sharma-Mittal and *Rényi* entropies, for each model of $W(\phi_i)$, we study the behavior of the generalized law and thermal equilibrium condition, using all entropies at Hubble horizon. We inspect that, the generalized law is valid and thermal equilibrium condition satisfies only for some cases.

Chapter 1
Introduction

Cosmological observations disclose that our universe is approximately homogeneous and isotropic at large scale described by the standard Friedmann-Robertson-Walker (FRW) model [80]. Also, several recent observations have confirmed the accelerated expansion (AE) of the universe but these observations do not offer any clear picture of this mysterious behavior of the universe [1]-[3]. The invention of this cosmic AE has promoted many researchers to explore the cause of this massive change in cosmic history [4]. However, a mysterious force known as dark energy (DE) is considered as the basic element which is responsible for expanding phase of the universe. In general relativity (GR), the most easy plan to cause the AE is to introduce the cosmological constant, but it has two basic fine-tuning and coincidence problems [5]. A few alternative models of DE in geometric part are $f(R)$ [6] and $f(R,T)$ [7] theories of gravity. From the different recognize cosmological evidence the current cosmic acceleration (CA) of the universe disclosed [3], [8]- [13], which is measured by certain analyses [14]-[18]. In the present age current perceptions fully recommend that the CA is experiencing by the universe [8, 19].

Scalar fields (SF's) have been studied comprehensively in cosmology during the last three decades in order to explain the current CA [20]-[22]. Harko et al. [7] described the $f(R,T)$ theories of gravity, allow one to scrutinize an optimistic replacement to DE, as a generalization of the $f(R)$ theories. It can also provided inflationary epoch explanation through its SF perspective, named as $f(R,T^{\phi})$ gravity. By progressing the $f(R,T^{\phi})$ approach which is submissive to $f(R,T)$ gravity, we expect here, to make the $f(R,T)$ theories accomplished to contribute additionally to inflationary and radiation-dominated epoch in a self-consistent technique. Baffou et al. [23] claimed that the high redshift $f(R,T)$ cosmological solutions lead to recover the standard model of cosmology if the $f(R,T)$ functional form is linear in R apparently because when $z \gg 1$, the radiation with equation of state parameter (EoS) $p=\rho/3$ influence the universe dynamics, disappearing the trace of the energy-momentum tensor. Myrzakulov [24] studied the generalization of $F(R)$, $F(T)$ and $F(R,T)$ theories. They found field equations of $F(R,T)$ gravity by presenting point-like Lagrangian explicitly. By using specific model $F(R,T) = \mu R + \nu T$, derived the exact solutions. Alves et al. [25] declared the presence of gravitational waves likewise for polarization approach in both $f(R,T)$ other than $f(R,T^{\phi})$ theories.

Moraes and Santos [26] showed how $f(R,T^{\phi})$ theories of gravity can broaden to the study of the primary stages of the universe, their results estimated a elegant exist form an inflationary stage to a radiation -dominated era. They also predicted a late time CA after a matter-dominated phase, allowing the $f(R,T^{\phi})$ theories to defined in a self-consistent way, all the distinctive phases of the universe dynamics. Without necessarily recovering to make $f(R,T)$ gravity able to describe a radiation universe without anyone else's input, the $f(R)$ formalism outcomes will make such a theory which is equipped for contributing all the distinctive stages of the universe dynamics. So as to do as such, they applied the first-order formalism [27, 28] to $f(R,T^{\phi})$ gravity. Vijay and singh [29] explored the SF behavior for flat FRW universe in modified $f(R,T)$ gravity. They examined the behavior of constructed model through the deceleration parameter.

In cosmology, many authors observed the significance of the SF. To clarify the CA it perhaps connected to generate the inflaton which is occurred in the early universe or as a DE applicant like quintessence [30]-[32]. SF's illustrate different development of the universe like the inflaton (inflationary era), the DE, the component of dark matter [33]-[41]. They are characterized to be coupled to the gravity, minimally or non-minimally [42] -[46]. In particular, to depict the advancement of the universe at least two SF's collaborate over their kinetic or potential state. The quintom is the most basic multi-SF theory in which quintessence and phantom SF devote dark portion and mostly DE portion of the region [47]-[50]. It is feasible to utilize just a single SF to interpret the DE and past inflation in quintessence inflation model [51]. Recently to analyze the possible CA, quintessence is conjure as alternate of the cosmological constant [52].

Quintessence originates from the SF models, which conclude the universe elements is represented by the SF. With two SF models, quintessence models additionally developed cosmological models [53]-[55]. Roy and Bamba [56] explored interacting quintessence model with the quintessence potential, by using the parametrization of interaction quintessence models they extended the quintessence SF. Hertzberg et al. [57] investigated the fine-tuning of quintessence model for DE in the framework of swampland conjectures. $D\acute{i}az$ [58] restricted to EoS to studied the problem of the quintessence potential, they acquired the statement of luminosity distance. By differing non-negative cosmological term they generalized the quintessence model, confining the SF energy density.

Zlatev et al. [59] presented a form of quintessence, tracker field. Including the new inspiration for the quintessence scheme, to demonstrate how it might clarify the occurrence. Roy and Banerjee [60] studied dynamical system consideration of SF. They checked for late time attractors and defined two examples, exponential and the power-law potentials. They examined the stable solutions for a few quintessence models. SF describe the late and early aspects of CA [61, 62]. Yang et al. [63] considered different quintessence SF models and they found that for early deceleration phase to the present CA all models carried out fine change. They also found a strong negative relation among the parameters for all quintessence SF models.

The determination of thermodynamical black hole (BH) recommended basic association between relativistic gravity and thermodynamics laws. Anyhow, people have been trying to find a remarkable way to develop such relation [64]. Jacobson [65] was first to find the Einstein field equations from the Clausius relation $T_hdS_h = \delta Q$, with the fact that the entropy is proportional to the horizon area (HA). BH act as thermodynamic system alongside temperature being related to surface gravity and entropy for HA [66]. Akbar and Cai [67] discovered that the differential form of Friedmann equations at the apparent horizon (AH) can be rearranged as $dE = TdS + WdV$ (E is the total energy of matter, V and W are the volume inside the AH and work density correspondingly). Mazumder and Chakraborty [68] analyzed the thermodynamics in scalar-tensor theory for homogeneous spherically symmetric FRW model and examined the GSLT at event horizon (EH) for the holographic DE and when universe filled with perfect fluid. The generalized second law of thermodynamics (GSLT) has a remarkable significance in modified theories of gravity.

Debnath et al. [69] explored GSLT for equilibrium and non-equilibrium phases at EH as well as AH in flat FRW universe filled with n-components. Sharif and Fatima [70] examined GSLT in modified Gauss-Bonnet gravity with power-law correction and logarithmic corrected entropy at EH and Hubble horizon (HH). Bamba and Geng [71] showed that inward and outward the AH for same temperature of the universe the second law of thermodynamics verified for phantom and non-phantom descriptions in $f(R)$ gravity. Sharif and Zubair [72] constructed that GSLT holds for both phantom as well as non-phantom phases of the universe likewise checked the thermodynamics for equilibrium and in addition non- equilibrium characterization at AH in $f(R,T)$ gravity. Sharif and Ikram [73] studied non-equilibrium thermodynamical behavior in homogeneous and isotropic universe at AH in $f(G,T)$ gravity. Chattopadhyay and Ghosh [74] investigated the validity of GSLT at AH, EH and particle horizon in modified $f(R)$ Horava-Lifshitz gravity [75]. They observed that GSLT is valid under this gravity.

The connection of the thermodynamic evolution is concerned to the idea of additional thermodynamical variables and entropy. Lymperis and Saridakis [76] utilized the tsallis entropy and through the application of the first law of thermodynamics (FLT) they constructed the cosmological scenarios. They showed that with the sequence of DE span and from the value of the parameter of DE, EoS δ, during the evolution, experience the phantom-divide crossing and can be quintessence or phantom-like. Debnath et al. [77] investigated the equilibrium and non-equilibrium picture of GSLT for EH and AH for flat FRW metric which is filled with n-component fluid. In quintessence and phantom regimes they acquired constraints on the power-law parameter α. Bamba [78] studied the GSLT in AH and future EH in $f(T)$ gravity. They also showed the conditions of the quintessence and phantom epoch in particular scenario by which GSLT will be valid. They also discussed validity of GSLT for logarithmic corrected entropy and power-law correction. Sharif and Zubair [79] checked the validity of GSLT in $f(R,T)$ gravity for AH. They focused on two specific models of $f(R,T)$ gravity and concluded that the derived models signified quintessence and phantom eras.

Motivated by the work of Sharif and Siddiqa [80], we discuss the validity of GSLT and thermal equilibrium condition in $f(R,T^{\phi})$ gravity for flat and non-flat FRW universe. We follow the first-order formalism and chose three models of Hubble parameter and for each model we construct the equation of GSLT and discuss the behavior of GSLT and thermal equilibrium condition at apparent horizon along with Bekenstein-Hawking entropy. We examine GSLT and thermal equilibrium condition graphically for constructed models.

We explore thermodynamical behavior in multi-quintessence SF, motivated by the work of Correa [81]. Here, we also implementing first-order formalism and chose three different superpotential models. We observe the behavior of GSLT and thermal equilibrium condition at Hubble horizon with Bekenstein-Hawking entropy for flat FRW universe. We also discuss the validity of GSLT and thermal equilibrium condition for Renyi, logarithmic corrected and Sharma-Mittal entropies at Hubble horizon. We observe the results graphically. We arrange the thesis work in the following configuration:

- In Chapter 2, we give some basic definitions to understand the thesis work easily.
- In Chapter 3, we discuss the thermodynamical behavior of $f(R,T^{\phi})$ gravity for (flat as well as non-flat) FRW universe by graphical representations.
- In Chapter 4, we discuss the thermodynamical behavior of multi-quintessence SF for flat FRW universe. And graphically shows the behavior of GSLT and thermal equilbrium condition for each case.
- In Chapter 5, we conclude the results.

Chapter 2
Preliminaries

In this chapter, we discuss some basic concepts which we use in this thesis and helpful to understand the thesis work.

2.1 Cosmology

The branch of astronomy interested about the investigation of the starting point and expansion of the cosmos is called cosmology. It is the logical analysis of the initiation, advancement and possible objective of the cosmos. It has made huge walks in the previous hundred years because of intellectual and informational evolution. Over Einstein's 1917 static model of the universe cosmology started as a part of hypothetical material science. The big bang theory is ruled by modern cosmology which unites observational astronomy and molecule material science. Instead of considering independently the galaxies, stars and BH that fill it, basically it thinks about the universe as one substance. To figure out the physical cosmos as a bound together entirely it unites the natural sciences, especially cosmology and material science. In general it is the logical investigation of the extensive scale properties of the cosmos.

2.2 Cosmic Acceleration

It is the perception that the development of the cosmos is to such an extent that the velocity at which a far off cosmic system is retreating from the spectator is consistently expanding with time. The expansion of the universe is accelerating. In 1925, Hubble demonstrated the expansion of the universe. The connection among the speed and distances of distant galaxies from Earth was demonstrated by him. Firstly, it was indicated by supernovae that the universe is expanding faster. The High-Z supernovae search team and the supernovae cosmology project, invented the expansion of the universe in 1998 and they used distant type Ia supernovae. This development more often indicated as the metric expansion. Mathematically

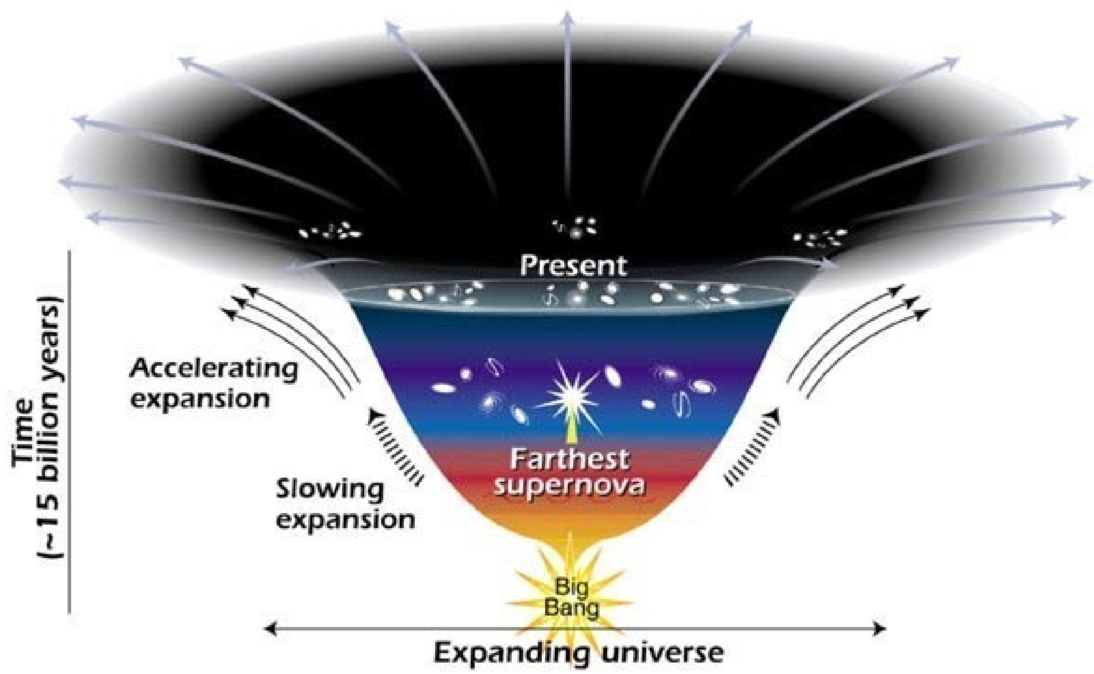


Figure 2.1: Expansion of the universe

and Physically, the metric signifies a proportion of distance and it suggest that the distance inside the universe is evolving itself. It is observed that, there is a force due to which universe is expanding, named as DE. As per Einstein's condition, the cosmic acceleration of the universe is represented by the sum and kind of energy in the cosmos, also through the geometry of the space.

2.3 Dark Energy

DE is that mysterious force which is responsible for the AE of the universe. It has repulsive nature with negatively large pressure but its complete characteristics are still unknown. It is not detected directly. The most recent discoveries show that over 70 percent of the cosmos is made out of DE. It is distinguished by its impact on the estimate at which large-scale arranges and by its impact on the estimate at which the cosmos extends for example milky way and bunches of the milky way structure through gravitational fluctuation.

2.4 ΛCDM Model

To coordinate with later examined data different DE models have been proposed and in this arrangement the most least complex model is the ΛCDM (Cold dark matter) model, which is in great concurrence with the ongoing observational information. In GR, this model is obtained by presenting the cosmological constant Λ, for which EoS is $w_\Lambda = -1$. In cosmology, this model can be enhanced by including quintessence, cosmological inflaton and different components that are flow regions of theory and research.

2.5 Quintessence

Quintessence is a theoretical type of DE, exactly a SF proposed as a clarification of the perception of CA of the universe. Relying upon the proportion of its dynamic and potential energy it can be either attractive or repulsive. Quintessence is a genuine type of energy particular from any typical issue, radiation or even dark matter. A few cosmologist state that, quintessence is a colorful sort of energy field that drives particles from one another, uncontrollable gravity and other essential powers. Steinhardt said, quintessence incorporates a broad range of potential outcomes. It is a time-advancing, forceful and spatially subordinate type of energy with negative pressure satisfactory to run the CA. He also recommend, while it was sufficiently cool for the atoms and ultimately stars to shape, from emission to matter -dominated cosmos, the quintessence diverted throughout the change. The coincidence problem and might be the fine-tuning problem can be solve by quintessence. The action for quintessence is [82]

$$S = \int d^4x\sqrt{-g}\left[-\frac{(\nabla\phi)^2}{2} - V(\phi)\right],$$

d^4x is invariant volume element in four-dimension, g is the determinant of metric, $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $V(\phi)$ represents potential of the field. The variation of the action with respect to scalar field in a flat FRW universe is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0,$$

where $V_\phi = \frac{dV}{d\phi}$. By varying the action, the energy-momentum tensor of the field is established as

$$T_{\mu\nu} = -\frac{2\delta S}{\sqrt{-g}\delta g^{\mu\nu}}.$$

Taking $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, we get

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right].$$

The energy density and pressure of the scalar field is given as

$$\begin{aligned} \rho &= \frac{\dot{\phi}^2}{2} + V(\phi) \\ p &= \frac{\dot{\phi}^2}{2} - V(\phi). \end{aligned}$$

The field equations are

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho, \\ \frac{\ddot{a}}{a} &= -\frac{8\pi G}{3}[\dot{\phi}^2 - V(\phi)]. \end{aligned}$$

2.6 Laws of Thermodynamics

A branch of science in which the relationship among energy, heat and work studied. In this branch, the quantities like pressure, volume, internal energy, temperature and entropy are discussed. Its interest is just about large-scale perceptions. It deals with the stability of the system. Firstly, the word thermodynamics utilized by Thomson [83], has Greek source and is translated as: Thermo means heat and dynamics means power. Sadi Carnot is known as Father of thermodynamics. The laws of thermodynamics are define below:

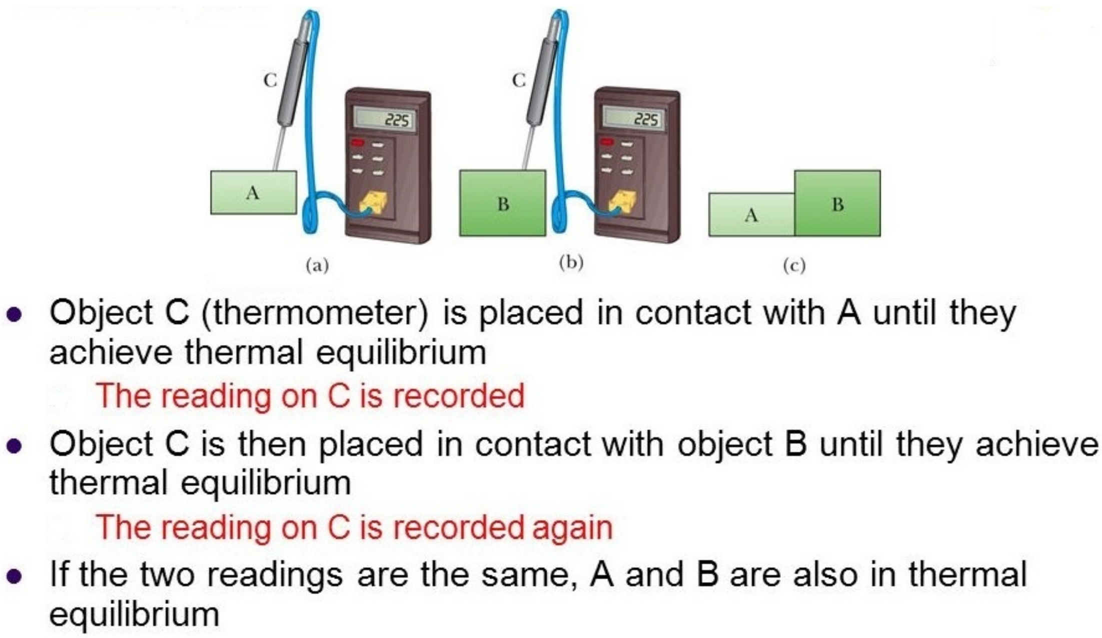


Figure 2.2: Example of Zero Law of Thermodynamics

2.6.1 Zeroth Law of Thermodynamics

As indicated by Sommerfeld [84] the title of the zeroth law of thermodynamics discovered by Ralph, the point at which he was talking about the 1935 content of Saha and Srivastava [85]. It states that if two systems are in thermal equilibrium with the third system then they all are in thermal equilibrium with one another.

2.6.2 First Law of Thermodynamics

In 1850, Clausius and Thomson discovered the FLT. This law is also known as the law of conservation. It states that the energy can be changed from one form to another form however it can be neither created nor destroyed. Mathematically, it is define as

$$\Delta U = Q - W,$$

where ΔU is the internal energy change, Q represents heat measure and W is the work done.

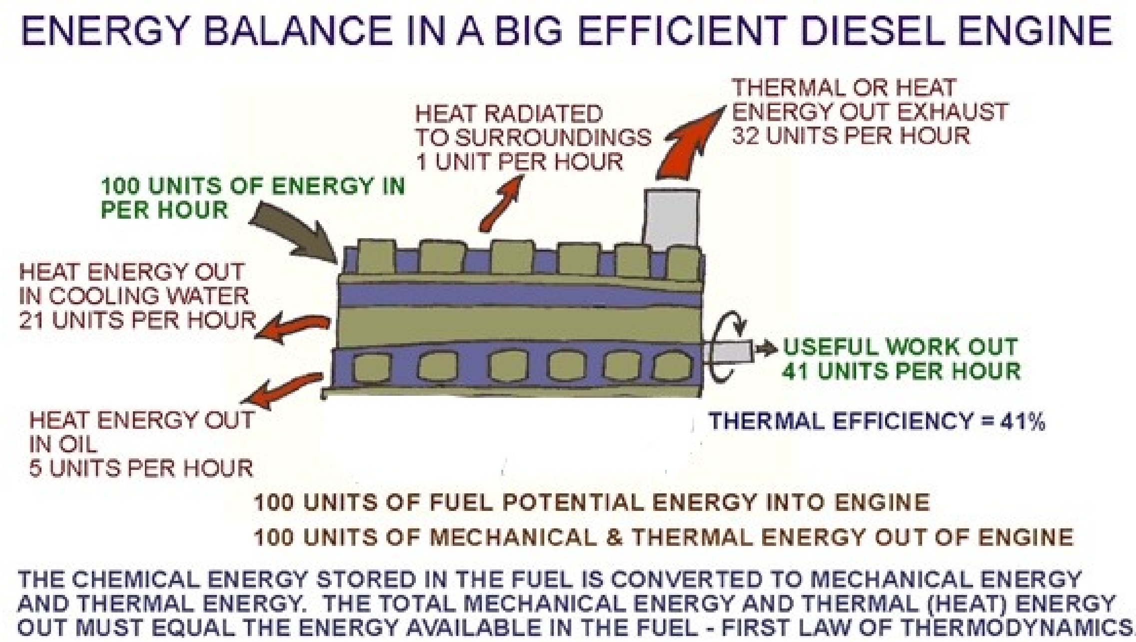


Figure 2.3: Example of First Law of Thermodynamics

The example of the FLT is the flow of energy in a diesel engine, at a point when a motor consumes fuel it changes over the energy gathered in the fuel's substance bonds into heat and fruitful mechanical work. According to the FLT when the majority of the fuel's energy is discharged by blazing in the chambers it does'nt vanish. For each hundred units of fuel energy that is blazed in the motor a hundred units of changed over energy needs to finish up some place. It does'nt vanish.

2.6.3 Second Law of Thermodynamics

In 1850, Clausius established the framework for the second law of thermodynamics by inspecting the connection between heat exchange and work [86]. His defination of the second law of thermodynamics was published in 1854 in German, which states that, without other same change heat can never go from a colder body to a hotter body, associated therewith happening in the meantime [87]. From conventional experience of refrigeration, heat can't suddenly spill out of cold areas to hot areas without outside work

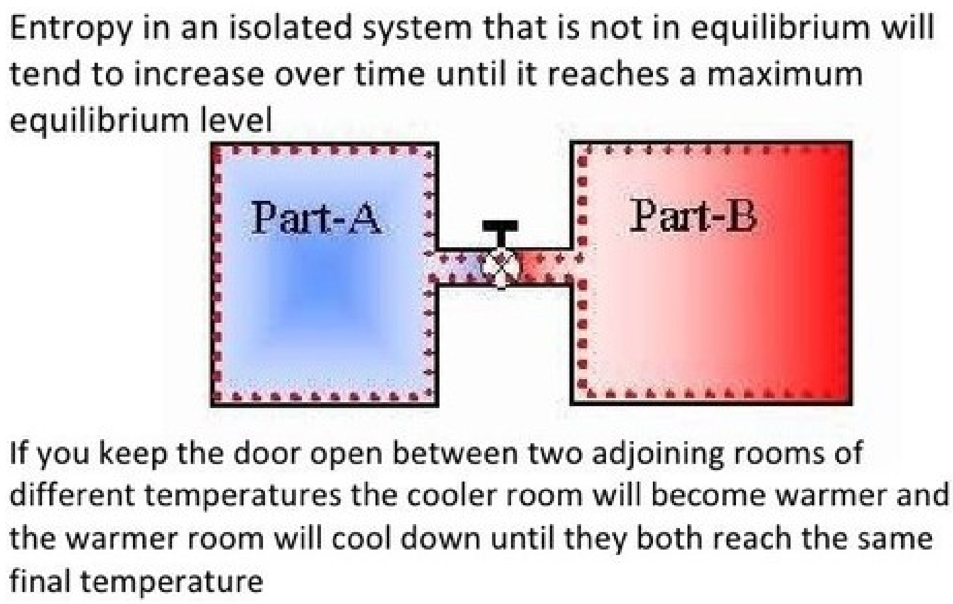


Figure 2.4: Example of Second Law of Thermodynamics

being performed on the system. For instance, in a refrigerator heat streams from cold to hot, however just when constrained by an outer assistant, the refrigeration system.

Thomson stated the second law of thermodynamics as, It is impractical, by means of extinct material firm, to get mechanical impact from any segment of matter by cooling it underneath the temperature of the coldest of the atmosphere substances [88].

2.6.4 Entropy

In 1865, Clausius formulated the term entropy. He had seen that a specific proportion was consistence in perfect or reversible heat cycles. The process was heat transaction to simple temperature. Clausius chose that the rationed proportion must compare to genuine physical amount and gave it name entropy [89]. We can write it mathematically as

$$S = \frac{Q}{T},$$

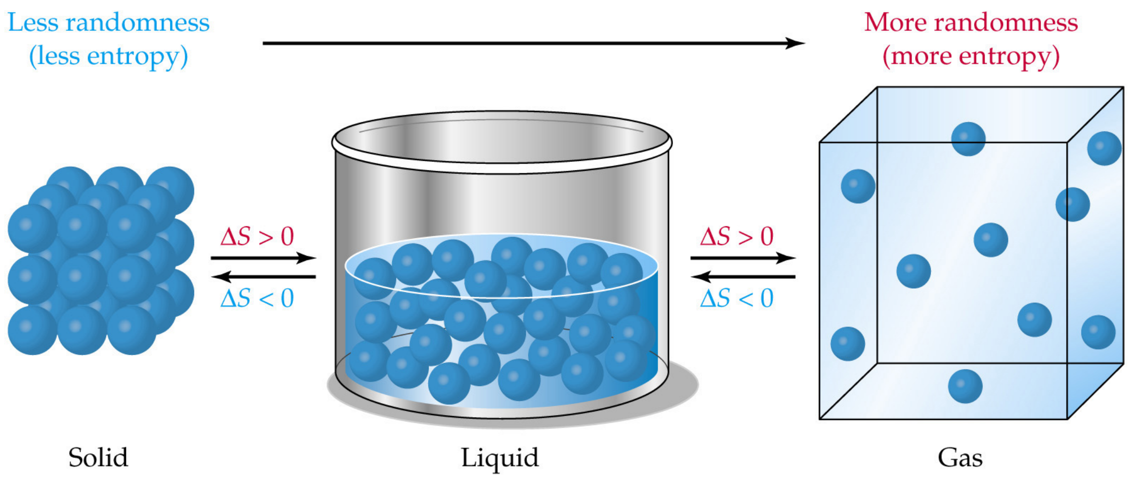


Figure 2.5: Example of Entropy

where S is the symbol of entropy, Q is the heat substance and T represents temperature. The change in entropy is always positive.

2.6.5 Third Law of Thermodynamics

The third law of thermodynamics was created by physicist Nernst (1906-1912) [90]. This law states that, for a limited number of physical processes it is not feasible for a physical system or object to have an absolute zero temperature. In 1923, Lewis and Randall expressed the alternative form of the third law of thermodynamics which states that, each substance has a limited positive entropy if the entropy of every component in a few crystalline state be taken as zero at the absolute zero of temperature, however without a doubt the zero of temperature the entropy may become zero and does as such become on account of perfect crystalline substances [91].

2.7 Thermodynamics in Black Hole

BH thermodynamics [92] is the territory of concentrate that tries to accommodate the laws of thermodynamics with the presence of BH EH's. The

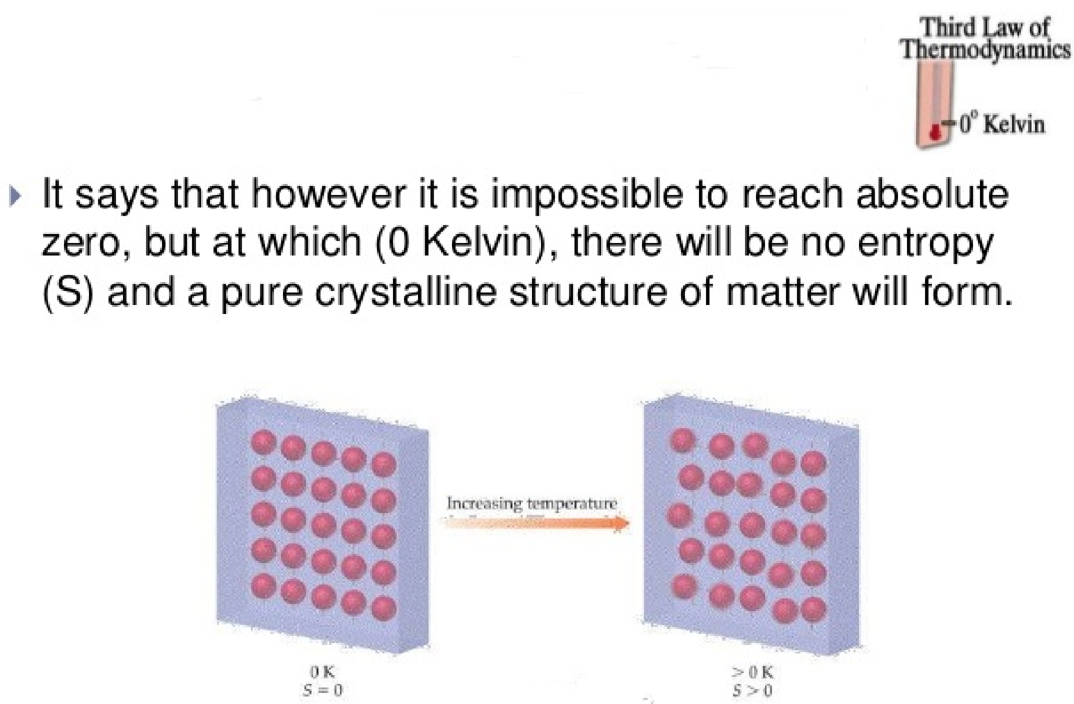


Figure 2.6: Example of Third Law of Thermodynamics

laws of BH mechanics are physical properties that BH are accepted to fulfill. In 1970, Hawking discovered the four laws of BH mechanics with Carter and Bardeen, drawing a similarity with thermodynamics [93, 94].

2.7.1 Zero Law of Black Hole Mechanics

The zero law of BH states that, for a static BH horizon has a constant surface gravity. i.e., it could never be achievable to twist a BH so fast that it would break separated.

2.7.2 First Law of Black Hole Mechanics

The difference in energy is identified with change of region, angular momentum and electric charge for perturbation of static BH

$$dE = \frac{k}{8\pi} dA + \Omega dJ + \Phi dQ,$$

where E represents energy, k is the surface gravity, A defines HA, Ω is angular velocity, J is angular momentum, electrostatic potential is represented as Φ and Q is the electric charge.

2.7.3 Second Law of Black Hole Mechanics

This law is the remark of Hawking's area theorem. It expresses that for a spontaneous process the change in entropy is always greater and equal to zero in an isolated system, recommending a connection among the HA of BH and entropy

$$\frac{dA}{dt} \geq 0.$$

2.7.4 Generalized Second Law of Thermodynamics

Bekenstein [95] proposed a generalized version of the second law of thermodynamics which states that the sum of BH entropy S_{BH}, and the entropy of matter in the BH outer region S, always increases. Mathematically, it can be represented as

$$\dot{S}_{BH} + \dot{S} \geq 0.$$

2.7.5 Third Law of Black Hole Mechanics

This law states that without surface gravity the BH cannot form. Expressing that k can't go to zero is practically equivalent to third law of BH thermodynamics which expresses that the entropy of the system at total zero is a well defined constant.

2.8 Thermal Equilibrium Condition

If there is no net progression of thermal energy among two physical systems when they are associated with each other and exchange no heat, then they are in thermal equilibrium. It obeys the zero law of thermodynamics. Two systems are in thermal equilibrium with each other, if the following condition holds:

$$\ddot{S}_{tot} \leq 0,$$

where double dot represents the double derivative with respect to time.

2.8.1 Bekenstein-Hawking Entropy

The Bekenstein-Hawking entropy [96] or BH entropy is the measure of entropy that must be appointed to BH with the end goal for it to consent to the laws of thermodynamics as they are explained by observers outside to that BH. BH entropy is an idea with geometric root however with numerous physical results. In Einstein's gravity Bekenstein-Hawking entropy relation is,

$$S_{BH} = \frac{A}{4G},$$

where A is the area of the AH, given as, $A = 4\pi r^2$.

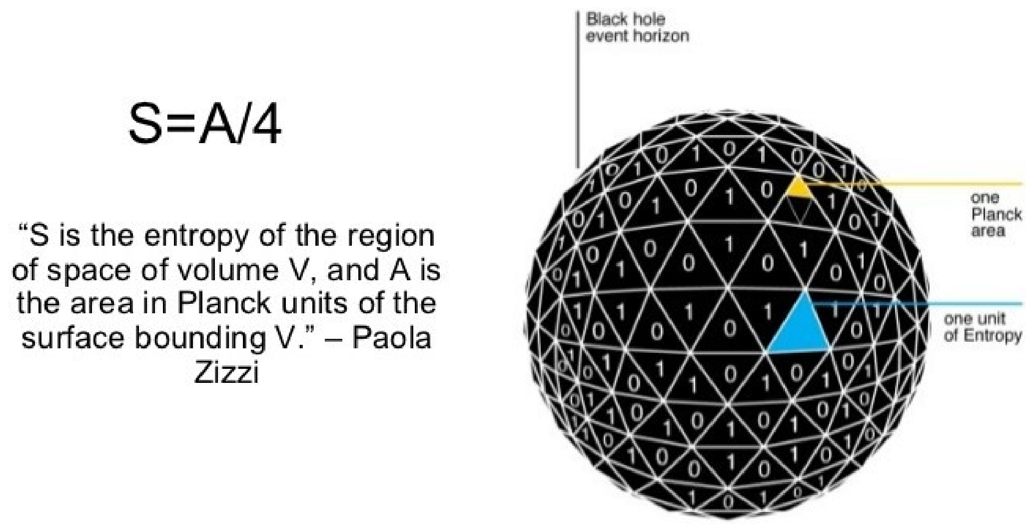


Figure 2.7: The Bekenstein-Hawking entropy is the entropy to be attributed to any BH: one fourth of its HA expressed in units of the Planck area

2.8.2 Rényi Entropy

Recently, Rényi generalized entropy have extensively used in order to study various gravitational and cosmological framework. The Rényi entropy is also important in quantum information where it can be used as a measure of tangle [97]. Here we define Rényi entropy to check the validity of GSLT, we have

$$S = \frac{1}{\delta}\ln(1+\frac{\delta A}{4}),$$

where $\delta = 1 - Q$.

2.8.3 Logarithmic Corrected Entropy

The entropy-area relation including quantum corrections conduct curvature corrections within Einstein-Hilbert action. The Bekenstein-Hawking logarithmic corrected entropy [98] is defined by the relation

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{4G} + \beta \frac{4G}{A} + \gamma,$$

where α, β and γ are dimensionless constants, apart from the exact values of these constants are still checked.

2.8.4 Sharma-Mittal Entropy

The unique entropy measure is the Sharma-Mittal entropy [99] that permits rise to a thermostatistics, which is

$$S_{SM} = \frac{1}{1-r}[(1 + \frac{\delta A}{4})^{\frac{1-r}{\delta}} - 1],$$

where r is the free parameter.

2.9 Modified Theories

The excessively interesting way is the modified gravity, for the DE and late time accelerating universe application. It gives the natural substitute for DE and useful in high energy physics. It defines the transition of non-phantom phase to phantom phase and from deceleration phase to acceleration phase [100]-[102]. To account the late acceleration and early inflation and to characterize the universe history in the general context of modified gravity, higher order gravities have been proposed [103]-[108]. A more profound comprehension of Einstein-Hilbert gravity arranged by higher order gravities and in general modified gravities . The development of modified theories begin from the Einstein Lagrangian and incorporate additional term for example, in Weyl gravity [109, 110], in $f(G)$ gravity [111, 112], in $f(R)$ gravity [113]-[115], in Lovelock gravity [116] etc.

In 1915, Einstein proposed the general theory of relativity and the present characterization of gravitation in modern physics. Giving a cooperative depiction of gravity as a geometric property of spacetime or space and time, it generalizes the special relativity and supplants Newton's law of universal gravitation. GR anticipate the gravitational curving of light by enormous body. In 1970, Buchdahl [6] proposed a form of modified theory $f(R)$ which is the extension of Einstein's theory. This theory is a group of family, each one characterized by a distinct function f of the Ricci scalar R. There might be opportunity to clarify the CA and structure arrangement of the cosmos without including unknown types of DE and dark matter, as a result of presenting an arbitrary function.

2.10 $f(R,T)$ Theory

The $f(R,T)$ gravity is the extension of $f(R)$ theory proposed by Harko [7], which is an explicit coupling of an arbitrary function of R with the trace of energy-momentum tensor T. The action for this gravity is

$$S = \frac{1}{16\pi}\int \sqrt{-g}d^4x f(R,T) + \int \sqrt{-g}d^4x\mathcal{L}_m,$$

where $\mathcal{L}_m$ is the matter Lagrangian and T is the trace of energy-momentum tensor. The stress-energy tensor is defined as

$$T_{\mu\nu} = -\frac{2\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}\sqrt{-g}},$$

its trace is $T = g^{\mu\nu}T_{\mu\nu}$. The variation of Christoffel symbols is

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{g^{\lambda\alpha}}{2}(\nabla_\mu\delta g_{\nu\alpha} + \nabla_\nu\delta g_{\alpha\mu} - \nabla_\alpha\delta g_{\mu\nu}).$$

The Ricci scalar variation gives the following expression

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}.$$

The field equation for this model is defined as

$$\begin{aligned}&f_R(R,T)R_{\mu\nu} - \frac{f(R,T)g_{\mu\nu}}{2} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T)\\ &= 8\pi T_{\mu\nu} - f_T(R,T)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu},\end{aligned}$$

where ∇_μ corresponds to the covariant derivative, $\Box \equiv \nabla_\mu\nabla^\mu$ is the d'Alembert operator and

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.$$

2.11 $f(R, T^{\phi})$ Theory

We examine the self-interacting case of $f(R,T)$ gravity which is $f(R,T^{\phi})$ gravity with a SF. The action for the $f(R,T^{\phi})$ gravity is [7]

$$S = \frac{1}{2}\int \sqrt{-g}d^4x[f(R,T^{\phi}) + 2\mathcal{L}_{\phi}],$$

where $\mathcal{L}_{\phi}$ represents matter Lagrangian density, is descibed as

$$\mathcal{L}_{\phi} = -\frac{\epsilon\dot{\phi}^2(t)}{2} + V(\phi).$$

The energy-momentum tensor of matter source is

$$T_{\mu\nu} = -\frac{2\delta(\mathcal{L}_{\phi}\sqrt{-g})}{\sqrt{-g}\delta g^{\mu\nu}}.$$

The field equations for this gravity are

$$\begin{aligned}
&f_R(R,T^{\phi})R_{\mu\nu} - \frac{f(R,T^{\phi})g_{\mu\nu}}{2} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f_R(R,T^{\phi})\\
&= T_{\mu\nu}f_T(R,T^{\phi})(T_{\mu\nu} + \Theta_{\mu\nu}),\\
&\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T^{\phi}_{\alpha\beta}}{\delta g^{\mu\nu}}.
\end{aligned}$$

The energy-momentum tensor of a SF with self-interacting scalar potential $V(\phi)$ is

$$T^{\phi}_{\mu\nu} = \epsilon\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left[\frac{\epsilon g^{\varrho\sigma}}{2}\phi_{,\varrho}\phi_{,\sigma} - V(\phi)\right],$$

where $\epsilon = \pm 1$ correspond to the phantom and quintessence SF. Its trace is $T^{\phi} = g^{\mu\nu}T^{\phi}_{\mu\nu}$.

Chapter 3
Thermodynamic Consequences of Specific Modified Gravity on the Apparent Horizon

In this chapter, we examine the GSLT and thermal equilibrium condition in $f(R,T^{\phi})$ gravity by implementing the first-order formalism. We take three distinct models of Hubble parameter. By using these models, we observe the GSLT and thermal equilibrium condition. We show our results graphically for flat as well as non-flat FRW universe at AH with Bekenstein-Hawking entropy. The results of this chapter are compiled and accepted in the form of a paper.

3.1 Basics of $f(R,T^{\phi})$ theory

The case of self-interacting SF in $f(R,T)$ gravity yields the $f(R,T^{\phi})$ gravity [117]. Here we introduce first-order formalism to evaluate analytical models corresponding to the $f(R,T^{\phi})$ theory. We use the model $f(R,T^{\phi}) = -R/4 + \lambda T^{\phi}$, [118] where λ is known as model parameter. The relevant action (The time integral of the Lagrangian is called the action denoted by S) in $f(R,T^{\phi})$ gravity is

$$S = \int d^4x\sqrt{-g}\left[\frac{-R}{4} + \lambda T^{\phi} + \mathcal{L}(\phi, \partial_{\nu}\phi)\right]. \tag{3.1.1}$$

Here, we choose $4\pi G = c = 1$ throughout this article. The related field equations are

$$G_{\alpha\beta} = 2(T^{\phi}_{\alpha\beta} - g_{\alpha\beta}\lambda T^{\phi} - 2\lambda\partial_{\alpha}\phi\partial_{\beta}\phi), \tag{3.1.2}$$

where $G_{\alpha\beta}$ is the Einstein tensor and $T^{\phi}_{\alpha\beta}$ is the energy-momentum tensor of the SF. In addition to, it is most convenient to define a field theory by specifying the Lagrange density, from which all equations of motion can be derived, here we are dealing with standard Lagrangian density for real SF, which is

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi), \tag{3.1.3}$$

where V(ϕ) assumed as self-interacting potential. However, the energy-momentum tensor is described as

$$T^{\phi}_{\alpha\beta} = \partial_{\alpha}\phi\partial_{\beta}\phi - g_{\alpha\beta}\mathcal{L}, \tag{3.1.4}$$

the trace of energy-momentum tensor is defined as $T^{\phi} = g^{\alpha\beta}T^{\phi}_{\alpha\beta}$ which becomes

$$T^{\phi} = \dot{\phi}^2 + 4V(\phi), \tag{3.1.5}$$

where dot represents derivative with respect to t. The line element of FRW model is

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{3.1.6}$$

where $a(t)$ is the scale factor and k represents curvature of the space:

- $k = 0 \quad \Rightarrow \quad$ flat space-time,
- $k = +1 \quad \Rightarrow \quad$ spherical curvature,
- $k = -1 \quad \Rightarrow \quad$ hyperbolic geometry.

From Eq.(3.1.2), the corresponding field equations are

$$H^2 = \frac{2}{3}\rho^{eff} - \frac{k}{a^2}, \tag{3.1.7}$$

$$H^2 + \dot{H} = -p^{eff} - \frac{k}{2a^2}, \tag{3.1.8}$$

where H is denoted as Hubble parameter, which depends on scale factor $a(t)$, the function of time t i.e ($H = \frac{\dot{a}}{a}$), here ρ^{eff} and p^{eff} are

$$\rho^{eff} = \left(\frac{1}{2} - \lambda\right)\dot{\phi}^2 + (4\lambda - 1)V(\phi), \tag{3.1.9}$$

$$p^{eff} = -\left(\frac{1}{2} - \lambda\right)\dot{\phi}^2 + (4\lambda - 1)V(\phi). \tag{3.1.10}$$

By using Eqs.(4.1.6) and (4.1.7), we have

$$\dot{H} = -(\rho^{eff} + p^{eff}) + \frac{k}{a^2}. \tag{3.1.11}$$

The equation of motion for SF is acquired as

$$(1 - 2\lambda)(\ddot{\phi} + 3H\dot{\phi}) + (1 - 4\lambda)V_{\phi} = 0, \tag{3.1.12}$$

where ϕ in subscript represents the derivative with respect to ϕ. With the assumption of first-order formalism [117], the Hubble parameter is given by

$$H = W(\phi). \tag{3.1.13}$$

From the field equations, the potential of SF is established as

$$V(\phi) = \frac{1}{4\lambda - 1}\left(\frac{3}{2}H^2 - (\frac{1}{2} - \lambda)\dot{\phi}^2 + \frac{3k}{2a^2}\right). \tag{3.1.14}$$

By substituting $H = W(\phi)$ into Eq. (4.2.2), we have

$$(1 - 2\lambda)\dot{\phi}^2 + W_\phi\dot{\phi} - \frac{k}{a^2} = 0, \tag{3.1.15}$$

which is a quadratic equation and has two roots

$$\dot{\phi} = \frac{-W_\phi \pm \sqrt{W_\phi^2 + (\frac{4k}{a^2})(1 - 2\lambda)}}{2(1 - 2\lambda)}, \tag{3.1.16}$$

each of which is first-order differential equation. For flat universe ($k = 0$), Eq.(4.2.6) yields the following two solutions

$$\dot{\phi} = 0, \quad \text{and} \quad \dot{\phi} = -\frac{W_\phi}{1 - 2\lambda}. \tag{3.1.17}$$

We here choose three models of the Hubble parameter, which are $W(\phi) = e^{b_1\phi}$ [117], $W(\phi) = b_2(\frac{\phi^3}{3} - \phi)$ [119] and $W(\phi) = b_3\sinh(\phi)$ [120], where b_1, b_2 and b_3 are real constants. Now, the first-order differential equation $\dot{\phi} = -\frac{W_\phi}{1-2\lambda}$, for the three different models of $W(\phi)$ is satisfied by

- $\phi(t) = \frac{1}{b_1}\ln[\frac{1-2\lambda}{b_1^2(t+c_1)}]$
- $\phi(t) = \tanh[\frac{b_2(t+c_2)}{1-2\lambda}]$
- $\phi(t) = 2\arctan h[\tan[\frac{b_3 t}{2-4\lambda} + \frac{c_3}{2}]]$

where c_1, c_2 and c_3 are constant parameters. However, in the case of closed and open universe ($k = \pm1$), Eq.(4.2.6) gives two roots of ϕ^+ and ϕ^- which are solved numerically for the Hubble parameter W(ϕ).

3.2 Thermodynamics Laws

Thermodynamics is the branch of physics which concerned with heat and temperature and their relation to energy and work. It is discussed on microscopic and macroscopic levels. The quantities like pressure, volume, internal energy, temperature and entropy are discussed in this branch. Now, we are interested to explore the generalized thermodynamics laws in the context of the $f(R,T^{\phi})$ gravity. The fundamental purpose of the next sections is to construct the FLT and GSLT in $f(R,T^{\phi})$ gravity.

3.2.1 First law of thermodynamics

Here, we are going to study the FLT in $f(R,T^{\phi})$ gravity at AH for FRW universe. To review this law, firstly we find the dynamical AH calculated by the condition [121] $h^{\mu\nu}\partial_{\mu}R_A\partial_{\nu}R_A=0$, $h^{\mu\nu}$ is a two-dimensional metric which is interpreted as $h_{\mu\nu}=diag(1,\frac{a^2}{1-kr^2})$. The above condition gives the AH as

$$R_A=\frac{1}{\sqrt{H^2+\frac{k}{a^2}}}.$$

From the above expression, we get

$$dR_A=HR_A^3(\rho^{eff}+p^{eff})dt, \tag{3.2.1}$$

where dR_A through small time interval dt clarify infinitesimal change in the radius of AH. Moreover, Bekenstein-Hawking entropy relation is given by $S=\frac{A}{4G}$, where A is the area of the AH given as, $A=4\pi r^2$ [122, 66]. We rewrite Eq.(4.2.8) as

$$\frac{G}{2\pi R_A}dS=HR_A^3(\rho^{eff}+p^{eff})dt. \tag{3.2.2}$$

The temperature of the AH is defined as [123]

$$\begin{aligned}T_h=\frac{|k_{sg}|}{2\pi},\quad k_{sg} &= \frac{1}{2\sqrt{-h}}\partial_{\mu}(\sqrt{-h}h^{\mu\nu}\partial_{\nu}R_A)\\ &= -\frac{1}{R_A}(1-\frac{\dot{R}_A}{2HR_A})=-\frac{R_A}{2}(2H^2+\dot{H}). \end{aligned}\tag{3.2.3}$$

By multiplying both sides of Eq.(4.2.9) with a term $-T_h = \frac{1}{2\pi R_A}(1-\frac{\dot{R_A}}{2HR_A})$, we can get

$$T_h dS = [-4\pi H R_A^3 dt + 2\pi R_A^2 dR_A](\rho^{eff} + p^{eff}). \tag{3.2.4}$$

Now, we are going to explain the energy content of the universe within the AH. The Misner-Sharp energy is described as $E = \frac{R_A}{2G}$. Defining the volume of the 3-dimensional sphere on the AH as $V = \frac{4\pi R_A^3}{3}$, the energy density can be written in term of volume as

$$\tilde{E} = \frac{3(H^2 + \frac{k}{a^2})}{2} V \equiv \rho^{eff} V, \tag{3.2.5}$$

Eq.(4.2.12) shows that $\tilde{E}$ is directly related to AH radius, so the small displacement dR_A in horizon radius will cause the infinitesimal change which is established as

$$d\tilde{E} = -4\pi R_A^3(\rho^{eff} + p^{eff})dt + 4\pi R_A^2 \rho^{eff} dR_A. \tag{3.2.6}$$

Additionally, putting together Eqs.(4.2.11) and (4.2.13), it follows that

$$T_h dS = d\tilde{E} - 2\pi R_A^2(\rho^{eff} - p^{eff})dR_A. \tag{3.2.7}$$

Introducing the work density, we find out

$$\tilde{W} = -\frac{1}{2}(T^{(M)\mu\nu}h_{\mu\nu} + \tilde{T}^{(de)\mu\nu}h_{\mu\nu}) = \frac{1}{2}(\rho^{eff} - p^{eff}), \tag{3.2.8}$$

here $T^{(M)\mu\nu}h_{\mu\nu}$ is the energy density of the matter and $\tilde{T}^{(de)\mu\nu}h_{\mu\nu}$ is the energy density of the dark components. Using the work density in Eq.(4.2.14), it results in

$$T_h dS = d\tilde{E} - \tilde{W} dV. \tag{3.2.9}$$

Thus, the FLT on the AH satisfies in $f(R, T^{\phi})$ gravity.

3.2.2 Generalized second law of thermodynamics

Bekenstein [95] proposed a generalized version of the second law of thermodynamics by utilizing the conjectured proportionality among BH entropy and HA. Thoroughly, the generalized entropy relation satisfies the condition

$$\dot{S}_{tot} = \dot{S} + \dot{S}_{in} \geq 0, \tag{3.2.10}$$

here $\dot{S}_{tot}$ is the entropy of all energy sources inside the horizon, $\dot{S}$ corresponds to the entropy associated with the horizon and $\dot{S}_{in}$ represents the sum of all entropy components inside the horizon. Let us continue with the modified FLT, to establish the GSLT in this formulation of $f(R, T^{\phi})$ gravity, we can write Gibb's equation as

$$T_i dS_i = dE_i + p_i dV. \tag{3.2.11}$$

The Gibb's equation relates the entropy of energy sources inside the horizon to density and pressure in the horizon is defined as $E = \rho V$ and $W = -p$. Eq.(4.2.18) can be expressed as

$$T_{in} dS_{in} = 4\pi R_A^2 (\dot{R}_A - HR_A)(\rho^i + p^i). \tag{3.2.12}$$

T_{in} signify the temperature for all the component insides the horizon. Here $\sum_i (\rho^i + p^i) = \rho^{eff} + p^{eff}$, combining the total entropy inside the horizon, it becomes

$$T_{in} dS_{in} = 4\pi R_A^2 (\dot{R}_A - HR_A)(\rho^{eff} + p^{eff}), \tag{3.2.13}$$

which leads to

$$\dot{S} = -\frac{8\pi^2 H(\dot{H} - \frac{k}{a^2})}{(H^2 + \frac{k}{a^2})^{\frac{4}{2}}}. \tag{3.2.14}$$

In case of thermal equilibrium, $T_{in} = T_h$, Eq. (4.2.20) implies

$$\dot{S}_{in} = \frac{4\pi R_A^2}{T_h} (\dot{R}_A - HR_A)(\rho^{eff} + p^{eff}). \tag{3.2.15}$$

After some calculations, we have

$$\dot{S}_{in} = 16\pi^2 \left(\frac{H\Big(-(\dot{H} - \frac{k}{a^2}) - (H^2 + \frac{k}{a^2}) \Big)(-\dot{H} + \frac{k}{a^2})}{\Big(2(H^2 + \frac{k}{a^2}) + (\dot{H} - \frac{k}{a^2}) \Big)(H^2 + \frac{k}{a^2})^{\frac{4}{2}}} \right). \tag{3.2.16}$$

Hence, by substituting Eqs.(4.2.21) and (4.2.23) in (4.2.17), it indicates the relation of GSLT of the form

$$\dot{S}_{tot} = 4\pi^2 \Bigg[\frac{4W\Big[-(W_{\phi}\dot{\phi} - \frac{k}{a^2}) - (W^2 + \frac{k}{a^2}) \Big](-W_{\phi}\dot{\phi} + \frac{k}{a^2})}{\Big[2(W^2 + \frac{k}{a^2}) + (W_{\phi}\dot{\phi} - \frac{k}{a^2}) \Big](W^2 + \frac{k}{a^2})^{\frac{4}{2}}}$$

$$- \quad \frac{2W(W_\phi \dot{\phi} - \frac{k}{a^2})}{(W^2 + \frac{k}{a^2})^{\frac{4}{2}}}\Bigg], \tag{3.2.17}$$

where $a = \frac{1}{1+z}$, W is in the form of Hubble parameter for which we choose three different forms and $\dot{\phi}$ defines in (4.2.7).

3.3 GSLT for Flat Spacetime: $k = 0$

In the following, we observe the behavior of GSLT and thermal equilibrium condition for Bekenstein entropy at AH for flat universe, with three different forms of $W(\phi)$.

$\underline{W = e^{b_1 \phi}}$**:**

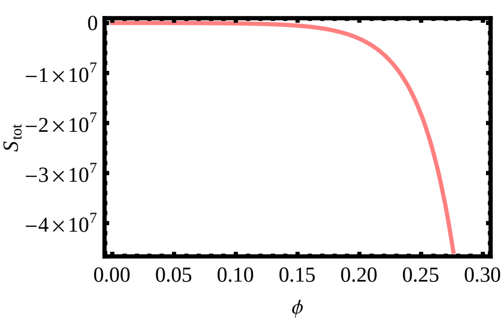


Figure 3.1: Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = e^{b_1 \phi}$ and $k = 0$.

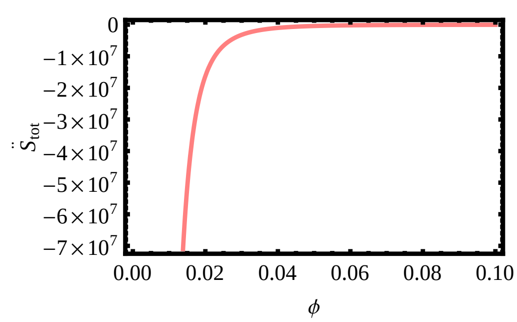
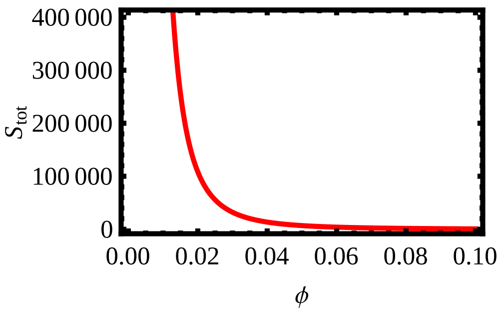


Figure 3.2: Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = e^{b_1 \phi}$ and $k = 0$.

In **Figure 3.1**, we plot $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy at AH with the values $b = 5$ and $\lambda = -5$. The trajectory of $\dot{S}_{tot}$ confirms the validity of GSLT at $\phi \leq 0.15$ with flat spacetime. In **Figure 3.2**, we plot $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy at AH with the same values of b and λ. The trajectory of $\ddot{S}_{tot} \leq 0$ shows the condition of thermal equilibrium satisfies with flat spacetime.



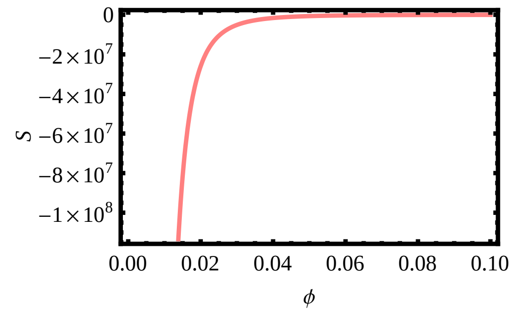


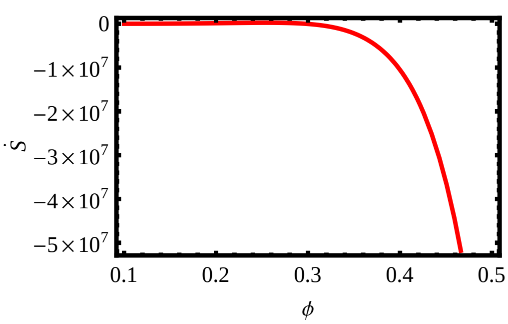
Figure 3.3: Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 0$.

Figure 3.4: Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 0$.

$W = b_2(\frac{\phi^3}{3} - \phi)$:

We plot $\dot{S}_{tot}$ versus ϕ by taking $b = 7$ and $\lambda = 7$ in **Figure 3.3** for Bekenstein entropy at AH. The trajectory of $\dot{S}_{tot}$ increases in positive direction which shows validity of GSLT for flat spacetime. In **Figure 3.4**, by taking the same values of b and λ, we plot $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy at AH. The trajectory of $\ddot{S}_{tot} \leq 0$ and satisfies the thermal equilibrium condition for flat spacetime.

$W = b_3 \sinh(\phi)$:



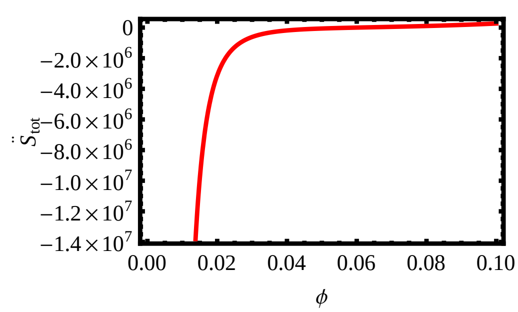


Figure 3.5: Plot of $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 0$.

Figure 3.6: Plot of $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 0$.

In **Figure 3.5**, we plot $\dot{S}_{tot}$ versus ϕ for Bekenstein entropy with $b = -6$ and $\lambda = 1.5$. The trajectory of $\dot{S}_{tot}$ confirms the validity of GSLT for early

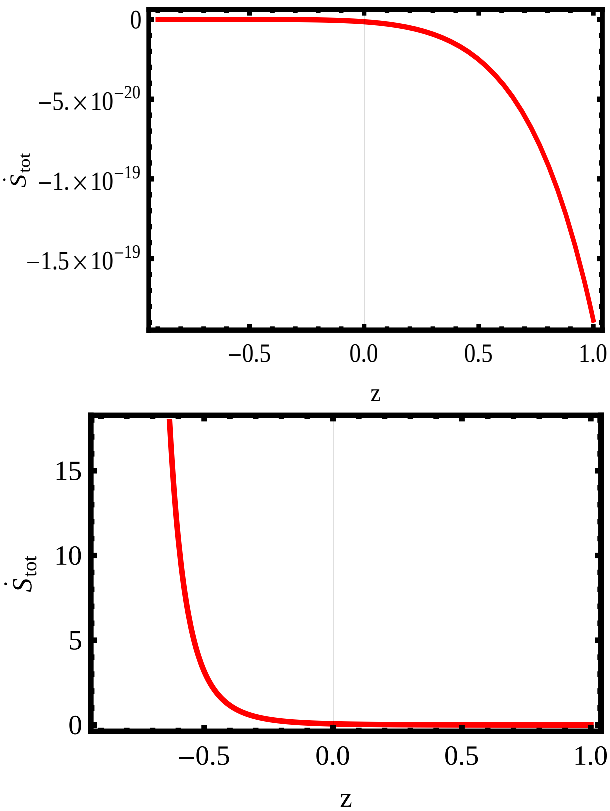


Figure 3.7: Plot of S' with z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

time at $\phi < 0.4$ with flat spacetime at AH. In **Figure 3.6**, for the same values of b and λ we plot $\ddot{S}_{tot}$ versus ϕ for Bekenstein entropy at AH. The trajectory of $\ddot{S}_{tot}$ shows the condition of thermal equilibrium satisfies for flat spacetime at early time.

3.4 GSLT for Closed Spacetime: $k = 1$

In this section, we observe the behavior of GSLT and thermal equilibrium condition for Bekenstein entropy at AH for closed universe, with different values of Hubble parameter $W(\phi)$.

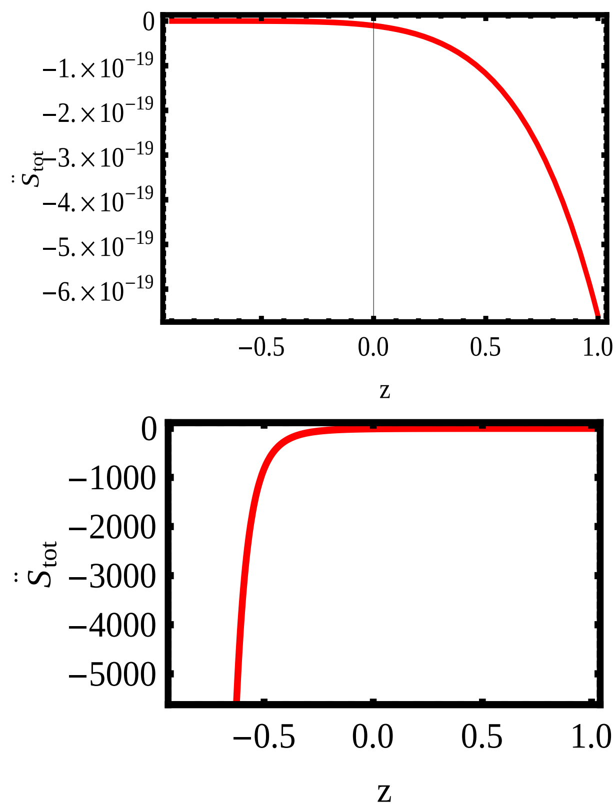


Figure 3.8: Plot of S'' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

$\underline{W = e^{b_1\phi}}$**:**

In **Figure 3.7**, we plot in terms of redshift and the axis label assign as $S' \equiv \frac{dS}{dz}$ with Bekenstein entropy at AH with the following values $b = 5$ and $\lambda = -5$. The entropy in left figure tends to zero at present and future times and remains negative at early times, for present and future times it fulfills the condition $S' \geq 0$ which shows GSLT is valid and the entropy in right figure remains zero at early and present times, tends to positive at future times which confirms the validity of GSLT. In **Figure 3.8**, we plot S'' versus z, the axis label assign as $S'' \equiv \frac{d^2S}{dz^2}$ by taking same values of b and λ the trajectory of left figure is remains negative at early time but it tends to zero at present and future times which confirms the thermal equilibrium condition ($S'' \leq 0$) at past, present and future times and the right side trajectory remains zero at past and present times and it tends to negative at future time so the condition of thermal equilibrium confirms at each time at AH for closed spacetime along with Bekenstein entropy.

$\underline{W = b_2(\frac{\phi^3}{3} - \phi)}$**:**

By using the following values of $b = 7$ and $\lambda = -7$ in **Figure 3.9**, the entropy (left side) remains positive at each time which shows the validity of GSLT for closed spacetime together with Bekenstein entropy at AH and the entropy (right side) remains negative for each time but it leading to zero for $z < -0.2$ at future time for which GSLT is valid. In **Figure 3.10**, with the same values of b and λ the left and right side trajectory gradually decreasing at early, present and future times and increasing at future time for $z \leq -0.3$, it tends to a negative $\frac{d^2S}{dz^2}$ which shows the thermal equilibrium condition satisfies at past, present and future times for closed spacetime at AH.

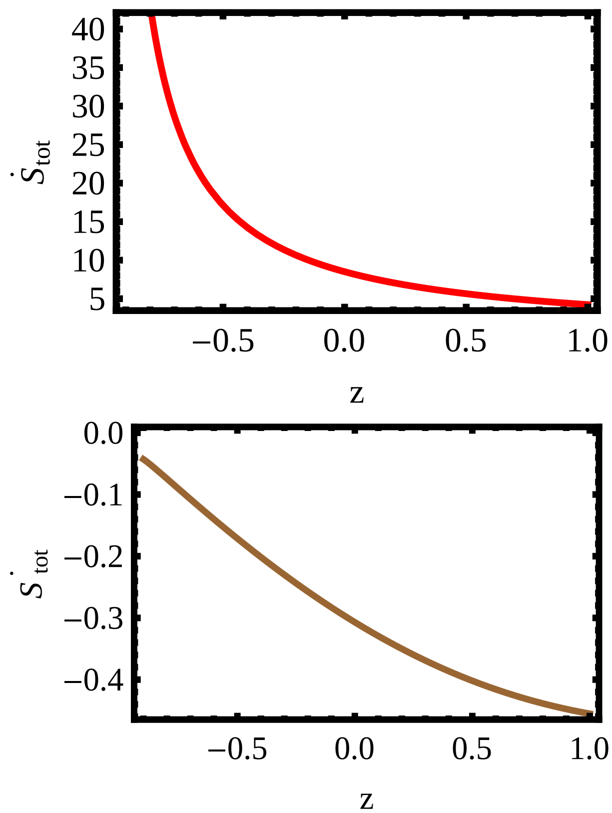


Figure 3.9: Plot of S' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

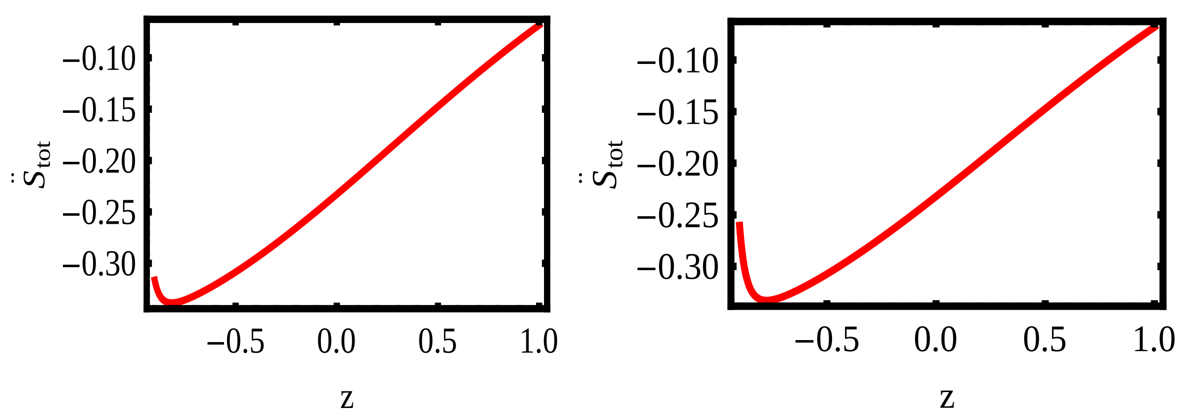


Figure 3.10: Plot of S'' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

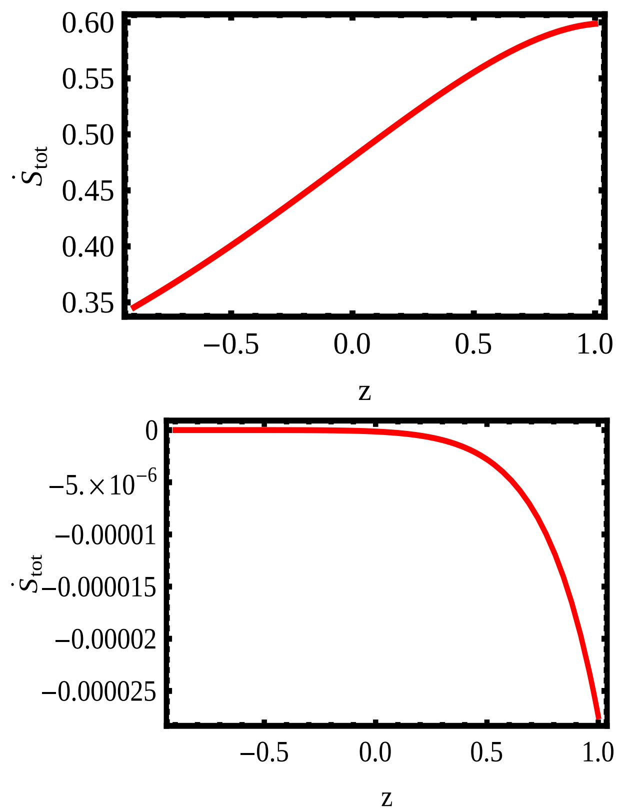


Figure 3.11: Plot of S' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

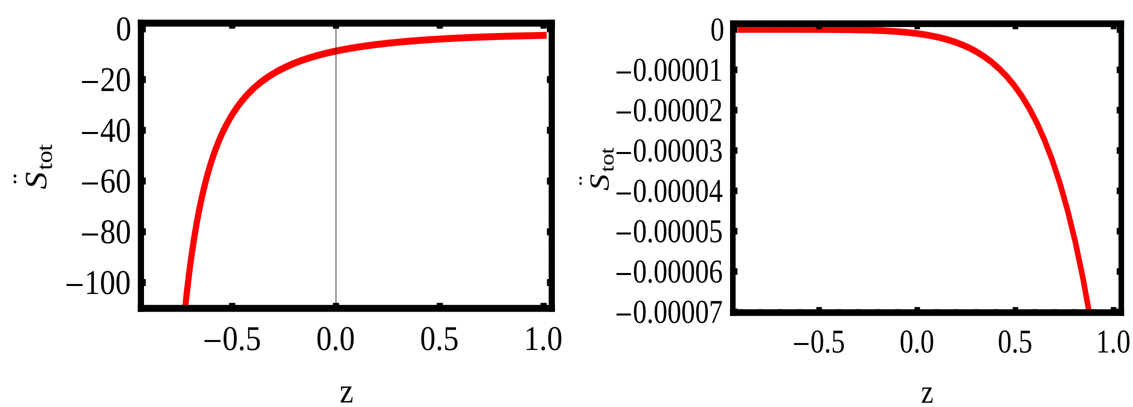


Figure 3.12: Plot of S'' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = 1$, for ϕ^+ on left and ϕ^- on right.

$\underline{W = b_3 \sinh(\phi)}$**:**

We use the same values of b and λ in **Figure 3.11** with Bekenstein entropy at AH. The left side entropy gradually decreasing towards present and future times and also at future time it tending to zero. GSLT is valid for closed spacetime at each time as $S' \geq 0$ and the right side entropy shows the validity of GSLT at present and future times for closed space time as for present and future times entropy leading to zero but at early time entropy decreases. In **Figure 3.12**, for Bekenstein entropy at AH, the trajectory of left side tends to a negative $\frac{d^2S}{dz^2}$ at present and future times and remains zero at early time and the right side trajectory shows S'' leading to zero at present and future times and remains negative at early time. For both the condition of thermal equilibrium satisfies for closed spacetime.

3.5 GSLT for Open Spacetime: $k = -1$

For Bekenstein entropy at AH for with different values of Hubble parameter $W(\phi)$, we observe the validity of GSLT and thermal equilibrium condition for open universe in the following.

$\underline{W = e^{b_1\phi}}$**:**

By utilizing Bekenstein entropy at AH with the same values of b and λ in **Figure 3.13**, the entropy (left figure) remains negative at early time but leads to zero at present and future times which confirms the validity of GSLT for open spacetime and the entropy (right figure) is positively moving towards future time and it tends to zero for early and present times. Hence, GSLT is valid at each time for open spacetime. In **Figure 3.14**, by using the same values of b and λ we plot S'' along redshift. The trajectories (left and right figures) show that the condition of thermal equilibrium satisfies for past, present and future times with open spacetime together with Bekenstein entropy at AH.

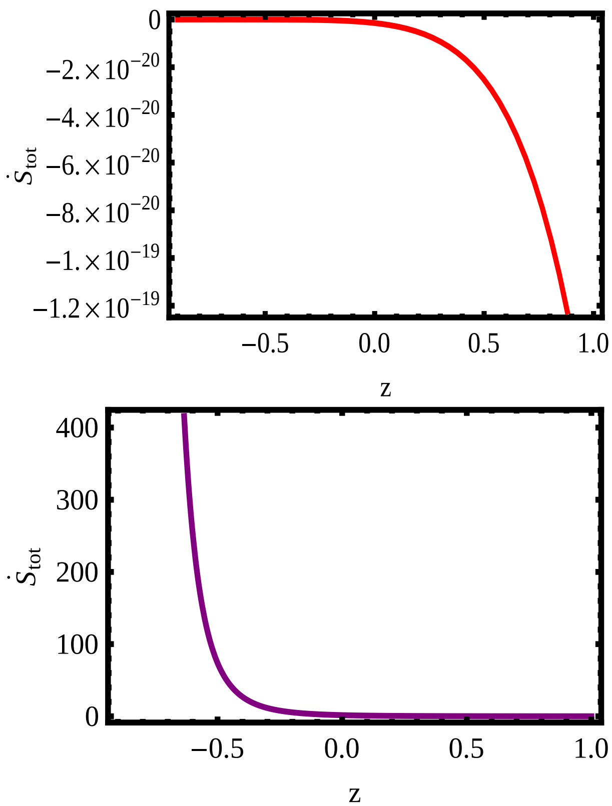


Figure 3.13: Plot of S' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

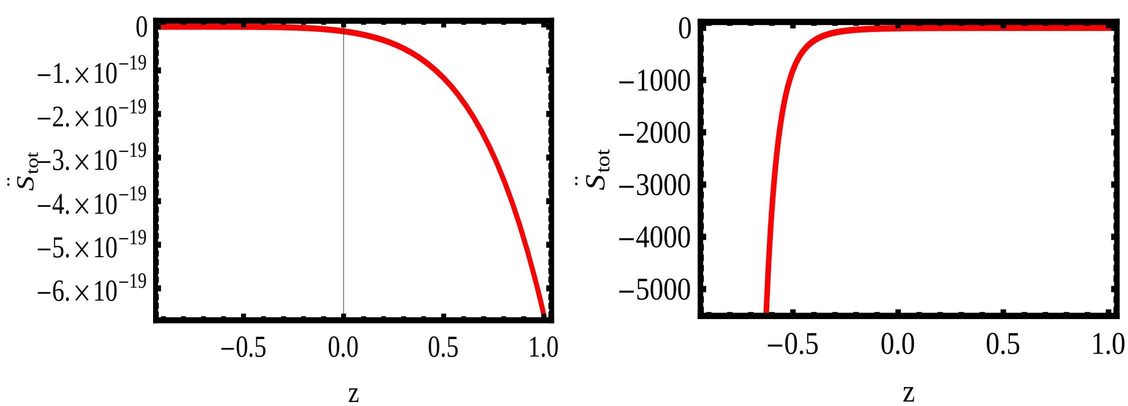


Figure 3.14: Plot of S'' versus z for Bekenstein entropy along with $W = e^{b_1\phi}$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

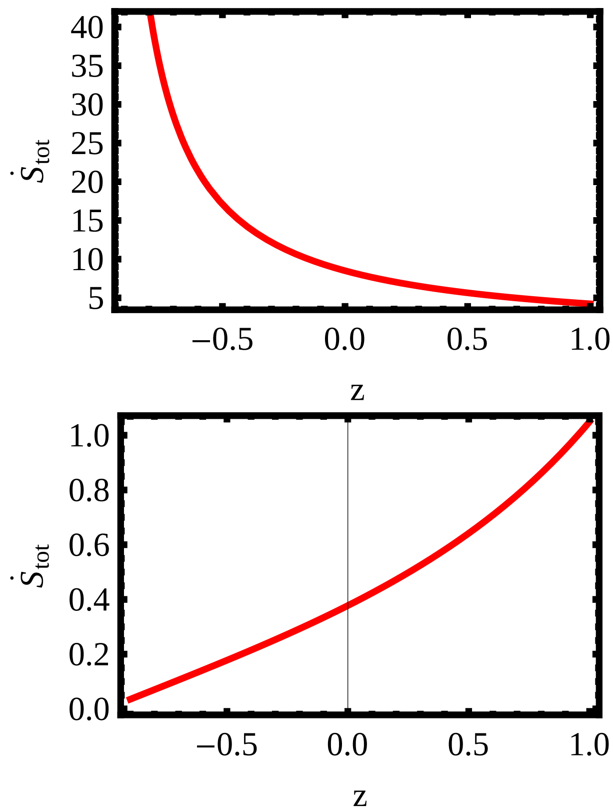


Figure 3.15: Plot of S' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

$W = b_2(\frac{\phi^3}{3} - \phi)$:

In **Figure 3.15**, by following the values $b = 7$ and $\lambda = 7$ we plot S' versus z for Bekenstein entropy at AH. The entropy of left side shows the validity of GSLT for open spacetime, S' is positively increases from early time to present and future times and the right side entropy positively decreases at early and present times and tends to zero at future time. Thus, the GSLT is valid for both. In **Figure 3.16**, for Bekenstein entropy at AH, left trajectory shows $S'' < 0$ for future time and gradually decreases and right trajectory shows S'' remains zero at early time and it tends to a negative $\frac{d^2S}{dz^2}$ at present and future times. Thus, it confirms the condition of thermal equilibrium for both trajectories with open spacetime.

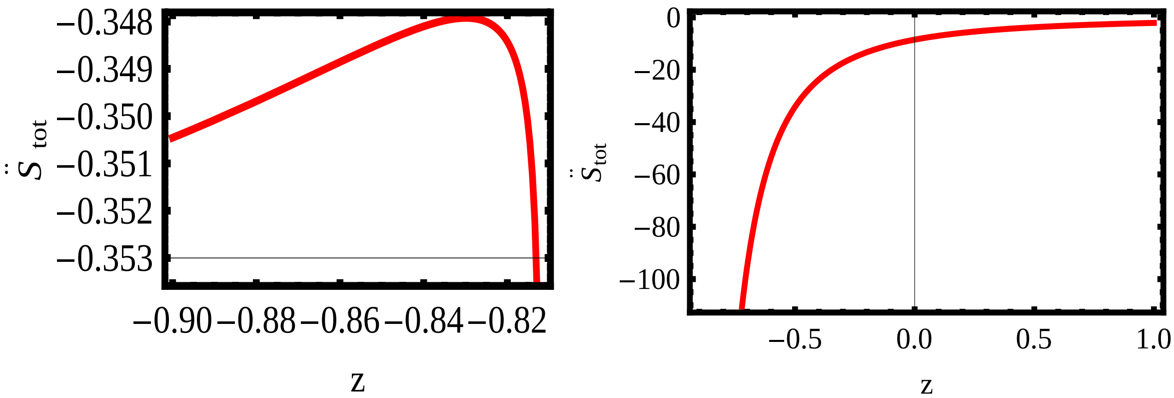


Figure 3.16: Plot of S'' versus z for Bekenstein entropy along with $W = b_2(\frac{\phi^3}{3} - \phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

$\underline{W = b_3 \sinh(\phi)}$:

At AH with the same values of b and λ in **Figure 3.17**, along with Bekenstein entropy. The entropy remains positive at each time and leading to zero at future time (left figure) and the entropy (right figure) remains negative at early time and tends to zero at present and future times. Hence, the GSLT is valid for both figures for open spacetime. In **Figure 3.18**, the left and right figures trajectory shows that the condition of thermal equilibrium satisfies for both at each time with open spacetime at AH together with Bekenstein entropy.

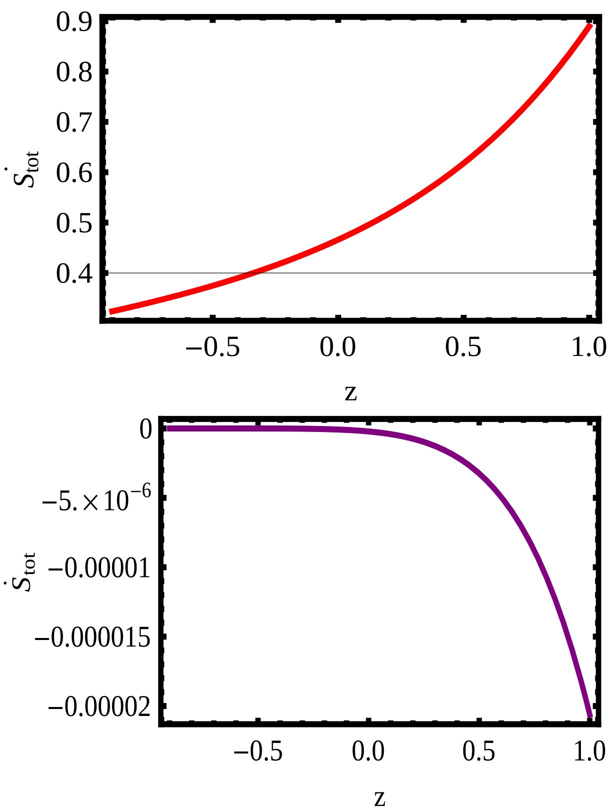


Figure 3.17: Plot of S' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

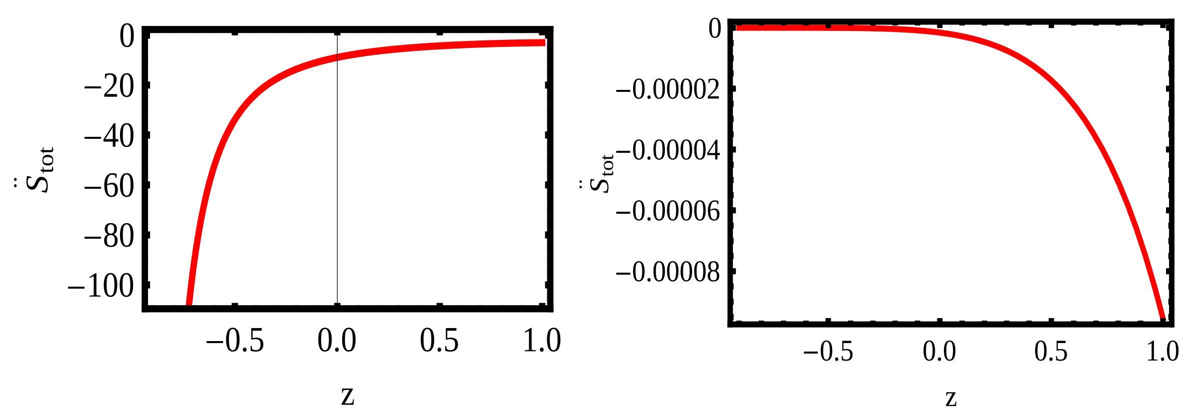


Figure 3.18: Plot of S'' versus z for Bekenstein entropy along with $W = b_3 \sinh(\phi)$ and $k = -1$, for ϕ^+ on left and ϕ^- on right.

Chapter 4
Thermodynamic Implications of Multiquintessence Scenario

In this chapter, we discuss the GSLT and thermal equilibrium condition for flat FRW universe in multi-quintessence SF. We here following the first-order formalism and chose three superpotential models of $W(\phi)$. We represent our results graphically at HH along with Bekenstein-Hawking entropy. Also, we observe the GSLT and thermal equilibrium condition for flat FRW universe at HH along with Bekenstein-Hawking entropy for three different entropies which are Renyi entropy, logarithmic-corrected entropy as well as Sharma-Mittal entropy. The results of this chapter are compiled and submitted in the form of a paper.

4.1 First-order formalism of multiquintessence scenario

We define the first-order formalism [117, 124] for the coupled SF's with gravity. The action [81] of four-dimensional gravity is given in the form

$$S = \int d^4x\sqrt{|g|}\left[\frac{-R}{4} + \frac{1}{2}g_{ab}\nabla^a\phi_i\nabla^b\phi_i - V(\phi_i)\right], \tag{4.1.1}$$

where ϕ_i, $i = 1, 2, ..., N$, defines real SF's, coupled to a set and $V(\phi) \equiv (\phi_1, \phi_2, ..., \phi_N)$ is the potential which characterize the theory on the subject of a limited arbitrary number of SF's, we here assuming that $c = 4\pi G = 1$.

The field equations for this model are

$$\begin{aligned} H^2 &= \frac{2\rho}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{\rho}{3} - p, \end{aligned} \tag{4.1.2}$$

where double dots represents the derivative with respect to time, ρ describes the energy density and p is the pressure of the system, respectively. The energy density and pressure are given as [81]

$$\rho = \sum_{j=1}^{N}\frac{\dot{\phi}_j^2}{3} + \frac{3W(\phi_i)^2}{2} - \frac{1}{2}\sum_{i=1}^{N}W_{\phi_i}^2,$$

From Eq.(4.1.1), the equation of motion for SF's is defined as

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_{\phi_i} = 0, \tag{4.1.4}$$

where ϕ_i in subscript defines the derivative with respect to ϕ_i, $V_{\phi_i} \equiv \frac{dV}{d\phi_i}$ and $\phi_i = \phi_i(t)$. By following the first-order formalism, the Hubble parameter $H = -W_{\phi_i}$, which leads to

$$\dot{\phi}_i = W_{\phi_i}, \quad i = 1, 2, ..., N. \tag{4.1.5}$$

By solving the field equation, the potential term we have

$$V(\phi_i) = \frac{3W(\phi_i)^2}{2} - \frac{1}{2}\sum_{i=1}^{N} W_{\phi_i}^2. \tag{4.1.6}$$

We defining the models of Hubble parameter and corresponding solutions of SF's. The models are:

4.1.1 Model 1

Firstly, we choose the superpotential that is Z_2 model [125] and the direct sum of sine-Gordon [126], given by

$$W(\phi_1, \phi_2) = \lambda_1\left(\phi_1 - \frac{\phi_1^3}{3}\right) + \lambda_2 \sinh(\phi_2) + \alpha_1, \tag{4.1.7}$$

where λ_1, λ_2 and α_1 are arbitrary constants. **Table 1**, shows the first-order equations and corresponding solutions of SF's.

Table 1:

$\dot{\phi}_i, i = 1, 2$	$\phi_i(t)$
$\lambda_1(1 - \phi_1^2)$	$\tanh(\lambda_1 t)$
$\lambda_2 \cosh(\phi_2)$	$\arcsin h[\tan(\lambda_2 t)]$

4.1.2 Model 2

The second superpotential is the combination of sine-Gordon, Z_2 and BNRT models [127], we have

$$W(\phi_1, \phi_2, \phi_3, \phi_4) = \lambda_1\left(\phi_1 - \frac{\phi_1^3}{3}\right) + \lambda_2 \sinh(\phi_2)$$

$$-\lambda_3\phi_3 + \frac{\lambda_3\phi_3^3}{3} + \mu_3\phi_3\phi_4^2 + \alpha_2. \tag{4.1.8}$$

The first-order equations and solutions of ϕ_1 and ϕ_2 are same as in **model 1**. In **Table 2**, we show the differential equations and their solutions.

Table 2:

$\dot{\phi}_i, i = 3, 4$	$\phi_i^{(k)}(t), k = 1, 2$
$-\lambda_3(1-\phi_3^2) + \mu_3\phi_4^2$	$\frac{\left(\sqrt{c_o^2-4}\right)\sinh(2\mu_3 t)}{(\sqrt{c_o^2-4})\cosh(2\mu_3 t)-c_o}$, $\frac{\left(\sqrt{1-16c_o}\right)\sinh(4\mu_3 t)}{(\sqrt{1-16c_o})\cosh(4\mu_3 t)+1}$, where $c_o < -2$ and $\lambda_3 = \mu_3$.
$2\mu_3\phi_3\phi_4$	$\frac{2}{\left(\sqrt{c_o^2-4}\right)\cosh(2\mu_3 t)-c_o}$, $-\frac{2}{\left(\sqrt{1-16c_o}\right)\cosh(4\mu_3 t)+1}$, where $c_o < \frac{1}{16}$ and $\lambda_3 = 4\mu_3$.

4.1.3 Model 3

For third model [128], the superpotential is as follows

$$W(\phi_1, \phi_2, \phi_3, \phi_4) = \lambda_1\left(\phi_1 - \frac{\phi_1^3}{3}\right) + \lambda_2\sin(\phi_2) - \frac{\lambda_3\phi_3^3}{3}$$
$$-\phi_3^2\phi_4 + \phi_4 - \frac{\phi_4^3}{3} + \alpha_3, \tag{4.1.9}$$

where α_3 is an arbitrary constant. The first-order differential equations and solutions of ϕ_1 and ϕ_2 are of **model 1**. **Table 3**, shows the first-order differential equations and solutions of the equations which satisfies the differential equations.

Table 3:

$\dot{\phi}_i, i = 3,4$	$\phi_i(t)$
$-\lambda_3\phi_3^2 - 2\phi_3\phi_4$	$4\Big((c_o^2(\lambda_3^2+4) - \lambda_3^2 - 5)\sinh(2t) + (c_o^2(\lambda_3^2+4) - \lambda_3^2 - 3)\cosh(2t)) + 2c_o\sqrt{\lambda_3^2+4}\Big)^{-1}.$
$-\phi_4^2 - \phi_3^2 + 1$	$\Big((c_o^2(\lambda_3^2+4) - \lambda_3^2 - 3)\sinh(2t) + (c_o^2(\lambda_3^2+4) - \lambda_3^2 - 5)\cosh(2t) - 2\lambda_3\Big)\times \Big(c_o^2(\lambda_3^2+4) - \lambda_3^2 - 5)\sinh(2t) + (c_o^2(\lambda_3^2+4) - \lambda_3^2 - 3)\cosh(2t)) + 2c_o\sqrt{\lambda_3^2+4}\Big)^{-1}.$

4.2 GSLT and Thermal Equilibrium Condition

In next sections, we study the validity of GSLT of the multiquintessence at HH for flat FRW universe. HH is given as

$$R_A = \frac{1}{H}.$$

From the above expression, we get

$$dR_A = HR_A^3(\rho + p)dt. \tag{4.2.1}$$

The Bekenstein entropy is defined as $S = \frac{A}{4G}$, where $A = 4\pi r^2$ is the area of the horizon [122, 66]. By using the Bekenstein entropy, Eq.(4.2.1) becomes

$$\frac{G}{2\pi R_A}dS = HR_A^3(\rho + p)dt. \tag{4.2.2}$$

The temperature of the horizon is defined as [129]

$$\begin{aligned} T_h = \frac{|k_{sg}|}{2\pi}, \quad k_{sg} &= \frac{1}{2\sqrt{-h}}\partial_\mu(\sqrt{-h}h^{\mu\nu}\partial_\nu R_A) \\ &= -\frac{1}{R_A}(1 - \frac{\dot{R}_A}{2HR_A}) = -\frac{R_A}{2}(2H^2 + \dot{H}). \end{aligned} \tag{4.2.3}$$

Multiplying $T_h = -\frac{1}{2\pi R_A}(1 - \frac{\dot{R_A}}{2HR_A})$ on both sides of Eq.(4.2.2), we have

$$T_h dS = [-4\pi H R_A^3 dt + 2\pi R_A^2 dR_A](\rho + p). \tag{4.2.4}$$

Now, we define the Misner-sharp energy which is $E = \frac{R_A}{4G}$, we describe the energy density in terms of the volume $V = \frac{4\pi R_A^3}{3}$, which becomes

$$\tilde{E} = \frac{3H^2}{8\pi G}V \equiv \rho V. \tag{4.2.5}$$

By taking the differential of the energy density, we easily find

$$d\tilde{E} = -4\pi H R_A^3(\rho + p)dt + 4\pi R_A^2 \rho dR_A. \tag{4.2.6}$$

Assembling Eqs.(4.2.4) and (4.2.6), we can obtain

$$T_h dS = d\tilde{E} - 2\pi R_A^2(\rho - p)dR_A. \tag{4.2.7}$$

The work density is defined as

$$\tilde{W} = -\frac{1}{2}(T^{(M)\mu\nu}h_{\mu\nu} + \tilde{T}^{(de)\mu\nu}h_{\mu\nu}) = \frac{1}{2}(\rho - p), \tag{4.2.8}$$

utilizing the work density in Eq.(4.2.7), we get

$$T_h dS = d\tilde{E} - \tilde{W}dV. \tag{4.2.9}$$

Eq.(4.2.9) shows that FLT is satisfied in multi-component SF.

However, GSLT states that sum of the BH entropy and the entropy of the BH external region can never be decreases [95]. The condition that satisfies the entropy relation, described as

$$\dot{S}_{tot} = \dot{S} + \dot{S}_{in} \geq 0, \tag{4.2.10}$$

where $\dot{S}_{tot}$ is the total entropy of the energy and matter inside the horizon, $\dot{S}$ relates to the horizon entropy and $\dot{S}_{in}$ correspond to the inner horizon with the sum of all entropy components. To study the GSLT, we now proceed with modified FLT,

$$T_i dS_i = dE_i + p_i dV, \tag{4.2.11}$$

which can be written as

$$T_{in}\dot{S}_{in} = 4\pi R_A^2(\dot{R}_A - HR_A)(\rho_i + p_i). \tag{4.2.12}$$

T_{in} denotes the temperature of the inner horizon for all components. Here $\sum_i(\rho^i + p^i) = \rho + p$, the total entropy inside the horizon becomes

$$T_{in}dS_{in} = 4\pi R_A^2(\dot{R}_A - HR_A)(\rho + p). \tag{4.2.13}$$

Taking the time derivative of Bekenstein entropy, we find out

$$\dot{S} = -\frac{2\pi\dot{H}}{H^3G} \tag{4.2.14}$$

The thermal equilibrium set with $T_{in} = T_h$ and Eq.(4.2.13) leads to

$$\dot{S}_{in} = \frac{1}{T_h}(\rho + p)4\pi R_A^2(\dot{R}_A - HR_A). \tag{4.2.15}$$

After some calculations, it results

$$\dot{S}_{in} = \frac{4\pi}{G}\left[\frac{\dot{H}(\dot{H} + H^2)}{H^3(\dot{H} + 2H^2)}\right]. \tag{4.2.16}$$

By putting Eqs.(4.2.14) and (4.2.16) in (4.2.10), we get

$$\dot{S}_{tot} = \frac{\pi}{W^2}\left[\frac{16\pi W_\phi^2(-W_\phi^2 + W^2)}{W(-W_\phi^2 + 2W^2)} - \frac{2W_\phi^2}{W}\right]. \tag{4.2.17}$$

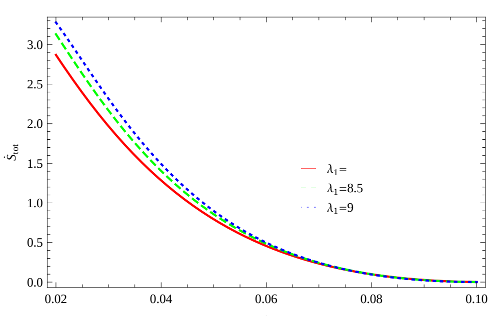


Figure 4.1: Plot of $\dot{S}_{tot}$ versus t for model 1.

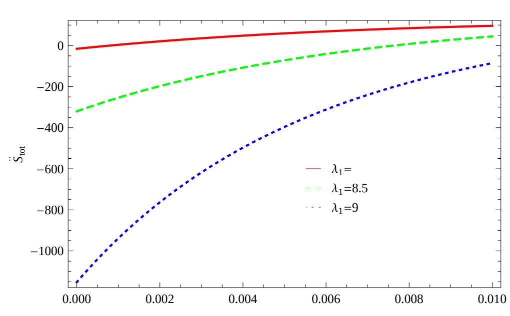


Figure 4.2: Plot of $\ddot{S}_{tot}$ versus t for model 1.

- **<u>For Model 1:</u>** In **Figure 4.1**, we plot graph of $\dot{S}_{tot}$ versus time for Bekenstein entropy at HH for flat spacetime. We choose values of parameter $\lambda_1 = 8,\ 8.5,\ 9,\ \lambda_2 = -4,\ \alpha = 5$. All trajectories are decreasing positively with the increasing value of t, which shows the validity of GSLT. **In Figure 4.2**, for $\lambda_1 = 8$, the trajectory is initially negative, while shows transition towards after some epoch. It means thermal condition holds at the present as well as early epoch but remains invalid in the later epoch. However, the trajectories remain in the negative phase which exhibits the validity of thermal condition for $\lambda_1 = 8.5,\ 9$.

- **<u>For Model 2:</u>** **Figure 4.3**, shows the graph of $\dot{S}_{tot}$ versus t. With the same values of $\lambda_1,\ \lambda_2,\ \lambda_3 = 5,\ c = 5,\ \mu = 5.5$ and $\alpha = 4$. All trajectories are gradually increasing in positive direction at present epoch as well as later epoch with the increasing value of t which leads to the validity of GSLT. **Figure 4.4**, shows that for $\lambda_1 = 8$ the trajectory remain in negative phase at later epoch which fulfills the thermal condition and for $\lambda_1 = 8.5,\ 9$, trajectories show decreasing behavior towards positive direction at later epoch and cannot maintain the stability of thermal condition.

- **<u>For Model 3:</u>** By taking the same values of $\lambda_1,\ \lambda_2,\ \lambda_3,\ c$ and $\alpha = -6.5$, **Figure 4.5**, demonstrate that for $\lambda_1 = 9$, GSLT preserved the validity in the later epoch while remains invalid for other two cases of $\lambda_1 = 8,\ 8.5$. **Figure 4.6**, shows that thermal equilibrium condition at the later epoch for $\lambda_1 = 8$. However, thermal stability occurs for other two cases $\lambda_1 = 8.5,\ 9$ at the present epoch as well as later epoch.

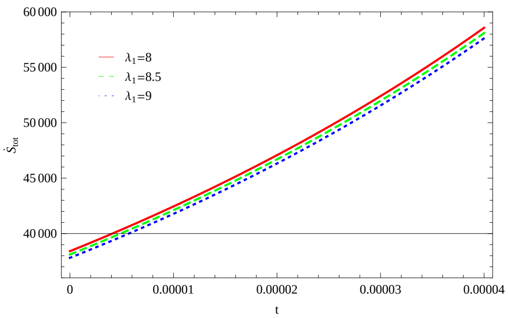


Figure 4.3: Plot of $\dot{S}_{tot}$ versus t for model 2.

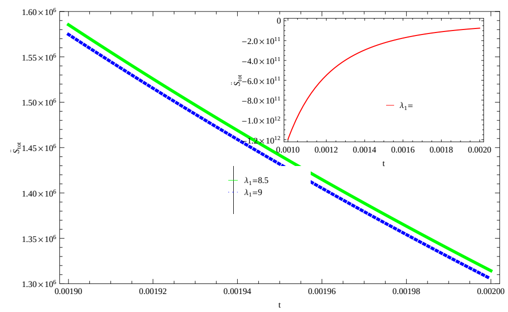


Figure 4.4: Plot of $\ddot{S}_{tot}$ versus t for model 2.

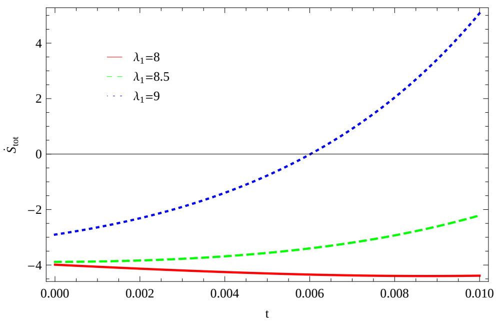


Figure 4.5: Plot of $\dot{S}_{tot}$ versus t for model 3.

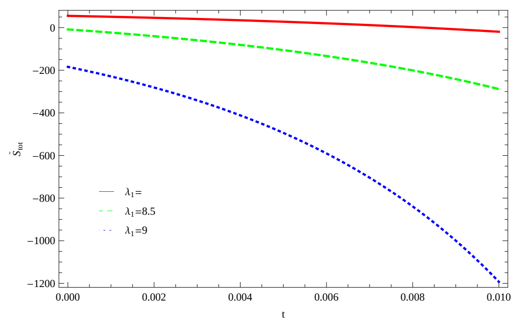


Figure 4.6: Plot of $\ddot{S}_{tot}$ versus t for model 3.

4.2.1 Sharma-Mittal Entropy

We here introduce Sharma-Mittal entropy to discuss the validity of GSLT, which is

$$S_{SM} = \frac{1}{1-r}[(1+\frac{\delta A}{4})^{\frac{1-r}{\delta}} - 1], \tag{4.2.18}$$

where r is the free parameter. By taking the time derivative of above entropy, we have

$$\dot{S}_{SM} = 2\pi R_A \dot{R}_A (1+\delta\pi R_A^2)^{\frac{1-r}{\delta}-1}. \tag{4.2.19}$$

For the case of HH, Eq.(4.2.19) becomes

$$\dot{S}_{SM} = -\frac{2\pi\dot{H}}{H^3}(1+\frac{\delta\pi}{H^2})^{\frac{1-r}{\delta}-1}. \tag{4.2.20}$$

Inserting Eqs.(4.2.16) and (4.2.20) in (4.2.10), we have

$$\dot{S}_{tot} = \frac{\pi}{W^2}\left[\frac{16\pi W_\phi^2(-W_\phi^2+W^2)}{W(-W_\phi^2+2W^2)} - \frac{2W_\phi^2}{W}(1+\frac{\delta\pi}{W^2})^{\frac{1-r}{\delta}-1}\right]. \tag{4.2.21}$$

We check the validity of GSLT and thermal equilibrium condition for Eq.(4.2.21) at HH by graphical representation.

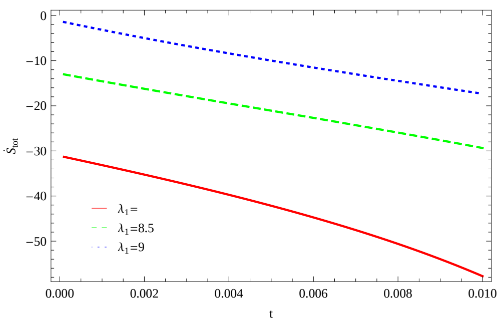


Figure 4.7: Plot of $\dot{S}_{tot}$ versus t for model 1.

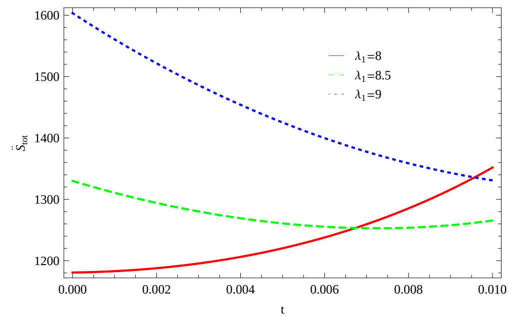


Figure 4.8: Plot of $\ddot{S}_{tot}$ versus t for model 1.

- **For Model 1:** In **Figure 4.7**, by taking the same values of $\lambda_1,\ \lambda_2,\ \alpha$ and $r = 1,\ \delta = 0.1$, the graph demonstrates that all trajectories gradually decreasing towards negative direction and remain in the negative phase at later epoch, which cannot fulfills the stability condition for GSLT. With the same values of $\lambda_1,\ \lambda_2,\ \alpha,\ r$ and δ and $\lambda_1 = 8,\ 8.5,\ 9,$ the thermal stability remain invalid as all trajectories remains in positive phase at present epoch as well as later epoch **(Figure 4.8)**.

- **For Model 2:**All trajectories are increasing in positive direction at present epoch as well as later epoch with the increasing value of t. By taking the same values of all parameters $\lambda_1,\ \lambda_2,\ \lambda_3,\ r,\ c,\ \mu,\ \alpha,$ which confirms the validity of GSLT **(Figure 4.9)**. In **Figure 4.10**, the thermal equilibrium condition satisfies with the same values of all parameters as all trajectories for $\lambda_1 = 8,\ 8.5$ 9 remain in the negative phase at present as well as later epoch.

- **For Model 3:**With the same values of all parameters, the left trajectories **(Figure 4.11)** shown that for $\lambda_1 = 8,\ 8.5$ remain in the positive phase and for $\lambda_1 = 9$ gradually decreasing towards negative phase at later epoch. Thus, validity of GSLT confirms only for two cases $\lambda_1 = 8,\ 8.5$. As we increases the value of λ_1 the stability condition cannot maintain. The right side trajectory $\lambda_1 = 8$ increasing towards positive phase at present as well as later epoch **(Figure 4.12)** and for $\lambda_1 = 8.5,\ 9$, trajectories show increasing behavior at later epoch and remain in the positive phase which cannot preserve the thermal condition.

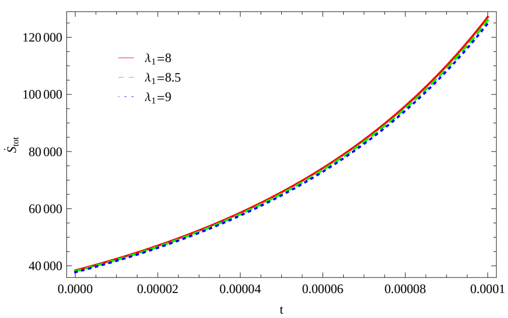


Figure 4.9: Plot of $\dot{S}_{tot}$ versus t for model 2.

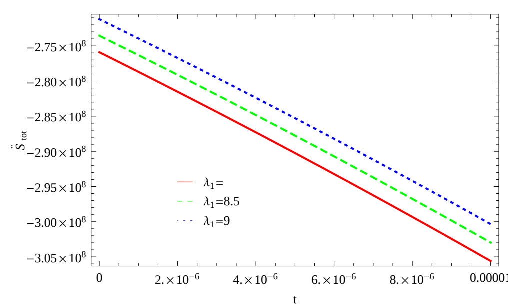


Figure 4.10: Plot of $\ddot{S}_{tot}$ versus t for model 2.

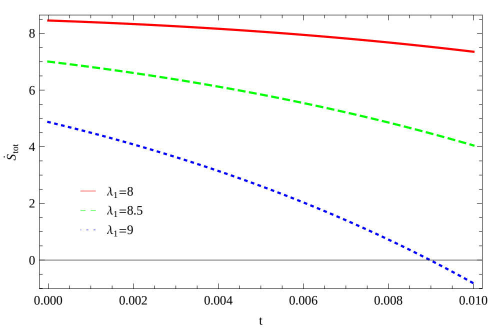


Figure 4.11: Plot of $\dot{S}_{tot}$ versus t for model 3.

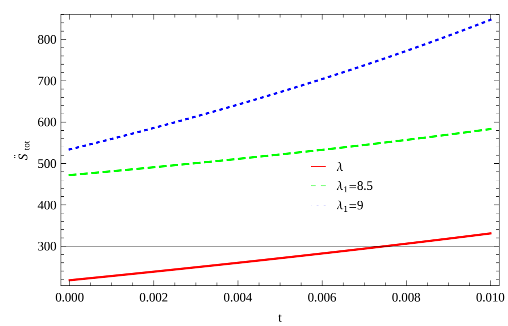


Figure 4.12: Plot of $\ddot{S}_{tot}$ versus t for model 3.

4.2.2 Logarithmic Corrected Entropy

The Bekenstein-Hawking logarithmic corrected entropy is defined by the relation

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{4G} + \beta \frac{4G}{A} + \gamma, \tag{4.2.22}$$

where $\alpha,\ \beta$ and γ are dimensionless constants, apart from the exact values of these constants are still checked. Now, by taking the time derivative of logarithmic corrected entropy, we easily get

$$\dot{S}=\left(\frac{\dot{A}}{A}\right)\frac{A}{4G}\left[1+\alpha\frac{4G}{A}-\beta\left(\frac{4G}{A}\right)^2\right]. \tag{4.2.23}$$

In the case of HH, Eq.(4.2.23) becomes

$$\dot{S}=-\frac{2\pi\dot{H}}{H^3}\left[1+\alpha\frac{GH^2}{\pi}-\beta(\frac{GH^2}{\pi})^2\right]. \tag{4.2.24}$$

Putting the values of $\dot{S}_{in}$ and $\dot{S}$ in Eq.(4.2.10), we have

$$\begin{aligned}\dot{S}_{tot} &= \frac{\pi}{W^2}\left[\frac{16\pi W_\phi^2(-W_\phi^2+W^2)}{W(-W_\phi^2+2W^2)}-\frac{2W_\phi^2}{W}\right.\\ &* \left.\left[1+\alpha\frac{W^2}{4\pi^2}-\beta\left(\frac{W^2}{4\pi^2}\right)^2\right]\right].\end{aligned} \tag{4.2.25}$$

Eq.(4.2.25) graphically represents to observe the validity of GSLT and thermal equilibrium condition at HH.

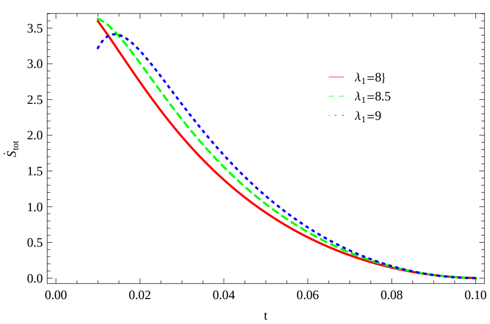


Figure 4.13: Plot of $\dot{S}_{tot}$ versus t for model 1.

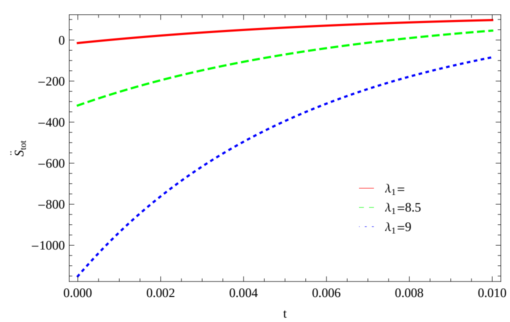


Figure 4.14: Plot of $\ddot{S}_{tot}$ versus t for model 1.

- **<u>For Model 1:</u>** In **Figure 4.13**, with the same values of $\lambda_1,\ \lambda_2,\ \alpha$ and $\alpha_1=2,\ \beta=1.5$, the trajectories decreasing towards positive direction at later epoch with the increasing value of t, which shows the validity of GSLT. The trajectories of **(Figure 4.14)** shown that for $\lambda_1=8$ trajectory increasing in positive phase at present epoch as well as later epoch. For $\lambda_1=8.5,\ 9$ the trajectory remains in the negative phase at present epoch and later epoch. Hence, with the increasing value of λ_1 the stability condition maintain and the thermal condition satisfies.

- **<u>For Model 2:</u>** The left side trajectories increasing positively at present epoch as well as later epoch with the increasing values of t, for all constant parameters, which shows that the GSLT is valid **(Figure 4.15)**. With the same values of all parameters, the thermal equilibrium condition satisfies, as all trajectories decreases at present as well as later epoch and remain in the negative phase for $\lambda_1 = 8,\ 8.5,\ 9$ **(Figure 4.16)**.

- **<u>For Model 3:</u>Figure 4.17**, demonstrate that by taking the same values of $\lambda_1,\ \lambda_2,\ \lambda_3,\ c$ and $\alpha = -6.5$, for $\lambda_1 = 9$, GSLT preserved the validity in the later epoch while remains invalid for other two cases of $\lambda_1 = 8,\ 8.5$. In **Figure 4.18**, we plot three graphs for $\lambda_1 = 8,\ 8.5,\ 9$, in first graph for $\lambda_1 = 8$ shows that trajectory decreasing towards positive direction at present as well as later epoch, for $\lambda_1 = 8.5$ the trajectory is remain in negative phase at present epoch as well as later epoch and for $\lambda_1 = 9$ the trajectory decreasing negatively. Hence, the thermal equilibrium condition satisfies for $\lambda_1 = 8.5,\ 9$ and is invalid for the case $\lambda_1 = 8$.

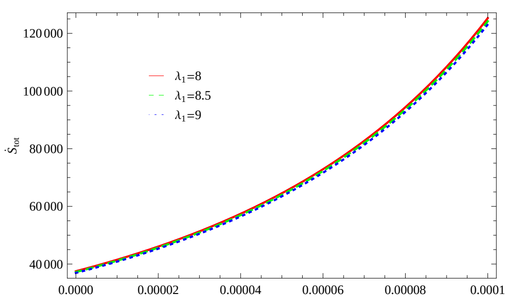


Figure 4.15: Plot of $\dot{S}_{tot}$ versus t for model 2.

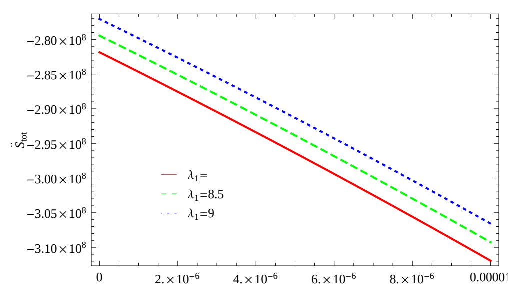


Figure 4.16: Plot of $\ddot{S}_{tot}$ versus t for model 2.

4.2.3 Rényi Entropy

We define here Rényi entropy to check the validity of GSLT, we have

$$S = \frac{1}{\delta}\ln(1 + \frac{\delta A}{4}), \tag{4.2.26}$$

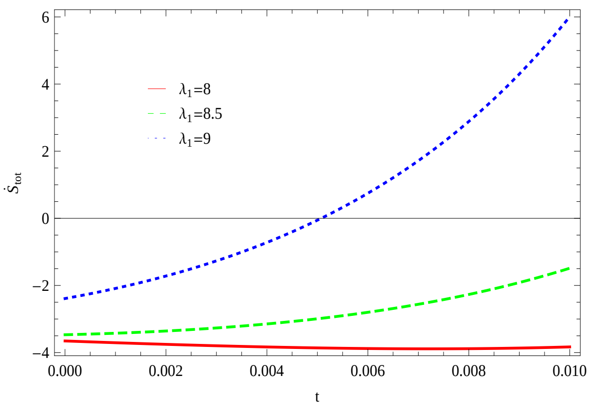


Figure 4.17: Plot of $\dot{S}_{tot}$ versus t for model 3.

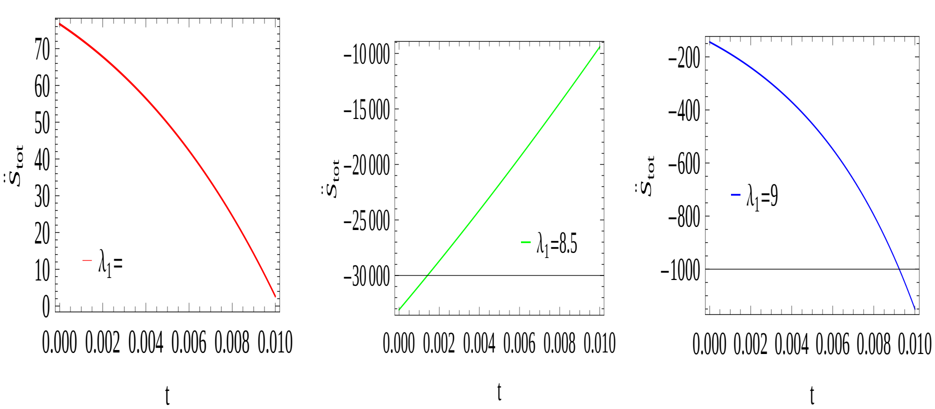


Figure 4.18: Plot of $\ddot{S}_{tot}$ versus t for model 3.

where $\delta = 1 - Q$, time derivative of Eq.(4.2.26) is

$$\dot{S} = \frac{2\pi\delta R_A \dot{R}_A}{\delta(1 + \delta\pi R_A^2)}, \tag{4.2.27}$$

Rényi entropy written in HH case as

$$\dot{S} = -\frac{2\pi\dot{H}}{H(H^2 + \delta\pi)}, \tag{4.2.28}$$

with given entropy Eq.(4.2.10) becomes

$$\dot{S}_{tot} = 16\pi^2 \left[\frac{W_\phi^2(-W_\phi^2 + W^2)}{W^3(-W_\phi^2 + 2W^2)}\right] - \frac{2\pi W_\phi^2}{W(W^2 + \delta\pi)}. \tag{4.2.29}$$

Form Eq.(4.2.29) we check the validity of GSLT and thermal equilibrium condition for Rényi entropy at HH.

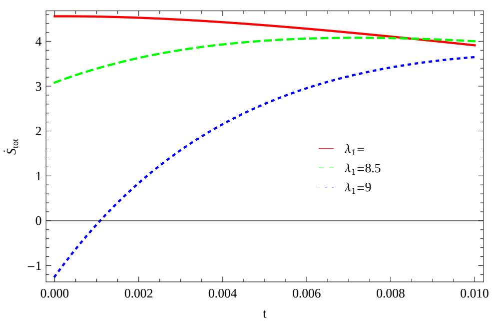


Figure 4.19: Plot of $\dot{S}_{tot}$ versus t for model 1.

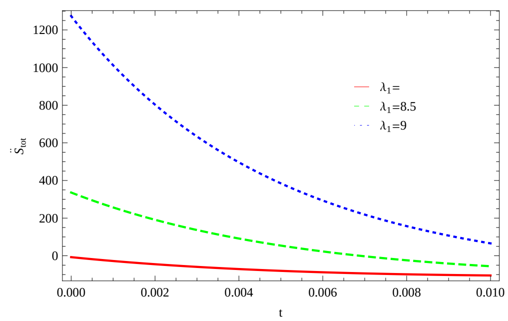


Figure 4.20: Plot of $\ddot{S}_{tot}$ versus t for model 1.

- **<u>For Model 1:</u>Figure 4.19**, shown that the trajectory decreases in positive phase for $\lambda_1 = 8$ and for other two cases $\lambda_1 = 8.5,\ 9$ trajectories increases toward positive direction at present as well as later epoch, which confirms the validity of GSLT. In **Figure 4.20**, the trajectories for $\lambda_1 = 8,\ 8.5,$ gradually decreases at present epoch as well as later epoch and preserved the thermal condition and for the decreasing trajectory in the positive phase for $\lambda_1 = 9$ the stability condition cannot maintain and the thermal equilibrium condition is invalid for this case.

- **<u>For Model 2:</u>**In **Figure 4.21**, by taking the same values of $\lambda_1,\ \lambda_2,\ \lambda_3,\ \alpha,$ $c,\ \mu\ and\ \delta$, for all cases of λ_1 all trajectories remain in the positive phase at present as well as later epoch which demonstrate that the GSLT is valid. In **Figure 4.22**, the trajectory remain constant in positive phase ($\lambda_1 = 8$) which cannot preserve the thermal condition and for other two cases ($\lambda_1 = 8.5,\ 9$) trajectories remain constant in negative phase which confirms the thermal condition.

- **<u>For Model 3:</u>Figure 4.23**, for $\lambda_1 = 9$, GSLT preserved the validity in the later epoch while remains invalid for other two cases of $\lambda_1 = 8,\ 8.5$, by taking the same values of all parameters. **Figure 4.24**, demonstrate that with the same values of all parameters the thermal condition cannot satisfies for all cases of λ_1 at present epoch as well as later epoch as all trajectories remain in the positive phase.

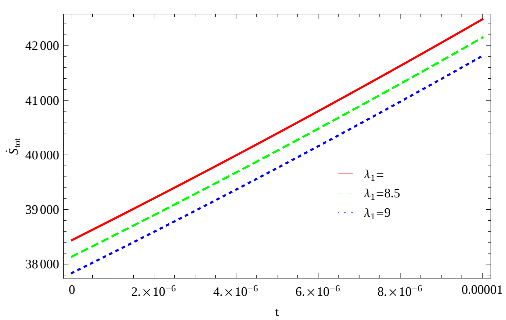


Figure 4.21: Plot of $\dot{S}_{tot}$ versus t for model 2.

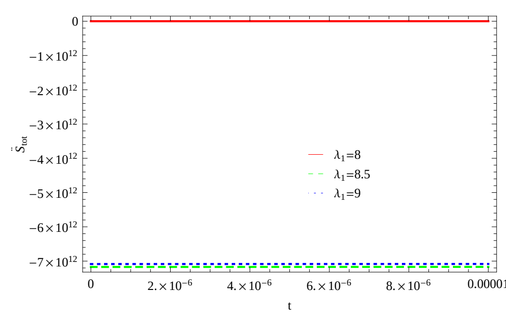


Figure 4.22: Plot of $\ddot{S}_{tot}$ versus t for model 2.

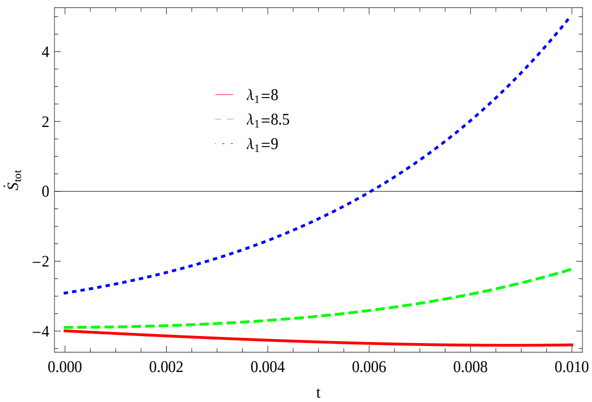


Figure 4.23: Plot of $\dot{S}_{tot}$ versus t for model 3.

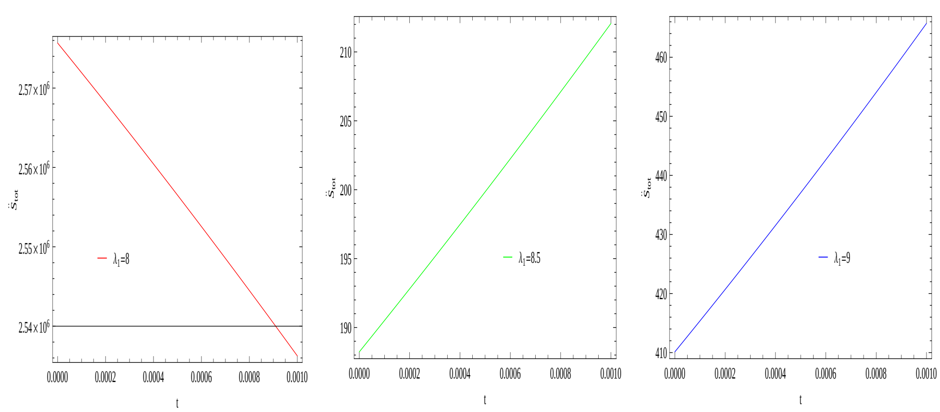


Figure 4.24: Plot of $\ddot{S}_{tot}$ versus t for model 3.

Chapter 5
Summary

In this thesis, we studied the thermodynamical laws in $f(R, T^{\phi})$ gravity for the expanding universe in the absence of matter. We researched that FLT and GSLT fulfilled in $f(R, T^{\phi})$ gravity and we checked the validity of GSLT at AH by utilizing Bekenstein-Hawking entropy. By considering the first-order formalism, the Hubble parameter becomes $H = W(\phi)$, where W is the function of SF. We choose three distinct values of the Hubble parameter given as $W = e^{b_1\phi}$, $W = b_2(\frac{\phi^3}{3} - \phi)$ and $W = b_3 \sinh(\phi)$. With the diverse values of the $W(\phi)$ for flat ($k = 0$), closed ($k = 1$) and open ($k = -1$) universe we checked the validity of GSLT and also analysed the condition of thermal equilibrium at AH with Bekenstein entropy. By graphical representation we observed that the thermal equilibrium condition is satisfied and the GSLT is valid for every different value of $W(\phi)$ together with Bekenstein-Hawking entropy at AH. The above analysis is summarized in the **Table**.

Also, we examined the validity of GSLT in multiquintessence at HH for flat FRW universe. Throughout we follow the first-order formalism and we take three different superpotential models of the Hubble parameter. **Model 1** is the direct sum of Z_2 and sine-Gordon, **model 2** is the combination of Z_2, BNRT and sine-Gordon and in **model 3** we consider the modified BNRT model along with Z_2 and sine-Gordon models. We also checked the stability condition of the thermal equilibrium and GSLT at present epoch and later epoch for Bekenstein entropy at HH along with three different Hubble parameter models for flat spacetime. We also choose three different entropies as an example which are Sharma-Mittal, logarithmic corrected and Rényi entropies, to observe the stability condition for each models of the $W(\phi_i)$ at HH for flat spacetime. We also observed that FLT satisfied in multiquintessence at HH for flat spacetime. To check the stability condition we plot graphs by taking the same values of all parameters and for different cases of λ_1 by choosing three different values for flat spacetime along with different $W(\phi_i)$ in the constructed model of each entropy at HH.

The stability of GSLT validity and thermal condition at HH for flat spacetime with all the entropies given by:

- **For Bekenstein entropy:**

  For **model 1** and **model 2** the stability condition preserved the validity of GSLT for Bekenstein entropy and thermal condition only satisfies for $\lambda_1 = 9$ for **model 1** and in **model 2** the stability condition for thermal equilibrium satisfies only for $\lambda_1 = 8$ at later epoch, while for **model 3** GSLT is valid at later epoch only for $\lambda_1 = 9$. Moreover, condition of thermal equilibrium satisfied for all the cases of λ_1.

- **For Sharma-Mittal entropy:**

  For **model 1** validity of GSLT cannot hold at later epoch as all trajectories remains in negative phase and the thermal equiibrium condition cannot occured for all cases, for **model 2** the stability condition of GSLT and the thermal equilibrium maintained at present as well as later epoch and for **model 3** GSLT is valid only for two cases $\lambda_1 = 8,\ 8.5$ and the thermal equilibrium condition cannot be satisfied for any case.

- **For Logarithmic corrected entropy:**

  The validity of GSLT at HH for **model 1** confirmed at later epoch and the stability condition of thermal equilibrium maintained for $\lambda_1 = 9$, for **model 2** GSLT validity and the thermal equilibrium condition confirmed at present epoch as well as later epoch and for **model 3** with the increasing value of λ_1, the stability condition of GSLT preserved and the thermal equilibrium condition is invalid for $\lambda_1 = 8$ and the thermal equilibrium condition occured for other two cases $\lambda_1 = 8.5,\ 9$.

- **For *Rényi* entropy:**

  The stability condition of GSLT for **model 1** is satisfied for all the cases at present as well as later epoch and the thermal equilibrium condition confirmed at present epoch as well as later epoch only for two cases $\lambda_1 = 8,\ 8.5$ and the stability of thermal equilibrium condition cannot be preserved with the increasing value of λ_1. For **model 2** GSLT is valid for all values of λ_1 at present as well as later epoch and with the increasing value of λ_1, the thermal equilibrium condition satisfied and for **model 3** GSLT maintained the validity at later epoch for $\lambda_1 = 9$ and the stability of thermal equilibrium condition cannot be satisfied for each case.

Thus, the stability of GSLT and the thermal equilibrium condition for each model satisfied only for some cases at present as well as later epoch.

$W(\phi)$	Space-time	Validity of GSLT	Thermal equilibrium
$e^{b_1\phi}$	$k=0$	Valid at present time and at early time for $z \leq 0.15$.	Satisfied at early time.
	$k=1$	Valid for ϕ^+ at present and future times and for ϕ^- at past, present and future times.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.
	$k=-1$	Valid for ϕ^+ at present and future times and for ϕ^- at past, present and future times.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.
$b_2(\frac{\phi^3}{3}-\phi)$	$k=0$	Valid at early time.	Satisfied at early time.
	$k=1$	Valid for ϕ^+ at past, present and future times and for ϕ^- at future time for $z < -0.2$.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.
	$k=-1$	Valid for ϕ^+ and ϕ^- at past, present and and future times.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.
$b_3\sinh(\phi)$	$k=0$	Valid at early time for $z < 0.4$.	Satisfied at early time.
	$k=1$	valid for ϕ^+ at past, present and future times and for ϕ^- at present and future times.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.
	$k=-1$	Valid for ϕ^+ at past, present and future times and for ϕ^- at present and future times.	Satisfied for ϕ^+ and ϕ^- at past, present and future times.

Chapter 6
References

[1] Roy, N. and Bhadra, N.: J. C. Astropart. Phys. **2018**(2018)11.

[2] Ade et al., P.A.R.: Astron. Astrophs. **594**(2016)A13.

[3] Riess et al., A.G.: Astron. J. **116**(1998)1009.

[4] Sharif, M. and Ikram, A.: Adv. High Ener. Phys. **2018**(2018)2563871.

[5] Velten, H.E.S., Vom Marttens, R.F. and Zimdahl, W.: Eur. Phys. J. C. **74**(2014)3160; Padilla, A.: arXiv:1502.05296.

[6] Buchdahl, H.A.: Phys. Rev. D 150(1970)1.

[7] Harko et al.: Phys. Rev. D **84**(2011)024020.

[8] Perlmutter S. et al.: Astrophys. J. **517**(1999)565.

[9] Riess, A.G. et al.: Astrophys. J. **607**(2004)665.

[10] Tegmark, M. et al.: Phys. Rev. D **69**(2004)103501.

[11] Benitez, N. et al.: Astrophys. J. **691**(2009)241.

[12] Parkinson D. et al.: Mon. Not. R. Astron. Soc. **401**(2010)2169.

[13] Spergel, D.N. et al.: Astrophys. J. Suppl. **170**(2007)377.

[14] Dunkley, J. et al.: Astrophys. J. **701**(2009)1804.

[15] Hinshaw, G. et al.: Astrophys. J. **208**(2013)19.

[16] Bennett, C.L. et al.: Astrophys. J. **208**(2013)20.

[17] Ade, P.A.R. et al.: Astron. Astrophys. **571**(2014)A16.

[18] Ade, P.A.R. et al.: Astron. Astrophys. **594**(2016)A13.

[19] Riess, A. G. et al.: Astron. J. **116**(1998)1009.

[20] Bazeia, D., Gomes, C.B., Losano, L. and Menezes, R.: Phys. Letter B **633**(2006)419.

[21] Frieman, J., Hill, C., Stebbins, A. and Waga, I.: Phys. Rev. Lett. **75**(1995)2077.

[22] Caldwell, R.R., Dave, R. and Steinhardt, P.J.: Phys. Rev. Lett. **80**(1998)1582.

[23] Baffou et al., E.H.: Astrophys. Space Sci. **356**(2015)173.

[24] Myrzakulov, R.: arXiv: 1205.5266v2 [Physics.gen-ph].

[25] Alves, M.E.S., Moraes, P.H.R.S., de Araujo, J.C.N. and Malheiro, M.: Phys. Rev. D **94**(2016)024032.

[26] Moraes, P.H.R.S. and Santos, J.R.L.: Eur. Phys. J.C. **76**(2016)60.

[27] Bazeia et al., D.: Phys. Lett. B **633**, (2006) 415.

[28] Bazeia et al., D.: Phys. Lett. B **743**(2015)98.

[29] Singh, V. and Singh, C.P.: Astrophys. Space Sci. **356**(2015)153.

[30] Frieman, J. et al.: Rev. Astron. Astrophys. **46**(2008)385.

[31] Caldwell, R.R. et al.: Phys. Rev. Lett. **80**(1998)1582.

[32] Liddle, A.R. and Lyth, D.H.: (Cambridge University Press, 2000).

[33] Linde, A.D.: Phys. Lett B **129**(1983)177.

[34] Barrow, J.D.: Phys. Rev. D **48**(1993)1585.

[35] Peebles, P.J. and Ratra, B.: Astrophys. Lett. J. **325**(1988)L17.

[36] Tsuwikawa, S.: Class. Quant. Grav. **30**(2013)214003.

[37] Liddle, A.R. and Scherrer, R.J.: Phys. Rev. D **59**(1999)023509.

[38] Sahni, V. and Wang, L.M.: Phys. Rev. D **62**(2000)103517.

[39] Matos, T. and Lopez, U.L.A.: Phys. Rev. D **63**(2001)063506.

[40] Lopez, U.L.A.: Phys. Conf. Series J. **761**(2016)012076.

[41] De Haro, J. et al.: Phys. Rev. D **93**(2016)084018.

[42] Peebles, P.J. and Ratra, B.: Phys. Rev. Mod. **75**(2003)559.

[43] Brans, C.H. and Dicke, R.H.: Phys. Rev. **124**(1961)925.

[44] Hanlon, J.: Phys. Rev. Lett. **29**(1972)137.

[45] Hordenski, G.W.: Phys. Int. Theory J. **10**(1974)363.

[46] Deffayet, C. and Steer, D.A.: Quan. Gra. Class **30**(2013)214006.

[47] Cai, Y.F. et al.: Phys. Rep. **493**(2010)1.

[48] Setare, M.R. and Saridakis, E.N.: Phys. Mod. Int. J. D **18**(2009)549.

[49] Lazkoz, R. et al.: Phys. Lett. B **649**(2007)103.

[50] Leon, G. et al.: Phys. Eur. J.C. **78**(2018)753.

[51] Dimopoulos, K. and Valle, J.W.F.: Phys. Astropart. **18**(2002)287.

[52] Huey, G. et al.: Phys. Rev. D **59**(1999)063005.

[53] Khurshudyan, M. et al.: Int. J. Theor. Phys. **53**(2014)2370.

[54] Bento, M.C. et al.: Phys. Rev. D **65**(2002)067301.

[55] Moraes, P.H.R.S. and Santos, J.R.L.: Phys. Rev. D **89**(2014)083516.

[56] Roy, N. and Bambda, K.: Astrophys. cos. **6**(2018)9.

[57] Hertzberg, M.P. et al.: arXiv:1812.03184v1 [hep-th].

[58] Diaz, P.F.G.: arXiv: Astro-phys/0004125v1.

[59] Zlatev, I. et al.: Phys. Rev. Lett. **82**(1999)5.

[60] Roy, N. and Banerjee, N.: Genr. Rel. Quan. Cos. **9**(2014)13.

[61] Guth, A.H.: Phys. Rev D **23**(1981)347.

[62] Linde, A.D.: Phys. Lett. B **108**(1982)389.

[63] Yang, W. et al.: arXiv:1810.08586v1[gr-qc].

[64] Hawking, S.W.: Commun. Math. Phys. **43**(1975)199; Bekenstein, J.D.: Phys. Rev. D **7**(1973)2333.

[65] Jacobson, T.: Phys. Lett. B **75**(1995)1260.

[66] Bardeen, J.M., Carter, B. and Hawking, S.W.: Commun. Math. Phys. **31**(1973)161.

[67] Akbar, M. and Cai, R.G.: Phys. Rev. D **75**(2007)084003.

[68] Mazumder, N. and Chakraborty, S.: Astrophys. Space Sci. **332**(2011)509-513.

[69] Debnath et al.: Eur. Phys. J. C. **72**(2012)1875.

[70] Sharif, M. and Fatima, H.I.: Astrophys. Space Sci. **354**(2014)507-515.

[71] Bamba, K. and Geng, C.Q.: J. Cosmol. Astropart. Phys. **11**(2011)008.

[72] Sharif, M. and Zubair, M.: J. Cosmo. Astropart. Phys. **03**(2012)028.

[73] Sharif, M. and Ikram, A.: Eur. Phys. J.C. **168**(2018)03007.

[74] Chattopadhyay, S. and Ghosh, R.: arXiv: 1204.3527v1 [Physics.gen-ph].

[75] Chaichian et al.: Class. Quantum Grav. **27**(2010)185021.

[76] Lymperis, A. and Saridakis, E. N.: Eur. Phys. J.C. **78**(2018)993.

[77] Debnath et al.: Eur. Phys. J.C. **72**(2012)1875.

[78] Bamba, K. et al.: Astrophys. Space Sci. **344**(2013)267.

[79] Sharif, M. and Zubair, M.: Phys. Soc. Jpn. J. **82**(2013)064001.

[80] Sharif, M. and Siddiqa, A.: Ad. High Energy Phys. **2018**(2018)9534279.

[81] Correa, R.A.C. et al.: Eur. Phys. J.C. **78**(2018)877.

[82] Copeland, E.J. et al.: Int. J. Mod. Phys. D **15**(2006)1753-1936.

[83] Thomson, W.: Transactions of the Royal Society of Edinburgh, **16**(1849)541-574.

[84] Sommerfeld, A.: (1951/1955)1.

[85] Saha, M.N. and Srivastava, B.N.: (1935)1.

[86] Clausius, R.: Annalen der Physik. **79**(1850)368-397.

[87] Clausius, R.: Annalen der Physik. xcii **12**(1854)481-506.

[88] Thomson, W.: Transactions of the Royal Society of Edinburgh. (1851)261268,289298.

[89] Cropper, William, H.: The Road to Entropy Rudolf Clausius. Great Physicists: The Life and Times of Leading Physicists from Galileo to Hawking. Oxford University Press. (2004)93105.

[90] Bailyn, M.: American Institute of Physics, New York, ISBN 0-88318-797-3, (1994)342.

[91] Kozliak, Evguenii, Lambert, and L. Frank: Residual Entropy, the Third Law and Latent Heat, Entropy. **10**(2008)27484.

[92] Carlip, S.: International J. of Mod. Phys. D **23**(2014)1430023.

[93] Ferguson, and Kitty: Stephen Hawking: His Life and Work, Transworld, ISBN 978-1-4481-1047-6(2011)65-67.

[94] Larsen, and Kristine: Stephen Hawking: a biography, ISBN 978-0-313-32392-8, (2005)38.

[95] Bekenstein, J.D.: Lett. Nuov. Cim. **4**(1972)737, Bekenstein, J.D.: Phys. Rev. D **7**(1973)2333, Bekenstein, J.D.: Phys. Rev. D **9**(1974)3292.

[96] Bekenstein, J.D.: Scholarpedia, **10**(2008)7375.

[97] Moradpur, H. et al.: Eur. Phys. J.C. **78**(2018)829.

[98] Zubair, M. et al.: Eur. Phys. J. C. **77**(2017)472.

[99] Jahromi, S.A. et al.: Phys. Lett. B **780**(2018)24.

[100] Nojiri, S.: Int. J. Geom. Meth. Mod. Phys. **4**(2007)115-145.

[101] Padmanabhan, T.: AIP Conf. Proc. **843**(2006)111-166.

[102] Edmund, J. et al.: Int. J. Mod. Phys. D **15**(2006)1753-1936.

[103] Nojiri, S. and Odintsov, S.D.: Phys. Rept. **505**(2011)59.

[104] Copeland, E.J. et al.: Int. J. Mod. Phys. D **15**(2006)1753.

[105] Cai, Y.F. et al.: Rept. Prog. Phys. **10**(2016)106901.

[106] Gross, D.J. and Sloan, J.H.: Nucl. Phys. B **291**(1987)41.

[107] Stelle, K.S.: Phys. Rev. D **16**(1977)953.

[108] Biswas, T. et al.: Phys. Rev. Lett. **108**(2012)031101.

[109] Mannheim, P.D. and Kazanas, D.: Astrophys. J. **342**(1989)635.

[110] Flanagan, E.E.: Phys. Rev. D **74**(2006)023002.

[111] Starobinsky, A.A.: Phys. Lett. B **91**(1980)99.

[112] De Felice, A. and Tsujikawa, S.: Living Rev. Rel. **13**(2010)3.

[113] Capozziello, S.: Int. J. Mod. Phys. D **11**(2002)483.

[114] Nojiri, S. and Odintsov, S.D.: Phys. Lett. B **631**(2005)1.

[115] De Felice, A. and Tsujikawa, S.: Phys. Lett. B **675**(2009)1.

[116] Deruelle, N. and Farina-Busto, L.: Phys. Rev. D **41**(1990)3696.

[117] Bazeia, D., Gomes, C.B., Losano, L. and Menezes, R.: Phys. Lett.B **633**(2006)415.

[118] Alves, M.E.S., Moraes, P.H.R.S., De Araujo, J.C.N. and Malheiro, M.: Phys. Rev. D **94**(2016)024032.

[119] Bazeia, D., Gonzalez Leon, M.A., Losano, L., Guilarte, J.M. and Santos, J.R.L: Phys. Scr. **87**,(2013) 045101.

[120] Bazeia, D.,Losano, L., Malbouisson, J.M.C. and Santos, J.R.L.: Eur. Phys. J.C. **71**(2011)1767.

[121] sheykhi, A.: Eur.Phys.J.C. **69**(2010)269.

[122] Hawking, S.W.: Commun. Math. Phys. **43**(1975)199

[123] Cai, R.G. and Kim, S.P.: JHEP **02**(2005)050.

[124] Moraes, P.H.R.S. and Santos, J.R.L.: Phys. Rev. D **89**(2014)083516.

[125] Vachaspati, T., Kinks and Domain walls: (Cambridge University Press, Cambridge, 2006).

[126] Rajaraman, R.: Phys. Astro. **15**(1987)418.

[127] Bazeia, D. et al.: Phys. J. A**30**(1997)8157.

[128] De Brito, G.P., and Dutra, A.S.: Phys. Lett. B **736**(2014)438.

[129] Cai, R.G. and Kim, S.P.: JHEP **0502**(2005)050.