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ELECTRODYNAMIC PROCESSES WITH NUCLEAR TARGETS*

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Abstract

It is known that two general form factors depending on energy loss and momentum transfer characterize inelastic electron scattering from nuclei in first Born approximation in $\alpha = 1/137$. The same two form factors appear in all electrodynamic processes connected by one photon exchange with nuclei. This observation is used to compute cross sections and to discuss experiments which are aimed at probing electrodynamics by scattering a pair producing electrons or muons from nuclear targets.

I. INTRODUCTION

It is well recognized[1] that the nuclear part of elastic electron-nucleon scattering in first Born approximation in $\alpha = 1/137$ can be summarized in terms of two scalar form factors, F_1 and F_2 depending on the invariant momentum transfer q^2. This is a consequence of the vector nature of the exchanged virtual photon in Fig. 1 and of electromagnetic current conservation. The nuclear part of this interaction can be isolated from whatever goes on at the other end of the photon line and therefore the same conclusion with the same form factors F_1 and F_2 can be drawn no matter what electrodynamic interactions occur at the lepton end of the photon line, be it scattering, bremsstrahlung, or pair production for electrons or muons as illustrated in Fig. 2. It was on the basis of this observation that various high energy tests of quantum electrodynamics were proposed.[2]

More recently it has been realized and emphasized[3] that there is an analogous general form for the single virtual photon exchange between inelastically scattered electrons and a nucleon or nucleus, leading to any final state as in Fig. 3. The two inelastic form factors F_1 and F_2 are now functions of two variables which may be taken as the invariant momentum transfer q^2 and the energy transfer

$$q \cdot P = \tfrac{1}{2}(q^2 + \mathcal{M}_f^2 - \mathcal{M}_T^2)$$

where $\mathcal{M}_f$ is the invariant mass of the final nuclear "anything" emerging from the lower vertex in Fig. 3.

The analogous extension to inelastic nuclear processes of the relation between the cross sections for Figs. 1 and 2 is the main point of the present note.[4] In reporting it here and presenting cross section calculations we wish to further emphasize its very great utility in planning and analyzing experiments at high energy electron accelerators. For experimental studies of electrodynamic behaviour of photons, electrons, and muons of high energies it frees one from both the limitations to proton targets and the difficult requirements of very high energy resolutions to assure that only elastic processes occur on the target. The same two general inelastic form factors appear in the pair production or bremsstrahlung events in Fig. 4 as in inelastic electron scattering (Fig. 3) for the same target. Therefore, for example, between measurements of inelastic electron scattering and of large angle lepton pair formation from the same targets with arbitrary nuclear excitation and pion formation, the nuclear unknowns $\mathcal{W}_1(q^2, q \cdot P)$ and $\mathcal{W}_2(q^2, q \cdot P)$ can be removed. All regions of the two dimensional plane in the phase space of the variables q^2 and $q \cdot P$ that are accessible in the pair production or bremsstrahlung experiments can be covered by inelastic scattering studies of Fig. 3. Moreover, if there are corrections to these assertions which we expect to be valid to order $\sim z\alpha = z/137$ where z is the nuclear charge, they can be detected by measuring any deviations from the Rosenbluth straight lines in the scattering analysis and by an analogous test given in the following for the Bethe-Heitler events of Fig. 4. It is expected that the present results will permit experimental tests of quantum electrodynamics to probe to regions of smaller distances because experiments can be performed

a) with targets of low z that are "easier" to work with than hydrogen, and

b) with more comfortable energy resolutions; comparison of different experiments permits $\mathcal{W}_1$, and $\mathcal{W}_2$ to be removed even though inelastic nuclear states are excited or pions are produced.

II. CALCULATION

A. Nuclear Form Factors

We adopt the following notations[†]: P denotes the initial four momentum of the target, $P^2 = -\mathcal{M}_T^2$. q is the four momentum of the virtual photon and $P' = P - q$ is the final four momentum of the target. We shall assume that the target is initially unoriented and that experimentally <u>all final nuclear states</u> consistent with the given kinematic conditions (give q and P) are summed over. In this case, the contribution of the nuclear part of the process indicated in Fig. 3 is given by

$$\mathcal{W}_{\mu\nu} = \frac{(2\pi)^3\Omega}{(ze)^2} \overline{\sum_{\substack{\text{initial}\\ \text{states}}}} \sum_{\substack{\text{final}\\ \text{states}}} \delta^{(4)}(P - P' - q) \langle P|J_\nu(0)|P'\rangle\langle P'|J_\mu(0)|P\rangle (E) \qquad (1)$$

where Ω is the normalization volume, E is the initial energy of the target, $\overline{\Sigma}$ indicates an average over the initial target states (i.e., M_J of the target), $|P\rangle$ and $|P'\rangle$ are the Heisenberg state vectors of the initial and final nuclear states (that is they are eigenstates of the nuclear four momentum operator $|P_\mu)$ and $J_\mu(0)$ is the electromagnetic current operator of the nucleus at the space time point $x_\mu = 0$. The four-dimensional delta function summarizes the translational invariance of the theory. Lorentz invariance tells us that $\mathcal{W}_{\mu\nu}$ must be a second rank tensor since the current operator is a 4-vector. Because of the sum over initial and final states, there are only two four vectors on which this tensor can depend, P and q. Since $P^2 = -\mathcal{M}_T^2$, there are

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[†]We use a metric such that $a_\mu = (\underline{a}, ia_o)$ and $a \cdot b = \underline{a} \cdot \underline{b} - a_o b_o$. In this metric $q^2 > 0$ for both scattering and pair production.

only two independent scalars which can be formed from these four vectors, q^2 and $q \cdot P$. Thus the most general form of the tensor $\mathcal{W}_{\mu\nu}$ is

$$\mathcal{W}_{\mu\nu} = A(q^2, q \cdot P)\delta_{\mu\nu} + B(q^2, q \cdot P)q_\mu q_\nu + C(q^2, q \cdot P)P_\mu P_\nu + D(q^2, q \cdot P)(q_\mu P_\nu + q_\nu P_\mu) + E(q^2, q \cdot P)(q_\mu P_\nu - q_\nu P_\mu) \; . \tag{2}$$

No term in $\varepsilon_{\mu\nu\rho\sigma} P_\rho q_\sigma$ can appear since the current operator is a polar vector under spatial reflections. We know further that the nuclear current operator must satisfy the continuity equation

$$\frac{\partial}{\partial x_\mu} J_\mu(x) = 0 \tag{3}$$

which implies that $q_\mu \mathcal{W}_{\mu\nu} = \mathcal{W}_{\mu\nu} q_\nu = 0$. These relations are sufficient to eliminate three of the invariant functions and one can thus write a symmetric tensor, with $\mathcal{W}_1$ and $\mathcal{W}_2$ both ≥ 0 according to the definition in Eq. (1)

$$\mathcal{W}_{\mu\nu} = \mathcal{W}_1(q^2, q \cdot P)\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \mathcal{W}_2(q^2, q \cdot P)\frac{1}{\mathcal{M}_T^2}\left(P_\mu - \frac{P \cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P \cdot q}{q^2} q_\nu\right). \tag{4}$$

This theorem is due to von Gehlen, Gourdin, and Bjorken.[3]

Some special cases of the above result will be interesting to us later and we include them for completeness. In the case that we have only elastic scattering from the nucleus, $P'^2 = -\mathcal{M}_T^2$ and $2q \cdot P = q^2$, and a spin zero target we have with the aid of Eq. (3)

$$\sqrt{\Omega^2 EE'} \left\langle P' = P - q \left| \frac{J_\mu(0)}{ze} \right| P \right\rangle = F(q^2) \left(P_\mu - \frac{P \cdot q}{q^2} q_\mu \right) \tag{5}$$

with $F(0) = 1$. By substituting in the equation for $\mathcal{W}_{\mu\nu}$ one finds

$$\mathcal{W}_1 = 0$$

$$\mathcal{W}_2 = |F(q^2)|^2 \frac{\mathcal{M}_T^2}{E'} \delta(E - E' - q_o) \tag{6}$$

In the case of a spin 1/2 target, and elastic scattering one finds

$$\sqrt{\Omega^2 EE'} \left\langle P' = P - q, \lambda' \left| \frac{J_\mu(0)}{ze} \right| P, \lambda \right\rangle = i\mathcal{M}_T \bar{u}_{\lambda'}(\vec{P}') \left[F_1(q^2)\gamma_\mu + F_2(q^2)\sigma_{\mu\nu} q_\nu \right] u_\lambda(\vec{P}) \tag{7}$$

in the usual Dirac notation with $F_1(0) = 1$, $F_2(0) = \frac{\lambda}{2\mathcal{M}_T}$ the anomalous magnetic moment. Again inserting in the equation for $\mathcal{W}_{\mu\nu}$ and carrying out the required sums one finds

$$\begin{aligned} \mathcal{W}_1 &= q^2 \left| \tfrac{1}{2} F_1(q^2) + \mathcal{M}_T F_2(q^2) \right|^2 \frac{1}{E'} \delta(E - E' - q_o) \\ \mathcal{W}_2 &= \mathcal{M}_T^2 \left[|F_1(q^2)|^2 + q^2 |F_2(q^2)|^2 \right] \frac{1}{E'} \delta(E - E' - q_o) \end{aligned} \tag{8}$$

Calculations of the contributions to $\mathcal{W}_1$ and $\mathcal{W}_2$ coming from nuclear excitation to discrete levels, quasi-elastic scattering from nucleons in the nucleus, and the production of a pion can be found in references 5.

B. Electron (Muon) Scattering

In terms of $\mathcal{W}_1$ and $\mathcal{W}_2$ the electron or muon scattering cross section for fixed electron energy and angle but summing over all else can be computed in standard fashion:

$$d\sigma = 2z^2\alpha^2 \frac{d\vec{p}'}{2\mathcal{E}'} \frac{1}{q^4} \mathcal{W}_{\mu\nu} \mathcal{N}_{\mu\nu} \frac{1}{\left[(p \cdot P)^2 - m^2 \mathcal{M}_T^2\right]^{\frac{1}{2}}} \tag{9}$$

where

$$\begin{aligned} \mathcal{N}_{\mu\nu} &= -\tfrac{1}{2}\,\mathrm{Tr}\gamma_\mu(m - i\not{p})\gamma_\nu(m - i\not{p}') \\ &= 2\left[p_\mu p'_\nu + p_\nu p'_\mu - \delta_{\mu\nu}(p \cdot p' + m^2)\right] . \end{aligned} \tag{10}$$

Combining gives

$$\begin{aligned} d\sigma = 2z^2\alpha^2 \frac{d\vec{p}'}{2\mathcal{E}'} \frac{1}{q^4} \frac{1}{\left[(p \cdot P)^2 - m^2 \mathcal{M}_T^2\right]^{\frac{1}{2}}} &\left\{ 2(q^2 - 2m^2)\mathcal{W}_1 \right. \\ &\left. + 2\left[\frac{2(p \cdot P)[(p + q) \cdot P]}{\mathcal{M}_T^2} - \tfrac{1}{2}q^2\right] \mathcal{W}_2 \right\} \end{aligned} \tag{11}$$

The three independent scalar functions in electron scattering can be taken as $\mathcal{E}$, $\mathcal{E}'$, θ in the laboratory system or as the three scalar variables q^2, $q \cdot P$, and $p \cdot P$. Measurements at fixed q^2 and $q \cdot P$ can separate $\mathcal{W}_1$ and $\mathcal{W}_2$ and check the one photon exchange form.

The cross section can be written in the laboratory frame as

$$\frac{d\sigma}{d\Omega' d|\vec{p}'|} = \frac{2z^2\alpha^2}{q^4} \left(\frac{\vec{p}'^2}{\mathcal{E}'|\vec{p}|}\right) \frac{1}{\mathcal{M}_T} \left[2(\mathcal{E}\mathcal{E}' - |\vec{p}||\vec{p}'|\cos\theta - 2m^2)\mathcal{W}_1 + (\mathcal{E}\mathcal{E}' + |\vec{p}||\vec{p}'|\cos\theta + m^2)\mathcal{W}_2\right]$$

where

$$\begin{aligned} \frac{q \cdot P}{\mathcal{M}_T} &= \mathcal{E} - \mathcal{E}' \\ q^2 &= 2\mathcal{E}\mathcal{E}' - 2|\vec{p}||\vec{p}'|\cos\theta - 2m^2 . \end{aligned} \tag{12}$$

These formula simplify if one can neglect the mass of the electron

$$\begin{aligned} \frac{d^2\sigma}{d\Omega' d\mathcal{E}'} &= \frac{4z^2\alpha^2}{q^4} \frac{\mathcal{E}'^2}{\mathcal{M}_T} \cos^2\frac{\theta}{2} \left[\mathcal{W}_2(q^2, q \cdot P) + 2\mathcal{W}_1(q^2, q \cdot P)\tan^2\frac{\theta}{2}\right] \\ q^2 &= 4\mathcal{E}\mathcal{E}' \sin^2\frac{\theta}{2} . \end{aligned} \tag{13}$$

We next, for completeness and convenience, give cross sections in some special cases, using Eqs. (6) and (8):

1.a) Elastic scattering, spin 0 target, laboratory system

$$\frac{d\sigma}{d\Omega'} = \left(\frac{|\vec{p}\,'|}{|\vec{p}|}\right)\frac{2z^2\alpha^2}{q^4}\left|F(q^2)\right|^2 \left(\mathcal{E}\mathcal{E}'+|\vec{p}||\vec{p}\,'| \cos\theta + m^2\right) \frac{1}{\frac{E'}{\mathcal{M}_T}+\frac{\mathcal{E}'}{\mathcal{M}_T}\left(1-\frac{|\vec{p}|}{|\vec{p}\,'|}\cos\theta\right)} \tag{14}$$

1.b) Relativistic limit (m = 0)

$$\frac{d\sigma}{d\Omega} = \frac{z^2\alpha^2\cos^2\frac{\theta}{2}}{4\mathcal{E}^2\sin^4\frac{\theta}{2}}\left|F(q^2)\right|^2 \frac{1}{\left(1+\frac{2\mathcal{E}}{\mathcal{M}_T}\sin^2\frac{\theta}{2}\right)} \tag{15}$$

2.a) Elastic scattering, spin 1/2 target, laboratory system

$$\frac{d\sigma}{d\Omega'} = \left(\frac{|\vec{p}\,'|}{|\vec{p}|}\right)\frac{2z^2\alpha^2}{q^4}\left\{\frac{q^2}{2\mathcal{M}_T^2}\left|F_1(q^2)+2\mathcal{M}_T F_2(q^2)\right|^2\left(\mathcal{E}\mathcal{E}'-|\vec{p}||\vec{p}\,'|\cos\theta-2m^2\right)\right.$$

$$\left.+\left(\left|F_1(q^2)\right|^2+q^2\left|F_2(q^2)\right|^2\right)\left(\mathcal{E}\mathcal{E}'+|\vec{p}||\vec{p}\,'|\cos\theta+m^2\right)\right\}\frac{1}{\frac{E'}{\mathcal{M}_T}+\frac{\mathcal{E}'}{\mathcal{M}_T}\left(1-\frac{|\vec{p}|}{|\vec{p}\,'|}\cos\theta\right)} \tag{16}$$

b) Relativistic limit (m = 0) — Rosenbluth Cross Section

$$\frac{d\sigma_R}{d\Omega'} = \frac{z^2\alpha^2}{4\mathcal{E}^2\sin^4\frac{\theta}{2}}\cos^2\frac{\theta}{2}\left\{\left(|F_1|^2+q^2|F_2|^2\right)+\frac{q^2}{2\mathcal{M}_T^2}\left|F_1+2\mathcal{M}_T F_2\right|^2\tan^2\frac{\theta}{2}\right\}\frac{1}{\left(1+\frac{2\mathcal{E}}{\mathcal{M}_T}\sin^2\frac{\theta}{2}\right)}$$

It is also interesting to see what one would get if only the Coulomb interaction is kept in the laboratory system.

3.a) Coulomb scattering in the laboratory system

$$\frac{d^2\sigma_c}{d|\vec{p}\,'|d\Omega'} = \frac{2z^2\alpha^2}{q^4}\left(\mathcal{E}\mathcal{E}'+|\vec{p}||\vec{p}\,'|\cos\theta+m^2\right)\frac{1}{\mathcal{M}_T}\left[\mathcal{W}_2-\frac{q^2}{\vec{q}^{\,2}}\mathcal{W}_1\right]\left(\frac{\vec{p}\,'^2}{\mathcal{E}'|\vec{p}|}\right) \tag{18}$$

3.b) Relativistic limit (m = 0)

$$\frac{d^2\sigma_c}{d|\vec{p}'|d\Omega'} = \frac{4z^2\alpha^2}{q^4}\cos^2\frac{\theta}{2}\frac{\mathcal{E}'^2}{\mathcal{M}_T}\left[\mathcal{W}_2 - \frac{q^2}{\vec{q}^2}\mathcal{W}_1\right] \tag{19}$$

c) Relativistic limit (m = 0) and $(q_0^2/\vec{q}^2) \ll 1$

$$\frac{d^2\sigma_c}{d|\vec{p}'|d\Omega'} = \frac{4z^2\alpha^2}{q^4}\cos^2\frac{\theta}{2}\frac{\mathcal{E}'^2}{\mathcal{M}_T}\left[\mathcal{W}_2 - \mathcal{W}_1\right] \tag{20}$$

C. Electron (Muon) Pair Production

We now turn to the central problem of calculating electron or muon pair production in terms of the two general nuclear form factors $\mathcal{W}_1$ and $\mathcal{W}_2$. First let us say a few words about some general properties of the process as pictured in Fig. 5. p_- and p_+ are the outgoing four-momenta of the lepton pair and k is the incident photon four momentum. For the top part of the process, the production of a pair by a real and a virtual photon, there are three independent scalars. These can be taken as $\ell^2 = -(p_- - k)^2$, $\underline{\ell}^2 = -(p_+ - k)^2$, and q^2. In computing the cross section for this process, the upper part of this diagram will again enter as a tensor, $M_{\mu\nu}$, just as does the nuclear part. There are three independent four vectors for constructing this tensor, which we take to be k_μ, q_μ and $\Delta_\mu \equiv (p_- - p_+)_\mu$. Only the symmetric part of the tensor will contribute since $\mathcal{W}_{\mu\nu}$ is symmetric. Also, any term which goes as q_μ will give zero since $q_\mu\mathcal{W}_{\mu\nu} = \mathcal{W}_{\mu\nu}q_\nu = 0$. Thus the most general form of $M_{\mu\nu}$ is (assuming parity conservation)

$$M_{\mu\nu} = M_1(\ell^2, \underline{\ell}^2, q^2)\delta_{\mu\nu} + M_2(\ell^2,\underline{\ell}^2,q^2)k_\mu k_\nu + M_3(\ell^2,\underline{\ell}^2,q^2)\Delta_\mu\Delta_\nu + M_4(\ell^2,\underline{\ell}^2,q^2)(\Delta_\mu k_\nu + \Delta_\nu k_\mu) \tag{21}$$

For the over-all process of pair production there are six independent scalars which may be thought of as $|\vec{k}|, \mathcal{E}_+, \mathcal{E}_-, \cos\theta_+, \cos\theta_-$, and φ in the lab where θ_+ and θ_- are the angles $\vec{p}_+$ and $\vec{p}_-$ make with $\vec{k}$ and φ is the angle between the $(\vec{p}_+\vec{k})$, $(\vec{p}_-\vec{k})$ planes. (See Fig. 6) Alternatively, one can work with $q^2, \ell^2, \underline{\ell}^2, k \cdot P$, $q \cdot P$, and $\Delta \cdot P$. For simplicity, we will rename them

$$\begin{aligned}
x_1 &= \ell^2 - m^2 \\
x_2 &= \underline{\ell}^2 - m^2 \\
x_3 &= k \cdot P/\mathcal{M}_T \\
x_4 &= \Delta \cdot P/\mathcal{M}_T \\
x_5 &= q \cdot P/\mathcal{M}_T \\
x_6 &= q^2
\end{aligned} \tag{22}$$

The other variables in the problem can be expressed in terms of these by the relations:

$$\begin{aligned}
2p_- \cdot k &= \ell^2 - m^2 = x_1 \\
2p_+ \cdot k &= \underline{\ell}^2 - m^2 = x_2 \\
2p_+ \cdot p_- &= q^2 + \ell^2 + \underline{\ell}^2 = x_1 + x_2 + x_6 + 2m^2 \\
\Delta \cdot k &= \tfrac{1}{2}(\ell^2 - \underline{\ell}^2) = \tfrac{1}{2}(x_1 - x_2) \\
\Delta \cdot q &= \tfrac{1}{2}(\underline{\ell}^2 - \ell^2) = \tfrac{1}{2}(x_2 - x_1) \\
\Delta^2 &= -(2m^2 + q^2 + \ell^2 + \underline{\ell}^2) = -(x_1 + x_2 + x_6 + 4m^2) \\
\mathcal{M}_f^2 &= \mathcal{M}_T^2 - x_6 + 2\mathcal{M}_T\, x_5
\end{aligned}$$

The prediction of quantum electrodynamics for the cross section for pair production can again be evaluated by standard techniques. There are two Feynman diagrams to be considered. One finds[†]

$$d\sigma = \frac{z^2\alpha^3}{\pi^2} \frac{d\vec{p}_+}{2\mathcal{E}_+} \frac{d\vec{p}_-}{2\mathcal{E}_-} \frac{1}{q^4} \mathcal{W}_{\mu\nu} M_{\mu\nu} \frac{1}{[(k \cdot P)^2]^{\frac{1}{2}}} \tag{23}$$

where

$$M_{\mu\nu} = -\frac{1}{2} \sum_{\text{pol.}} \left\{ \mathrm{Tr} \left[\not{e} \frac{1}{i(\not{p}_- - \not{k}) + m} \gamma_\mu + \gamma_\mu \frac{1}{i(\not{k} - \not{p}_+) + m} \not{e} \right] \left[m + i\not{p}_+ \right] \left[\not{e} \frac{1}{i(\not{k} - \not{p}_+) + m} \gamma_\nu + \gamma_\nu \frac{1}{i(\not{p}_- - \not{k}) + m} \not{e} \right] \left[m - i\not{p}_- \right] \right\} \tag{24}$$

e is the polarization of the incident photon. [Note $M_{\mu\nu} = M_{\nu\mu}$ is already symmetric.] The result for $M_{\mu\nu}$ can be written in the general form discussed above with

$$\begin{aligned}
M_1 &= \frac{4}{x_1 x_2} \left\{ x_6^2 + x_6(x_1 + x_2) \left[\frac{m^2}{x_1 x_2} (x_1 + x_2) + 1 \right] + \frac{1}{2}(x_1^2 + x_2^2) \right\} \\
M_2 &= \frac{4}{x_1 x_2} \left\{ x_6 + \frac{m^2}{x_1 x_2} (x_1 - x_2)^2 \right\} \\
M_3 &= \frac{4}{x_1 x_2} \left\{ x_6 + \frac{m^2}{x_1 x_2} (x_1 + x_2)^2 \right\} \\
M_4 &= \frac{4}{x_1 x_2} \left\{ \frac{m^2}{x_1 x_2} (x_1^2 - x_2^2) \right\}
\end{aligned} \tag{25}$$

[†] One can immediately obtain the cross section for electron bremsstrahlung from Eqs. (25,26) by use of the substitution rule,

$$d\sigma_{\text{brem.}} = \frac{z^2\alpha^3}{\pi^2} \frac{d\vec{k}}{2\omega_k} \frac{d\vec{p}\,'}{2\mathcal{E}'} \left\{ \frac{1}{\left[(p \cdot P)^2 - m^2\eta_T^2 \right]^{\frac{1}{2}}} \right\} \left[-\frac{1}{q^4} \mathcal{W}_{\mu\nu} M_{\mu\nu} \right]_{\substack{p_- \to p' \\ p_+ \to -p \\ k \to -k}}$$

where p' and p are the final and initial electron four-momentum and k becomes the outgoing photon four momentum.

and Eqs. (21) and (4) can be contracted together to form $\mathcal{W}_{\mu\nu}M_{\mu\nu}$ in Eq. (23). More directly we form the product of Eq. (4) with Eq. (24), observing that the terms in $\mathcal{W}_{\mu\nu}$ proportional to q_μ or q_ν vanish by current conservation, and obtain[†]

$$\mathcal{W}_{\mu\nu}\,\mathcal{M}_{\mu\nu} = \frac{4}{x_1x_2}\Bigg[\Bigg\{2(x_6 - 2m^2)(x_6 + x_1 + x_2 + 2m^2) + (x_1^2 + x_2^2)\left(1 + \frac{2m^2}{x_1x_2}\left[x_6 - 2m^2\right]\right)\Bigg\}\mathcal{W}_1(x_5x_6)$$
$$+\Bigg\{\frac{m^2}{x_1x_2}\left[(x_1^2 + x_2^2)(-x_6 - x_5^2 + x_3^2 + x_4^2) + (x_1^2 - x_2^2)(2x_3x_4)\right] \tag{26}$$
$$-(x_6 + x_5^2)(x_1 + x_2 + x_6 + 2m^2) - \tfrac{1}{2}(x_1^2 + x_2^2) - x_3x_5(x_1 + x_2)$$
$$+ x_4x_5(x_1 - x_2) + x_3^2(x_6 - 2m^2) + x_4^2(x_6 + 2m^2)\Bigg\}\mathcal{W}_2(x_5x_6)\Bigg]$$

The dependence on x_3 and x_4 comes from joining the electrodynamic and nuclear parts of the process together and can be used to experimentally establish the validity of the one photon exchange mechanism just as in electron scattering. It is most concisely exhibited by re-expressing Eq. (26) in terms of the $M_1 \ldots M_4$ constructed in Eq. (25) and which depend only on the energy, momentum transfer, and virtual photon mass in the Compton scattering at the upper vertex through the variables x_1, x_2, and x_6:

$$\mathcal{W}_{\mu\nu}M_{\mu\nu} = \mathcal{W}_1\Bigg\{3M_1 - \frac{(x_1 + x_2)^2}{4x_6}M_2 - \left[x_1 + x_2 + x_6 + 4m^2 + \frac{(x_1 - x_2)^2}{4x_6}\right]M_3$$
$$+ (x_1 - x_2)\left[1 + \frac{x_1 + x_2}{2x_6}\right]^2 M_4\Bigg\}$$
$$+\mathcal{W}_2\Bigg\{-\left(1 + \frac{x_5^2}{x_6}\right)M_1 + \left(x_3 - \frac{x_5(x_1 + x_2)}{2x_6}\right)^2 M_2 + \left(x_4 + \frac{x_5(x_1 - x_2)}{2x_6}\right)^2 M_3$$
$$+ 2\left(x_3 - \frac{x_5(x_1 + x_2)}{2x_6}\right)\left(x_4 + \frac{x_5(x_1 - x_2)}{2x_6}\right)M_4\Bigg\}.$$

[†] Although some of the individual factors are dimensional, their product $\mathcal{W}_{\mu\nu}M_{\mu\nu}$ is dimensionless.

The invariant scalar products can be evaluated in the laboratory system to give

$$
\begin{aligned}
x_1 &= -2\mathcal{E}_- |\vec{k}| \left(1 - \frac{|\vec{p}_-|}{\mathcal{E}_-} \cos\theta_-\right) \\
x_2 &= -2\mathcal{E}_+ |\vec{k}| \left(1 - \frac{|\vec{p}_+|}{\mathcal{E}_+} \cos\theta_+\right) \\
x_3 &= -|\vec{k}| \\
x_4 &= (\mathcal{E}_+ - \mathcal{E}_-) \\
x_5 &= (|\vec{k}| - \mathcal{E}_+ - \mathcal{E}_-) \\
x_6 &= -2m^2 - x_1 - x_2 - 2\mathcal{E}_+\mathcal{E}_- \left[1 - \frac{|\vec{p}_+||\vec{p}_-|}{\mathcal{E}_+\mathcal{E}_-} (\cos\theta_+ \cos\theta_- + \sin\theta_+ \sin\theta_- \cos\varphi)\right].
\end{aligned}
\tag{27}
$$

These variables are illustrated in Fig. 6.

Of special interest is the "symmetric case" where $\mathcal{E}_+ = \mathcal{E}_-$ and $\theta_+ = \theta_-$ since interference terms with two photon exchange corrections identically vanish for this condition.[2]

In terms of the general scalars this situation is characterized by

$$
\left.\begin{aligned}
\ell^2 &= \underline{\ell}^2 \\
\Delta \cdot P &= 0
\end{aligned}\right\} \quad \text{symmetric case} \tag{28}
$$

or

$$
\left.\begin{aligned}
x_1 &= x_2 \\
x_4 &= 0
\end{aligned}\right\} \quad \text{symmetric case}
$$

The formula simplify considerably in this case and one has

$$M_1 = \frac{4}{x_1^2}\left[x_1^2 + 2x_6x_1 + x_6(x_6 + 4m^2)\right]$$

$$M_2 = \frac{4x_6}{x_1^2} \tag{29}$$

$$M_3 = \frac{4}{x_1^2}(x_6 + 4m^2)$$

$$M_4 = 0$$

$$\mathcal{W}_{\mu\nu}\mathcal{M}_{\mu\nu} = \frac{4}{x_1^2}\left[\left\{2(x_6 - 2m^2)(x_6 + 2x_1 + 4m^2) + 2x_1^2\right\}\mathcal{W}_1(x_5x_6) + \left\{-(x_6 + x_5^2)(2x_1 + x_6 + 4m^2) - x_1^2 - 2x_1x_3x_5 + x_6x_3^2\right\}\mathcal{W}_2(x_5x_6)\right] \tag{30}$$

For fixed q^2 and $q \cdot P$ (x_5 and x_6) so that the nuclear physics doesn't change, the entire dependence on incident energy $k \cdot P$ (x_3) is contained in the term $\propto \left(x_3 - \frac{x_1x_5}{x_6}\right)^2 \mathcal{W}_2$. This dependence can be used to test the validity of the one photon exchange mechanism and to separate the form factors $\mathcal{W}_1$ and $\mathcal{W}_2$. In order to test electrodynamics one again programs experiments at fixed x_5 and x_6 but with variable $x_1 = \ell^2 - m^2$, which is just the mass of either virtual intermediate lepton line in the symmetric case. The kinematic variables in the lab in the symmetric case are:

$$\begin{aligned} x_1 &= \ell^2 - m^2 = -2|\vec{k}|\mathcal{E}\left(1 - \frac{|\vec{p}|}{\mathcal{E}}\cos\theta\right) \\ x_3 &= P \cdot k/\mathcal{M}_T = -|\vec{k}| \\ x_5 &= P \cdot q/\mathcal{M}_T = |\vec{k}| - 2\mathcal{E} \\ x_6 &= q^2 = -4\vec{p}^2\sin^2\theta\sin^2\frac{\varphi}{2} + 4|\vec{k}|\mathcal{E}\left(1 - \frac{|\vec{p}|}{\mathcal{E}}\cos\theta\right) - 4m^2 \end{aligned} \tag{31}$$

We again discuss some special cases of the symmetric case:

1. Relativistic limit ($m = 0$), lab system

$$x_1 = - 4|\vec{k}|\mathcal{E} \sin^2 \frac{\theta}{2}$$
$$x_3 = - |\vec{k}|$$
$$x_5 = |\vec{k}| - 2\mathcal{E}$$
$$x_6 = 8\mathcal{E} \sin^2 \frac{\theta}{2} \left[|\vec{k}| - 2\mathcal{E} \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2}\right] = q^2$$

$$d\sigma = d\mathcal{E}_+ d\mathcal{E}_- d\Omega_+ d\Omega_- \frac{z^2\alpha^3}{\pi^2} \frac{\mathcal{E}^2}{q^4} \frac{1}{|\vec{k}|^3 \mathcal{M}_T} \left\{ \left[\vec{k}^2 - q^2 \operatorname{ctn}^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2}\right] \left[2\mathcal{W}_1 - \left(1 + \frac{(P \cdot q)^2}{q^2 \mathcal{M}_T^2}\right) \mathcal{W}_2\right] + q^2 \vec{k}^2 \left[\frac{(P \cdot q)}{\mathcal{M}_T q^2} - \frac{1}{4\mathcal{E} \sin^2 \frac{\theta}{2}}\right]^2 \mathcal{W}_2 \right\} \tag{32}$$

2.a) Elastic scattering from Fixed Spin Zero Target — Bethe-Heitler Cross Section

In this case, from our previous discussion

$$\mathcal{W}_1 = 0$$
$$\mathcal{W}_2 = |F(\vec{q}^2)|^2 \mathcal{M}_T \delta(|\vec{k}| - \mathcal{E}_+ - \mathcal{E}_-)$$

and we have for a fixed target $|\vec{k}| = 2\mathcal{E}$ and $x_5 = P \cdot q = 0$:

$$d\sigma_{B.H.} = \left\{ \delta(\mathcal{E}_+ + \mathcal{E}_- - |\vec{k}|) d\mathcal{E}_+ d\mathcal{E}_- d\Omega_+ d\Omega_- \frac{z^2\alpha^3}{4\pi^2} \frac{\vec{p}^2}{\vec{q}^4} \frac{|F(\vec{q}^2)|^2}{|\vec{k}|} \cdot \frac{4}{x_1^2} \left[- x_1^2 - 2x_6 x_1 - x_6(x_6 + 4m^2) + x_6 x_3^2\right] \right\} \tag{33}$$

$$d\sigma_{B.H.} = \left\{ \delta(\mathcal{E}_+ + \mathcal{E}_- - |\vec{k}|) d\mathcal{E}_+ d\mathcal{E}_- d\Omega_+ d\Omega_- \frac{z^2\alpha^3}{4\pi^2} \frac{\vec{p}^2}{\vec{q}^4} \frac{|F(q^2)|^2}{|\vec{k}|^3 \left(1 - \frac{|\vec{p}|}{\mathcal{E}} \cos \theta\right)^2} \cdot \left[2 \frac{\vec{p}^2}{\mathcal{E}^2} \sin^2\theta\right] \left[\vec{q}^2(1 - \cos \varphi) + \vec{k}^2(1 + \cos \varphi)\right] \right\} \tag{34}$$

b) Relativistic limit ($m = 0$)

$$d\sigma_{B.H.} = \left\{ \delta(\mathcal{E}_+ + \mathcal{E}_- - |\vec{k}|) d\mathcal{E}_+ d\mathcal{E}_- d\Omega_+ d\Omega_- \frac{z^2\alpha^3}{64\pi^2 |\vec{k}|^3} \frac{|F(\vec{q}^2)|^2 \cos^2 \frac{\theta}{2}}{\sin^6 \frac{\theta}{2}} \cdot \frac{\left[\cos^2 \frac{\varphi}{2} + 4 \sin^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2} \left(1 - \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2}\right)\right]}{\left[1 - \cos^2 \frac{\theta}{2} \sin^2 \frac{\varphi}{2}\right]^2} \right\} \tag{35}$$

c) Relativistic limit $(m = 0)$ $\varphi = \pi$

$$d\sigma_{B.H.} = \delta(\mathcal{E}_+ + \mathcal{E}_- - |\vec{k}|) d\mathcal{E}_+ d\mathcal{E}_- d\Omega_+ d\Omega_- \frac{Z^2\alpha^3}{16\pi^2 |\vec{k}|^3} |F(\vec{q}^2)|^2 \frac{\cos^2 \frac{\theta}{2}}{\sin^6 \frac{\theta}{2}} \tag{36}$$

D. Regions of Kinematic Variables

Since the main argument of the present paper is that one can use the general inelastic form factors as measured in electron scattering to eliminate the nuclear physics in pair production experiments, one must face the question of what regions of kinematic variables q^2 and $q \cdot P$ are covered in each of these experiments. It turns out that exactly the same regions of kinematic variables can be covered in both experiments. This is indicated in Fig. 7.

We have used the fact that $2q \cdot P = q^2 + \mathcal{M}_f^2 - \mathcal{M}_T^2$. Elastic scattering is a straight line $P \cdot q = \frac{1}{2} q^2$. There will be a series of discrete lines corresponding to the excited bound states of the nuclear system with discrete $\mathcal{M}_f^2$ and then a continuum of values corresponding to particle emission from the nucleus. To see that the above is the allowed region, one can ask what values of q^2 are accessible for a given $\mathcal{M}_f^2 - \mathcal{M}_T^2$. In both electron scattering and pair production q^2 is space-like so $q^2 \geq 0$. In both cases, q^2 can go to infinity merely by fixing all angles at finite values and letting the incident energy go to infinity. The only question then is what is the minimum value of q^2. In electron scattering q^2 goes to zero if one looks in the forward direction and lets the incident energy go to infinity. In pair production one can also have q^2 going to zero in the case where all the particles come off in the forward direction and the incident energy goes to infinity.

III. PHOTO ABSORPTION ON THE NUCLEUS

The same general nuclear tensor $\mathcal{W}_{\mu\nu}$ can be used to calculate the cross section for the absorption of a real photon. This process is indicated in Fig. 8. k is the incoming four momentum of the incident photon and corresponds to -q with our previous definitions. Thus, in this case, the form factors are evaluated for $q^2 = k^2 = 0$ and

$$\frac{P \cdot q}{\mathcal{M}_T} = \frac{-P \cdot k}{\mathcal{M}_T} = |\vec{k}| \ .$$

$\hat{e}_{\vec{k}\lambda}$ is the polarization of the incident photon. The cross section is

$$\sigma_\gamma = \frac{1}{2} \sum_{\text{pol.}} \frac{(2\pi)^2 Z^2 \alpha}{\left[(k \cdot P)^2\right]^{\frac{1}{2}}} (\hat{e}_{\vec{k}\lambda})_\mu \mathcal{W}_{\mu\nu} (\hat{e}_{\vec{k}\lambda})_\nu \tag{37}$$

$$= \frac{(2\pi)^2 Z^2 \alpha}{\left[(k \cdot P)^2\right]^{\frac{1}{2}}} \frac{1}{2} \mathcal{W}_{\mu\mu} \tag{38}$$

The question now is what is $\mathcal{W}_{\mu\mu}$ under the conditions $q^2 = 0$, $\frac{P \cdot q}{\mathcal{M}_T} = |\vec{k}|$. We recall

$$\mathcal{W}_{\mu\nu} = \mathcal{W}_1(k^2, -k \cdot P)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \mathcal{W}_2(k^2, -k \cdot P)\left(P_\mu - \frac{(P \cdot k)k_\mu}{k^2}\right)\left(P_\nu - \frac{(P \cdot k)k_\nu}{k^2}\right) \tag{39}$$

There are no singularities when one sets $k^2 = 0$ as can be seen from Eq. (1), since the matrix elements are just the physical amplitudes for photo-absorption to individual final states. The apparent singularities in Eq. (39) must then cancel as $k^2 \to 0$ which gives the following relations on the inelastic form factors

$$\mathcal{W}_1(k^2, -k \cdot P) - \frac{(P \cdot k)^2}{k^2} \mathcal{W}_2(k^2, -k \cdot P) \underset{k^2 \to 0}{=} O(k^2)$$

$$\mathcal{W}_2(k^2, -k \cdot P) \underset{k^2 \to 0}{=} O(k^2) \tag{40}$$

Therefore one finds

$$\mathcal{W}_{\mu\mu}(k^2,-k \cdot P) \underset{k^2 \to 0}{=} 3\mathcal{W}_1(k^2,-k \cdot P) + \mathcal{W}_2(k^2,-k \cdot P)\left(-\mathcal{M}_T^2 - \frac{(P \cdot k)^2}{k^2}\right) \tag{41}$$

or

$$\mathcal{W}_{\mu\mu}(0,-k \cdot P) = 2\mathcal{W}_1(0,-k \cdot P) \quad .$$

This leads to

$$\sigma_\gamma = \frac{(2\pi)^2 z^2 \alpha}{\left[(k \cdot P)^2\right]^{\frac{1}{2}}} \mathcal{W}_1(0,-k \cdot P) = \frac{(2\pi Z)^2 \alpha}{K M_T} W_1(0,|k|M_T) \tag{42}$$

Photo absorption thus measures the form factor $\mathcal{W}_1(0,|\vec{k}|\mathcal{M}_T)$. This relation together with Eq. (40) above gives a useful approximation to the form factors which includes all the inelastic processes contained in the photoabsorption process. This is[3,6]

$$\begin{aligned} \mathcal{W}_1(q^2, q \cdot P) &= \frac{\left[(q \cdot P)^2\right]^{\frac{1}{2}}}{(2\pi)^2 z^2 \alpha}\, \sigma_\gamma\left(\frac{q \cdot P}{\mathcal{M}_T}\right) + O(q^2) \\ \mathcal{W}_2(q^2, q \cdot P) &= \frac{q^2}{\left[(q \cdot P)^2\right]^{\frac{1}{2}}}\, \frac{\sigma_\gamma\left(\frac{q \cdot P}{\mathcal{M}_T}\right)}{(2\pi)^2 z^2 \alpha} + O(q^4) \end{aligned} \tag{43}$$

The exact range of validity of these formulas is not so easily established.

ELECTRODYNAMIC PROCESSES WITH NUCLEAR TARGETS

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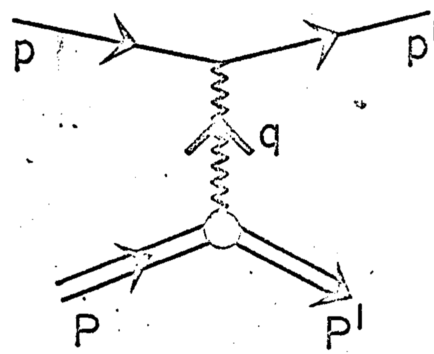
6. S. M. Berman, unpublished.

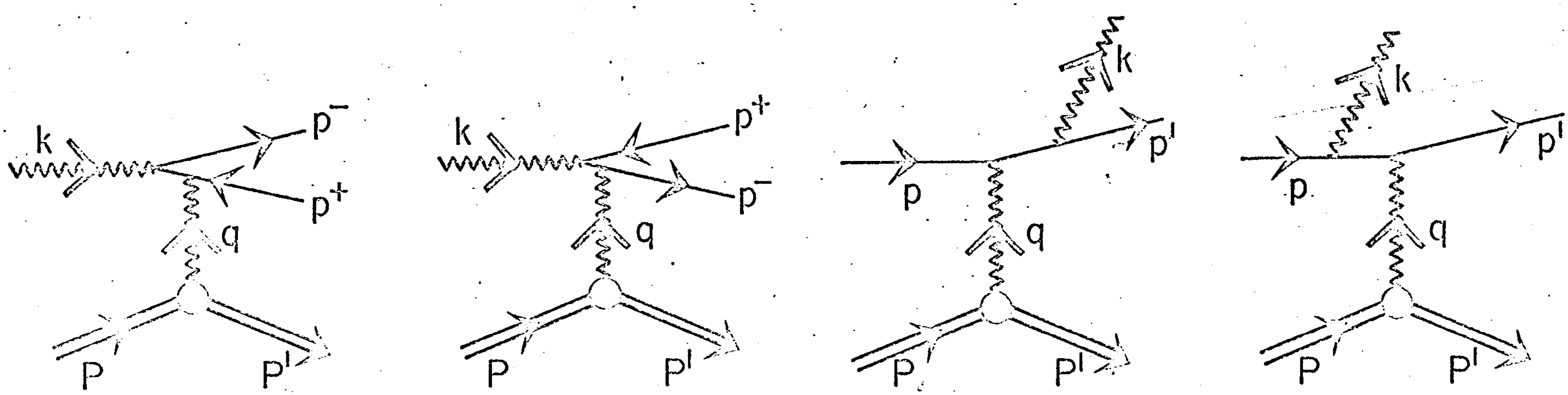
FIGURE CAPTIONS

1. Electron-nucleon scattering (in first Born approximation).
2. Lepton pair production and bremsstrahlung from nucleons.
3. Electron scattering from a nucleus leading to arbitrary final nuclear states.
4. Lepton pair production and bremsstrahlung from a nucleus leading to arbitrary final nuclear states.
5. General diagram for pair production with single photon exchange to the nucleus.
6. Kinematic variables in the laboratory system.
7. Ranges of invariant kinematic variables in scattering and pair production experiments.
8. Photo-absorption on a nucleus.



$$q = p' - p = P - P'$$

FIG. 1



PAIR PRODUCTION

$$q = p^{+} + p^{-} - k$$
$$= P - P'$$

BREMSSTRAHLUNG

$$q = k' + p' - p$$
$$= P - P'$$

FIG. 2

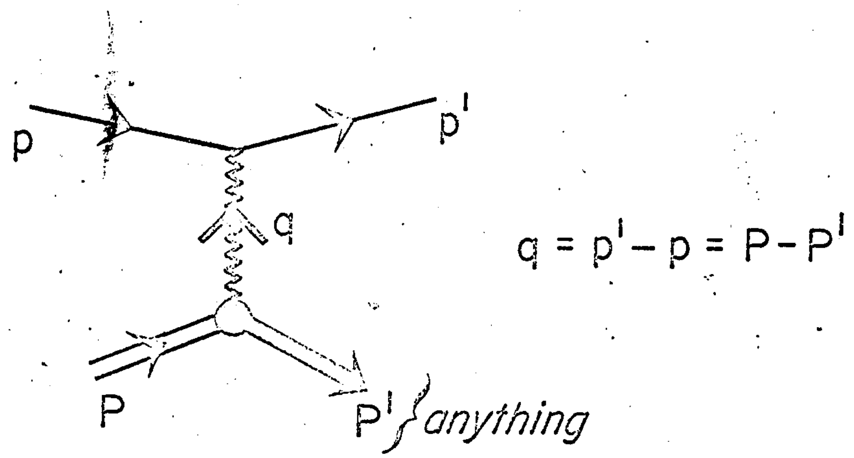
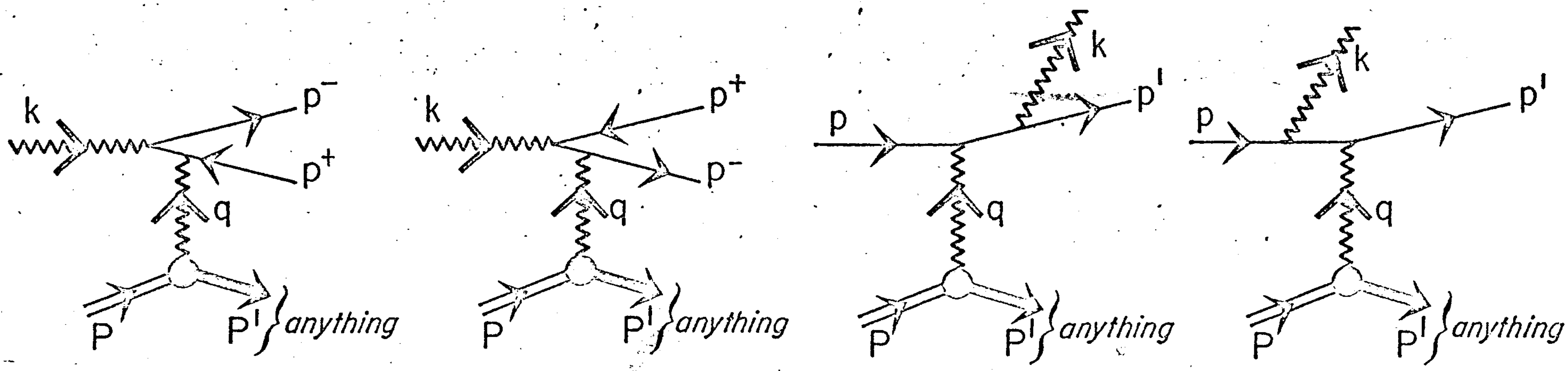


FIG. 3



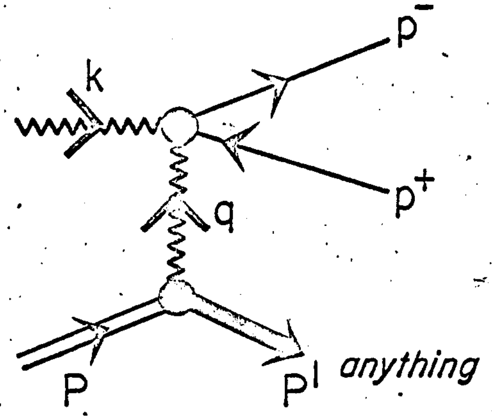
PAIR PRODUCTION

$$q = p^+ + p^- - k$$
$$= P - P'$$

BREMSSTRAHLUNG

$$q = p' + k - p$$
$$= P - P'$$

FIG. 4



$$q = p^{+} + p^{-} - k = P - P'$$

FIG. 5

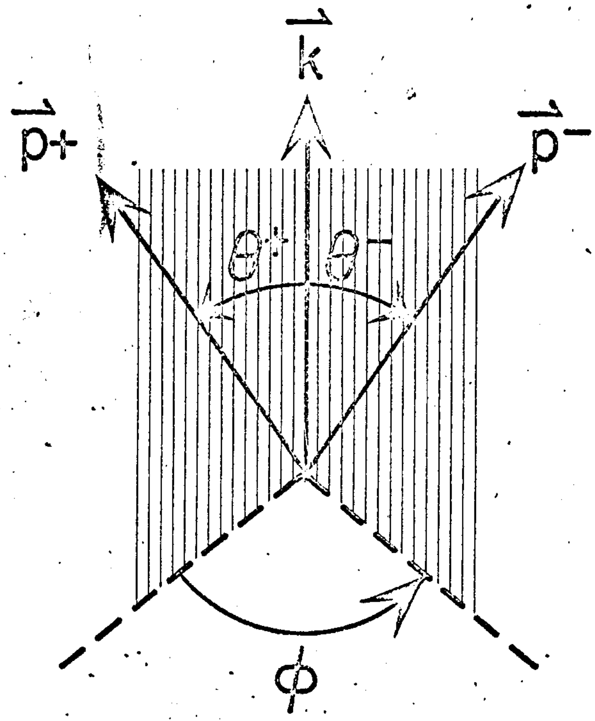


FIG. 6

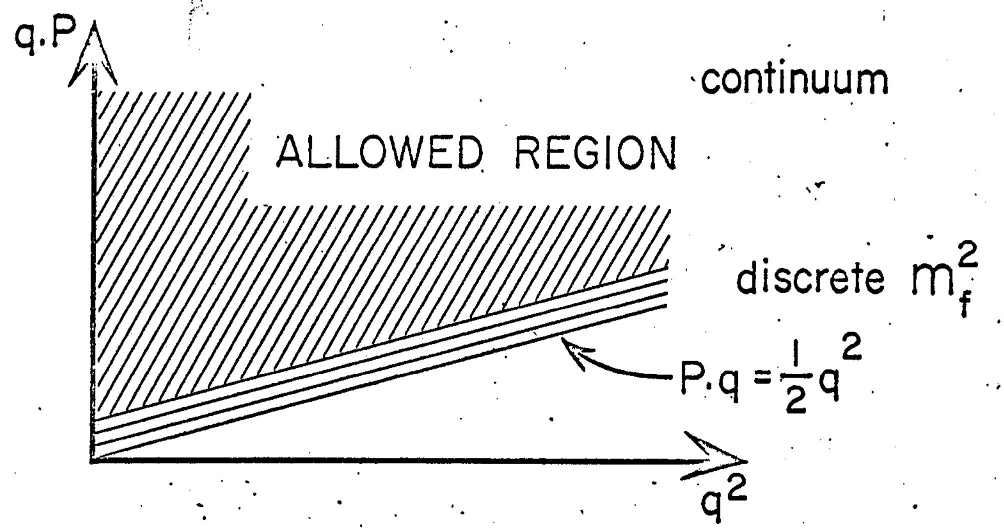


FIG. 7

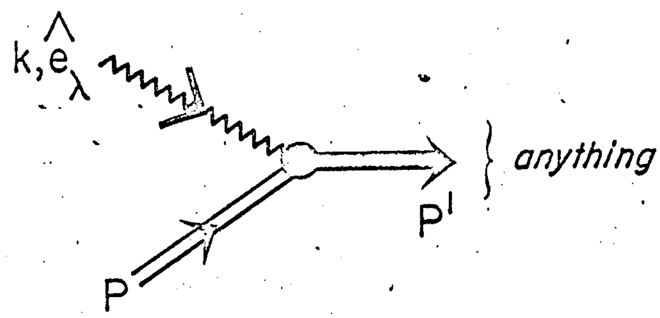


FIG. 8