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Glueball Masses as a Test of the $1/N$ Expansion

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ABSTRACT

We compute the scalar glueball mass $m(0^{++})$ in units of the square-root of the string tension $\sqrt{\sigma}$, for SU(N) gauge theories on the lattice, with $N = 2, 3, 5, 6$. We identify a general-scaling window in which the glueball mass is approximately independent of the lattice spacing, yielding an estimate of $m(0^{++})$ in the continuum. The estimate is corroborated by the excellent agreement between Hamiltonian and Lagrangian results for $N = 2, 3$. The continuum values of $m(0^{++})$ thus obtained for various values of N are remarkably close to each other, indicating a rapid convergence of the $1/N$ expansion.

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The $1/N$ expansion[1-3] provides an appealing conceptual framework for understanding many qualitative features of hadronic physics as consequences of QCD. The most notable of these are[3] :

(a) the suppression of quark loop effects in hadronic physics and the absence of exotic mesons; (b) Zweig's rule; (c) the approximate validity of Regge description of hadronic S-matrix as sum over tree diagrams involving exchange of physical hadrons only; (d) the relative importance of resonant two-body final states in multiparticle decays of unstable mesons. In addition, the large-N picture of QCD provides a basis for understanding the phenomenological success of Skyrmion physics in describing the static and dynamic properties of baryons[4] .

The qualitative arguments in favor of the large-N approximation are therefore very compelling. On the other hand, the question whether the $1/N$ expansion can become a practical calculational tool remains open. The two main reasons for this are: first, although SU(N) gauge theory is greatly simplified in the large-N limit, it is still very difficult to solve for physical observables in $3+1$ dimensions.* Second, even if the solution of the large-N theory were known, one would still need to determine whether for physical observables the $1/N$-expansion converges fast enough to make large-N a quantitatively reliable approximation to the real world, with $N = 3$. The most straightforward way of answering this question would be to compute the coefficients of some $1/N$ terms in the large-N expansion. This has proven to be exceedingly difficult, since such corrections involve all the complexity of summing non-planar diagrams.

In this work we estimate the importance of $1/N$ corrections by a different

* For one promising approach see however Ref. 5 and references therein.

approach: we numerically evaluate the scalar glueball mass as a function of N, thus providing the first direct evidence that these corrections are small. Our strategy is as follows. Given some physical observable $\langle O\rangle_N$ for a family of SU(N) theories, $N=2,3,\ldots$ we first require that $\langle O\rangle_N$ converges to a definite value:

$$\langle O\rangle_N \xrightarrow[N\to\infty]{} \langle O\rangle_\infty . \tag{1}$$

In the leading order in $1/N$ expansion the various observables typically scale like some power of N; for example, pion-nucleon cross-section $\sim N^0$, $f_\pi \sim N^{\frac{1}{2}}$, $g_A \sim N^1$, etc.[3] Eq. (1) is trivially satisfied if $\langle O\rangle_N \sim N^0$. If $\langle O\rangle_N \sim N^\alpha$, with $\alpha \neq 0$, then one can always form a ratio in which the leading dependence on N cancels out. As an example, consider f_π and g_A. These quantities have been calculated[6] in the Skyrme model, which can be thought of as a rough approximation to the effective low-energy Lagrangian of large-N QCD[4] . While both f_π and g_A differ substantially from experiment (by 30% and 50%, respectively), their ratio is independent of N: to the leading order, $f_\pi^2/g_A \sim N^0$, and agrees with the experiment to 3%.[7]

In order for $\langle O\rangle_\infty$ to serve as a fairly accurate estimate of $\langle O\rangle_3$, we further require the convergence to be fast:

$$\left|\frac{\langle O\rangle_N - \langle O\rangle_3}{\langle O\rangle_3}\right| \ll 1 \qquad \text{for} \qquad N \gg 3. \tag{2}$$

In practice, for SU(N) in four dimensions, there is no rigorous way of testing the validity of (2), since we have no way of calculating $\langle O\rangle_N$ analytically, nor do we know how to compute the $1/N$ corrections explicitly. We can, however, calculate

$\langle O \rangle_N$ *approximately* for several values of N. If (2) is valid for the approximants to $\langle O \rangle_N$ and $\langle O \rangle_3$, then we have at least a good indication that it might be true for the exact solution of the theory as well.

The validity of (2) has been previously studied analytically, in the context of two-dimensional field theories,[8] and numerically for the plaquette determinant $\det U(p)$ in four-dimensional lattice gauge theory.[9] As far as we know, in the existing literature there is no direct test of (2) for continuum observables with direct physical significance in $3+1$ dimensions. In the following we provide such a test by demonstrating that (2) is indeed valid for the approximate mass of the scalar glueball in pure-gauge SU(N) theories.

The standard method for calculating glueball masses in QCD is lattice gauge theory, which can be defined either in the Lagrangian form on a Euclidean space-time lattice,[10] or in the Hamiltonian form with continuous time and a three-dimensional spatial lattice. Our work will mainly concentrate on the latter and is based on the Kogut-Susskind SU(N) Hamiltonian:[11]

$$H = \frac{g^2}{a} \left\{ \sum_l \tfrac{1}{2} E_l^\alpha E_l^\alpha + \tfrac{2N}{g^4} \sum_p \left[1 - \frac{1}{2N} \mathrm{Tr} \left(U_p + U_p^\dagger \right) \right] \right\}. \tag{3}$$

where E_l^α is the chromoelectric field on the link l and U_p is the gauge invariant, oriented product of the link field variables U_l taken around a plaquette p.

The masses calculated from (3) are functions of the dimensionless coupling constant g^2, expressed in physical units by means of the inverse lattice constant $1/a$. In order for any dimensional observable m_i to have a fixed value in the continuum limit, g^2 must vary with a in a well defined manner governed by the

β-function. When the continuum is approached by letting $a \rightarrow 0$, asymptotic freedom requires that $g^2 \rightarrow 0$ as well. Consequently, the β-function in the weak coupling limit is determined by the continuum perturbation theory. Up to two loops, it gives the scaling of the lattice scale parameter Λ_L as[12] :

$$\Lambda_L = \frac{1}{a}\left[\frac{48\pi^2}{11}\xi\right]^{51/121} \exp\left[-\frac{24\pi^2}{11}\xi\right]. \tag{4}$$

where $\xi \equiv 1/Ng^2$ and Λ_L can be perturbatively related to the usual QCD scale parameter Λ.[13] If (4) holds for a certain range of ξ, usually referred to as the (asymptotic) *scaling window*, the lattice theory is said to exhibit *asymptotic scaling.* All masses, and all observables with dimensions of mass m_i, must scale in the same fashion and be proportional to Λ_L with coefficients $\widetilde{m}_i$ which are independent of ξ in the weak coupling limit:

$$m_i(\xi) = \widetilde{m}_i \, \Lambda_L(\xi). \tag{5}$$

As an obvious consequence of (5), dimensionless ratios of physical observables evaluated inside the scaling window do not depend on g^2, nor on the lattice spacing and reproduce the mass ratios in the continuum:

$$\frac{m_i(\xi)}{m_j(\xi)} = \frac{\widetilde{m}_i}{\widetilde{m}_j} \equiv R_{ij}. \tag{6}$$

It is important to point out that a lattice theory can exhibit a more *general scaling* in a wider scaling window, for which (5) remains valid but for which Λ_L <u>is not</u> given by (4). In that regime, scaling is governed by a *non-perturbative* β

function, which differs substantially from the continuum one* but mass ratios (6) are still independent of ξ, and reproduce continuum physics.[17,18]

Whether or not (5) is true in the intermediate coupling regime is an empirical question for a given lattice calculation. One should plot the appropriate ratios R_{ij} as functions of the lattice spacing (or lattice coupling constant) and see whether they are approximately constant over a range of values of a or g^2. A generic case exhibiting such a "scaling window" is schematically depicted in Figure 1: the ratio R starts from the strong coupling regime (no scaling), exhibits a scaling window and eventually diverges. The absence of scaling in the extreme weak coupling limit is usually due to the breakdown of various approximation methods, resulting from dominance of finite size effects.

We have tested the convergence of $1/N$ expansion by computing the ratio $R_N(\xi) \equiv m(0^{++})/\sqrt{\sigma}$ of the scalar glueball mass $m(0^{++})$ to the square-root of the string tension $\sqrt{\sigma}$, for SU(N) theories on the lattice with $N = 2, 3, 5, 6$. It is interesting to note that $m^2(0^{++})/\sigma$ is not just an arbitrary ratio of two masses: it is the intercept of the Regge trajectory corresponding to the 0^{++} state. For $N > 3$ there are no results from Lagrangian Monte Carlo calculations because of the prohibitively large amount of computer time required. Instead, we base our work on some recent Hamiltonian calculations. For $N \geq 3$, we use the variational estimates of $m(0^{++})$ obtained in Ref. 19. The variational method employed there gives an excellent estimate of the exact ground state energy for SU(3) and reproduces the critical value of ξ at which a phase transition in the $N \to \infty$ limit

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⋆ For a clear discussion of this point see Refs. 14 and 15. For some recent results on the non-perturbative β function see Ref. 16.

takes place. The corresponding expressions for σ up to $\mathcal{O}(\xi^8)$ are taken from the strong coupling expansion of Ref. 20. Where higher order terms are available, they have little effect on $R_N(\xi)$ in the region of interest. For $N = 2$ the ratio $m(0^{++})/\sqrt{\sigma}$ has been computed directly, using the t-expansion.[17] For $N = 3$ a recently obtained t-expansion result for this ratio[21] is in good agreement with the variational calculation, thus providing a valuable consistency test for the various approximation methods.

The curves showing $R_N(\xi)$ are shown in Fig. 2. They all exhibit the behavior schematically depicted in Fig. 1; thus provide a good indication of the onset of a "scaling window" as required by eq. (5) and (6). Further evidence that the results shown in Fig. 2 do indeed represent continuum physics is supplied by Euclidean Monte Carlo results for SU(2) and SU(3), for which extensive numerical simulations have been performed (see Refs. 22-26 for the most recent Monte Carlo results). If one assumes $\sqrt{\sigma} \approx 0.4$ GeV then all these different calculations predict $m(0^{++}) \approx 1.2$ GeV, provided that the effect of the fermion loops is small.

Since R is a ratio of two physical masses, for a given N its value should be the same, independent of the details of lattice regularization. Indeed, Euclidean Monte Carlo results for R_2 and R_3, as bracketed by the two horizontal lines in Fig. 2, are in excellent agreement with the Hamiltonian calculation, both variational and t-expansion.* We find it especially gratifying that very different approximation methods do indeed yield the same continuum physics.

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* The Monte Carlo estimates of R were obtained as functions of the *Euclidean* coupling constant g_E^2 and therefore cannot be compared to the Hamiltonian results on a coupling by coupling basis. On the other hand, the continuum values of R are the same in both cases.

The most interesting physical result in Fig. 2 is that the continuum values of R_N are remarkably close to one another† for all N and that the large-N limit is effectively reached for $N \gtrsim 5$. To our knowledge, this provides the first direct evidence, in the sense of eq. (2), for the rapid convergence of $1/N$ expansion for physical observables in SU(N) . A caveat is however also necessary at that this point: Figure 2 shows a *ratio* of two physical quantities. It is possible (as would be suggested from two dimensions by Ref. 8) that the $1/N$ corrections to the glueball mass and string tension taken separately are not very small. Their values may be very close however, so that in the ratio the $1/N$ terms cancel out. It would very interesting to find out whether this is indeed the case in four dimensions and why such $1/N$ corrections might be close.

There are a large number of observables which are independent of N in the large-N limit.[3] If the fast convergence of the large-N expansion is true not only for glueball masses, but for the latter physical observables as well, then an approximate solution of the large-N theory might reasonably be expected to yield a good quantitative estimate of $N = 3$ physics.

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† The close values of R_2 and R_3 have been already commented upon in Ref. 23.

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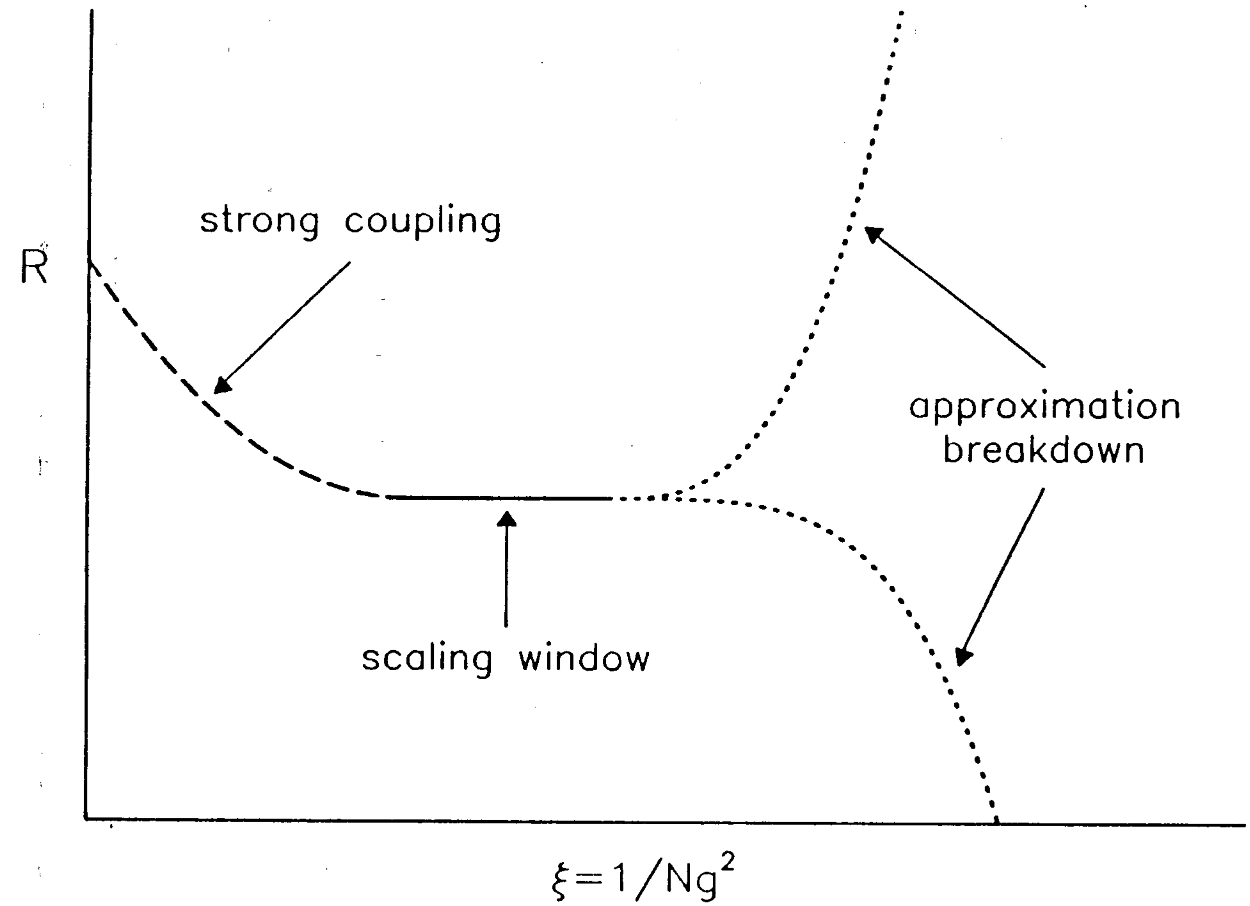

Fig. 1. A generic case illustrating the different regimes in a typical calculation of mass ratios on the lattice.

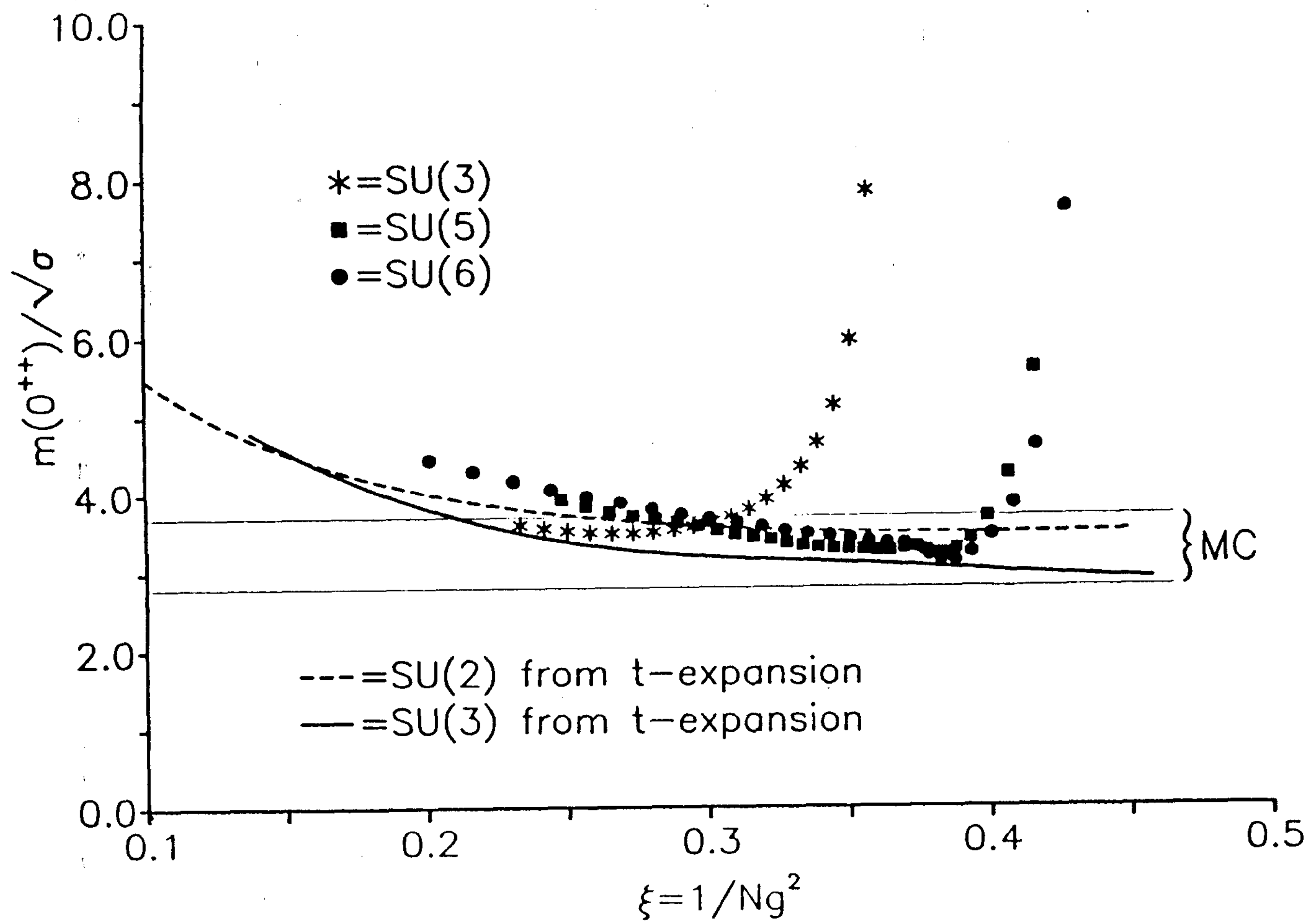

Fig. 2. The ratio $m(0^{++})/\sqrt{\sigma}$ for SU(N) lattice gauge theories, N=2[17] and 3,5,6.[19,20,21] SU(3) t-expansion curve is an average of several Padé approximants in ref. 21. The two thin horizontal lines bracket recent Euclidean Monte-Carlo results for SU(2) and SU(3)[22−26] .