

Gravitational waves from extreme mass-ratio inspirals

Takahiro Tanaka

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

E-mail: tama@scphys.kyoto-u.ac.jp

Abstract. Extreme mass-ratio inspiral (EMRI) is the main target of Laser Interferometer Space Antenna (LISA). We give a short review on this system as test-bench of general relativity. We briefly mention the recent progress in our study to obtain sufficiently accurate gravitational waveforms from this type of inspiralling binaries.

1. Introduction

There are various types of sources of gravitational waves. Among them, inspiralling binaries are the most promising candidate. This is because reasonably high event rate is expected and because the wave form, which is called a chirp signal, can be presumably theoretically predicted in advance. Besides the inspiralling binaries, there are semi-periodic sources such as binaries with large separation or pulsars, sources possibly correlated with optical counter part such as supernovae or γ -ray bursts, and stochastic background.

We expect that the observation of gravitational waves from inspiralling binaries in general will bring various information; event rate, binary parameters and possible deviation from general relativity. Binaries composed of stellar mass black holes and/or neutron stars are targets of ground based gravitational wave detectors [1, 2, 3, 4]. From these binaries we may be able to study the equation of state of neutron stars, possible correlation with the short γ -ray bursts and the abundance of primordial black hole binaries if they exist. Massive or intermediate mass black hole binaries are the target of the future space missions [5, 6, 7]. From the observation of gravitational waves from these sources, we will obtain information about the formation history of super massive black holes at the galactic centers. The last that we mention is the extreme (or intermediate) mass-ratio inspirals (EMRI), which is the main topic of my talk. This target is expected to be a good probe of the spacetime geometry around the black hole [8, 9].

Binary evolution is divided into three phases. The first is the inspiral phase, where the separation of binary is large compared with the horizon radius. In this phase, one can neglect the effect of internal structure of each object, and hence the evolution of binary can be approximated by that of two point particles [10]. To extract the maximum information from the observation, accurate prediction of the wave form is requested. First of all, detection of weak gravitational wave signal is possible only when we know the possible pattern of wave form in advance. To extract the binary parameters as accurately as possible, the correspondence between the wave form and the binary parameters must be also determined very precisely. When we use the observation of gravitational waves for the test of general relativity [11], we will use the event

with the largest signal to noise ratio. In this case more precise theoretical prediction might be required. The inspiral phase is followed by the merging phase, where strongly non-linear dynamics can be understood only by numerical approach. Recently, there has been big progress in the area of numerical relativity, and we are now ready to simulate what happens at this merging phase. After the merging phase, there is the phase of ringing tail corresponding to the quasi-normal oscillations of the black hole formed after the coalescence.

2. Extreme mass ratio inspirals

The main frequency band of LISA (the Laser Interferometer Space Antenna) [5] is between 0.003Hz to 0.03 Hz. These frequencies correspond to the typical time scale of the super-massive black holes with mass between $10^5 M_\odot$ and $5 \times 10^6 M_\odot$. When various kinds of objects like white dwarfs, neutron stars and stellar mass black holes orbit such a super-massive black hole, gravitational waves in LISA band will be generated. Such a binary system with extreme mass-ratio is called EMRI. The basic scenario of the formation of such binaries is as follows. Most of galaxies are thought to have a central black hole, and there is a star cluster around the central black hole. Occasionally a stellar encounter which leads to a large angle scattering occurs in the star cluster. Sometimes the orbit after the scattering comes into an extremely eccentric orbit which passes the very vicinity of the central black hole. If the orbit is close enough to the central black hole, the gravitational radiation reaction is efficient before the star is rescattered by the surrounding star cluster. Then, the orbit is gradually circularized and the star is captured by the central black hole. Although the orbit is very eccentric ($e \sim 1$) at the beginning of the formation of this system, the eccentricity is reduced to a moderate value ($1 - e = O(1)$), say, at the last three years before the coalescence. The event rate is estimated and a few $\times 10^2$ events for 3 year observation by LISA are expected[12, 13], although the estimate of the event rate is still quite uncertain.

Since EMRI has a large mass ratio, the reduced mass μ is much smaller than the total mass M . In this case the small mass star, which we shall call satellite, follows very close to the geodesic in the spacetime governed by the large central black hole. The deviation from the geodesic motion is caused by the radiation reaction due to gravitational wave emission. This effect is suppressed by the mass ratio μ/M and is weak for EMRI. This means that a large number of cycles N are spent before plunge in the strong field region near the black hole horizon. From the comparison with the observational date, roughly speaking, difference in the number of cycles ΔN greater than 1 is detectable. Hence, a large number of cycles are advantageous for precise comparison between the theoretical templates and the observational data. The waveform depends on the metric of the background spacetime governed by the central super-massive black hole. Therefore it is said that gravitational waves from EMRI map the strong field region of the black hole spacetime. It is unlikely that the rotation of the central black hole is slow enough to be able to neglect its effect. Hence, we need to understand the waveform generated by a satellite orbiting a rotating super-massive black hole.

The precise theoretical prediction is possible for a binary system in which the effects other than the gravitational interaction can be neglected. If some other interaction gives a significant effect on the waveform, we cannot use this system as an ideal source for testing general relativity. Here “a significant effect” means that the induced phase shift in the gravitational waveform is greater than $O(1)$ after a large number of cycles over the whole observational period. Hence, even if the effect is a tiny minor correction to the equation of motion of the satellite, it can be harmful.

The super-massive black hole will be associated with the accretion disk. The disk-satellite interaction may cause a significant effect. This was studied by Narayan [14]. Roughly speaking, the interaction time scale of the satellite with the disk surrounding the central black hole is

estimated using the usual formula for the dynamical friction time scale:

$$t_{df} = \frac{v_{rel}^3}{4\pi G^2 \mu \rho \log \Lambda}. \quad (1)$$

Here $\log \Lambda$ is the coulomb logarithm of $O(1)$ and G is the Newton's constant. The relative velocity between the satellite and the disk gas is almost at the Keplerian velocity in the case of advection dominated accretion flow (ADAF). The assumption of ADAF will be valid for the most of galactic centers which are not very active like AGN. In the case of ADAF, the accretion flow is almost spherically symmetric. Thus the density of the disk gas ρ will be estimated by using the accretion rate \dot{M} as $\rho = \dot{M}/4\pi r^2 v_r$, where the radial velocity v_r is estimated as $v_r \approx \alpha v_k$ with $\alpha \approx 0.1$ using the phenomenological α -viscosity prescription. Substituting these estimates, we have

$$t_{df} \approx 4.5 \times 10^{12} \frac{M}{10^6 M_\odot} \left(\frac{\mu}{10 M_\odot} \right)^{-1} \left(\frac{\dot{M}}{10^{-2} \dot{M}_{Edd}} \right)^{-1} \text{ yr}, \quad (2)$$

where \dot{M}_{Edd} is the Eddington accretion rate.

Using the above formula we can estimate the shift of the orbital frequency Δf due to the dynamical friction as

$$\frac{\Delta f}{f} \approx \frac{T_{obs}}{t_{df}}. \quad (3)$$

T_{obs} is the duration of the observation, which is typically $O(1)$ year). This shift of the orbital frequency propagates to the change in the number of cycles as

$$\Delta N \approx \Delta f T_{obs} \approx f T_{obs} \frac{T_{obs}}{t_{df}}. \quad (4)$$

The observational frequency band f is $\approx 10^{-3}$ Hz, and hence $f T_{obs}$ is at most 10^5 or so. On the other hand, the last factor T_{obs}/t_{df} is $\approx 10^{-12}$ from Eq. (2). Therefore the effect of dynamical friction would be safely neglected in most cases.

3. Templates and test of general relativity

The theoretical prediction of the gravitational wave form from a binary in a quasi-circular orbit takes the form in Fourier space as

$$\tilde{h}(f) \approx A f^{-7/6} e^{i\Psi(f)}, \quad (5)$$

with $A = \mathcal{M}^{5/6}/\sqrt{20\pi^3} D_L$, $\mathcal{M} = \mu^{3/5} M^{2/5}$ and D_L is the luminosity distance. The phase $\Psi(f)$ is obtained in the post-Newtonian expansion as

$$\Psi = 2\pi f t_c - \phi_c + \frac{3}{128} (\pi \mathcal{M} f)^{-5/3} \left[1 + \frac{20}{9} \left(\frac{743}{331} + \frac{11}{4} \frac{\mu}{M} \right) (\pi \mathcal{M} f)^{2/3} - (16\pi - \beta) (\pi \mathcal{M} f) + \dots \right], \quad (6)$$

where t_c is the coalescence time and ϕ_c is the phase at $t = t_c$. β is a parameter related to the spins of the component stars. The first term in the square brackets is the Newtonian term. The next terms are post-Newtonian corrections at 1PN order and 1.5PN order, respectively. We know how higher order expansion formally continues although the explicit coefficients in general cases are known only up to 3.5PN order. Only for detection, one may introduce a new search parameter at each order of expansion. In this sense the detailed knowledge about the higher order corrections may not be necessary for detection. However, for the precise extraction of the binary parameters or for the test of general relativity, we need higher order templates which are

accurate within the observational error. Since the observational error is inversely proportional to the signal to noise ratio, better templates are requested when we observe by chance a golden event whose signal to noise ratio is extra-ordinary high.

For the test of general relativity, so far two possibilities of modified gravity are most investigated. One is the scalar tensor theory and the other is massive gravity. In these two cases the waveform is modified as

$$\Psi = \dots + \frac{3}{128}(\pi M f)^{-5/3} \left[\alpha u^{-2/3} + 1 + \left(\frac{3715}{756} + \frac{55}{9} \frac{\mu}{M} + \frac{128}{3} \frac{\mu}{M} \beta_g \right) u^{2/3} - (16\pi - \beta) u + \dots \right], \quad (7)$$

where we have introduced $u = \pi M f$.

α is the parameter related to the scalar degree of freedom. α is inversely proportional to the so-called Brans-Dicke parameter ω_{BD} . The current bound on the Brans-Dicke parameter coming from the dipole radiation is $\omega_{BD} > 140$ for 4U 1820-30 (Neutron star + white dwarf binary in NGC6624) [15]. Constraint from future observation by LISA for $1.4M_{\odot}$ neutron star + $400M_{\odot}$ black hole can be $\omega_{BD} > 2 \times 10^4$ [11]. If we can use higher frequency band ($\sim 0.1\text{Hz}$) for observation which is the target of DECIGO project [6], much stronger constraint is obtained like $\omega_{BD} > 5 \times 10^9$ by observing $1.4M_{\odot}$ neutron star + $10M_{\odot}$ black hole binary at a cosmological distance. $\beta_g = (\pi^2 M / \lambda_g^2 \int a^3 dt)$ is the parameter related to the graviton mass, where λ_g is the Compton wavelength of the massive graviton, a is the scale factor and η is the conformal time. The expected constraint obtained by future observations by LISA is $\lambda_g > 1\text{kpc}$, which is brought by observing the merger of $10^7 M_{\odot}$ black holes [11].

We said that EMRI is best suited as a probe of the black hole spacetime. The spacetime governed by the central black hole is expected to be given by Kerr geometry. In this case the quadrupole moment Q (and also higher order moment) of the spacetime is related to the spin S like $Q = -S^2/M$. However, in modified gravity theory some deviation from the Kerr geometry may arise. In Ref. [9] it was shown that Q/M^3 can be measured to within $\sim 10^{-2} - 10^{-4}$ based on the last year of data assuming $SNR = 100$.

4. Black hole perturbation

As we mentioned earlier, the template of gravitational wave in the inspiralling phase is obtained by using the standard post-Newtonian expansion. However, the higher order expansion is not easy in this method. In the case of EMRI we may need a very precise template due to the large number of cycles spent near the black hole. To obtain more accurate templates for EMRI, there is a more suitable method, which is the black hole perturbation. When the mass-ratio is large, we can treat the satellite as a small perturber of the central black hole. Then one can solve the Einstein equations

$$G^{\mu\nu}[\mathbf{g}] = 8\pi G T^{\mu\nu},$$

by expanding the metric as

$$g_{\mu\nu} = g_{\mu\nu}^{BH} + h_{\mu\nu}.$$

The method to compute \mathbf{h} at the lowest order of expansion is well-established. Here we do not need to assume the slow motion of the satellite $v/c \ll 1$ as in the case of standard post-Newtonian expansion. At linear order, the equations $G^{\mu\nu}[\mathbf{g}^{(BH)} + \mathbf{h}] = 8\pi G T^{\mu\nu}$ can be reduced to a single master equation $L\psi = 4\pi\sqrt{-g}T$ for a single complex master variable ψ . L is a second order differential operator. For a non-rotating central black hole, the procedure is known as Regge-Wheeler-Zerilli formalism. (There are two real variables in this case.) For a rotating case, one can use the Teukolsky formalism. Recently, a systematic post-Newtonian expansion

method was developed for these master equations [16]. Hence, we can systematically derive higher order analytic formulae in the post-Newtonian expansion.

The source term T is a function composed of the energy momentum tensor of the satellite. Schematically, $T^{(1)} = \tau_{\mu\nu} T^{\mu\nu}$ in the case of Teukolsky formalism, and $\tau_{\mu\nu}$ is a second order differential operator. The master variable ψ is a projected component of Weyl curvature. The master equation is usually solved by means of Green function method. First we solve the homogeneous equation $L\Omega = 0$. This equation allows separation of variables by setting $\Omega = \sum R_\Lambda(r) Y_\Lambda(\theta, \varphi) e^{-i\omega t}$, where $\Lambda = \{\ell, m, \omega\}$ represents a set of separation constants. $Y_\Lambda(\theta, \varphi)$ is the angular harmonic function. Substituting this decomposition, we obtain a radial equation $[\partial_r^2 + \dots] R(r) = 0$. Using thus obtained solutions of the homogeneous equation, we have $\psi = \sum \Omega_\Lambda^{up} Z_\Lambda$, and Z_Λ is obtained by the convolution of the appropriately normalized mode function and the source term. Since the master variable ψ reduces to the usual two polarizations of gravitational waves as $\psi \sim (\ddot{h}_+ - i\ddot{h}_\times)/2$ in the limit of large radius, the energy and the angular momentum loss rates due to gravitational wave emission are written in term of the amplitude of gravitational waves, Z_Λ . The formulas are

$$\frac{dE}{dt} = - \sum_{\Lambda} |Z_\Lambda|^2, \quad \frac{dL_z}{dt} = - \sum_{\Lambda} \frac{m}{\omega} |Z_\Lambda|^2.$$

5. Adiabatic evolution of orbits and wave form

What we want to obtain is the phase evolution of gravitational waves. As a simple example, let's consider quasi-circular orbits. Basically the evolution of the frequency of gravitational waves is determined by the evolution of the satellite orbital period. Equating the energy lost by the gravitational wave emission with the minus of the change in the binding energy, the change rate of the orbital frequency f is evaluated as

$$\frac{df}{dt} = - \frac{d\mathcal{E}_{GW}}{dt} \left(\frac{d\mathcal{E}_{orbit}}{df} \right)^{-1}. \quad (8)$$

The leading order of $d\mathcal{E}_{GW}/dt$ starts with $O(\mu^2)$. This is because the energy was exactly conserved, if the radiation reaction were turned off. On the other hand, the leading order of the relation between the binding energy and the orbital frequency, $d\mathcal{E}_{orbit}/df$, is determined by the geodesic motion on a given background black hole spacetime, and it is $O(\mu)$. The effect of the self-force, which is the force acting on a particle due to the metric perturbation caused by the particle itself, is higher order correction of $O(\mu^2)$ in $d\mathcal{E}_{orbit}/df$. Roughly speaking, to obtain the leading order correction to the waveform, we therefore have only to know the effect of the self-force on $d\mathcal{E}_{GW}/dt$.

The extension to more general orbits is straight forward in the case of Schwarzschild background. In this case one can assume that the orbit is on the equatorial plane without loss of generality. Then, the geodesics are completely specified by the energy \mathcal{E} and the z -component of the angular momentum \mathcal{L} . Both "constants of motion" have the associated timelike and rotational Killing vectors, $\eta_{(t)}^\mu \equiv (\partial_t)^\mu$ and $\eta_{(\varphi)}^\mu \equiv (\partial_\varphi)^\mu$. Therefore we can define conserved current by $\int d\Sigma^\mu \eta^\nu t_{\mu\nu}$ using the effective energy momentum tensor of gravitational waves $t_{\mu\nu}$. Here $d\Sigma^\mu$ is the volume element of a constant time hypersurface.

In the case of Kerr background, we do not have spherical symmetry. Hence, one cannot say that the orbits are on the equatorial plane in general. To specify geodesics off the equatorial plane, we need to consider another "constant of motion". It is well known that the geodesics on the Kerr background possess the third "constant of motion" called Carter constant. However, Carter constant is not associated with any Killing vector field. Instead, it is related to a rank two Killing tensor. Killing tensor satisfies the equation

$$K_{(\mu\nu;\rho)} = 0. \quad (9)$$

Using this Killing tensor, the Carter constant is defined as

$$Q \equiv \mu K_{\alpha\beta} u^\alpha u^\beta, \quad (10)$$

where $u^\alpha := dz^\alpha/d\tau$ is the four velocity of the satellite.

To evaluate dQ/dt , we need to compute the self-force acting on the particle directly [17]. Though the prescription to calculate the self-force is formally established [18, 19, 20], performing explicit calculation is not so straight forward. However, it was found to be much easier to compute the averaged value of dQ/dt . As an approximation, we may use the averaged values of the change rates for the “constants of motion” instead of those evaluated by using the instantaneous self-force. We call it adiabatic approximation. This approximation will be as good as an estimate using the balance argument mentioned above, and moreover is also applicable to dQ/dt .

Gal'tsov has shown that the radiative field correctly reproduces the results obtained by using the balance argument for $d\mathcal{E}/dt$ and $d\mathcal{L}/dt$ when they are averaged over an infinitely long time interval [21]. Here the radiative field is defined by half retarded field minus half advanced one. Since the radiative field is free from divergence, we do not have to worry about how to regularize the self-force.

However, there had been no justification for applying the same scheme to dQ/dt until very recently. The breakthrough was brought by Mino, who gave a justification for applying the same scheme to dQ/dt [22].

Finally a simplified formula for the averaged value of dQ/dt was obtained by [23]. The self-force is explicitly written in term of the metric perturbation as

$$f^\mu[\mathbf{h}] := -\frac{\mu}{2}(g^{\mu\nu} + u^\mu u^\nu)(h_{\nu\rho;\sigma} + h_{\nu\sigma;\rho} - h_{\rho\sigma;\nu})u^\rho u^\sigma.$$

Using this force, the evolution of Carter constant is given by

$$\frac{dQ}{d\tau} = 2K_\mu^\nu u^\mu f_\nu \quad (11)$$

Some complicated operation is necessary to compute all the metric component from the master variable ψ , although the prescription is given by Chrzanowski [24]. If we naively combine all the formulas, the final expression becomes very messy and hard to handle. But we found that, after several non-trivial manipulations, the final expression is greatly simplified. It turned out to be as easy to evaluate as the averaged values of $d\mathcal{E}/dt$ and $d\mathcal{L}/dt$. Some demonstrative calculations have been performed in [25, 26].

6. Summary

Among the various gravitational sources, extreme mass-ratio binaries are the most suitable ones to prove the geometry of a black hole spacetime. For high-precision test of general relativity, we need both space-borne gravitational antenna and accurate theoretical templates.

As a tool for computing the gravitational waveform, black hole perturbation theory and the radiation reaction due to the emission of gravitational waves have been developed. Here we reported the recent progress in the area of the adiabatic approximation to the radiation reaction. Within this approximation, we are now ready to compute the evolution of the satellite orbit taking into account the gravitational reaction. In this approximation we treat only the averaged rates of change of the “constants of motion”. So far, it is not clear if we need to calculate not the averaged effect but the instantaneous self-force for the use of precision test of general relativity in reality. This must be clarified soon. If the computation of instantaneous self-force is really necessary, it is an urgent problem which must be solved before the launch of LISA.

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