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Shifman-Vainshtein-Zakharov sum rules

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Prof. Mikhail A. Shifman accepted the invitation on 12 December 2008 (self-imposed deadline: 12 June 2009).

Figure 1: When the separation between the probe color charges exceeds critical the nonlinearity becomes so strong that it makes no sense to speak about individual gluons. The color fields between the probe charges form a flux tube with transverse dimensions $\sim \Lambda^{-1}$.

The **Shifman-Vainshtein-Zakharov (SVZ) sum rules** relate hadronic parameters, such as meson masses and coupling constants, baryon magnetic moments, etc., to a few characteristics of the vacuum of quantum chromodynamics (QCD): gluon and quark condensates. The method is based on Wilson's operator product expansion (OPE) which was adapted by the authors to QCD in the mid-1970s. Quark confinement is assumed rather than proved. The SVZ method was applied, with remarkable success, for (approximate) calculations of a large variety of properties of all low-lying hadronic states. In the 1980s the SVZ method was developed in various directions shifting the emphasis from calculation of masses and coupling constants of "classic" resonances to such problems as magnetic moments, form factors at intermediate momentum transfer, weak decays, structure functions of deep inelastic scattering at intermediate x , heavy quarkonim systems, and many others. At the initial stages the method was essentially unchallenged, it had no competitors. Crucial elements of the method laid the foundation or were incorporated in heavy quark theory, light-cone sum rules and the recently developed holographic (AdS/QCD correspondence) framework. Qualitative insights into the vacuum structure provided by the SVZ method gave rise to instanton liquid and other models of the QCD vacuum.

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Introduction

The basic microscopic degrees of freedom of quantum chromodynamics (QCD) are quarks and gluons. Their interaction is described by the Lagrangian

(1)

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \sum_f \bar{q} i \not{D} q$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, f stands for flavor, and the color indices of the quark fields q are suppressed. (For simplicity the quark mass terms are omitted in the expression above).

Neither quarks nor gluons are asymptotic states showing up in the physically observed spectrum. Experimentally observed are hadrons — color-singlet composite states "built" of quarks and gluons. This phenomenon, the most salient feature of non-Abelian Yang-Mills theory, is known as quark (or color) confinement. If one takes a heavy probe quark and an antiquark separated by a large distance, the potential energy between them grows linearly with distance, leading to unavoidable formation of colorless hadrons.

In order to study hadronic properties it is convenient to start from the empty space — the vacuum — inject there a quark-antiquark pair, and then follow the evolution of the valence quarks injected in the "vacuum medium." The injection is achieved by external currents. The most popular are the vector and axial currents. Their popularity is due to the fact that they actually exist in nature: virtual photons and W bosons couple to the vector and axial quark currents. Therefore, they are experimentally accessible in the e^+e^- annihilation into hadrons or hadronic τ decays.

Identification of microscopic constituents inside hadrons — quarks and gluons — became possible because of weakness of their interactions at short distances (large Euclidean momenta). At short distances perturbation theory applies. QCD explains this phenomenon by celebrated asymptotic freedom (Gross and Wilczek 1973, Politzer 1973) stating that the effective coupling constant α_s falls off at large (Euclidean) momenta as

(2)

$$\alpha_s(p^2) \approx \frac{4\pi}{b \ln(p^2/\Lambda^2)}, \qquad b = \frac{11}{3} N_c - \frac{2}{3} N_f,$$

where N_c is the number of colors, N_f is the number of flavors, Λ is the dynamical scale parameter of QCD, and, finally, b is the first coefficient in the Gell-Mann—Low function. The same formula shows that the effective color interaction becomes stronger as separation between the color charges increases (i.e. p decreases). When the distance becomes larger than some number times Λ^{-1}, and exceeds a critical one, the "branchings" of gluons become so intensive (Fig.1) that it makes no sense to speak about individual gluons. Rather, one should phrase one's consideration in terms of the chromoelectric and chromomagnetic fields.

In infrared-free QED, the field induced by the probe charges separated by a large distance, is dispersed all over space, in accordance with the Coulomb law. In QCD a specific organization of the QCD vacuum makes such a dispersed configuration energetically inexpedient (Nambu 1974, 't Hooft 1975, Mandelstam 1976). Rather, the chromoelectric field between the probe color charges squeezes itself into a sausage-like (or string-like) configuration. The situation is reminiscent of the Meissner effect in superconductivity. The superconducting media do not tolerate the magnetic field. If one imposes, as an external boundary condition, a certain flux of the magnetic field through such a medium, the magnetic field will be squeezed into a thin tube carrying all the magnetic flux (the Abrikosov vortices). The flux tube gives rise to a linear potential between the magnetic sources at its endpoints.

Superconductivity is caused by condensation of the Cooper pairs — pairs of electric charges. A phenomenologically acceptable picture of large distance dynamics in QCD requires *chromoelectric* flux tubes, rather than magnetic; hence the phenomenon is referred to as the *dual Meissner effect*. The dual Meissner effect was proven to take place in a (distant) relative of QCD, $\mathcal{N}=2$ supersymmetric Yang-Mills theory, leading to linear potential and quark confinement (Seiberg and Witten 1994). However, the spectrum and properties of composite mesons in supersymmetric gauge theories are very different from those one observes in QCD.

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Basic idea of the SVZ method

There is a long way to go from the flux-tube picture (which is not yet established in QCD, in spite of vigorous efforts) to a quantitative framework allowing one to *calculate* a variety of hadronic properties for real mesons and baryons. The basic idea lying behind the SVZ method (Shifman 1979) is as follows. In the states comprising the low-lying part of the hadronic spectrum, both mesonic (e.g. ρ) and baryonic (e.g. p or n), quarks are not asymptotically far from each other, on average. The distance between them is of the order of Λ^{-1}. Under these circumstances, the string-like long chromoelectric flux tubes have no chance to be developed. The valence quark pair injected in the vacuum, in a sense, perturbs it only slightly. Then one does not need the full QCD string theory to approximately obtain the properties of the hadronic states. Their basic features depend on how the valence quarks of which they are built interact with typical *vacuum* field fluctuations.

It was suggested (and then established) that the QCD vacuum is sufficiently characterized by a number of condensates (Shifman 1979): the gluon condensate $\langle G^2_{\mu\nu}\rangle$, the quark condensate $\langle\bar{q}q\rangle$, the mixed condensate $\langle\bar{q}\sigma Gq\rangle$, the four-quark condensate, and a few others. Using the SVZ method one can first find the numerical values of the above condensates and then determine, with sufficient accuracy, parameters of a large number of mesons and baryons.

Without invoking the entire infinite set of condensates one can capture only gross features of the vacuum medium. Correspondingly, any calculation of the hadronic parameters based on the SVZ method is approximate in nature. The usefulness of the method lies in its analytic capabilities: analytic analyses become possible in a wide range of problems from hadronic physics, including those where the answer was not known *a priori*. In addition to static properties of mesons and baryons, the SVZ method was applied in problems with three-point functions, external electromagnetic and other auxiliary fields, light-cone kinematics, and so on. This allowed one to approximately calculate magnetic moments, form factors at intermediate momentum transfers, weak decays constants, structure functions of deep inelastic scattering at intermediate x, transition amplitudes of the $B\to\rho$ type, and many other quantities.

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Wilson's OPE

OPE allows one to consistently separate the short and large distance contributions. The former are then represented by the vacuum condensates while the latter are accounted for in the coefficient functions. In constructing OPE one must introduce a somewhat artificial boundary, μ, usually referred to as *normalization point*. All fluctuations with frequencies higher than μ are supposed to be hard and are included in the coefficient functions. Those with frequencies lower than μ are soft. Only soft modes are to be retained in the condensates. Thus, the separation principle of OPE is "soft versus hard." Both, the OPE coefficients and condensates are explicitly μ dependent. All physical quantities are μ independent; the normalization point dependence of the condensates is compensated by that of the coefficient functions.

The theoretical basis of any calculation within the SVZ method is the operator product expansion for correlation functions of two or more currents. Say, for two currents, in the coordinate space, one can write

(3)

$$\left\langle T\{J(x)\;J(0)\}\right\rangle_{x\to 0}=\sum_n C_n(x;\mu)\langle\mathcal{O}_n(\mu)\rangle$$

where the sum runs over a complete set of local gauge invariant (quark-gluon) operators with appropriate quantum numbers. The degree of locality is set by μ. The series on the right-hand side is infinite. The operators $\mathcal{O}_n(\mu)$ can be ordered according to their normal dimensions. The coefficient functions $C_n(x;\mu)$ are determined mainly, but not exclusively, by perturbation theory. Without the condensates the perturbative series cannot be consistently defined because of ambiguities associated with the high-order terms in the perturbative series.

In many instances it is more convenient to use OPE for correlation functions in the momentum space. For instance, for the conserved vector current $\bar{u}\gamma^\mu d$

(4)

$$\Pi_{\mu\nu}=i\int e^{iqx}d^4x\langle 0|T\{J_\mu(x)J^\dagger_\nu(0)\}|0\rangle\,,$$

where q is the total momentum of the quark-antiquark pair injected in the vacuum. Due to the current conservation $\Pi_{\mu\nu}$ is transversal,

(5)

$$\Pi_{\mu\nu}=(q_\mu q_\nu-q^2g_{\mu\nu})\Pi(q^2)\,,\qquad D(Q^2)\equiv-(4\pi^2)Q^2(d\Pi/dQ^2)\,.$$

The operator product expansion takes the form

$$D(Q^2)=\sum_n C_n(q;\mu)\langle\mathcal{O}_n(\mu)\rangle\,,$$

where, in addition to the unit operator of dimension zero, the leading dimension-4 and 6 condensates on the right-hand side are

(6)

$$\mathcal{O}_G=\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}\,,\qquad\langle\mathcal{O}_G\rangle\sim 0.012\ \mathrm{GeV}^4\,,$$

and

(7)
$\mathcal{O}_{4q} = (\bar{q}_1\Gamma_1 q_2)(\bar{q}_3\Gamma_2 q_4)\,.$

To the leading order in $1/N_c^2$ the four-quark condensate can be factorized,

(8)
$\langle\mathcal{O}_{4q}\rangle = \langle\bar{q}_1\Gamma_1 q_2\rangle\langle\bar{q}_3\Gamma_2 q_4\rangle\,, \qquad \langle\bar{q}q\rangle \sim -(250\ \mathrm{MeV})^3$

The dimension-6 operators $\mathcal{O}_{qG} = ig\,\bar{q}\sigma_{\mu\nu}G_{\mu\nu}q$ and $\mathcal{O}_{3G} = G^3$ can be neglected. The operator product expansion in its general form was engineered by Wilson (Wilson 1969, Wilson and Kogut 1974). Wilson's idea was adapted for the QCD environment in (Novikov et. al. 1985).

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Borel transformation

The condensate series for $D(q^2)$ is asymptotic. Convergence is achieved by the Borel transformation which, in the case at hand, can be performed as follows. For the functions obeying dispersion relations one applies the limiting procedure

(9)
$$\hat{\mathcal{B}} = \lim \frac{1}{(n-1)!}(Q^2)^n\left(-\frac{d}{dQ^2}\right)^n\,,\quad Q^2\to\infty,\ n\to\infty,\ \frac{Q^2}{n}\equiv M^2\ \text{fixed}\,.$$

M^2 is called the Borel parameter. One can show that applying $\hat{\mathcal{B}}$ to $(1/Q^2)^n$ one gets

$$\hat{\mathcal{B}}\left(\frac{1}{Q^2}\right)^n = \frac{1}{(n-1)!}\left(\frac{1}{M^2}\right)^n\,,\qquad \hat{\mathcal{B}}(\ln Q^2) = -1\,,$$

which entails, in turn

$$\hat{\mathcal{B}}\left(\frac{1}{s+Q^2}\right) = \frac{1}{M^2}\,e^{-s/M^2}\,.$$

The Borel-transformed dispersion relation takes the form

(10)
$$\tilde{\Pi}(M^2) \equiv \{\hat{\mathcal{B}}\Pi(Q^2)\} = \frac{1}{\pi M^2}\int ds\,\mathrm{Im}\,\Pi(s)\,e^{-s/M^2}\,.$$

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Sample application: SVZ for ρ mesons

Analysis of the ρ meson channel was among the first applications of the SVZ method which proved its predictive power (Shifman et al. 1979). In this channel we have

(11)
$$I(M^2) \equiv \frac{12\pi^2}{N_c}\tilde{\Pi}(M^2) = \frac{1}{M^2}\int ds e^{-s/M^2}\rho(s)$$
$$= 1 + I_{\rm pt} + I_{\rm cond}$$

where $\rho(s)$ is the spectral density and

$$I_{\rm pt}(M^2) = k_1 a\left(\frac{M^2}{e^\gamma}\right) + k_2\left[a\left(\frac{M^2}{e^\gamma}\right)\right]^2 + (k_3 + \frac{\pi^2}{6}k_1)\left[a\left(\frac{M^2}{e^\gamma}\right)\right]^3 + \dots,$$

$$I_{\rm cond}(M^2) = \frac{\pi^2}{3M^4}\langle\mathcal{O}_G\rangle - \frac{8\pi^3}{M^6}\left\langle\alpha_s\left(\bar{u}\gamma_\alpha\gamma_5 T^a u - \bar{d}\gamma_\alpha\gamma_5 T^a d\right)^2\right\rangle$$
$$-\frac{16\pi^3}{9M^6}\left\langle\alpha_s\left(\bar{u}\gamma_\alpha T^a u + \bar{d}\gamma_\alpha T^a d\right)\sum_q\left(\bar{q}\gamma_\alpha T^a q\right)\right\rangle + \dots.$$

The perturbative data $k_{1,2,3}$ and a are

$$k_1 = \frac{4}{9}, \quad k_2 = 0.729 k_1, \quad k_3 = -2.03 k_1; \qquad a \equiv \frac{b}{4}\frac{\alpha_s}{\pi}.$$

To enhance the ρ meson contribution in (11) one must choose the value of the Borel parameter M^2 as small as possible. However, the condensate series explodes at small M^2. A balance is achieved in an intermediate domain called the window of stability. The spectral density is parametrized by three parameters to be fitted, the ρ meson mass m_ρ, its coupling constant to the vector current F_ρ, and s_0 marking the onset of continuum in the given channel,

(12)
$$\rho(s) = F_\rho^2\,\delta(s - m_\rho^2) + \theta(s - s_0)\,,$$

see Fig. 2. In fact, one is interested only in the ρ meson characteristics m_ρ and F_ρ. Since inside the window of stability the ρ meson saturates the integral in (11) at the level $\sim 90\%$, the fact that the above spectral density is inaccurate in the continuum is not important. A factor of two error in the continuum (the actual error is smaller) affects determination of the ρ meson parameters m_ρ^2 and F_ρ^2 at the level of $\lesssim 10\%$ accuracy. It is quite clear that the truncated condensate series does not allow one to go to the limit $M \to 0$ where determination of the ρ meson parameters would be exact. One must stay inside the window where the expansion is still under control. Correspondingly, the hadron parameters can only be determined approximately within the SVZ method. This is the price one has to pay for limited knowledge of the QCD vacuum encoded in a few condensates.

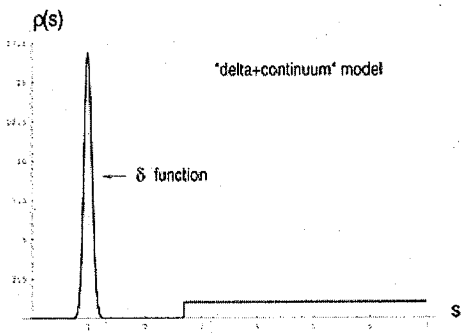

Figure 2: The original SVZ "delta+continuum" model for the spectral density.

Numerically, the theoretical and phenomenological parts of (11) are shown in Fig. 3.

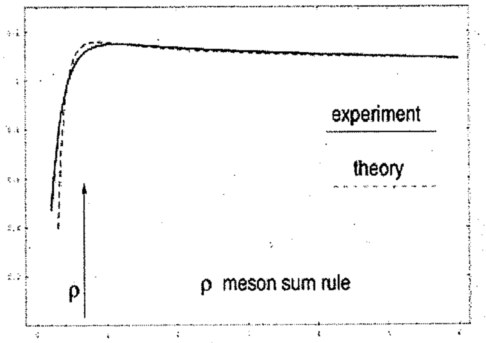

Figure 3: I *versus* M^2 (in GeV^2) in the ρ meson channel: confronting experiment and theory.

The center of the window of stability is indicated by a vertical arrow. The ρ meson example is quite typical. A similar situation takes place for the majority of low-lying hadrons: those built from the light quarks, heavy, and light and heavy.

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Exceptional cases

There are a few cases in which the SVZ method does not provide one with access to the lowest-lying mesons (Novikov et. al. 1981). This happens for specific, "nonclassical" hadrons, with a very strong coupling to the vacuum fluctuations, which are very different from, say, the ρ meson or nucleon. The most known example of this type is the $O^{\pm}$ glueball. In such channels one can reverse the argument and convert the failure of the SVZ sum rules into success by saying that the SVZ method predicts that not all hadrons are alike (Novikov et.

al. 1981).

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Further reading

- Pedagogical presentation of the SVZ sum rules and related topics can be found in
  - M. A. Shifman, *Snapshots of hadrons or the story of how the vacuum medium determines the properties of the classical mesons which are produced, live and die in the QCD vacuum,* Prog. Theor. Phys. Suppl. **131**, 1 (1998) [arXiv:hep-ph/9802214],
  - M. A. Shifman, *Quark-hadron duality,* arXiv:hep-ph/0009131, in the Boris Ioffe Festschrift *At the Frontier of Particle Physics* Ed. M. Shifman (World Scientific, Singapore, 2001).
- Two review papers were published in the mid-1980's.
  - B.L. Ioffe, *Acta Phys. Polon.* **B16** (1985) 543;
  - L.J. Reinders, H. Rubinstein, and S. Yazaki, *Phys. Reports* **127** (1985) 1
- A collection of the original papers with an extended commentary reflecting the state of the art in the late 1980's can be found in
  - *Vacuum Structure and QCD Sum Rules,* Ed. M. Shifman (North-Holland, Amsterdam, 1992).
- A brief survey of the light-cone sum rules is given in
  - V. Braun, hep-ph/9801222.
- SVZ sum rules in the heavy quark theory are discussed in
  - M. Neubert, *Phys. Reports* **245** (1994) 259.
- For a review of the sum rule applications in weak decays see
  - A. Khodjamirian and R. Rückl, hep-ph/9801443.
- A review on heavy quark theory based on OPE is
  - I. Bigi, M. Shifman and N. Uraltsev, *Aspects of heavy quark theory,* Ann. Rev. Nucl. Part. Sci. **47**, 591 (1997) [arXiv:hep-ph/9703290].

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- Author's homepage

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