Imperial College London
Department of Theoretical Physics

Supergravity as Yang-Mills Squared

Michele Zoccali

November 28, 2018

Submitted in part fulfilment of the requirements for the degree of Doctor of Philosophy in Physics of Imperial College London and the Diploma of Imperial College London.

Abstract

This thesis is concerned with the study of various aspects of the Yang-Mills Squared construction, which aims at furthering our understanding of the provocative idea that gravity may be regarded, in some sense, as the “square” of gauge theory. By assuming a convolutive tensor product and introducing a bi-adjoint scalar field as one of the factors, the Yang-Mills Squared formalism describes a purely field-theoretic realisation of this correspondence: the content as well as the global U-duality symmetries of a wide variety of supergravity theories may be shown to originate in the product of the contents and R-symmetries of two super Yang-Mills theories. Here we apply these ideas to twin supergravities, pairs of theories with identical bosonic sectors but different supersymmetric completions, to demonstrate that they are related in a controlled fashion through the underlying Yang-Mills factors. This has the additional advantage of giving a prescription for constructing new examples of factorisable supergravities from known ones. The second part of the thesis is devoted to the study of the gauge symmetries of linearised axion-dilaton gravity by adopting a Becchi-Rouet-Stora-Tyutin (BRST) covariant formulation of Yang-Mills. The content, BRST and anti-BRST transformations as well as the equations of motion of the gravitational side are shown to be related to those of the gauge theory side by means of a dictionary building the gravity fields as sums of convolutions of Yang-Mills potentials and ghosts, thus providing a fully Lorentz-covariant version of the Yang-Mills Squared map. The anti-BRST transformations of the Kalb-Ramond 2-form sector are shown to naturally anti-commute with BRST in this formalism.

Declaration

Unless properly referenced, the work presented in this thesis is the result of research by the author alone or in collaboration with Alexandros Anastasiou, Leron Borsten, Michael J. Duff, Mia Hughes, Alessio Marrani and Silvia Nagy. The material presented is based on the following papers:

- A. Anastasiou, L. Borsten, M. J. Duff, M. Hughes, A. Marrani, S. Nagy, M. Zoccali, *Twin supergravities from Yang-Mills theory squared*, Physical Review D **96** (2017)
- A. Anastasiou, L. Borsten, M. J. Duff, S. Nagy, M. Zoccali, *Gravity as Gauge Theory Squared: a Ghost Story*, Phys. Rev. Lett. **121**, 211601

Acknowledgments

I would like to thank my supervisor Michael Duff for allowing me to spend the last four years exploring the meanders of theoretical physics and for guiding me through them with passion and invaluable insight.

Thank you to my collaborators, you have been immensely helpful and supportive: Alexandros Anastasiou, for everything you have taught me, for returning my stolen backpack and for introducing me to Panikkos; Leron Borsten, for your patience, your insights and your passion for knowledge; Alessio Marrani, belli capelli, we survived a fire together in the name of physics; Silvia Nagy, for all those great, endless discussions and for apparently setting my route all the way from Queen Mary!

Thank you to all my fellow PhD students, you are a great bunch. Even though I might not have showed it enough, you have made my time here a much more enjoyable one. Thank you to my friends, wherever you may be. To Dimitrios and Gianluca, my favourite PhD students when I was an undergrad, for a great deal of inspiration early on. To the VIP room of Elimo, for some of the best moments of the last six years. To the Philip Walk gang: Andrea, Claire, Clara, Federico, Minima and especially Ramiro; let's do it again sometimes. To Kazuma, because "one does not simply". To Giulia, for ever trying to convince me to "fly": I will try one day. To SzeYing, for being the best partner I could ever ask for, all my love.

To my parents and my siblings, for your unfaltering support, encouragement and love: non ce l'avrei fatta senza di voi.

Contents

List of Tables

Chapter 1

Overview

A deeper understanding of the quantum-mechanical properties of the gravitational field and its interactions is arguably one of the most intriguing enterprises which theoretical physics has embarked on. Aside for its importance, one reason why it is so interesting to think about this is the apparent chasm separating our currently most successful model of gravity, as a theory of the geometry of a dynamical spacetime, and those describing the other fundamental forces, which are best understood in terms of gauge theories. These are models which describe the dynamics and interactions of the fundamental fields of nature with the aid of additional, unphysical degrees of freedom which help maintaining many desirable features, such as Lorentz covariance and locality, manifest. The redundancy in such a description is then accounted for by ensuring that the action of such theories is invariant under a set of *gauge* transformations, which heuristically help keep track of the unphysical modes. The quantisation of gauge theories represents an extremely successful chapter of theoretical physics, culminating with the Standard Model; however, the attempts to apply the insights gained there to models of the gravitational field fail to accomplish a similar level of success.

As it is often the case, however, it has become clear that other, perhaps more indirect, relations between models of gravity and quantum gauge theories are poised to teach us a great amount; of these, the AdS/CFT correspondence [1–3] represents, without a doubt, the prime example. Another interesting idea, more closely related to the spirit of this thesis, is that gravity may be thought of, in some way, as arising from a "product" of two gauge theories. After first appearing in string theory with the KLT relations [4] as well as in string field theory [5, 6], this provocative idea has gained new traction in recent years when Bern, Carrasco and Johansson (BCJ) formalised it in the context of scattering amplitudes [7, 8]. They taught us that, whenever the kinematic factors (contractions of momenta and polarisation vectors) of a certain gauge theory amplitude obey the same algebraic relations as the colour factors (products of structure constants) according to the so-called *color-kinematic duality* [9], it is possible to construct amplitudes describing graviton scattering as the *double copy* of the gauge theory one, that is by replacing the colour factors with a second copy of the kinematics. For a nice review, see [10]. In addition to sparking great interest with this radical proposal, they also showed that the existence of such a connection renders previously intractable gravitational calculations much simpler to handle: rather than evaluating the extremely complicated gravity amplitudes head-on, it is possible to "simply" compute the gauge theory amplitudes and construct the gravity result as a by-product. Indeed, calculations at high loop orders have been challenging our state-of-the-art understanding of the quantum-mechanical nature of supergravity theories, particularly with regards to the emergence of ultraviolet divergences or lack thereof [11–19]. Recently, a number of works have generalised the double copy procedure aiming at a classification of double copy constructible theories [20–29]. The double copy construction of gauged supergravities has been considered in [30]. Other, a priori independent lines of research have been converging somewhat on similar results. Examples include studies concerning the Cachazo-He-Yuan (CHY) formulae [31–39] and the related ambitwistor string theories [40–45]; string theory [46–55]; kinematic algebras [56–60]. Alternatively, a double-copy prescription valid directly at the level of classical solutions has been proposed for various models [61–72].

In this thesis, we address the "Gravity = Gauge x Gauge" idea from a different standpoint, exploiting a formalism which is a priori independent from the double copy prescription in the scattering amplitudes context, albeit surely related in some way; we refer to it as *Yang-Mills Squared*. This focusses primarily on the relationship between the symmetries of supersymmetric gauge theories of the Yang-Mills type and those of supergravity; more specifically, it aims at constructing the latter from (a product of) the former, thus finding a "Yang-Mills origin" to the invariances of supergravity. This program is conceptually unified, but practically formed of two halves: indeed, the analysis of the global symmetries on one hand, and the local ones on the other, necessitate two distinct treatments, a fact which is reflected in the structure of this thesis: after a review of the general context in Chapter 2, Chapters 3 and 4 are devoted to global symmetries, while Chapters 5 and 6 are concerned with gauge symmetries.

In Chapter 2, we remind the reader about the two sides in this game: (supersymmetric) Yang-Mills gauge theory and supergravity. After a brief exposition of the necessary concepts and notation relative to *classical* Yang-Mills theory [73], Section 2.2 introduces the idea of supersymmetry by directly discussing minimal $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions: the Yang-Mills gauge potential and one adjoint-valued spinor field come together in a single entity, called a supermultiplet, bound by symmetry transformations which, being generated by a fermionic Noether charge Q carrying half-integer helicity, turns bosons into fermions, and vice versa. For an introduction, see for instance [74–76] The idea of extended (that is, $\mathcal{N} > 1$ in $D = 4$) supersymmetry, generated by multiple supercharges, Q^i, leads naturally to the notion of R-symmetry, formally the automorphism group of the supersymmetry algebra, which extends to an internal symmetry of the whole supermultiplet. This is then used to review the classification [77] of supermultiplets in $D = 4$ in Section 2.2.2.

Yang-Mills theory and, even more so, supergravities are often studied in dimensions different from $D = 4$ and usually ranging between $3 \leq D \leq 11$ for a variety of reasons: arguably the most important of these is the observation that the five consistent (anomaly free) supersymmetric string theories live in 10 spacetime dimensions and their conjectured mysterious "mother" theory, known as *M-theory*, lives in $D = 11$. Careful consideration of the properties of minimal spinors in different dimensions leads to the important classification of the allowed R-symmetry groups, which in turn allows one to classify all possible supermultiplets in these dimensions. As far as the super Yang-Mills multiplets are concerned, the highest dimension admitting any is $D = 10$, where in fact one may find two inequivalent minimal super Yang-Mills theories, owing to the chiral nature of the minimal spinors in this dimension; these have supersymmetry $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (0, 1)$ respectively. Again, see [77].

Finally, in Section 2.3 we introduce the idea of supergravity [78–80] by means of the $\mathcal{N} = 1$ theory in four dimensions, namely the field theoretic realisation of the smallest supergravity multiplet comprising only the metric $g_{\mu\nu}$ and its supersymmetric partner, the gravitino Ψ_μ. The particular simplicity of this model arise, among other things, from the absence of scalar fields in the spectrum; the consequences of the presence of scalar fields in a supergravity theory are presented in Section 2.3.1: specifically, they span a coset space G/H, known as the *scalar manifold*, where G is the group of a non-compact global symmetry acting non-linearly on the $\dim(G/H)$ scalars, while H is its maximal compact subgroup [81–83]. In supergravity, the group G is usually referred to as the U-duality group, owing to its affinity to that of the U-duality in string theory [84], although the latter is the discrete analog of the former. The actions of G and H usually extend to invariances of (the equations of motion of) the whole multiplet, thus not just of the scalar sector, and are thus central in our understanding of supergravity theories; whilst H is the maximal symmetry which is linearly realised on all fields, G only charges the bosons: the scalars, as mentioned, trasform non-linearly, while the vector fields and higher rank p-forms (possibly together with their duals) transform in linear representations. The graviton is invariant under both. The classification of the scalar manifolds appearing in supergravities with at least 8 real supercharges, where they are homogeneous symmetric spaces, is reviewed to conclude this Section. For less supersymmetric theories, the metric of the scalar manifold depends on one or more arbitrary parameters which define a particular geometry on the manifold.

Chapter 3 reviews the Yang-Mills Squared construction proper [85–91]. In an identical fashion as with all incarnations of the double copy, the tensor product of the on-shell states of two (in general different) Yang-Mills vector potentials is decomposed into the rank-2 symmetric, rank-2 antisymmetric and singlet irreducible representations of the little group, which are interpreted as the helicity states for a graviton, a Kalb-Ramond 2-form and a dilaton, namely the states constituting the universal bosonic sector of supergravity theories. We refer to the Yang-Mills theories entering the product as the *Left* and *Right* factors. However, it would be entirely unsatisfactory if the relationship between gauge theory and supergravity were to stop at the simple identification of on-shell helicity states.

In an attempt to go beyond this and towards a Lorentz covariant field theoretic description of the double copy, one expands the tensor product to include a so-called *spectator* scalar field valued in the bi-adjoint of the Left and Right gauge groups, $G \times \tilde{G}$, in such a way that the bona-fide gravitational states may be made singlets under the Yang-Mills gauge groups, as required for consistency. A second important refinement to the standard tensor product adopted in the relevant literature is the assumption that it is a convolution, rather than a simple product, which combines the spacetime dependence of the factors, first introduced in [90]; this is shown in Section 3.1. The convolutive structure determines that it is in fact the convolution inverse of the bi-adjoint scalar which enters the product. The convolution and the spectator are necessary ingredients in deriving the *local* symmetries of supergravity at linear level, which include diffeomorphisms, 2-form gauge invariance and supersymmetry, from the gauge symmetries of the Yang-Mills factors. These results will be expanded upon in Chapter 6.

Similarly, taking the tensor product of two supersymmetric Yang-Mills multiplets with $\mathcal{N}$ and $\tilde{\mathcal{N}}$, respectively, consistently yields the states of a $(\mathcal{N}+\tilde{\mathcal{N}})$-extended supergravity. In Section 3.2, we review how one may relate the existence of the G and H symmetries of supergravities to the underlying internal symmetries of the Yang-Mills factors, often coinciding with their R-symmetry. The tensor product of little group representations determines the content of some supergravity theory; these states are first labelled as representations of $\mathfrak{int} \oplus \tilde{\mathfrak{int}}$, the Lie algebras of the internal Yang-Mills groups; then, whenever needed, there exists an enhancement to a bigger Lie algebra $\mathfrak{h}$ corresponding, in all cases, to the correct H group. The non-compact group G corresponding to H is then uniquely determined if one assumes that the supergravity scalars span a symmetric space. We refer to this procedure as *squaring*. After performing all possible squaring products of pure (as in, not coupled to matter) super Yang-Mills multiplets in dimensions $3 \leq D \leq 10$, which are organised in a *pyramid of supergravities*, we give the explicit example of how, in four spacetime dimensions, $\mathcal{N}=4$ super Yang-Mills squares to $\mathcal{N}=8$ supergravity in Section 3.2.4. Finally, in Section 3.3 we show, through the specific example of half-maximal supergravity, how a generalisation of the squaring procedure is achieved by considering, as inputs in the tensor product, non-supersymmetric Yang-Mills factors and even matter couplings, namely chiral or hypermultiplets. Employing this sort of generalised squaring is then possible to show that a wide variety of theories admit a Yang-Mills factorisation, including virtually all ungauged supergravities coupled to vector [27] and hypermultiplets with homogeneous scalar spaces [91]. Indeed, one of the most interesting open questions of this program, as well as of the double copy in general, is to establish the generality of this phenomenon, which may be summarised in the single question [91]: *are all supergravities Yang-Mills Squared?*

Chapter 4 is devoted to twin supergravities, first classified in [92] and further discussed in [93, 94]. These are pairs of supergravity theories with identical bosonic sectors, but different supersymmetric completions; they are referred to as the big $\mathcal{N}_+$ twin and the little $\mathcal{N}_-$ twin. After reviewing their classification as well as their construction from a common

($\mathcal{N}_+ + \mathcal{N}_-$)-extended "parent" theory in Section 4.1, we show in Section 4.2, following [95], how all supergravity theories admitting a twin also admit a Yang-Mills factorisation. Furthermore, we argue that the *twinness* relation tying them together may be recast in terms of the Yang-Mills factors, thus providing a general prescription for constructing twin theories from Yang-Mills: if the parent supergravity admits a factorisation (it typically lies in the pyramid mentioned above), the big and little twins are obtained by breaking the R-symmetries of the Left and Right factors, respectively. The way this is carried out is by decomposing the R-symmetry to the appropriate smaller group, and by replacing the resulting adjoint-valued spinor multiplet with an identical one living in some different representation of the gauge group. In order to illustrate things, we provide an explicit example, again in four dimensions: the unique $\mathcal{N} = 6$ supergravity twin to $\mathcal{N} = 2$ coupled to 15 vector multiplets. We conclude with a discussion of the few isolated cases which fall outside of the general prescription; this is usually attributed to their not having a parent supergravity lying in the pyramid.

Chapter 5 morally divides this thesis in two halves. While Chapter 3 introduces the framework which we use to study the Yang-Mills origin of the global symmetries of supergravity and Chapter 4 presents one application thereof, the remainder of the present text is devoted to the study of the local invariances of supergravity, at least inasmuch as they may be shown to originate from those of the underlying gauge theories. We argue that the correct way to understand these necessarily requires one to deviate from the purely classical Yang-Mills squared map of the previous chapters and consider instead the Becchi-Rouet-Stora-Tyutin (BRST) form of Yang-Mills [96–102]. Chapter 5 is purely intended as a review of BRST invariance and related concepts: we chose to present this with a historical flavour, namely arriving at showing the "original" BRST invariance of the gauge-fixed Yang-Mills action in Section 5.3 after having performed Faddeev and Popov's gauge-fixing in the path integral formalism [103, 104]. Once BRST invariance is established so, we introduce the related anti-BRST transformations [105–112], which play an interesting role in our construction. Finally, in Section 5.5, we present the BRST-fixed version of the Kalb-Ramond 2-form field [113–125], which is extremely interesting since it constitutes perhaps the least complicated example of a gauge theory not of the Yang-Mills type: while the latter has an irreducible, closed gauge algebra, the former is first-stage reducible. It turns out that the Faddeev-Popov gauge-fixing and associated BRST procedure fail for most theories whose gauge algebra is not closed and irreducible. Then, those with algebras which are open (closed only on-shell), soft (structure functions instead of constants) or reducible (gauge invariances for gauge parameters) are best treated by the more general and encompassing field-antifield or Batalin-Vilkovisky (BV) formalism [126–133]. We give a brief review of it in Appendix B based upon the excellent [134], where we also sketch the derivation of the gauge-fixed action for Yang-Mills and the 2-form, following [135].

Chapter 6 is devoted to the Yang-Mills origin of the local gauge symmetries of $\mathcal{N} = 0$ supergravity, at linearised approximation, as well as of its dynamics. By considering a more general squaring product involving the Faddeev-Popov ghosts, we review the result [6] that the spectrum contains, in addition to the physical graviton, 2-form and dilaton, also the ghosts and ghosts-for-ghosts required for consistency. Going beyond on-shell states, we construct the supergravity *fields* as sums of convolutions of Yang-Mills fields (physical and ghost); we refer to these collectively as the *dictionary*. On acting on these expressions with the Yang-Mills BRST and anti-BRST charges and using the corresponding Yang-Mills transformations and properties of the convolution, we reproduce the expected BRST and anti-BRST variations of the gravitational side. Furthermore, we show that the equations of motion of the gauge theory factors combine through the dictionary to yield the equations of motion describing the gravitational dynamics.

Chapter 7 contains some final remarks about the results, as well as hopes and speculations about future directions. Finally, Appendix A collects some tables which didn't find their place in the main text, Appendix B, as mentioned, briefly discusses the BV formalism and Appendix C contains some more detailed computations relevant to Chapter 6.

Chapter 2

Gauge theory and supergravity

2.1 Yang-Mills theory

Yang-Mills theory and the principle of non-Abelian gauge invariance lie at the heart of our understanding of three of the fundamental forces: the electromagnetic, the weak and the strong forces. Albeit a standard topic, let us briefly remind the reader of its most salient features, to set the stage for the more involved developments of the ensuing chapters and to fix the notation.

The idea of Yang and Mills was to extend the principle of local gauge invariance of Maxwell's theory from the simple local phase rotation enjoyed by electrically charged matter fields, $\psi \to e^{i\alpha(x)}\psi$, to more general, continuously generated groups of transformations, $\mathcal{G}$, acting on collections of fields in some (irreducible) representation. For example, $\mathcal{G}$ acts on a set of N fields ψ^i and their conjugates via its fundamental and conjugate-fundamental representations,

$$\psi \to \psi' = U(x)\psi, \tag{2.1}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}[U(x)]^\dagger \tag{2.2}$$

where, crucially, the action of the group is a function of spacetime, labelled by x^μ. A priori, the matrices $U(x)$ may be elements of any Lie group. Certain requirements imposed by physical considerations – such as real and positive-definite kinetic terms in the resulting Lagrangian density – lead one to restrict to *compact* and *semi-simple* Lie groups. Let us consider here Yang-Mills theory based on the (N^2-1)-dimensional (compact, simple) group $SU(N)$ of $N \times N$ unitary matrices with unit determinant.

Some facts about $SU(N)$ and $\mathfrak{su}(N)$

The matrices $U(x) = U(x)^i{}_j$ furnish the fundamental representation of $SU(N)$, with the additional continuous dependence on x required by the gauge principle. The corresponding (fundamental representation of the) Lie algebra, $\mathfrak{su}(N)$, is spanned by a set of N^2-1

traceless anti-hermitian $N \times N$ matrices $(t_A)^i{}_j$, known as *generators*, related to the corresponding representation of the group by

$$U(x)^i{}_j = e^{-\theta^A(x)(t_A)^i{}_j}. \tag{2.3}$$

For simple Lie algebras, the generators of any representation r, denoted t^r_A, can be chosen to be trace orthogonal with some representation-dependent coefficient, $C(r)$. For the fundamental representation,

$$\mathrm{Tr}(t_A t_B) = -\tfrac{1}{2}\delta_{AB} \tag{2.4}$$

Furthermore, the space of generators is closed under the Lie bracket operation,

$$[t^r_A, t^r_B] = f_{AB}{}^C t^r_C \tag{2.5}$$

where the collection of real numbers $f_{AB}{}^C = -f_{BA}{}^C$, being the same for all representations, is known as the *structure constants* of the algebra. They famously satisfy the Jacobi identity

$$f_{AD}{}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}{}^D + f_{CD}{}^E f_{AB}{}^D = 0. \tag{2.6}$$

Among the infinite number of finite-dimensional inequivalent irreducible group representations, a central role is played by the so-called *adjoint* representation,

$$\begin{aligned} \mathrm{Ad} : SU(N) &\to \mathrm{Aut}(\mathfrak{su}(N)) \\ U &\mapsto \mathrm{Ad}_U \end{aligned} \tag{2.7}$$

mapping elements of $SU(N)$ to automorphisms of the Lie algebra $\mathrm{Ad}_U : \mathfrak{su}(N) \to \mathfrak{su}(N)$, defined by

$$\mathrm{Ad}_U(t_A) = U t_A U^{-1} \in \mathfrak{su}(N). \tag{2.8}$$

The statement that the above lies in the Lie algebra is equivalent to finding some matrix $R(U)^B{}_A$ such that we can write it as a linear combination of generators,

$$U t_A U^{-1} = t_B R(U)^B{}_A \tag{2.9}$$

which implies that the $R(U)^B{}_A$ are the matrices of the adjoint representation. The relevance of all this for field theory lies in the fact that, given a set of $\dim(\mathcal{G})$ fields X^A, we can form the matrices $X = X^A t_A$ as linear combinations of the Lie algebra generators. Then, using (2.9), one can write down the adjoint action of the group, either in matrix notation or in components, as

$$X \to X' = U X U^{-1} \tag{2.10}$$

$$X^A \to X'^A = R(U)^A{}_B X^B. \tag{2.11}$$

The Yang-Mills field, $A^A_\mu(x)$

The analogue of the $U(1)$ gauge field of Maxwell's theory is a collection of $\dim(\mathcal{G})$ Lorentz vectors, A^A_μ. As such, it is convenient to consider them as the entries of the $N \times N$ matrix $A_\mu = A^A_\mu t_A$, which transforms as a gauge field in the adjoint representation of $SU(N)$,

$$A_\mu \to A'_\mu = \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x) A_\mu U(x)^{-1} \tag{2.12}$$

where the Yang-Mills coupling constant, g, is inserted for later convenience. Notice that (2.12) reduces to the known QED transformation in the limit $U \in U(1)$. As in the Abelian case, the non-tensorial gauge transformation of A_μ perfectly interplays with the gauge transformation of derivatives of fields, such that one may define the covariant derivative for a field Φ in a generic representation,

$$D_\mu \Phi = \partial_\mu \Phi + g(\mathrm{d}_A \Phi) \tag{2.13}$$

with the aid of the Lie algebra representations $\mathrm{d}_A = A^A_\mu t_A$. In particular, the above is adapted for matter fields in the fundamental and adjoint representations to

$$D_\mu \psi = \partial_\mu \psi + g A_\mu \psi, \tag{2.14}$$

$$D_\mu X = \partial_\mu X + g[A_\mu, X], \tag{2.15}$$

themselves transforming according as the representation of the field they act on. Thus, they belong to the same representation space and their Lie bracket (itself belonging to the same space) can be written as a linear combination of the generators. The coefficients in this basis are the Lorentz components of an $\mathfrak{su}(N)$-valued 2-form on spacetime $\mathcal{M}$,

$$g^{-1}[D_\mu, D_\nu] \equiv F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{2.16}$$

$$= F^A_{\mu\nu} t_A \tag{2.17}$$

namely, the non-Abelian field strength tensor. This satisfies the Bianchi identity,

$$D_{[\mu} F^A_{\nu\rho]} = 0, \tag{2.18}$$

which is trivially solved by (2.16), as a consequence of the Jacobi identity obeyed by the Lie bracket. Albeit no longer gauge-invariant as in the $U(1)$ case, the field strength may be used to construct an invariant of $SU(N)$, by virtue of its tensorial transformation. The invariant, which relies on the Cartan-Killing form, reads

$$\mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{2} F^A_{\mu\nu} F^{\mu\nu}_A \tag{2.19}$$

where we used (2.4). Finally, we can write the gauge-invariant Yang-Mills action:

$$S[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^D x \left[\frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) - \bar{\psi}(\gamma^\mu D_\mu - m)\psi \right]. \tag{2.20}$$

Infinitesimal (Lie algebra) transformations

It is often easier to work with Lie algebra, rather than group, representations. Thus, expanding $U = \exp(-\theta^A(x)t_A)$ to first order in $\theta(x)$, we get the transformations of the fundamental matter fields under group elements infinitesimally close to the identity,

$$\delta\psi^i = -\theta^A(x)(t_A)^i{}_j\psi^j \tag{2.21}$$

$$\delta\psi^*_i = \theta^A(x)\psi^*_j(t_A)^j{}_i. \tag{2.22}$$

For the adjoint representation, we perform the same expansion in (2.9), to obtain

$$\begin{aligned} R(U)^A{}_B &= \delta^A{}_B + \theta^C(x) f_{BC}{}^A + \mathcal{O}(\theta^2) \\ &\overset{!}{=} e^{-\theta^C(x)(t^{\mathrm{adj}}_C)^A{}_B} \end{aligned} \tag{2.23}$$

i.e. the structure constants generate the adjoint representation, $(t^{\mathrm{adj}}_C)^A{}_B = f_{CB}{}^A$. Consequently, under infinitesimal (Lie algebra) unitary transformations, the X^A transform as

$$\delta X^A = f_{BC}{}^A X^B \theta^C(x). \tag{2.24}$$

The gauge components of the gauge field get rotated by

$$\delta A^A_\mu = \frac{1}{g}\partial_\mu\theta^A(x) + f_{BC}{}^A A^B_\mu\theta^C(x) \tag{2.25}$$

$$= \frac{1}{g}\Big(D_\mu\theta(x)\Big)^A. \tag{2.26}$$

where the appearance of the covariant derivative of the Lie algebra-valued function $\theta(x)$ manifestly ensures that this gauge transformation consistently maps $\mathfrak{su}(N) \to \mathfrak{su}(N)$.

2.2 Global supersymmetry

2.2.1 Super-Yang-Mills in four dimensions

A very interesting extension of the theory defined by the action functional (2.20) occurs[1] when one takes the fermions coupled to the Yang-Mills gauge potential to be massless minimal spinors transforming in the adjoint representation of the gauge group $\mathcal{G}$. Since the properties of spinors are particular to the specific spacetime dimension chosen, let us specialise for clarity to $D = 4$ where a minimal spinor may be taken to be either a four-component $\mathbb{R}$-valued Majorana spinor or, equivalently, a (conjugate) pair of 2-dimensional $\mathbb{C}$-valued Weyl spinors. Since, as explained at the end of this section, the spinors in this theory ought to be in the same gauge group representation as the Yang-Mills potential,

[1] This statament is strictly true in dimensions $D = 3, 4, 6, 10$, where supersymmetry may be established with just a vector and a spinor. In the remaining dimensions, this requires additional scalars in the multiplet.

namely the (real) adjoint, let us write the following using Majorana spinors. Thus, the action reads

$$S[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left[-\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A - \frac{1}{2} \bar{\psi}^A \gamma^\mu (D_\mu \psi)_A \right]. \tag{2.27}$$

where we have re-instated the gauge indices for clarity. Crucially, this can be seen to be invariant, up to a total derivative, under the *supersymmetry* transformations

$$\delta_\epsilon A^A_\mu = -\frac{1}{2} \bar{\epsilon} \gamma_\mu \psi^A, \tag{2.28}$$

$$\delta_\epsilon \psi^A = \frac{1}{4} \gamma^{\rho\sigma} F^A_{\rho\sigma} \epsilon, \tag{2.29}$$

where ϵ is a *global* anticommuting Majorana spinor. The novelty lies in the fact that they "rotate" bosons into fermions, and viceversa. This may be understood by realising that they are generated, in the sense explained in (2.32), by spinorial charges carrying half-integer helicity. Indeed, varying the action under these with ϵ a spacetime-dependent function yields, as usual, the form of the Noether *super*current, the vector-spinor quantity

$$J^\mu = \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu \psi_A, \tag{2.30}$$

which is conserved by virtue of the Bianchi identity of the potential and the Fierz spinor identities in $D = 4$. Its spatial integral yields the conserved spinor supercharge,

$$Q_\alpha = \int \mathrm{d}^3x \; J^0_\alpha(\vec{x}, t), \tag{2.31}$$

which generates the infinitesimal supersymmetry variations via Poisson brackets at the classical level or, with an eye towards quantisation, via the algebraic commutators as in

$$\begin{aligned} \delta_\epsilon A^A_\mu &= [\bar{\epsilon} Q, A^A_\mu] \\ \delta_\epsilon \psi^A &= [\bar{\epsilon} Q, \psi^A]. \end{aligned} \tag{2.32}$$

The fact that the supercharge transforms in the spinor representation of the Lorentz group suggests that they generate neither a conventional spacetime symmetry nor an internal one, commuting with the Poincaré generators. This statement is formalised by giving the explicit form of the algebra spanned by the symmetry generators. Very conveniently, its exact form may be computed directly from the infinitesimal transformations: on applying two such variations in succession on any field ϕ in the two possible orderings, namely $[\delta_1, \delta_2]\phi = \delta_1(\delta_2 \phi) - \delta_2(\delta_1 \phi)$, where we defined $\delta_i := \delta_{\epsilon_i}$. One computes, for the theory at hand,

$$\begin{aligned} [\delta_1, \delta_2] A^A_\mu &= -\frac{1}{2} \bar{\epsilon}_1 \gamma^\nu \epsilon_2 F^A_{\nu\mu}, \\ [\delta_1, \delta_2] \psi^A &= -\frac{1}{2} \bar{\epsilon}_1 \gamma^\nu \epsilon_2 (D_\nu \psi)^A + \alpha \frac{\delta S}{\delta \psi}(A). \end{aligned} \tag{2.33}$$

Thus, the commutator of successive transformations yields, in both cases, a covariant translation[2] by $a^\nu \equiv -\bar{\epsilon}_1\gamma^\nu\epsilon_2/2$ on ϕ, as required by the gauge covariance of the LHS, up to a term proportional, for $\alpha \in \mathbb{R}$, to the equations of motion of the spinor. An algebra of this type, which requires using the equations of motion, is said to be *on-shell closed*. Using (2.32) to re-express $[\delta_1, \delta_2]$, one reads off the algebra obeyed by the supercharges,

$$\{Q_\alpha, \overline{Q}^\beta\} = -\frac{1}{2}(\gamma^\mu)_\alpha{}^\beta P_\mu, \tag{2.34}$$

where the anti-commutator appears on stripping away the anticommuting spinor parameters and P_μ is the conserved Noether charge associated with translations in spacetime. The other (anti)-commutators involving Q are

$$[P_\mu, Q_\alpha] = 0, \tag{2.35}$$

$$[J_{\mu\nu}, Q_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta, \tag{2.36}$$

where the first one implies that a supersymmetry transformation does not change the 4-momentum, and hence the mass, of the field it acts on, while the second one re-iterates the fact that the supercharge Q indeed transforms in the spinor representation of the Lorentz group, generated by the skew-symmetric product of Clifford gamma matrices, $\gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$. Together with the usual commutators defining the bosonic Poincaré algebra, the conditions (2.34) - (2.36) define the so-called $\mathcal{N} = 1$ super-Poincaré algebra, where $\mathcal{N}$ counts the number of conserved supercharges. It is an instance of a graded algebra, with even (viz. bosonic, B) and odd (viz. fermionic, F) elements whose commutators obey the pattern $[B, B] = B$, $[B, F] = F$ and $\{F, F\} = B$. The algebra above remains invariant under the chiral $\mathfrak{u}(1)$ transformation of Q,

$$\delta Q_\alpha = [T_R, Q_\alpha] = -i(\gamma_*)_\alpha{}^\beta Q_\beta \tag{2.37}$$

where γ_* is the highest rank gamma matrix and T_R, the generator of the symmetry motion, commutes with the generators of the Poincaré subalgebra, thus describing an internal symmetry. This symmetry is known as *R-symmetry*, owing to its status as the standalone internal symmetry with a non vanishing commutator with the super-Poincaré generators. It is realised in the field theory of the $\mathcal{N} = 1$ Yang-Mills multiplet, as a $\mathfrak{u}(1)$ rotation of the fermions,

$$\delta\psi^A = ir\gamma_*\psi^A \tag{2.38}$$

under which the action is invariant. Notice the fact, which is true in general, that the Yang-Mills potential is in the trivial representation of the R-symmetry. Because of this, one can set its R-charge to zero, and consistency with (2.37) implies that $r = 1$ in (2.38). All other internal symmetries, commuting with Poincaré, also commute with the supercharge; since this includes the internal gauge group, this explains why we had to take the fermions in the same gauge group representation as A_μ^A.

[2]Namely, the result is gauge-covariant and it differs from the usual translation by a field-dependent gauge transformation.

Extended supersymmetry

Equation (2.34) represents the simplest possible example of a supersymmetry algebra. In general, there may be multiple sets of supersymmetry transformations leaving the action invariant, leading to the existence of a number of conserved supercharges, denoted by Q^i_α with $i = 1, \dots, \mathcal{N}$. If this is the case, the theory is said to possess *extended supersymmetry* [77]. In $D = 4$, the superalgebra for $\mathcal{N} > 1$ is most conveniently described in terms of complex Weyl spinors (in four component notation), denoted by $P_{L/R}Q_i$ making use of the projection operators

$$P_{L/R} = \frac{1}{2}\left(\mathbb{1} \pm \gamma_*\right). \tag{2.39}$$

The anticommutator is only non-vanishing between supercharges of different chirality; thus, lowering the spinor index, the (Q, Q) anticommutators may be written as

$$\{Q_{\alpha i}, Q^j_\beta\} = -\frac{1}{2}\delta^j_i (P_L\gamma^\mu)_{\alpha\beta} P_\mu \tag{2.40}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = 0 \tag{2.41}$$

where the compact notation for the chiral supercharges, $Q_i \equiv P_L Q_i$ and $Q^i \equiv P_R Q^i$, has been adopted. The remaning two anti-commutators follow by charge conjugation. Unsurprisingly, the $\mathfrak{u}(1)$ R-symmetry discussed above is but the limiting $\mathcal{N} = 1$ case of a more general internal symmetry rotating the charges, which acts as

$$[T_R, Q_{\alpha i}] = (U_A)_i{}^j Q_{\alpha j}, \qquad [T_R, Q^i_\alpha] = (U_A)^i{}_j Q^j_\alpha \tag{2.42}$$

on the two types of charges. Demanding that these commutators are compatible with the superalgebra, in particular with its super-Jacobi identities, imposes that the matrices U_A are anti-Hermitian and indeed span a representation of the R-symmetry algebra, which is thus fixed to $\mathfrak{u}(\mathcal{N})$.

2.2.2 Particle representations

Having restricted to $D = 4$ is also convenient to discuss the nature of the massless[3] representations of the supersymmetry algebra. Although we arrived at the supersymmetry algebra basing our discussion on the Yang-Mills theory of a gauge vector, we will see that the following general considerations lead to a classification of supersymmetric theories in $D = 4$ much beyond those of the Yang-Mills type. The key observation is that, upon quantisation, the generators $Q_{\alpha i}$ become operators acting on a Hilbert space. In particular, one finds a subset of them carrying helicity $\pm 1/2$ and acting as creation/annihilation operators on the space of states $|h\rangle$, labelled only by the helicity since the momentum/energy of all states may be fixed from the onset, owing to it being invariant under the action of Q. Thus, fixed a "highest helicity state" $|h_0\rangle$, in the kernel of the annihilation operator, one

[3] This thesis will be concerned exclusively with massless fields.

may create states with helicity $h_0 - m/2$ by acting m times with the creation operators, denoted here by $Q^{\dagger i}$, which lower the helicity by $1/2$:

$$\begin{aligned}
&|h_0\rangle \\
&|h_0 - \tfrac{1}{2}; i\rangle = Q^{\dagger i}\,|h_0\rangle \\
&|h_0 - 1; [ij]\rangle = Q^{\dagger i} Q^{\dagger j}\,|h_0\rangle\,, \\
&\ \vdots
\end{aligned} \tag{2.43}$$

where the R-symmetry indices are antisymmetrised as a consequence of the anticommutativity of alike charges, cf. equation (2.41): therefore, the states $|h_0 - m/2\rangle$ transform in the rank-m antisymmetric irreducible representation of the R-symmetry algebra. Thus, there are $\binom{\mathcal{N}}{m}$ states of helicity $h_0 - m/2$, and the sequence ends when the single lowest helicity state $|h_0 - \mathcal{N}/2; [ij \ldots \mathcal{N}]\rangle$ is reached. Notice that there is a total of $2^{\mathcal{N}}$ states in a massless irreducible representation, which goes by the name of *supermultiplet*. These are always divided equally into $2^{\mathcal{N}-1}$ bosonic states with integer helicity and the same number of fermionic states with half-integer helicity.

Furthermore, in general, CPT symmetry demands that the conjugate sequence is also added, which amounts to repeating the steps above starting with the lowest helicity state and using the operator which raising the helicity. Exceptions to this rule are the cases for which $\mathcal{N} = 4|h_0|$, whereby $|-h_0\rangle = |h_0 - \mathcal{N}/2\rangle$, that is the negative helicity counterpart of the initial state coincides with the singlet obtained after $\mathcal{N}$ steps. This highlights the all-important fact that for $\mathcal{N} > 4|h_0|$ this algorithm would produce states with $|h| > |h_0|$, contradicting the original assumption of $|h_0\rangle$ as the maximal helicity state. Therefore, in $D = 4$,

- multiplets with $|h_0\rangle = |\frac{1}{2}\rangle$ exist up to $\mathcal{N} = 2$,
- multiplets with $|h_0\rangle = |1\rangle$ exist up to $\mathcal{N} = 4$,
- multiplets with $|h_0\rangle = |2\rangle$ exist up to $\mathcal{N} = 8$.

In principle, this series could continue for all values of h_0. However, fields with spins higher than $h = 2$ are usually not included in physical models due to the apparently insurmountable difficulty in writing down a consistent theory describing their interactions; on physical grounds, then, the series stops here.

Let us illustrate this more concretely. We will denote states by their representations under $\mathfrak{u}(1)_{st} \oplus \mathfrak{u}(\mathcal{N})$, with the superscript denoting the helicity and the subscript the $\mathfrak{u}(1) \subset \mathfrak{u}(\mathcal{N})$ R-symmetry charge. Thus, for notational clarity, we will henceforth only consider integer values of $\mathfrak{u}(1)$ charges in order to avoid fractions appearing as subscripts or superscripts, which calls for a redefinition of the charge under $\mathfrak{u}(1)_{st}$ as twice the helicity, $s = 2h$.

- **Vector multiplets**, $|s_0\rangle = |2h_0\rangle = 2$

Consider the case in which a vector A_μ^A is taken as the highest helicity state, which in our modified notation means $|s_0\rangle = |2\rangle$, and where the number of real conserved charges is $\mathcal{Q} = 8$, which in four dimensions is equivalent to $\mathcal{N} = 2$. In turn, by the discussion above, this fixes the R-symmetry to $\mathfrak{u}(2)$. The multiplet is built by acting with $Q^{\dagger i}$, transforming as the $\mathbf{2}_1^{-1}$ of $\mathfrak{u}(1)_{st} \oplus \mathfrak{u}(2)$, on the positive helicity state of the Yang-Mills vector, transforming as the $\mathbf{1}_0^2$, and extracting the antisymmetric part of the tensor product. Pictorially, this reads

$$\begin{array}{rcccl} & \mathbf{1}_0^2 & & \mathbf{1}_0^{-2} & \\ (\Lambda Q^\dagger) & \mathbf{2}_1^1 & + & \mathbf{2}_{-1}^{-1} & (\Lambda Q) \\ (\Lambda^2 Q^\dagger) & \mathbf{1}_2^0 & & \mathbf{1}_{-2}^0 & (\Lambda^2 Q) \end{array} \tag{2.44}$$

Note that the supermultiplets, in most of the literature and thus in this thesis, owe their name to the state with highest helicity. Consequently, the multiplet above is referred to as $\mathcal{N} = 2$ *vector multiplet*, or concisely as $\mathbf{V}_2$. On gathering the states with the same $|s|$, one obtains

$$\begin{aligned} \mathbf{V}_2 &= (A_\mu^A, \lambda^i, \phi^a) \\ &= \{\mathbf{1}_0^2 + \mathbf{1}_0^{-2}, \mathbf{2}_1^1 + \mathbf{2}_{-1}^{-1}, \mathbf{1}_2^0 + \mathbf{1}_{-2}^0\}. \end{aligned} \tag{2.45}$$

Another example of interest is furnished by $\mathcal{N} = 4$ super-Yang-Mills theory, or $\mathbf{V}_4$, which exemplifies the self-conjugate property exhibited by multiplets with $\mathcal{N} = 4|h_0| = 2|s_0|$ discussed above. Repeating the procedure above leads to

$$\begin{array}{rc} & \mathbf{1}_0^2 \\ (\Lambda Q^\dagger) & \mathbf{4}_1^1 \\ (\Lambda^2 Q^\dagger) & \mathbf{6}_2^0 \\ (\Lambda^3 Q^\dagger) & \overline{\mathbf{4}}_3^{-1} \\ (\Lambda^4 Q^\dagger) & \mathbf{1}_4^{-2} \end{array} \tag{2.46}$$

where the CPT conjugate states need not be added. Notice how, as a consequence of its self-conjugacy, the $\mathfrak{u}(1) \subset \mathfrak{u}(2)$ charges are not consistent, which implies that this multiplet cannot support the Abelian factor, which needs to be dropped. The R-symmetry in this case is reduced to $\mathfrak{su}(4)$ and the multiplet reads

$$\mathbf{V}_4 = \{\mathbf{1}^2 + \mathbf{1}^{-2}, \mathbf{4}^1 + \overline{\mathbf{4}}^{-1}, \mathbf{6}^0\}. \tag{2.47}$$

- **Supergravity multiplets**, $|s_0\rangle = 4$

Everything goes through as above if the initial "vacuum" state is taken with helicity 2, or $|s_0\rangle = 4$, which describes the positive helicity state of a graviton, $h_{\mu\nu}$. Doing so results in the so-called *supergravity multiplets*, containing a single graviton, invariant under the R-symmetry, $\mathcal{N}$ gravitini in the defining, as well as vectors, spinors and scalars in the rank-m antisymmetric representations. These theories will be introduced in detail in Section 2.3. Let us emphasise, in particular, the case of $\mathcal{N} = 8$ supergravity, which is maximal in four dimensions: as such, by the same argument as for $\mathbf{V}_4$, it possesses $\mathfrak{su}(8)$ R-symmetry, rather than $\mathfrak{u}(8)$, under which its content is organised as

$$\mathbf{G}_8 = \{\mathbf{1}^4 + \mathbf{1}^{-4}, \mathbf{8}^3 + \overline{\mathbf{8}}^{-3}, \mathbf{28}^2 + \overline{\mathbf{28}}^{-2}, \mathbf{56}^1 + \overline{\mathbf{56}}^{-1}, \mathbf{70}^0\}. \tag{2.48}$$

- **Chiral and hyper-multiplets**, $|s_0\rangle = 1$

  Finally, it is worth spending a few words on those cases where the highest helicity state is a spinor, $|s_0\rangle = 1$. As indicated earlier, the maximal amount of supersymmetry allowed is $\mathcal{Q} = 8$; therefore, the only possible cases are for $\mathcal{N} = 1, 2$. Since they present some subtleties, it is convenient to discuss them separately. The $\mathcal{N} = 1$ multiplet, known as the *chiral* multiplet[4], contains one Majorana spinor and a complex scalar, transforming under $\mathfrak{u}(1)_{st} \oplus \mathfrak{u}(1)$ as

$$\begin{aligned}\mathbf{C}_1 &= (\lambda, \phi) \\ &= \{(1, r) + (-1, -r), (0, r-1) + (0, -r+1)\},\end{aligned} \tag{2.49}$$

  where the $\mathfrak{u}(1)$ charge is not determined by the content alone, but an analysis of the specific model and its interactions (in particular its superpotential) is needed to determine the exact value of r. The $\mathcal{N} = 2$ case, known as the *hypermultiplet*, possesses more structure: despite being maximal, hence self-conjugate, it admits a doubling (and the extra $\mathfrak{u}(1)$ of the R-symmetry) owing to the fact that the singlet spinor may carry an arbitrary r charge, as opposed to the singlet vector of $\mathbf{V}_4$ which cannot be charged under the R-symmetry. Therefore, it contains two Majorana spinors and two complex scalars, transforming under $\mathfrak{u}(1)_{st} \oplus \mathfrak{u}(2)$ as

$$\mathbf{H}_2 = \{\mathbf{1}^1_r + \mathbf{1}^{-1}_{-r},\ \mathbf{2}^0_{r+1} + \mathbf{2}^0_{-(r+1)}, \mathbf{1}^{-1}_{r+2} + \mathbf{1}^1_{-(r+2)}\}. \tag{2.50}$$

  Notice, however, that in the case with $r = -1$ the content simply looks like two copies of $\{\mathbf{1}^1_{-1},\ \mathbf{2}^0_0,\ \mathbf{1}^{-1}_1\}$, which calls for the enhancement of the global symmetries to include an extra $\mathfrak{sp}(1)$ rotating the two identical submultiplets. This extends to an $\mathfrak{sp}(n)$ for n hypermultiplets. Here, one is confronted with a choice: (i) interpret the additional $\mathfrak{sp}(n)$ as a global symmetry, whereby the multiplet trasforms in the overall real representation under $\mathfrak{u}(1)_{st} \oplus \mathfrak{sp}(n) \oplus \mathfrak{u}(2)$,

$$\mathbf{H}^A_2 = \{(\mathbf{n}, \mathbf{1})^1_{-1},\ (\mathbf{n}, \mathbf{2})^0_0,\ (\mathbf{n}, \mathbf{1})^{-1}_1\}. \tag{2.51}$$

  For consistency, then, $\mathbf{H}_2$ must be in a real representation of the gauge group, e.g. the adjoint as indicated by the upper case index A above. Otherwise, (ii) interpret

[4]Notably, this is also known as the Wess-Zumino multiplet. They described it in 1974 [136], when it represented the first instance of a supersymmetric field theory.

the $\mathfrak{sp}(n)$ as the gauge group, under whose action the multiplet transforms in the pseudoreal defining representation $\mathbf{n}$, denoted by the lowercase index a. This implies that, this time, the overall real multiplet is simply

$$\mathbf{C}_2^a = \{\mathbf{1}_{-1}^{1},\ \mathbf{2}_0^0,\ \mathbf{1}_1^{-1}\}, \tag{2.52}$$

known as the *half-hypermultiplet* in the literature, with the same content as the chiral $\mathbf{C}_1$, only this time transforming under an internal $\mathfrak{u}(2)$ rather than a $\mathfrak{u}(1)$.

The complete classification of massless irreducible supermultiplets in four dimensions is given in Table 2.1, while the original and complete analysis can be found in [77].

$\mathcal{Q}$	Name	Content	R-symmetry
32	$\mathbf{G}_8$	$\mathbf{1}^4 + \mathbf{8}^3 + \mathbf{28}^2 + \mathbf{56}^1 + \mathbf{70}^0 + \overline{\mathbf{56}}^{-1} + \overline{\mathbf{28}}^{-2} + \overline{\mathbf{8}}^{-3} + \mathbf{1}^{-4}$	$\mathfrak{su}(8)$
24	$\mathbf{G}_6$	$\mathbf{1}_0^4 + \mathbf{6}_1^3 + (\mathbf{15}_2^2 + \mathbf{1}_6^{-2}) + (\mathbf{20}_3^1 + \overline{\mathbf{6}}_5^{-1}) + \overline{\mathbf{15}}_4^0 + c.c.$	$\mathfrak{u}(6)$
20	$\mathbf{G}_5$	$\mathbf{1}_0^4 + \mathbf{5}_1^3 + \mathbf{10}_2^2 + (\overline{\mathbf{10}}_3^1 + \mathbf{1}_5^{-1}) + \overline{\mathbf{5}}_4^0 + c.c.$	$\mathfrak{u}(5)$
16	$\mathbf{G}_4$	$\mathbf{1}_0^4 + \mathbf{4}_1^3 + \mathbf{6}_2^2 + \overline{\mathbf{4}}_3^1 + \mathbf{1}_4^0 + c.c.$	$\mathfrak{u}(4)$
″	$\mathbf{V}_4$	$\mathbf{1}^2 + \mathbf{4}^1 + \mathbf{6}^0 + \overline{\mathbf{4}}^{-1} + \mathbf{1}^{-2}$	$\mathfrak{su}(4)$
12	$\mathbf{G}_3$	$\mathbf{1}_0^4 + \mathbf{3}_1^3 + \overline{\mathbf{3}}_2^2 + \mathbf{1}_3^1 + c.c.$	$\mathfrak{u}(3)$
8	$\mathbf{G}_2$	$\mathbf{1}_0^4 + \mathbf{2}_1^3 + \mathbf{1}_2^2 + c.c.$	$\mathfrak{u}(2)$
″	$\mathbf{V}_2$	$\mathbf{1}_0^2 + \mathbf{2}_1^1 + \mathbf{1}_2^0 + c.c.$	$\mathfrak{u}(2)$
″	$\mathbf{H}_2$	$\mathbf{1}_r^1 + \mathbf{2}_{r+1}^0 + \mathbf{1}_{r+2}^{-1} + c.c.$	$\mathfrak{u}(2)$
″	$\mathbf{C}_2^a$	$\mathbf{1}_{-1}^1 + \mathbf{2}_0^0 + \mathbf{1}_1^{-1}$	$\mathfrak{u}(2)$
4	$\mathbf{G}_1$	$(4,0) + (3,1) + c.c.$	$\mathfrak{u}(1)$
″	$\mathbf{V}_1$	$(2,0) + (1,1) + c.c.$	$\mathfrak{u}(1)$
″	$\mathbf{C}_1$	$(1,r) + (0,r+1) + c.c.$	$\mathfrak{u}(1)$

Table 2.1: All allowed supermultiplets in $D = 4$. Note that $\mathbf{G}_7 = \mathbf{G}_8$ and $\mathbf{V}_3 = \mathbf{V}_4$, and the half-hypermultiplet $\mathbf{C}_2^a$ exists on its own only for pseudoreal representations of the gauge group.

Representations in various dimensions

The supermultiplets discussed in $D = 4$ are related, through toroidal dimensional reduction, to their counterparts in various other dimensions. Since the central object in the construction of irreducible representations is the spinorial supercharge Q_α, one needs to be mindful of the properties of (minimal) spinors in each dimension, in order to correctly re-distribute the degrees of freedom (read $\mathcal{Q}$). The type and number of components of minimal spinors in dimensions $3 \leq D \leq 11$, given in Table 2.2, are obtained by checking, for each D, which conditions may be applied to the Dirac representation in order to obtain a more fundamental one.

- In odd dimensions, one may impose (a version of) the Majorana reality condition, which halves the number of independent real components. In $D = 3, 9, 11$, this is the whole story and the minimal spinors are referred to simply as *Majorana*. In $D = 5, 7$, this condition can only be imposed on pairs of spinors: interpreting these pairs as the "minimal spinors" leads to these being referred to as *symplectic*.

- In even dimensions, to complicate things, the Dirac spinor representation is fully reducible, in an entirely analogous fashion to the Weyl representation in $D = 4$. Here, looking for minimal spinors amounts to checking whether the two chiral components are independently Majorana or, conversely, whether the two Weyl component are rotated into each other under charge conjugation, that is

$$(P_{L/R}\psi)^C = P_{L/R}\psi, \qquad \text{vs} \qquad (P_{L/R}\psi)^C = P_{R/L}\psi. \tag{2.53}$$

  The first case is true in its simplest form in $D = 10$ (hence a minimal spinor is *Majorana-Weyl*), while in $D = 6$ it is true again on a pair of (Weyl) spinors (hence *symplectic-Weyl*). On the other hand, the second is true in $D = 4, 8$, where working with a 4-component Majorana spinor or two 2-component Weyl spinors gives rise to equivalent physics.

Note that, although we restrict to $3 \leq D \leq 11$ on physical grounds, the reality properties of minimal spinors are the same for D and $D + 8$, giving rise to an infinite sequence $M, M, S, SW, S, M, M, MW, \ldots$ displaying the so-called *Bott periodicity*. This analysis leads to finding the number of independent components listed in Table (2.2). For an extensive treatment of the properties of spinors in various dimensions, see [80]. In any given spacetime dimension, once the reality properties of the minimal spinors, and thus of the supersymmetry generator Q_α, have been established, one may construct the relevant supersymmetry algebra and read off the conditions on the matrices U_R of the defining representation of the R-symmetry via $[T_R, Q_{\alpha i}] = (U_R)_i{}^j Q_{\alpha j}$. Doing so yields the R-symmetry algebras listed in the last column of Table (2.2). Notice that in $D = 6, 10$, where truly chiral spinors exist, the supersymmetries are denoted as $(\mathcal{N}_L, \mathcal{N}_R)$. For notational consistency, we keep using $\mathcal{N}$ to denote the (chiral) supersymmetries also in $D = 5, 6, 7$, counting the number of minimal spinors, rather than the number of symplectic pairs; indeed, the symplectic nature of the spinors has favoured the use of $\hat{\mathcal{N}} := \mathcal{N}/2$ in the literature. At this point, the knowledge of the $D = 4$ supermultiplets, together with the relations between multiplets in different dimensions implied by dimensional reduction, is enough to characterise all allowed theories. In particular,

D	Type	$\mathcal{Q}$	R-symmetry
11	M	32	$\mathfrak{so}(\mathcal{N})$
10	MW	16	$\mathfrak{so}(\mathcal{N}_L) \oplus \mathfrak{so}(\mathcal{N}_R)$
9	M and D odd	16	$\mathfrak{so}(\mathcal{N})$
8	M and D even	16	$\mathfrak{u}(\mathcal{N})$
7	S	16	$\mathfrak{usp}(\mathcal{N})$
6	SW	8	$\mathfrak{usp}(\mathcal{N}_L) \oplus \mathfrak{usp}(\mathcal{N}_R)$
5	S	8	$\mathfrak{usp}(\mathcal{N})$
4	M and D even	4	$\mathfrak{u}(\mathcal{N})$
3	M and D odd	2	$\mathfrak{so}(\mathcal{N})$

Table 2.2: Properties of minimal spinors in $3 \leq D \leq 11$. The last column lists the type of internal symmetry algebra that may act on them in the defining representation.

- *Supergravity multiplets* $\mathbf{G}_{\mathcal{N}}$ *exist up to* $D = 11$.

  In $D = 12$, where a Dirac spinor carries $2^6 = 64$ complex components and only *either* the Majorana *or* the Weyl conditions may be imposed (same as $D = 4$ by Bott periodicity), a minimal spinor carries 64 independent real components. The multiplet obtained by dimensionally reducing to $D = 4$ would thus violate the physical bound of maximum helicity. This latter "physical" upper bound of $\mathcal{Q} = 32$ real supercharges, saturated in four dimensions by $\mathcal{N} = 8$ supergravity, is saturated in $D = 11$ by the minimal *unique* $\mathcal{N} = 1$ theory, $\mathbf{G}_1$, where a single Majorana spinor carries 32 independent real components.

- *Vector multiplets* $\mathbf{V}_{\mathcal{N}}$ *exist up to* $D = 10$.

  By the same reasoning, uplift of the $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions, with $\mathcal{Q} = 16$, is possible all the way to $D = 10$, where a single Majorana-Weyl spinor carries 16 independent real components. Since the supersymmetries are truly chiral, there exist two inequivalent vector multiplets in $D = 10$: the $\mathbf{V}_{(1,0)}$ and the $\mathbf{V}_{(0,1)}$.

- *Chiral multiplets* $\mathbf{C}_1$ *exist up to* $D = 4$*, (half-)hypermultiplets* $\mathbf{H}$ *up to* $D = 6$*.*

  Since chiral multiplets are defined for $\mathcal{Q} = 4$ and a minimal spinor in $D = 5$ already carries 8 components, chiral multiplets cannot uplift beyond $D = 4$. On the other hand, as it is clear from the entries of Table (2.2), hypermultiplets exist up to $D = 6$. Similarly to the case of vector multiplets, the maximal dimensions admitting hypermultiplets has chiral spinors, which in turn implies that there exist two distinct, inequivalent hypermultiplets, denoted $\mathbf{H}_{(2,0)}$ and $\mathbf{H}_{(0,2)}$.

- *Tensor multiplets* $\mathbf{T}_{(\mathcal{N}_L,\mathcal{N}_R)}$ *exist only in* $D = 6$*.*

  These are multiplets containing antisymmetric tensors $T_{\mu\nu}$. While in principle these exist in lower dimensions as well, the rank-2 antisymmetric representation of the little group is dual to the singlet and the vector representations in $D = 4, 5$ respectively. Hence, under certain assumptions (such as standard kinetic terms, or the requirement that they be in the adjoint representation of the gauge group in $D = 5$) these are already captured by the matter and vector multiplets above. In $D = 6$, however, a 2-form is (anti) self-dual, living in the $(\mathbf{3}, \mathbf{1})$ or in the $(\mathbf{1}, \mathbf{3})$ of the little group algebra $2\mathfrak{usp}(2)$, thus distinct from other representations. Tensor multiplets may carry 16 or 8 real supercharges, resulting in the multiplets $\mathbf{T}_{(4,0)}$, $\mathbf{T}_{(2,0)}$ and their chiral counterparts.

All the possible supermultiplets[5] in $4 \leq D \leq 11$ are listed in Tables A.5, A.6 and A.7. Since the Yang-Mills multiplets will play a privileged role in the remainder of this thesis, they are also listed together in Table 2.3.

2.3 Supergravity

The existence of such representations of the supersymmetry algebra does not necessarily imply the existence of a field theory realisation with the corresponding content. For example, we did not include multiplets with highest helicity state carrying $h = 3/2$ in the classification of $D = 4$ multiplets above, since these are not realised as field theories on their own. The spin 3/2 field, also known as the *Rarita-Schwinger* field, has flat-space dynamics specified by the action,

$$S = -\int \mathrm{d}^4x \; \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho. \tag{2.54}$$

It constitutes the four-dimensional realisation of the *gravitino* field, whose name helps anticipate its role in physical models as the supersymmetric partner of the spin-2 field, the graviton. Together, they form the backbone of what we defined previously as a

[5]There is the additional possibility of so-called *conformal multiplets* in $D = 6$, which are non-gravitational analogs of the supergravity multiplets containing higher tensor fields. We do not include these here.

D	Little group	$\mathcal{Q}=16$	$\mathcal{Q}=8$	$\mathcal{Q}=4$	$\mathcal{Q}=2$
10	$\mathfrak{so}(8)$	$\mathbf{V}_{(1,0)}$ $\varnothing$			
9	$\mathfrak{so}(7)$	$\mathbf{V}_1$ $\varnothing$			
8	$\mathfrak{su}(4)$	$\mathbf{V}_1$ $\mathfrak{u}(1)$			
7	$\mathfrak{usp}(4)$	$\mathbf{V}_2$ $\mathfrak{usp}(2)$			
6	$\mathfrak{usp}(2)\oplus\mathfrak{usp}(2)$	$\mathbf{V}_{(2,2)},\ \mathbf{T}_{(4,0)}$ $2\mathfrak{usp}(2),\mathfrak{usp}(4)$	$\mathbf{V}_{(2,0)},\ \mathbf{T}_{(2,0)}$ $\mathfrak{usp}(2)$		
5	$\mathfrak{usp}(2)$	$\mathbf{V}_4$ $\mathfrak{usp}(4)$	$\mathbf{V}_2$ $\mathfrak{usp}(2)$		
4	$\mathfrak{u}(1)$	$\mathbf{V}_4$ $\mathfrak{su}(4)$	$\mathbf{V}_2$ $\mathfrak{u}(2)$	$\mathbf{V}_1$ $\mathfrak{u}(1)$	
3	$\varnothing$	$\mathbf{V}_8$ $\mathfrak{so}(8)$	$\mathbf{V}_4$ $3\mathfrak{so}(3)$	$\mathbf{V}_2$ $2\mathfrak{so}(2)$	$\mathbf{V}_1$ $\varnothing$

Table 2.3: All super Yang-Mills multiplets in $3 \leq D \leq 10$, together with their R-symmetry algebra. Note that the $D=3$ entries are valid after dualisation of the vector.

supergravity multiplet. As field theories, these are characterised by the property that they realise *local supersymmetry*, namely their action and equations of motion are invariant under a set of supersymmetry transformations in which the fermionic parameter is allowed to be an arbitrary function of the spacetime coordinates, $\epsilon(x)$. Noting that the action above is invariant under the local fermionic transformation

$$\Psi_\mu \to \Psi'_\mu = \Psi_\mu + \partial_\mu \epsilon(x) \tag{2.55}$$

hints at the fact that the gravitino acts as the gauge field for local SUSY, much in the same way as the Yang-Mills field does for local internal symmetries. The fact that there exists a connection between local supersymmetry and the presence of the gravitational field may be understood as a consequence of the supersymmetry algebra itself: the commutator of two global supersymmetry transformations closes on translations, as seen in (2.34); when the former are made local transformations, the latter must be "gauged" as well, yielding diffeomorphisms[6]. The converse is also true: if a theory exhibiting supersymmetry need also contain gravitational interactions, the notion of a global spinorial parameter becomes ill-defined. Thus, one should really let it depend on x^μ.

Fermions are defined according as their transformation properties under the action of the Lorentz group. Therefore, when coupling fermions to gravity, hence to a curved manifold, it is necessary to somehow recover the familiar flat-space Minkowski structure where one knows how to properly define them, namely a tangent space at each spacetime point. This is achieved by expressing the metric as

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x) \tag{2.56}$$

where, for a given D-dimensional spacetime, e^a_μ is a $D \times D$ matrix known as the vielbein, whose definition may be taken to be the above. Note that (2.56) is invariant under the transformations

$$e'^a_\mu(x) = (\Lambda^{-1})^a{}_b e^b_\mu(x) \tag{2.57}$$

for any x-dependent $\Lambda^a{}_b$ leaving η_{ab} invariant (that is, a Lorentz transformation); this subtracts $D(D-1)/2$ degrees of freedom from e^a_μ, leaving $D(D+1)/2$ independent components. In this sense, e^a_μ carries the same amount of information as the rank-2 symmetric metric tensor $g_{\mu\nu}$. The indices $a = 1, \ldots, D$ are referred to as those of *local Lorentz* frames.

The gauge field of this newly found local transformation is known as the *spin connection*, denoted by $\omega_\mu{}^a{}_b$, transforming under local Lorentz transformations as

$$\omega'_\mu{}^a{}_b = (\Lambda^{-1})^a{}_c\, \partial_\mu \Lambda^c{}_b + (\Lambda^{-1})^a{}_c\, \omega_\mu{}^c{}_d \Lambda^d{}_b \tag{2.58}$$

that is, just like the Yang-Mills field in (2.12). This transformation, among other things, is tailored so that the spacetime components of the spacetime 2-form T^a, known as the *torsion*,

$$T^a_{\mu\nu} = 2\left(\partial_{[\mu} e^a_{\nu]} + \omega_{[\mu}{}^{ab} e_{\nu]b}\right) = 2 e^a_\rho \Gamma^\rho_{[\mu\nu]} \tag{2.59}$$

indeed transform like a Lorentz vector. Depending on the connection, the torsion may or may not vanish. As the last equality makes manifest, affine connections whose coordinate space components are symmetric in their lower indices are, by definition, torsion-less. It may be proven that the usual Levi-Civita connection is the unique torsion-free connection which, in addition, is covariantly constant. Furthermore, the spin connection allows to construct the local Lorentz covariant derivative in the usual way: for instance, on a Lorentz vector V^a, one has

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \tag{2.60}$$

[6]It is possible to show that the gauging of the translation generators correctly reproduces diffeomorphisms, by imposing the right constraints. See [80].

while the coordinate space covariant derivative may be obtained by replacing Lorentz with spacetime indices using the vielbein,

$$e^\nu_a D_\mu V^a := \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \tag{2.61}$$

which establishes the relation between the spin connection and the more familiar affine connection,

$$\Gamma^\rho_{\mu\nu} = e^\rho_a \left(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu \right). \tag{2.62}$$

When acting on spinor fields, transforming in a representation generated by $M_{ab} = \frac{1}{2}\gamma_{ab}$, one has

$$D_\mu \psi = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} \right) \psi. \tag{2.63}$$

Therefore, the covariant derivative on the gravitino Ψ_μ, with spinor indices suppressed, is given by

$$\nabla_\mu \Psi_\nu = \partial_\mu \Psi_\nu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\Psi_\nu - \Gamma^\rho_{\mu\nu}\Psi_\rho. \tag{2.64}$$

This is sufficient to write down some simple supergravity Lagrangians. For example, there is only one case in dimensions $4 \leq D \leq 11$ where a graviton and a gravitino are sufficient to establish supersymmetry without additional fields, namely in the $D = 4$, $\mathcal{N} = 1$ supergravity theory based on the multiplet $\mathbf{G}_1 = \{g_{\mu\nu}, \Psi_\mu\}$. In the first order formalism, that is treating the vielbein and the general torsionful spin connection as two independent variables, the action is given by

$$S = \int \mathrm{d}^4x \; e \left(R(\omega) - \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho \right), \tag{2.65}$$

where the first term is the Ricci scalar built as a contraction of the curvature 2-form, with components

$$R_{\mu\nu}{}^{ab} = 2\left(\partial_{[\mu}\omega_{\nu]}^{ab} + \omega_{[\mu}{}^a{}_c \omega_{\nu]}^{cb} \right). \tag{2.66}$$

The action is invariant under the supersymmetry transformations

$$\delta_\epsilon e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a \Psi_\mu, \tag{2.67}$$

$$\delta_\epsilon \Psi_\mu = D_\mu \epsilon, \tag{2.68}$$

where that of the gravitino is the natural curved space generalisation of (2.55). Solving for the spin connection, and substituting the result back into the action (2.65), one may re-express it in terms of a more familiar torsion-free connection plus additional terms as

$$S = \int \mathrm{d}^4x \; e \left(R - \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \hat{D}_\nu \Psi_\rho + \mathcal{L}_{torsion} \right), \tag{2.69}$$

where the covariant derivative acting on the gravitino now contains the torsion-free connection and the last term describes 4-gravitino contact terms.

2.3.1 Scalars and supergravity

More generally, however, the discussion in the previous section shows that supergravity multiplets need contain a variety of fields on lower spins. Particularly interesting is the bosonic sector of the resulting field theories, described by the general Lagrangian,

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{1}{4}\mathcal{M}_{AB}F^{A}_{\mu\nu}F^{\mu\nu B} - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \dots \tag{2.70}$$

where the second term describes a collection of Abelian gauge fields A^A_μ coupled to n scalar fields ϕ^i exhibiting a generalisation of the $D = 4$ electromagnetic duality, the third encodes the dynamics of the scalar fields themselves and the dots represent the kinetic terms of higher p-form gauge fields which may be present, depending on the dimension. The metric appearing in the scalar kinetic term, g_{ij}, is that of a coset space G/H, known as the *scalar manifold*, where G is a non-compact group and H its maximal compact subgroup. The former is not a symmetry of the Lagrangian in general, but a duality transformation: in $D = 4$, for example, it rotates the vector fields' equations of motion into their Bianchi identities and acts non-linearly on the scalar fields of the theory, while it leaves all fermionic fields (and the metric) invariant. The latter is the largest symmetry group which is realised linearly on all fields.

Dimensional reduction and $SL(2,\mathbb{R})$

One convenient way to understand the origin of G and H is by means of the Kaluza-Klein dimensional reduction approach. We do not give a detailed review here: in particular, we focus on the global symmetries possessed by the lower dimensional theory without much explanation of how they originate in the higher dimensional theory's diffeomorphisms, which is an extremely fascinating idea at the core of the unification program; the interested reader is referred to the excellent [137]. The simplest non-trivial example is provided by the reduction of the metric in $D+2$ dimensions on a 2-torus: the components of the higher dimensional metric, g_{MN} with $M, N = 1, \dots, D+2$, yield

$$\begin{aligned} G_{MN} \to G_{\mu\nu},\ G_{\mu i},\ G_{(ij)} \\ \equiv g_{\mu\nu},\ A^i_\mu,\ \phi^i + \chi \end{aligned} \tag{2.71}$$

that is from a D-dimensional perspective, there are one graviton $g_{\mu\nu}$, two vectors A^i_μ with $i = 1, 2$ labelling the dimensions of the torus, and three scalars packaged in the symmetric matrix $G_{(ij)}$. These may be split into the ϕ^i, arising from the metric tensor at each circle reduction, and the single scalar χ, originating from the reduction of the $D+1$ dimensional vector A^1_μ. Restricting to the scalar sector, in line with the aim of this discussion, one sees that under a suitable set of field redefinitions, $\phi^i \to (\varphi, \phi)$, the lower dimensional Lagrangian reads

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial^\mu\chi\partial_\mu\chi. \tag{2.72}$$

The first is the unique term containing φ; it is easy to see that it is invariant under constant scalings $\delta\varphi = c$. The remaining part of the Lagrangian describes the dynamics of the other two scalars; after performing the redefinition

$$\tau = \chi + ie^{-\phi}, \tag{2.73}$$

it is re-written as

$$\mathcal{L}(\phi,\chi) = \frac{\partial_\mu \tau \partial^\mu \overline{\tau}}{2(\mathrm{Im}\tau)^2}. \tag{2.74}$$

This is invariant under the transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.75}$$

with the constraint $ad - bc = 1$, which amounts to a non-linearly realised $SL(2,\mathbb{R})$ transformation on τ, since the constraint may be seen as the unimodular condition on the otherwise unconstrained 2×2 matrix with a, b, c, d as entries,

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.76}$$

Therefore, the Lagrangian (2.72) is invariant under $GL(2,\mathbb{R}) \cong SL(2,\mathbb{R}) \times \mathbb{R}$, where the extra $\mathbb{R}$ factor stands for the constant shifts of φ by an arbitrary real number c.

Furthermore, it is possible to express all of the above in a different fashion, which helps clarify the action of the $SL(2,\mathbb{R})$ as well as the origin of the coset structure of the scalar space. Consider representing the two scalars (ϕ,χ) as the 2×2 matrix

$$\mathcal{V} = e^{\frac{1}{2}\phi H} e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} \in SL(2,\mathbb{R}), \tag{2.77}$$

known as the *coset representative*, where H and E_+ are the Cartan and the positive root generators of $SL(2,\mathbb{R})$ in the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.78}$$

Then, defining the matrix $M = \mathcal{V}^T\mathcal{V}$, one may rewrite the (ϕ,χ) part of the Lagrangian (2.72) as

$$\mathcal{L}(\phi,\chi) = \frac{1}{4}\mathrm{Tr}\left(\partial M^{-1}\partial M\right). \tag{2.79}$$

This is left invariant by an $SL(2,\mathbb{R})$ transformations of the representative, since $\mathcal{V}' = \mathcal{V}S$ induces

$$\mathcal{M}' = S^T\mathcal{M}S \tag{2.80}$$

which, in turn, leaves the trace above invariant. However, in general, the transformed $\mathcal{V}'$ is not of the upper-triangular form as the original matrix, which implies that one may not read off the variations on the scalars directly, and the treatment is incomplete. One ought to consider, in addition to the $SL(2,\mathbb{R})$, a compensating transformation which preserves the upper-triangular form of the representative. It may be proven that there is a unique unimodular orthogonal matrix $\mathcal{O} \in SO(2)$ that accomplishes this, namely

$$\mathcal{O} = \left(c^2 + e^{2\phi}(c\chi + a)^2\right)^{-1/2} \begin{pmatrix} e^{\phi}(c\chi + a) & c \\ -c & e^{\phi}(c\chi + a) \end{pmatrix}. \tag{2.81}$$

It is important that such a transformation be orthogonal since then, under the simultaneous variation $\mathcal{V}' = \mathcal{O}\mathcal{V}S$, one has

$$\mathcal{M}' = S^T \mathcal{V}^T \mathcal{O}^T \mathcal{O}\mathcal{V}S = S^T \mathcal{M} S \tag{2.82}$$

as before, hence the invariance of the Lagrangian is preserved. Notice that the $SO(2)$ transformation is local, as it depends on the two scalar fields. Nonetheless, it is not associated to any propagating gauge field, so one may gauge away the $SO(2)$ part of $SL(2,\mathbb{R})$, leaving the physical scalars ϕ and χ to parameterise the non-compact coset $SL(2,\mathbb{R})/SO(2)$.

Scalar conspiracy and enhancements

The more general case of a dimensional reduction of the metric tensor on an d-torus T^d works analogously, with few adjustments and refinements to the mathematical machinery required in order to construct the explicit expressions leading to the correct kinetic term. The result is that the lower dimensional theory contains $d(d+1)/2$ scalar fields in the symmetric matrix $G_{(ij)}$, now with $i, j = 1, \ldots, d$. All but one of them parameterise the coset manifold $SL(d,\mathbb{R})/SO(d)$, where the global symmetry is enhanced to $GL(d,\mathbb{R})$ if the higher-dimensional theory is in addition scaling invariant. On the other hand, if the bosonic sector of the original theory contains additional fields other than the metric, a further important enhancement is possible. The typical example is that of the dimensional reduction of $D = 11$ supergravity, with on-shell content

$$\mathbf{G}_1 = \{G_{MN}, \Psi_M, C_{MNR}\} = \{\mathbf{44}, \mathbf{128}, \mathbf{84}\} \tag{2.83}$$

that is a graviton, a gravitino and a 3-form potential. The reduction of the 3-form does not yield additional scalars until $D = 9$. When the internal space is a 3-torus, however, leaving an external 8-dimensional spacetime, there is one scalar, C_{123}, coming from the 3-form, in addition to the 6 scalars (3 dilatons and 3 axions) coming from the reduction of the metric on T^3. Five of the latter, with the exception of one dilaton as in the T^2 reduction, span a $SL(3,\mathbb{R})/SO(3)$ coset space, whilst the remaining dilaton couples to C_{123} to once again yield a dilaton/axion $SL(2,\mathbb{R})/SO(2)$. Therefore, the additional scalar is responsible for enhancing the expected $GL(3,\mathbb{R})/SO(3)$ to

$$\frac{G}{H}=\frac{SL(3,\mathbb{R})}{SO(3)}\times\frac{SL(2,\mathbb{R})}{SO(2)}. \tag{2.84}$$

Things work in a similar fashion for $d\leq 5$, namely reduction to six dimensions. For higher dimensional tori, one ought to take into account that higher dimensional forms may be dual to scalars. A first hint comes from counting on-shell degrees of freedom: a p-form potential in D spacetime dimensions has $\binom{D-2}{p}=\binom{D-2}{D-p-2}$ on-shell degrees of freedom. Restricting to the case at hand, namely that of the 3-form and hence setting $p\leq 3$, the cases of higher p-forms dual to a scalar are given by those (D,p) pairs yielding just one dof. As may be verified in Table 2.4, one finds $(5,3)$, $(4,2)$ and $(3,1)$, symbolising that a scalar is dual to a 3-form in 5 dimensions, a 2-form in 4-dimensions and a 1-form in 3-dimensions. Thus, in these dimensions, the space of scalar fields after toroidal dimensional

D	$C_{(3)}$		$C_{(2)}$		$C_{(1)}$		$C_{(0)}$	
	$\binom{D-2}{3}$	on T^d	$\binom{D-2}{2}$	on T^d	$\binom{D-2}{1}$	on T^d	dof	on T^d
5	1	1	3	6	3	15	1	20
4	$\varnothing$	1	1	7	2	21	1	35
3	$\varnothing$	1	$\varnothing$	8	1	28	1	56

Table 2.4: Which components of C_{MNR} are dual to scalars in each $D=3,4,5$? For each p-form, with $p=0,\ldots,3$, the left column indicates the on-shell degrees of freedom carried in D dimensions, whilst the right column reports the number of forms of that type coming from the dimensional reduction of C_{MNR} in $D=11$. For instance, in $D=4$, this contributes seven 2-forms which are dual to scalars.

reduction consists of $d(d+1)/2$ dilatons and axions coming from the metric, $\binom{d}{3}$ axionic scalars coming from the 3-form, as well as some D-dependent number of scalars coming from dualisation of the appropriate p-form.

For example, the all-important reduction of $D=11$ supergravity to four dimensions, on T^7, includes $28+35+7$ scalars in $G_{(ij)}$, $C_{[ijk]}$ and $C_{[\mu\nu]i}$, for a total of 70 scalars. This is precisely the correct number of scalars of the maximal supergravity multiplet in $D=4$, which is also singled out by the T^7 reduction of the single 32-component Majorana gravitino Ψ_M which yields 8 number of 4-component Majorana gravitini Ψ_μ and 56 Majorana spinors. Earlier, the scalars were shown to transform irreducibly under the R-symmetry $SU(8)$; indeed, decomposing $SU(8)\supset SO(7)$ one obtains

$$\mathbf{70}\rightarrow\mathbf{1}+\mathbf{7}+\mathbf{27}+\mathbf{35} \tag{2.85}$$

which shows the enhancement of the local invariance from the naive $SO(7)$ to $SU(8)$, under which all fields transform in linear representations. This is accompanied by a corresponding enhancement of the global symmetry, namely $GL(7,\mathbb{R}) \to E_{7(7)}$. Indeed, under $E_{7(7)} \supset SU(8)$, the adjoint splits according as

$$\mathbf{133} \to \mathbf{63} + \mathbf{70} \tag{2.86}$$

which shows that the scalars parameterise the 70-dimensional coset space $E_{7(7)}/SU(8)$.

Classification

All three cases analysed above exhibit homogeneous symmetric scalar manifolds, where a homogeneous space G/H is one which admits a transitive action of G, with $H \subset G$ the isotropy group keeping a point invariant. A homogeneous space is said to be symmetric if any element g of $\mathrm{Lie}(G) = \mathfrak{g}$ may be written as $g = h + p$ with $h \in \mathfrak{h}$ and $p \in \mathfrak{p}$, where $\mathfrak{p}$ is the complement of $\mathfrak{h}$ in $\mathfrak{g}$, respecting the commutator structure

$$[h_1, h_2] \in \mathfrak{h}, \qquad [h_1, p_1] \in \mathfrak{p}, \qquad [p_1, p_2] \in \mathfrak{h}. \tag{2.87}$$

In fact, all supergravities obtained from $D = 11$ upon toroidal reduction, representing the maximally supersymmetric case in each respective dimension, admit symmetric scalar cosets. Non-maximal supergravities may be obtained by reducing on some compact spaces other than the torus, or by consistent truncation. We will see an example of the latter in Chapter 4. In general, it is possible to classify the type of scalar manifold as a function of the number of real supercharges:

- Supergravities with $\mathcal{Q} > 16$ cannot be coupled to any vector or matter multiplets, and are therefore unique. Their scalar manifolds are always homogeneous symmetric spaces.

- Supergravities with $8 < \mathcal{Q} \leq 16$ may be coupled to vector/tensor multiplets. In this range, the kinetic terms of both the supergravity and the field theory of the vector/tensor multiplets are uniquely fixed once the content is given. The scalar manifolds are again homogeneous and symmetric.

- Supergravities with $\mathcal{Q} \leq 8$ may be coupled to both vector multiplets and chiral/hypermultiplets. In this range, the kinetic terms of both local and rigid supersymmetric theories are not uniquely fixed as a function of the field content. In particular, the scalar space metric may in general depend on one or more arbitrary functions, which specify a geometry on the scalar manifold. One notices that supergravity multiplets with this number of supercharges exist only for $D \leq 6$. Furthermore, postponing the discussion of $D = 3$ theories to Chapter 4 in order to avoid repetitions, such multiplets do not contain any scalars. Therefore, the scalar manifolds are determined by the content of the remaining coupled multiplets. In particular,

- For $\mathcal{Q} = 8$, the scalar manifolds spanned by both vector and hyperscalars are understood in terms of so-called *special geometries*. The scalars in the vector multiplets span *very special real* manifolds in $D = 5$ and *special Kahler* in $D = 4$. The hyperscalars span *quaternionic-Kahler* manifolds in all cases $D = 4, 5, 6$ (as well as $D = 3$), since hypers are insensitive to dimensional reduction.
- For $\mathcal{Q} = 4$, in $D = 4$, the vector multiplet $\mathbf{V}_1$ does not contain scalars. Scalars may exclusively come from chiral multiplets $\mathbf{C}_1$, and span a *Kahler-Hodge* manifold.

Notice that these need not be homogeneous or symmetric spaces and that, in all cases, the full scalar space is given by the product of the those spanned by the vector multiplet scalars and the hyperscalars separately.

All the symmetric scalar spaces for $\mathcal{Q} > 8$ and $4 \leq D \leq 10$ are given in Table 2.5.

D	$\mathcal{Q}=32$	$\mathcal{Q}=24$	$\mathcal{Q}=20$	$\mathcal{Q}=16$	$\mathcal{Q}=12$
10	$O(1,1)$ or $\frac{SL(2,\mathbb{R})}{SO(2)}$			$O(1,1)\times\frac{O(n)}{O(n)}$	
9	$O(1,1)\times\frac{SL(2,\mathbb{R})}{SO(2)}$			$O(1,1)\times\frac{O(1,n)}{O(n)}$	
8	$\frac{SL(3,\mathbb{R})}{SO(3)}\times\frac{SL(2,\mathbb{R})}{SO(2)}$			$O(1,1)\times\frac{O(2,n)}{SO(2)\times O(n)}$	
7	$\frac{SL(5,\mathbb{R})}{USp(4)}$			$O(1,1)\times\frac{O(3,n)}{SO(3)\times O(n)}$	
6	$\frac{SO(5,5)}{USp(4)\times USp(4)}$	$\frac{SU^*(4)}{USp(4)}\times\frac{USp(2)}{USp(2)}$		$O(1,1)\times\frac{O(4,n)}{USp(2)^2\times O(n)}$ or $\frac{O(5,n)}{USp(4)\times O(n)}$	
5	$\frac{E_{6(6)}}{USp(8)}$	$\frac{SU^*(6)}{USp(6)}$		$O(1,1)\times\frac{O(5,n)}{USp(4)\times O(n)}$	
4	$\frac{E_{7(7)}}{SU(8)}$	$\frac{SO^*(12)}{U(6)}$	$\frac{SU(1,5)}{U(5)}$	$\frac{SL(2,\mathbb{R})}{SO(2)}\times\frac{SO(6,n)}{SU(4)\times SO(n)}$	$\frac{U(3,n)}{U(3)\times U(n)}$

Table 2.5: Symmetric scalar manifolds of supergravities in $3\leq D\leq 10$ with $\mathcal{Q}>8$, following [80]. There are two cases with two entries: (i) in $D=10$ they describe $\mathbf{G}_{(1,1)}$ and $\mathbf{G}_{(2,0)}$, whilst (ii) in $D=6$ they describe $\mathbf{G}_{(2,2)}+n\mathbf{V}_{(2,2)}$ and $\mathbf{G}_{(4,0)}+n\mathbf{T}_{(0,4)}$, respectively. The $D=3$ theories are reported in Table 4.1. For half-maximal theories and below, n is the number of vector multiplets coupled to the supergravity multiplet.

Chapter 3

Supergravity as Yang-Mills Squared

3.1 Basics

In the absence of matter, the metric satisfies the vacuum Einstein equations $R_{\mu\nu} = 0$, whose simplest solution is the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. Then, the typical approach is to consider this as a flat background and set up a perturbative expansion in powers of the coupling constant κ,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) + \mathcal{O}(\kappa^2) \tag{3.1}$$

where the coefficient of the linear power of κ is regarded as a fluctuation about the Minkowski vacuum and is referred to as *graviton*. The linearised Ricci tensor reads

$$R^{lin}_{\mu\nu} = -\frac{\kappa}{2}\Big(\Box h_{\mu\nu} - \partial^\rho(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho}) + \partial_\mu\partial_\nu h^\rho_\rho\Big) \tag{3.2}$$

which gives the form of the equation of motion for the graviton in vacuum, namely

$$\Box h_{\mu\nu} - 2\partial_{(\mu}\partial^\rho h_{\nu)\rho} + \partial_\mu\partial_\nu h = 0. \tag{3.3}$$

These are invariant under the local transformation of the graviton

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{3.4}$$

by an arbitrary vector $\xi_\mu(x)$, which we refer to invariably as *spin-2 gauge transformation* or *linear diffeomorphisms*. Then, very simply put, the "squaring" approach has its roots in the observation that the on-shell states of the spin-2 graviton may be identified as the rank-2 symmetric traceless component of the tensor product of two defining little group representations describing the on-shell states of a gauge boson,

$$A_\mu \otimes \tilde{A}_\nu = \begin{cases} h_{\mu\nu} \equiv A_{(\mu} \otimes \tilde{A}_{\nu)} - \frac{\eta_{\mu\nu}}{D} A^\rho \otimes \tilde{A}_\rho \\ B_{\mu\nu} \equiv A_{[\mu} \otimes \tilde{A}_{\nu]} \\ \varphi \equiv A^\rho \otimes \tilde{A}_\rho \end{cases} \tag{3.5}$$

with the on-shell states of a 2-form and a scalar going along for the ride as the remaining irreducible representations. It is natural to extend the identification of gravitational states to these: they describe the (on-shell states of the) Kalb-Ramond antisymmetric tensor [113] and a dilaton, thus completing the interpretation of the full tensor product as the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of supergravity (or string theory). We view the above as a tensor product of gauge bosons belonging to *distinct* Yang-Mills theories, referred to as the *Left* and *Right* factors respectively, thus $A_\mu \in \mathrm{YM}_L$ and $\tilde{A}_\nu \in \mathrm{YM}_R$.

The careful reader would have noticed, at this point, that something is missing in (3.5): the gauge indices of the left and right Yang-Mills gauge groups, $\mathcal{G}$ and $\tilde{\mathcal{G}}$, have been omitted. Having re-instated them, it is easy to see that if we wish to maintain our identifications, we ought to account for them in such a way that the gravitational states be singlets under the combined gauge group $\mathcal{G} \times \tilde{\mathcal{G}}$, as they ought to be. In the scattering amplitude literature, the two gauge groups are taken to be the same, and only tensor products containing a singlet are allowed. Here, in order to work with generic gauge groups, we formally introduce a scalar quantity, valued in the adjoint of both gauge groups, $\Phi^{AA'}$, whose primary role is to contract the gauge indices to yield gauge invariant objects, such that the tensor product now reads

$$A_\mu^A \otimes \Phi^{-1}_{AA'} \otimes \tilde{A}_\nu^{A'}, \tag{3.6}$$

where the presence of the inverse of the scalar will be motivated shortly. The appearence of this "spectator" scalar is roughly consistent with the results in the scattering amplitude literature, whereby gravitational amplitudes are equivalent, in a precise fashion, to the product of two gauge theory amplitudes together with a third amplitude pertaining to the field theory of a bi-adjoint scalar field with a cubic Lagrangian [64, 65]. The bi-adjoint scalar also emerges as the "zeroth" copy of gauge theory amplitudes, that is the converse of the double copy: instead of replacing color factors by kinematic ones, one eliminates the latter in favour of a second copy of the former, thus obtaining a doubly-coloured amplitude. Another interesting perspective directly relating the bi-adjoint scalar theory to the $\alpha' = 0$ limit of the KLT kernel of string theory is given in [138].

Field theoretic aspects

The above construction is restricted to little group representations; despite this, we will devote two chapters in this thesis to showing how this version of the gauge/gravity map may be considerably extended, to reveal a rich structure and a web of relations between different (super)gravity theories. A different, perhaps more ambitious view could be adopted, whereby a similar map or "dictionary" may be built at the level of spacetime fields. Although this will be the subject of Chapter 6, it is worth introducing some of the basic concepts here. Firstly, one should specify how the spacetime dependence of the Yang-Mills (and spectator) fields carries over to those of gravity. It turns out to be very effective to assume the precise form of the map to be a convolution in spacetime: for two functions on $\mathbb{R}^{1,D-1}$, $f(x)$ and $g(x)$, their convolution is given by

$$[f \star g](x) := \int \mathrm{d}^D y \; f(y)g(x-y) = \int \mathrm{d}^D y \; f(x-y)g(y), \tag{3.7}$$

heuristically consistent with the observation that the double copy relations are multiplicative in momentum space[1]. The convolution is both commutative, $f \star g = g \star f$ and associative, $f \star (g \star h) = (f \star g) \star h$. Adjoining the aforementioned gauge index contraction, the triplet of states above may be promoted to a triplet of *fields* defined by

$$(h_{\mu\nu}, B_{\mu\nu}, \varphi)(x) = [A_\mu \circ \tilde{A}_\nu](x) = [A_\mu^A \star \Phi^{-1}_{AA'} \star \tilde{A}_\nu^{A'}](x), \tag{3.8}$$

where the notation $\circ$ has been introduced to denote convolution and index contraction with the spectator scalar field, which allows us to omit the gauge indices from here on.

Aside from the comparison with the amplitude double copy, one compelling reason to assume the convolutive nature of the product from our perspective is that only after assuming a convolution, as opposed to a simple product, is one able to derive the (linearised) local diffeomorphism and 2-form gauge symmetries of graviton and Kalb-Ramond field from the gauge invariance of the underlying Yang-Mills fields. The reader is referred to Section 6.1 for a more complete discussion, and to [139] for the full derivation of the symmetries of $\mathcal{N} = 1$ supergravity which include, aside from those already mentioned, also local Lorentz and supersymmetry transformations of the whole multiplet. Suffice here to say that these results crucially rely on the non-Leibniz property of the convolution, namely

$$\partial_\mu (f \star g) = \partial_\mu f \star g = f \star \partial_\mu g, \tag{3.9}$$

which practically allows one to "pull out" a derivative, acting on just one factor inside the integral, to act on the whole product. The derivative rule (3.9) is not valid for the convolution of any two functions on $\mathbb{R}^{1,D-1}$: it is possible to show [72] that it suffices to restrict the allowed domain to functions in the image of the Green function for the d'Alembert operator, namely to functions $f(x)$ such that

$$f(x) = [G \star j](x) := \frac{1}{\Box} j(x) \tag{3.10}$$

where $G = G(x-y)$ is the usual Green function for the d'Alembertian, $\Box_x G(x-y) = \delta^D(x-y)$ and where we use the streamlined notation $\Box^{-1}$ to denote convolution with G, as in [72]. We also demand that the Green function commutes with the d'Alembertian,

$$\Box^{-1}\Box = \Box\Box^{-1} = \mathrm{Id} \tag{3.11}$$

[1]Recall that the (inverse) Fourier transform of a product is given by the convolution of the individual (inverse) Fourier transforms.

on this domain. Notice, in particular, how this implies that plane wave solutions, such that $\Box f(x) = 0$, are therefore excluded. This is relevant, among other things, to the double copy of classical solutions which, as the name suggests, aims at finding a gauge theory interpretation of the solutions of the classical Einstein equations of motion by relating them to a double copy of e.g. Maxwell or Yang-Mills solutions. In Section 6.3, we will show that such careful consideration of the properties of the convolution is necessary in order to obtain a proper understanding of the gauge/gravity map, especially with regards to the Yang-Mills origin of the gravitational dynamics.

Finally, having been granted the status of spacetime fields rather than on-shell states, the quantities in (3.8) should inherit a number of properties qualifying them as such. Among these, they ought to have a specific mass dimension, which fixes how they might enter a Lagrangian density encoding their symmetries and dynamics. In this regard, notice how the introduction of the spectator scalar, $\Phi^{-1}_{AA'}$, gives a way out[2] of the formal difficulty in matching the mass dimensions of the three fields $(h_{\mu\nu}, B_{\mu\nu}, \varphi)$, each of which has standard mass dimension $(D-2)/2$ in D-dimensional spacetime. Obviously the naive tensor product $A_\mu \otimes \tilde{A}_\nu$ has twice the required value, that is $D-2$. If we had introduced the convolution, but not the spectator, things would be even worse since, in that case, $A_\mu \star \tilde{A}_\nu = \int \mathrm{d}^D y \; A_\mu(y)\tilde{A}_\nu(x-y)$ has fixed mass dimension $-D + D - 2 = -2$. On the other hand, it is possible to require that the dictionary prescribed in (3.8), which explicitly reads

$$[A^A_\mu \star \Phi^{-1}_{AA'} \star \tilde{A}^{A'}_\nu](x) = \int \mathrm{d}^D z \mathrm{d}^D y \; A_\mu(x-y)\Phi^{-1}_{AA'}(z)\tilde{A}_\nu(y-z), \tag{3.12}$$

has the correct mass dimension: the "spectator" scalar must be assigned mass dimension

$$[\Phi^{-1}_{AA'}] = \frac{3D+2}{2}, \tag{3.13}$$

which shows that it cannot be a conventional scalar field. Indeed, this value is typical of the *convolution inverse* of a usual scalar field, defined by the relation

$$[\Phi^{-1} \star \Phi](x) = \int \mathrm{d}^D y \; \Phi^{-1}(y)\Phi(x-y) = \delta^{(D)}(x), \tag{3.14}$$

which motivates the notation. Postponing further discussions on the field theoretic aspects of the squaring map until Chapter 6, let us now focus on extending the above construction from a purely group theoretic perspective.

3.2 Squaring pure super-Yang-Mills

The remainder of this chapter is devoted to a review of the results in [85–88], which laid out the Yang-Mills squared construction. We present this here as it constitutes the

[2]Aside from the ad hoc inclusion of dimensionful parameters.

foundation on which the results on twin supergravities of Chapter 4 build on. Compared to the amplitudes double copy, this represents an a priori independent, albeit surely related, group theoretic approach to the idea that gravity is the "square" of gauge theory. Excellent reviews, containing a much more detailed exposition of this material, are [139–141].

3.2.1 Overview

Experimental questions aside, the inclusion of supersymmetry to a physical model is theoretically very compelling, inasmuch as it provides a larger framework which very often exhibits a much richer structure than its non-supersymmetric counterpart. Being no exception, the gauge/gravity map briefly sketched in the preceding section greatly benefits from a supersymmetric extension, as one could anticipate from the appearence of the whole NS-NS sector - reminiscent of supergravity - in the simple (i.e. $\mathcal{N} = 0$) product.

Thus, rather than the tensor product of two Yang-Mills potentials, it is tempting to consider that of two pure[3] super-Yang-Mills (sYM) multiplets, where we denote the amount of supersymmetry by $\mathcal{N}$ and $\tilde{\mathcal{N}}$ for the Left and Right sYM theories, respectively[4]. The number of distinct possible cases is a function of the number of allowed multiplets in any given spacetime dimension, which are summarised in Table 2.3. Naturally, one expects the resulting states to organise themselves into allowed supergravity multiplets, giving a precise (and hopefully unique) mapping from a pair of sYM theories on one side, which we denote by $(\mathbf{V}_{\mathcal{N}}, \mathbf{V}_{\tilde{\mathcal{N}}})$, and a specific supergravity theory on the other. The simplest instance in which this can be seen to work is in 10-dimensional spacetime (the highest admitting a sYM multiplet), squaring two vector multiplets containing spinors of opposite chirality, that is

$$\begin{aligned}\mathbf{V}_{(1,0)} \otimes \mathbf{V}_{(0,1)} &= (\mathbf{8}_v + \mathbf{8}_s) \otimes (\mathbf{8}_v + \mathbf{8}_c) \\ &= (\mathbf{35}_v + \mathbf{28} + \mathbf{1}) + (\mathbf{56}_s + \mathbf{8}_s) + (\mathbf{56}_c + \mathbf{8}_c) + (\mathbf{56}_v + \mathbf{8}_v) \\ &= \mathbf{G}_{(1,1)},\end{aligned} \tag{3.15}$$

resulting precisely in the states of $\mathcal{N}_{(1,1)}$ supergravity in ten dimensions, with the two gravitini (also of opposite chirality) transforming in the $\mathbf{56}_s$ and $\mathbf{56}_c$. The simplicity of this example owes to the triviality of the R-symmetry of the sYM factors, which correctly square to a supergravity theory with an equally trivial R-symmetry (or H group altogether, in this case): the content on both sides is labelled simply by $\mathfrak{so}(8)_{\text{st}}$ representations[5]. In general, however, the fields belonging to a sYM multiplet carry, alongside (adjoint) gauge and Lorentz indices, also those of the relevant global internal symmetry,

[3] That is, not coupled to matter. This case will be considered in Section 3.3.2.

[4] The slightly awkward notation is due to the fact that we prefer to reserve $\mathcal{N}_{L/R}$ to indicate the chirality of the supercharges, and $\mathcal{N}_{\pm}$ for the twin theories of Chapter 4.

[5] This statement is marginally incomplete: the similar product of $D = 10$ vector multiplets with the *same* chirality leads to $\mathbf{G}_{(2,0)}$ with R-symmetry $SO(2)$, as may be verified in Table 3.6. This type of enhancement is explained in Section 3.2.3 and an explicit example is given in Section 3.2.4.

with Lie algebra $\mathfrak{int}(\mathcal{N}, D)$, which always acts trivially on the vector, but rotates the spinors in its defining representation and the scalars among themselves, as it is apparent if we write the components of the vector multiplet as $\mathbf{V}_\mathcal{N} = (A_\mu, \lambda^i, \phi^a)^A$, where $i = 1, \dots, \mathcal{N}$ while $a = 1, \dots, \mathcal{Q}/2 + 2 - D$ and A the adjoint index of the gauge group. Thus, if this is to work, the question naturally arises of how the R-symmetry representations ought to combine under the squaring map, in order to obtain a consistent and meaningful set of results.

With the luxury of hindsight, we could imagine a natural guess to be that *the internal global symmetries of the sYM factors square to the maximal compact symmetry H of the corresponding supergravity, the largest global internal symmetry linearly realised on the fields in the multiplets.* Indeed, this may be shown to be the case. Before presenting a sketch of the proof, however, let us discuss why it makes sense that this should be true. Heuristically, one could visualise the tensoring of two supersymmetric multiplets as in the following table, where the bona fide gravitational states, defined as irreducible $\mathfrak{so}(D-2)_{\text{st}}$

	$\tilde{A}_\nu$	$\tilde{\lambda}^{i'}$	$\tilde{\phi}^{a'}$
A_μ	$h_{\mu\nu} + B_{\mu\nu} + \varphi$	$\Psi^{i'}_\nu + \chi^{i'}$	$V^{a'}_\mu$
λ^i	$\Psi^i_\mu + \chi^i$	$\varphi^{ii'} + \dots$	$\chi^{ia'}$
ϕ^a	V^a_ν	$\chi^{ai'}$	$\varphi^{aa'}$

Table 3.1: Generic form of the squaring of two SUSY multiplets, $\mathbf{V}_\mathcal{N} \otimes \mathbf{V}_{\tilde{\mathcal{N}}}$.

representations due to the identification of the left and right little groups, directly inherit both representations under $\mathfrak{int}(\mathcal{N}, D)$ and $\mathfrak{int}(\tilde{\mathcal{N}}, D)$, which ought to be kept independent. Overall, then, they are labelled as representations of

$$\mathfrak{so}(D-2)_{\text{st}} \oplus \mathfrak{int}(\mathcal{N}, D) \oplus \mathfrak{int}(\tilde{\mathcal{N}}, D) \oplus \delta_{4D}\mathfrak{u}(1)_d \tag{3.16}$$

where the Kronecker delta symbolises that the additional Abelian factor is present only in 4 dimensions, where the generators of the Lorentz transformations in the tensor product representations of the little group correspond to the sum of those in the Left and Right factors, labelled by the helicity of the states. Thus, the generators corresponding to the orthogonal combination $\mathfrak{u}(1)_d \equiv \mathfrak{u}(1)_{\text{st}} - \tilde{\mathfrak{u}}(1)_{\text{st}}$ commute with those of the little group and may contribute as an internal symmetry of the supergravity states.

The tensoring thus yields states ranging from spin-2, the graviton $h_{\mu\nu}$, all the way down to spin-0 scalars, in half-integral decrements, as expected. In particular, notice how the spectrum includes $\mathcal{N}$ gravitini coming from $\Psi^i_\mu \sim \lambda^i \circ \tilde{A}_\mu$ and $\tilde{\mathcal{N}}$ gravitini coming from $\Psi^{i'}_\mu \sim A_\mu \circ \tilde{\lambda}^{i'}$, thus suggesting the general result that the corresponding (pure or matter coupled) supergravity will possess $(\mathcal{N}+\tilde{\mathcal{N}})$-extended supersymmetry, that is

$$\mathbf{V}_{\mathcal{N}} \otimes \mathbf{V}_{\tilde{\mathcal{N}}} = \mathbf{G}_{\mathcal{N}+\tilde{\mathcal{N}}} \oplus \mathbf{M}_{\mathcal{N}+\tilde{\mathcal{N}}} \tag{3.17}$$

where $\mathbf{M}_{\mathcal{N}}$ denotes possible supersymmetric matter multiplets coupled to the supergravity multiplet. Note also that the tensor product of two sYM spinors, that is the central entry of Table 3.1, must be expanded onto the basis of the Clifford algebra provided by the skew-symmetric products of the generating gamma matrices; this implies that, in general, the result will depend on the spacetime dimension. Notice, however, how the notation of the table betrays the fact that, irrespective of D, there always will be a set of $\mathcal{N}\tilde{\mathcal{N}}$ scalar fields $\varphi^{ii'}$, usually referred to as Ramond-Ramond (RR) scalars. In addition to those in the RR sector, scalar fields belonging to the gravity side are also produced in other types of tensor products, namely $\varphi \subset A_\mu \circ \tilde{A}_\nu$ and $\varphi^{aa'} \subset \phi^a \circ \tilde{\phi}^{a'}$ and the dualised 2-form $B_{\mu\nu}$ in four dimensions, where the corresponding 3-form field strength is Hodge-dual to a 1-form field strength, hence to a scalar. We will discuss shortly how the knowledge of the sYM origin of the scalar space of supergravity can help pin down a unique result for each squaring. How, though, can this set of states be shown to correspond to those of honest supergravity?

- *Compact symmetries*

  The key observation is that there always exists, for all dimensions and combinations of supercharges, a maximal embedding of the (direct sum of) Lie algebras inherited directly from Yang-Mills into a larger one,

  $$\mathfrak{h}(D,\mathcal{N},\tilde{\mathcal{N}}) \supset \mathfrak{int}(\mathcal{N},D) \oplus \mathfrak{int}(\tilde{\mathcal{N}},D) \oplus \delta_{4D}\mathfrak{u}(1)_d \tag{3.18}$$

  such that the bona fide gravitational states above may be seen to consistently uplift to irreducible representations of $\mathfrak{h}$. As suggested by the notation, the remarkable fact is that, without exception, the latter corresponds to the Lie algebra of H, the maximal compact subgroup of the U-duality group G of the precise supergravity theory whose content is fixed by the tensor product. In this sense, the $\mathfrak{so}(D-2)_{\text{st}}$ tensor product and the embedding (3.18) conspire to pin down a particular supergravity theory.

- *Non-compact symmetries*

  The fact that the scalar states as computed from the product of the Yang-Mills factors transform irreducibly under the linear action of $\mathfrak{h}$ is enough to establish that they parameterise a locally homogeneous space [27, 89]. If we additionally assume that said space is symmetric, we are led to the conclusion that they span a coset manifold, G/H, with the structure defined in (2.87), with tangent space at each point given by the vector space $\mathfrak{p} = \mathfrak{g} - \mathfrak{h}$. Thus, the scalars span $\mathfrak{p}$ and transform in the $\dim(\mathfrak{p})$-dimensional representation of $\mathfrak{h}$. This fixes the Lie algebra $\mathfrak{g}$, which in all cases matches the expected non-compact U-duality group G.

3.2.2 Construction in $D = 3$

Let us now briefly present the essence of the proof. This relies on the mathematics of the four normed division algebras (NDAs) $\mathbb{A}_n$ of real dimension $n = 1, 2, 4, 8$, namely the reals $\mathbb{R}$, the complexes $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. Since the original work presented in this thesis does not require knowledge of the division algebras, we will only mention certain results which are relevant to our purposes without much justification. The interested reader is referred to the original works [85–88] and to the excellent discussions contained in [139–141]. At the core of the construction is a formula, the (reduced) *magic square* formula,

$$\mathfrak{L}_2 (\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + (\mathbb{A}_L \otimes \mathbb{A}_R) \tag{3.19}$$

which builds a 4×4 array of Lie algebras from a set of subalgebras, the so-called *triality* algebras of the division algebras, denoted $\mathfrak{tri}(\mathbb{A})$. Remarkably, the entries of the reduced magic square correspond exactly to the $\mathfrak{h}$ algebras of the supergravities whose content arises from tensoring the allowed $\mathcal{N} = 1, 2, 4, 8$ multiplets in $D = 3$, as reported in Table (3.2).

	$\mathbf{V}_8$ $\mathfrak{so}(8)$	$\mathbf{V}_4$ $3\mathfrak{so}(3)$	$\mathbf{V}_2$ $2\mathfrak{so}(2)$	$\mathbf{V}_1$ $\varnothing$
$\mathbf{V}_8$ $\mathfrak{so}(8)$	$\mathbf{G}_{16}$ $\mathfrak{g} = \mathfrak{e}_{8(8)}$ $\mathfrak{h} = \mathfrak{so}(16)$	$\mathbf{G}_{12}$ $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ $\mathfrak{h} = \mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$\mathbf{G}_{10}$ $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ $\mathfrak{h} = \mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathbf{G}_9$ $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ $\mathfrak{h} = \mathfrak{so}(9)$
$\mathbf{V}_4$ $3\mathfrak{so}(3)$	$\mathbf{G}_{16}$ $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ $\mathfrak{h} = \mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$\mathbf{G}_8 + 4\mathbf{V}_8$ $\mathfrak{g} = \mathfrak{so}(8,4)$ $\mathfrak{h} = \mathfrak{so}(8) \oplus 2\mathfrak{so}(3)$	$\mathbf{G}_6 + 2\mathbf{V}_6$ $\mathfrak{g} = \mathfrak{su}(4,2)$ $\mathfrak{h} = \mathfrak{so}(6) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)$	$\mathbf{G}_5 + \mathbf{V}_5$ $\mathfrak{g} = \mathfrak{usp}(4,2)$ $\mathfrak{h} = \mathfrak{so}(5) \oplus \mathfrak{so}(3)$
$\mathbf{V}_2$ $2\mathfrak{so}(2)$	$\mathbf{G}_{10}$ $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ $\mathfrak{h} = \mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathbf{G}_6 + 2\mathbf{V}_6$ $\mathfrak{g} = \mathfrak{su}(4,2)$ $\mathfrak{h} = \mathfrak{so}(6) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)$	$\mathbf{G}_4 + \mathbf{V}_4 + \mathbf{C}_4$ $\mathfrak{g} = 2\mathfrak{su}(2,1)$ $\mathfrak{h} = \mathfrak{so}(4) \oplus 2\mathfrak{so}(2)$	$\mathbf{G}_3 + \mathbf{V}_3$ $\mathfrak{g} = \mathfrak{su}(2,1)$ $\mathfrak{h} = \mathfrak{so}(3) \oplus \mathfrak{so}(2)$
$\mathbf{V}_1$ $\varnothing$	$\mathbf{G}_9$ $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ $\mathfrak{h} = \mathfrak{so}(9)$	$\mathbf{G}_5 + \mathbf{V}_5$ $\mathfrak{g} = \mathfrak{usp}(4,2)$ $\mathfrak{h} = \mathfrak{so}(5) \oplus \mathfrak{so}(3)$	$\mathbf{G}_3 + \mathbf{V}_3$ $\mathfrak{g} = \mathfrak{su}(2,1)$ $\mathfrak{h} = \mathfrak{so}(3) \oplus \mathfrak{so}(2)$	$\mathbf{G}_2 + \mathbf{V}_2$ $\mathfrak{g} = \mathfrak{so}(2,1)$ $\mathfrak{h} = \mathfrak{so}(2)$

Table 3.2: The magic square of supergravities in $D = 3$. The U-duality algebras $\mathfrak{g}$ correspond to the entries of the Freudenthal-Rosenfeld-Tits (FRT) magic square of division algebras, while their maximal compact subalgebras $\mathfrak{h}$ correspond to those of the reduced magic square.

It is possible to show that there exists a parameterisation of super-Yang-Mills theories in $D = 3$ in terms of $\mathbb{A}_n$, exploiting a relation between the division algebras and supersymmetry whereby the triality algebras correspond to the internal symmetries of the $D = 3$ Yang-Mills multiplets, $\mathfrak{tri}(\mathbb{A}_{\mathcal{Q}/2}) \cong \mathfrak{int}(\mathcal{N}, 3)$, namely $\mathfrak{tri}(\mathbb{A}_{\mathcal{Q}/2}) = \varnothing, 2\mathfrak{so}(2), 3\mathfrak{so}(3), \mathfrak{so}(8)$ for $\mathcal{N} = 1, 2, 4, 8$, to be compared with the relevant entries of Table 2.3. As a consequence of this, the reduced magic square formula (3.19) acquires a physical interpretation! Namely, it formalises, for $D = 3$, the enhancement (3.18) of the (direct sum of) internal symmetries of the sYM factors to the full $\mathfrak{h}$ algebra. Indeed, part of the reformulation of sYM in terms of the NDAs involves realising that, as vector spaces, the division algebra $\mathbb{A}_n$ is isomorphic to the total sYM spinor space for $\mathbb{A}_{\mathcal{Q}/2} \cong \mathbf{s}^{\mathcal{N}}$ which, in addition, always carries the fundamental representation of the internal symmetry algebra, e.g.

- $\mathbf{V}_2^{D=3}$ carries 4 supercharges, so $\mathbb{A}_{\mathcal{Q}/2} = \mathbb{C}$, and has two real spinors, $\mathbb{R}^2 \cong \mathbb{C}$, transforming as the $(1, -1)$ and $(-1, 1)$ of $\mathfrak{int}(2, 3) = 2\mathfrak{so}(2)$;
- $\mathbf{V}_8^{D=3}$ carries 16 supercharges, so $\mathbb{A}_{\mathcal{Q}/2} = \mathbb{O} \cong \mathbb{R}^8$, and has eight real spinors, $\mathbb{R}^8$, transforming as the $\mathbf{8}$ of $\mathfrak{int}(8, 3) = \mathfrak{so}(8)$.

Thus, the formula (3.19) may suggestively be re-written as

$$\mathfrak{h}(3, \mathcal{N}, \tilde{\mathcal{N}}) = \mathfrak{int}(\mathcal{N}, 3) \oplus \mathfrak{int}(\tilde{\mathcal{N}}, 3) + \mathbf{s}^{\mathcal{N}} \otimes \mathbf{s}^{\tilde{\mathcal{N}}}, \tag{3.20}$$

which makes manifest the Yang-Mills Squared origin of the $\mathfrak{h}$ algebras in Table 3.2: with the knowledge of the internal symmetries of both sYM factors together with their spinorial content and representations, one may build $\mathfrak{h}$ uniquely.

Non-compact symmetry

Let us now clarify why the magic square defined by (3.19) is referred to as "reduced". It turns out that there is an even bigger algebra which can act on the tensor product of two NDAs, namely that of the Freudenthal-Rosenfeld-Tits *magic square*, defined by

$$\begin{aligned} \mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R) &= \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R) \\ &= \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) + 2(\mathbb{A}_L \otimes \mathbb{A}_R). \end{aligned} \tag{3.21}$$

The commutation relations between generators in $\mathfrak{L}_2$ and the complement may be computed to be those of (2.87), which establishes the reduced magic square as the maximal compact subalgebra of $\mathfrak{L}_{1,2}$. It will not surprise, then, that the entries of the magic square (3.21) correspond to the non-compact Lie algebras $\mathfrak{g}$ of the $D = 3$ supergravities appearing in Table 3.2. With an eye to the Yang-Mills origin of these, we then re-write the above as

$$\mathfrak{g}(3, \mathcal{Q}, \tilde{\mathcal{Q}}) = \mathfrak{h}(3, \mathcal{Q}, \tilde{\mathcal{Q}}) + \mathfrak{p}(3, \mathcal{Q}, \tilde{\mathcal{Q}}), \tag{3.22}$$

through the identifications

$$\mathfrak{g}(3,\mathcal{Q},\tilde{\mathcal{Q}}) \sim \mathfrak{L}_{1,2}(\mathbb{A}_{\mathcal{Q}/2},\mathbb{A}_{\tilde{\mathcal{Q}}/2}) \tag{3.23}$$

$$\mathfrak{h}(3,\mathcal{Q},\tilde{\mathcal{Q}}) \sim \mathfrak{L}_{2}(\mathbb{A}_{\mathcal{Q}/2},\mathbb{A}_{\tilde{\mathcal{Q}}/2}) \tag{3.24}$$

$$\mathfrak{p}(3,\mathcal{Q},\tilde{\mathcal{Q}}) \sim 2\left(\mathbb{A}_{\mathcal{Q}/2}\otimes\mathbb{A}_{\tilde{\mathcal{Q}}/2}\right). \tag{3.25}$$

Having already shown the Yang-Mills origin of $\mathfrak{h}$, all is left to do is to find one for the non-compact complement, $\mathfrak{p}$. As we have seen, for supergravities with homogeneous symmetric scalar manifolds (which we assume), this space is spanned by the scalars. From a Yang-Mills perspective, in three dimensions these come from only two kinds of tensor products, since the vectors are always dualised to scalars as a consequence of working on-shell: left with right scalars, $\phi\circ\tilde{\phi}$ and (the Fierz decomposition of) left with right[6] spinors, $\lambda\circ\tilde{\lambda}$. The non-compact space is thus seen to be isomorphic, as a vector space, to

$$\mathfrak{p}(3,\mathcal{Q},\tilde{\mathcal{Q}}) \cong \mathbf{S}(3,\mathcal{Q})\otimes\mathbf{S}(3,\tilde{\mathcal{Q}})+\mathbf{s}^{\mathcal{N}}\otimes\mathbf{s}^{\tilde{\mathcal{N}}}, \tag{3.26}$$

where $\mathbf{S}(3,\mathcal{Q})$ represents the vector space spanned by the scalars of the Left Yang-Mills multiplet, with identical expressions for the Right factor. Notice that, in $D=3$, supersymmetry implies $\mathbf{S}(3,\mathcal{Q})\cong\mathbf{s}^{\mathcal{N}}$. It may be checked that, in all cases, the scalars obtained from squaring carry precisely the irreducible $\mathfrak{h}$ representations necessary for the enhancement $\mathfrak{h}\subset\mathfrak{g}$ to occur, which can also be checked explicitly by computing the commutators of $\mathfrak{h}$ and $\mathfrak{p}$.

3.2.3 Squaring in all dimensions

All this may be generalised to all dimensions $3\leq D\leq 10$, to obtain the Lie algebras of the groups appearing in the scalar cosets G/H of all supergravities arising from $\mathbf{V}_{\mathcal{N}}\otimes\mathbf{V}_{\tilde{\mathcal{N}}}$ in these dimensions. Denoting by $\mathbb{A}[n]$ the set of $n\times n$ matrices with entries in the division algebras $\mathbb{A}_n=\mathbb{R},\mathbb{C},\mathbb{H}$, the first observation to be made is that the set of anti-Hermitian elements in $\mathbb{A}[n]$,

$$\mathfrak{a}(n,\mathbb{A}) := \{x\in\mathbb{A}[n] : x^{\dagger}=-x\} \tag{3.27}$$

yields the classical Lie algebras

$$\mathfrak{a}(n,\mathbb{A}) = \begin{cases} \mathfrak{so}(n), & \mathbb{A}=\mathbb{R}, \\ \mathfrak{u}(n), & \mathbb{A}=\mathbb{C}, \\ \mathfrak{sp}(n), & \mathbb{A}=\mathbb{H}. \end{cases} \tag{3.28}$$

[6] We remind the reader, to avoid possible confusion, that "left" and "right" in this thesis usually refer to the two sYM factors entering the squaring product, not to the chirality of the spinors. We tried to pay due care and make it very clear where we actually refer to the left and right Weyl components of a spinor in even dimensions, for instance by using projector operators $P_{L/R}$.

corresponding to the R-symmetry algebras in various dimensions, according to the pattern in Table 2.2, which justifies the identification,

$$\mathfrak{r}(\mathcal{N}, D) = \mathfrak{a}(\mathcal{N}, \mathbb{D}) \tag{3.29}$$

where $\mathbb{D}$ is the (direct sum of) division algebras associated to the reality properties of minimal spinors, and hence to the R-symmetry, in each dimension, e.g. in $D = 10$, $\mathbb{D} = \mathbb{R}_L \oplus \mathbb{R}_R$ corresponding to the R-symmetry $\mathfrak{so}(\mathcal{N}_L) \oplus \mathfrak{so}(\mathcal{N}_R)$. One may build larger algebras using the fact that

$$\mathfrak{a}(\mathcal{N} + \tilde{\mathcal{N}}, \mathbb{D}) = \mathfrak{a}(\mathcal{N}, \mathbb{D}) \oplus \mathfrak{a}(\tilde{\mathcal{N}}, \mathbb{D}) + \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}] \tag{3.30}$$

where the term $\mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]$ may be thought of as a $\dim_{\mathbb{D}} = \mathcal{N}\tilde{\mathcal{N}}$ vector space spanned by $\mathcal{N} \times \tilde{\mathcal{N}}$ matrices valued in $\mathbb{D}$. With an eye towards the Yang-Mills Squared interpretation, using (3.29), one may re-write the above as

$$\mathfrak{r}(\mathcal{N} + \tilde{\mathcal{N}}, D) = \mathfrak{r}(\mathcal{N}, D) \oplus \mathfrak{r}(\tilde{\mathcal{N}}, D) + \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}] \tag{3.31}$$

which may be interpreted as constructing the R-symmetry of the supergravity from those of the sYM factors, as follows: upon tensoring two sYM theories, one can form the doublet of Left and Right supercharges $(Q, \tilde{Q}) \in \mathbb{D}^{\mathcal{N}} \oplus \mathbb{D}^{\tilde{\mathcal{N}}}$. Each one is acted upon by the R-symmetries of the Left and Right sYM theories independently; however, there is now the possibility of the additional "off-diagonal" contributions, corresponding to $\mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]$. The action of $\mathfrak{r}(\mathcal{N} + \tilde{\mathcal{N}}, D)$ in this basis is given explicitly by

$$X = \begin{pmatrix} X_L & 0 \\ 0 & X_R \end{pmatrix} + \begin{pmatrix} 0 & M \\ -M^\dagger & 0 \end{pmatrix} \tag{3.32}$$

where $X_L \in \mathfrak{r}(\mathcal{N}, D)$, $X_R \in \mathfrak{r}(\tilde{\mathcal{N}}, D)$ and $M \in \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]$. This may be used to check that the commutators are indeed the correct ones, cf. [88] for the explicit calculation.

The expression (3.31) is enough for those cases when the whole set of internal symmetries of *both* sYM factors coincides with their R-symmetries. This is not always the case: in fact, it is convenient to define

$$\mathfrak{int}(\mathcal{N}, D) = \mathfrak{r}(\mathcal{N}, D) \oplus \mathfrak{q}(\mathcal{N}, D) \ominus \delta_{D4}\delta_{\mathcal{N}4}\mathfrak{u}(1)_r \tag{3.33}$$

for $\mathfrak{q} = \mathfrak{so}(2), \mathfrak{so}(3)$ for $\mathbf{V}_2, \mathbf{V}_4$ in $D = 3$. These enhancements of the internal symmetry in three dimensions are due to the dualisation of the vector to a scalar, while the $D = 4$ factor symbolises the fact that the maximal $\mathcal{N} = 4$ sYM multiplet is CPT self-conjugate and thus cannot support the $\mathfrak{u}(1)_r \subset \mathfrak{u}(4)$ part of the R-symmetry. Then, in all dimensions, the formula which builds the algebras of the maximal compact supergravity group H ought to account for the above discrepancy, as well as for the extra Abelian factor $\mathfrak{u}(1)_d$ mentioned in the discussion around (3.16), with charge given by the difference of the Left and Right helicities. It is given by

$$\mathfrak{h}(D, \mathcal{N}, \tilde{\mathcal{N}}) = \mathfrak{int}(\mathcal{N}, D) \oplus \mathfrak{int}(\tilde{\mathcal{N}}, D) \oplus \delta_{D4}\mathfrak{u}(1)_d + \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]. \tag{3.34}$$

The compact expression (3.34) is valid in all dimensions thanks to the unifying potential of the division algebraic notation, which captures at once the nature of the minimal spinors, hence the R-symmetry algebras, in all dimensions.

Naturally, it may be simplified by restricting to a specific spacetime dimension, where it then depends simply on $\mathcal{N}$ and $\tilde{\mathcal{N}}$, thus yielding a sequence of algebras corresponding to supergravities with different amounts of supersymmetry. In $D = 8$, for instance, the internal symmetry of the $\mathcal{N}$-extended Yang-Mills multiplet is $\mathfrak{u}(\mathcal{N})$, which fixes $\mathbb{D} = \mathbb{C}$ and implies

$$\begin{aligned} \mathfrak{h}(8, \mathcal{N}, \tilde{\mathcal{N}}) &= \mathfrak{int}(\mathcal{N}, 8) \oplus \mathfrak{int}(\tilde{\mathcal{N}}, 8) + \mathbb{C}[\mathcal{N}, \tilde{\mathcal{N}}] \\ &= \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(\tilde{\mathcal{N}}) \oplus \tilde{\mathfrak{u}}(1) + \mathbb{C}[\mathcal{N}, \tilde{\mathcal{N}}] \\ &= \mathfrak{u}(1)' \oplus \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{su}(\tilde{\mathcal{N}}) \oplus \mathfrak{u}(1)'' + \mathbb{C}[\mathcal{N}, \tilde{\mathcal{N}}] \\ &= \mathfrak{u}(1)' \oplus \mathfrak{su}(\mathcal{N} + \tilde{\mathcal{N}}) \end{aligned} \tag{3.35}$$

where the difference between the second and the third line highlights that a rotation of the $\mathfrak{u}(1)$ charges may be needed in order to find the correct embedding; two examples of such rotations will be presented in Section 4.2. In fact, as it is clear from Table 2.3, there is a unique $D = 8$, $\mathcal{N} = 1$ sYM multiplet with R-symmetry algebra $\mathfrak{u}(1)$; therefore, according to (3.35), the supergravity resulting from tensoring it with a second copy of itself should have $\mathfrak{h} = \mathfrak{u}(2)$, which is indeed the (Lie algebra of the) correct compact symmetry group of the unique $\mathcal{N} = 2$ supergravity in that dimension, as may be verified in Table 2.5. The results for all D, $\mathcal{N}$ and $\tilde{\mathcal{N}}$ constitute the $\mathfrak{h}$ entries of the so-called *magic pyramid* of supergravity theories. At the base of the pyramid is the 4×4 magic square of supergravities in $D = 3$, which was the subject of the previous section and whose entries are tabulated in Table 3.2. Then follow a 3×3 square in $D = 4$, 2×2 squares in $D = 5, 6$ and a single theory in each dimension $7 \leq D \leq 10$, whose entries are summarised in Tables 3.3 through 3.6.

Note that equation (3.34) may be thought of as the decomposition of the adjoint representation of $\mathfrak{h}$ under the branching $H \supset Int(\mathcal{N}, D) \times Int(\tilde{\mathcal{N}}, D)$, expressing the $\dim(\mathfrak{h})$ generators in a manifest $\mathfrak{int} \oplus \mathfrak{int}$ basis, which in turn suggests the Yang-Mills origin of the enhancement: in addition to the generators of the left and right internal symmetry algebras, which live in the $(\mathbf{adj}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{adj})$ respectively, thus manifestly only acting on the left (resp. right) sYM factor in the tensor product, the formula suggests that the missing generators necessary to obtain a full $\mathfrak{h}$-transformation should live in the space $\mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]$ and carry the representation $(\mathbf{def}, \mathbf{def})$. It is then tempting to suppose that these correspond to the scalar part[7] of the tensor product of supersymmetry generators, $Q_L \otimes \tilde{Q}_R$, which indeed spans the correct on-shell vector space,

$$Q \otimes \tilde{Q} \in \mathbb{D}^{\mathcal{N}} \otimes \mathbb{D}^{\tilde{\mathcal{N}}} \cong \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}], \tag{3.36}$$

and transforms in the bi-defining as required. Indeed, said product of Grassmann-odd generators can be shown to yield a bosonic transformation of the supergravity states: it

[7] In the sense that it commutes with the little group generators and can thus be considered internal.

corresponds precisely to the off-diagonal components of the $\mathfrak{h}$-transformation when decomposed according to $\mathfrak{int} \oplus \mathfrak{int}$, *provided that* the momentum factors arising (in momentum space) from the derivatives in the supersymmetry variations are dropped. Because of this fact, however, this identification remains more of a useful heuristic argument, rather than a solid proof. An example displaying all this is the subject of Section 3.2.4.

Non-compact symmetry

As was done for $D = 3$ before, it is possible to concoct a similarly unified formula which yields the Lie algebras $\mathfrak{g}$ of the corresponding non-compact symmetry groups, G. One constructs the non-compact complement $\mathfrak{p}$ to be adjoined to $\mathfrak{h}$ using the knowledge of the space spanned by the supergravity scalars as it arises from the tensor product. It is given, as a vector space, by

$$\mathfrak{p}(D, \mathcal{N}, \tilde{\mathcal{N}}) = (1 + \delta_{D4} - \delta_{D3})\mathbb{R} \otimes \mathbb{R} + \mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}] + \mathbf{S}(D, \mathcal{N}) \otimes \mathbf{S}(D, \tilde{\mathcal{N}}), \tag{3.37}$$

where the three terms symbolise the three distinct origins of supergravity scalar fields in the tensor product of two sYM theories, namely:

- The tensor product of the two vectors, which always gives one scalar (the trace). In $D = 4$ there is an extra scalar contribution from the dualisation of the Kalb-Ramond 2-form, $B_{\mu\nu}$, while in $D = 3$ we must subtract one degree of freedom since, having dualised both sYM vectors to scalars, their contribution is already accounted for in the third term.

- The tensor product of two gaugini which, just like for the supercharges in (3.36), is expanded on the Clifford algebra basis. The rank-0 contribution of the Fierz expansion may be succintly represented as $\mathbb{D}[\mathcal{N}, \tilde{\mathcal{N}}]$, which automatically takes into account the reality properties of the two gaugini entering the formula.

- The tensor product of the scalars of the Left and Right theories, each set spanning a space $\mathbf{S}(D, \mathcal{N}) \cong \mathbb{R}^{\mathcal{Q}/2 - D + 2 + \delta_{D3}}$.

Again, a Lie algebra structure may be given to this vector space by checking the commutators [88]. All the resulting $\mathfrak{g}$ algebras are summarised, together with their maximal compact subalgebras, in the tables below.

Finally, a comment. It is not clear, to date, what to say about the Yang-Mills origin of the generators in $\mathfrak{p}$: all known symmetry generators of the underlying sYM theories conspire to form the linear action of $\mathfrak{h}$ on the supergravity states, with the supersymmetry generators heuristically acting as the driving force behind the enhancement $\mathfrak{int} \oplus \mathfrak{int} \subset \mathfrak{h}$. Which combination, if any, of gauge theory transformations (not necessarily leaving the sYM action invariant) should come together to form the non-linear action of the full U-duality group Lie algebra, $\mathfrak{g}$, is a very intriguing outstanding question.

$D=4$ $\mathfrak{u}(1)_{st}$	$\mathbf{V}_4$ $\mathfrak{su}(4)$	$\mathbf{V}_2$ $\mathfrak{u}(2)$	$\mathbf{V}_1$ $\mathfrak{u}(1)$
$\mathbf{V}_4$ $\mathfrak{su}(4)$	$\mathbf{G}_8$ $\mathfrak{g}=\mathfrak{e}_{7(7)}$ $\mathfrak{h}=\mathfrak{su}(8)$	$\mathbf{G}_6$ $\mathfrak{g}=\mathfrak{so}^\star(12)$ $\mathfrak{h}=\mathfrak{u}(6)$	$\mathbf{G}_5$ $\mathfrak{g}=\mathfrak{su}(1,5)$ $\mathfrak{h}=\mathfrak{u}(5)$
$\mathbf{V}_2$ $\mathfrak{u}(2)$	$\mathbf{G}_6$ $\mathfrak{g}=\mathfrak{so}^\star(12)$ $\mathfrak{h}=\mathfrak{u}(6)$	$\mathbf{G}_4+2\mathbf{V}_4$ $\mathfrak{g}=\mathfrak{su}(1,1)\oplus\mathfrak{so}(2,6)$ $\mathfrak{h}=\mathfrak{u}(4)\oplus\mathfrak{u}(1)$	$\mathbf{G}_3+\mathbf{V}_3$ $\mathfrak{g}=\mathfrak{u}(1,3)$ $\mathfrak{h}=\mathfrak{u}(3)\oplus\mathfrak{u}(1)$
$\mathbf{V}_1$ $\mathfrak{u}(1)$	$\mathbf{G}_5$ $\mathfrak{g}=\mathfrak{su}(1,5)$ $\mathfrak{h}=\mathfrak{u}(5)$	$\mathbf{G}_3+\mathbf{V}_3$ $\mathfrak{g}=\mathfrak{u}(1,3)$ $\mathfrak{h}=\mathfrak{u}(3)\oplus\mathfrak{u}(1)$	$\mathbf{G}_2+\mathbf{H}_2$ $\mathfrak{g}=\mathfrak{u}(1,2)$ $\mathfrak{h}=\mathfrak{u}(2)\oplus\mathfrak{u}(1)$

$D=5$ $\mathfrak{sp}(1)_{st}$	$\mathbf{V}_4$ $\mathfrak{sp}(2)$	$\mathbf{V}_2$ $\mathfrak{sp}(1)$
$\mathbf{V}_4$ $\mathfrak{sp}(2)$	$\mathbf{G}_8$ $\mathfrak{g}=\mathfrak{e}_{6(6)}$ $\mathfrak{h}=\mathfrak{sp}(4)$	$\mathbf{G}_6$ $\mathfrak{g}=\mathfrak{su}^\star(6)$ $\mathfrak{h}=\mathfrak{sp}(3)$
$\mathbf{V}_2$ $\mathfrak{sp}(1)$	$\mathbf{G}_6$ $\mathfrak{g}=\mathfrak{su}^\star(6)$ $\mathfrak{h}=\mathfrak{sp}(3)$	$\mathbf{G}_4+\mathbf{V}_4$ $\mathfrak{g}=\mathfrak{o}(1,1)\oplus\mathfrak{o}(5,1)$ $\mathfrak{h}=\mathfrak{sp}(2)$

$D=6$ $2\mathfrak{sp}(1)_{st}$	$\mathbf{V}_{(2,2)}$ $2\mathfrak{sp}(1)$	$\mathbf{V}_{(0,2)}$ $\mathfrak{sp}(1)$
$\mathbf{V}_{(2,2)}$ $2\mathfrak{sp}(1)$	$\mathbf{G}_{(4,4)}$ $\mathfrak{g}=\mathfrak{o}(5,5)$ $\mathfrak{h}=\mathfrak{sp}(2)\oplus\mathfrak{sp}(2)$	$\mathbf{G}_{(2,4)}$ $\mathfrak{g}=\mathfrak{sp}(1)\oplus\mathfrak{su}^*(4)$ $\mathfrak{h}=\mathfrak{sp}(1)\oplus\mathfrak{sp}(2)$
$\mathbf{V}_{(2,0)}$ $\mathfrak{sp}(1)$	$\mathbf{G}_{(4,2)}$ $\mathfrak{g}=\mathfrak{su}^\star(4)\oplus\mathfrak{sp}(1)$ $\mathfrak{h}=\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$	$\mathbf{G}_{(2,2)}$ $\mathfrak{g}=\mathfrak{o}(1,1)\oplus 2\mathfrak{sp}(1)$ $\mathfrak{h}=\mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$

Table 3.3: Tensoring pure super Yang-Mills multiplets in $D = 4, 5, 6$. The first yields the 4×4 layer of the pyramid, while the remaining two yield 3×3 layers. Each vector multiplet entering the product is given with its R-symmetry group algebra, while the resulting supergravity theory is given with the algebras associated to the non-compact U-duality group G and its maximal compact subgroup H.

$D = 6$ $2\mathfrak{sp}(1)_{st}$	$\mathbf{V}_{(2,0)}$ $\mathfrak{sp}(1)$
$\mathbf{V}_{(2,0)}$ $\mathfrak{sp}(1)$	$\mathbf{G}_{(4,0)} + \mathbf{T}_{(4,0)}$ $\mathfrak{g} = \mathfrak{o}(5,1)$ $\mathfrak{h} = \mathfrak{sp}(2)$

$D = 7$ $\mathfrak{sp}(2)_{st}$	$\mathbf{V}_2$ $\mathfrak{sp}(1)$
$\mathbf{V}_2$ $\mathfrak{sp}(1)$	$\mathbf{G}_4$ $\mathfrak{g} = \mathfrak{sl}(5;\mathbb{R})$ $\mathfrak{h} = \mathfrak{sp}(2)$

Table 3.4: On the left, the remaining $D = 6$ product, that of two super Yang-Mills multiplets of the same chirality. On the right, the unique $D = 7$ squaring product.

$D = 8$ $2\mathfrak{su}(4)_{st}$	$\mathbf{V}_1$ $\mathfrak{u}(1)$
$\mathbf{V}_1$ $\mathfrak{u}(1)$	$\mathbf{G}_2$ $\mathfrak{g} = \mathfrak{sl}(2;\mathbb{R}) \oplus \mathfrak{sl}(3;\mathbb{R})$ $\mathfrak{h} = \mathfrak{u}(2)$

$D = 9$ $\mathfrak{so}(7)_{st}$	$\mathbf{V}_1$ $\varnothing$
$\mathbf{V}_1$ $\varnothing$	$\mathbf{G}_2$ $\mathfrak{g} = \mathfrak{o}(1,1) \oplus \mathfrak{sl}(2;\mathbb{R})$ $\mathfrak{h} = \mathfrak{so}(2)$

Table 3.5: The unique squaring products in $D = 8, 9$. Note that in the latter case, $Q \otimes \tilde{Q}$ constructs the single generator of $\mathfrak{so}(2)$.

$D = 10$ $\mathfrak{so}(8)_{st}$	$\mathbf{V}_{(0,1)}$ $\varnothing$
$\mathbf{V}_{(1,0)}$ $\varnothing$	$\mathbf{G}_{(1,1)}$ $\mathfrak{g} = \varnothing$ $\mathfrak{h} = \varnothing$

$D = 10$ $\mathfrak{so}(8)_{st}$	$\mathbf{V}_{(1,0)}$ $\varnothing$
$\mathbf{V}_{(1,0)}$ $\varnothing$	$\mathbf{G}_{(2,0)}$ $\mathfrak{g} = \mathfrak{sl}(2;\mathbb{R})$ $\mathfrak{h} = \mathfrak{so}(2)$

Table 3.6: The $D = 10$ products: squaring multiplets of opposite chirality yields Type IIA supergravity (left), whilst same chirality multiplets give Type IIB (right).

3.2.4 Squaring $\mathcal{N}=4$ sYM gives $\mathcal{N}=8$ supergravity

Let us illustrate this by means of an example. We choose the prototypical case of $D=4$, $\mathcal{N}=8$ supergravity as the square of $\mathcal{N}=4$ sYM because it is one of the most studied in the amplitude literature, because the reader may be already familiar with it and, finally, because we will need it again in Chapter 4. Consider the $D=4$, $\mathcal{N}=4$ vector multiplet, whose content under $\mathfrak{u}(1)_{\text{st}}\oplus\mathfrak{su}(4)$ is given by

$$\mathbf{V}_4=\{A_\mu,\lambda^i,\phi^a\}=\{\mathbf{1}^2+\mathbf{1}^{-2},\mathbf{4}^1+\overline{\mathbf{4}}^{-1},\mathbf{6}^0\}, \tag{3.38}$$

reflecting the fact that it is CPT self-conjugate and thus cannot support the $\mathfrak{u}(1)\subset\mathfrak{u}(4)$. As a consequence, we have one less internal $\mathfrak{u}(1)$ charge to work with, which renders this example particularly amenable[8]. Thus, the states of the squaring product $\mathbf{V}_4\otimes\mathbf{V}_4$ are labelled by $\mathfrak{u}(1)_{\text{st}}\oplus 2\mathfrak{su}(4)\oplus\mathfrak{u}(1)_d$ and are given in Table 3.7.

	$\mathbf{1}^2+c.c.$	$\mathbf{4}^1+c.c.$	$\mathbf{6}^0$
$\mathbf{1}^2+c.c.$	$(\mathbf{1},\mathbf{1})^4_0+(\mathbf{1},\mathbf{1})^0_4+c.c.$	$(\mathbf{1},\mathbf{4})^3_1+(\mathbf{1},\overline{\mathbf{4}})^1_3+c.c.$	$(\mathbf{1},\mathbf{6})^2_2+c.c.$
$\mathbf{4}^1+c.c$	$(\mathbf{4},\mathbf{1})^3_{-1}+(\overline{\mathbf{4}},\mathbf{1})^1_{-3}+c.c.$	$(\mathbf{4},\mathbf{4})^2_0+(\mathbf{4},\overline{\mathbf{4}})^0_2+c.c.$	$(\mathbf{4},\mathbf{6})^1_1+c.c.$
$\mathbf{6}^0$	$(\mathbf{6},\mathbf{1})^2_{-2}+c.c.$	$(\mathbf{6},\mathbf{4})^1_{-1}+c.c.$	$(\mathbf{6},\mathbf{6})^0_0$

Table 3.7: Squaring $\mathbf{V}_4\otimes\mathbf{V}_4$ in four dimensions. The positive helicity states are presented explicitly.

According to (3.18), it should be possible to find an enhancement of the internal symmetries. This may be easily seen to be the case upon organising the states according as their little group representation: in $D=4$ it suffices to show this for either the positive or the negative helicity states. Choosing the former here we obtain, for helicities ranging from (4) to (0),

Under $\quad\mathfrak{su}(8)\supset\mathfrak{su}(4)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)_d$,

$$\begin{aligned}
\mathbf{1}&\to(\mathbf{1},\mathbf{1})_0\\
\mathbf{8}&\to(\mathbf{4},\mathbf{1})_{-1}+(\mathbf{1},\mathbf{4})_1\\
\mathbf{28}&\to(\mathbf{6},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{6})_2+(\mathbf{4},\mathbf{4})_0\\
\mathbf{56}&\to(\overline{\mathbf{4}},\mathbf{1})_{-3}+(\mathbf{1},\overline{\mathbf{4}})_3+(\mathbf{6},\mathbf{4})_{-1}+(\mathbf{4},\mathbf{6})_1\\
\mathbf{70}&\to(\mathbf{1},\mathbf{1})_4+(\mathbf{1},\mathbf{1})_{-4}+(\mathbf{6},\mathbf{6})_0+(\mathbf{4},\overline{\mathbf{4}})_2+(\overline{\mathbf{4}},\mathbf{4})_{-2}
\end{aligned} \tag{3.39}$$

[8]In general, if more than one internal $\mathfrak{u}(1)$ is inherited by the states after squaring, a rotation of the charges is needed to match the precise representation content of supergravity.

showing that all states in a particular set, specified by helicity, may be consistently put in a representation of $\mathfrak{h} = \mathfrak{su}(8)$. Expressing the adjoint of the latter in a $\mathfrak{su}(4)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)$ basis,

$$\mathbf{63} \longrightarrow (\mathbf{15},\mathbf{1})_0 + (\mathbf{1},\mathbf{15})_0 + (\mathbf{1},\mathbf{1})_0 + (\mathbf{4},\overline{\mathbf{4}})_2 + (\overline{\mathbf{4}},\mathbf{4})_{-2}, \tag{3.40}$$

helps understand the Yang-Mills origin of the $\mathfrak{su}(8)$ generators. Consider, for instance, the two negative helicity gravitino states coming from squaring, namely

$$\Psi_1^i = \lambda^i \otimes \tilde{V} \qquad (\mathbf{4},\mathbf{1})_1^{-3} \tag{3.41}$$
$$\Psi_2^{i'} = V \otimes \tilde{\lambda}^{i'} \qquad (\mathbf{1},\mathbf{4})_{-1}^{-3} \tag{3.42}$$

They may be put together in a 8-dimensional column, $\Psi^I = (\Psi_1^i, \Psi_2^{i'})^T$, which should transform under the full $\mathfrak{su}(8)$. The index $I = 1,\ldots,8$ is that of the bona fide defining representation of $\mathfrak{su}(8)$. On-shell and at the infinitesimal level, the states of the two super Yang-Mills factors are transformed under the spacetime and R-symmetries as

$$\delta_{st}V = 2i\alpha V, \qquad \delta_R V = 0, \tag{3.43}$$
$$\delta_{st}\lambda^i = i\alpha\lambda^i, \qquad \delta_R\lambda^i = \theta^m (T_m)^i{}_j \lambda^j, \tag{3.44}$$

where $(T_m)^i{}_j \in \mathfrak{su}(4)$, and with a similar set for the states of the Right factor (i.e. the states with tildas). Then, simply exploiting the R-symmetries of the two factors, and denoting transformations of the supergravity theory with a prime, δ', the doublet inherits the transformation

$$\begin{aligned}\delta'_R\Psi^I = \delta'_R\begin{pmatrix}\Psi_1^i\\ \Psi_2^{i'}\end{pmatrix} &= \begin{pmatrix}\delta_R\lambda^i\otimes\tilde{V} + \lambda^i\otimes\delta_R\tilde{V}\\ \delta_R V\otimes\tilde{\lambda}^{i'} + V\otimes\delta_R\tilde{\lambda}^{i'}\end{pmatrix}\\ &= \begin{pmatrix}\theta^m(T_m)^i{}_j\Psi_1^j\\ \tilde{\theta}^{m'}(\tilde{T}_{m'})^{i'}{}_{j'}\Psi_2^{j'}\end{pmatrix}\\ &= \begin{pmatrix}\theta^m(T_m)^i{}_j & 0\\ 0 & \tilde{\theta}^{m'}(\tilde{T}_{m'})^{i'}{}_{j'}\end{pmatrix}\begin{pmatrix}\Psi_1^j\\ \Psi_2^{j'}\end{pmatrix}\end{aligned} \tag{3.45}$$

corresponding to the $(\mathbf{15},\mathbf{1})$ and $(\mathbf{1},\mathbf{15})$. The singlet contribution $(\mathbf{1},\mathbf{1})$ is due to the peculiar feature present only in four dimensions, which allows to construct two mutually commuting transformations on the tensor product (read supergravity) states, namely $\delta'_{st} = \delta_{st} + \tilde{\delta}_{st}$ and $\delta'_d = \delta_{st} - \tilde{\delta}_{st}$. Since the former is identified with the spacetime little group algebra transformation, the latter is necessarily interpreted, by virtue of its commutativity with δ'_{st}, as an internal symmetry on the supergravity states. In this case,

$$\begin{aligned}\delta'_d\Psi^I = \delta'_d\begin{pmatrix}\Psi_1^i\\ \Psi_2^{i'}\end{pmatrix} &= \begin{pmatrix}\delta_{st}\lambda^i\otimes\tilde{V} - \lambda^i\otimes\delta_{st}\tilde{V}\\ \delta_{st}V\otimes\tilde{\lambda}^{i'} - V\otimes\delta_{st}\tilde{\lambda}^{i'}\end{pmatrix}\\ &= i\alpha\begin{pmatrix}-\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\begin{pmatrix}\Psi_1^j\\ \Psi_2^{j'}\end{pmatrix}.\end{aligned} \tag{3.46}$$

This exhausts the effect of varying the sYM pieces under their respective bosonic symmetry transformations. Finally, one may consider the simultaneous variation of both sYM factors under a supersymmetry transformation. Schematically, at the Yang-Mills level one has that

$$\delta_\epsilon V \sim \epsilon_i^* \lambda^i, \tag{3.47}$$

$$\delta_\epsilon \lambda^i \sim V \epsilon^i + \ldots, \tag{3.48}$$

where we are only interested in the piece of the supersymmetry variation of the spinors proportional to the vector. Then, the simultaneous application on the states in the tensor product induces the supergravity transformation,

$$\delta' \Psi_1^i \equiv \delta_\epsilon \lambda^i \otimes \delta_{\tilde{\epsilon}} \tilde{V} \sim (\epsilon^j \tilde{\epsilon}^*_{j'})(V \otimes \tilde{\lambda}^{j'}) + \cdots \equiv T^j{}_{j'} \Psi_2^{j'}, \tag{3.49}$$

$$\delta' \Psi_2^{i'} \equiv \delta_\epsilon V \otimes \delta_{\tilde{\epsilon}} \tilde{\lambda}^{i'} \sim (-\epsilon^*_j \tilde{\epsilon}^{j'})(\lambda^j \otimes \tilde{V}) + \cdots \equiv U_j{}^{j'} \Psi_1^j, \tag{3.50}$$

which shows how the $(\mathbf{4},\mathbf{1})$ and the $(\mathbf{1},\mathbf{4})$ get rotated into each other. The dots correspond to pieces proportional to (scalar)×(spinor) terms, which are discarded on the basis that they do not commute with spacetime. Using $(\epsilon^j \tilde{\epsilon}^*_{j'})^* = \epsilon^*_j \tilde{\epsilon}^{j'}$, one identifies $U_j{}^{j'} \equiv (-T^\dagger)^{j'}{}_j$. When represented on the doublet,

$$\delta' \Psi^I = \delta' \begin{pmatrix} \Psi_1^i \\ \Psi_2^{i'} \end{pmatrix} = \begin{pmatrix} 0 & T^j{}_{j'} \\ (-T^\dagger)^{j'}{}_j & 0 \end{pmatrix} \begin{pmatrix} \Psi_1^j \\ \Psi_2^{j'} \end{pmatrix}, \tag{3.51}$$

these off-diagonal entries are the "missing" generators in the $(\mathbf{4},\overline{\mathbf{4}})+(\overline{\mathbf{4}},\mathbf{4})$, and completes the set of $\mathfrak{su}(8)$ generators. Notice, as mentioned before, that this is only a heuristic argument, whose role is to give an idea of what kind of transformation of the two factors indeed reproduces the required missing generators. The reason it fails to be more than this is that the variation of the spinor is really given by (2.28) which may be seen to contain a derivative of the vector field, in the gauge-invariant combination $F_{\mu\nu}$. When transported in momentum space and reduced on-shell, the derivative becomes a factor of energy, E, which need to be dropped by hand.

Finally, focussing on the space spanned by the scalars, one sees from (3.39) that it carries the action of the $\mathbf{70}$ of $\mathfrak{h}$; recalling its interpretation as the space of the non-compact generators of G, one is able to derive $\mathfrak{g}$ as the Lie algebra whose adjoint representation decomposes, under $G \supset H$, as $\mathfrak{g} = \mathfrak{h} + \mathfrak{p} = \mathbf{63} + \mathbf{70}$. Thus, G should be a group with real dimension 133 and maximal compact subgroup $SU(8)$. This fixes $\mathfrak{g} = \mathfrak{e}_{7(7)}$ and, consequently, we identify the resulting scalar coset as $E_{7(7)}/SU(8)$. The result of this squaring is denoted by

$$\mathbf{V}_4 \otimes \mathbf{V}_4 = \mathbf{G}_8. \tag{3.52}$$

3.3 Generalised squaring

The previous section has been devoted to "squaring" pure $\mathcal{N}$-extended super Yang-Mills theories. All possible products in $3 \leq D \leq 10$ form the so-called magic pyramid, a sequence of $n \times n$ arrays, one in each dimension, with n indicating the number of allowed sYM multiplets in each specific dimension, decreasing as D increases. Each of the arrays' entries are (the $\mathfrak{g}$ and $\mathfrak{h}$ algebras of) supergravity theories, whose content and symmetries are related in a consistent fashion to a pair of super Yang-Mills factors. However, it is clear from the tables in Section 3.2.3 that only a small subset of all supergravities are produced this way: while virtually all more-than-half-maximal $16 < \mathcal{Q} \leq 32$ theories admit a factorisation and hence lie in the pyramid, the wealth of supergravities theories with half-maximal or even smaller number of real supercharges, $4 \leq \mathcal{Q} \leq 16$, is far from being captured by the pyramid alone. One question naturally comes to mind:

Are the supergravities in the pyramid the only cases where the factorisation into two super Yang-Mills factors is possible?

The answer is negative. It turns out that a much larger class of supergravities is constructible in a similar way, once the correct extension of the procedure above is found. Thankfully, the issue is clear. Squaring pure super Yang-Mills multiplets gives rise to only a fixed (very small) number of supergravity theories with very specific spectra, and thus a very limited number of matter couplings, while in general supergravities with low supersymmetry may be coupled to matter multiplets in a (sometimes infinite) number of ways. Thus, the question becomes: *what is the Yang-Mills origin of general matter couplings in supergravity?*

3.3.1 Squaring hypers and single vectors

The resolution of the problem hinges on the realisation that new products should be considered, alongside those of the $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$ type. In particular, it is necessary to include

- $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_0$: the product of a vector multiplet with a single gauge vector, here treated as an "$\mathcal{N} = 0$ multiplet" and denoted as $\mathbf{V}_0$. These are summarised in Table 3.8.

- $\mathbf{H}_{\mathcal{N}} \otimes \tilde{\mathbf{H}}_{\tilde{\mathcal{N}}}$: the product of two hypermultiplets (or chirals, in $D = 3, 4$), which may result in either vector or tensor multiplets. In the same spirit as for the vector multiplets, we also extend this kind of product to the $\mathcal{N} = 0$ case, namely when one factor is a minimal spinor, λ. These are summarised in Table 3.9.

Notice that cross products of the type $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{H}}_{\tilde{\mathcal{N}}}$ produce non-dynamical "gravitino multiplets", e.g. the $\left(\frac{3}{2}, 1\right)$ multiplet in $D = 4$, known not to correspond to independent interacting field theories. While these products yield sensible results[9], they are omitted

[9]In the sense that, just like all other products, they consistently map the pair to (a sum of) known multiplets.

D	$\mathbf{V}_{\mathcal{N}} \otimes \mathbf{V}_0$	Supergravity	D	$\mathbf{V}_{\mathcal{N}} \otimes \mathbf{V}_0$	Supergravity
10	$\mathbf{V}_{(1,0)} \otimes \mathbf{V}_{(0,0)}$	$\mathbf{G}_{(1,0)}$	5	$\mathbf{V}_4 \otimes \mathbf{V}_0$	$\mathbf{G}_4$
9	$\mathbf{V}_1 \otimes \mathbf{V}_0$	$\mathbf{G}_1$		$\mathbf{V}_2 \otimes \mathbf{V}_0$	$\mathbf{G}_2 + \mathbf{V}_2$
8	$\mathbf{V}_1 \otimes \mathbf{V}_0$	$\mathbf{G}_1$	4	$\mathbf{V}_4 \otimes \mathbf{V}_0$	$\mathbf{G}_4$
7	$\mathbf{V}_2 \otimes \mathbf{V}_0$	$\mathbf{G}_2$		$\mathbf{V}_2 \otimes \mathbf{V}_0$	$\mathbf{G}_2 + \mathbf{V}_2$
6	$\mathbf{V}_{(2,2)} \otimes \mathbf{V}_{(0,0)}$	$\mathbf{G}_{(2,2)}$		$\mathbf{V}_1 \otimes \mathbf{V}_0$	$\mathbf{G}_1 + \mathbf{C}_1$
	$\mathbf{V}_{(2,0)} \otimes \mathbf{V}_{(0,0)}$	$\mathbf{G}_{(2,0)} + \mathbf{T}_{(2,0)}$			

Table 3.8: Squaring vector multiplets with a single gauge vector.

D	$\mathbf{H}_{\mathcal{N}} \otimes \mathbf{H}_{\mathcal{N}}$	Supergravity	D	$\mathbf{H}_{\mathcal{N}} \otimes \mathbf{H}_{\mathcal{N}}$	Supergravity
6	$\mathbf{H}_{(2,0)} \otimes \mathbf{H}_{(2,0)}$	$4\mathbf{T}_{(4,0)}$	4	$\mathbf{H}_2 \otimes \mathbf{H}_2$	$4\mathbf{V}_4$
	$\mathbf{H}_{(2,0)} \otimes \mathbf{H}_{(0,2)}$	$4\mathbf{V}_{(2,2)}$		$\mathbf{H}_2 \otimes \mathbf{C}_1$	$2\mathbf{V}'_3$
	$\mathbf{H}_{(2,0)} \otimes \lambda_L$	$2\mathbf{T}_{(2,0)}$		$\mathbf{H}_2 \otimes \lambda$	$2\mathbf{V}_2$
	$\mathbf{H}_{(2,0)} \otimes \lambda_R$	$2\mathbf{V}_{(2,0)}$		$\mathbf{C}_1 \otimes \mathbf{C}_1$	$\mathbf{V}_2 + \mathbf{H}_2$
5	$\mathbf{H}_2 \otimes \mathbf{H}_2$	$4\mathbf{V}_4$		$\mathbf{C}_1 \otimes \lambda$	$\mathbf{V}_1 + \mathbf{C}_1$
	$\mathbf{H}_2 \otimes \lambda$	$2\mathbf{V}_2$			

Table 3.9: Tensor products of two hyper- or chiral multiplets. For pseudoreal representations of the gauge group, the corresponding products involving half-hypermultiplets $\mathbf{C}_2$ are obtained from those of full hypers via the identification $\mathbf{C}_2 = \mathbf{H}_2/2$.

from the analysis here as we are ultimately interested in the field theoretic realisations of the corresponding multiplets. In $D = 6$ there is the additional possibility of squaring tensor multiplets, $\mathbf{T}_{\mathcal{N}} \otimes \tilde{\mathbf{T}}_{\tilde{\mathcal{N}}}$. This is by all means allowed, but we do not include a detailed presentation here as this is not relevant to the developments of the present text; consult e.g. [139] for more information. However, let us note that a subset of these products[10] yield all possible supergravity multiplets in $D = 6$, all of which however may be alternatively constructed from squaring $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$. A subtle difference between the two types of products is the following: the tensor multiplets must be used if one requires the product to yield only $\mathbf{G}_{(2,0)}$ or $\mathbf{G}_{(4,0)}$ since, in vector multiplet products, these arise along with additional tensor multiplets, e.g. for the latter

$$\mathbf{V}_{(2,0)} \otimes \tilde{\mathbf{V}}_{(2,0)} = \mathbf{G}_{(4,0)} + \mathbf{T}_{(4,0)} \qquad \text{vs} \qquad \mathbf{T}_{(4,0)} \otimes \tilde{\mathbf{T}}_{(0,0)} = \mathbf{G}_{(4,0)}. \tag{3.53}$$

[10]The remaining products yield conformal multiplets instead, possibly coupled to tensors.

Squaring non-adjoint matter

A few more comments are in order. Notice that it is not strictly true in all cases that the product should be limited to adjoint representations of the gauge group, with a spectator scalar valued in the bi-adjoint representation of $\mathcal{G} \times \tilde{\mathcal{G}}$, as in (3.6). In the present context, while this is still obviously valid for $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_0$ products, it need not be when squaring hyper or chiral multiplets, which may live in any arbitrary representation. Given two such multiplets in non-adjoint gauge group representations ρ and $\tilde{\rho}$, we similarly postulate the existence of a scalar field valued in the $\rho \otimes \tilde{\rho}$ representation which mediates the product as

$$\mathbf{H}_{\mathcal{N}} \circ \tilde{\mathbf{H}}_{\tilde{\mathcal{N}}} \equiv \mathbf{H}^a_{\mathcal{N}} \otimes \Phi^{-1}_{aa'} \otimes \tilde{\mathbf{H}}^{a'}_{\tilde{\mathcal{N}}}. \tag{3.54}$$

with indices $a = 1, \ldots, \dim\rho$ and $a' = 1, \ldots, \dim\tilde{\rho}$. Notice that it does not suffice for the two representations to be non-adjoint; it is also necessary that they share the same reality properties. In other words, it is not possible to take the product of a real and a complex representation, say. All of this is relevant because, in order to couple arbitrary matter to the supergravity multiplets, we will be led to consider more general squaring products where each factor is given by a vector multiplet coupled to a non-adjoint multiplet $\mathbf{H}^a_{\mathcal{N}}$. In this case, consistency implies the "sum of squares" rule,

$$(\mathbf{V}^A_{\mathcal{N}} \oplus \mathbf{H}^a_{\mathcal{N}}) \otimes (\tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}} \oplus \tilde{\mathbf{H}}^{a'}_{\tilde{\mathcal{N}}}) = (\mathbf{V}^A_{\mathcal{N}} \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}}) \oplus (\mathbf{H}^a_{\mathcal{N}} \otimes \tilde{\mathbf{H}}^{a'}_{\tilde{\mathcal{N}}}), \tag{3.55}$$

corresponding, heuristically, to the spectator scalar taking the diagonal form,

$$\Phi_{\mathcal{A}\mathcal{A}'} = \begin{pmatrix} \Phi_{AA'} & 0 \\ 0 & \Phi_{aa'} \end{pmatrix}. \tag{3.56}$$

Indeed, if this were not the case, the products $\mathbf{V}^A_{\mathcal{N}} \otimes \tilde{\mathbf{H}}^{a'}_{\tilde{\mathcal{N}}}$ would yield too many gravitini for the result to correspond to a supergravity theory. This type of squaring, with non-adjoint matter coupled to one or both Yang-Mills factors, is also at the core of our understanding of twin supergravities from Yang-Mills, as discussed in Chapter 4.

3.3.2 Coupling matter to supergravity

These new tools allow one to understand the Yang-Mills origin of matter couplings in supergravity. As highlighted in Section 2.3, supergravity theories are unique for $\mathcal{Q} > 16$. Below this, vector multiplets may be coupled for $\mathcal{Q} \leq 16$ and, finally, hyper and chiral multiplets for $\mathcal{Q} = 8, 4$ respectively. Similarly, on the Yang-Mills side on things, a maximal vector multiplet with $\mathcal{Q} = 16$ does not admit couplings to further matter multiplets, while for half- and quarter-maximal, $\mathcal{Q} = 8, 4$ respectively, it may couple to hyper- and chiral multiplets. Let us discuss here how to obtain vector multiplet couplings to supergravities with $\mathcal{Q} = 16$, to introduce the reader to the general ideas of matter couplings from Yang-Mills. For more details on couplings with lower supersymmetry see [139], while a full classification of vector and hypermultiplet couplings to $\mathcal{Q} = 8$ supergravity with homogeneous scalar manifolds may be found in [27, 91].

First factorisation

The most straightforward way to construct half-maximal supergravities coupled to n vector multiplets consists in

$$\begin{aligned}\mathbf{V}^{A}_{\mathcal{Q}=16}\otimes(\tilde{\mathbf{V}}^{A'}_{0}+n\tilde{\phi}^{A'}) &= (\mathbf{V}^{A}_{\mathcal{Q}=16}\otimes\tilde{\mathbf{V}}^{A'}_{0})+n(\mathbf{V}^{A}_{\mathcal{Q}=16}\otimes\tilde{\phi}^{A'})\\ &=\mathbf{G}_{\mathcal{Q}=16}+n\mathbf{V}_{\mathcal{Q}=16}.\end{aligned} \tag{3.57}$$

The Left factor is simply a maximal vector multiplet with R-symmetry algebra $\mathfrak{r}(\mathcal{N}_{\max}, D)$, while the Right factor is non-supersymmetric, comprising of a single gauge vector and n scalars, both in the adjoint representation of the gauge group. The Right theory possesses an $\mathfrak{so}(n)$ invariance, under which the vector is a singlet and the scalars transform in the defining. In this case, the number of adjoint scalar fields coupled to the non-supersymmetric Yang-Mills theory is a direct measure of the number of vector multiplets in the resulting supergravity, since the term $\mathbf{V}_{\mathcal{Q}=16}\otimes\mathbf{V}_0$ always contributes just the supergravity multiplet, as may be verified in Table 3.8. No enhancement occurs, which is in full agreement with the observation that it is driven by the simultaneous supersymmetry variation of both sYM factors: this cannot happen here as the Right factor is non-supersymmetric. Thus, the compact algebra is simply the sum of the Left and Right symmetries (up to the Abelian factor in four dimensions, which may of course still be present),

$$\mathfrak{h}=\mathfrak{r}(\mathcal{N},D)\oplus\mathfrak{so}(n)\oplus\delta_{D4}\mathfrak{u}(1)_d, \tag{3.58}$$

and the corresponding scalar coset reads

$$\frac{G}{H}=\frac{SO(\phi_L,n)}{SO(\phi_L)\times SO(n)}\times\mathcal{M}_{AA'}, \tag{3.59}$$

with

$$\mathcal{M}_{AA'}=\begin{cases}\varnothing & D=3\\ \frac{SU(1,1)}{U(1)} & D=4\\ O(1,1) & D=5,\ldots,10.\end{cases} \tag{3.60}$$

This matches the correct scalar spaces for $\mathcal{Q}=16$ supergravities coupled to n vectors, see Table 2.5. Notice it is the direct product of two terms: in honest supergravity, this reflects the split between the scalars of the vector multiplets and those belonging to the supergravity multiplet. As it is apparent from (3.60), there is usually only one scalar parameterising $\mathcal{M}_{AA'}$, with two exceptions: in $D=3$ there is none, as the supergravity multiplet is taken to comprise just the graviton and $\mathcal{N}$ gravitini, hence all scalars fall in the matter multiplets; in $D=4$, due to the structure of the little group, there are two such scalars.

Returning to the Yang-Mills origin of the coset (3.59), as the notation suggests, the first term is generated by the scalars arising from $n(A_\mu\circ\tilde{\phi})$, while the second is generated by the scalars in the $A_\mu \circ \tilde{A}_\nu$ sector, which may be checked to indeed belong to the supergravity multiplet. As a consistency check, notice that $A_\mu \circ \tilde{A}_\nu$ yields only one scalar field (the trace $A^\rho \circ \tilde{A}_\rho$) in all cases, with two exceptions: in $D = 3$, since we dualise the vector to a scalar on the Yang-Mills side, we have none; in $D = 4$, after dualisation of the 2-form $A_{[\mu} \circ \tilde{A}_{\nu]}$, we have two such scalars. This exactly matches the situation in honest supergravity discussed above.

Second factorisation

While the factorisation in (3.57) remains valid in all dimensions, there exists another one in dimensions $D = 4, 5, 6$, which exploits the "sum of squares" rule (3.55), as well as highlighting the important role of the half-hypermultiplet in the Yang-Mills Squared construction[11]. It is given by

$$\begin{aligned}(\mathbf{V}^A_{\mathcal{Q}=8} + \mathbf{C}^a_{\mathcal{Q}=8}) \otimes (\tilde{\mathbf{V}}^{A'}_{\mathcal{Q}=8} + (n - n_D)\tilde{\mathbf{C}}^{a'}_{\mathcal{Q}=8}) &= (\mathbf{V}^A_{\mathcal{Q}=8} \otimes \tilde{\mathbf{V}}^{A'}_{\mathcal{Q}=8}) + (n - n_D)(\mathbf{C}^a_{\mathcal{Q}=8} \otimes \tilde{\mathbf{C}}^{a'}_{\mathcal{Q}=8}) \\ &= (\mathbf{G}_{\mathcal{Q}=16} + n_D \mathbf{V}_{\mathcal{Q}=16}) + (n - n_D)\mathbf{V}_{\mathcal{Q}=16} \\ &= \mathbf{G}_{\mathcal{Q}=16} + n\mathbf{V}_{\mathcal{Q}=16},\end{aligned} \tag{3.61}$$

where in the second line the first set of brackets indicate the result of the $\mathbf{V} \otimes \tilde{\mathbf{V}}$ sector, which changes according as the dimension it is being performed in: one has $n_D = 2, 1, 0$ for $D = 4, 5, 6$ respectively, as one can see from Table 3.3. Consequently, in order for the resulting supergravity to have exactly n vector multiplets, one must tune the number of vectors required in the $\mathbf{C} \otimes \tilde{\mathbf{C}}$ sector. Observing from Table 3.9 that $\mathbf{H}_{\mathcal{Q}=8} \otimes \tilde{\mathbf{H}}_{\mathcal{Q}=8} = 4\mathbf{V}_{\mathcal{Q}=16}$ in all dimensions, then $\mathbf{C}_{\mathcal{Q}=8} \otimes \tilde{\mathbf{C}}_{\mathcal{Q}=8} = \mathbf{V}_{\mathcal{Q}=16}$. It is obvious that this type of factorisation can only exist for those dimensions where a hypermultiplet, and hence the half-hyper, is defined, namely up to $D = 6$. Recall that the half-hypermultiplet makes sense on its own only in a pseudoreal representation of the gauge group. The remainder of the internal symmetry representation space is real and may thus be rotated under $\mathfrak{so}(n - n_D)$. In this case, the symmetry directly inherited from squaring is not the full compact symmetry $\mathfrak{h}$ of the supergravity: the two sets of n_D and $(n-n_D)$ vector multiplets coming from the $\mathbf{V} \otimes \tilde{\mathbf{V}}$ and $\mathbf{C} \otimes \tilde{\mathbf{C}}$ sectors, respectively, are brought together under the enhancement

$$\mathfrak{h} \supset \mathfrak{r}(\mathcal{N}_{\text{half}}, D) \oplus \mathfrak{r}(\tilde{\mathcal{N}}_{\text{half}}, D) \oplus \mathfrak{so}(n - n_D), \tag{3.62}$$

occurring according to the same logic as before, since now both sides are supersymmetric.

[11]The half-hypermultiplet C_2 allows for the construction of more general theories in our framework, as well as in the amplitude double copy literature, see for instance [27].

D	Left theory		Right theory		Supergravity
	Content	Symmetry	Content	Symmetry	Content
10	$\mathbf{V}^{A}_{(1,0)}$	$\varnothing$	$\mathbf{V}^{A'}_{(0,0)} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{1,0} + n\mathbf{V}_{1,0}$
9	$\mathbf{V}_{1}$	$\varnothing$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{1} + n\mathbf{V}_{1}$
8	$\mathbf{V}^{A}_{1}$	$\mathfrak{u}(1)$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{1} + n\mathbf{V}_{1}$
7	$\mathbf{V}^{A}_{2}$	$\mathfrak{sp}(1)$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{2} + n\mathbf{V}_{2}$
6	$\mathbf{V}^{A}_{(2,2)}$	$2\mathfrak{sp}(1)$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{(2,2)} + n\mathbf{V}_{(2,2)}$
	$\mathbf{V}^{A}_{(2,0)} + \mathbf{C}^{a}_{(2,0)}$	$\mathfrak{sp}(1)$	$\mathbf{V}^{A'}_{(0,2)} + n\mathbf{C}^{a'}_{(0,2)}$	$\mathfrak{sp}(1) \oplus \mathfrak{so}(n)$	$\mathbf{G}_{(2,2)} + n\mathbf{V}_{(2,2)}$
5	$\mathbf{V}^{A}_{4}$	$\mathfrak{sp}(2)$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{4} + n\mathbf{V}_{4}$
	$\mathbf{V}^{A}_{2} + \mathbf{C}^{a}_{2}$	$\mathfrak{sp}(1)$	$\mathbf{V}^{A'}_{2} + (n-1)\mathbf{C}^{a'}_{2}$	$\mathfrak{sp}(1) \oplus \mathfrak{so}(n-1)$	$(\mathbf{G}_{4} + \mathbf{V}_{4}) + (n-1)\mathbf{V}_{4}$
4	$\mathbf{V}^{A}_{4}$	$\mathfrak{su}(4)$	$\mathbf{V}^{A'}_{0} + n\phi^{A'}$	$\mathfrak{so}(n)$	$\mathbf{G}_{4} + n\mathbf{V}_{4}$
	$\mathbf{V}^{A}_{2} + \mathbf{C}^{a}_{2}$	$\mathfrak{u}(2)$	$\mathbf{V}^{A'}_{2} + (n-2)\mathbf{C}^{a'}_{2}$	$\mathfrak{u}(2) \oplus \mathfrak{so}(n-2)$	$(\mathbf{G}_{4} + 2\mathbf{V}_{4}) + (n-2)\mathbf{V}_{4}$

Table 3.10: Factorisation of $\mathcal{Q} = 16$ supergravity coupled to vector multiplets.

Chapter 4

Twins from Yang-Mills

4.1 Twin supergravities

In Section 2.3 we have reviewed the classification of supergravities, focussing mainly of the symmetries of such theories. Of particular interest was the fact that the scalar fields belonging to a supergravity theory parameterise a homogeneous (and often symmetric) space $\mathcal{M} = G/H$, with G the non-compact U-duality group and H its maximal compact subgroup. The latter corresponds to the R-symmetry for pure supergravities while, if the theory is coupled to matter multiplets, it includes an additional factor rotating these. Whilst all fields in a supergravity theory carry linear representations of H, only (a subset of) the bosons "feel" the action of G: the graviton is invariant, the p-forms (possibly together with their duals where this is appropriate) transform in linear representations, the scalars transform non-linearly. In most cases, knowledge of the field content and of the scalar manifold go a long way in determining the Lagrangian. Furthermore, it is often useful to look purely at the bosonic subsector of supergravity theories, for instance when working out particular background solutions, or to discuss certain aspects of their gaugings.

It is thus a very intriguing fact that certain supergravity theories, despite enjoying completely different amounts of supersymmetry, denoted here by $\mathcal{N}_-$ and $\mathcal{N}_+$ with the obvious assignment $\mathcal{N}_- < \mathcal{N}_+$, share an identical bosonic sector, all the way from the content to interactions and to the scalar manifold itself. After being discussed variously in the literature, they were classified by Samtleben and Roest in [92] and further analysed in [93, 94]. Going further, they may be regarded as the same bosonic theory admitting two distinct fermionic supersymmetric completions. Their classification is carried out in $D = 3$, where all vectors may be dualised to scalars, which implies that the bosonic sector of all supergravities is entirely described by the geometry of the scalar manifold G/H. The considerations at the end of Section 2.3.1 concerning the properties of the scalar manifolds as a function of $\mathcal{Q}$ may be seen to be reflected by the entries of Table 4.1 describing the cosets of $D = 3$ supergravities: unique and one-parameter[1] families of symmetric spaces

[1]For n the number of vector multiplets coupled to the supergravity multiplet.

for $Q > 16$ and $8 < \mathcal{Q} \leq 16$ respectively; described by a certain geometry for $\mathcal{Q} \leq 8$.

$\mathcal{Q}$	Theory$_{\mathcal{N}}$	$\mathcal{M}_{scalar(3)}$
2	$\mathbf{G}_1 + \ldots$	Riemannian
4	$\mathbf{G}_2 + \ldots$	Kähler
6	$\mathbf{G}_3 + \ldots$	Quaternionic
8	$\mathbf{G}_4 + \ldots$	Quaternionic×Quaternionic
10	$\mathbf{G}_5 + n\mathbf{M}_5$	$\frac{\mathrm{Sp}(1,n)}{\mathrm{Sp}(1)\times\mathrm{Sp}(n)}$
12	$\mathbf{G}_6 + n\mathbf{M}_6$	$\frac{\mathrm{SU}(4,n)}{\mathrm{SU}(4)\times\mathrm{SU}(n)\times\mathrm{U}(1)}$
16	$\mathbf{G}_8 + n\mathbf{M}_8$	$\frac{\mathrm{SO}(8,n)}{\mathrm{SO}(8)\times\mathrm{SO}(n)}$
18	$\mathbf{G}_9$	$\frac{\mathrm{F}_{4(-20)}}{\mathrm{SO}(9)}$
20	$\mathbf{G}_{10}$	$\frac{\mathrm{E}_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{SO}(2)}$
24	$\mathbf{G}_{12}$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SO}(3)}$
32	$\mathbf{G}_{16}$	$\frac{\mathrm{E}_{8(8)}}{\mathrm{SO}(16)}$

Table 4.1: Scalar manifolds in $D = 3$.

Thus, in $D = 3$, the only criterion for two theories to be "twins" is that they have the same scalar coset: from the table, it is apparent that it is not possible to find pairs of identical coset both in the range $10 \leq \mathcal{Q} \leq 32$. Therefore, identifying twin theories becomes equivalent to finding those cosets with $\mathcal{Q} \geq 10$ which also happen to satisfy (at least) one among the geometric conditions of those with $2 \leq \mathcal{Q} \leq 8$. For instance, the fact that all scalar manifolds of supergravity theories with extended supersymmetry are Riemannian implies that all admit an $\mathcal{N}_- = 1$ twin. Notice, however, that a number of these will feel more natural than others in our analysis of the Yang-Mills origin of the "twinness" relation, which is the subject of the next section. Furthermore, all theories with $\mathcal{N}_- = 3$ are twin to $\mathcal{N}_+ = 4$ theories with a trivial second quaternionic factor. The remaining (non-trivial) twin pairs, denoted by $(\mathcal{N}_+, \mathcal{N}_-)$, are given in Table 4.2. Notice, using that last two rows of the table, that the scalar manifold

$$\mathcal{M} = \frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)} \tag{4.1}$$

admits three supersymmetric completions, for $\mathcal{N} = 2, 4, 6$. We refer to this as the *triplet*, and denote it by $(\mathcal{N}_+, \mathcal{N}_-^+, \mathcal{N}_-^-)$.

Once the twin pairs have been established in $D = 3$, those in higher dimensions are obtained by dimensional oxidation (up to $D = 6$), as shown in the last column of the

$(\mathcal{N}_+, \mathcal{N}_-)$	$\mathcal{M}_{scalar(3)}$	Type	D_{max}
$(12, 4)$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SO}(3)}$	Q	6
$(8, 4)$	$\frac{\mathrm{SO}(8,4)}{\mathrm{SO}(8)\times\mathrm{SO}(4)}$	Q	6
$(5, 4)$	$\frac{\mathrm{Sp}(2,1)}{\mathrm{Sp}(2)\times\mathrm{Sp}(1)}$	Q	3
$(10, 2)$	$\frac{\mathrm{E}_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{SO}(2)}$	K	4
$(8, 2)$	$\frac{\mathrm{SO}(8,2)}{\mathrm{SO}(8)\times\mathrm{SO}(2)}$	K	4
$(6, 2)$	$\frac{\mathrm{SU}(4,p)}{\mathrm{SU}(4)\times\mathrm{U}(p)}$	K	4*
$(4, 2)$	$\frac{\mathrm{SU}(2,p)}{\mathrm{SU}(2)\times\mathrm{U}(p)} \times \frac{\mathrm{SU}(2,q)}{\mathrm{SU}(2)\times\mathrm{U}(q)}$	QK	4

Table 4.2: All non-trivial pairs of twin supergravities in $D = 3$. D_{max} is the maximum dimension to which each pair can be uplifted. * The (6,2) sequence may only be uplifted to $D = 4$ for $p = 2$, where the two theories oxidise to $D = 4$ supergravities with holomorphic kinetic vector matrices, which can be twins to $\mathcal{N}_- = 1$.

table. However, this requires some care: while all higher dimensional twins are obtained by oxidation, not all oxidised pairs form twins. An equivalent statement is that, in $D > 3$, it is no longer sufficient, albeit of course necessary, for two theories to share the same bosonic content and scalar manifold to be considered twins. Consider, for instance, the scalar manifold

$$\mathcal{M} = \frac{SU(3,3)}{U(3) \times SU(3)}, \qquad \dim_{\mathbb{R}}(\mathcal{M}) = 18 \tag{4.2}$$

spanned, in $D = 4$, by the 18 scalars of three distinct supergravities with different amounts of supersymmetry, namely $\mathcal{N} = 1, 2, 3$, with contents $\mathbf{G}_1 + 6\mathbf{V}_1 + 9\mathbf{C}_1$, $\mathbf{G_2} + 9\mathbf{V}_2$ and $\mathbf{G}_3 + 3\mathbf{V}_3$, respectively. Despite sharing the same scalar manifold, which represents the necessary and sufficient condition in $D = 3$, they clearly are distinct (bosonic) theories:

- The $\mathcal{N} = 2$ theory has total degrees of freedom $f = 80$, in contrast with the $f = 64$ of the remaining two and is thus ruled out as a candidate twin.

- The $\mathcal{N} = 1$ and $\mathcal{N} = 3$ theories thus share the same manifold and bosonic content $(h_{\mu\nu}, 6A_\mu, 18\phi)$, however the latter has a non-holomorphic kinetic vector matrix, which implies the impossibility of (re-)interpreting it as an $\mathcal{N}_- = 1$ theory, where all kinetic vector matrices are holomorphic.

Another example in $D = 4$ is given by $\mathcal{N} = 2$ supergravity minimally coupled to a single vector multiplet and the T^3 model: both share the same content $\mathbf{G}_2 + \mathbf{V}_2$ and scalar coset

$$\mathcal{M} = \frac{SU(1,1)}{U(1)} \times \frac{SU(2)}{SU(2)}, \qquad \dim_{\mathbb{R}}(\mathcal{M}) = 2. \tag{4.3}$$

Their U-duality groups, however, differ by a $U(1)$ factor, namely $G = U(1,1)$ for the minimally coupled case and $G = SU(1,1)$ for the T^3, under which the Abelian 2-form field strengths of each theory together with their duals transform in the representations $\mathbf{2}_1 + \mathbf{2}_{-1}$ and $\mathbf{4}$, respectively. Hence, the two theories are not twins. With the benefit of hindsight, this was to be expected since they have the same amount of supersymmetry. As it turns out, twin supergravities always have distinct supersymmetries, a fact which may be attributed to their origin as complementary truncations of a common $(\mathcal{N}_+ + \mathcal{N}_-)$-extended parent theory, as we will demostrate shortly.

Nonetheless, the $\mathcal{N} = 2$ minimally coupled theory does admit a little twin[2] with $\mathcal{N}_- = 1$, namely $\mathbf{G}_1 + 2\mathbf{V}_1 + \mathbf{C}_1$ with coset $SU(1,1)/U(1)$. Together, they form a $(2,1)$ twin pair in $D = 4$, which is the oxidation of one, among the infinite family of $(4,2)$ pairs in $D = 3$ (cf. last row of Table 4.2), namely the one with scalar manifold given by the c-map of (4.3),

$$\mathcal{M} = \frac{SU(2,2)}{SU(2) \times SU(2) \times U(1)}, \qquad \dim_{\mathbb{R}}(\mathcal{M}) = 8. \tag{4.4}$$

The pairs of allowed twins which do uplift from $D = 3$ are summarised in Table 4.3. One may check that the twin pair just discussed corresponds to the $(2,1)$ entry in the last row with $p = 2$ and $q = 0$.

4.1.1 Parent theory and complementary truncations

Following [92], it is possible in all cases to interpret two supergravity theories in a twin pair $(\mathcal{N}_+, \mathcal{N}_-)$ as two complementary consistent truncations of a single "parent" theory with $\mathscr{N} = \mathcal{N}_+ + \mathcal{N}_-$ number of supersymmetries. Since this will prove convenient in the Yang-Mills Squared construction of twins of the next section, we will review this here. While the idea remains valid for all cases, it takes its simplest form, as it is often the case, when enough supersymmetry is present. Thus, let us focus of the case[3] where the little twin has $\mathcal{Q} = 8$: the existence of a twin pair is guaranteed if, given a supergravity theory with scalar manifold $\hat{G}/\hat{H}$, one can find two groups, G and H, such that

$$\hat{G} \supset G \times SU(2), \qquad \hat{H} \supset H \times SU(2), \tag{4.5}$$

[2]Notice that the T^3 model does not have a twin. This ties in nicely with the observation that it also does not admit a Yang-Mills factorisation [91], at least not in any conventional sense. If the T^3 model had a twin, it would be the only case of twin supergravity which cannot be constructed from Yang-Mills.

[3]The other possible case is for $\mathcal{Q} = 4$; the argument is very similar, although the definition of the truncation involves Abelian $U(1)$ factors, which are messier than the $SU(2)$.

$(\mathcal{N}_+,\mathcal{N}_-)_{(4)}$	$\mathcal{M}_{scalar(4)}$	$(\mathcal{N}_+,\mathcal{N}_-)_{(5)}$	$\mathcal{M}_{scalar(5)}$	$(\mathcal{N}_+,\mathcal{N}_-)_{(6)}$	$\mathcal{M}_{scalar(6)}$
$(6,2)$	$\frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)}$	$(6,2)$	$\frac{\mathrm{SU}^\star(6)}{\mathrm{Sp}(3)}$	$((4,2),(0,2))$	$\frac{\mathrm{SU}^\star(4)}{\mathrm{Sp}(2)}$
$(4,2)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,2)}{\mathrm{U}(4)}$	$(4,2)$	$\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(5,1)}{\mathrm{Sp}(2)}$	$((2,2),(0,2))$	$\mathrm{O}(1,1)$
				$((4,0),(0,2))$	$\frac{\mathrm{SU}^\star(4)}{\mathrm{Sp}(1)}$
$(5,1)$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}$				
$(4,1)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$				
$(3,1)$	$\frac{\mathrm{U}(3,1)}{\mathrm{U}(3)\times\mathrm{U}(1)}$				
$(2,1)$	$\frac{\mathrm{U}(1,p-1)}{\mathrm{U}(p-1)\times\mathrm{U}(1)}\times\frac{\mathrm{SU}(2,q)}{\mathrm{SU}(2)\times\mathrm{U}(q)}$				

Table 4.3: Pairs of twin supergravities in $D = 4, 5, 6$. Notice that all scalar cosets include an empty $SU(2)/SU(2)$ or $U(1)/U(1)$ factor, for $\mathcal{N}_- = 2, 1$ respectively, which contains the R-symmetry of the little twin theory, and is related to the truncation from the parent theory.

with the additional requirement that H be the maximal compact subgroup of G. Then, big and little twin are defined by two complementary consistent truncations, referred to as $\mathbb{T}_\pm$, which are engineered to retain identical subsets of the original bosonic spectrum of the parent theory, but different fermionic ones (both, of course, carrying the same number of degrees of freedom as their bosonic counterpart). This is enforced by requiring that

- $\mathbb{T}_+$ keeps only fields in bosonic representations of $SU(2)$
- $\mathbb{T}_-$ keeps only bosonic fields in bosonic representations and fermionic fields in fermionic representations of $SU(2)$

Example in three dimensions

In $D = 3$, for example, the U-duality group of the unique maximal $\mathcal{N} = 16$ theory and its compact subgroup may be decomposed as

$$E_{8(8)} \supset E_{7(-5)} \times SU(2), \tag{4.6}$$

$$SO(16) \supset SO(12) \times SO(3) \times SU(2), \tag{4.7}$$

so that one identifies the new scalar manifold,

$$\frac{G}{H} = \frac{E_{7(-5)}}{SO(12)\times SO(3)} \times \frac{SU(2)}{SU(2)}, \qquad \dim_{\mathbb{R}}\left(\frac{G}{H}\right) = 64 \tag{4.8}$$

which may be verified to be that of the $(12,4)$ twin pair in Table 4.2. Recalling that in $D=3$ the R-symmetry groups are of the orthogonal type, mixing real Majorana spinors in the defining representation, one sees that in the new basis provided by equation (4.7) the 16 gravitini of the $\mathscr{N}=16$ parent split according as

$$\mathbf{16} \longrightarrow \underbrace{(\mathbf{12},\mathbf{1},\mathbf{1})}_{\mathbb{T}_+} + \underbrace{(\mathbf{1},\mathbf{2},\mathbf{2})}_{\mathbb{T}_-} . \tag{4.9}$$

Thus, the $\mathbb{T}_+$ truncation, which discards the fermionic **2** of the second $SU(2)$, keeps the 12 gravitini of the $\mathcal{N}_+ = 12$ theory mixing under the $SO(12)$ R-symmetry, while $\mathbb{T}_-$ preserves the complement four gravitini, gauging the $\mathcal{N}_- = 4$ supersymmetry of the little twin, identifying the $SO(4) \cong SU(2) \times SU(2)$ factor as the R-symmetry in this case. In the bosonic sector, comprising only scalars after dualisation, both truncations keep the $SU(2)$ singlet contribution in the decomposition of the adjoint of $E_{8(8)}$ under (4.6)

$$\mathbf{248} \longrightarrow (\mathbf{133},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + \cancel{(\mathbf{56},\mathbf{2})}, \tag{4.10}$$

thus discarding the last term on the right[4]. Modding out the 3 compact scalars, one is left only with the first term representing the 64 physical scalars spanning the coset (4.8), as it is apparent from the decomposition under $E_{7(-5)} \supset SO(12) \times SO(3)$,

$$\mathbf{133} \longrightarrow (\mathbf{66},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + \underbrace{(\mathbf{32},\mathbf{2})}_{\phi_{phys}}, \tag{4.11}$$

where they appear as the only non-compact contribution in the last term. Thus, counting degrees of freedom (64 bosonic plus 64 fermionic), one may identify the twin pairs as $\mathbf{G}_{12}$ and $\mathbf{G}_4 + 16\mathbf{V}_4$.

It is rather important to understand the role played by the $SU(2)$ appearing in both (4.6) and (4.7). All fields in the big twin theory are, by construction, invariant under it. As for the little twin, all bosons are again singlets, while the fermions transform in some (fermionic) representation. In fact, as seen below (4.9), the $SU(2)$ constitutes part of the $SO(4)$ R-symmetry of the $\mathcal{N}_-$ theory. A hint that the exact same prescription is still valid to identify twin pairs in higher dimensions $D=4,5,6$ comes from the fact that $SO(3) \cong SU(2) \cong USp(2)$, implying that $SU(2)$ may act as part of the R-symmetry for the little twin in all dimensions $3 \leq D \leq 6$ which admit pairs of twin supergravities.

Example in four dimensions

In order to illustrate this last point, and given that we will need this example again later, let us see how this works for the maximal $(6,2)$ pair in $D=4$. The parent theory is given by $\mathscr{N}=8$ supergravity, $\mathbf{G}_8$, whose content as $\mathfrak{h} = \mathfrak{su}(8)$ representations was derived in

[4]Notice, we reserve the strikethrough notation only for those representations which are excluded by both truncations $\mathbb{T}_\pm$.

(2.48), with scalar manifold $E_{7(7)}/SU(8)$. Following the prescription in (4.29), one finds the following branchings,

$$E_{7(7)} \supset SO^\star(12) \times SU(2), \tag{4.12}$$
$$SU(8) \supset U(6) \times SU(2), \tag{4.13}$$

which identify the scalar manifold shared by the twins as

$$\frac{G}{H} = \frac{SO^\star(12)}{U(6)} \times \frac{SU(2)}{SU(2)}, \qquad \dim_{\mathbb{R}}\left(\frac{G}{H}\right) = 30. \tag{4.14}$$

Indeed, the 8 gravitini of the parent split under (4.13) as

$$\mathbf{8} \longrightarrow \underbrace{(\mathbf{6},\mathbf{1})_1}_{\mathbb{T}_+} + \underbrace{(\mathbf{1},\mathbf{2})_{-3}}_{\mathbb{T}_-}, \tag{4.15}$$

which immediately shows that the $\mathbb{T}_+$ truncation will keep 6 gravitini mixing in the fundamental representation of the $U(6)$ R-symmetry, while the $\mathbb{T}_-$ will keep the remaining 2 gravitini, mixing under $U(2)$ R-symmetry. The spinors, which carry the **56** of $SU(8)$, decompose as

$$\mathbf{56} \longrightarrow \underbrace{(\mathbf{20},\mathbf{1})_3 + (\mathbf{6},\mathbf{1})_{-5}}_{\mathbb{T}_+} + \underbrace{(\mathbf{15},\mathbf{2})_{-1}}_{\mathbb{T}_-}. \tag{4.16}$$

Notice the crucial fact, at the core of the twin phenomenon, that both truncations lead to 64 fermionic degrees of freedom, albeit distributed differently: the $\mathcal{N}_+$ theory has $6\Psi_\mu$ and 26χ, while the $\mathcal{N}_- = 2$ theory has $2\Psi_\mu$ and 30χ. This, together with the fact that in $D = 4$ both a gravitino and a spinor carry 2 on-shell degrees of freedom (the two helicity states), proves the statement correct.

As for the bosonic sector, while the graviton is a singlet under $E_{7(7)}$, the scalars transform in the **133** while the vectors and their duals transform in the **56**. However, it is convenient to work directly at the level of $SU(8)$ representations, where the physical scalars and the vectors transform linearly as the **70** and **28**. Under the branching (4.13), these decompose as

$$\mathbf{28} \longrightarrow (\mathbf{1},\mathbf{1})_{-6} + (\mathbf{15},\mathbf{1})_2 + \cancel{(\mathbf{6},\mathbf{2})_{-2}} \tag{4.17}$$
$$\mathbf{70} \longrightarrow (\mathbf{15},\mathbf{1})_{-4} + (\overline{\mathbf{15}},\mathbf{1})_4 + \cancel{(\mathbf{20},\mathbf{2})_0} \tag{4.18}$$

where we crossed out the contributions which get discarded by both truncations, once again highlighting the fact that these preserve identical subsets of the original bosonic space. Notice that these are exactly the 15 complex scalars parameterising the scalar manifold (4.14), as one may check by decomposing the adjoint of $\hat{G} = E_{7(7)}$ according to (4.12),

$$\mathbf{133} \longrightarrow (\mathbf{66},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + \cancel{(\mathbf{32},\mathbf{2})}. \tag{4.19}$$

Discarding the $SU(2)$ doublet as always, and modding out the 3 compact scalars, one finds the 30 physical scalar degrees of freedom in the **66**, under the decomposition $SO^\star(12) \supset U(6)$,

$$\mathbf{66} \longrightarrow \mathbf{35}_0 + \mathbf{1}_0 + \underbrace{\mathbf{15}_{-4} + \overline{\mathbf{15}}_4}_{\phi_{phys}}. \tag{4.20}$$

The degrees of freedom of the graviton (2) together with those of the 16 vectors (32) and of the 15 complex scalars (30) add up 64 as in the fermionic sector, thus establishing supersymmetry for both twins, as required. Piecing everything together, the resulting twin theories are

- $\mathcal{N}_+ = 6$ pure supergravity, $\mathbf{G}_6$, with R-symmetry $U(6)$ and a trivial action under $SU(2)$. The content transforms, under $\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(6)_{\text{R-sym}} \oplus \overline{\mathfrak{su}}(2) \oplus \mathfrak{u}(1)_{\text{R-sym}}$, as

$$\begin{aligned}
&(\mathbf{1},\mathbf{1})^4_0 + (\mathbf{1},\mathbf{1})^{-4}_0 \\
&(\mathbf{6},\mathbf{1})^3_1 + (\overline{\mathbf{6}},\mathbf{1})^{-3}_{-1} \\
&(\mathbf{15},\mathbf{1})^2_2 + (\overline{\mathbf{15}},\mathbf{1})^{-2}_{-2} + (\mathbf{1},\mathbf{1})^2_{-6} + (\mathbf{1},\mathbf{1})^{-2}_6 \\
&(\mathbf{20},\mathbf{1})^1_3 + (\mathbf{20},\mathbf{1})^{-1}_{-3} + (\mathbf{6},\mathbf{1})^1_{-5} + (\overline{\mathbf{6}},\mathbf{1})^{-1}_5 \\
&(\overline{\mathbf{15}},\mathbf{1})^0_4 + (\mathbf{15},\mathbf{1})^0_{-4}
\end{aligned} \tag{4.21}$$

  where we use a bar to denote the Lie algebra of the $SU(2) \subset SU(8)$ which drives the truncations. Here, the trivial representations under $\overline{\mathfrak{su}}(2)$ are kept to highlight the origin of this theory as a truncation of $\mathbf{G}_8$.

- $\mathcal{N}_- = 2$ supergravity coupled to 15 vector multiplets, $\mathbf{G_2} + 15\mathbf{V}_2$, belonging to the magic sequence [142, 143], with R-symmetry $U(2)$ and isotropy group $SU(6)_{iso}$ rotating the 15 vector multiplets. Under $\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(6)_{\text{iso}} \oplus \overline{\mathfrak{su}}(2)_{\text{R-sym}} \oplus \mathfrak{u}(1)_{\text{R-sym}}$

$$\begin{aligned}
&(\mathbf{1},\mathbf{1})^4_0 + (\mathbf{1},\mathbf{1})^{-4}_0 \\
&(\mathbf{1},\mathbf{2})^3_{-3} + (\mathbf{1},\mathbf{2})^{-3}_3 \\
&(\mathbf{1},\mathbf{1})^2_{-6} + (\mathbf{1},\mathbf{1})^{-2}_6
\end{aligned}
\quad + \quad
\begin{aligned}
&(\mathbf{15},\mathbf{1})^2_2 + (\overline{\mathbf{15}},\mathbf{1})^{-2}_{-2} \\
&(\mathbf{15},\mathbf{2})^1_{-1} + (\overline{\mathbf{15}},\mathbf{2})^{-1}_1 \\
&(\mathbf{15},\mathbf{1})^0_{-4} + (\overline{\mathbf{15}},\mathbf{1})^0_4
\end{aligned} \tag{4.22}$$

Notice how the (decomposition of the) parent's R-symmetry has a very precise interpretation: (i) the $\overline{\mathfrak{su}}(2)$ acts trivially for the big twin and as part of the R-symmetry $\mathfrak{u}(2)$ for the little twin; (2) the $\mathfrak{su}(6)$ acts as part of the R-symmetry $\mathfrak{u}(6)$ for the big twin and as an isotropy group rotating the 15 vector multiplets for the little twin. This behaviour is a general feature of all twin pairs.

4.2 Yang-Mills origin of twins

Given the classification of twin supergravities, reviewed in the previous section, and the construction of generic supergravity theories as products of two super Yang-Mills theories presented in Chapter 3, two questions naturally arise:

1. Is there a relationship between the set of supergravity theories admitting a twin and that of those which are constructible as double copies?

2. Is there a Yang-Mills Squared analog of twinness? If so, what is it? Said otherwise, is there a prescription which, given a supergravity as a double copy, consistently yields its twin?

In the following, we will argue in favour of a positive answer to both questions. The idea consists in regarding the supergravity theories in the Yang-Mills Squared pyramid, namely those admitting a factorisation of the $\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$ type, as the parent theory discussed in the previous section. Heuristically, then, the two truncations defining the twins, $\mathbb{T}_{\pm}$, are obtained by means of corresponding "truncations" of the underlying super Yang-Mills factors. More precisely, the prescription may be summarised as follows:

1. Find the factorisation of the parent theory,

$$\mathbf{G}_{\mathscr{N}} + \mathbf{M}_{\mathscr{N}} = \mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \tag{4.23}$$

where $\mathscr{N} = \mathcal{N} + \tilde{\mathcal{N}}$, in addition to the relation $\mathscr{N} = \mathcal{N}_+ + \mathcal{N}_-$ by definition of parent theory.

2. Effect the first truncation $\mathbb{T}_+$ to the big $\mathcal{N}_+$ twin by decomposing one of the two Yang-Mills factors, here always taken to be the Left one without loss of generality, by expressing its content in a manifest $\mathfrak{r}(\mathcal{N}', D)$ basis, with $\mathcal{N}' + \tilde{\mathcal{N}} = \mathcal{N}_+$,

$$\mathbf{V}^A_{\mathcal{N}} = \mathbf{V}^A_{\mathcal{N}'} + \mathbf{C}^A_{\mathcal{N}'} \tag{4.24}$$

and by exchanging the resulting adjoint-valued spinor multiplet $C^A_{\mathcal{N}'}$ with one valued in a non-adjoint representation of the gauge group, that is $\mathbf{C}^A_{\mathcal{N}'} \to \mathbf{C}^a_{\mathcal{N}'}$. Given the "sum of squares" rule (3.55), this has the effect of reducing the factorisation of the parent to

$$(\mathbf{V}^A_{\mathcal{N}'} + \mathbf{C}^a_{\mathcal{N}'}) \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}} = \mathbf{V}^A_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}} \tag{4.25}$$

which yields the big twin $\mathbf{G}_{\mathcal{N}_+} + \mathbf{M}_{\mathcal{N}_+}$. One could interpret this as a way to discard the $\mathcal{N} - \mathcal{N}'$ gravitini contained in the (non-dynamical) product $\mathbf{C}^a_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}}$. This step is exactly analogous to truncating according as the decomposition $\hat{H} \supset H \times SU(2)$.

3. Effect the second truncation $\mathbb{T}_-$ to the little $\mathcal{N}_-$ twin by decomposing the Right Yang-Mills factor as above. Once the adjoint spinor multiplet is once again replaced by a non-adjoint one, $\tilde{\mathbf{C}}^{A'}_{\tilde{\mathcal{N}}'} \to \tilde{\mathbf{C}}^{a'}_{\tilde{\mathcal{N}}'}$, the supersymmetry of the Right theory is effectively reduced $\tilde{\mathcal{N}} \to \tilde{\mathcal{N}}'$ and the Right side becomes

$$\tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}} \to \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}'} + \tilde{\mathbf{C}}^{a'}_{\tilde{\mathcal{N}}'} + \dots \tag{4.26}$$

In many instances $\tilde{\mathcal{N}}' = 0$, thus

$$\tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}} \to \tilde{A}^{A'}_{\mu} + \tilde{\chi}^{a'} + \tilde{\phi}^{A'}, \tag{4.27}$$

where a single gaugino is treated as an $\tilde{\mathcal{N}} = 0$ spinor multiplet. This implies that the full product now reads

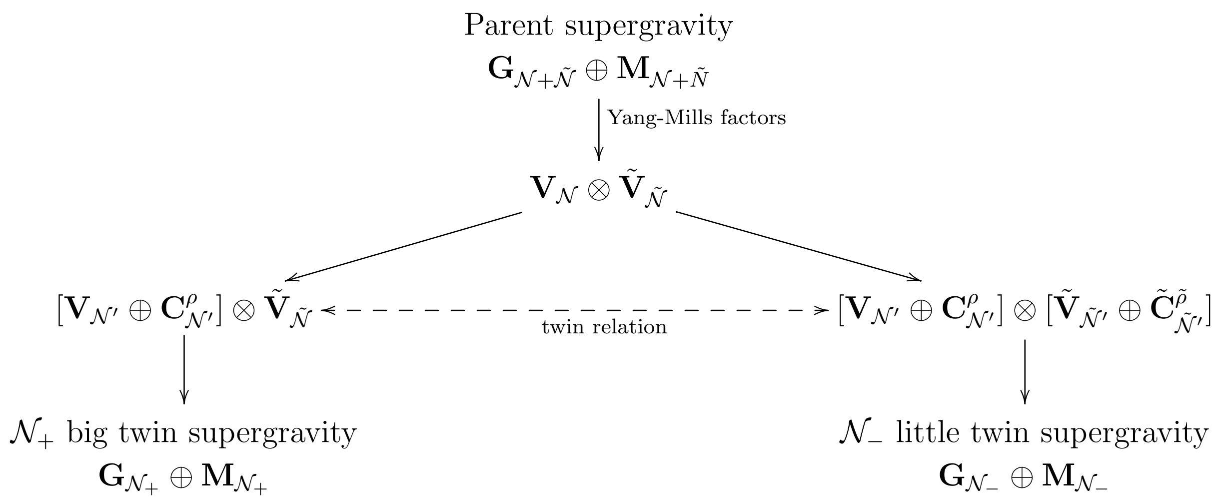
$$(\mathbf{V}^{A}_{\mathcal{N}'} + \mathbf{C}^{a}_{\mathcal{N}'}) \otimes (\mathbf{V}^{A'}_{\tilde{\mathcal{N}}'} + \mathbf{C}^{a'}_{\tilde{\mathcal{N}}'} + \dots) = (\mathbf{V}^{A}_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}'}) + (\mathbf{C}^{a}_{\mathcal{N}'} \otimes \tilde{\mathbf{C}}^{a'}_{\tilde{\mathcal{N}}'}) + (\mathbf{V}^{A}_{\mathcal{N}'} \otimes \dots), \tag{4.28}$$

which, compared to the factorisation of the big twin (4.25), discards a further $\tilde{\mathcal{N}} - \tilde{\mathcal{N}}'$ gravitini (as well as some spinors). It does, however, reinstate some spinor states through the now "active" product $\mathbf{C}^{a} \otimes \tilde{\mathbf{C}}^{a'}$. From the Yang-Mills Squared perspective, this is what makes it possible for the two truncations to have the same number of fermionic states, which ought to be true if they are to represents twin theories. This is also the reason why it is convenient to keep track of the "truncated" states $\mathbf{C}^{a}_{\mathcal{N}'}$ in (4.25): they come back into play in the factorisation of the little twin.

To summarise, then, the relation between twin theories is given, in terms of their respective factorisations, by

Big twin		Little twin	
$(\mathbf{V}^{A}_{\mathcal{N}'} + \mathbf{C}^{a}_{\mathcal{N}'}) \otimes \tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}}$	$\Longleftrightarrow$	$(\mathbf{V}^{A}_{\mathcal{N}'} + \mathbf{C}^{a}_{\mathcal{N}'}) \otimes (\tilde{\mathbf{V}}^{A'}_{\tilde{\mathcal{N}}'} + \tilde{\mathbf{C}}^{a'}_{\tilde{\mathcal{N}}'} + \dots)$	(4.29)

This leads to the realisation that all the supergravity theories which sit in the Yang-Mills Squared pyramid possess the correct features to be a big twin, *provided that* they are also consistent truncations of another supergravity in the pyramid, acting as the parent. We will see below that it is indeed true that all pyramid supergravities, *with the exception of those on the maximal spine*, possess in fact a smaller twin theory. Finally, the whole process is pictured below.



4.2.1 The (6,2) twins in four dimensions

Let us illustrate this by means of an example: the maximal $(6,2)$ pair in $D = 4$. The parent theory should be a $\mathscr{N} = 6+2 = 8$ theory, which leads us to the unique possibility of maximal supergravity $\mathbf{G}_8$. Indeed, this was shown to be the case using the method of complementary truncations in Section 4.1.1. Furthermore, we also know from the discussion of Section 3.2.4 that the theory admits the factorisation

$$\mathbf{G}_8 = \mathbf{V}_4 \otimes \tilde{\mathbf{V}}_4. \tag{4.30}$$

Let us see how the prescription outlined above reproduces the two twin theories.

The $\mathcal{N}_+ = 6$ big twin

The big twin was derived as a truncation of $\mathbf{G}_8$ in (4.21), where its content was expressed in terms of representations of $\mathfrak{su}(6)_{\text{R-sym}} \times \overline{\mathfrak{su}}(2) \times \mathfrak{u}(1)_{\text{R-sym}}$ where the representation under the $\overline{\mathfrak{su}}(2) \subset \mathfrak{su}(8)$ was kept, even though entirely trivial, to show the origin of the theory as a truncation. It is convenient to further decompose $\mathfrak{su}(6)_{\text{R-sym}} \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2)' \oplus \mathfrak{u}(1)'$ to compare with the result of the squaring calculation below, which as usual is written in terms of the representations under the symmetries directly inherited from the Left and Right factors. Not only this is convenient to understand the Yang-Mills origin of all gravitational states, it is usually necessary when the result of the squaring product is charged under more than one $\mathfrak{u}(1)$, as we will show below. Under the relevant branchings,

$$\mathfrak{su}(6)_{\text{R-sym}} \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2)' \oplus \mathfrak{u}(1)'$$

$$\begin{aligned}
\mathbf{6} &\to (\mathbf{1},\mathbf{2})_{-2} + (\mathbf{4},\mathbf{1})_{1} \\
\mathbf{15} &\to (\mathbf{1},\mathbf{1})_{-4} + (\mathbf{4},\mathbf{2})_{-1} + (\mathbf{6},\mathbf{1})_{2} \\
\mathbf{20} &\to (\mathbf{4},\mathbf{1})_{-3} + (\overline{\mathbf{4}},\mathbf{1})_{3} + (\mathbf{6},\mathbf{2})_{0}
\end{aligned} \tag{4.31}$$

and their conjugates, the content of $\mathbf{G}_6$, restricting to just the positive helicity states since the rest may be obtained by conjugation, reads

$$\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(4) \oplus \mathfrak{su}(2)' \oplus \overline{\mathfrak{su}}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)$$

$$\begin{array}{l}
(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00}\\
(\mathbf{1},\mathbf{2},\mathbf{1})^3_{-21} + (\mathbf{4},\mathbf{1},\mathbf{1})^3_{11}\\
(\mathbf{1},\mathbf{1},\mathbf{1})^2_{0-6} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-42} + (\mathbf{4},\mathbf{2},\mathbf{1})^2_{-12} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{22}\\
(\mathbf{1},\mathbf{2},\mathbf{1})^1_{-2-5} + (\mathbf{4},\mathbf{1},\mathbf{1})^1_{1-5} + (\mathbf{4},\mathbf{1},\mathbf{1})^1_{-33} + (\overline{\mathbf{4}},\mathbf{1},\mathbf{1})^1_{33} + (\mathbf{6},\mathbf{2},\mathbf{1})^1_{03}\\
(\mathbf{1},\mathbf{1},\mathbf{1})^0_{-4-4} + (\mathbf{4},\mathbf{2},\mathbf{1})^0_{-1-4} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{2-4}
\end{array} \tag{4.32}$$

where it is obvious that all states are invariant under the $\overline{\mathfrak{su}}(2) \subset \mathfrak{su}(8)$ part of the parent's R-symmetry, as expected.

This ought to be compared with the corresponding squaring calculation. According to the prescription above, we should decompose $\mathbf{V}_4$ into a $\mathbf{V}_2$ and keep track of the remnant states. Denote the R-symmetry of the Left factor by $\mathfrak{su}(4)_L$ and its decomposition by $\mathfrak{su}(4)_L \supset \mathfrak{su}(2)_L \oplus \overline{\mathfrak{su}}(2)_L \oplus \mathfrak{u}(1)_L$, where the notation of the second factor indicates its affinity to the $\overline{\mathfrak{su}}(2) \subset \mathfrak{su}(8)$, in that both drive the respective truncations. Then,

$$\begin{array}{llllll}
\mathbf{1}^2 & \to & (\mathbf{1},\mathbf{1})^2_0 & & \\
\mathbf{4}^1 & \to & (\mathbf{2},\mathbf{1})^1_1 & + & (\mathbf{1},\mathbf{2})^1_{-1}\\
\mathbf{6}^0 & \to & (\mathbf{1},\mathbf{1})^0_2 + (\mathbf{1},\mathbf{1})^0_{-2} & + & (\mathbf{2},\mathbf{2})^0_0\\
\overline{\mathbf{4}}^{-1} & \to & (\mathbf{2},\mathbf{1})^{-1}_{-1} & + & (\mathbf{1},\mathbf{2})^{-1}_1\\
\mathbf{1}^2 & \to & (\mathbf{1},\mathbf{1})^{-2}_0 & & \\
 & & & & \\
\mathbf{V}_4 & = & \mathbf{V}_2 & + & \mathbf{H}_2
\end{array} \tag{4.33}$$

where $\mathfrak{u}(2)_L \cong \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_L$ represents the R-symmetry in the $\mathcal{N}=2$ basis, while the $\overline{\mathfrak{su}}(2)_L$ acts as a flavour group. Notice that the set of states which are singlets under the latter group form a $\mathcal{N}=2$ vector multiplet, while those which are not form a hypermultiplet. The latter would normally be truncated away by demanding that only $\overline{\mathfrak{su}}(2)_L$ singlets are kept, however we keep it here for future convenience. Then, according as (4.29), the big twin theory is constructed in two steps: (1) "effectively truncate" the hypermultiplet, by replacing it with one in a non-adjoint gauge group representation ρ which does not multiply the adjoint-valued Right Yang-Mills factor; and (2) perform the tensor product

$$(\mathbf{V}^A_2 + \mathbf{H}^a_2) \otimes \tilde{\mathbf{V}}^{A'}_4, \tag{4.34}$$

which is carried out in Table 4.4 with the $\mathbf{V}_2$ in an explicit $\mathfrak{u}(2)_L \oplus \overline{\mathfrak{su}}(2)_L$ basis. Here $a = 1, \ldots, \dim\rho$ is the index of the new gauge group representation. Notice that the latter

	$\mathbf{1}^2 + c.c.$	$\mathbf{4}^1 + c.c.$	$\mathbf{6}^0$
$(\mathbf{1},\mathbf{1})^2_0 + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00} + (\mathbf{1},\mathbf{1},\mathbf{1})^0_{04} + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{4})^3_{01} + (\mathbf{1},\mathbf{1},\overline{\mathbf{4}})^1_{03} + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{6})^2_{02} + c.c.$
$(\mathbf{2},\mathbf{1})^1_1 + c.c.$	$(\mathbf{2},\mathbf{1},\mathbf{1})^3_{1-1} + (\mathbf{2},\mathbf{1},\mathbf{1})^1_{-1-3} + c.c.$	$(\mathbf{2},\mathbf{1},\mathbf{4})^2_{10} + (\mathbf{2},\mathbf{1},\overline{\mathbf{4}})^0_{12} + c.c.$	$(\mathbf{2},\mathbf{1},\mathbf{6})^1_{11} + c.c.$
$(\mathbf{1},\mathbf{1})^0_2 + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-2-2} + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{4})^1_{2-1} + (\mathbf{1},\mathbf{1},\mathbf{4})^1_{-2-1} + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{6})^0_{20} + c.c.$
$[(\mathbf{1},\mathbf{2})^1_{-1} + c.c.]^a$			
$[(\mathbf{2},\mathbf{2})^0_0]^a$			

Table 4.4: The factorisation of the big $\mathcal{N}_+ = 6$ twin of the $(6, 2)$ pair in $D = 4$. It amounts to the squaring $\mathbf{V}_2 \otimes \mathbf{V}_4 = \mathbf{G}_6$. The positive helicity states are presented explicitly and the states of the hypermultiplet $\mathbf{H}^a_2$ are included, albeit not contributing to this product, to facilitate the comparison with the little twin case in Table 4.5.

need be real if the $\mathfrak{su}(2) \oplus \overline{\mathfrak{su}}(2)$ symmetry is to be preserved. On collecting the states by (positive) helicity, one obtains

$$\begin{array}{l}
\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(4)_R \oplus \mathfrak{su}(2)_L \oplus \overline{\mathfrak{su}}(2)_L \oplus \mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_d \\
\\
(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00} \\
(\mathbf{1},\mathbf{2},\mathbf{1})^3_{1-1} + (\mathbf{4},\mathbf{1},\mathbf{1})^3_{01} \\
(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-2-2} + (\mathbf{4},\mathbf{2},\mathbf{1})^2_{10} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{02} \\
(\mathbf{1},\mathbf{2},\mathbf{1})^1_{-1-3} + (\mathbf{4},\mathbf{1},\mathbf{1})^1_{2-1} + (\mathbf{4},\mathbf{1},\mathbf{1})^1_{-2-1} + (\overline{\mathbf{4}},\mathbf{1},\mathbf{1})^1_{03} + (\mathbf{6},\mathbf{2},\mathbf{1})^1_{11} \\
(\mathbf{1},\mathbf{1},\mathbf{1})^0_{04} + (\mathbf{4},\mathbf{2},\mathbf{1})^0_{-1-2} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{20} + c.c.
\end{array} \tag{4.35}$$

where the complex conjugate has been formally added to the scalars to remind that it is not clear, a priori, which scalars should "correspond" to the rest of the positive helicity states. It should be clear, however, that there are in total 30 scalars in this example. As far as the representations under the non-Abelian factors are concerned, this matches the content of $\mathbf{G}_6$ on identifying $\mathfrak{su}(2)_L \cong \mathfrak{su}(2)'$ and $\overline{\mathfrak{su}}(2)_L \cong \overline{\mathfrak{su}}(2)$, which amounts to saying that it is the former which contributes to the R-symmetry of the $\mathcal{N}_+ = 6$ theory. However, this is not immediately true for the $\mathfrak{u}(1)$ charges! It is a feature of squaring that, in cases when more than one Abelian factor is present, a rotation of the corresponding charges may be needed to recover the form which makes the enhancement manifest. In this case, denoting the charges of $\mathfrak{u}(1)'$ and $\mathfrak{u}(1)$ in (4.32) by h' and h, and those of $\mathfrak{u}(1)_L$

and $\mathfrak{u}(1)_d$ in (4.35) by h_L and h_d, the correct rotation relating the two pairs is

$$\begin{pmatrix} h' \\ h \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} h_L \\ h_d \end{pmatrix}. \tag{4.36}$$

In other words, the symmetries of the $\mathcal{N}_+ = 6$ big twin supergravity are built from those of its factors as

$$\mathfrak{su}(6)_{\text{R-sym}} \oplus \overline{\mathfrak{su}}(2) \oplus \mathfrak{u}(1)_{\text{R-sym}} \supset \Big(\mathfrak{su}(4)_R \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)'\Big) \oplus \overline{\mathfrak{su}}(2)_L \oplus \mathfrak{u}(1). \tag{4.37}$$

This completes the construction of the $\mathcal{N}_+ = 6$ theory.

The $\mathcal{N}_- = 2$ little twin

Once again, let us begin by decomposing the content of the little $\mathcal{N}_- = 2$ twin given in (4.22) by breaking its isotropy group as $\mathfrak{su}(6)_{\text{iso}} \supset \mathfrak{su}(4) \oplus \mathfrak{su}(2)' \oplus \mathfrak{u}(1)'$ exactly as in (4.31). This results in the content $\mathbf{G}_2 + 15\mathbf{V}_2$ begin labelled as

$$\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(4) \oplus \mathfrak{su}(2)' \oplus \overline{\mathfrak{su}}(2) \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)$$

$$\begin{array}{rl}
\mathbf{G}_2 & \left\{ \begin{array}{l} (\mathbf{1},\mathbf{1},\mathbf{1})^4_{00} \\ (\mathbf{1},\mathbf{1},\mathbf{2})^3_{0-3} \\ (\mathbf{1},\mathbf{1},\mathbf{1})^2_{0-6} \end{array} \right. \\
15\mathbf{V}_2 & \left\{ \begin{array}{l} (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-42} + (\mathbf{4},\mathbf{2},\mathbf{1})^2_{-12} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{22} \\ (\mathbf{1},\mathbf{1},\mathbf{2})^1_{-4-1} + (\mathbf{4},\mathbf{2},\mathbf{2})^1_{-1-1} + (\mathbf{6},\mathbf{1},\mathbf{2})^1_{2-1} \\ (\mathbf{1},\mathbf{1},\mathbf{1})^0_{-4-4} + (\mathbf{4},\mathbf{2},\mathbf{1})^0_{-1-4} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{2-4} \end{array} \right.
\end{array} \tag{4.38}$$

to be compared with the corresponding squaring product.

Starting from the factorisation of the big twin (4.34), the little $\mathcal{N}_- = 2$ twin theory is obtained by decomposing the Right factor $\tilde{\mathbf{V}}_4^{A'}$. Since the little twin ought to have just 8 real supercharges, which are already accounted for by the Left factor, the Right must necessarily be an $\mathcal{N} = 0$ theory. This is achieved by breaking the $\mathfrak{su}(4)_R$ R-symmetry[5], which practically means re-purposing it as an internal rotation of the spinors and scalars, without mention of its action on any conserved supercharges. Furthermore, moving the $\mathcal{N} = 0$ spinor multiplet, namely the 4 gaugini, to a non-adjoint gauge group representation, $\chi^{iA'} \to \chi^{ia'}$, where $i = 1, \dots, 4$ is the index of the remnant $\mathfrak{su}(4)_R$, the Right factor becomes

$$\tilde{\mathbf{V}}_4^{A'} = \tilde{A}_\mu^{A'} + \tilde{\chi}^{ia'} + \tilde{\phi}^{[ij]A'}. \tag{4.39}$$

Then, the product yielding the little twin theory is

$$(\mathbf{V}_2^A + \mathbf{H}_2^a) \otimes (\tilde{A}_\mu^{A'} + \tilde{\chi}^{ia'} + \tilde{\phi}^{[ij]A'}) = \mathbf{V}_2^A \otimes (\tilde{A}_\mu^{A'} + \tilde{\phi}^{[ij]A'}) + \mathbf{H}_2^a \otimes \tilde{\chi}^{ia'}. \tag{4.40}$$

	$\mathbf{1}^2 + c.c.$	$[\mathbf{4}^1 + c.c.]^{a'}$	$\mathbf{6}^0$
$(\mathbf{1},\mathbf{1})^2_0 + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00} + (\mathbf{1},\mathbf{1},\mathbf{1})^0_{04} + c.c.$		$(\mathbf{1},\mathbf{1},\mathbf{6})^2_{02} + c.c.$
$(\mathbf{2},\mathbf{1})^1_1 + c.c.$	$(\mathbf{2},\mathbf{1},\mathbf{1})^3_{1-1} + (\mathbf{2},\mathbf{1},\mathbf{1})^1_{-1-3} + c.c.$		$(\mathbf{2},\mathbf{1},\mathbf{6})^1_{11} + c.c.$
$(\mathbf{1},\mathbf{1})^0_2 + c.c.$	$(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-2-2} + c.c.$		$(\mathbf{1},\mathbf{1},\mathbf{6})^0_{20} + c.c.$
$[(\mathbf{1},\mathbf{2})^1_{-1} + c.c.]^a$		$(\mathbf{1},\mathbf{2},\mathbf{4})^2_{-10} + (\mathbf{1},\mathbf{2},\mathbf{4})^0_{1-2} + c.c.$	
$[(\mathbf{2},\mathbf{2})^0_0]^a$		$(\mathbf{2},\mathbf{2},\mathbf{4})^1_{0-1} + c.c.$	

Table 4.5: The factorisation of the little $\mathcal{N}_- = 2$ twin of the $(6, 2)$ pair in $D = 4$. The positive helicity states are presented explicitly. Comparison with Table 4.4 shows the complementarity of the two truncations.

It is carried out explicitly in Table 4.5, which makes manifest the complementary nature of this product compared to that of the big twin, in Table 4.4. In this case, the squaring leads to

$$\mathfrak{u}(1)_{st} \oplus \mathfrak{su}(4)_R \oplus \mathfrak{su}(2)_L \oplus \overline{\mathfrak{su}}(2)_L \oplus \mathfrak{u}(1)' \oplus \mathfrak{u}(1)$$

$$\begin{aligned} &\mathbf{G}_2 \left\{ \begin{array}{l} (\mathbf{1},\mathbf{1},\mathbf{1})^4_{00} \\ (\mathbf{1},\mathbf{2},\mathbf{1})^3_{1-1} \\ (\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} \end{array} \right. \\ &15\mathbf{V}_2 \left\{ \begin{array}{l} (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-2-2} + (\mathbf{4},\mathbf{1},\mathbf{2})^2_{-10} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{02} \\ (\mathbf{1},\mathbf{2},\mathbf{1})^1_{-1-3} + (\mathbf{4},\mathbf{2},\mathbf{2})^1_{0-1} + (\mathbf{6},\mathbf{2},\mathbf{1})^1_{11} \\ (\mathbf{1},\mathbf{1},\mathbf{1})^0_{04} + (\mathbf{4},\mathbf{1},\mathbf{2})^0_{1-2} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{20} \end{array} \right. \end{aligned} \tag{4.41}$$

Again up to the $\mathfrak{u}(1)$ charges, this matches (4.38) on identifying $\overline{\mathfrak{su}}(2)_L \cong \mathfrak{su}(2)'$ and $\mathfrak{su}(2)_L \cong \overline{\mathfrak{su}}(2)$, namely the opposite identification as before. This is tantamount to saying that $\mathfrak{su}(2)_L \subset \mathfrak{su}(4)_L$ originating from the Left factor's R-symmetry is mapped *in both cases* to the R-symmetry of the resulting supergravity, which is $\mathfrak{su}(2)' \subset \mathfrak{su}(6)$ for the big twin and $\overline{\mathfrak{su}}(2) \subset \mathfrak{su}(8)$ for the little one. The rotation which aligns the charges is given, this time, by

$$\begin{pmatrix} h' \\ h \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} h_L \\ h_d \end{pmatrix}, \tag{4.42}$$

[5]In this context, the subscript R stands for Right, as opposed to R-symmetry.

namely a similar, yet distinct rotation from its big twin counterpart in (4.36). That two rotations should be needed is immediately obvious. For instance, consider the two gravitino states produced in both squaring products: since they originate in the $\mathbf{V}_2^A \otimes \tilde{\mathbf{V}}_0^{A'}$ sector, which is shared by both factorisations, they are identical and read $(\mathbf{1},\mathbf{2},\mathbf{1})^3_{1-1}$. These, however, must match the corresponding states in the decompositions of $\mathbf{G}_6$ in (4.32) and of $\mathbf{G}_2 + 15\mathbf{V}_2$ in (4.38), which read $(\mathbf{1},\mathbf{2},\mathbf{1})^3_{-21}$ and $(\mathbf{1},\mathbf{2},\mathbf{1})^3_{0-3}$ respectively. These are in general different since the truncations $\mathbb{T}_\pm$ from the parent pick complementary sets of fermions. Since the same pair of charges must match two other distinct pairs, two rotations are required.

The interesting fact is the following: one might (and should) worry about those *bosonic* states with a boson×boson origin, namely the graviton, a subset of the vectors and of the scalars. These are common to both squaring products since they come from the $\mathbf{V}_2^A \otimes \tilde{\mathbf{V}}_0^{A'}$ and $\mathbf{V}_2^A \otimes \tilde{\phi}^{A'}$ sectors. However, *unlike for the gravitino states*, their "honest supergravity" counterparts are identical in the decompositions of the big and little twins, as both truncations keep the same bosonic states. After all, an identical bosonic sector (all the way down to $\mathfrak{u}(1)$ charges) is the "defining" property of twins. For the case at hand, again for the positive helicity states (plus the complex conjugate of the scalars), these are

Squaring (for both $\mathcal{N}_+$ and $\mathcal{N}_-$):	Honest (for both $\mathcal{N}_+$ and $\mathcal{N}_-$):
$(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00}$	$(\mathbf{1},\mathbf{1},\mathbf{1})^4_{00}$
$(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-2-2} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{02}$	$(\mathbf{1},\mathbf{1},\mathbf{1})^2_{0-6} + (\mathbf{1},\mathbf{1},\mathbf{1})^2_{-42} + (\mathbf{6},\mathbf{1},\mathbf{1})^2_{22}$
$(\mathbf{6},\mathbf{1},\mathbf{1})^0_{20} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{-20}$	$(\mathbf{6},\mathbf{1},\mathbf{1})^0_{2-4} + (\mathbf{6},\mathbf{1},\mathbf{1})^0_{-24}$

(4.43)

Fortunately, one may check that these states come with just the right representations and charges so as to admit two distinct rotations. In particular, the degeneracy in the representation spaces of the vectors and scalars produces just enough freedom so that, by modifying the identification of the states on both sides, one may accommodate the two rotations. For instance, for the big twin one identifies

$$\begin{aligned}(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} &\longrightarrow (\mathbf{1},\mathbf{1},\mathbf{1})^0_{-42}\\ (\mathbf{6},\mathbf{1},\mathbf{1})^0_{-20} &\longrightarrow (\mathbf{6},\mathbf{1},\mathbf{1})^0_{2-4}\end{aligned} \qquad \text{with} \begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}, \tag{4.44}$$

while the little twin requires one to identify

$$\begin{aligned}(\mathbf{1},\mathbf{1},\mathbf{1})^2_{2-2} &\longrightarrow (\mathbf{1},\mathbf{1},\mathbf{1})^0_{0-6}\\ (\mathbf{6},\mathbf{1},\mathbf{1})^0_{-20} &\longrightarrow (\mathbf{6},\mathbf{1},\mathbf{1})^0_{-24}\end{aligned} \qquad \text{with} \begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}. \tag{4.45}$$

This should clarify why the complex conjugate of the scalars has been added in (4.43). This concludes the construction of the little $\mathcal{N}_- = 2$ twin theory. It is worth noting that this provides a new double copy construction of this theory, known in the literature as the quaternionic magic $D = 4$, $\mathcal{N} = 2$ supergravity, which was previously reproduced from Yang-Mills, using a different factorisation, in [27]. The reader interested in the complete classification of the double copy construction of $\mathcal{N} = 2$ supergravities is referred to that paper, which deals with vector multiplet couplings, or [91], where the couplings to hypermultiplets are also clarified.

4.2.2 Pyramid twins and triplets

All twin pairs in $D = 3, 4, 5$ follow precisely the same pattern as the $D = 4$, $(6, 2)$ pair just described. Their factorisations are summarised in Tables 4.6, 4.7 and 4.8. Note that, similarly to the $\mathcal{N}_- = 2$ theory above, the factorisation is, in general, non-unique. Another example is provided by the $D = 4$, $(4, 2)$ pair, with coset

$$\frac{G}{H} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6,2)}{U(4)} \times \frac{SU(2)}{SU(2)} \tag{4.46}$$

which, in line with its twin nature, may be interpreted either as the $n = 2$ element of the one-parameter family of scalar spaces for $\mathbf{G}_4 + n\mathbf{V}_4$ in $D = 4$, see Table 2.5, or the $m = 6$ element of the $D = 4$ Generic Jordan series[6] with content $\mathbf{G}_2 + (1 + m)\mathbf{V}_2$. The big twin admits the two types of factorisations already discussed in Section 3.3.2, namely

$$\mathbf{G}_4 + 2\mathbf{V}_4 = \begin{cases} (\mathbf{V}_2^A + \mathbf{H}_2^a) & \otimes \quad \mathbf{V}_2^{A'} \\ \mathbf{V}_4^A & \otimes \quad (A_\mu^{A'} + 2\phi^{A'}), \end{cases} \tag{4.47}$$

while the little twin may be factorised as

$$\mathbf{G}_2 + 7\mathbf{V}_2 = \begin{cases} (\mathbf{V}_2^A + \mathbf{H}_2^a) & \otimes \quad (A_\mu^{A'} + 2\chi^{a'} + 2\phi^{A'}) \\ \mathbf{V}_2^A & \otimes \quad (A_\mu^{A'} + 6\phi^{A'}). \end{cases} \tag{4.48}$$

The first factorisation in either case is that dictated by the twin prescription above; the second is, for the $\mathcal{N}_+ = 4$ case, the generic factorisation for $\mathcal{Q} = 16$ supergravities coupled to vector multiplets discussed in Section 3.3.2 while, for the $\mathcal{N}_- = 2$ case, it is the $q = 4$, $r = 0$ element of the general factorisation of the Generic Jordan sequence in the vector multiplet sector of [91]. The possibility of multiple factorisations is an interesting one, which calls for a thorough investigation. As mentioned before, this is still work in progress, and as such is not included in this thesis.

The reader might have noticed that we did not include $D = 6$ in the comment above, the reason being that it deviates slightly from the cases in $D = 3, 4, 5$, due to the chiral nature of spinors in six dimensions. While the truncation from the parent to the big twin follows as before from reducing the degree of supersymmetry of the Left factor, the little twin needs one extra ingredient. In addition to decomposing the R-symmetry of the Right

[6] See [91] for a discussion of this sequence and its Yang-Mills factorisation.

Yang-Mills, a flip of the chirality of the Left factor is also required. Thus, schematically, the factorisations of parent, big and little twin theories are respectively given by

$$\begin{array}{lcccl} \mathbf{V}_{(\mathcal{N}_L,\mathcal{N}_R)} & \otimes & \tilde{\mathbf{V}}_{(\tilde{\mathcal{N}}_L,\tilde{\mathcal{N}}_R)} & = & \mathbf{G}_{(\mathcal{N}_L,\mathcal{N}_R)}+\mathbf{M}_{(\mathcal{N}_L,\mathcal{N}_R)} \\ \mathbf{V}_{(\mathcal{N}'_L,\mathcal{N}'_R)}+\mathbf{H}^{\rho}_{(\mathcal{N}'_L,\mathcal{N}'_R)} & \otimes & \tilde{\mathbf{V}}_{(\tilde{\mathcal{N}}_L,\tilde{\mathcal{N}}_R)} & = & \mathbf{G}_{(\mathcal{N}_{L+},\mathcal{N}_{R+})}+\mathbf{M}_{(\mathcal{N}_{L+},\mathcal{N}_{R+})} \\ \underbrace{\mathbf{V}_{(\mathcal{N}'_R,\mathcal{N}'_L)}+\mathbf{H}^{\tilde{\rho}}_{(\mathcal{N}'_R,\mathcal{N}'_L)}}_{\text{Chirality flipped}} & \otimes & \tilde{\mathbf{V}}_{(\tilde{\mathcal{N}}'_L,\tilde{\mathcal{N}}'_R)}+\tilde{\mathbf{H}}^{\rho}_{(\tilde{\mathcal{N}}'_L,\tilde{\mathcal{N}}'_R)} & = & \mathbf{G}_{(\mathcal{N}_{L-},\mathcal{N}_{R-})}+\mathbf{M}_{(\mathcal{N}_{L-},\mathcal{N}_{R-})} \end{array} \tag{4.49}$$

This property is universal to all twin pairs in $D=6$, as may be verified in Table 4.8. For example, for the maximal $((4,2),(0,2))$ twin pair one obtains

$$\begin{array}{lcccl} \mathbf{V}_{(2,2)} & \otimes & \tilde{\mathbf{V}}_{(2,2)} & = & \mathbf{G}_{(4,4)} \\ \mathbf{V}_{(2,0)}+\mathbf{H}^{\rho}_{(2,0)} & \otimes & \tilde{\mathbf{V}}_{(2,2)} & = & \mathbf{G}_{(4,2)} \\ \underbrace{\mathbf{V}_{(0,2)}+\mathbf{H}^{\rho}_{(0,2)}}_{\text{Chirality flipped}} & \otimes & \tilde{A}+2(\tilde{\chi}^{\rho}_L,\tilde{\chi}^{\rho}_R)+4\tilde{\phi} & = & \mathbf{G}_{(0,2)}+8\mathbf{V}_{(0,2)}+5\mathbf{T}_{(0,2)}. \end{array} \tag{4.50}$$

Apart from this additional step, the construction of $D=6$ twins from Yang-Mills respects the prescription above. As may be checked in the tables, it is apparent that every supergravity appearing in the pyramid, with the exception of those on the maximal "spine", indeed possess a twin. Furthermore, from this perspective, it is natural for the sequence of maximal theories not to admit a twin, since their parent theory may not be found among the pyramid supergravities. Albeit not a one-to-one correspondence, this establishes an interesting connection between the pyramid and the twinness property:

$$\text{Lies in pyramid (except spine)} \qquad \Rightarrow \qquad \text{Has a twin} \tag{4.51}$$

while the converse is not true. Indeed, there exist a handful of twins which lie outside of the Yang-Mills Squared pyramid, as we will see below. Furthermore, note that all twin theories admit a Yang-Mills factorisation:

$$\text{Has a twin} \qquad \Rightarrow \qquad \text{Has a Yang-Mills origin} \tag{4.52}$$

while the converse is again not true.

Triplets

As mentioned in the discussion around equation (4.1), the scalar manifold

$$\mathcal{M}=\frac{SU(4,2)}{SU(4)\times SU(2)\times SU(1)}\times\frac{SU(2)}{SU(2)}\times\frac{U(1)}{U(1)} \tag{4.53}$$

is shared by three $D = 3$ supergravity theories, with $\mathcal{N} = 6, 4, 2$. As was remarked in [92], the presence of two empty factors corresponding to the R-symmetries of two distinct little twins highlights this fact. These three theories, as well as the twin relation which relates them, uplift to $D = 4$, where they constitute a $(3, 2, 1)$ *triplet* of supergravities, with common coset

$$\mathcal{M} = \frac{SU(3,1)}{SU(3)\times U(1)} \times \frac{SU(2)}{SU(2)} \times \frac{U(1)}{U(1)}. \tag{4.54}$$

While sharing the same scalar manifold represents the original observation behind twin theories, these are in fact more precisely classified in terms of their parent theory, from which they descend by consistent truncation. Denoting the three as $(\mathcal{N}_+, \mathcal{N}_-^+, \mathcal{N}_-^-)$ serves to indicate at once that there exists a standard twin relation between the three distinct sub-pairs $(\mathcal{N}_+, \mathcal{N}_-^+)$, $(\mathcal{N}_+, \mathcal{N}_-^-)$ and $(\mathcal{N}_-^+, \mathcal{N}_-^-)$, in the sense that they may be understood as truncations of a common parent theory. Their content is

$$\begin{aligned} \mathcal{N}_+ &: \quad \mathbf{G}_3 + \mathbf{V}_3 \\ \mathcal{N}_-^+ &: \quad \mathbf{G}_2 + 3\mathbf{V}_2 \\ \mathcal{N}_-^- &: \quad \mathbf{G}_1 + 4\mathbf{V}_1 + 3\mathbf{C}_1. \end{aligned} \tag{4.55}$$

Since the $\mathcal{N}_+ = 3$ theory lies in the pyramid, namely the $\mathbf{V}_2 \otimes \tilde{\mathbf{V}}_1$ entry, and it is not maximal, one is able to find theories in the pyramid which act as parents to the $(3, 2)$ and $(3, 1)$ sub-pairs. We will see that this is not possible for the $(2, 1)$ case.

Proceeding in order, the $(3, 2)$ pair can be derived from a $\mathscr{N} = 5$ parent, which is given by the unique pyramid product $\mathbf{V}_4 \otimes \tilde{\mathbf{V}}_1$. Then, one has the following sequence of Yang-Mills factorisations,

$$\begin{array}{ccccc} \mathbf{V}_4 & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_5 \\ \mathbf{V}_2 + \mathbf{H}_2^{\rho} & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_3 + \mathbf{V}_3 \\ \mathbf{V}_2 + \mathbf{H}_2^{\rho} & \otimes & \tilde{A}_\mu + \tilde{\chi}^{\tilde{\rho}} & = & \mathbf{G}_2 + 3\mathbf{V}_2 \end{array} \tag{4.56}$$

As far as the big twin is concerned, it inherits a $\mathfrak{u}(2)_L + \overline{\mathfrak{su}}(2)_L$ from the Left factor, a $\mathfrak{u}(1)_R$ from the Right one and the diagonal $\mathfrak{u}(1)_d$ typical of $D = 4$. Its $\mathcal{N}_+ = 3$ algebra is constructed as

$$\mathfrak{u}(3)_{\text{R-sym}} \oplus \mathfrak{u}(1)_{\text{iso}} \supset \Big(\mathfrak{u}(2)_L \oplus \mathfrak{u}(1)_R\Big) \oplus \mathfrak{u}(1)_d \tag{4.57}$$

up to a possible rotation of the $\mathfrak{u}(1)$ charges. As it is apparent from (4.57), the flavour $\overline{\mathfrak{su}}(2)_L$ symmetry is trivial on all states of the supergravity theory, since it only acts on the hypermultiplet of the Left factor, which does not contribute to the product in this case. On the other hand, the little $\mathcal{N}_-^+ = 2$ twin inherits the same symmetries (with the difference that $\mathfrak{u}(1)_R$ acts as a flavour, rather than R-symmetry), which conspire to form its $\mathcal{N}_-^+ = 2$ algebra as

$$\mathfrak{u}(2)_{\text{R-sym}} \oplus \mathfrak{su}(3)_{\text{iso}} \oplus \mathfrak{u}(1)_{\text{iso}} \supset \mathfrak{u}(2)_L \oplus \Big(\overline{\mathfrak{su}}(2)_L \oplus \mathfrak{u}(1)_R\Big) \oplus \mathfrak{u}(1)_d, \tag{4.58}$$

which shows that the R-symmetry of the Left factor carries over as the R-symmetry of the resulting supergravity, unscathed.

Next, for the (3, 1) pair, starting from the $\mathscr{N} = 4$ parent in the pyramid,

$$\begin{array}{lclcl} \mathbf{V}_2 & \otimes & \tilde{\mathbf{V}}_2 & = & \mathbf{G}_4 + 2\mathbf{V}_4 \\ \mathbf{V}_1 + \mathbf{C}_1^\rho & \otimes & \tilde{\mathbf{V}}_2 & = & \mathbf{G}_3 + \mathbf{V}_3 \\ \mathbf{V}_1 + \mathbf{C}_1^\rho & \otimes & \tilde{A}_\mu + 2\tilde{\chi}^{\tilde{\rho}} + 2\tilde{\phi} & = & \mathbf{G}_2 + 3\mathbf{V}_2 \end{array} \tag{4.59}$$

In this case, for the big twin, the Left factor carries $\mathfrak{u}(1)_L \oplus \overline{\mathfrak{u}}(1)_L$ coming from the original $\mathfrak{u}(2)$ R-symmetry of $\mathbf{V}_2$, while the Right factor still has a $\mathfrak{u}(2)_R$ R-symmetry intact. These join to form the $\mathcal{N}_+ = 3$ algebra just as in (4.57), the only difference being the (inverted) origin of the $\mathfrak{u}(3)$ R-symmetry, namely

$$\mathfrak{u}(3)_{\text{R-sym}} \oplus \mathfrak{u}(1)_{\text{iso}} \supset \Big(\mathfrak{u}(2)_R \oplus \mathfrak{u}(1)_L\Big) \oplus \mathfrak{u}(1)_d. \tag{4.60}$$

Once again, the flavour group responsible for the "truncation" of the Left factor, here $\overline{\mathfrak{u}}(1)_L$ only charges the non-adjoint spinor multiplet $\mathbf{C}_1$, hence acts trivially on all supergravity states. For the little $\mathcal{N}_-^- = 1$ twin, the symmetries conspire to form the algebra as

$$\mathfrak{u}(1)_{\text{R-sym}} \oplus \mathfrak{u}(3)_{\text{iso}} \oplus \mathfrak{u}(1)_{\text{iso}} \supset \mathfrak{u}(1)_L \oplus \Big(\mathfrak{u}(2)_R \oplus \overline{\mathfrak{u}}(1)_L\Big) \oplus \mathfrak{u}(1)_d \tag{4.61}$$

where the $\mathfrak{u}(1)_L$ R-symmetry of the Left factor becomes the R-symmetry of supergravity, while the flavour groups $\mathfrak{u}(2)_R$ and $\overline{\mathfrak{u}}(1)_L$ form the $\mathfrak{u}(3)$ isotropy rotating the various multiplets.

Finally, the (2, 1) pair is not a truncation of a pyramid supergravity, hence the twin relation devised above does not apply straightforwardly. Nonetheless, it is possible to obtain all three (parent, big and little twin) from Yang-Mills, as

$$\begin{array}{lclcl} \mathbf{V}_2 + \mathbf{H}_2^\rho & \otimes & \tilde{\mathbf{V}}_1 + \mathbf{C}_1^{\tilde{\rho}} & = & \mathbf{G}_3 + 3\mathbf{V}_3 \\ \mathbf{V}_2 & \otimes & \tilde{A}_\mu + 2\tilde{\chi}^{\tilde{\rho}} + 2\tilde{\phi} & = & \mathbf{G}_2 + 3\mathbf{V}_2 \\ \mathbf{V}_1 + \mathbf{C}_1^\rho & \otimes & \tilde{A}_\mu + 2\tilde{\chi}^{\tilde{\rho}} + 2\tilde{\phi} & = & \mathbf{G}_1 + 4\mathbf{V}_1 + 3\mathbf{C}_1 \end{array} \tag{4.62}$$

4.2.3 Non-pyramid twin pair

In addition to the (2, 1) sub-pair of the $D = 4$ triplet, there is another twin pair in $D = 4$ which does not follow the pattern above, since the big twin does not reside in the pyramid. The two theories have supersymmetries $(\mathcal{N}_+, \mathcal{N}_-) = (4, 1)$ and content

$$\begin{array}{ll} \mathcal{N}_+ : & \mathbf{G}_4 \\ \mathcal{N}_- : & \mathbf{G}_1 + 6\mathbf{V}_1 + \mathbf{C}_1, \end{array} \tag{4.63}$$

while the common scalar manifold is

$$\frac{G}{H} = \frac{SU(1,1)}{U(1)} \times \frac{SO(6)}{SO(6)} \times \frac{U(1)}{U(1)}. \tag{4.64}$$

In this case, the three factorisations for parent, big and little twin are

$$\begin{array}{lllll} \mathbf{V}_4 & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_5 \\ \mathbf{V}_4 & \otimes & \tilde{A}_\mu + \tilde{\chi}^{\tilde{\rho}} & = & \mathbf{G}_4 \\ A_\mu + 4\chi^\rho + 6\phi & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_1 + 6\mathbf{V}_1 + \mathbf{C}_1 \end{array} \tag{4.65}$$

This deviates from the behaviour of the pyramid twins in that the factorisations of the big and little twin are obtained by breaking *either* the Right factor *or* the Left, respectively, but not both.

Left theory		Right theory		Supergravity			
Content	Symmetry	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
$\mathbf{V}_4+\mathbf{C}_4^\rho$	$\mathfrak{so}(4)\oplus\overline{\mathfrak{so}}(3)$	$\tilde{\mathbf{V}}_8$ $\tilde{A}(\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{8})+\tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	12 4	$\mathbf{G}_{12}$ $\mathbf{G}_4+16\mathbf{V}_4$	$\mathfrak{so}(12)_{\text{R-sym}}\oplus\mathfrak{so}(3)$ $\mathfrak{so}(4)_{\text{R-sym}}\oplus\mathfrak{so}(12)_{\text{iso}}\oplus\mathfrak{so}(3)$	$\frac{E_{7(-5)}}{SO(12)\times SO(3)}$
$\mathbf{V}_4+\mathbf{C}_4^\rho$	$\mathfrak{so}(4)\oplus\overline{\mathfrak{so}}(3)$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1},\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2})+\tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	8 4	$\mathbf{G}_8+4\mathbf{V}_8$ $\mathbf{G}_4+8\mathbf{V}_4$	$\mathfrak{so}(8)_{\text{R-sym}}\oplus\mathfrak{so}(4)_{\text{iso}}$ $\mathfrak{so}(4)_{\text{R-sym}}\oplus\mathfrak{so}(8)_{\text{iso}}\oplus\mathfrak{so}(4)$	$\frac{SO(8,4)}{SO(8)\times SO(4)}$
$\mathbf{V}_2+\mathbf{C}_2^\rho$	$\mathfrak{so}(2)\oplus\overline{\mathfrak{so}}(2)$	$\tilde{\mathbf{V}}_8$ $\tilde{A}(\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{8})+\tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	10 2	$\mathbf{G}_{10}$ $\mathbf{G}_2+\mathbf{V}_2+10\mathbf{V}_2+5\mathbf{C}_2$	$\mathfrak{so}(10)_{\text{R-sym}}\oplus\mathfrak{so}(2)$ $\mathfrak{so}(2)_{\text{R-sym}}\oplus\mathfrak{so}(10)_{\text{iso}}\oplus\mathfrak{so}(2)$	$\frac{E_{6(-14)}}{SO(10)\times SO(2)}$
$\mathbf{V}_2+\mathbf{C}_2^\rho$	$\mathfrak{so}(2)\oplus\overline{\mathfrak{so}}(2)$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1},\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2})+\tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	6 2	$\mathbf{G}_6+2\mathbf{V}_6$ $\mathbf{G}_2+\mathbf{V}_2+4\mathbf{V}_2+3\mathbf{C}_2$	$\mathfrak{so}(6)_{\text{R-sym}}\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(2)_{\text{iso}}$ $\mathfrak{so}(2)_{\text{R-sym}}\oplus[\mathfrak{su}(4)\oplus\mathfrak{u}(2)]_{\text{iso}}$	$\frac{SU(4,2)}{SU(4)\times U(2)}$
$\mathbf{V}_2+\mathbf{C}_2^\rho$	$\mathfrak{so}(2)\oplus\overline{\mathfrak{so}}(2)$	$\tilde{\mathbf{V}}_2$ $\tilde{A}+2\tilde{\chi}^{\tilde{\rho}}+\tilde{\phi}$	$\mathfrak{so}(2)$	4 2	$\mathbf{G}_4+\mathbf{V}_4+\mathbf{C}_4$ $\mathbf{G}_2+\mathbf{V}_2+\mathbf{V}_2+2\mathbf{C}_2$	$\mathfrak{so}(4)_{\text{R-sym}}\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2)_{\text{iso}}$ $\mathfrak{so}(2)_{\text{R-sym}}\oplus[\mathfrak{u}(2)\oplus\mathfrak{u}(2)]_{\text{iso}}$	$\frac{SU(2,1)\times SU(2,1)}{U(2)\times U(2)}$
$\mathbf{V}_1+\mathbf{C}_1^\rho$	$\varnothing$	$\tilde{\mathbf{V}}_8$ $\tilde{A}(\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{8})+\tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	9 1	$\mathbf{G}_9$ $\mathbf{G}_1+16\mathbf{V}_1$	$\mathfrak{so}(9)_{\text{R-sym}}$ $\mathfrak{so}(9)_{\text{iso}}$	$\frac{F_{4(-20)}}{SO(9)}$
$\mathbf{V}_1+\mathbf{C}_1^\rho$	$\varnothing$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1},\mathbf{1})+\tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2})+\tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	5 1	$\mathbf{G}_5+\mathbf{V}_5$ $\mathbf{G}_1+8\mathbf{V}_1$	$\mathfrak{so}(5)_{\text{R-sym}}+\mathfrak{so}(3)$ $\mathfrak{so}(5)_{\text{iso}}+\mathfrak{so}(3)$	$\frac{Sp(2,1)}{Sp(2)\times Sp(1)}$
$\mathbf{V}_1+\mathbf{C}_1^\rho$	$\varnothing$	$\tilde{\mathbf{V}}_2$ $\tilde{A}+2\tilde{\chi}^{\tilde{\rho}}+\tilde{\phi}$	$\mathfrak{so}(2)$	3 1	$\mathbf{G}_3+\mathbf{V}_3$ $\mathbf{G}_1+4\mathbf{V}_1$	$\mathfrak{so}(3)_{\text{R-sym}}+\mathfrak{so}(2)$ $\mathfrak{so}(3)_{\text{iso}}+\mathfrak{so}(2)$	$\frac{SU(2,1)}{U(2)}$
$\mathbf{V}_1+\mathbf{C}_1^\rho$	$\varnothing$	$\tilde{\mathbf{V}}_1$ $\tilde{A}+\tilde{\chi}^{\tilde{\rho}}$	$\varnothing$	2 1	$\mathbf{G}_2+\mathbf{V}_2$ $\mathbf{G}_1+2\mathbf{V}_1$	$\mathfrak{so}(2)_{\text{R-sym}}$ $\mathfrak{so}(2)_{\text{iso}}$	$\frac{SL(2,\mathbb{R})}{SO(2)}$

Table 4.6: Twin supergravities from Yang-Mills in $D=3$.

Left theory		Right theory		Supergravity			
Content	Symmetry	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
$\mathbf{V}_2 + \mathbf{H}_2^\rho$	$\mathfrak{u}(2) \oplus \overline{\mathfrak{su}}(2)$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) + \tilde{\phi}(\mathbf{6})$	$\mathfrak{su}(4)$	6 2	$\mathbf{G}_6$ $\mathbf{G}_2 + 15\mathbf{V}_2$	$\mathfrak{u}(6)_{\text{R-sym}}$ $\mathfrak{u}(2)_{\text{R-sym}} \oplus \mathfrak{u}(6)_{\text{iso}}$	$\frac{SO^*(12)}{U(6)}$
$\mathbf{V}_2 + \mathbf{H}_2^\rho$	$\mathfrak{u}(2) \oplus \overline{\mathfrak{su}}(2)$	$\tilde{\mathbf{V}}_2$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) + 2\tilde{\phi}(\mathbf{1})$	$\mathfrak{u}(2)$	4 2	$\mathbf{G}_4 + 2\mathbf{V}_4$ $\mathbf{G}_2 + \mathbf{V}_2 + 6\mathbf{V}_2$	$\mathfrak{u}(4)_{\text{R-sym}} \oplus \mathfrak{so}(2)_{\text{iso}}$ $\mathfrak{u}(2)_{\text{R-sym}} \oplus \mathfrak{u}(4)_{\text{iso}}$	$\frac{SL(2,\mathbb{R})}{U(1)}$
$\mathbf{V}_1 + \mathbf{C}_1^\rho$	$\mathfrak{u}(1) \oplus \overline{\mathfrak{u}}(1)$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) + \tilde{\phi}(\mathbf{6})$	$\mathfrak{so}(4)$	5 1	$\mathbf{G}_5$ $\mathbf{G}_1 + 10\mathbf{V}_1 + 5\mathbf{C}_1$	$\mathfrak{u}(5)_{\text{R-sym}}$ $\mathfrak{u}(1)_{\text{R-sym}} \oplus \mathfrak{u}(5)_{\text{iso}}$	$\frac{SU(5,1)}{U(5)}$
$\mathbf{V}_1 + \mathbf{C}_1^\rho$	$\mathfrak{u}(1) \oplus \overline{\mathfrak{u}}(1)$	$\tilde{\mathbf{V}}_2$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) + 2\tilde{\phi}(\mathbf{1})$	$\mathfrak{u}(2)$	3 1	$\mathbf{G}_3 + \mathbf{V}_3$ $\mathbf{G}_1 + \mathbf{V}_1 + 3\mathbf{V}_1 + 3\mathbf{C}_1$	$\mathfrak{u}(3)_{\text{R-sym}} \oplus \mathfrak{u}(1)$ $\mathfrak{u}(1)_{\text{R-sym}} \oplus \mathfrak{u}(3)_{\text{iso}}$	$\frac{U(3,1)}{U(3)\times U(1)}$
$\mathbf{V}_1 + \mathbf{C}_1^\rho$	$\mathfrak{u}(1) \oplus \overline{\mathfrak{u}}(1)$	$\tilde{\mathbf{V}}_1$ $\tilde{A} + \tilde{\chi}^{\tilde{\rho}}$	$\mathfrak{u}(1)$	2 1	$\mathbf{G}_2 + \mathbf{H}_2$ $\mathbf{G}_1 + \mathbf{V}_1 + 2\mathbf{C}_1$	$\mathfrak{u}(2)_{\text{R-sym}} \oplus \mathfrak{u}(1)$ $\mathfrak{u}(1)_{\text{R-sym}} \oplus \mathfrak{u}(2)_{\text{iso}}$	$\frac{U(2,1)}{U(2)\times U(1)}$

Table 4.7: Twin supergravities from Yang-Mills in $D = 4$.

Left theory		Right theory		Supergravity			
Content	Symmetry	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
$\mathbf{V}_2 + \mathbf{H}_2^\rho$	$\mathfrak{sp}(1) \oplus \overline{\mathfrak{sp}}(1)$	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) + \tilde{\phi}(\mathbf{5})$	$\mathfrak{sp}(2)$	6 2	$\mathbf{G}_6$ $\mathbf{G}_2 + 14\mathbf{V}_2$	$\mathfrak{sp}(3)_{\text{R-sym}}$ $\mathfrak{sp}(1)_{\text{R-sym}} \oplus \mathfrak{sp}(3)_{\text{iso}}$	$\frac{SU^*(6)}{Sp(3)}$
$\mathbf{V}_2 + \mathbf{C}_2^\rho$	$\mathfrak{sp}(1) \oplus \overline{\mathfrak{sp}}(1)$	$\tilde{\mathbf{V}}_2$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) + \tilde{\phi}(\mathbf{1})$	$\mathfrak{sp}(1)$	4 2	$\mathbf{G}_4 + \mathbf{V}_4$ $\mathbf{G}_2 + \mathbf{V}_2 + 5\mathbf{V}_2$	$\mathfrak{sp}(2)_{\text{R-sym}}$ $\mathfrak{sp}(1)_{\text{R-sym}} \oplus \mathfrak{sp}(2)_{\text{iso}}$	$\frac{O(1,1)\times O(5,1)}{Sp(2)}$
$\mathbf{V}_{(2,0)} + \mathbf{H}^\rho_{(2,0)}$ $\mathbf{V}_{(0,2)} + \mathbf{H}^\rho_{(0,2)}$	$\mathfrak{sp}(1) \oplus \overline{\mathfrak{sp}}(1)$ $\overline{\mathfrak{sp}}(1) \oplus \mathfrak{sp}(1)$	$\tilde{\mathbf{V}}_{(2,2)}$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}_-^{\tilde{\rho}}(\mathbf{2}) + \tilde{\chi}_+^{\tilde{\rho}}(\mathbf{2}) + \tilde{\phi}(\mathbf{4})$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$	$(4,2)$ $(0,2)$	$\mathbf{G}_{(4,2)}$ $\mathbf{G}_{(0,2)} + (4+4)\mathbf{V}_{(0,2)} + 5\mathbf{T}_{(0,2)}$	$\mathfrak{sp}(2)_{\text{R-sym}} \oplus \mathfrak{sp}(1)_{\text{R-sym}}$ $\mathfrak{sp}(1)_{\text{R-sym}} \oplus \mathfrak{sp}(2)_{\text{iso}}$	$\frac{SO(5,1)}{SO(5)}$
$\mathbf{V}_{(2,0)} + \mathbf{H}^\rho_{(2,0)}$ $\mathbf{V}_{(0,2)} + \mathbf{H}^\rho_{(0,2)}$	$\mathfrak{sp}(1) \oplus \overline{\mathfrak{sp}}(1)$ $\overline{\mathfrak{sp}}(1) \oplus \mathfrak{sp}(1)$	$\tilde{\mathbf{V}}_{(2,0)}$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}_-^{\tilde{\rho}}(\mathbf{2})$	$\mathfrak{sp}(1)$ $\mathfrak{sp}(1)$	$(4,0)$ $(0,2)$	$\mathbf{G}_{(4,0)} + \mathbf{T}_{(4,0)}$ $\mathbf{G}_{(0,2)} + 5\mathbf{T}_{(0,2)}$	$\mathfrak{sp}(2)_{\text{R-sym}}$ $\mathfrak{sp}(1)_{\text{R-sym}} \oplus \mathfrak{sp}(2)_{\text{iso}}$	$\frac{SO(5,1)}{SO(5)}$
$\mathbf{V}_{(2,0)} + \mathbf{H}^\rho_{(2,0)}$ $\mathbf{V}_{(0,2)} + \mathbf{H}^\rho_{(0,2)}$	$\mathfrak{sp}(1) \oplus \overline{\mathfrak{sp}}(1)$ $\overline{\mathfrak{sp}}(1) \oplus \mathfrak{sp}(1)$	$\tilde{\mathbf{V}}_{(0,2)}$ $\tilde{A}(\mathbf{1}) + \tilde{\chi}_+^{\tilde{\rho}}(\mathbf{2})$	$\mathfrak{sp}(1)$ $\mathfrak{sp}(1)$	$(2,2)$ $(0,2)$	$\mathbf{G}_{(2,2)}$ $\mathbf{G}_{(0,2)} + 4\mathbf{V}_{(0,2)} + \mathbf{T}_{(0,2)}$	$\mathfrak{sp}(1)_{\text{R-sym}} + \mathfrak{sp}(1)_{\text{R-sym}}$ $\mathfrak{sp}(1)_{\text{R-sym}} \oplus \mathfrak{sp}(1)_{\text{iso}}$	$O(1,1)$

Table 4.8: Twin supergravities from Yang-Mills in $D = 5$ and $D = 6$.

Chapter 5

Gauge symmetries and their quantum realisation

Noether identities

A fascinating feature of gauge theories, and certainly one of great importance, is the fact that gauge invariance of the action functional implies $\dim(\mathcal{G})$ relations, known as *Noether identities*, among the equations of motion. For a nice discussion, see [134]. As a consequence, the latter no longer impose enough independent conditions to fix the dynamics uniquely. Usually, one proceeds by imposing $\dim(\mathcal{G})$ conditions on the gauge field by requiring a so-called *gauge-fixing* functional of A_μ to vanish, $F[A] = 0$. Demanding that the theory makes sense quantum-mechanically, however, allows us to better delineate the seemingly damaging consequences of gauge invariance and find a way around them. In the process, a much richer structure underpinning general gauge theories is revealed which, among other things, leads to the discovery of very elegant and powerful mathematical tools. In particular, here we shall mainly be concerned with the "emerging" BRST symmetry[1] and its consequences.

It turns out that most of this structure is accessible from but a corner of Yang-Mills theory, namely the kinematics and self-interactions of the gauge field. Consequently, for the remainder of this discussion, we will neglect any coupling to matter fields and their interactions with A_μ, and restrict our attention solely to the theory defined by the set of all Lie algebra-valued gauge fields, denoted by $\mathcal{A} = \{A_\mu(x)\}$, whose dynamics are encoded in the action

$$S_0[A] = -\frac{1}{4}\int \mathrm{d}^D x\; F^A_{\mu\nu} F^{\mu\nu}_A, \tag{5.1}$$

where we append the subscript "0" to signify that (5.1) is to be regarded simply as the classical starting point, obtainable as a limit of a more fundamental, quantum-mechanically

[1]It is "emergent" only from a historical perspective; indeed, it is possible to elevate it as the fundamental principle of general gauge theories, which may be then constructed with the only assumption that the BRST symmetry is present in a larger, more fundamental phase space. This is the core idea behind the field-antifield, or Batalin-Vilkoviskiy (BV), formalism, see Appendix B.

consistent action, S, derived in what follows. Varying with respect to the gauge field yields the equations of motion,

$$\frac{\delta S_0[A]}{\delta A^\nu_A} = D^\mu F^A_{\mu\nu} = 0. \tag{5.2}$$

The Noether identity for Yang-Mills can be written as

$$D^\nu D^\mu F^A_{\mu\nu} = 0, \tag{5.3}$$

which can be checked using (2.16) and the antisymmetry of the structure constants. Notice how, in the zero coupling limit $g = 0$, this reduces to $\dim(\mathcal{G})$ copies of the electromagnetic Noether identity, $\partial^\nu \partial^\mu F_{\mu\nu} = 0$, which holds simply because $[\partial_\mu, \partial_\nu] = 0$.

What are, though, the difficulties hindering a smooth quantisation?

- *No more Hilbert space?*

  As long as one is interested (as we are in this thesis) in maintaining manifest Lorentz covariance in the quantised theory, it is reasonable to expect issues analogous to those arising in the Abelian case: there, timelike photon states had negative norm and the Hilbert space character of state space was lost. The solution to this problem is usually attributed to Gupta and Bleuer [144, 145]; an important refinement thereof, wherein the subsidiary condition is imposed as an operator identity rather than directly on the state space, is due to Nakanishi [146] and Lautrup [147]. Similarly, the full spectrum of Yang-Mills theory contains many unwanted states, threatening the unitarity of the theory. Happily, there exist a similar, albeit more involved, resolution, found by Kugo and Ojima [98–100]. We refer the reader to Section 5.3 for a more detailed explanation.

- *Noether identities and no propagators*

  The very existence of the Noether identities (5.3) has a second (related, of course) striking consequence, which poses a serious obstacle to the correct definition of a perturbative expansion: for the naive classical action (5.1), propagators do not exist! This may be proven by differentiating the (general form of the) Noether identities and showing that the Hessian of the classical action is non-invertible, as done in [134]. Indeed, using the definition of the field stregths in terms of A^A_μ, the quadratic part of the action becomes

  $$S_0[A] = \int \mathrm{d}^D x \left[\frac{1}{2} A^{\mu A} \Big(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu \Big) A^\nu_A \right] + \text{(self-interactions)} \tag{5.4}$$

  where we notice that the operator $K_{\mu\nu} \equiv \Box \eta_{\mu\nu} - \partial_\mu \partial_\nu$, whose inverse would yield the propagator, is non-invertible since it has a non-trivial kernel,

  $$(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) \partial^\nu \chi = 0 \tag{5.5}$$

consisting of the zero modes $\partial^\nu \chi$, for any scalar function χ. These are gauge transformations of the vacuum, $A_\mu = 0$, with gauge parameter $\theta^A = \chi^A$ and correspond to longitudinal modes of the quantised excitation of the gauge field, since their Fourier components are proportional to the momentum k_μ.

- *Physically distinct configurations are fewer than it seems*

  Another issue to keep in mind when attempting quantisation is that the space $\mathcal{A}$ is foliated, under the action of gauge transformations, into elements of the coset space $\mathcal{A}/\mathcal{G}$, namely the equivalence classes[2]

$$\left\{ A_\mu \sim A_\mu^U \mid A_\mu^U = \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x) A_\mu U(x)^{-1} \right\} \in \mathcal{A}/\mathcal{G} \tag{5.6}$$

  known as *gauge orbits*. Elements belonging to the same orbit represent the exact same physical configuration[3], so the action functional returns the same value when evaluated on any of them. If our quantisation scheme of choice involves summing over different configurations, then we are at risk of grossly overcounting, unless a way is found to cleverly disentagle this redundancy and eliminate it.

5.1 Path integral quantisation

Imagine defining the quantum theory via the path integral over all possible gauge field configurations in $\mathcal{A}$,

$$Z = \int_{\mathcal{A}} \mathrm{D}[A] e^{iS_0[A]} \tag{5.7}$$

where the integration measure is formally defined as $\mathrm{D}[A] = \prod_{\mu,A,x} dA_\mu^A(x)$ and is assumed to be invariant under arbitrary gauge transformations, $\mathrm{D}[A^U] = \mathrm{D}[A]$. The classical action is gauge invariant as well, $S_0[A^U] = S_0[A]$; thus Z is too.

In accordance to the discussion above, we would like to isolate the contributions of the (infinitely degenerate) gauge orbits and restrict our functional integral so as to sum only over physically distinct configurations. Heuristically, we'd like to be able to write

$$Z = \int_{\mathcal{G}} \mathrm{D}[U] \int_{\mathcal{A}/\mathcal{G}} \mathrm{d}\mu[A] e^{iS_0[A]}. \tag{5.8}$$

where $\mathrm{D}[U]$ and $\mathrm{d}\mu[A]$ are some measures on the gauge group and on $\mathcal{A}/\mathcal{G}$, respectively, yet to be properly defined. It turns out that the correct definition of the latter requires some work, as shown below.

[2]Only in this section, we change notation slightly, $A'_\mu \to A^U_\mu$, since here we want a shorthand notation to specify which gauge transformation U is being performed. We will return to the standard notation for transformations, A'_μ, in the remainder of this thesis.

[3]As gauge transformations do not take a physical state into another; in a Hamiltonian description, they are generated by Noether charges which vanish on solutions of the equations of motion.

One essential ingredient is to "cut out" the redundant *gauge* degrees of freedom by defining a surface in $\mathcal{A}$, known as the *gauge slice*, which intersects all gauge orbits and is specified by the vanishing of a local, Lie algebra-valued gauge-fixing functional

$$F^A[A] = \{F^A(A_\mu, \partial_\mu A_\nu, \dots)(x),\ x \in \mathcal{M}\} = 0. \tag{5.9}$$

Our aim, then, is to localise the integration over the gauge slice. For this to correspond to honest integration over orbit space, however, the gauge-fixing has to pick a unique representative for each orbit[4]: this is achieved by *assuming* that, for each A_μ lying on some orbit, there exist a unique gauge transformation U taking A_μ onto the gauge slice. Let us denote these group elements by U_F, since they depend on the choice of gauge-fixing function. In short

$$\forall A_\mu,\ \exists \text{ unique } U_F \in \mathcal{G} \text{ such that } F[A^{U_F}] = 0, \tag{5.10}$$

thus ensuring that every orbit be intercepted exactly once.

Now for the subtle point: even though $F[A]$ is well-defined, it turns out that we would be committing a mistake if we tried to restrict the integral in (5.7) by adding a (functional) Dirac delta such as $\delta\big[F[A]\big]$, without thinking twice. Indeed, we would be failing to notice that now the integration (incorrectly) depends on our choice of $F[A]$. The solution to this puzzle, known as the *Faddeev-Popov trick* or *insertion of unity*, was discovered in 1967 due to an intuition of L. Faddeev, while V. Popov provided the mathematical justification[5].

5.1.1 Faddeev-Popov determinant

The idea of Faddeev and Popov [103] was to average the gauge condition over the gauge group $\mathcal{G}$. Define the quantity $\Delta[A]$ by

$$1 = \Delta[A] \int_{\mathcal{G}} \mathrm{D}[U]\delta\big[F[A^U]\big] \tag{5.11}$$

where $\mathrm{D}[U]$ is the measure on the gauge group briefly mentioned above. It is formally defined as the product of the group measure at each $x \in \mathcal{M}$, and it is assumed to be invariant, namely $\mathrm{D}[U] = \mathrm{D}[UU']$. Notice that $\Delta[A]$ is gauge invariant owing to the invariance of the measure, since

$$\begin{aligned} \Delta^{-1}[A^U] &= \int_{\mathcal{G}} \mathrm{D}[U']\delta\big[F[A^{UU'}]\big] \\ &= \int_{\mathcal{G}} \mathrm{D}[UU']\delta\big[F[A^{UU'}]\big] \\ &= \Delta^{-1}[A] \end{aligned} \tag{5.12}$$

[4]We ignore here the subtleties related to the global definition of the gauge-fixing function, the Gribov ambiguity, etc., since perturbation theory is well defined within the first Gribov region.

[5]According to the historical account given by Faddeev in his Scholarpedia entry.

i.e. it is a function only of the equivalence classes, not of A_μ itself. We proceed by inserting this fancy "1" in the path integral (5.7) and change the order of integration, to obtain

$$Z = \int_{\mathcal{G}} \mathrm{D}[U] \int_{\mathcal{A}} \mathrm{D}[A]\ \Delta[A]\delta\big[F[A^U]\big]e^{iS_0[A]}. \tag{5.13}$$

Next, using the gauge-invariance of $\mathrm{D}[A]$, $\Delta[A]$ and $S_0[A]$ and relabeling $A^U \to A$, the above becomes

$$\begin{aligned} Z &= \int_{\mathcal{G}} \mathrm{D}[U] \int_{\mathcal{A}} \mathrm{D}[A^U]\ \Delta[A^U]\delta\big[F[A^U]\big]e^{iS_0[A^U]} \\ &= \left(\int_{\mathcal{G}} \mathrm{D}[U]\right) \int_{\mathcal{A}} \mathrm{D}[A]\ \Delta[A]\delta\big[F[A]\big]e^{iS_0[A]} \\ &= \mathrm{Vol}(\mathcal{G}) \int_{\mathcal{A}} \mathrm{D}[A]\ \Delta[A]\delta\big[F[A]\big]e^{iS_0[A]}, \end{aligned} \tag{5.14}$$

which shows that the A_μ-dependent integrand in (5.13) was, in fact, independent of U and, as a consequence, the integration over $\mathcal{G}$ factorises and simply yields the volume of the gauge group. Thus, we have successfully restricted the integral to the gauge slice by virtue of the delta function, such as it does not depend on the particular choice of gauge slice, $F[A]$. Furthermore, in doing so, we have isolated the integration over all gauge transformations which bothered us in the discussion around (5.6). Effectively, we have found a way to express the measure on orbit space by modifying the measure on $\mathcal{A}$,

$$\int_{\mathcal{A}/\mathcal{G}} \mathrm{d}\mu[A] = \int_{\mathcal{A}} \mathrm{D}[A]\Delta[A]\delta\big[F[A]\big]. \tag{5.15}$$

But what is $\Delta[A]$*?* It is more convenient to perform the calculation near the identity, $U_F = \mathrm{Id}$, where the gauge field is shifted as $A^\theta_\mu = A_\mu + D_\mu\theta$. Expanding the gauge-transformed gauge-fixing condition in θ,

$$F[A^\theta] = F[A] + M[A]\theta + \mathcal{O}(\theta^2) \tag{5.16}$$

we define, *on the gauge slice* $F[A] = 0$, the operator $M[A]$, with components

$$\big(M[A]\big)^A{}_B = \frac{\delta F^A[A^\theta]}{\delta\theta^B}\bigg|_{\theta=0} \tag{5.17}$$

as the linear approximation of the function $F[A^\theta]$ at $\theta = 0$, i.e. its Jacobian. Then, from the definition (5.11),

$$\begin{aligned} \Delta^{-1}[A] &= \int_{\mathcal{G}} \mathrm{D}[U]\delta\big[F[A^U]\big] \\ &= \int_{\mathfrak{g}} \mathrm{D}[\theta]\delta\big[M[A]\theta\big] \\ &= \big|\mathrm{det}M[A]\big|^{-1}. \end{aligned} \tag{5.18}$$

The last equality holds due to there being, by assumption, only one solution to the equation $F[A^U] = 0$, the U_F of (5.10). Thus, on the gauge slice, *the operator* $\Delta[A]$ *is the (absolute value of the functional) determinant of the Jacobian of the gauge-fixing condition*, conventionally written as

$$\Delta[A] = \big|\det M[A]\big| = \left|\det\left(\frac{\delta F^A[A^\theta]}{\delta\theta^B}\right)\right| \qquad \text{if} \qquad F[A] = 0. \tag{5.19}$$

This is known as the *Faddeev-Popov determinant.* Notice that, in the path integral (5.14), $F[A] = 0$ is always satisfied due to the delta function; therefore, the identification (5.19) is always valid.

5.2 Faddeev-Popov ghosts and gauge-fixing

Let us refine slightly the expression for the path integral obtained so far, for general gauge-fixing condition $F[A]$. The reason for wanting to do so is that the expression

$$Z = \mathrm{Vol}(\mathcal{G}) \int_{\mathcal{A}} \mathrm{D}[A] \; \big|\det M[A]\big| \delta\big[F[A]\big] e^{iS_0[A]}, \tag{5.20}$$

despite embodying our successful attempt at factorising the action of the gauge group and fixing the gauge redundancy (among which also the zero modes of (5.5)), is not immediately useful to a perturbative approach. However, one realises that *both* the Faddeev-Popov determinant and the delta function may be expressed as functional integrals over some field. The idea is that this will introduce new A_μ-dependent terms in the Yang-Mills action, thus effectively modifying the perturbative expansion in such a way as to solve the issues anticipated above. It is best to use the representation of the determinant[6] as a Gaussian integral over a pair of adjoint, Grassmann-valued scalar fields: the Faddeev-Popov *ghost* c^A and *antighost* $\bar{c}^A$. That is, suppressing the gauge indices,

$$\det M[A] = \int \mathrm{D}[c]\mathrm{D}[\bar{c}] \; e^{-i\int \mathrm{d}^D x \; \bar{c} M[A] c}. \tag{5.21}$$

The delta function can be lifted to the exponent by employing a little trick: first, the gauge-fixing condition is generalised to $F[A] - \omega = 0$ for arbitrary Lie algebra-valued functions $\omega(x)$, noticing that this is allowed because this modification is immaterial to $\det M[A]$, since $\omega(x)$ does not transform under the gauge group. Then, averaging over ω with a weight function, normally chosen to be a Gaussian centered at $\omega = 0$ with width $w = -\xi$, we obtain instead

$$\int \mathrm{D}[\omega] \; \delta\big[F[A] - \omega\big] e^{\frac{i}{2\xi}\int \mathrm{d}^D x \; \omega^2} = e^{\frac{i}{2\xi}\int \mathrm{d}^D x \; (F[A])^2}. \tag{5.22}$$

[6]We omit here the absolute value, thus assuming the positivity of the determinant. This is fine for our purposes, in perturbation theory, but is not a global property on $\mathcal{A}$. For large enough A_μ, the determinant can indeed change sign, leading to the notion of Gribov regions.

Heuristically, this "blurs" the gauge slice, effectively allowing gauge configurations to fluctuate about it, with a suppression factor controlled by ξ. At last, we should be able to show that a well-defined perturbative expansion is indeed possible. After all, this was the original motivation for all of the above manipulations. Although one is not forced to, it is very convenient at this stage to let go of generality in favour of concreteness, by specifying the gauge slice. In this thesis, we will mainly be concerned with the *linear covariant gauge*, defined by

$$F^A[A] = \partial^\mu A^A_\mu. \tag{5.23}$$

In addition to this gauge choice, there exist a good number of distinct gauges one might choose to employ: for a well-written and rather complete review, see [148]. In the linear covariant gauge, the Faddeev-Popov determinant reads

$$\det M[A] = \det\left\{\frac{\delta}{\delta\theta}\left[\partial^\mu\left(A_\mu + \frac{1}{g}D_\mu\theta\right)\right]\right\} = \frac{1}{g}\det\left(\partial^\mu D_\mu\right). \tag{5.24}$$

Notably, the A_μ-dependence of the Faddeev-Popov determinant, entering through the covariant derivative, is characteristic of Yang-Mills theory – as opposed to electromagnetism – in the sense that it is a consequence of the non-Abelian nature of the gauge group. For $U(1)$, the covariant derivative in (5.24) reduces to the usual partial derivative and the determinant, $\det(\partial^\mu\partial_\mu)$, can be factored out of the path integral, where it merely contributes to the normalisation. This is the reason why, as long as we commit to a specific gauge choice, we can ignore the ghosts in electromagnetism.

5.2.1 The Faddeev-Popov Lagrangian

Combining all of the above, we see that we have reformulated the path integral over $\mathcal{A}/\mathcal{G}$ in (5.20) as one over the enlarged phase space $\mathcal{P} = \{A_\mu, c, \overline{c}\}$, namely

$$Z = \mathrm{Vol}(\mathcal{G}) \int_{\mathcal{A}} \mathrm{D}[A]\mathrm{D}[c]\mathrm{D}[\overline{c}]\; e^{iS[A,c,\overline{c}]}, \tag{5.25}$$

with the new action, modified by the addition of the term $2\xi^{-1}(\partial^\mu A^A_\mu)^2$ due to the integrated delta function (5.22), given by

$$S[A,c,\overline{c}] = \int \mathrm{d}^D x\; \left[-\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A + \frac{1}{2\xi}(\partial^\mu A^A_\mu)^2 - \overline{c}^A\partial^\mu(D_\mu c)_A\right]. \tag{5.26}$$

This manifestly solves the "zero modes" issue, since the quadratic part of the above,

$$S_{\mathrm{quad}}[A] = \int \mathrm{d}^D x\; \left[\frac{1}{2}A^{\mu A}\left(\Box\eta_{\mu\nu} - \left(1+\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^\nu_A\right], \tag{5.27}$$

has no zero modes and can be inverted to give the (momentum space) gauge field propagator,

$$\tilde{D}^{\mu\nu}_F(p) = \frac{-i}{p^2+i\epsilon}\left(g^{\mu\nu} - (1+\xi)\frac{p^\mu p^\nu}{p^2}\right). \tag{5.28}$$

Somewhat confusingly, different choices of ξ are referred to as gauges as well. Keeping in mind that we are working *within* the linear covariant gauge, it should be noted that some popular choices include $\xi = 0$ (*Landau gauge*) and $\xi = -1$ (*Feynman-'t Hooft gauge*). The former, since the width of the Gaussian vanishes, is equivalent to the linear covariant gauge for $\omega = 0$, namely the usual Lorenz gauge. The latter has the advantage of simplifying the kinetic term, hence the propagator, of the gauge field.

In conclusion, the price to pay in exchange for an invertible operator was a gauge-dependent Green function (see Chapter 4 of [80] for an elegant derivation in this spirit, purely at the classical level). The consistency of the theory demands, however, that physical quantities be gauge-independent, as a choice of measuring sticks should not affect the results of an experiment. Remarkably, it is indeed true (and it can be proven rigorously, using the BRST symmetry of the ensuing section) that physical observables of Yang-Mills theory do not, in fact, depend on ξ.

5.2.2 The Lautrup-Nakanishi field

Very often, especially so if one cares about keeping manifest Lorentz covariance, it is convenient to introduce an adjoint auxiliary field, $b(x) = b(x)^A t_A$, known in the literature as the *Lautrup-Nakanishi* auxiliary field. It is named after the authors who first realised its usefulness in improving some aspects of the quantisation of Abelian gauge theory in covariant gauges [146, 147] where, of course, the index A is simply one-dimensional. In particular, as mentioned earlier, their work provided a refinement of the Gupta-Bleuer formalism which were to prove itself better suited for a non-Abelian generalisation, whereby the physical state condition of Gupta-Bleuer $\partial^\mu A_\mu^+ \left|\psi_{phys}\right\rangle$ is replaced by $b^+ \left|\psi_{phys}\right\rangle$, making use of the positive frequency (annihilation) part of the newly introduced field b. Notice, however, that the unambiguous separation into positive and negative frequency parts, consistent with time evolution for all t, is only possible due to the free nature of b in the Abelian theory, $\Box b^{(Ab)} = 0$. As we will see shortly, this is no longer true in Yang-Mills, where the need arises for a suitable extension of the Lautrup-Nakanishi condition.

The auxiliary field b also finds its place within the path integral formulation of Yang-Mills, via

$$\delta\big[F[A]\big] = \int \mathrm{D}[b]\ e^{i\int \mathrm{d}^D x\ b(x)\cdot F[A]}, \tag{5.29}$$

the functional analogue of the familiar $\delta(g(x)) = \int \mathrm{d}p\ e^{ip\cdot g(x)}$. Thus, b is recognised as a Lagrange multiplier enforcing the gauge-fixing condition $F[A] = 0$, namely Landau gauge (that is, $\xi = 0$). In line with (5.22), it is preferable to add a (BRST-exact, see next section) quadratic term,

$$\int \mathrm{D}[\omega]\ \delta\big[F[A] - \omega\big] e^{\frac{i}{2\xi}\int \mathrm{d}^D x\ \omega^2} = \int \mathrm{D}[\omega] \int \mathrm{D}[b]\ e^{i\int \mathrm{d}^D x\ b(F[A]-\omega)+\frac{i}{2\xi}\omega^2} \tag{5.30}$$

$$= \int \mathrm{D}[b]\ e^{i\int \mathrm{d}^D x\ bF[A]-\frac{\xi}{2}b^2}. \tag{5.31}$$

With this addition, the phase space is enlarged yet again to $\mathcal{P}_b = \{A, c, \bar{c}, b\}$ and the Yang-Mills action functional becomes

$$S[A, c, \bar{c}, b] = \int \mathrm{d}^D x \left[-\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A + b_A \left(F^A[A] - \frac{\xi}{2} b^A \right) - \bar{c}^A M[A]_{AB} c^B \right]. \tag{5.32}$$

The Lautrup-Nakanishi field is auxiliary since it appears without time derivatives in the Lagrangian and, consequently, its equations of motion are algebraic. In fact, the latter simply pose that the auxiliary field be equal to the gauge-fixing condition,

$$b^A = \frac{1}{\xi} F^A[A], \tag{5.33}$$

a fact, due to the addition of the quadratic term, which greatly simplifies some calculations in Yang-Mills, such as the proof that observables are ξ-independent. The auxiliary field can be integrated out by completing the square and performing a Gaussian integral (or, equivalently, by substituting (5.33) back into the action), a procedure which returns (5.26). As we will see shortly, the Lautrup-Nakanishi auxiliary field plays yet one more important role: it is exactly what is needed to close the gauge (read BRST) algebra off-shell.

5.3 BRST invariance

Let us come back to the form of the Yang-Mills action defined in (5.32), that is the result of the Faddeev-Popov procedure and the insertion of the auxiliary field b. Choosing for concreteness the linear covariant gauge (5.23), it reads

$$S[A, c, \bar{c}, b] = \int \mathrm{d}^D x \left[-\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A + b_A \left(\partial^\mu A^A_\mu - \frac{\xi}{2} b^A \right) - \bar{c}^A (\partial^\mu D_\mu)_{AB} c^B \right]. \tag{5.34}$$

Crucially, although the classical gauge invariance is lost in the quantum theory, the action functional (5.34) enjoys a new invariance, found by Becchi, Rouet, Stora and, independently, by Tyutin (hence the acronym BRST). Define the *BRST operator* Q by its action on all fields,

$$QA^A_\mu = (D_\mu c)^A \qquad Q\bar{c}^A = b^A \tag{5.35}$$
$$Qc^A = -\tfrac{1}{2}[c, c]^A \qquad Qb^A = 0, \tag{5.36}$$

noticing, in particular, how Q maps bosonic into fermionic fields, *mixes gauge field configurations together with the ghosts and the Lautrup-Nakanishi field* and, above all, is nilpotent on all fields, $Q^2 = 0$. The fact that the action is Q-invariant, i.e.

$$QS[A, c, \bar{c}, b] = 0, \tag{5.37}$$

allows us to define an infinitesimal BRST "symmetry" transformation acting on any $\Psi \in \mathcal{P}_b = \{A, c, \bar{c}, b\}$ as $\delta_B \Psi := \epsilon Q \Psi$, with ϵ a *global* Grassmann-valued number, that is the parameter of a global *super*symmetry transformation. Notice that (5.34) is further invariant under an additional global scale transformation involving exclusively the Faddeev-Popov ghosts,

$$\delta_\alpha c = \alpha c \tag{5.38}$$

$$\delta_\alpha \bar{c} = -\alpha \bar{c}. \tag{5.39}$$

5.3.1 The physical Hilbert space

Since both the BRST transformations and the ghost rescalings are global, one ought to be able to find the corresponding conserved Noether charges, $\mathbb{Q}_B$ and $\mathbb{Q}_c$, known as the BRST and ghost charge, respectively. These generate the transformations they correspond to via commutators, e.g. $\delta_B \Psi = [\epsilon \mathbb{Q}_B, \Psi]$. The BRST charge inherits its properties from Q: it is *fermionic* and it carries a non-vanishing *ghost number*, since $[\mathbb{Q}_B, \mathbb{Q}_c] = \mathbb{Q}_B$. In formulae,

$$\varepsilon(\mathbb{Q}_B) = 1, \qquad\qquad gh(\mathbb{Q}_B) = 1, \tag{5.40}$$

with ε indicating the parity of an operator. Its most important quality, however, is its nilpotency,

$$\mathbb{Q}_B^2 = 0, \tag{5.41}$$

which is instrumental in the characterisation of the physical subspace. Our quest toward it is, in fact, more complicated than its Abelian counterpart: in addition to the states of negative norm already present for $U(1)$, the spectrum of Yang-Mills contains the unphysical (violating the spin-statistics relation) ghost fields as well as the non-dynamical Lautrup-Nakanishi field. As shown by Kugo and Ojima [98–100], the correct generalisation of the Lautrup-Nakanishi condition is to identify[7] the physical states with elements of the cohomology (i.e. the *cohomology classes*) of the BRST charge[8], i.e. the quotient space

$$\mathcal{H}_{phys} = \frac{\ker \mathbb{Q}_B}{\mathrm{im}\ \mathbb{Q}_B}, \tag{5.42}$$

having defined the kernel and the image of the BRST charge as

$$\ker \mathbb{Q}_B := \{|\psi\rangle : \mathbb{Q}_B |\psi\rangle = 0\} \tag{5.43}$$

$$\mathrm{im}\mathbb{Q}_B := \{|\phi\rangle : \exists\, |\lambda\rangle \;\; s.t. \;\; |\phi\rangle = \mathbb{Q}_B |\lambda\rangle\}. \tag{5.44}$$

Heuristically, the cohomology measures the amount by which closed forms fail to be exact in the topological space at hand. The reason this identification is well-motivated is

[7] Rigorously, one should construct the cohomology of the *free* BRST operator first, at $g = 0$, and only later extend it to the interacting theory, see the Scholarpedia entry by Becchi.

[8] Which is an operator on the Fock space in the canonical quantisation scheme. We will not be emphasising this distinction here.

because the kernel of $\mathbb{Q}_B$ is a subspace with positive semi-definite norm, consisting (for Yang-Mills' one-particle states) of the two transverse gluons and the ghost. Among these, the offending zero-norm states, the ghost $|c\rangle$ in this case, also live in $\mathrm{im}\mathbb{Q}_B$ owing to the nilpotency of the charge. In order to further decouple these and thus restrict $\ker \mathbb{Q}_B$ to a Hilbert space, one considers elements of the kernel differing by elements of the image as physically equivalent[9], i.e.

$$|\psi\rangle \sim |\psi'\rangle = |\psi\rangle + |\phi\rangle\,, \qquad \forall\, |\psi\rangle \in \ker \mathbb{Q}_B,\ \forall\, |\phi\rangle \in \mathrm{im}\mathbb{Q}_B \tag{5.45}$$

which is exactly the description of elements of (5.42). Notice that $|\phi\rangle$, in addition to having zero norm, must be orthogonal to states $|\psi\rangle$ in the kernel. Thus, the norms are correctly preserved under the equivalence relation, $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle$. It is common to further demand that physical states have vanishing ghost number, i.e. satisfy $\mathbb{Q}_c\,|\psi\rangle = 0$. This, however, is often superfluous since states with non-vanishing ghost number tend not to be elements of the cohomology. Thus, we learn that:

"The gauge invariance of the classical theory, whereby physical states correspond to equivalence classes of the gauge group $\mathcal{G}$, is fixed and its consequences encoded in the quantum theory by BRST invariance, which implies a similar identification of the physical states with equivalence classes, albeit this time under $\mathbb{Q}_B$ and in the enlarged phase space $\mathcal{P}_b$."

5.3.2 BRST, Lagrangians and gauge-fixing

Note that, in light of this new symmetry, we can revisit our interpretation of the Yang-Mills Lagrangian in (5.34), as follows: the first classical term, $\mathcal{L}_0 = F^A_{\mu\nu}F^{\mu\nu}_A$, is Q-closed owing to its original gauge invariance; the second and third terms, albeit not gauge invariant, may be written as a Q-exact form,

$$Q\Psi_{gf} := Q\left[\bar{c}_A\left(\partial^\mu A^A_\mu - \frac{\xi}{2}b^A\right)\right] = b_A\left(\partial^\mu A^A_\mu - \frac{\xi}{2}b^A\right) - \bar{c}^A(\partial^\mu D_\mu)_{AB}c^B, \tag{5.46}$$

using (5.35) and noting that the BRST operator is a graded derivation on the space of fields, that is[10]

$$Q(ab) = (Qa)b + (-1)^{\varepsilon(a)}a(Qb). \tag{5.47}$$

The field Ψ_{gf}, carrying $\varepsilon(\Psi_{gf}) = 1$ and $gh(\Psi_{gf}) = -1$, is sometimes referred to as the *gauge-fixing fermion.* Thus, we learn that the Yang-Mills Lagrangian (5.34) may be written compactly as

$$\mathcal{L}[A, c, \bar{c}, b] = -\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A + Q\Psi_{gf} \tag{5.48}$$

[9] A more rigorous treatment involves defining an indefinite inner product on the Fock space, see [101] or Becchi's Scholarpedia entry for elegant treatments.

[10] Here, we let it act from the left. Most texts on the field/antifield formalism adopt the opposite convention, where Q acts from the right, see Appendix B.

and the proof of its Q-invariance reduces to recalling that Q is nilpotent. We say that it *lies in the same BRST cohomology class* as the classical one, $\mathcal{L}_0 \in \ker Q$. Note that adding Q-exact terms to $\mathcal{L}_0$, as above, does not change the expectation value of gauge-invariant operators. In particular, we could add a second term of the form $Q\Psi_2$, which amounts to choosing a different gauge-fixing functional (and, correspondingly, a modified ghost term), without affecting the physical sector and, thus, the result of any experiment. In this sense, the BRST formalism is manifestly independent of the choice of gauge-fixing.

In conclusion, imagine starting with the classical Lagrangian (which we know is ill-defined quantum-mechanically) and looking for possible extensions which will be consistent at the quantum level. Simply demanding that the result be invariant under the nilpotent operator Q heavily constrains the candidate terms to the form (5.48) which, when considering Lorentz invariance and mass dimension, becomes essentially unique[11] and the same as that of Faddeev-Popov (modulo quartic ghost interactions, see [111]).

5.4 Anti-BRST and OSp(2)-invariance

A close look at the Faddeev-Popov Lagrangian (5.32) appears to suggest a marked asymmetry in the roles of the ghost and antighost. The antighost, in fact, is more akin to the Lautrup-Nakanishi field: the latter is a Lagrange multiplier enforcing the gauge-fixing condition, while the former enforces the BRST-invariance of the gauge-fixing condition itself, $QF[A] = M[A]c = 0$. Indeed, $\overline{c}$ and b form a doublet representation of the BRST algebra, see [101]. Thus, perhaps, it may come as a surprise that the Faddeev-Popov Lagrangian is invariant under a second set of operations, known as the *extended* or *anti-BRST transformations*, very similar in structure to those of BRST, albeit with the roles of the ghost and antighost being, roughly speaking, interchanged. These are implemented by the fermionic operator $\overline{Q}$, carrying negative ghost number $gh(\overline{Q}) = -1$. We give them here next to their BRST counterparts, to facilitate a direct comparison:

$$\begin{aligned} QA_\mu^A &= (D_\mu c)^A & \qquad \overline{Q}A_\mu^A &= (D_\mu \overline{c})^A \\ Qc^A &= -\tfrac{1}{2}[c,c]^A & \overline{Q}\overline{c}^A &= -\tfrac{1}{2}[\overline{c},\overline{c}]^A \\ Q\overline{c}^A &= b^A & \overline{Q}c^A &= -b^A - [\overline{c},c]^A \\ Qb^A &= 0 & \overline{Q}b^A &= -\overline{Q}[\overline{c},c]^A \end{aligned} \tag{5.49}$$

Anti-BRST was discussed first in [105] and "re-discovered" in [106]. The anti-BRST operator is also off-shell[12] nilpotent and so is, in addition, the sum $Q+\overline{Q}$, which amounts to the statement that $\overline{Q}$ anti-commutes with the BRST operator. Overall, Q and $\overline{Q}$ satisfy

$$Q^2 = 0, \qquad \overline{Q}^2 = 0, \qquad \{\overline{Q}, Q\} = 0 = (Q+\overline{Q})^2. \tag{5.50}$$

[11] Up to shifts of Ψ_{gf}, of course, corresponding to different gauge choices.

[12] By "off-shell" here we mean that the Lautrup-Nakanishi multiplier has not been eliminated. As we will see, this implies that the BRST algebra closes without using the equations of motion.

Furthermore, it is possible to consider an $OSp(2)$ symmetry [107, 108, 110, 112] rotating the ghost into the antighost, and viceversa, which unifies the BRST and anti-BRST operators into a doublet, denoted Q^α, with $\alpha = 1,2$ and $Q^1 = Q$ and $Q^2 = \overline{Q}$. The properties (5.50) may then be written more compactly as

$$Q^\alpha Q^\beta + Q^\beta Q^\alpha = 0. \tag{5.51}$$

This implies a refinement of the general form of the Lagrangian (5.48), in that we now have the various possibilities of invariance under BRST, anti-BRST or both, resulting in (see [130])

$$\mathcal{L} = \mathcal{L}_0 + Q^\alpha \Psi_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta} Q^\alpha Q^\beta \chi, \tag{5.52}$$

where $\mathcal{L}_0$ is the classical piece, Ψ_α is an $OSp(2)$ doublet of gauge-fixing fermions Ψ_1 and Ψ_2 at ghost number -1 and 1, respectively, while χ is a gauge-fixing boson, at ghost number 0. Special gauge choices, whereby $\Psi_\alpha = 0$, result in theories which are simultaneously BRST and anti-BRST invariant; these are usually referred to as "$OSp(2)$ invariant" theories. Less ambitiously, if only one component of Ψ_α vanishes, the theory retains invariance under either BRST or anti-BRST. Notice that a consistent quantisation only requires that the Lagrangian be invariant under either of the two [130]. Interesting general results about anti-BRST quantisation in the field-antifield formalism are derived in [131].

5.4.1 Curci-Ferrari condition

Note that it is quite common to find a slightly different form of the anti-BRST transformations in the literature, namely

$$\begin{aligned} \overline{Q} A_\mu^A &= (D_\mu \overline{c})^A \\ \overline{Q}\overline{c}^A &= -\tfrac{1}{2}[\overline{c},\overline{c}]^A \\ \overline{Q} c^A &= \overline{b}^A \\ \overline{Q}\overline{b}^A &= 0 \end{aligned} \tag{5.53}$$

which have the obvious advantage of mimicking the BRST transformations exactly, upon exchange of $c \to \overline{c}$ and $b \to \overline{b}$, owing to having defined the new field $\overline{b}^A$ as the anti-BRST transformation of the ghost (recall, on the other hand, that the field b^A is the BRST transformation of the antighost). In order to close the whole (BRST plus anti-BRST) algebra, these variations must be supplemented with the transformations

$$Q\overline{b}^A = [\overline{b}, c]^A, \qquad \overline{Q} b^A = [b, \overline{c}]^A. \tag{5.54}$$

Furthermore, the requirement that the BRST and anti-BRST operators anticommute (or, equivalently, that the sum of the operators is nilpotent) results in the additional condition

$$\overline{b}^A = -b^A - [\overline{c}, c]^A \tag{5.55}$$

which is exactly the transformation of the ghost in (5.49). This expression is sometimes referred to as the Curci-Ferrari condition; interesting discussions concerning its geometrical origin can be found in [124] and [149].

5.4.2 Off-shell vs on-shell nilpotency

The nilpotency of Q corresponds to the closure of the algebra of infinitesimal BRST transformations,

$$\begin{aligned} [\delta_1, \delta_2]\Psi &= -\epsilon_1\epsilon_2\{Q, Q\}\Psi \\ &= -2\epsilon_1\epsilon_2 Q^2\Psi \\ &= 0, \end{aligned} \tag{5.56}$$

for $\Psi \in \mathcal{P}_b$. As mentioned above, it is the auxiliary field b which helps close the algebra. A posteriori, this is one of the main motivations for including it from the onset. Indeed, were we to remove it using its algebraic equation of motion (5.33), we would find the Lagrangian of (5.26), reported here for convenience,

$$\mathcal{L}[A, c, \overline{c}] = -\frac{1}{4}F^A_{\mu\nu}F^{\mu\nu}_A + \frac{1}{2\xi}(\partial^\mu A^A_\mu)^2 - \overline{c}^A\partial^\mu(D_\mu c)_A, \tag{5.57}$$

which is invariant under the restricted BRST and anti-BRST transformations,

$$\begin{aligned} QA^A_\mu &= (D_\mu c)^A & \overline{Q}A^A_\mu &= (D_\mu\overline{c})^A \\ Qc^A &= -\tfrac{1}{2}[c, c]^A & \overline{Q}\overline{c}^A &= -\tfrac{1}{2}[\overline{c}, \overline{c}]^A \\ Q\overline{c}^A &= \tfrac{1}{\xi}\partial^\mu A^A_\mu & \overline{Q}c^A &= -\tfrac{1}{\xi}\partial^\mu A^A_\mu - [\overline{c}, c]^A. \end{aligned} \tag{5.58}$$

Note the general feature that, in the absence of the auxiliary field, the antighost is rotated into the gauge-fixing functional, $F^A[A] = \partial^\mu A^A_\mu$ in the linear gauge chosen. This, of course, is due to the equation of motion of the Lautrup-Nakanishi field (5.33). This will still be true for the more complicated theories we will deal with later on. The BRST (anti-BRST) operator remains nilpotent on A^A_μ and c^A ($\overline{c}^A$), as their transformations are unchanged and they do not involve either $\overline{c}^A$ (c^A) or b^A; however, when applied to the antighost (ghost), it is only nilpotent modulo equations of motion. For instance,

$$\begin{aligned} Q^2\overline{c}^A &= \frac{1}{\xi}\partial^\mu(D_\mu c)^A \\ &\approx 0 \end{aligned} \tag{5.59}$$

where the symbol $\approx$ denotes weak equality (i.e. on the surface in phase space specified by the equations of motion). One says in this case that the BRST charge is only *on-shell nilpotent* or, equivalently, that the BRST algebra closes on-shell.

5.5 Kalb-Ramond 2-form

Recall the action for a p-form gauge field in D-dimensional spacetime[13],

$$S = -\frac{1}{4}\int *F^{(p+1)} \wedge F^{(p+1)}, \qquad F^{(p+1)} = \mathrm{d}A^{(p)} \tag{5.60}$$

This type of action is invariant under the gauge transformations $\delta A^p = \mathrm{d}\Lambda^{(p-1)}$, as they leave the field strength $F^{(p+1)}$ invariant. Since, however, these gauge transformations are unaffected by a shift $\Lambda^{(p-1)} \to \mathrm{d}\Lambda^{(p-2)}$, not all components of $\Lambda^{(p-1)}$ provide independent gauge symmetries of the action. Of the $\binom{D}{p}$ Lorentz components of the p-form gauge field, we subtract $\binom{D}{p-1}$ corresponding to the first gauge transformation, while we compensate for their not being independent by adding back the $\binom{D}{p-2}$ components of $\Lambda^{(p-2)}$. The same logic continues to apply at each level, such that the original p-form carries a total of $\binom{D-1}{p}$ off-shell components, that is Lorentz components minus (the independent) gauge transformations, thus furnishing the rank-p antisymmetric representation of $SO(D-1)$. To add some terminology, these are usually referred to as $(p-1)$-*stage reducible gauge theories*. Let us specialise to the case with $p = 2$. Let us denote the potential as $B_{\mu\nu}$, which we refer to as the *Kalb-Ramond 2-form*, and its field strength by $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$. Its action, this time given in components (and flat spacetime), reads

$$S = -\frac{1}{24}\int \mathrm{d}^D x\ H^{\mu\nu\rho}H_{\mu\nu\rho}, \tag{5.61}$$

invariant under the gauge transformation $\delta B_{\mu\nu} = \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu$, with $\delta\zeta_\mu = \partial_\mu\zeta$ representing the gauge transformation of the gauge parameter. In four dimensions, for example, this field carries $6-4+1=3$ degrees of freedom off-shell and just 1 on-shell, being dual to a scalar.

What we are really after, however, is the BRST-fixed version of this action. It turns out that the correct quantisation of a first-stage reducible gauge theory such as this one requires more work that the "simple" Yang-Mills case, which is the stereotypical example of a theory with an irreducible gauge algebra. It can be demonstrated that the standard Faddeev-Popov procedure (as well as the BRST "trick" of Section 5.3.2) fails to work for most gauge theories whose algebra is not closed and irreducible. These ought to be analysed in the context of a much more general formalism, whose core idea consists in elevating the BRST (and anti-BRST) symmetry to the fundamental principle underlying all gauge theories, known as *field-antifield* or *Batalin-Vilkovisky* (BV) formalism. In Appendix B we provide a very brief review of the formalism, based on the excellent [134], together with a derivation of the "BRST-fixed" Lagrangian for the Kalb-Ramond field, which we now discuss. Indeed, if the reader prefers to skip the detour to the Appendix (or to the more complete and much recommended review paper), he or she must only accept the introduction of some new ingredients which are required to fix a set of reducible gauge invariances. Namely,

[13]Note, we use a perhaps unconventional factor of 1/4, instead of the more popular 1/2.

- Two Lorentz-vector, fermionic anticommuting ghosts, d_μ and $\overline{d}_\mu$, involved in fixing the first-stage gauge redundancy, $\delta B_{\mu\nu} = 2\partial_{[\mu}\zeta_{\nu]}$. These have ghost number 1 and -1, respectively.

- Three Lorentz-scalar, bosonic *commuting* ghosts, $d, \overline{d}$ and η. These arise when fixing the second-stage gauge invariance, $\delta\zeta_\mu = \partial_\mu\zeta$. They are known as *second-generation ghosts* or *ghosts-for-ghosts* and have ghost number 2, -2 and 0 respectively.

- One Lorentz-vector bosonic commuting Lautrup-Nakanishi multiplier, $b_{(B)\mu}$, enforcing the vanishing of the gauge-fixing functional relative to $\delta B_{\mu\nu}$.

- Two Lorentz-scalar fermionic *anticommuting* Lautrup-Nakanishi multipliers, $b_{(d)}$ and $\overline{b}_{(d)}$, enforcing the vanishing of the gauge-fixing functional relative to $\delta\zeta_\mu$. They have ghost number 1 and -1.

The properties of all these are summarised in the tables of Appendix A. Then, the full Lagrangian is given by

$$\mathcal{L}_B = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2,$$

where the various pieces are

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{24} H^{\mu\nu\rho} H_{\mu\nu\rho}, \\
\mathcal{L}_1 &= b^\mu_{(B)} \left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta - \frac{\xi_{(B)}}{2} b_{(B)\mu} \right) - \overline{d}_\nu \left(\Box d^\nu - \partial^\nu \partial^\mu d_\mu \right), \\
\mathcal{L}_2 &= m_{(d)} \overline{b}_{(d)} \partial^\mu d_\mu - b_{(d)} \left(\partial^\mu \overline{d}_\mu - \xi_{(d)} \overline{b}_{(d)} \right) - m_{(d)} \overline{d} \Box d.
\end{aligned}$$

The first is simply the classical starting point. The second, $\mathcal{L}_1$, involves the gauge-fixing functional for the first-stage invariance and the first-generation ghost and antighost. Note, we choose here the standard linear covariant gauge for the Kalb-Ramond field, which necessarily ought to involve the scalar ghost η. Finally, $\mathcal{L}_2$ consists of the gauge-fixing of the second-stage invariance, as well as the kinetic term for the second-generation ghosts d and $\overline{d}$. The parameters $\xi_{(B)}$ and $\xi_{(d)}$, as in the Yang-Mills case, are the weights of the Gaussian averaging over different gauge conditions, while $m_{(d)}$ is an additional a priori arbitrary parameter which is allowed by the BRST invariance of the Lagrangian above. Indeed, the action is left invariant by the set of BRST transformations

$$\begin{aligned}
QB_{\mu\nu} &= 2\partial_{[\mu} d_{\nu]} \qquad & Q\overline{d}_\mu &= b_{(B)\mu} \qquad & Qb_{(B)\mu} &= 0 \\
Qd_\mu &= \partial_\mu d & Q\overline{d} &= \overline{b}_{(d)} & Q\overline{b}_{(d)} &= 0 \\
Qd &= 0 & Q\eta &= b_{(d)} & Qb_{(d)} &= 0
\end{aligned} \tag{5.62}$$

In particular, the fact that the first and third term in $\mathcal{L}_2$ cancel against each other allows for the presence of an arbitrary $m_{(d)}$. It turns out that only careful consideration of the anti-BRST transformations and their interplay with those in (5.62) can lead to constraining $m_{(d)}$. Indeed, requiring that the anti-BRST charge anticommutes with that of BRST *on all fields* leads to the formulation of [124]. There, the authors construct a set of anti-BRST variations with the property that $\{Q, \overline{Q}\} = 0$ by considering an additional auxiliary field and a condition analogous to the Curci-Ferrari relation given in (5.55). Eliminating the auxiliary fields from their model, one sees that

$$m_{(d)} = \xi_{(d)}. \tag{5.63}$$

The reason for highlighting this fact is the following: as we will show in Chapter 6, squaring two BRST-fixed Yang-Mills theories yields, among other things, the 2-form theory just described. The BRST and anti-BRST symmetries of Yang-Mills get mapped to those of the Kalb-Ramond field, which are necessarily anticommuting on all fields; as a consistency check, the constraint (5.63) may be seen to arise naturally in that context. Finally, another interesting feature is related to the value $m_{(d)}$ chooses to take: consider eliminating the auxiliary fields through their algebraic equations,

$$b_{(B)\mu} = \frac{1}{\xi_{(B)}} \left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right) \tag{5.64}$$

$$b_{(d)} = \frac{m_{(d)}}{\xi_{(d)}} \partial^\mu d_\mu \tag{5.65}$$

$$\overline{b}_{(d)} = \frac{1}{\xi_{(d)}} \partial^\mu \overline{d}_\mu \tag{5.66}$$

which results in the Lagrangian

$$\mathcal{L}_B = -\frac{1}{24} H^{\mu\nu\rho} H_{\mu\nu\rho} + \frac{1}{2\xi_{(B)}} \left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right)^2 - \overline{d}^\mu \Box d_\mu + \left(\frac{\xi_{(d)} - m_{(d)}}{\xi_{(d)}}\right) \overline{d}_\mu \partial^\mu \partial^\nu d_\nu + m_d \overline{d} \Box d. \tag{5.67}$$

It is invariant under the reduced BRST transformations

$$\begin{aligned} QB_{\mu\nu} &= 2\partial_{[\mu} d_{\nu]} & Qd &= 0 \\ Qd_\mu &= \partial_\mu d & Q\overline{d} &= \frac{1}{\xi_{(d)}} \partial^\mu \overline{d}_\mu \\ Q\overline{d}_\mu &= \frac{1}{\xi_{(B)}} \left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right) & Q\eta &= \frac{m_{(d)}}{\xi_{(d)}} \partial^\mu d_\mu, \end{aligned} \tag{5.68}$$

whose algebra closes only on-shell. It is easy to see, then, that the constraint (5.63) implies the vanishing of the term $\overline{d}_\mu \partial^\mu \partial^\nu d_\nu$, which simplifies the equations of motion of the first-generation ghosts to $\Box d^\alpha_\mu = 0$. The anti-BRST transformations will be discussed in Section 6.2.3.

Chapter 6

Gravitational gauge symmetries and dynamics

6.1 Classical picture

Recall the fundamental tensor product at the heart of the Yang-Mills Squared formalism, concentrating for the present discussion on the Lorentz representations, thus neglecting internal global and gauge indices. The tensor product of two defining irreducible representation of $SO(D-2)$ may be always decomposed into a 2^{nd}-rank symmetric traceless, antisymmetric and trace parts; in field theory, these represent the on-shell states of a graviton, Kalb-Ramond 2-form and dilaton. The product may be pictorially written as

$$A_\mu \circ \tilde{A}_\nu = h_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi \qquad \text{(on-shell)} \tag{6.1}$$

These collectively constitute the physical content of axio-dilaton gravity, also sometimes referred to $\mathcal{N}=0$ supergravity due to its appearance as the universal bosonic (NS-NS, for string theorists) sector of supergravity theories.

We are interested in promoting this mapping from on-shell helicity states to covariant fields. One could imagine extending the $SO(D-2)$ tensor product above to one involving the two physical as well as the single auxiliary degrees of freedom, carrying a representation of $SO(D-1)$, or all the way to a product of the Lorentz components as $SO(1,D-1)$ representations, which would include tensoring the gauge degrees of freedom as well. These, however, are not entirely satisfactory: indeed, while being sufficient for mapping the gauge symmetries of Yang-Mills into those of a gravitational theory maintaining covariance, this type of squaring fails at a consistent *covariant* map of the dynamics. This may be understood, for example, by counting the degrees of freedom generated by the $SO(1,3)$ tensor product: $A_\mu \otimes \tilde{A}_\nu$ is a 4×4 matrix with 16 components, which is not enough to describe the components of a symmetric rank-2 tensor (10), 2-form (6) and dilaton (1). Something is missing.

Let us illustrate this by considering the theory of free fields, neglecting the (self) interactions. In this limit, note that the connection "loses" the non-linear part of its local gauge transformation (as it is proportional to g), but this may be still imposed as a global invariance, namely

$$\delta A^A_\mu(x) = \partial_\mu \theta^A(x) + f_{BC}{}^A A^B_\mu \hat{\theta}^C, \tag{6.2}$$

where $\hat{\theta}$ is a spacetime-independent parameter. Note that it is possible to view the latter as the leading term in an expansion of the full gauge parameter in powers of the coupling constant, $\sigma^A(x) = g^0\hat{\theta}^A + g\theta^A(x) + \mathcal{O}(g^2)$. This is a very convenient expedient in constructing the non-linear gauge transformation from the linear theory using Noether's iterative method [150]. The transformation (6.2) informs us that, in this limit, the Yang-Mills field behaves like a collection of $\dim(\mathcal{G})$ Abelian gauge fields, rotated into each other by a *global* $\mathcal{G}$ transformation. Notice that the spectator scalar contributes the global $\mathcal{G}\times\tilde{\mathcal{G}}$ adjoint transformations

$$\delta\Phi^{-1}_{AA'} = -f^C{}_{AB}\Phi^{-1}_{CA'}\theta^B - \tilde{f}^{C'}{}_{A'B'}\Phi^{-1}_{AC'}\tilde{\theta}^{B'}, \tag{6.3}$$

where $\tilde{f}^A{}_{BC}$ are the structure constants of the Right theory's gauge group $\tilde{\mathcal{G}}$. Then, the $SO(1, D-1)$ route has two avenues:

1. We could choose to decompose the tensor product in terms of the quantities,

$$A_\mu \circ \tilde{A}_\nu = H_{(\mu\nu)} \oplus B_{[\mu\nu]} \tag{6.4}$$

where $H_{\mu\nu}$ and $B_{\mu\nu}$ have $D(D+1)/2$ and $D(D-1)/2$ independent components prior to imposing the equations of motion, respectively. When considered on its own, the decomposition (6.4) is not problematic: indeed, given the classical gauge transformation of the (linearised) Yang-Mills theory in (6.2), one can derive the correct gauge transformations of a graviton and a 2-form,

$$\delta H_{\mu\nu} = \delta\left(A_{(\mu} \circ \tilde{A}_{\nu)}\right) = 2\partial_{(\mu}\xi_{\nu)} \tag{6.5}$$

$$\delta B_{\mu\nu} = \delta\left(A_{[\mu} \circ \tilde{A}_{\nu]}\right) = 2\partial_{[\mu}\zeta_{\nu]} \tag{6.6}$$

using the definitions of Section 3.1, equation (6.3) and the parameter dictionary

$$\xi_\mu := \frac{1}{2}\left(A_\mu \circ \tilde{\theta} + \theta \circ \tilde{A}_\mu\right) \tag{6.7}$$

$$\zeta_\mu := \frac{1}{2}\left(A_\mu \circ \tilde{\theta} - \theta \circ \tilde{A}_\mu\right). \tag{6.8}$$

However, insisting that $H_{\mu\nu}$ be the off-shell graviton necessarily implies that the dilaton finds no room off-shell, hence the assignment (6.4) is manifestly at odds with (6.1). This is a signal that this type of squaring is inconsistent with a Lorentz covariant map of the dynamics, as mentioned above. Note that this disparity does not cause any headaches if one is willing to break covariance with a gauge-fixing such as Coulomb gauge on the gauge theory side and physical gauge $\partial^i h_{i\mu} = 0$

on the gravity side: the tensor product then reduces to that of the two physical, propagating degrees of freedom $A_i,\ i = 1, \ldots, D-2$, yields $A_i \otimes \tilde{A}_j = h_{ij} + B_{ij} + \phi$, where we separate the symmetric traceless, antisymmetric and trace parts. Then, for example, the Klein-Gordon equations of motion of the graviton's propagating degrees of freedom follow from those of the Yang-Mills degrees of freedom. Of course, this is nothing but a rewriting of the on-shell state squaring.

Indeed, even ignoring the absence of the dilaton, one has trouble deriving the correct dynamics for the would-be gravitational fields in a covariant fashion: given the (linearised) equations of motion of the gauge connection in the absence of sources, $\Box A_\mu = \partial_\mu \partial^\rho A_\rho$, one obtains

$$\Box H_{\mu\nu} = \partial_{(\mu} \partial^\rho H_{\nu)\rho} \tag{6.9}$$

$$\Box B_{\mu\nu} = \partial_{[\mu} \partial^\rho B_{\nu]\rho} \tag{6.10}$$

instead of the correct

$$\Box h_{\mu\nu} = 2\partial_{(\mu} \partial^\rho h_{\nu)\rho} - \partial_\mu \partial_\nu h \tag{6.11}$$

$$\Box B_{\mu\nu} = 2\partial_{[\mu} \partial^\rho B_{\nu]\rho}. \tag{6.12}$$

The only way to find agreement between the dynamics of the squaring fields, $H_{\mu\nu}$ and $B_{\mu\nu}$, with those of the correct graviton and 2-form is by fixing the gauge on the gauge theory side, choosing $\partial^\rho A_\rho$ to vanish (Lorenz gauge). In this gauge, one has that $\Box A_\mu = 0$; this implies, through the dictionary, that

$$\Box H_{\mu\nu} = 0 \tag{6.13}$$

$$\Box B_{\mu\nu} = 0, \tag{6.14}$$

which are the equations of motion of a graviton in de-Donder gauge (and a similar choice of gauge for the 2-form). Having to restrict to a specific gauge on both the Yang-Mills and the gravitational side is, however, entirely unsatisfactory. It is possible to arrive at the same conclusion following a sort of reverse argument, which nicely exemplifies the problem at hand: consider the correct equations of motion of the Kalb-Ramond 2-form given in (6.12). If one assumes, as was done here, the Yang-Mills Squared dictionary for this field to simply be the antisymmetric part of the tensor product (6.4), then it follows that

$$\begin{aligned} \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\mu\rho} &= \frac{1}{2}\left(\partial_\mu \partial^\rho A_\rho \circ \tilde{A}_\nu - A_\nu \circ \partial_\mu \partial^\rho \tilde{A}_\rho + (\mu \leftrightarrow \nu)\right) \\ &= \frac{1}{2}\left(\Box A_\mu \circ \tilde{A}_\nu - A_\nu \circ \Box \tilde{A}_\mu + (\mu \leftrightarrow \nu)\right) \\ &= 2\Box\left(A_{[\mu} \circ \tilde{A}_{\nu]}\right) \end{aligned} \tag{6.15}$$

where in the first line we substitute the dictionary for $B_{\mu\nu}$, in the second we use the unsourced Yang-Mills equations, and in order to reach the final line we pull the d'Alembertian out of the convolution using (3.9) (assuming it is acting on well-behaved fields). Then, plugging this back in (6.12), one must require $\Box B_{\mu\nu} = 0$ once again, which is true if we demand the Yang-Mills fields are in Lorenz gauge. Notice that here we have "illicitly" assumed that the derivative rule holds on both arguments of the convolution, which is not guaranteed for all fields. In line with the discussion of Section 3.1, one should, as we will shortly, introduce effective source terms for all fields, and work with sourced equations of motion. Whilst this represents an improvement compared to the naive example just provided here, it may be shown to yield a graviton-axion-dilaton system whose dynamics are not fully general [72]. Thus, we keep looking.

2. The previous discussion leaves open an obvious possibility: to define a candidate physical dilaton by artificially extracting the trace from $H_{\mu\nu}$, and explore the consequences. In this case, the dictionary would read

$$\mathfrak{h}_{\mu\nu} := A_{(\mu} \circ \tilde{A}_{\nu)} - \frac{\eta_{\mu\nu}}{D} A^\rho \circ \tilde{A}_\rho \tag{6.16}$$

$$B_{\mu\nu} := A_{[\mu} \circ \tilde{A}_{\nu]} \tag{6.17}$$

$$\varphi := A^\rho \circ \tilde{A}_\rho. \tag{6.18}$$

where $\mathfrak{h}_{\mu\nu}$ denotes the symmetric traceless product. The resulting gauge transformations, however, are less familiar than in the previous case. Leaving aside the Kalb-Ramond 2-form, as it is the same as above, one has

$$\delta\mathfrak{h}_{\mu\nu} = \partial_{(\mu}\xi_{\nu)} - \frac{2}{D}\eta_{\mu\nu}\partial^\rho\xi_\rho \tag{6.19}$$

$$\delta\varphi = 2\partial^\rho\xi_\rho, \tag{6.20}$$

that is, neither of them transforms with the linearised diffeomorphism expected for a graviton and a dilaton. This is because the combination $A^\rho \circ \tilde{A}_\rho$ does not generally describe the physical dilaton[1]. One could (mistakenly) argue that the interplay of the two gauge transformations resembles that between the graviton and a compensating scalar, of the type used in the conformal construction of Poincaré (super)gravity[2]. Indeed, the combination

$$H_{\mu\nu} := \mathfrak{h}_{\mu\nu} - \frac{\eta_{\mu\nu}}{D}\varphi, \tag{6.21}$$

which we denoted by $H_{\mu\nu}$ since it corresponds to the symmetric part of the tensor product as in (6.4), transforms as we know with the correct linear diffeomorphism, see (6.5), and as such it could represent the Weyl dilatation invariant graviton of a construction of that sort. This idea may appear to be corroborated by yet another

[1]Under certain assumptions, adopting this kind of dictionary leads to a constrained graviton-dilaton system whereby their respective sources are not independent [72].

[2]See, for instance, chapters 8 and 15 in [80].

argument: we can compute the equations of motion inherited by $\mathfrak{h}_{\mu\nu}$ and φ via the dictionary:

$$\Box\mathfrak{h}_{\mu\nu} = \partial_{(\mu}\partial^{\rho}\mathfrak{h}_{\nu)\rho} + \frac{1}{D}\left(\partial_{\mu}\partial_{\nu}\varphi - \eta_{\mu\nu}\Box\varphi\right) \tag{6.22}$$

$$\Box\varphi = \left(\frac{D}{D-1}\right)\partial^{\rho}\partial^{\sigma}\mathfrak{h}_{\rho\sigma}. \tag{6.23}$$

One can check that the tensor equation is traceless, which implies that the equations above fix one fewer degree of freedom than if a physical scalar were coupled to gravity, in full agreement with the identification of φ as a compensating scalar.

However, if $\mathfrak{h}_{\mu\nu}$ and φ truly were to be an off-shell graviton and compensator, the Weyl invariant combination $H_{\mu\nu}$ should obey the usual linearised Einstein equation (i.e. the Pauli-Fierz equation for a spin-2 field), which is contradicted by (6.9)! Thus, the failure of $SO(1, D-1)$ squaring is exemplified by (at least) the following two features:

- The impossibility to identify a combination of Yang-Mills fields with the physical dilaton field, due to the "missing" degree of freedom.
- The fact that a covariant map of the dynamics of the degrees of freedom actually generated by squaring is not possible, if not when trivialised by the Lorentz gauge.

We will see in the next section how recourse to a BRST-covariant formalism provides exactly what is needed to remedy this.

6.2 BRST squaring

As anticipated at the end of the previous section, it would appear that adopting a "BRST basis" for Yang-Mills, i.e. including at least the two ghosts (if not the auxiliary field) in addition to the gauge connection, may provide a covariant description of the squaring map.

6.2.1 Gauge theory side

Let us first consider the case in which the Lautrup-Nakanishi fields have been integrated out (we will reinstate them in Section 6.4). Furthermore, as for the classical case above, we will restrict our attention to the asymptotic (free) fields, in order to establish a baseline on which to build on. Thus, we will be concerned with the theory at vanishing coupling $g = 0$, resembling a collection of dim$\mathcal{G}$ Maxwell fields. Omitting the gauge indices, now trivial, the theory is defined by the action functional,

$$\mathcal{L}_{lin}[A, c, \bar{c}] = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2\xi}(\partial^{\mu}A_{\mu})^2 - \bar{c}\Box c, \tag{6.24}$$

where, compared to the full Lagrangian in (5.57), the Faddeev-Popov operator $\partial^\mu D_\mu$ has reduced to the d'Alembertian $\Box := \partial^\mu \partial_\mu$, whilst the field strength lost its non-Abelian piece and reads

$$f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.25}$$

The Lagrangian is invariant, besides the global $\mathcal{G}$ transformations of (6.2), under the BRST/anti-BRST transformations

$$\begin{aligned} QA_\mu &= \partial_\mu c & \qquad \overline{Q}A_\mu &= \partial_\mu \overline{c} \\ Qc &= 0 & \overline{Q}\overline{c} &= 0 \\ Q\overline{c} &= \tfrac{1}{\xi}\partial^\mu A_\mu & \overline{Q}c &= -\tfrac{1}{\xi}\partial^\mu A_\mu \end{aligned} \tag{6.26}$$

which follow from (5.58) by setting $g = 0$. Note that, in this limit, $Q\overline{c}$ and $\overline{Q}c$ become more symmetric than in the full theory, owing to the disappearance of the commutator term. Equivalently, the Curci-Ferrari condition in this limit is simply $\overline{b} = -b$. Notice that these are such that

$$\{Q, \overline{Q}\} = 0 \tag{6.27}$$

on all fields. The equations of motion following from the action (6.24) are

$$\Box A_\mu - \left(\frac{\xi+1}{\xi}\right)\partial_\mu \partial^\rho A_\rho = j_\mu \tag{6.28}$$

$$\Box c^\alpha = j^\alpha. \tag{6.29}$$

where we introduce effective source terms encoding the boundary conditions for the linearised fields, following [72]: the idea is that the linearisation of the fields holds in a region of spacetime Σ where the sources of the full non-linear theory, denoted by J, are negligible; thus, in the interior of such region, the fields obey the familiar vacuum field equations, *together with some appropriate boundary conditions on* $\partial\Sigma$. However, the latter may be re-expressed in terms of effective sources j, of the sort appearing above. Doing so will allow us to explicitly work with functions on which the derivative rule of the convolution in (3.9) applies, in line with the discussion in Section 3.1.

Notice that the analog of current conservation after BRST quantisation, then, is the Lorentz covariant expression

$$\partial^\rho j_\rho = -\frac{1}{\xi}\Box\partial^\rho A_\rho. \tag{6.30}$$

6.2.2 Counting states

We wish to perform the squaring

$$\{A_\mu, c, \overline{c}\} \otimes \{\tilde{A}_\mu, \tilde{c}, \tilde{\overline{c}}\}, \tag{6.31}$$

which generates, along with the usual ones, a proliferation of states involving the ghosts. Naturally, we ought to make sure that the totality of the degrees of freedom produced by tensoring two Yang-Mills theories is in one-to-one correspondence with the states of the gravitational theory we are mapping to, rather than an arbitrary subset thereof. This is best achieved by counting the gravitational degrees of freedom in a "graded" fashion, according as their ghost number inherited directly from the product. Indeed, for any two fields a and b, the convolutive tensor product defined in (3.8) satisfies

$$gh(a \circ \tilde{b}) = gh(a) + gh(b), \tag{6.32}$$

$$\varepsilon(a \circ \tilde{b}) = \varepsilon(a) + \varepsilon(b), \qquad (\text{mod}2). \tag{6.33}$$

Counting in $D = 4$ for concreteness, the resulting states can be divided according to ghost number and parity:

- $gh(\psi) = 0$ and $\varepsilon(\psi) = 0$

  A total of 18 bosonic dof. In addition to the obvious product of gauge potentials, $A_\mu \circ \tilde{A}_\nu$, we have the orthogonal combinations $c \circ \tilde{\bar{c}} \pm \bar{c} \circ \tilde{c}$.

- $gh(\psi) = \pm 1$ and $\varepsilon(\psi) = 1$

  A total of 16 fermionic dof. The pair of orthogonal combinations $A_\mu \circ \tilde{c} \pm c \circ \tilde{A}_\mu$ and $A_\mu \circ \tilde{\bar{c}} \pm \bar{c} \circ \tilde{A}_\mu$.

- $gh(\psi) = \pm 2$ and $\varepsilon(\psi) = 0$

  A total of 2 bosonic dof. The two products $c \circ \tilde{c}$ and $\bar{c} \circ \tilde{\bar{c}}$.

Thus, a first, naive look suggests two potential candidates for the missing degree of freedom: either one of the two Lorentz-scalar, ghost number zero, bosonic products $c \circ \tilde{\bar{c}} \pm \bar{c} \circ \tilde{c}$ could do the job.

As first proposed in [5], a good way to begin discriminating between the two is to adopt an $OSp(2)$-covariant quantisation of the two Yang-Mills theories, such that the ghost and antighost in each can be regarded as a doublet, c^α, with $c^1 = c$ and $c^2 = \bar{c}$. Then, the tensor product of the ghost sectors, $c^\alpha \otimes \tilde{c}^\beta$, is merely the $\underline{2} \otimes \underline{2}$ of $OSp(2)$. The result may be decomposed into irreducible $OSp(2)$ representations,

$$\begin{aligned} &\underline{1}: &\quad c^\alpha \circ \tilde{c}_\alpha = \varepsilon_{\alpha\beta} c^\alpha \circ \tilde{c}^\beta &= c \circ \tilde{\bar{c}} - \bar{c} \circ \tilde{c} && (0) \\ &\underline{3}: & c^{(\alpha} \circ \tilde{c}^{\beta)} &= \begin{cases} c \circ \tilde{c} & (2) \\ c \circ \tilde{\bar{c}} + \bar{c} \circ \tilde{c} & (0) \\ \bar{c} \circ \tilde{\bar{c}} & (-2) \end{cases} \end{aligned} \tag{6.34}$$

where $\varepsilon_{\alpha\beta}$ is the $OSp(2)$ invariant and round brackets denote symmetrisation. As the reader might have guessed at this point, it will be the case that the "missing" degree of freedom of the physical sector is indeed the singlet. We will show that this is the correct assignment with the aid of the algebra of BRST transformations. Furthermore, a complete understanding of its role in the field theoretic incarnation of squaring, that is beyond just on-shell state identification, requires the analysis of gravitational dynamics. Both BRST transformations and dynamics are studied in Section 6.3 below. The content of the squaring is summarised schematically in Table 6.1.

	$\tilde{A}_\nu$ (0)	$\tilde{c}^\beta$ (± 1)
A_μ (0)	$A_{(\mu} \circ \tilde{A}_{\nu)} - \frac{\eta_{\mu\nu}}{D} A_\rho \circ \tilde{A}^\rho$ $A_{[\mu} \circ \tilde{A}_{\nu]}$ $A_\rho \circ \tilde{A}^\rho$	$A_\mu \circ \tilde{c}^\beta$
c^α (± 1)	$c^\alpha \circ \tilde{A}_\nu$	$c^\alpha \circ \tilde{c}_\alpha$ $c^{(\alpha} \circ \tilde{c}^{\beta)}$

Table 6.1: The content resulting from tensoring two Yang-Mills theories in a "BRST basis". The ghost number is indicated in brackets (+1 for c^1 and -1 for c^2).

Next, we ought to identify the remaining products. Taking a cue from equation (6.1), the natural guess is to attempt to match a graviton/2-form/dilaton system, only this time furnished with the plethora of additional, unphysical fields typical of the BRST formalism.

- The BRST theory of linear gravity is a straightforward extension of the Yang-Mills case, for a gauge field carrying a spin-2 transformation, rather than spin-1. The most notable difference is that the ghost and antighost are (still Grassmann) Lorentz vectors, c_μ and $\overline{c}_\mu$, instead of scalars.

- As we have seen in Section 5.5, the theory of the Kalb-Ramond 2-form is an example of a first stage reducible Abelian gauge theory; its BRST/BV quantisation thus involves two first-generation fermionic vector ghosts, d_μ and $\overline{d}_\mu$, and three second-generation bosonic scalar ghosts, d, $\overline{d}$ and η.

- Finally, the dilaton has no classical gauge invariance (in the Einstein frame), so it does not inherit any (non-trivial) BRST transformation and does not involve any ghosts.

At this point, matching Lorentz representation, parity and ghost number (summarised in Tables A.1 - A.3), it is not difficult to realise what the further identifications (at least at the level of states) should be:

$$c^\alpha \circ \tilde{A}_\mu \pm A_\mu \circ \tilde{c}^\alpha \quad \Longleftrightarrow \quad c^\alpha_\mu,\ d^\alpha_\mu \tag{6.35}$$

$$c^{(\alpha} \circ \tilde{c}^{\beta)} \quad \Longleftrightarrow \quad d,\ \overline{d},\ \eta \tag{6.36}$$

Furthermore, *the BRST operator defined on the gauge theory side, Q, may be used to construct that on the gravity side*: having defined the action of Q on the convolutive product as that of a graded derivation acting from the left,

$$Q(a \circ \tilde{b}) = (Qa) \circ \tilde{b} + (-1)^{\varepsilon(a)} a \circ (Q\tilde{b}), \tag{6.37}$$

where the contribution from the spectator scalar is trivial and can be omitted, we can derive the BRST transformations of the bona-fide gravitational fields from those of the underlying Yang-Mills fields. Then, for example, we can impose that the 2-form ghost, d_μ, satisfies the BRST transformation given in (5.62), while simultaneously reading off a dictionary for its ghost-for-ghost, d:

$$\begin{aligned} Qd_\mu &= Q(A_\mu \circ \tilde{c} + c \circ \tilde{A}_\mu) \\ &= \partial_\mu(2c \circ \tilde{c}) \\ &:= \partial_\mu d, \end{aligned} \tag{6.38}$$

having used (6.37) and the Yang-Mills transformations (6.26). Of course, the mapping is to be considered well-defined only if all the field dictionaries as well as the BRST transformations on the gravity side collude to form a consistent theory with a closed BRST algebra (viz. nilpotent Q).

In [5], a recipe is given to directly identify the gravitational states with the irreducible representations of $OSp(D-1,1|2)$, the orthosymplectic supergroup generalising the Lorentz group in a spacetime with two anticommuting dimensions. There, the Yang-Mills field and the ghosts are unified into a single $OSp(D-1,1|2)$ vector with a graded index $i = \{\mu, \alpha\}$, so that $A_i = \{A_\mu, c_\alpha\}$. The tensor product is then decomposed into its (graded) symmetric traceless, antisymmetric and trace parts. In our notation, it would read

$$T^{t\!/\!r}_{(ij]} = \left\{ A_{(\mu} \circ \tilde{A}_{\nu)} - \frac{\eta_{\mu\nu}}{D} A^\rho \circ \tilde{A}_\rho;\ c^\alpha \circ \tilde{A}_\mu + A_\mu \circ \tilde{c}^\alpha;\ 2A^\rho \circ \tilde{A}_\rho + Dc^\alpha \circ \tilde{c}_\alpha \right\} \tag{6.39}$$

$$T_{[ij)} = \left\{ A_{[\mu} \circ \tilde{A}_{\nu]};\ c^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{c}^\alpha;\ c^{(\alpha} \circ \tilde{c}^{\beta)} \right\} \tag{6.40}$$

$$T^i{}_i = \left\{ A^\rho \circ \tilde{A}_\rho + c^\alpha \circ \tilde{c}_\alpha \right\}. \tag{6.41}$$

According to [5], one should identify these combinations with the gravitational fields

$$T^{t\!/\!r}_{(ij]} = \{\mathfrak{h}_{\mu\nu};\ c^\alpha_\mu;\ \chi\} \tag{6.42}$$

$$T_{[ij)} = \{B_{\mu\nu};\ d^\alpha_\mu;\ (d,\ \overline{d},\ \eta)\} \tag{6.43}$$

$$T^i{}_i = \{\varphi\}, \tag{6.44}$$

and regard φ and χ as two scalars conformally coupled to the metric in the full non-linear theory. The assumption of such a coupling is called for because the fields so defined neither possess the correct BRST transformations for the simple Lagrangian with diagonal kinetic terms, nor do they obey the simple equations of motion characteristic of such a Lagrangian.

For instance, consider the field φ in (6.44): it inherits the BRST transformation $Q\varphi = \frac{\xi-1}{\xi}\partial^\rho c_\rho$, instead of simply being BRST invariant, and it obeys an equation similar to (6.23), rather than a Klein-Gordon equation.

However, the expressions above cannot correspond to the dictionary of the sort we are after here, since they may be shown to be insufficient for deriving the BRST transformations and the covariant equations of motion of the gravitational theory from those of the Yang-Mills pieces. Thus, we follow a different route: since, at linear level, the various field redefinitions taking one from a conformally coupled theory to the uncoupled theory are sums of linear terms, it should be possible to obtain a dictionary which directly corresponds to the physical, free dilaton, such that $Q\varphi = 0$ and $\Box\varphi = 0$. This requires careful consideration of the dynamics on the gravity side, which we study below.

6.2.3 Gravitational side

At linear approximation, the 2-form cannot be coupled to the graviton or the dilaton in a local, Lorentz covariant fashion. We work, as on the gauge theory side, after the elimination of the Lagrange multiplier fields. The Lagrangian describing its dynamics is thus given by (5.67), obtained previously with the aid of the field-antifield formalism. On the other hand, even though the dilaton may be coupled to the graviton, through dimension-D terms such as $h_{\mu\nu}\partial^\mu\partial^\nu\varphi$ (the linearised version of the string frame), we choose to perform all calculations in the Einstein frame, where no such coupling is present and computations are marginally simpler. Thus, we define the full Lagrangian of the gravitational theory to be

$$\mathcal{L} = \mathcal{L}_h + \mathcal{L}_B + \mathcal{L}_\varphi, \tag{6.45}$$

where

$$\begin{aligned}
\mathcal{L}_h &= -\frac{1}{4}h^{\mu\nu}R^{lin}_{\mu\nu} + \frac{1}{2\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right)^2 - \bar{c}^\mu\Box c_\mu,\\
\mathcal{L}_B &= -\frac{1}{24}H^{\mu\nu\rho}H_{\mu\nu\rho} + \frac{1}{2\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu\eta\right)^2 - \bar{d}^\mu\Box d_\mu + \left(\frac{\xi_{(d)} - m_{(d)}}{\xi_{(d)}}\right)\bar{d}_\mu\partial^\mu\partial^\nu d_\nu + m_d\bar{d}\Box d,\\
\mathcal{L}_\varphi &= -\frac{1}{4}(\partial\varphi)^2,
\end{aligned}$$

where $R^{lin}_{\mu\nu}$ is the linearised Ricci tensor given in (3.2), which contributes a second minus sign so that the kinetic terms for the spatial components h_{ij} are positive as required. Furthermore, we denote by $\xi_{(\phi)}$ the parameter of the gauge-fixing averaging relative to some field ϕ; it should be understood that $\xi_{(d)}$ corresponds to the vector ghost d_μ, rather than the scalar d, as only the former possesses a gauge invariance that requires fixing. We leave manifest the seemingly superfluous parameter $m_{(d)}$, which descends to the Lagrangian (6.46) from the term $m_{(d)}\bar{b}_{(d)}\partial^\mu d_\mu$ of its formulation with Lautrup-Nakanishi fields, since it will be convenient when discussing the features of BRST and anti-BRST invariance in the 2-form sector.

Dynamics

From the Lagrangian above, we obtain the equations of motion of the physical fields,

$$\Box h_{\mu\nu} - \frac{\xi_{(h)}+2}{\xi_{(h)}}\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h\right) = j_{\mu\nu}(h) \tag{6.46}$$

$$\Box B_{\mu\nu} + 2\frac{\xi_{(B)}+2}{\xi_{(B)}}\partial^\rho\partial_{[\mu}B_{\nu]\rho} = j_{\mu\nu}(B) \tag{6.47}$$

$$\Box\varphi = j(\varphi), \tag{6.48}$$

supplemented by those of the various ghosts,

$$\Box c^\alpha_\mu = j^\alpha_\mu(c) \tag{6.49}$$

$$\Box d^\alpha_\mu + \left(1 - \frac{m_{(d)}}{\xi_{(d)}}\right)\partial_\mu\partial^\rho d^\alpha_\rho = j^\alpha_\mu(d) \tag{6.50}$$

$$\Box d^i = j(d^i) \tag{6.51}$$

where i is merely a label, denoting the three second-generation ghosts together as d^i, $i = 1, 2, 3$.

BRST and anti-BRST transformations

The Lagrangian and the equations of motion are left invariant by the corresponding sets of BRST transformations: for the graviton/dilaton sector, these are

$$\begin{aligned} Qh_{\mu\nu} &= 2\partial_{(\mu}c_{\nu)} & Q\bar{c}_\mu &= \frac{1}{\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right) \\ Qc_\mu &= 0 & Q\varphi &= 0, \end{aligned} \tag{6.52}$$

while those of the 2-form sector read

$$\begin{aligned} QB_{\mu\nu} &= 2\partial_{[\mu}d_{\nu]} & Qd &= 0 \\ Qd_\mu &= \partial_\mu d & Q\bar{d} &= \frac{1}{\xi_{(d)}}\partial^\mu\bar{d}_\mu \\ Q\bar{d}_\mu &= \frac{1}{\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu\eta\right) & Q\eta &= \frac{m_{(d)}}{\xi_{(d)}}\partial^\mu d_\mu. \end{aligned} \tag{6.53}$$

As for Yang-Mills, there exist a second set of transformations under which the Lagrangian is invariant. Their form for the graviton/dilaton sector, owing to the simplicity gained by working at linear approximation, is exactly analogous to that of Maxwell theory, or linearised Yang-Mills; namely

$$\begin{aligned} \overline{Q}h_{\mu\nu} &= 2\partial_{(\mu}\bar{c}_{\nu)} & \overline{Q}\bar{c}_\mu &= 0 \\ \overline{Q}c_\mu &= -\frac{1}{\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right) & \overline{Q}\varphi &= 0, \end{aligned} \tag{6.54}$$

where one notices the familiar switch in the roles of the ghost and the antighost: in particular, it is the former that is now mapped to the gauge-fixing functional under the action of the symmetry, while the latter is invariant. Finally, the anti-BRST transformations in the 2-form sector deserve some more care, as their form is not so immediately obvious as for Yang-Mills or linear gravity. They are given by

$$\begin{aligned} \overline{Q}B_{\mu\nu} &= 2\partial_{[\mu}\bar{d}_{\nu]} & \overline{Q}d &= -\frac{1}{\xi_{(d)}}\partial^{\mu}d_{\mu} \\ \overline{Q}d_{\mu} &= -\frac{1}{\xi_{(B)}}\left(\partial^{\nu}B_{\nu\mu}+\partial_{\mu}\eta\right)+\frac{1}{\xi_{(d)}}\partial_{\mu}\eta & \overline{Q}\bar{d} &= 0 \\ \overline{Q}\bar{d}_{\mu} &= \partial_{\mu}\bar{d} & \overline{Q}\eta &= -\partial^{\mu}\bar{d}_{\mu}. \end{aligned} \tag{6.55}$$

Notice the perhaps unexpected additional term in the variation of d_{μ}. It is related to the requirement that the BRST and anti-BRST charges anticommute on all fields, that is $\{Q,\overline{Q}\}=0$, and is understood most elegantly in terms of a Curci-Ferrari type condition for the gauge algebra of the Abelian 2-form theory [124], as discussed in Section 5.5. Although many different versions of the anti-BRST transformations for the 2-form exist in the literature, let us conclude this section with a heuristic argument as to why it is this particular form that one should expect to obtain as the "square" of Yang-Mills. Consider applying the anti-commutator on the convolution of two Yang-Mills fields, schematically denoted by a,b here; then, using (6.37), one can show that

$$\begin{aligned} \{Q,\overline{Q}\}(a\circ b) &= \{Q,\overline{Q}\}a\circ b+\left((-)^{\epsilon(a)+1}+(-)^{\epsilon(a)}\right)\left(Qa\circ\overline{Q}b+\overline{Q}a\circ Qb\right)+a\circ\{Q,\overline{Q}\}b \\ &= 0, \end{aligned} \tag{6.56}$$

where the middle term is obviously zero, while the first and last term vanish owing to the fact that the BRST and anti-BRST charges anticommute in Yang-Mills, see (6.27). This implies that, regardless of the specific form of the convolution, the result will always be such that the charges anticommute. The transformations (6.55) accomplish just this.

6.3 Dictionary, symmetries and dynamics

We wish to construct a dictionary general enough to consistently map both the BRST transformations and the dynamics of the gauge theory side to their gravitational counterparts, without resorting to an ad hoc choice of gauge on either side but rather incorporating in the correspondence a map between distinct gauge choices. As mentioned above, we work under the assumption that, at linear level, it should be possible to build a dictionary for the gravitational fields such that their dynamics and symmetries as derived from Yang-Mills be consistent with the simple Lagrangian (6.45). Then, the dictionary for the gravitational fields in a different frame, obtained from these basic ones through field redefinitions would be simple sums of existing ones.

6.3.1 Dilaton

To clarify the main points, let us begin with the dilaton since, while being the simplest possible example, it already exhibits all the features we wish to discuss. It needs to be closed with respect to both the BRST and anti-BRST operators and satisfy equation (6.48), while also keeping the gauge-fixing parameter ξ of the Yang-Mills factors general (i.e. this should work manifestly in any linear covariant gauge, and implicitly in all gauges). The only Yang-Mills products which can contribute to it are the scalars $A^\rho \circ \tilde{A}_\rho$ and $c^\alpha \circ \tilde{c}_\alpha$, however their most general combination $\varphi = A^\rho \circ \tilde{A}_\rho + \alpha\, c^\alpha \circ \tilde{c}_\alpha$, for arbitrary α, will not suffice: while it could be made BRST closed by choosing $\alpha = \xi$, applying the d'Alembertian reproduces (6.48) only for a specific value of ξ. Instead, consider the most general ansatz compatible with the required tensor structure, mass dimension and ghost number[3],

$$\varphi = A^\rho \circ \tilde{A}_\rho + \alpha_1\, c^\alpha \circ \tilde{c}_\alpha + \frac{\alpha_2}{\Box} \partial A \circ \partial \tilde{A}, \tag{6.57}$$

which makes use of the Green operator, $\Box^{-1}$, defined in (3.10), in order to allow for additional terms with higher mass dimension. In this case, the *unique* possibility is the (dimension 3) term $\partial A \circ \partial \tilde{A}$, where we use the shorthand notation for the Lorentz invariant divergence $\partial A = \partial^\rho A_\rho$. Note that this is exactly the same term that was introduced in [72] at the classical (that is, without BRST) level. The expansion in powers $\Box^{-n}$ stops naturally at $n = 1$ since no non-trivial Lorentz invariant combination of Yang-Mills fields and derivatives exists with the required properties. By non-trivial, we mean that it does not reduce to an existing term by the identity $\Box\Box^{-1} = \Box^{-1}\Box = 1$ introduced earlier in (3.11). For example, at $n = 2$, a trivial term would be

$$\frac{1}{\Box^2} \partial^\rho \partial A \circ \partial_\rho \partial \tilde{A} = \frac{\Box}{\Box^2} \partial A \circ \partial \tilde{A} = \frac{1}{\Box} \partial A \circ \partial \tilde{A}, \tag{6.58}$$

[3]Note that, without loss of generality, we set one of the parameters of the field dictionaries equal to 1, since what matters is their relative values. This corresponds to a global normalisation of the dictionary. Notice this will not be true for source dictionaries, which cannot be normalised independently from the respective fields.

where the first equality follows by moving the derivatives outside the convolution integral, using the non-Leibniz property of the convolution (3.9). Let us check whether, given the dictionary (6.57), we now have enough freedom to satisfy all our demands above.

Symmetries

Indeed, after having computed the following,

$$Q(A^\rho \circ \tilde{A}_\rho) = \left(c \circ \partial\tilde{A} + \partial A \circ \tilde{c}\right), \tag{6.59}$$

$$\begin{aligned} Q(c^\alpha \circ \tilde{c}_\alpha) &= \left(-c \circ Q\tilde{\bar{c}} - Q\bar{c} \circ \tilde{c}\right) \\ &= \xi^{-1}\left(c \circ \partial\tilde{A} + \partial A \circ \tilde{c}\right), \end{aligned} \tag{6.60}$$

$$\begin{aligned} Q(\Box^{-1}\partial A \circ \partial\tilde{A}) &= \Box^{-1}\left(\Box c \circ \partial\tilde{A} + \partial A \circ \Box\tilde{c}\right) \\ &= \left(c \circ \partial\tilde{A} + \partial A \circ \tilde{c}\right), \end{aligned} \tag{6.61}$$

we see that demanding BRST invariance implies

$$\begin{aligned} 0 = Q\varphi &= Q\left(A^\rho \circ \tilde{A}_\rho + \alpha_1\ c^\alpha \circ \tilde{c}_\alpha + \frac{\alpha_2}{\Box}\partial A \circ \partial\tilde{A}\right) \\ &= \left(1 - \frac{\alpha_1}{\xi} + \alpha_2\right)\left(c \circ \partial\tilde{A} + \partial A \circ \tilde{c}\right). \end{aligned} \tag{6.62}$$

When facing an equation of this sort, we choose to require that the polynomial of parameters $p(\alpha_i, \xi)$ be made to vanish, rather than to constrain the (convolutions of) fields themselves, if this can be avoided. In fact, choosing the latter option quickly leads one to nonsense. If it were forced upon us to constrain the fields, that would signal a breakdown of the model. Luckily, we will see below that this is not the case. Thus, we can impose BRST invariance of the dilaton by the single constraint

$$\alpha_2 = \frac{\alpha_1 - \xi}{\xi}. \tag{6.63}$$

Note that, in the simple case of the dilaton, demanding anti-BRST invariance yields the same constraint (for the different combination $\bar{c} \circ \partial\tilde{A} + \partial A \circ \tilde{\bar{c}}$). In other words, the requirement (6.63) of BRST invariance automatically implies anti-BRST invariance, $\overline{Q}\varphi = 0$.

Equations of motion

Notice that requiring the field to satisfy the BRST/antiBRST transformations has not fixed all the arbitrary parameters of its dictionary, a fact which will be generally true for all fields on the gravity side. The hope is that the remaining free parameters are fixed uniquely by demanding that it also satisfies the correct equations of motion, $\Box\varphi = j(\varphi)$. To this end, we ought to define a dictionary for the source of the dilaton as well. Again based on Lorentz invariance, mass dimension and ghost number, and under the assumption that it be composed exclusively out of Yang-Mills sources, the most general ansatz for the dilaton source would be

$$j(\varphi) = \frac{\hat{\alpha}_0}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{\hat{\alpha}_1}{\Box} j^\alpha \circ \tilde{j}_\alpha + \frac{\hat{\alpha}_2}{\Box^2} \partial j \circ \partial \tilde{j}. \tag{6.64}$$

where we denote the parameters of source dictionaries with hats. However, recall from equation (6.30) that $\partial j = -\xi^{-1} \Box \partial A$. Thus, the last term in (6.64) is not independent from the last term in (6.57), and will only contribute to a redefinition[4] of the numerical parameter α_2. The dictionary for the dilaton source is then given just by

$$j(\varphi) = \frac{\hat{\alpha}_0}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{\hat{\alpha}_1}{\Box} j^\alpha \circ \tilde{j}_\alpha. \tag{6.65}$$

Then, we ought to solve the following equation:

$$\begin{aligned} 0 &= \Box \varphi - j(\varphi) \\ &= \Box \left(A^\rho \circ \tilde{A}_\rho + \alpha_1 \, c^\alpha \circ \tilde{c}_\alpha + \frac{\alpha_2}{\Box} \partial A \circ \partial \tilde{A} \right) - \left(\frac{\hat{\alpha}_0}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{\hat{\alpha}_1}{\Box} j^\alpha \circ \tilde{j}_\alpha \right). \end{aligned} \tag{6.66}$$

At this stage, it is crucial to realise that various a priori independent "squared" tensor structures, e.g. $A^\rho \circ \tilde{A}_\rho$ and $\Box^{-1} \partial A \circ \partial \tilde{A}$, may no longer be independent under the action of the equations of motion *and* the non-Leibniz property of the convolution: for example, these two terms, using the linear Yang-Mills equation (6.28), can be shown to mix as

$$\begin{aligned} \Box(A^\rho \circ \tilde{A}_\rho) &= \Box A^\rho \circ \tilde{A}_\rho \\ &= \left(\frac{\xi + 1}{\xi} \right) \partial A \circ \partial \tilde{A} + j^\rho \circ \tilde{A}_\rho \\ &= \left(1 - \frac{1}{\xi^2} \right) \partial A \circ \partial \tilde{A} + \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho, \end{aligned} \tag{6.67}$$

$$\Box(\Box^{-1} \partial A \circ \partial \tilde{A}) = \partial A \circ \partial \tilde{A}, \tag{6.68}$$

where we may act with the d'Alembert operator on whichever side of the convolution. Of course, the final result should not depend on this choice at all, and indeed it is easy to check that it does not. Furthermore, notice how (6.67) establishes that $\Box(A^\rho \circ \tilde{A}_\rho)$ is in fact also related to the $\Box^{-1} j^\rho \circ j_\rho$ structure, where to show this we have modified the $j^\rho \circ \tilde{A}_\rho$ term by inserting a "1" as

$$\begin{aligned} j^\rho \circ \tilde{A}_\rho &= \frac{1}{\Box} j^\rho \circ \Box \tilde{A}_\rho \\ &= \left(\frac{\xi + 1}{\xi} \right) \frac{1}{\Box} \partial j \circ \partial \tilde{A} + \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho \\ &= - \left(\frac{\xi + 1}{\xi^2} \right) \partial A \circ \partial \tilde{A} + \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho, \end{aligned} \tag{6.69}$$

[4] An alternative way to show that the $\hat{\alpha}_2$ term is redundant is by demanding that the sources fall into a BRST multiplet. The last term then drops out, as it cannot be supported.

where the last line follows by using (6.30). Using the above, equation (6.66) can be recast in the form

$$0 = \left(1 - \frac{1}{\xi^2} + \alpha_2\right) \partial A \circ \partial \tilde{A} + \frac{(1-\hat{\alpha}_0)}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{(\alpha_1 - \hat{\alpha}_1)}{\Box} j^\alpha \circ \tilde{j}_\alpha. \tag{6.70}$$

These, together with the constraint (6.63), imply that all the parameters in the dictionary of the dilaton and its source are fixed uniquely in terms of the Yang-Mills ξ:

$$\begin{aligned} \alpha_1 &= \frac{1}{\xi} & \hat{\alpha}_0 &= 1 \\ \alpha_2 &= \frac{1}{\xi^2} - 1 & \hat{\alpha}_1 &= \alpha_1. \end{aligned} \tag{6.71}$$

Minimal bases

Despite the nice result, one should wonder about its uniqueness: given the inter-dependence of the tensor structures exemplified by (6.69), it is clear that this computation will in general depend on which "basis"[5] may have been chosen, and it is a priori far from obvious that the result (6.71) carries with it any real meaning, or that it was legitimate for us to set each coefficient to zero separately in (6.70). The crucial observation here is that the basis used above,

$$B_1 = \left\{ \partial A \circ \partial \tilde{A},\ \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho,\ \frac{1}{\Box} j^\alpha \circ \tilde{j}_\alpha \right\} \tag{6.72}$$

constitutes a *minimal basis*, in the sense that no further use of the equations of motion can decrease its size. Indeed, it is obvious from (6.69) that attempting to trade either one of the first two "basis elements" still results in a 3-dimensional basis, while the last element cannot be rotated into any of the others by the equations of motion, since at linear level the ghosts are free and, in particular, their dynamics are completely independent of the gauge field. Thus, for instance, using (6.69) to solve for $\partial A \circ \partial \tilde{A}$, one may swap (6.72) for the new minimal basis,

$$B_2 = \left\{ j^\rho \circ \tilde{A}_\rho,\ \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho,\ \frac{1}{\Box} j^\alpha \circ \tilde{j}_\alpha \right\}. \tag{6.73}$$

Notice that, in general, *the existence and size of the minimal basis may be proved by trading all effective source terms for the expression in terms of fields and derivatives which they correspond to via the equations of motion. The totality of the resulting independent tensor structures then correspond to the minimal basis, as no action of the Yang-Mills equations may relate them. The coefficients of each of these may then be required to vanish independently.* However, in practice, it is often more useful to use different minimal bases, depending on the specific case at hand.

[5] We use inverted commas here because of our somewhat loose usage of the word.

As it is easy to imagine, there are a multitude of such "changes of basis" in the space defined by the convolution. For instance, starting with the basis B_1 and using the alternative identity

$$\partial A \circ \partial \tilde{A} = \frac{\xi^2}{\xi^2 - 1} \left(\Box(A^\rho \circ \tilde{A}_\rho) - \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho \right) \tag{6.74}$$

we obtain yet another 3-dimensional basis,

$$B_3 = \left\{ \Box(A^\rho \circ \tilde{A}_\rho),\ \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho,\ \frac{1}{\Box} j^\alpha \circ \tilde{j}_\alpha \right\}. \tag{6.75}$$

It can be checked that, while the details of the calculation differ, the result (6.71) is independent of which specific basis is chosen, as long as this is minimal.

We postulate that a minimal basis should be employed when solving equations such as (6.70). Although a more formal proof of why this should be true, such as the definition of orthogonality with respect to some inner product structure, is lacking, it seems sensible to assume that using a bigger basis introduces an additional, unnecessary redundancy. For example, starting once again from B_1, making the valid substitution

$$\partial A \circ \partial \tilde{A} = \frac{\xi}{\xi + 1} \left(\Box(A^\rho \circ \tilde{A}_\rho) - j^\rho \circ \tilde{A}_\rho \right) \tag{6.76}$$

leads to the 4-dimensional set

$$\left\{ \Box(A^\rho \circ \tilde{A}_\rho),\ j^\rho \circ \tilde{A}_\rho,\ \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho,\ \frac{1}{\Box} j^\alpha \circ \tilde{j}_\alpha \right\} \qquad \text{(bad!)} \tag{6.77}$$

which indeed yields different, often self-contradictory results for the parameters a_i. In using this set as a basis, one would be failing to notice that one among the four elements can be solved in terms of (a subset of) the others. Thus, for any minimal basis, the BRST/anti-BRST invariant dilaton and its source, satisfying $\Box\varphi = j(\varphi)$, are given in terms of Yang-Mills quantities as

$$\varphi = A^\rho \circ \tilde{A}_\rho + \frac{1}{\xi} c^\alpha \circ \tilde{c}_\alpha + \left(\frac{1}{\xi^2} - 1 \right) \frac{1}{\Box} \partial A \circ \partial \tilde{A}, \tag{6.78}$$

$$j(\varphi) = \frac{1}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{1}{\xi} \frac{1}{\Box} j^\alpha \circ \tilde{j}_\alpha. \tag{6.79}$$

6.3.2 Kalb-Ramond 2-form

Repeating this exercise for the 2-form sector is straightforward, if only more tedious owing to a larger BRST/anti-BRST multiplet and the tensorial nature of the equations of motion. Although most of the lengthy computations are omitted in the main text (the interested reader is referred to Appendix C), we would like to present it in some detail, given the very straightforward nature of the manipulations, so as to follow the logic.

The strategy remains the same: we write down the most general ansätze for the fields, this time including the 2-form field and its first and second generation ghosts, purely based on tensor structure, mass dimension and ghost number:

$$B_{\mu\nu} = A_{[\mu} \circ \tilde{A}_{\nu]} + \frac{\alpha_1}{\Box} \left(\partial A \circ \partial_{[\mu} \tilde{A}_{\nu]} - \partial_{[\mu} A_{\nu]} \circ \partial \tilde{A} \right) \tag{6.80}$$

$$d^\alpha_\mu = \beta_1 \left(c^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{c}^\alpha \right) + \beta_2 \frac{\partial_\mu}{\Box} \left(c^\alpha \circ \partial \tilde{A} - \partial A \circ \tilde{c}^\alpha \right) \tag{6.81}$$

$$d = \gamma_1 \ (c \circ \tilde{c}) \tag{6.82}$$

$$\overline{d} = \gamma_2 \ (\overline{c} \circ \tilde{\overline{c}}) \tag{6.83}$$

$$\eta = \gamma_3 \ (c \circ \tilde{\overline{c}} - \overline{c} \circ \tilde{c}) \tag{6.84}$$

Notice that we have streamlined the notation by writing the first generation ghost and antighost as a doublet, d^α_μ: their dictionaries need not have distinct parameters as these would end up being identified in any case by virtue of the BRST and anti-BRST transformations.

Symmetries

As for the dilaton, if the map is to be consistent, there must be a choice of parameters (α, β, γ) such that acting with Q^α on the (convolutions of) Yang-Mills fields relates the gravitational fields in the expected way. For example, to illustrate the point, consider the case of the 2-form itself: applying the doublet of BRST operators on the dictionary (6.80) yields

$$\begin{aligned} Q^\alpha B_{\mu\nu} &= \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right) \\ &\quad + \frac{\alpha_1}{\Box} \left(\Box c^\alpha \circ \partial_{[\mu} \tilde{A}_{\nu]} + \partial A \circ \partial_{[\mu} \partial_{\nu]} \tilde{c}^\alpha - \partial_{[\mu} \partial_{\nu]} c^\alpha \circ \partial \tilde{A} - \partial_{[\mu} A_{\nu]} \circ \Box \tilde{c}^\alpha \right) \\ &= (1 + \alpha_1) \, \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right), \end{aligned} \tag{6.85}$$

which may be brought to the more suggestive form $Q^\alpha = 2\partial_{[\mu} d^\alpha_{\nu]}$ by identifying the ghost doublet with the combination

$$d^\alpha_\mu := \frac{1 + \alpha_1}{2} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right) + \beta_2 \partial_\mu K \tag{6.86}$$

where the last term arises because the ghosts d^α_μ should be read off equation (6.85) only up to a gradient term due to the antisymmetry. Comparison with our ansätz (6.81), then, suggests that one should identify $2\beta_1 = 1 + \alpha_1$ and should clarify the choice of notation for the arbitrary parameter extracted from K above. One may continue like so, acting with Q and reading off the rest of the multiplet, to find that this procedure does indeed produce the correct results throughout. An equivalent yet faster way is to directly compare the various ansätze with the help of the BRST variations (5.68): the above becomes

$$\begin{aligned} 0 &= QB_{\mu\nu} - 2\partial_{[\mu} d_{\nu]} \\ &= (1 + \alpha_1 - 2\beta_1) \, \partial_{[\mu} \left(c \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c} \right) \end{aligned} \tag{6.87}$$

which fixes β_1 as expected, while β_2 remains free, corresponding to the fact that the second term drops out because of antisymmetry. It is easy to see that imposing $\overline{Q}B_{\mu\nu} = 2\partial_{[\mu}\bar{d}_{\nu]}$ has the same implications for $\beta_{1,2}$. Then, the ghost must satisfy

$$\begin{aligned} 0 &= Qd_\mu - \partial_\mu d \\ &= -\left(2(\beta_1+\beta_2)+\gamma_1\right)\partial_\mu(c\circ\tilde{c}) \end{aligned} \tag{6.88}$$

which fixes γ_1. The dictionary of the ghost-for-ghost, d, satisfies $Qd = 0$ identically, owing to the Yang-Mills transformation $Qc = 0$. Next in line is the slightly less trivial case of the vector antighost; we have that

$$\begin{aligned} 0 &= Q\bar{d}_\mu - \frac{1}{\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right) \\ &= \beta_1\left(\frac{1}{\xi}-\frac{1}{\xi_{(B)}}\right)\left(\partial A\circ\tilde{A}_\mu - A_\mu\circ\partial\tilde{A}\right) - \left(\beta_1+\beta_2-\frac{\gamma_3}{\xi_{(B)}}\right)\partial_\mu\left(c\circ\tilde{\bar{c}}+\bar{c}\circ\tilde{c}\right) \end{aligned} \tag{6.89}$$

which fixes γ_3 and, in particular, is enough to fix the gravitational parameter $\xi_{(B)}$ uniquely in terms of that of Yang-Mills, ξ, with the extremely simple relation

$$\xi_{(B)} = \xi. \tag{6.90}$$

Thus, the consistency of the mapping (read closure of the BRST algebra) induces a map between gauge choices on the gauge and gravity sides, e.g. Feynman gauge into Feynman gauge. As far as BRST transformations are concerned, there remain to check two of the second generation ghosts:

$$0 = Q\bar{d} - \frac{1}{\xi_{(d)}}\partial^\mu\bar{d}_\mu = \left(\frac{\gamma_2}{\xi}+\frac{\beta_1+\beta_2}{\xi_{(d)}}\right)\left(\partial A\circ\tilde{\bar{c}} - \bar{c}\circ\partial\tilde{A}\right) \tag{6.91}$$

$$0 = Q\eta - \frac{m_{(d)}}{\xi_{(d)}}\partial^\mu d_\mu = \left(\frac{\gamma_3}{\xi}+\frac{m_{(d)}}{\xi_{(d)}}(\beta_1+\beta_2)\right)\left(\partial A\circ\tilde{c} - c\circ\partial\tilde{A}\right). \tag{6.92}$$

While the first fixes γ_2, the second, in conjuction with (6.89), implies an interesting relation: it demands that we set

$$m_{(d)} = \xi_{(d)}, \tag{6.93}$$

which we know from the discussion in Section 5.5 reduces the equations of motion of the vector ghost and antighost to

$$\Box d^\alpha_\mu = j^\alpha_\mu. \tag{6.94}$$

So far, we have mostly required the field dictionaries to satisfy some BRST transformations: with the exception of $B_{\mu\nu}$, we have not demanded anti-BRST yet. It is easy to check that the whole set of anti-BRST transformations yields just one new constraint on the parameters, namely

$$\xi_{(d)} = \frac{\xi}{2}. \tag{6.95}$$

Thus, after (6.90), also the weight controlling the gauge-fixing of the ghost d_μ enjoys a very simple relation with its Yang-Mills counterpart, ξ. This exhausts the (extended) BRST transformations which, by construction, are also nilpotent on-shell.

Equations of motion

Similarly to the case of the dilaton, requiring that the dictionaries satisfy both sets of BRST transformations is not enough to constrain all parameters. Luckily, we still have to impose that they satisfy the prescribed equations of motion. In order to study how the dynamics of the gauge theory give rise, through the dictionaries, to those in the gravity theory, we write down the ansätze for the gravitational source terms, up to "current conservation",

$$j_{\mu\nu}(B) = \frac{\hat{\alpha}}{\Box} j_{[\mu} \circ \tilde{j}_{\nu]} \tag{6.96}$$

$$j^{\alpha}_{\mu}(d) = \frac{\hat{\beta}}{\Box} \left(j^{\alpha} \circ \tilde{j}_{\mu} - j_{\mu} \circ \tilde{j}^{\alpha} \right) \tag{6.97}$$

$$j(d) = \frac{\hat{\gamma}_1}{\Box} j(c) \circ \tilde{j}(c) \tag{6.98}$$

$$j(\bar{d}) = \frac{\hat{\gamma}_2}{\Box} j(\bar{c}) \circ \tilde{j}(\bar{c}) \tag{6.99}$$

$$j(\eta) = \frac{\hat{\gamma}_3}{\Box} \left(j(c) \circ \tilde{j}(\bar{c}) - j(\bar{c}) \circ \tilde{j}(c) \right) \tag{6.100}$$

where, this time, we must remain agnostic and not normalise the dictionaries so that one parameter is equal to unity, as this might be inconsistent with the normalisation of the respective field, through the equations of motion. The equations to be solved this time are

$$\Box B_{\mu\nu} + \xi'_{(B)} \partial^{\rho} \partial_{[\mu} B_{\nu]\rho} - j_{\mu\nu}(B) = 0, \qquad \xi'_{(B)} := 2\frac{\xi_{(B)} + 2}{\xi_{(B)}} \tag{6.101}$$

$$\Box d^{\alpha}_{\mu} - j^{\alpha}_{\mu}(d) = 0 \tag{6.102}$$

$$\Box d^{i} - j(d^{i}) = 0 \tag{6.103}$$

where we define a new parameter $\xi'_{(B)}$ purely for notational clarity in what follows. The first of these equations fixes $\hat{\alpha}$ and α_1. Indeed, direct application of the differential operators $\Box$ and $\partial^{\rho}\partial_{\mu}$ yields

$$\begin{aligned} 0 &= \Box B_{\mu\nu} + \xi'_{(B)} \partial^{\rho} \partial_{[\mu} B_{\nu]\rho} - j_{\mu\nu}(B) \\ &= \Box \left(A_{[\mu} \circ \tilde{A}_{\nu]} \right) + \frac{1}{4} \left(\xi'_{(B)}(1 + \alpha_1) - 2\alpha_1 \right) \left(F_{\mu\nu} \circ \partial \tilde{A} - \partial A \circ \tilde{F}_{\mu\nu} \right) - \frac{\hat{\alpha}}{\Box} j_{[\mu} \circ \tilde{j}_{\nu]}, \end{aligned} \tag{6.104}$$

however, as before, we should pay due care to the size of the basis: if the expression above were to represent a minimal basis, we would be in trouble, as we would be forced by the presence of the first term to restrict to field configurations satisfying $\Box(A_{[\mu} \circ \tilde{A}_{\nu]}) = 0$. Luckily, this is not the case: as mentioned above, one possible way to show that the above may be reduced is to substitute the (source)×(source) term in favour of (field)×(field) type of terms using the gauge field equations; in this case,

$$\begin{aligned}\frac{1}{\Box} j_{[\mu} \circ \tilde{j}_{\nu]} &= \frac{1}{\Box}\left(\Box A_{[\mu} - \frac{\xi+1}{\xi}\partial_{[\mu}\partial A\right) \circ \left(\Box \tilde{A}_{\nu]} - \frac{\xi+1}{\xi}\partial_{\nu]}\partial\tilde{A}\right) \\ &= \Box(A_{[\mu} \circ \tilde{A}_{\nu]}) + \frac{\xi+1}{\xi}\left(F_{\mu\nu} \circ \partial\tilde{A} - \partial A \circ \tilde{F}_{\mu\nu}\right).\end{aligned} \tag{6.105}$$

Thus, in the minimal basis $\{\Box(A_{[\mu}A_{\nu]}),\ F_{\mu\nu}\partial A - \partial A F_{\mu\nu}\}$, equation (6.104) becomes

$$(1-\hat{\alpha})\,\Box\left(A_{[\mu} \circ \tilde{A}_{\nu]}\right) + \frac{1}{4}\left(\xi'_{(B)}(1+\alpha_1) - 2\left(\alpha_1 + \frac{\xi+1}{\xi}\hat{\alpha}\right)\right)\left(F_{\mu\nu} \circ \partial\tilde{A} - \partial A \circ \tilde{F}_{\mu\nu}\right) = 0. \tag{6.106}$$

Notice that, in this (or any equivalent) minimal basis, we are no longer forced to constrain the fields; instead, we read off the constraints on $\hat{\alpha}$ and α_1. Furthermore, the equations of the ghosts (6.102) read, in terms of Yang-Mills products,

$$\begin{aligned}0 &= \Box d^\alpha_\mu - j^\alpha_\mu(d) \\ &= \left(\hat{\beta} - \beta_1\right)\left(j^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{j}^\alpha\right) + \left(\beta_2 + \frac{\xi+1}{\xi}\hat{\beta}\right)\partial_\mu\left(c^\alpha \circ \partial\tilde{A} - \partial A \circ \tilde{c}^\alpha\right).\end{aligned} \tag{6.107}$$

The 2-dimensional basis displayed above is already minimal, so we can read off the constraints on $\hat{\beta}$ and β_2. Finally, equations (6.103) merely fix the remaining parameters in $j(d^i)$. Thus, in summary, the choices of gauge-fixing on both sides are related through

$$\xi_{(B)} = 2\xi_{(d)} = \xi. \tag{6.108}$$

and all dictionary parameters are uniquely fixed in terms of ξ, as shown in Table 6.2.

Field	Parameters	Source	Parameters
$B_{\mu\nu}$	$\alpha_1 = -\frac{1}{2}$	$j_{\mu\nu}(B)$	$\hat{\alpha} = 1$
d^α_μ	$\beta_1 = \frac{1}{4}, \quad \beta_2 = -\frac{\xi+1}{4\xi}$	$j^\alpha_\mu(d)$	$\hat{\beta} = \frac{1}{4}$
$d, \overline{d}, \eta$	$\gamma_1 = \gamma_2 = \frac{1}{2\xi}, \quad \gamma_3 = \frac{1}{4}$	$j(d), j(\overline{d}), j(\eta)$	$\hat{\gamma}_i = \gamma_i$

Table 6.2: Parameters of the dictionaries of the Kalb-Ramond sector, completely fixed in terms of ξ.

6.3.3 Graviton

Last, but far from least, gravity. The general ansätze for the graviton and its ghosts,

$$\begin{aligned} h_{\mu\nu} = A_{(\mu} \circ \tilde{A}_{\nu)} + \frac{\partial_\mu \partial_\nu}{\Box}\left(a_1 A^\rho \circ \tilde{A}_\rho + a_2 c^\alpha \circ \tilde{c}_\alpha\right) + \frac{a_3}{\Box}\left(\partial A \circ \partial_{(\mu}\tilde{A}_{\nu)} + \partial_{(\mu}A_{\nu)} \circ \partial\tilde{A}\right) \\ + \frac{a_4}{\Box^2}\partial_\mu\partial_\nu \partial A \circ \partial\tilde{A} + \eta_{\mu\nu}\left(b_1 A^\rho \circ \tilde{A}_\rho + b_2 c^\alpha \circ \tilde{c}_\alpha + \frac{b_3}{\Box}\partial A \circ \partial\tilde{A}\right) \end{aligned} \tag{6.109}$$

$$c^\alpha_\mu = \alpha_1\left(c^\alpha \circ \tilde{A}_\mu + A_\mu \circ \tilde{c}^\alpha\right) + \alpha_2 \frac{\partial_\mu}{\Box}\left(c^\alpha \circ \partial\tilde{A} + \partial A \circ \tilde{c}^\alpha\right) \tag{6.110}$$

where this time we denoted a_i and b_i the parameters for the graviton (with the latter reserved for those terms proportional to $\eta_{\mu\nu}$, which play a slightly separate role) and α_i for the ghost and antighost. Notice the appearance, for the first time, of a $\Box^{-2}$ term, the *unique* instance (in the whole graviton/2-form/dilaton theory) where this is allowed in a field dictionary[6] by tensor structure and mass dimension considerations.

Symmetries

As before, we begin by imposing both BRST transformations of the physical field,

$$\begin{aligned} 0 &= Q^\alpha h_{\mu\nu} - 2\partial_{(\mu}c^\alpha_{\nu)} \\ &= (1 + a_3 - 2\alpha_1)\,\partial_{(\mu}\left(c^\alpha \circ \tilde{A}_{\nu)} + A_{\nu)} \circ \tilde{c}^\alpha\right) \\ &\quad + \left(a_1 - \frac{a_2}{\xi} + a_3 + a_4 - 2\alpha_2\right)\frac{\partial_\mu\partial_\nu}{\Box}\left(c^\alpha \circ \partial\tilde{A} + \partial A \circ \tilde{c}^\alpha\right) \end{aligned} \tag{6.111}$$

which fixes α_1 and α_2. Notice that none of the parameters b_i enter (6.111); as mentioned above, this is due to the $\eta_{\mu\nu}$ terms in the graviton behaving differently: since no Q^α-transformation can bring them to the required form $\partial_{(\mu}Y^\alpha_{\nu)}$ for some[7] $Y^\alpha_\mu \subset c^\alpha_\mu$, their BRST transformation ought to cancel among each other, yielding one constraint on the parameters. We proceed by checking that

$$\begin{aligned} Qc_\mu &= \alpha_1\left(-c \circ \partial_\mu\tilde{c} + \partial_\mu c \circ \tilde{c}\right) + \alpha_2\frac{\partial_\mu}{\Box}\left(-c \circ \Box\tilde{A} + \Box A \circ \tilde{c}\right) \\ &= \alpha_1\partial_\mu\left(-c \circ \tilde{c} + c \circ \tilde{c}\right) + \alpha_2\partial_\mu\left(-c \circ \tilde{A} + A \circ \tilde{c}\right) \\ &= 0, \end{aligned} \tag{6.112}$$

as required. Since it is satisfied identically, with the tensor structures cancelling owing to the minus signs coming from the anticommutativity $\{Q, c\} = 0$, this imposes no further constraints on the parameters. Next in line is the more complicated BRST transformation of the antighost, $Q\bar{c}_\mu$. In order for the BRST algebra to close (albeit only on-shell since we are working without the Lautrup-Nakanishi field, $b_{(h)\mu}$), it ought to satisfy

$$Q\bar{c}_\mu = \frac{1}{\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right). \tag{6.113}$$

[6] As we will see, the only other allowed $\Box^{-2}$ term is in the dictionary of the graviton source.

[7] Here we mean that Y^α_μ contains a subset of Yang-Mills Squared products relevant to c^α_μ.

The LHS, computed by acting with Q on the $\alpha = 2$ component of the dictionary (6.110), yields

$$Q\bar{c}_\mu = (\alpha_1 + \alpha_2)\,\partial_\mu\,(c^\alpha \circ \tilde{c}_\alpha) + \frac{\alpha_1}{\xi}\left(A_\mu \circ \partial\tilde{A} + \partial A \circ \tilde{A}_\mu\right) + \frac{2\alpha_2}{\xi}\frac{\partial_\mu}{\Box}\partial A \circ \partial\tilde{A}, \tag{6.114}$$

which must be equal to the RHS, as computed from the graviton dictionary (6.109):

$$\begin{aligned}\frac{1}{\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right) &= \left(\frac{a_2 + (2-D)b_2}{2\xi_{(h)}}\right)\partial_\mu\,(c^\alpha \circ \tilde{c}_\alpha) + \left(\frac{1+a_3}{2\xi_{(h)}}\right)\left(A_\mu \circ \partial\tilde{A} + \partial A \circ \tilde{A}_\mu\right) \\ &+ \left(\frac{a_4 + (2-D)b_3}{2\xi_{(h)}}\right)\frac{\partial_\mu}{\Box}\partial A \circ \partial\tilde{A} + \left(\frac{a_1 + (2-D)b_1 - 1}{2\xi_{(h)}}\right)\partial_\mu(A^\rho \circ \tilde{A}_\rho).\end{aligned} \tag{6.115}$$

On comparing (6.114) and (6.115), the gauge-fixing parameter on the gravity side is simply set to be equal to its gauge theory analogue,

$$\xi_{(h)} = \xi. \tag{6.116}$$

This exhausts the BRST transformations; the remaining anti-BRST variations, $\overline{Q}\bar{c}_\mu$ and $\overline{Q}c_\mu$, do not impose new constraints.

Equations of motion

The ansätze for the sources read

$$j_{\mu\nu}(h) = \hat{a}_0\frac{1}{\Box}j_{(\mu} \circ \tilde{j}_{\nu)} + \frac{\partial_\mu\partial_\nu}{\Box^2}\left(\hat{a}_1 j^\rho \circ \tilde{j}_\rho + \hat{a}_2 j^\alpha \circ \tilde{j}_\alpha\right) + \eta_{\mu\nu}\left(\frac{\hat{b}_1}{\Box}j^\rho \circ \tilde{j}_\rho + \frac{\hat{b}_2}{\Box}j^\alpha \circ \tilde{j}_\alpha\right) \tag{6.117}$$

$$j^\alpha_\mu(c) = \frac{\hat{\alpha}_1}{\Box}\left(j_\mu \circ \tilde{j}^\alpha + j^\alpha \circ \tilde{j}_\mu\right). \tag{6.118}$$

Notice how the symmetry of $j_{\mu\nu}(h)$ allows it to have a more complicated structure than its antisymmetric analogue, $j_{\mu\nu}(B)$, in (6.96). These ought to be plugged in the following equations of motion, the first of which is worked out in Appendix C:

$$\Box h_{\mu\nu} - \frac{\xi_{(h)} + 2}{\xi_{(h)}}\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h\right) - j_{\mu\nu}(h) = 0 \tag{6.119}$$

$$\Box c^\alpha_\mu - j^\alpha_\mu(c) = 0. \tag{6.120}$$

Working with any allowed 8-dimensional (2-dimensional) basis[8] for the first (second) of these equations, we read off the remaining constraints on the parameters. Overall, we find that there exists a consistent choice of parameters such that both the BRST/anti-BRST

[8] We give these in Appendix C.

transformations and the equations of motion above are satisfied: for the graviton,

$$a_1 = \frac{\xi^2}{1-\xi^2}a_4 + \frac{1}{1-\xi} \qquad b_1 = \frac{1}{2-D}\left(\frac{\xi^2}{\xi^2-1}a_4 + \frac{\xi}{\xi-1}\right) \tag{6.121}$$

$$a_2 = \frac{\xi}{1-\xi^2}a_4 + \frac{1+\xi}{2(1-\xi)} \qquad b_2 = \frac{b_1}{\xi} \tag{6.122}$$

$$a_3 = -\frac{1}{2} \qquad b_3 = \left(\frac{1}{\xi^2} - 1\right) b_1 \tag{6.123}$$

$$a_4 = a_4 \tag{6.124}$$

the ghosts,

$$\alpha_1 = \frac{1}{4} \qquad \alpha_2 = -\frac{\xi+1}{4\xi} \tag{6.125}$$

and the sources

$$\hat{a}_0 = 1 \qquad \hat{b}_1 = b_1 \tag{6.126}$$

$$\hat{a}_1 = a_1 \qquad \hat{b}_2 = \frac{b_1}{\xi} \tag{6.127}$$

$$\hat{a}_2 = \frac{a_1}{\xi} \qquad \hat{\alpha}_1 = \alpha_1. \tag{6.128}$$

The reader should have noticed that, with the exception of the ghost sector whose parameters $\alpha_{1,2}$ and $\hat{\alpha}_1$ are uniquely fixed in terms of ξ, most parameters depend additionally on the unfixed a_4. Despite this excessive freedom, which brings along with it concerns of overfitting, it is a non-trivial fact that no contradiction arises throughout the whole calculation. There are, however, two noteworthy facts about a_4:

1. It is the parameter of the unique $\Box^{-2}$ term in the graviton dictionary.
2. It can be consistently removed, in the sense that no contradictions arise if one sets $a_4 = 0$ by hand.

No apparent good reason exists at linear level for discarding the $\Box^{-2}$ term; nonetheless, we are free to do so, following (ii), thus restricting to the (sub)set of parameter values

$$\begin{aligned} a_1 &= \frac{1}{1-\xi} & b_1 &= \frac{1}{2-D}\left(\frac{\xi}{\xi-1}\right) \\ a_2 &= \frac{1+\xi}{2(1-\xi)} & b_2 &= \frac{b_1}{\xi} \\ a_3 &= -\frac{1}{2} & b_3 &= \left(\frac{1}{\xi^2}-1\right) b_1, \end{aligned} \tag{6.129}$$

while (6.125) - (6.128) remain the same. This set of constraints represents the *minimal graviton dictionary*, in the sense that it cannot be reduced further in such a way as to still satisfy BRST and equations of motion without being forced to fix ξ. All the various dictionaries are summarised in Table 6.3 for convenience.

To conclude, it would be extremely interesting to study whether this slightly awkward freedom is merely a by-product of having worked at linearised approximation. It seems plausible that, as it often happens, non-linear effects could impose stricter constraints; this arbitrariness in the dictionaries could then prove vital in allowing enough flexibility to accommodate such changes. Another interesting observation is that the dictionary resulting from setting $a_4 = 0$, given in (6.129), appears to be singular when the Yang-Mills gauge-fixing parameter approaches the value $\xi = 1$, thus rendering the positive integer valued branch of gauge choices seemingly inaccessible. This awkward feature disappears when one considers the most general dictionary allowed by the squaring procedure, namely that where a_4 may assume values other than zero. The singularity at $\xi = 1$ is removed and thus seen to be unphysical for any function of ξ, $a_4 = -2f(\xi)$, with the simplest example being the constant function $f(\xi) = 1$.

Field	Dictionary
$h_{\mu\nu}$	$A_{(\mu}\circ\tilde{A}_{\nu)}+\frac{1}{2(1-\xi)}\frac{\partial_\mu\partial_\nu}{\Box}\left(2A^\rho\circ\tilde{A}_\rho+(1+\xi)c^\alpha\circ\tilde{c}_\alpha\right)-\frac{1}{2\Box}\left(\partial A\circ\partial_{(\mu}\tilde{A}_{\nu)}+\partial_{(\mu}A_{\nu)}\circ\partial\tilde{A}\right)$ $+\frac{\eta_{\mu\nu}}{(2-D)(\xi-1)}\left(\xi A^\rho\circ\tilde{A}_\rho+c^\alpha\circ\tilde{c}_\alpha+\left(\frac{1-\xi^2}{\xi}\right)\frac{1}{\Box}\partial A\circ\partial\tilde{A}\right)$
$B_{\mu\nu}$	$A_{[\mu}\circ\tilde{A}_{\nu]}-\frac{1}{2\Box}\left(\partial A\circ\partial_{[\mu}\tilde{A}_{\nu]}-\partial_{[\mu}A_{\nu]}\circ\partial\tilde{A}\right)$
φ	$A^\rho\circ\tilde{A}_\rho+\frac{1}{\xi}c^\alpha\circ\tilde{c}_\alpha+\left(\frac{1}{\xi^2}-1\right)\frac{1}{\Box}\partial A\circ\partial\tilde{A}$
c^α_μ	$\frac{1}{4}\left[\left(c^\alpha\circ\tilde{A}_\mu+A_\mu\circ\tilde{c}^\alpha\right)-\frac{\xi+1}{\xi}\frac{\partial_\mu}{\Box}\left(c^\alpha\circ\partial\tilde{A}+\partial A\circ\tilde{c}^\alpha\right)\right]$
d^α_μ	$\frac{1}{4}\left[\left(c^\alpha\circ\tilde{A}_\mu-A_\mu\circ\tilde{c}^\alpha\right)-\frac{\xi+1}{\xi}\frac{\partial_\mu}{\Box}\left(c^\alpha\circ\partial\tilde{A}-\partial A\circ\tilde{c}^\alpha\right)\right]$
d	$\frac{1}{2\xi}(c\circ\tilde{c})$
$\overline{d}$	$\frac{1}{2\xi}(\overline{c}\circ\tilde{\overline{c}})$
η	$\frac{1}{4}(c\circ\tilde{\overline{c}}+\overline{c}\circ\tilde{c})$
$j_{\mu\nu}(h)$	$\frac{1}{\Box}j_{(\mu}\circ\tilde{j}_{\nu)}+\frac{1}{1-\xi}\frac{\partial_\mu\partial_\nu}{\Box^2}\left(j^\rho\circ\tilde{j}_\rho+\frac{1}{\xi}j^\alpha\circ\tilde{j}_\alpha\right)+\frac{1}{(2-D)(\xi-1)}\frac{\eta_{\mu\nu}}{\Box}\left(\xi j^\rho\circ\tilde{j}_\rho+j^\alpha\circ\tilde{j}_\alpha\right)$
$j_{\mu\nu}(B)$	$\frac{1}{\Box}j_{[\mu}\circ\tilde{j}_{\nu]}$
$j(\varphi)$	$\frac{1}{\Box}j^\rho\circ\tilde{j}_\rho+\frac{1}{\xi}\frac{1}{\Box}j^\alpha\circ\tilde{j}_\alpha$
$j^\alpha_\mu(c)$	$\frac{1}{4\Box}\left(j_\mu\circ\tilde{j}^\alpha+j^\alpha\circ\tilde{j}_\mu\right)$
$j^\alpha_\mu(d)$	$\frac{1}{4\Box}\left(j_\mu\circ\tilde{j}^\alpha-j^\alpha\circ\tilde{j}_\mu\right)$
$j(d)$	$\frac{1}{2\xi\Box}j(c)\circ\tilde{j}(c)$
$j(\overline{d})$	$\frac{1}{2\xi\Box}j(\overline{c})\circ\tilde{j}(\overline{c})$
$j(\eta)$	$\frac{1}{4\Box}\left(j(c)\circ\tilde{j}(\overline{c})-j(\overline{c})\circ\tilde{j}(c)\right)$

Table 6.3: Summary of the final dictionaries which map both BRST/antiBRST transformations and the dynamics of Yang-Mills to those of gravity. The horizontal lines separate physical, ghost and source sectors. The graviton dictionary is the minimal one, with $a_4 = 0$.

6.4 Squaring the Lautrup-Nakanishi field

Up to now, we have been working without the auxiliary Lautrup-Nakanishi field. The reader may wonder why, given that Yang-Mills (or the linearised version thereof) is not only equivalent, but in some ways simpler when the field b is included. However, as it turns out, the Yang-Mills/gravity map is slightly less obvious in this case and a full dictionary has not been worked out yet and it is considered as work in progress. Nonetheless, we would like to present here the main new ideas starting, as was done for the previous case, with the identification of the degrees of freedom in the image of the tensor product, schematically presented in Table 6.4.

	$\tilde{A}_\nu$ (0)	$\tilde{c}^\beta$ (± 1)	$\tilde{b}$ (0)
A_μ (0)	$A_{(\mu} \circ \tilde{A}_{\nu)} - \frac{\eta_{\mu\nu}}{D} A_\rho \circ \tilde{A}^\rho$ $A_{[\mu} \circ \tilde{A}_{\nu]}$ $A_\rho \circ \tilde{A}^\rho$	$A_\mu \circ \tilde{c}^\beta$	$A_\mu \circ \tilde{b}$
c^α (± 1)	$c^\alpha \circ \tilde{A}_\nu$	$c^\alpha \circ \tilde{c}_\alpha$ $c^{(\alpha} \circ \tilde{c}^{\beta)}$	$c^\alpha \circ \tilde{b}$
b (0)	$b \circ \tilde{A}_\nu$	$b \circ \tilde{c}^\beta$	$b \circ \tilde{b}$

Table 6.4: Extended squaring table including the Lautrup-Nakanishi field.

Compared to the previous case, given in Table 6.1, we notice the appearance of the new states on the outer row and column. Of course, we expect some of these to correspond to the Lautrup-Nakanishi multipliers of the gravity theory, as it seems only natural that squaring Yang-Mills theory formulated with b and with an off-shell closed algebra would result in a similar construction on the gravity side. The graviton/2-form/dilaton theory discussed above requires one vector multiplier for the spin-2 gauge-fixing, $b_{(h)\mu}$, which enters the Lagrangian as

$$\mathcal{L}_{gf}(h) = b^\mu_{(h)} \left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h - \frac{\xi_{(h)}}{2} b_{(h)\mu} \right), \tag{6.130}$$

as well as one bosonic vector and two fermionic scalar multipliers for the 2-form (cf. Section 5.5), denoted by $b_{(B)\mu}$, $b_{(d)}$ and $\overline{b}_{(d)}$. As the bar notation suggests in this context (i.e. BRST quantised theories), the latter two fields are conjugate with respect to the ghost grading, i.e. $gh(b_{(d)}) = 1$ while $gh(\overline{b}_{(d)}) = -1$. A part of the map is very straightforward, and does not require much thought. It is easy to convince oneself that, at the level of states,

$$b \circ \tilde{A}_\mu \pm A_\mu \circ \tilde{b} \qquad \Longleftrightarrow \qquad b_{(h)\mu},\ b_{(B)\mu} \tag{6.131}$$

$$b \circ \tilde{c}^\alpha - c^\alpha \circ \tilde{b} \qquad \Longleftrightarrow \qquad b^\alpha_{(d)} \tag{6.132}$$

where $b^\alpha_{(d)}$ is the doublet of fermionic scalar multipliers $b_{(d)}$ and $\overline{b}_{(d)}$. In fact, one could go a little further than this and, as before, exploit the BRST transformations to start refining the various dictionaries. This time, the appropriate set of variations on the gauge theory side is given by the linear approximation of (5.49), namely

$$\begin{aligned} QA_\mu &= \partial_\mu c & \qquad \overline{Q}A_\mu &= \partial_\mu \overline{c} \\ Qc &= 0 & \overline{Q}\overline{c} &= 0 \\ Q\overline{c} &= b & \overline{Q}c &= -b \\ Qb &= 0 & \overline{Q}b &= 0, \end{aligned} \tag{6.133}$$

while on the gravitational side one expects to map to the off-shell BRST transformations, namely those involving the auxiliary fields. For instance, as far as the Kalb-Ramond field is concerned, we require that the variations in (5.62) are reproduced. As a simple example to illustrate things, consider the modified transformations of the 2-form antighost, resulting in the auxiliary field $Q\overline{d}_\mu = b_{(B)\mu}$, rather than in its equation of motion. Taking a reduced dictionary for the 2-form antighost $\overline{d}_\mu$ for clarity, one obtains

$$\begin{aligned} b_{(B)\mu} = Q\overline{d}_\mu &= Q\left(\overline{c} \circ \tilde{A}_\mu - A_\mu \circ \tilde{\overline{c}}\right) \\ &= b \circ \tilde{A}_\mu - A_\mu \circ \tilde{b} - \partial_\mu \left(c \circ \tilde{\overline{c}} + \overline{c} \circ \tilde{c}\right). \end{aligned} \tag{6.134}$$

Thus, the ghost number zero combination of ghosts (already shown to contribute to η) is seen to quite naturally enter the dictionary for $b_{(B)\mu}$ by means of the BRST transformations. In fact, this should not surprise us, given that the two fields are related to each other through the algebraic equation of motion for the multiplier (5.64),

$$b_{(B)\mu} = \frac{1}{\xi_{(B)}} \left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right). \tag{6.135}$$

However, a brief look at Table 6.4 reveals that more states are generated in the tensor product than those one would naively expect, and one is left wondering which quantities on the gravity side may be identified with the three remaining products, which we denote as

$$b \circ \tilde{c}^\alpha + c^\alpha \circ \tilde{b} \qquad \Longleftrightarrow \qquad \zeta^\alpha \tag{6.136}$$

$$b \circ \tilde{b} \qquad \Longleftrightarrow \qquad b_{(\zeta)} \tag{6.137}$$

and whose properties, such as ghost number and mass dimension as they come from Yang-Mills, are summarised in A.2.

6.4.1 Weyl rescaling

As we have learnt in the preceding examples, the symmetric combinations such as ζ^α tend to belong to the graviton sector, while we have already suggested in (6.131) that the orthogonal antisymmetric combination $b \circ \tilde{c}^\alpha - c^\alpha \circ \tilde{b}$ appears to be well suited to describe the fermionic auxiliary fields $b^\alpha_{(d)}$ in the Kalb-Ramond sector. A clue that the other one, $b \circ \tilde{b}$, should also belong in the graviton sector is the fact that it emerges as the BRST (anti-BRST) transformation of the $\alpha = 2$ ($\alpha = 1$) component of ζ^α, that is

$$Q\overline{\zeta} = Q\left(b \circ \tilde{\overline{c}} + \overline{c} \circ \tilde{b}\right) = b \circ \tilde{b} = b_{(\zeta)}, \tag{6.138}$$

$$\overline{Q}\zeta = \overline{Q}\left(b \circ \tilde{c} + c \circ \tilde{b}\right) = -b \circ \tilde{b} = -b_{(\zeta)}. \tag{6.139}$$

Our chosen notation is suggestive: we interpret ζ^α as a doublet of (fermionic, scalar) ghosts for some gauge symmetry. As it is customary, the antighost (ghost) of a certain gauge symmetry is rotated, under the action of Q ($\overline{Q}$), into the relevant Lautrup-Nakanishi auxiliary field, which suggests the identification of $b \circ \tilde{b}$. *But which symmetry?*

Once again, the BRST transformations provide a great deal of information; it suffices to notice that ζ is the result of a BRST transformation of the ghost singlet, that is

$$Q\left(c^\alpha \circ \tilde{c}_\alpha\right) = -\left(b \circ \tilde{c} + c \circ \tilde{b}\right) = -\zeta \tag{6.140}$$

to realise that this symmetry is a Weyl dilatation acting on the graviton and the dilaton, since these are the only fields whose dictionary contain the combination $c^\alpha \circ \tilde{c}_\alpha$, as may be verified in Table 6.3. The effects of such a symmetry were briefly described around equation (6.21). Indeed, for simplicity, consider the BRST variation of the "toy graviton", a subset of the full dictionary, denoted here by $\hat{h}_{\mu\nu}$,

$$\begin{aligned} Q\hat{h}_{\mu\nu} &= Q\left(A_{(\mu} \circ \tilde{A}_{\nu)} + \eta_{\mu\nu} c^\alpha \circ \tilde{c}_\alpha\right) \\ &= \partial_{(\mu}\left(c \circ \tilde{A}_{\nu)} + A_{\nu)} \circ \tilde{c}\right) - \eta_{\mu\nu}\left(b \circ \tilde{c} + c \circ \tilde{b}\right) \\ &:= 2\partial_{(\mu}\hat{c}_{\nu)} - \eta_{\mu\nu}\zeta \end{aligned} \tag{6.141}$$

where we denoted the diffeomorphism ghost with a hat to again indicate that it is not the most general one identified in the previous section, but rather just a toy example to present the idea. This suggests the possibility that the graviton inherits, through the dictionary, both (the BRST analog of) linear diffeomorphisms and a Weyl dilatation owing to the BRST transformations of the Yang-Mills gauge field A_μ and ghosts c^α. Similarly, the dilaton would inherit a shift by ζ: consider its dictionary[9] in Table 6.3 and apply the

[9] When working with Lautrup-Nakanishi fields, this is no longer the most general, since we could include terms containing b in the dictionary.

off-shell BRST transformations (6.133) to the factors,

$$
\begin{aligned}
Q\varphi &= Q\left(A^\rho \circ \tilde{A}_\rho + \frac{1}{\xi}c^\alpha \circ \tilde{c}_\alpha + \left(\frac{1}{\xi^2} - 1\right)\frac{1}{\Box}\partial A \circ \partial \tilde{A}\right) \\
&= \frac{1}{\xi^2}\left(c \circ \partial \tilde{A} + \partial A \circ \tilde{c}\right) - \frac{1}{\xi}\left(b \circ \tilde{c} + c \circ \tilde{b}\right) \\
&:= \frac{1}{\xi^2}\partial^\rho \hat{c}_\rho - \frac{1}{\xi}\zeta.
\end{aligned}
\tag{6.142}
$$

The important fact to highlight here is that the dilaton *can* inherit a shift by ζ through its Yang-Mills factors, in line with the interpretation that ζ parameterises the spurious[10] conformal symmetry. The other factor is not too important at the moment, since it could be canceled by picking a different dictionary. In that case, of course, the question of the compatibility of the new "off-shell" dictionary and that of Table 6.3 upon elimination of b ought to be addressed.

While in principle one should encounter no hindrance in reproducing all the correct BRST/antiBRST off-shell closed algebra of the gravitational side from the Yang-Mills factors in this (A, c^α, b) basis, doing so in the most general way (checking for uniqueness) presents two problems: the first, of very practical nature, is that it is computationally quite involved and lengthy; the second, more fundamental, is that it appears implausible that the BRST/antiBRST variations alone would fix all the a priori arbitrary parameters, as was the case for the analogous calculation without auxiliary fields. If this supposition were to hold, then one could resort to imposing the equations of motion: morally, however, this feels like the incorrect route to take, since then the gravitational fields so constructed would necessarily "know" about the equations of motion they ought to satisfy, moving away from the purely off-shell description of the BRST system that the introduction of the Lautrup fields hoped to achieve to begin with. In addition, we note that there are other ways to constrain these bona fide off-shell dictionaries containing the auxiliary fields b, such as requiring that they match the results of the previous section upon eliminating these through their algebraic equations at the level of the dictionaries themselves.

[10] By spurious we mean that the compensating scalar is pure gauge. Its single degree of freedom can be completely gauged away by the Weyl shift.

Chapter 7

Conclusions

It might well be that gravity is not, in any physically meaningful sense, the “square” or the “double-copy” of Yang-Mills gauge theory. Regardless of whether this is true, or not, the fast growing number of instances where this structure is shown to emerge in our mathematical descriptions of nature is a reality which we should not overlook. It is apparent from the discussion and perhaps even more clearly from the tables of Chapter 3 that a vast number of ungauged (as well as some gauged) supergravity theories, at least to the extent to which their content, symmetries and interactions are concerned, may be shown to have a factorisation in terms of a pair of Yang-Mills theories. Much effort is indeed being devoted to achieve a complete classification.

Furthermore, the Yang-Mills Squared construction offers an alternative perspective, and hopefully an important computational tool, to familiar and omnipresent manipulations in supergravity, such as dimensional reduction and truncation. Effecting these in a specific supergravity of interest simply corresponds to reducing or truncating the simpler Yang-Mills theories which enter the factorisation. It is possible to understand why a supergravity theory may or may not uplift to a specific dimension by inspection of the properties of its factors [139]. Chapter 4 shows one application of these ideas: the twin relation linking pairs of supergravities together is recast in Yang-Mills lingo through a controlled truncation of the factors. Twin supergravities are theorised to be a useful testing ground for many manipulations such as gaugings or other types of deformations [92]: it is possible that some computations in this direction could be simplified by considering the Yang-Mills factorisation of such theories.

It is conceptually provocative that the “hidden” global U-dualities of supergravity should be related through the Yang-Mills Squared map to such “manifest” symmetries as the R-symmetries of the gauge theory factors. For example, this sort of construction gives one possible a posteriori justification for the appearence of the exceptional groups in supergravity. It is equally as provocative that the local diffeomorphism invariance, p-form gauge invariance as well as the supersymmetry transformations on the gravity side should all be constructible from Yang-Mills's gauge invariance and supersymmetry variations, as shown in [90].

The main motivation behind this work as well as the ideas presented in Chapter 6 was two-fold: (1) to obtain, if at all possible, a manifestly Lorentz-covariant formulation of the Yang-Mills Squared map, going beyond on-shell state identification and on-shell scattering amplitudes, and (2) to give a map between the dynamics of gauge theory and those of supergravity without having to restrict to a specific gauge on both sides. We show how to achieve this in Chapter 6, at linear level, by recasting the classical gauge invariance as BRST invariance, with the ghost fields encoding gauge transformations in a covariant fashion. In this context, it is the BRST and anti-BRST transformations of Yang-Mills which map to those of supergravity. It turns out that resorting to such a "semi-classical" BRST treatment is a necessary ingredient if one is interested in deriving the correct gravitational equations of motion, whereby BRST allows for a better handle on the various physical/unphysical degrees of freedom which otherwise render the identification of the gravitational fields rather obscure. The properties of the convolutive tensor product become indispensable in this regard: being able to move derivatives around without picking boundary terms is crucial to the derivation of the BRST variations and the dynamics. It is very important to notice that these desirable properties of the convolution do not hold in general, but only on certain functional domains: a thorough investigation of their nature and the consequences of this restriction on the allowed space of fields constitute one of the natural continuations of our study.

The results of Chapter 6 may be summarised with the term of *on-shell covariant squaring*: the definition of the map depends heavily on the use of the equations of motion, where these are paramount in proving that the a priori arbitrary parameters in the dictionary are uniquely fixed in terms of the quantity ξ, the weight controlling the gauge-fixing in Yang-Mills. In fact, there is one isolated exception: whilst the totality of the parameters in the dilaton and 2-form sectors are fixed, the graviton dictionary exhibits a sort of redundancy, whereby the BRST, anti-BRST transformations and the equations of motion are correctly mapped with some freedom to spare. One parameter, the coefficient of the lone $\square^{-2}$ term, is not fixed, although it may be set to zero without affecting the consistency of the overall dictionary. As remarked above, this seemingly superfluous freedom is seen to be necessary in allowing ξ to assume any integer value, specifically eliminating a singular point at $\xi = 1$ arising when the parameter is set to zero by hand. Another intriguing possibility is that this freedom will be fixed when going to higher orders in perturbation theory. Moreover, the triplet of gravitational gauge-fixing weights, $(\xi_{(h)}, \xi_{(B)}, \xi_{(d)})$, is also fixed uniquely as a function of ξ, with the extremely simple relation $\xi_{(h)} = \xi_{(B)} = 2\xi_{(d)} = \xi$. A BRST-fixed theory encodes the information about all possible gauge-fixing choices in its gauge-fixing and ghost terms in the action, and thus in its equations of motion modified by ξ-dependent functions. In this sense, our construction is independent of the gauge-fixing functional chosen: choosing different gauge-fixing functionals in any of the theories involved would yield a different relation between the weights $\xi_{(i)}$, but would not affect the validity of the formalism. We have simply chosen the Lagrangians which yield the simplest relation.

We stress again that the map obtained is strictly on-shell. One might argue that it would be desirable to have a purely off-shell formulation of these ideas. We attempted to solve this problem in Section 6.4 by considering squaring the auxiliary Lautrup-Nakanishi fields along with the Yang-Mills potentials and ghost fields; while it is not entirely clear how to do this as of yet, there are some indications that the supergravity resulting from such a generalised squaring would possess interesting features such as a spurious Weyl dilatation symmetry acting on the graviton and the dilaton, of the type usually seen in the conformal constructions of supergravity actions. In other words, such squaring products could land us somewhat in the midst of the procedure of gauge-fixing the gauged superconformal group to obtain Poincaré supergravity.

Finally, such a generalisation to include the auxiliary fields could be a stepping stone towards a yet more general formulation of the Yang-Mills Squared construction, covariant with respect to the whole field-antifield formalism's phase space and extended BV transformations.

Appendix A

Some more tables

Field	Ghost number	Mass dimension	Parity
A_μ	0	1	0
c	1	0	1
$\bar{c}$	-1	2	1
b	0	2	0

Table A.1: The spectrum of Yang-Mills theory after BRST quantisation.

Field	Ghost number	Mass dimension	Parity
$h_{\mu\nu}$	0	1	0
c_μ	1	0	1
$\bar{c}_\mu$	-1	2	1
φ	0	1	0
ζ	1	1	1
$\bar{\zeta}$	-1	3	1
$b_{(h)\mu}$	0	2	0
$b_{(\zeta)}$	0	3	0

Table A.2: The properties of the various fields appearing in the graviton-dilaton sector of Yang-Mills Squared. Those below the central line are only present when squaring with Lautrup-Nakanishi fields. Notice that the Weyl ghosts have similar properties to the Lautrup fields $b_{(d)}$ of the Kalb-Ramond sector, thus we would expect a similar Lagrangian, i.e. $\bar{\zeta}\zeta$.

Field	Ghost number	Mass dimension	Parity
$B_{\mu\nu}$	0	1	0
d_μ	1	0	1
$\bar{d}_\mu$	-1	2	1
d	2	-1	0
$\bar{d}$	-2	3	0
η	0	1	0
$b_{(B)}$	0	2	0
$b_{(d)}$	1	1	1
$\bar{b}_{(d)}$	-1	3	1

Table A.3: The properties of the various fields appearing in the Kalb-Ramond sector of Yang-Mills Squared. Those below the central line are only present when squaring with Lautrup-Nakanishi fields.

D; Little group	ϕ	A_μ	$A_{\mu\nu}$	$A_{\mu\nu\rho}$	λ	Ψ_μ	$g_{\mu\nu}$
$11; \mathfrak{so}(9)$	$\mathbf{1}$	$\mathbf{9}$	$\mathbf{36}$	$\mathbf{84}$	$\mathbf{16}$	$\mathbf{128}$	$\mathbf{44}$
$10; \mathfrak{so}(8)$	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{28}_v$	$\mathbf{56}_v$	$\mathbf{8}_s,\ \mathbf{8}_c$	$\mathbf{56}_s,\ \mathbf{56}_c$	$\mathbf{35}_v$
$9; \mathfrak{so}(7)$	$\mathbf{1}$	$\mathbf{7}$	$\mathbf{21}$	$\mathbf{35}$	$\mathbf{8}$	$\mathbf{48}$	$\mathbf{27}$
$8; \mathfrak{so}(6)$	$\mathbf{1}$	$\mathbf{6}$	$\mathbf{15}$	$\mathbf{10},\ \overline{\mathbf{10}}$	$\mathbf{4},\ \overline{\mathbf{4}}$	$\mathbf{20}'$	$\mathbf{20}$
$7; \mathfrak{so}(5)$	$\mathbf{1}$	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{4}$	$\mathbf{16}$	$\mathbf{14}$
$6; \mathfrak{so}(4)$	$\mathbf{1}$	$\mathbf{4}$	$(\mathbf{3},\mathbf{1}),\ (\mathbf{1},\mathbf{3})$	$(\mathbf{2},\mathbf{2})$	$(\mathbf{2},\mathbf{1}),\ (\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{2}),\ (\mathbf{2},\mathbf{3})$	$(\mathbf{3},\mathbf{3})$
$5; \mathfrak{so}(3)$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{4}$	$\mathbf{5}$
$4; \mathfrak{so}(2)$	(0)	$(2)+(-2)$	(0)	$\varnothing$	$(1)+(-1)$	$(3)+(-3)$	$(4)+(-4)$

Table A.4: Little group representations carries by the fields appearing in this thesis. Recall that we double the helicity in $D = 4$ to avoid fractions. Comparison of the columns for the different p-forms (including the scalar) suggests which forms are dual to each other in $D = 4, 5, 6, 7$.

D; Little group	$\mathcal{Q}$	Name	Reps under $\mathfrak{so}(D-2)\oplus R$	R
11; $\mathfrak{so}(9)_{st}$	32	$\mathbf{G}_1$	$\mathbf{44}+\mathbf{128}+\mathbf{84}$	$\varnothing$
10; $\mathfrak{so}(8)_{st}$	32	$\mathbf{G}_{(1,1)}$	$\mathbf{35}_v+\mathbf{56}_s+\mathbf{56}_c+\mathbf{56}_v+\mathbf{28}+\mathbf{8}_v+\mathbf{8}_s+\mathbf{8}_c+\mathbf{1}$	$\varnothing$
	″	$\mathbf{G}_{(2,0)}$	$\mathbf{35}_v^0+\mathbf{56}_s^1+\mathbf{56}_s^{-1}+\mathbf{35}_s^0+\mathbf{28}^2$ $+\mathbf{28}^{-2}+\mathbf{8}_s^1+\mathbf{8}_s^{-1}+\mathbf{1}^4+\mathbf{1}^{-4}$	$\mathfrak{so}(2)$
	16	$\mathbf{G}_{(1,0)}$	$\mathbf{35}_v+\mathbf{56}_s+\mathbf{28}+\mathbf{8}_c+\mathbf{1}$	$\varnothing$
	″	$\mathbf{V}_{(1,0)}$	$\mathbf{8}_v+\mathbf{8}_s$	$\varnothing$
9; $\mathfrak{so}(7)_{st}$	32	$\mathbf{G}_2$	$\mathbf{27}_0+\mathbf{48}_1+\mathbf{48}_{-1}+\mathbf{35}_0+\mathbf{21}_2$ $+\mathbf{21}_{-2}+\mathbf{7}_0+\mathbf{7}_2+\mathbf{7}_{-2}+\mathbf{8}_3+\mathbf{8}_{-3}$ $+\mathbf{8}_1+\mathbf{8}_{-1}+\mathbf{1}_0+\mathbf{1}_4+\mathbf{1}_{-4}$	$\mathfrak{so}(2)$
	16	$\mathbf{G}_1$	$\mathbf{27}+\mathbf{48}+\mathbf{21}+\mathbf{7}+\mathbf{8}+\mathbf{1}$	$\varnothing$
	″	$\mathbf{V}_1$	$\mathbf{7}+\mathbf{8}+\mathbf{1}$	$\varnothing$
8; $\mathfrak{su}(4)_{st}$	32	$\mathbf{G}_2$	$(\mathbf{20}';\mathbf{1})_0+(\mathbf{20};\mathbf{2})_1+(\overline{\mathbf{20}};\mathbf{2})_{-1}+(\mathbf{10};\mathbf{1})_2$ $+(\overline{\mathbf{10}};\mathbf{1})_{-2}+(\mathbf{15};\mathbf{3})_0+(\mathbf{6};\mathbf{3})_2+(\mathbf{6};\mathbf{3})_{-2}$ $+(\mathbf{4};\mathbf{2})_{-3}+(\overline{\mathbf{4}};\mathbf{2})_3+(\mathbf{4};\mathbf{4})_{-1}+(\mathbf{4};\mathbf{4})_1$	$\mathfrak{u}(2)$
	16	$\mathbf{G}_1$	$\mathbf{20}'_0+\mathbf{20}_{-1}+\overline{\mathbf{20}}_1+\mathbf{15}_0+\mathbf{6}_2$ $+\mathbf{6}_{-2}+\mathbf{4}_1+\overline{\mathbf{4}}_{-1}+\mathbf{1}_0$	$\mathfrak{u}(1)$
	″	$\mathbf{V}_1$	$\mathbf{6}_0+\mathbf{4}_1+\overline{\mathbf{4}}_{-1}+\mathbf{1}_2+\mathbf{1}_{-2}$	$\mathfrak{u}(1)$
7; $\mathfrak{sp}(2)_{st}$	32	$\mathbf{G}_4$	$(\mathbf{14};\mathbf{1})+(\mathbf{16};\mathbf{4})+(\mathbf{10};\mathbf{5})+(\mathbf{5};\mathbf{10})+(\mathbf{4};\mathbf{16})+(\mathbf{1};\mathbf{14})$	$\mathfrak{sp}(2)$
	16	$\mathbf{G}_2$	$(\mathbf{14};\mathbf{1})+(\mathbf{16};\mathbf{2})+(\mathbf{10};\mathbf{1})+(\mathbf{5};\mathbf{3})+(\mathbf{4};\mathbf{2})+(\mathbf{1};\mathbf{1})$	$\mathfrak{sp}(1)$
	″	$\mathbf{V}_2$	$(\mathbf{5};\mathbf{1})+(\mathbf{4};\mathbf{2})+(\mathbf{1};\mathbf{3})$	$\mathfrak{sp}(1)$

Table A.5: All allowed supermultiplets in $7\leq D\leq 11$, with their content given as representations of the little group algebra $\mathfrak{so}(D-2)$ and the internal symmetry algebra, corresponding to R-symmetry for matter and $\mathfrak{h}$ for supergravity.

D; Little group	$\mathcal{Q}$	Name	Reps under $\mathfrak{so}(D-2) \oplus R$	R
6; $2\mathfrak{sp}(1)_{st}$	32	$\mathbf{G}_{(4,4)}$	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{4}) + (\mathbf{2},\mathbf{3};\mathbf{4},\mathbf{1})$ $(\mathbf{2},\mathbf{2};\mathbf{4},\mathbf{4}) + (\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{3};\mathbf{5},\mathbf{1})$ $(\mathbf{2},\mathbf{1};\mathbf{4},\mathbf{5}) + (\mathbf{1},\mathbf{2};\mathbf{5},\mathbf{4}) + (\mathbf{1},\mathbf{1};\mathbf{5},\mathbf{5})$	$2\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$
	24	$\mathbf{G}_{(4,2)}$	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{3};\mathbf{4},\mathbf{1})$ $(\mathbf{2},\mathbf{2};\mathbf{4},\mathbf{2}) + (\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{5},\mathbf{1})$ $(\mathbf{2},\mathbf{1};\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{5},\mathbf{2}) + (\mathbf{1},\mathbf{1};\mathbf{5},\mathbf{1})$	$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$
	16	$\mathbf{G}_{(2,2)}$	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{2};\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{3};\mathbf{2},\mathbf{1})$ $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2}) + (\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$ $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$
	$''$	$\mathbf{V}_{(2,2)}$	$(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$
	$''$	$\mathbf{G}_{(4,0)}$	$(\mathbf{3},\mathbf{3};\mathbf{1}) + (\mathbf{2},\mathbf{3};\mathbf{4}) + (\mathbf{1},\mathbf{3};\mathbf{5})$	$\mathfrak{sp}(2) \oplus \varnothing$
	$''$	$\mathbf{T}_{(4,0)}$	$(\mathbf{3},\mathbf{1};\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{4}) + (\mathbf{1},\mathbf{1};\mathbf{5})$	$\mathfrak{sp}(2) \oplus \varnothing$
	8	$\mathbf{G}_{(2,0)}$	$(\mathbf{3},\mathbf{3};\mathbf{1}) + (\mathbf{2},\mathbf{3};\mathbf{2}) + (\mathbf{1},\mathbf{3};\mathbf{1})$	$\mathfrak{sp}(1) \oplus \varnothing$
	$''$	$\mathbf{V}_{(2,0)}$	$(\mathbf{2},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{2})$	$\mathfrak{sp}(1) \oplus \varnothing$
	$''$	$\mathbf{T}_{(2,0)}$	$(\mathbf{3},\mathbf{1};\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{2}) + (\mathbf{1},\mathbf{1};\mathbf{1})$	$\mathfrak{sp}(1) \oplus \varnothing$
	$''$	$\mathbf{H}_{(2,0)}$	$2 \times [(\mathbf{2},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{2})]$	$\mathfrak{sp}(1) \oplus \varnothing$
	$''$	$\mathbf{C}_{(2,0)}$	$(\mathbf{2},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{2})$	$\mathfrak{sp}(1) \oplus \varnothing$
5; $\mathfrak{sp}(1)_{st}$	32	$\mathbf{G}_8$	$(\mathbf{5};\mathbf{1}) + (\mathbf{4};\mathbf{8}) + (\mathbf{3};\mathbf{27}) + (\mathbf{2};\mathbf{48}) + (\mathbf{1};\mathbf{42})$	$\mathfrak{sp}(4)$
	24	$\mathbf{G}_6$	$(\mathbf{5};\mathbf{1}) + (\mathbf{4};\mathbf{6}) + (\mathbf{3};\mathbf{14}+\mathbf{1}) + (\mathbf{2};\mathbf{6}+\mathbf{14}') + (\mathbf{1};\mathbf{14})$	$\mathfrak{sp}(3)$
	16	$\mathbf{G}_4$	$(\mathbf{5};\mathbf{1}) + (\mathbf{4};\mathbf{4}) + (\mathbf{3};\mathbf{5}+\mathbf{1}) + (\mathbf{2};\mathbf{4}) + (\mathbf{1};\mathbf{1})$	$\mathfrak{sp}(2)$
	$''$	$\mathbf{V}_4$	$(\mathbf{3};\mathbf{1}) + (\mathbf{2};\mathbf{4}) + (\mathbf{1};\mathbf{5})$	$\mathfrak{sp}(2)$
	8	$\mathbf{G}_2$	$(\mathbf{5};\mathbf{1}) + (\mathbf{4};\mathbf{2}) + (\mathbf{3};\mathbf{1})$	$\mathfrak{sp}(1)$
	$''$	$\mathbf{V}_2$	$(\mathbf{3};\mathbf{1}) + (\mathbf{2};\mathbf{2}) + (\mathbf{1};\mathbf{1})$	$\mathfrak{sp}(1)$
	$''$	$\mathbf{H}_2$	$2(\mathbf{2};\mathbf{1}) + 2(\mathbf{1};\mathbf{2})$	$\mathfrak{sp}(1)$
	$''$	$\mathbf{C}_2^a$	$(\mathbf{2};\mathbf{1}) + (\mathbf{1};\mathbf{2})$	$\mathfrak{sp}(1)$

Table A.6: All allowed supermultiplets in $D = 5, 6$, with their content given as representations of the little group algebra $\mathfrak{so}(D-2)$ and the internal symmetry algebra, corresponding to R-symmetry for matter and $\mathfrak{h}$ for supergravity. We do not give the conformal multiplets here; if interested, see [77].

D; Little group	$\mathcal{Q}$	Name	Content	R-symmetry
$4; \mathfrak{u}(1)_{st}$	32	$\mathbf{G}_8$	$\mathbf{1}^4 + \mathbf{8}^3 + \mathbf{28}^2 + \mathbf{56}^1 + \mathbf{70}^0 + \overline{\mathbf{56}}^{-1} + \overline{\mathbf{28}}^{-2} + \overline{\mathbf{8}}^{-3} + \mathbf{1}^{-4}$	$\mathfrak{su}(8)$
	24	$\mathbf{G}_6$	$\mathbf{1}_0^4 + \mathbf{6}_1^3 + (\mathbf{15}_2^2 + \mathbf{1}_6^{-2}) + (\mathbf{20}_3^1 + \overline{\mathbf{6}}_5^{-1}) + \overline{\mathbf{15}}_4^0 + c.c.$	$\mathfrak{u}(6)$
	20	$\mathbf{G}_5$	$\mathbf{1}_0^4 + \mathbf{5}_1^3 + \mathbf{10}_2^2 + (\overline{\mathbf{10}}_3^1 + \mathbf{1}_5^{-1}) + \overline{\mathbf{5}}_4^0 + c.c.$	$\mathfrak{u}(5)$
	16	$\mathbf{G}_4$	$\mathbf{1}_0^4 + \mathbf{4}_1^3 + \mathbf{6}_2^2 + \overline{\mathbf{4}}_3^1 + \mathbf{1}_4^0 + c.c.$	$\mathfrak{u}(4)$
	$''$	$\mathbf{V}_4$	$\mathbf{1}^2 + \mathbf{4}^1 + \mathbf{6}^0 + \overline{\mathbf{4}}^{-1} + \mathbf{1}^{-2}$	$\mathfrak{su}(4)$
	12	$\mathbf{G}_3$	$\mathbf{1}_0^4 + \mathbf{3}_1^3 + \overline{\mathbf{3}}_2^2 + \mathbf{1}_3^1 + c.c.$	$\mathfrak{u}(3)$
	8	$\mathbf{G}_2$	$\mathbf{1}_0^4 + \mathbf{2}_1^3 + \mathbf{1}_2^2 + c.c.$	$\mathfrak{u}(2)$
	$''$	$\mathbf{V}_2$	$\mathbf{1}_0^2 + \mathbf{2}_1^1 + \mathbf{1}_2^0 + c.c.$	$\mathfrak{u}(2)$
	$''$	$\mathbf{H}_2$	$\mathbf{1}_r^1 + \mathbf{2}_{r+1}^0 + \mathbf{1}_{r+2}^{-1} + c.c.$	$\mathfrak{u}(2)$
	$''$	$\mathbf{C}_2^a$	$\mathbf{1}_{-1}^1 + \mathbf{2}_0^0 + \mathbf{1}_1^{-1}$	$\mathfrak{u}(2)$
	4	$\mathbf{G}_1$	$(4,0) + (3,1) + c.c.$	$\mathfrak{u}(1)$
	$''$	$\mathbf{V}_1$	$(2,0) + (1,1) + c.c.$	$\mathfrak{u}(1)$
	$''$	$\mathbf{C}_1$	$(1,r) + (0,r+1) + c.c.$	$\mathfrak{u}(1)$

Table A.7: All allowed supermultiplets in $D = 4$. Note that $\mathbf{G}_7 = \mathbf{G}_8$ and $\mathbf{V}_3 = \mathbf{V}_4$, and the half-hypermultiplet $\mathbf{C}_2^a$ exists on its own only for pseudoreal representations of the gauge group.

Appendix B

Field-antifield formalism redux

As mentioned in Chapter 5, BRST and anti-BRST invariance provide a powerful tool for the analysis of quantum gauge theories, where they simplify, both conceptually and computationally, the study of various of their aspects, among which unitarity and renormalisability usually stand out. Furthermore, as shown above, they place strong constraints on the allowed forms of the action, thus becoming useful at pinning down the (hopefully unique) quantum-mechanically consistent theory itself. In fact, it so happens that invariance under BRST and anti-BRST transformations may be taken as the fundamental principle underlying all gauge theories, of which Yang-Mills constitutes one of the simplest available examples. The most successful approach in this regard is the so-called *field-antifield*, or *Batalin-Vilkovisky* (BV), formalism: this succeeds at obtaining a BRST (and/or anti-BRST) invariant theory *prior to* gauge-fixing, at the price of a doubled configuration space whereby each field (physical, ghost and Lautrup-Nakanishi) is assigned a "canonical momentum" variable (its *antifield*) relative to a new symplectic bilinear form, known as the *antibracket*. We briefly describe here, closely following the exposition of [130], the field-antifield formalism for BRST invariance, for clarity; its extension to include anti-BRST and $Sp(2)$ invariance is well-established [127–129, 131].

For any two functionals on the space of fields, Φ^A, and antifields, Φ^*_A, the antibracket is defined as

$$(F, G) = \frac{\partial_r F}{\partial \Phi^A} \frac{\partial_l G}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi^*_A} \frac{\partial_l G}{\partial \Phi^A} \tag{B.1}$$

where the subscripts l/r denote left and right derivatives. The *master action* $\mathcal{S}$ is the (minimal) solution to the master equation,

$$(\mathcal{S}, \mathcal{S}) = 2 \frac{\partial_r \mathcal{S}}{\partial \Phi^A} \frac{\partial_l \mathcal{S}}{\partial \Phi^*_A} = 0, \tag{B.2}$$

provided that it coincides with the classical action when antifields are set to zero, that is it satisfies the condition $\mathcal{S}(\Phi, 0) = S_0$. The master action is invariant under the "gauge-unfixed", or generalised, BRST transformations[1],

$$\delta' F(\Phi^A, \Phi^*_A) = (F, \mathcal{S}) = \epsilon(\mathcal{Q}F), \tag{B.3}$$

as a direct consequence of the master equation (B.2). In particular, fields and antifields transform according as

$$\mathcal{Q}\Phi^A = \frac{\partial_l \mathcal{S}}{\partial \Phi^*_A}, \qquad\qquad \mathcal{Q}\Phi^*_A = -\frac{\partial_l \mathcal{S}}{\partial \Phi^A}. \tag{B.4}$$

The generalised BRST transformations are off-shell nilpotent, $\mathcal{Q}^2 = 0$, by construction. The minimal solution $\mathcal{S}$ above still contains gauge invariances, which makes it unsuitable for quantisation. One needs to fix the antifields in a specific way[2]: usually, one enlarges the configuration space yet again by introducing so-called trivial pairs (think antighost + Lautrup-Nakanishi), in such a way that the resulting non-minimal action, $\mathcal{S}_{nm}$, is still a solution to the master equation[3]. One proceeds by choosing a gauge-fixing fermion Ψ, much like (5.46), and eliminate the antifields by specifying the surface,

$$\Sigma_\Psi = \left\{ \Phi^*_A : \Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A} \right\}, \tag{B.5}$$

where using left or right derivatives is equivalent since $\varepsilon(\Psi) = 1$. The expressions for the antifields so obtained are substituted back into $\mathcal{S}_{nm}$, which yields the gauge-fixed action

$$\mathcal{S}_\Psi := \mathcal{S}|_{\Sigma_\Psi} = \mathcal{S}\left(\Phi^A, \frac{\partial \Psi}{\partial \Phi^A}\right). \tag{B.6}$$

The gauge-fixed action is invariant under the generalised BRST transformations restricted to Σ_Ψ, that is

$$\mathcal{Q}_\Psi \Phi^A = \left.\frac{\partial_l \mathcal{S}}{\partial \Phi^*_A}\right|_{\Sigma_\Psi}. \tag{B.7}$$

Both the gauge-fixed action $\mathcal{S}_\Psi$ and BRST transformations $\mathcal{Q}_\Psi$ so defined coincide with their counterparts in the usual BRST formalism, S and Q, for theories in which the classical BRST transformations are off-shell nilpotent, such as those in (5.35). On the other hand, theories with more complicated gauge algebras, which do not admit a BRST treatment, can only be dealt with in the framework of the field-antifield formalism. Next, we briefly revisit Yang-Mills theory to exemplify the main features of the above construction, before treating the (slightly) more complicated Kalb-Ramond 2-form field.

[1]Note that, in this Appendix, we will assume a right action of the BRST charge, instead of the left action assumed in the main text. This is to align with the usual conventions of the field-antifield formalism.

[2]Which cannot be $\Phi^*_A = 0$ as this, by virtue of the boundary condition to the master equation, returns the classical action, which as we know is no good at quantum level.

[3]Note that this extra step is not required in the full, $ISp(2)$ symmetric field-antifield formalism, as the minimal $ISp(2)$ set automatically includes all the fields which belong to the non-minimal sector in the only-BRST-symmetric version, see [130].

B.1 Yang-Mills, again

Yang-Mills theory is an example of gauge theory with a closed, irreducible algebra. Thus, we expect the BV formalism to produce, upon gauge-fixing, the same results as the Faddeev-Popov procedure.

We begin by defining the content of the theory: the configuration space comprises the fields $\Phi^A = \{A_\mu^A, c^A\}$ and corresponding antifields $\Phi_A^* = \{A_{\mu A}^*, c_A^*\}$. The minimal solution is given by

$$\mathcal{S} = \int \mathrm{d}^D x \, \left[-\frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} + A_{\mu A}^* (D^\mu c)^A - \frac{1}{2} c_A^* [c, c]^A \right] \tag{B.8}$$

Using (B.3), we compute at once the unfixed BRST transformations, which leave the above invariant, to be

$$\mathcal{Q} A_\mu^A = (D_\mu c)^A \qquad \mathcal{Q} A_{\mu A}^* = -(D^\nu F_{\nu\mu})_A - [A_\mu^*, c]_A \tag{B.9}$$

$$\mathcal{Q} c^A = -\tfrac{1}{2}[c, c]^A \qquad \mathcal{Q} c_A^* = -(D^\mu A_\mu^*)_A - [c^*, c]_A, \tag{B.10}$$

Since the Yang-Mills algebra is irreducible, we need only add one trivial pair $(\bar{c}^A, b^A)$, together with the respective antifields $(\bar{c}_A^*, b_A^*)$, to build the non-minimal solution

$$\mathcal{S}_{nm} = \mathcal{S} + \int \mathrm{d}^D x \; b^A \bar{c}_A^*, \tag{B.11}$$

from which we compute the BRST transformations of the non-minimal sector,

$$\mathcal{Q} \bar{c}^A = b^A \qquad \mathcal{Q} \bar{c}_A^* = 0 \tag{B.12}$$

$$\mathcal{Q} b^A = 0 \qquad \mathcal{Q} b_A^* = -\bar{c}_A^*. \tag{B.13}$$

Choosing the same gauge-fixing fermion as in (5.46),

$$\Psi = \int \bar{c}_A \left(\partial^\mu A_\mu^A - \frac{\xi}{2} b^A \right), \tag{B.14}$$

we may solve for the antifields making use of (B.5); we get,

$$A_{\mu A}^* = -\partial_\mu \bar{c}_A \qquad \bar{c}_A^* = \partial^\mu A_\mu^A - \frac{\xi}{2} b^A \tag{B.15}$$

$$c_A^* = 0 \qquad b_A^* = -\frac{\xi}{2} \bar{c}_A. \tag{B.16}$$

The gauge-fixed action is obtained by plugging these back into (B.11),

$$\mathcal{S}_\Psi = \int \mathrm{d}^D x \; -\frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} - \partial_\mu \bar{c}_A (D^\mu c)^A + b_A \left(\partial^\mu A_\mu^A - \frac{\xi}{2} b^A \right) \tag{B.17}$$

invariant under the gauge-fixed BRST transformations $\mathcal{Q}_\Psi = \mathcal{Q}|_{\Sigma_\Psi}$,

$$Q A_\mu^A = (D_\mu c)^A \qquad Q \bar{c}^A = b^A \tag{B.18}$$

$$Q c^A = \tfrac{1}{2}[c, c]^A \qquad Q b^A = 0. \tag{B.19}$$

Small sign adjustments are needed to obtain the form of the Lagrangian (5.34) and transformations (5.35) of Section 5.3.

B.2 Abelian 2-form

The original gauge-invariant action of the Kalb-Ramond field is

$$S_0 = \int -\frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} \tag{B.20}$$

The minimal solution to the classical master equation [135] is

$$S^{\min} = \int -\frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} + B^{*\mu\nu}(\partial_\mu d_\nu - \partial_\nu d_\mu) + d^{*\mu}\partial_\mu d \tag{B.21}$$

At this stage the space of fields is $\Phi^A = \{B_{\mu\nu}, d_\mu, d\}$ only, with respective antifields $\Phi^*_A = \{B^{*\mu\nu}, d^{*\mu}, d^*\}$. Using (B.3), one can compute the (gauge-unfixed) BRST transformations of the gauge-unfixed action $S^{\min}$:

$$\begin{aligned} \delta_B B_{\mu\nu} &= \partial_\mu d_\nu - \partial_\nu d_\mu & \delta_B B^{*\mu\nu} &= -\frac{1}{4}\partial_\rho H^{\rho\mu\nu} \\ \delta_B d_\mu &= \partial_\mu d & \delta_B d^*_\mu &= 2\partial^\nu B^*_{\mu\nu} \\ \delta_B d &= 0 & \delta_B d^* &= \partial^\mu d^*_\nu \end{aligned} \tag{B.22}$$

In order to make contact with BRST, use path integral methods and compute correlations functions in standard perturbation theory, we need to "gauge-fix" the field-antifield action. Schematically, this is done by a two-step modification of the action

$$S^{\min} \to S^{\text{non-min}} \to S^{\text{non-min}}_\psi \tag{B.23}$$

For this theory, one needs to introduce 3 trivial pairs, $(\overline{d}_\mu, b^\mu)$, $(\overline{d}, \overline{b}_{(d)})$ and $(\eta, b_{(d)})$. Using elements of the trivial pairs (or their antifields), construct the *non-minimal* solution

$$S^{\text{non-min}} = \int -\frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} + B^{*\mu\nu}(\partial_\mu d_\nu - \partial_\nu d_\mu) + d^{*\mu}\partial_\mu d + \underbrace{\overline{d}^*_\mu b^\mu + \overline{d}^* \overline{b}_{(d)} + \eta^* b_{(d)}}_{\text{non-minimal sector}} \tag{B.24}$$

At this stage the field space is $\Phi^A = \{B_{\mu\nu}, d_\mu, d, \overline{d}_\mu, b_\mu, \overline{d}, \overline{b}_{(d)}, \eta, b_{(d)}\}$ as well as the antifields. So we can compute the (unfixed) BRST transformations of the extra fields and antifields, which leave the extended action $S^{\text{non-min}}$ invariant:

$$\delta_B \overline{d}_\mu = b_\mu \qquad \delta_B b^*_\mu = -\overline{d}^*_\mu \tag{B.25}$$

$$\delta_B \overline{d} = \overline{b}_{(d)} \qquad \delta_B \overline{b}^*_{(d)} = \overline{d}^* \tag{B.26}$$

$$\delta_B \eta = b_{(d)} \qquad \delta_B b^*_{(d)} = \eta^* \tag{B.27}$$

$$\delta_B(b_\mu, b_{(d)}, \overline{b}_{(d)}) = 0 \qquad \delta_B(\overline{d}^*_\mu, \overline{d}^*, \eta^*) = 0 \tag{B.28}$$

There are (an infinite number of) gauge-fixing fermions. Two appropriate choices [135] are

$$\Psi_1 = \int \overline{d}^\mu \partial^\nu B_{\nu\mu} + \overline{d}\partial^\mu d_\mu + \overline{d}^\mu \partial_\mu \eta \tag{B.29}$$

$$\Psi_2 = \int \overline{d}^\mu (\partial^\nu B_{\nu\mu} + \alpha b_\mu) + \overline{d}(\partial^\mu d_\mu + \beta \overline{b}_{(d)}) + \eta(-\partial^\mu \overline{d}_\mu + \gamma b_{(d)}) \tag{B.30}$$

where the second corresponds to performing a Gaussian average. We can now eliminate the antifields[4] through $\Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A}$. The antifields for Ψ_1 (left) or Ψ_2 (right) are

$$\begin{aligned}
B^{*\mu\nu} &= -\frac{1}{2}(\partial^\mu \overline{d}^\nu - \partial^\nu \overline{d}^\mu) & B^{*\mu\nu} &= -\frac{1}{2}(\partial^\mu \overline{d}^\nu - \partial^\nu \overline{d}^\mu) \\
d^{*\mu} &= -\partial^\mu \overline{d} & d^{*\mu} &= -\partial^\mu \overline{d} \\
\overline{d}^{*\mu} &= \partial^\nu B_{\nu\mu} + \partial_\mu \eta & \overline{d}^{*\mu} &= \partial^\nu B_{\nu\mu} + \partial_\mu \eta + \alpha b_\mu \\
\overline{d}^* &= \partial^\mu d_\mu & \overline{d}^* &= \partial^\mu d_\mu + \beta \overline{b}_{(d)} \\
\eta^* &= -\partial^\mu \overline{d}_\mu & \eta^* &= -\partial^\mu \overline{d}_\mu + \gamma b_{(d)}
\end{aligned} \tag{B.31}$$

as well as expressions for $(b^*_\mu, \overline{b}^*_{(d)}, b^*_{(d)})$ which we can ignore here since they do not appear in $S^{\text{non-min}}$. After eliminating the antifields with Ψ_1, the *gauge-fixed* action reads

$$\begin{aligned}
S^{\text{non-min}}_{\Psi_1} = \int &-\frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2}(\partial^\mu \overline{d}^\nu - \partial^\nu \overline{d}^\mu)(\partial_\mu d_\nu - \partial_\nu d_\mu) - \partial^\mu \overline{d} \partial_\mu d \\
&+ (\partial^\nu B_{\nu\mu} + \partial_\mu \eta) b^\mu + (\partial^\mu d_\mu) \overline{b}_{(d)} - (\partial^\mu \overline{d}_\mu) b_{(d)}
\end{aligned} \tag{B.32}$$

while Ψ_2 yields

$$\begin{aligned}
S^{\text{non-min}}_{\Psi_1} = \int &-\frac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{2}(\partial^\mu \overline{d}^\nu - \partial^\nu \overline{d}^\mu)(\partial_\mu d_\nu - \partial_\nu d_\mu) - \partial^\mu \overline{d} \partial_\mu d \\
&+ (\partial^\nu B_{\nu\mu} + \partial_\mu \eta + \alpha b_\mu) b^\mu + (\partial^\mu d_\mu + \beta b_{(d)}) \overline{b}_{(d)} - (\partial^\mu \overline{d}_\mu + \gamma \overline{b}_{(d)}) b_{(d)}
\end{aligned} \tag{B.33}$$

Finally, the full set of gauge-fixed BRST transformations for the second, more general, case (B.33) read, for minimal fields (left), antighosts (centre) and auxiliaries (right),

$$\begin{aligned}
\delta_{B_\Psi} B_{\mu\nu} &= \partial_\mu d_\nu - \partial_\nu d_\mu & \delta_{B_\Psi} \overline{d}_\mu &= b_\mu & \delta_{B_\Psi} b_\mu &= 0 \\
\delta_{B_\Psi} d_\mu &= \partial_\mu d & \delta_{B_\Psi} \overline{d} &= b & \delta_{B_\Psi} b &= 0 \\
\delta_{B_\Psi} d &= 0 & \delta_{B_\Psi} \eta &= \pi & \delta_{B_\Psi} \pi &= 0
\end{aligned} \tag{B.34}$$

Again, small sign adjustments are needed to obtain the form of the transformations given in (5.62), compatible with a BRST charge acting from the left rather than on the right.

[4] Using left or right functional derivatives is equivalent in this case, since $\epsilon(\Psi) = 1$.

Appendix C

Computing the dictionaries

C.1 Kalb-Ramond 2-form

Consider the most general ansatze for the fields,

$$B_{\mu\nu} = A_{[\mu} \circ \tilde{A}_{\nu]} + \frac{\alpha_1}{\Box}\left(\partial A \circ \partial_{[\mu}\tilde{A}_{\nu]} - \partial_{[\mu}A_{\nu]} \circ \partial\tilde{A}\right) \tag{C.1}$$

$$d^\alpha_\mu = \beta_1\left(c^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{c}^\alpha\right) + \beta_2\frac{\partial_\mu}{\Box}\left(c^\alpha \circ \partial\tilde{A} - \partial A \circ \tilde{c}^\alpha\right) \tag{C.2}$$

$$d = \gamma_1\ (c \circ \tilde{c}) \tag{C.3}$$

$$\overline{d} = \gamma_2\ (\overline{c} \circ \tilde{\overline{c}}) \tag{C.4}$$

$$\eta = \gamma_3\ (c \circ \tilde{\overline{c}} - \overline{c} \circ \tilde{c}) \tag{C.5}$$

and their BRST/antiBRST transformations,

$$\begin{aligned}
QB_{\mu\nu} &= 2\partial_{[\mu}d_{\nu]} & \overline{Q}B_{\mu\nu} &= 2\partial_{[\mu}\overline{d}_{\nu]} \\
Qd_\mu &= \partial_\mu d & \overline{Q}d_\mu &= -\frac{1}{\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu\eta\right) + \frac{1}{\xi_{(d)}}\partial_\mu\eta \\
Q\overline{d}_\mu &= \frac{1}{\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu\eta\right) & \overline{Q}\overline{d}_\mu &= \partial_\mu\overline{d} \\
Qd &= 0 & \overline{Q}d &= -\frac{1}{\xi_{(d)}}\partial^\mu d_\mu \\
Q\overline{d} &= \frac{1}{\xi_{(d)}}\partial^\mu\overline{d}_\mu & \overline{Q}\overline{d} &= 0 \\
Q\eta &= \frac{m_{(d)}}{\xi_{(d)}}\partial^\mu d_\mu & \overline{Q}\eta &= -\partial^\mu\overline{d}_\mu.
\end{aligned}$$

We will attempt to fix all arbitrary parameters in the dictionaries by reproducing these using the variations of the underlying Yang-Mills factors,

$$\begin{aligned}
QA_\mu &= \partial_\mu c & \overline{Q}A_\mu &= \partial_\mu\overline{c} \\
Qc &= 0 & \overline{Q}\overline{c} &= 0 \\
Q\overline{c} &= \tfrac{1}{\xi}\partial^\mu A_\mu & \overline{Q}c &= -\tfrac{1}{\xi}\partial^\mu A_\mu
\end{aligned} \tag{C.6}$$

assuming a left action of the BRST charges,

$$Q(a \circ \tilde{b}) = (Qa) \circ \tilde{b} + (-1)^{\varepsilon(a)} a \circ (Q\tilde{b}). \tag{C.7}$$

First in line are

$$\begin{aligned} Q^\alpha B_{\mu\nu} &= \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right) \\ &\quad + \frac{\alpha_1}{\Box} \left(\Box c^\alpha \circ \partial_{[\mu} \tilde{A}_{\nu]} + \partial A \circ \partial_{[\mu} \partial_{\nu]} \tilde{c}^\alpha - \partial_{[\mu} \partial_{\nu]} c^\alpha \circ \partial \tilde{A} - \partial_{[\mu} A_{\nu]} \circ \Box \tilde{c}^\alpha \right) \\ &= (1 + \alpha_1)\, \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right), \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} \partial_{[\mu} d^\alpha_{\nu]} &= \beta_1 \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right) + \beta_2 \frac{\partial_{[\mu} \partial_{\nu]}}{\Box} \left(c^\alpha \circ \partial \tilde{A} - \partial A \circ \tilde{c}^\alpha \right) \\ &= \beta_1 \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right), \end{aligned} \tag{C.9}$$

which imply

$$\begin{aligned} 0 &= Q^\alpha B_{\mu\nu} - 2\partial_{[\mu} d^\alpha_{\nu]} \\ &= (1 + \alpha_1 - 2\beta_1)\, \partial_{[\mu} \left(c^\alpha \circ \tilde{A}_{\nu]} - A_{\nu]} \circ \tilde{c}^\alpha \right). \end{aligned} \tag{C.10}$$

This fixes

$$2\beta_1 = 1 + \alpha_1. \tag{C.11}$$

Then,

$$\begin{aligned} 0 &= Q d_\mu - \partial_\mu d \\ &= \beta_1 \left(-c \circ \partial_\mu \tilde{c} - \partial_\mu c \circ \tilde{c} \right) + \beta_2 \frac{\partial_\mu}{\Box} \left(-c \circ \Box \tilde{c} - \Box c \circ \tilde{c} \right) - \gamma_1 \partial_\mu \left(c \circ \tilde{c} \right) \\ &= -\left(2(\beta_1 + \beta_2) + \gamma_1 \right) \partial_\mu (c \circ \tilde{c}) \end{aligned} \tag{C.12}$$

which implies

$$\gamma_1 = -2(\beta_1 + \beta_2). \tag{C.13}$$

The BRST transform of d is trivial,

$$Qd = \gamma_1 Q(c \circ \tilde{c}) = 0, \tag{C.14}$$

owing to $Qc = 0$ on the Yang-Mills side, so γ_1 does not pick up any further constraints. Then, compute

$$Q\overline{d}_\mu = \beta_1 \left(\frac{1}{\xi} \left(\partial A \circ \tilde{A}_\mu - A_\mu \circ \partial \tilde{A} \right) - \partial_\mu \left(c \circ \tilde{\overline{c}} + \overline{c} \circ \tilde{c} \right) \right) - \beta_2 \partial_\mu \left(c \circ \tilde{\overline{c}} + \overline{c} \circ \tilde{c} \right) \tag{C.15}$$

as well as the quantity,

$$
\begin{aligned}
\partial^\nu B_{\nu\mu} + \partial_\mu \eta &= \frac{1}{2}\left(\partial A \circ \tilde{A}_\mu - A_\mu \circ \partial \tilde{A}\right) + \gamma_3 \partial_\mu \left(c \circ \tilde{\bar{c}} + \bar{c} \circ \tilde{c}\right) \\
&\quad + \frac{\alpha_1}{2\Box}\left(\partial A \circ \Box \tilde{A}_\mu - \cancel{\partial A \circ \partial_\mu \partial \tilde{A}} - \Box A_\mu \circ \partial \tilde{A} + \cancel{\partial_\mu \partial A \circ \partial \tilde{A}}\right) \\
&= \left(\frac{1+\alpha_1}{2}\right)\left(\partial A \circ \tilde{A}_\mu - A_\mu \circ \partial \tilde{A}\right) + \gamma_3 \partial_\mu \left(c \circ \tilde{\bar{c}} + \bar{c} \circ \tilde{c}\right) \\
&= \beta_1 \left(\partial A \circ \tilde{A}_\mu - A_\mu \circ \partial \tilde{A}\right) + \gamma_3 \partial_\mu \left(c \circ \tilde{\bar{c}} + \bar{c} \circ \tilde{c}\right),
\end{aligned}
\tag{C.16}
$$

corresponding to the gauge-fixing functional which fixes the 2-form gauge freedom. The two crossed-out terms cancel against each other after moving the derivative, while the remaining d'Alembertians are factorised out of the convolutions and cancel with the Green function. The last line follows using (C.11). Then,

$$
\begin{aligned}
0 &= Q\bar{d}_\mu - \frac{1}{\xi_{(B)}}\left(\partial^\nu B_{\nu\mu} + \partial_\mu \eta\right) \\
&= \beta_1 \left(\frac{1}{\xi} - \frac{1}{\xi_{(B)}}\right)\left(\partial A \circ \tilde{A}_\mu - A_\mu \circ \partial \tilde{A}\right) - \left(\beta_1 + \beta_2 - \frac{\gamma_3}{\xi_{(B)}}\right)\partial_\mu \left(c \circ \tilde{\bar{c}} + \bar{c} \circ \tilde{c}\right).
\end{aligned}
\tag{C.17}
$$

The second term implies the constraint

$$
\gamma_3 = -\xi(\beta_1 + \beta_2), \tag{C.18}
$$

whilst the first term yields the first of the relations between the Yang-Mills and 2-form gauge-fixing parameters,

$$
\xi_{(B)} = \xi. \tag{C.19}
$$

Next, we directly compute

$$
\begin{aligned}
0 &= Q\bar{d} - \frac{1}{\xi_{(d)}}\partial^\mu \bar{d}_\mu \\
&= \gamma_2 \left(\frac{1}{\xi}\partial A \circ \tilde{\bar{c}} - \bar{c} \circ \frac{1}{\xi}\partial \tilde{A}\right) - \frac{1}{\xi_{(d)}}\left(\beta_1 \left(\bar{c} \circ \partial \tilde{A} - \partial A \circ \tilde{\bar{c}}\right) + \beta_2 \frac{\Box}{\Box}\left(\bar{c} \circ \partial \tilde{A} - \partial A \circ \tilde{\bar{c}}\right)\right) \\
&= \left(\frac{\gamma_2}{\xi} + \frac{\beta_1 + \beta_2}{\xi_{(d)}}\right)\left(\partial A \circ \tilde{\bar{c}} - \bar{c} \circ \partial \tilde{A}\right),
\end{aligned}
\tag{C.20}
$$

which results in

$$
\gamma_2 = -\frac{\xi}{\xi_{(d)}}(\beta_1 + \beta_2). \tag{C.21}
$$

The calculation for the ghost-number zero ghost η is similar:

$$
\begin{aligned}
0 &= Q\eta - \frac{m_{(d)}}{\xi_{(d)}}\partial^\mu d_\mu \\
&= \gamma_3\left(-c\circ\frac{1}{\xi}\partial A + \frac{1}{\xi}\partial A\circ\tilde{c}\right) - \frac{m_{(d)}}{\xi_{(d)}}\left(\beta_1\left(c\circ\partial\tilde{A} - \partial A\circ\tilde{c}\right) + \beta_2\frac{\Box}{\Box}\left(c\circ\partial\tilde{A} - \partial A\circ\tilde{c}\right)\right) \\
&= \left(\frac{\gamma_3}{\xi} + \frac{m_{(d)}}{\xi_{(d)}}(\beta_1+\beta_2)\right)\left(\partial A\circ\tilde{c} - c\circ\partial\tilde{A}\right).
\end{aligned}
\tag{C.22}
$$

Thus,

$$
\gamma_3 = -m_{(d)}\frac{\xi}{\xi_{(d)}}(\beta_1+\beta_2), \tag{C.23}
$$

but given (C.18), we get the important constraint

$$
m_{(d)} = \xi_{(d)}, \tag{C.24}
$$

whose consequences were discussed in the main text. Demanding that the whole set of antiBRST transformations is mapped by the dictionary induces only one additional constraint. Consider

$$
\begin{aligned}
0 &= \overline{Q}\bar{d}_\mu - \partial_\mu\overline{d} \\
&= \beta_1\left(-\overline{c}\circ\partial_\mu\tilde{\overline{c}} - \partial_\mu\overline{c}\circ\tilde{\overline{c}}\right) + \beta_2\frac{\partial_\mu}{\Box}\left(-\overline{c}\circ\Box\tilde{\overline{c}} - \Box\overline{c}\circ\tilde{\overline{c}}\right) - \gamma_2\partial_\mu(\overline{c}\circ\tilde{\overline{c}}) \\
&= -2(\beta_1+\beta_2)\partial_\mu(\overline{c}\circ\tilde{\overline{c}}) - \gamma_2\partial_\mu(\overline{c}\circ\tilde{\overline{c}}) \\
&= -\left(2(\beta_1+\beta_2)+\gamma_2\right)\partial_\mu(\overline{c}\circ\tilde{\overline{c}}),
\end{aligned}
\tag{C.25}
$$

which imposes

$$
\gamma_2 = -2(\beta_1+\beta_2). \tag{C.26}
$$

Because γ_2 already satisfies (C.21), this constrains the ξ's as

$$
\xi_{(d)} = \frac{\xi}{2}. \tag{C.27}
$$

Not only the remaining relations impose no new constraints; they provide non trivial consistency checks in that they reproduce, sometimes after quite a few lines of algebra, exactly the same constraints found above. Thus, so far, we have shown that the ansatze in (C.1) are general enough so as to allow one to reproduce the BRST/antiBRST transformations and algebra of the 2-form theory. In fact, they do so with some freedom to spare: a quick look shows that not all parameters have been fixed uniquely in terms of purely Yang-Mills quantities (as opposed to other parameters appearing in the dictionary). We show below that all these are fixed uniquely once we require that the dictionary maps the equations of motion of Yang-Mills to those of the 2-form theory. Recall, these are

$$\Box B_{\mu\nu} + \xi'_{(B)}\partial^\rho\partial_{[\mu}B_{\nu]\rho} - j_{\mu\nu}(B) = 0, \qquad \xi'_{(B)} := 2\frac{\xi_{(B)}+2}{\xi_{(B)}} \tag{C.28}$$

$$\Box d^\alpha_\mu - j^\alpha_\mu(d) = 0, \tag{C.29}$$

$$\Box d^i - j(d^i) = 0. \tag{C.30}$$

We wish to reproduce these using their Yang-Mills counterparts, namely

$$\Box A_\mu - \left(\frac{\xi+1}{\xi}\right)\partial_\mu\partial^\rho A_\rho = j_\mu, \tag{C.31}$$

$$\Box c^\alpha = j^\alpha, \tag{C.32}$$

and

$$\partial^\rho j_\rho = -\frac{1}{\xi}\Box\partial^\rho A_\rho. \tag{C.33}$$

Let us begin with the physical field, the 2-form $B_{\mu\nu}$. We hit the dictionary with the appropriate differential operators in order to form $\Box B_{\mu\nu} + \xi'_{(B)}\partial^\rho\partial_{[\mu}B_{\nu]\rho}$. The result is summarised in Table C.1.

$B_{\mu\nu}$	$\Box B_{\mu\nu}$	$\xi'_{(B)}\partial^\rho\partial_{[\mu}B_{\nu]\rho}$
$A_{[\mu}\circ\tilde{A}_{\nu]}$	$\Box\left(A_{[\mu}\circ\tilde{A}_{\nu]}\right)$	$\frac{\xi'_{(B)}}{4}\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right)$
$\frac{\alpha_1}{\Box}\left(\partial A\circ\partial_{[\mu}\tilde{A}_{\nu]} - \partial_{[\mu}A_{\nu]}\circ\partial\tilde{A}\right)$	$-\frac{\alpha_1}{2}\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right)$	$\frac{\xi'_{(B)}\alpha_1}{4}\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right)$

Table C.1: Yang-Mills factorisation of the equations of motion for $B_{\mu\nu}$.

Let us work out just one entry, to illustrate how this works: consider

$$\begin{aligned}&\xi'_{(B)}\partial^\rho\partial_{[\mu}\left[\frac{\alpha_1}{2\Box}\left(\partial A\circ\partial_{[\nu}\tilde{A}_{\rho} - \partial A\circ\partial_\rho\tilde{A}_{\nu]}\right) - \frac{\alpha_1}{2\Box}\left(\partial_{\nu]}A_\rho\circ\partial\tilde{A} - \partial_\rho A_{\nu]}\circ\partial\tilde{A}\right)\right]\\ &= \frac{\xi'_{(B)}\alpha_1}{2\Box}\left(\cancel{\partial A\circ\partial_{[\mu}\partial_{\nu]}\partial\tilde{A}} - \partial A\circ\Box\partial_{[\mu}\tilde{A}_{\nu]} - \cancel{\partial_{[\mu}\partial_{\nu]}\partial A\circ\partial\tilde{A}} + \Box\partial_{[\mu}A_{\nu]}\circ\partial\tilde{A}\right)\\ &= \frac{\xi'_{(B)}\alpha_1}{4}\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right),\end{aligned} \tag{C.34}$$

which reproduces the bottom right entry of the table. At this point, recalling that the ansatz for the effective source of the 2-form is

$$j_{\mu\nu}(B) = \frac{\hat{\alpha}}{\Box}j_{[\mu}\circ\tilde{j}_{\nu]}, \tag{C.35}$$

we may write

$$\begin{aligned} 0 &= \Box B_{\mu\nu} + \xi'_{(B)}\partial^\rho\partial_{[\mu}B_{\nu]\rho} - j_{\mu\nu}(B) \\ &= \Box\left(A_{[\mu}\circ\tilde{A}_{\nu]}\right) + \frac{1}{4}\left(\xi'_{(B)}(1+\alpha_1) - 2\alpha_1\right)\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right) - \frac{\hat{\alpha}}{\Box}j_{[\mu}\circ\tilde{j}_{\nu]}. \end{aligned} \tag{C.36}$$

As it was explained in the main text, this form for (the factorisation of) the equations for $B_{\mu\nu}$ is naive: it does not account for the fact that the basis

$$\left\{\Box(A_{[\mu}\circ\tilde{A}_{\nu]}),\ F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu},\ \frac{1}{\Box}j_{[\mu}\circ\tilde{j}_{\nu]}\right\} \tag{C.37}$$

is not *minimal*. Said otherwise, it would lead to wrong conclusions, that is fixing the first term to zero, because one would be failing to notice that

$$\begin{aligned} \frac{1}{\Box}j_{[\mu}\circ\tilde{j}_{\nu]} &= \frac{1}{\Box}\left(\Box A_{[\mu} - \frac{\xi+1}{\xi}\partial_{[\mu}\partial A\right)\circ\left(\Box\tilde{A}_{\nu]} - \frac{\xi+1}{\xi}\partial_{\nu]}\partial\tilde{A}\right) \\ &= \Box(A_{[\mu}\circ\tilde{A}_{\nu]}) + \frac{\xi+1}{\xi}\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right), \end{aligned} \tag{C.38}$$

which implies that one possible minimal basis is in fact $\{\Box(A_{[\mu}A_{\nu]}),\ F_{\mu\nu}\partial A - \partial A F_{\mu\nu}\}$. Then the equations of motion in the (Yang-Mills)×(Yang-Mills) "basis" read

$$(1-\hat{\alpha})\Box\left(A_{[\mu}\circ\tilde{A}_{\nu]}\right) + \frac{1}{4}\left(\xi'_{(B)}(1+\alpha_1) - 2\left(\alpha_1 + \frac{\xi+1}{\xi}\hat{\alpha}\right)\right)\left(F_{\mu\nu}\circ\partial\tilde{A} - \partial A\circ\tilde{F}_{\mu\nu}\right) = 0. \tag{C.39}$$

This implies the constraints $\hat{\alpha} = 1$ and $\alpha_1 = -1/2$. Finally, we ought to impose the equations of motion of the first and second generation ghosts: recall that their general form is

$$\Box d^\alpha_\mu = \left(\frac{m_{(d)}}{\xi_{(d)}} - 1\right)\partial_\mu\partial^\rho d^\alpha_\rho + j^\alpha_\mu(d) \tag{C.40}$$

but that, given the constraint (C.24), the first term on the RHS vanishes and they reduce to (C.29) and (C.30). Then, recalling the ansatz

$$j^\alpha_\mu(d) = \frac{\hat{\beta}}{\Box}\left(j^\alpha\circ\tilde{j}_\mu - j_\mu\circ\tilde{j}^\alpha\right), \tag{C.41}$$

we have

$$\begin{aligned} 0 &= \Box d^\alpha_\mu - j^\alpha_\mu(d) \\ &= \beta_1\left(j^\alpha\circ\tilde{A}_\mu - A_\mu\circ\tilde{j}^\alpha\right) + \beta_2\partial_\mu\left(c^\alpha\circ\partial\tilde{A} - \partial A\circ\tilde{c}^\alpha\right) - \frac{\hat{\beta}}{\Box}\left(j^\alpha\circ\tilde{j}_\mu - j_\mu\circ\tilde{j}^\alpha\right). \end{aligned} \tag{C.42}$$

This basis is once again non-minimal, so let us use the identity

$$j^\alpha\circ\tilde{j}_\mu - j_\mu\circ\tilde{j}^\alpha = \Box\left(j^\alpha\circ\tilde{A}_\mu - A_\mu\circ\tilde{j}^\alpha\right) - \frac{\xi+1}{\xi}\Box\partial_\mu\left(c^\alpha\circ\partial\tilde{A} - \partial A\circ\tilde{c}^\alpha\right) \tag{C.43}$$

to rewrite it as

$$\begin{aligned}0 &= \Box d^\alpha_\mu - j^\alpha_\mu(d) \\ &= \left(\beta_1 - \hat{\beta}\right)\left(j^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{j}^\alpha\right) + \left(\beta_2 + \frac{\xi+1}{\xi}\hat{\beta}\right)\partial_\mu\left(c^\alpha \circ \partial\tilde{A} - \partial A \circ \tilde{c}^\alpha\right).\end{aligned} \tag{C.44}$$

This basis is minimal, since attempting to eliminate one of the two elements for a multiple of the other yields a third element as well, namely

$$j^\alpha \circ \tilde{A}_\mu - A_\mu \circ \tilde{j}^\alpha = \Box c^\alpha \circ \tilde{A}_\mu - A_\mu \circ \Box\tilde{c}^\alpha \tag{C.45}$$

$$= c^\alpha \circ \left(\frac{\xi+1}{\xi}\partial_\mu\partial\tilde{A} + \tilde{j}_\mu\right) - \left(\frac{\xi+1}{\xi}\partial_\mu\partial A\right) \circ \tilde{c}^\alpha \tag{C.46}$$

$$= \frac{\xi+1}{\xi}\partial_\mu\left(c^\alpha \circ \partial\tilde{A} - \partial A \circ \tilde{c}^\alpha\right) + \left(c^\alpha \circ \tilde{j}_\mu - j_\mu \circ \tilde{c}^\alpha\right). \tag{C.47}$$

Therefore, we can read off the constraints: together, the first and the second term imply that $\hat{\beta} = \beta_1 = -\frac{\xi}{\xi+1}\beta_2$. One may check that the equations of motion for the second generation ghosts very trivially set $\hat{\gamma}_i = \gamma_i$, which explains a posteriori the notation chosen. This exhausts the relations to be checked: combining all the constraints obtained above, one sees that they are all constrained uniquely in terms of ξ, as it is shown in Table C.2.

Field	Parameters	Source	Parameters
$B_{\mu\nu}$	$\alpha_1 = -\frac{1}{2}$	$j_{\mu\nu}(B)$	$\hat{\alpha} = 1$
d^α_μ	$\beta_1 = \frac{1}{4}, \quad \beta_2 = -\frac{\xi+1}{4\xi}$	$j^\alpha_\mu(d)$	$\hat{\beta} = \frac{1}{4}$
$d, \overline{d}, \eta$	$\gamma_1 = \gamma_2 = \frac{1}{2\xi}, \quad \gamma_3 = \frac{1}{4}$	$j(d), j(\overline{d}), j(\eta)$	$\hat{\gamma}_i = \gamma_i$

Table C.2: Parameters of the dictionaries of the Kalb-Ramond sector, completely fixed in terms of ξ.

C.2 Graviton's equations of motion

First, let us briefly compute the equations of motion. From the full BRST-fixed Lagrangian for linearised gravity, one obtains

$$-\frac{1}{2}R^{lin}_{\mu\nu} = \partial_{(\mu}b_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho b_\rho, \tag{C.48}$$

where the Lautrup-Nakanishi field looks like a source term for the linearised Ricci tensor, not unlike in Yang-Mills, where it "sources" the field strength, namely $\partial^\nu F_{\nu\mu} = -\partial_\mu b$. Using the algebraic equation for b_μ which follows from the Lagrangian, that is

$$b_\mu = \frac{1}{\xi_{(h)}}\left(\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\right), \tag{C.49}$$

and using the expression for the Ricci tensor given in (3.2), we may rewrite the above as

$$\Box h_{\mu\nu} = \left(\frac{\xi+2}{\xi}\right)\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma}\right) + \left(\frac{\xi+1}{\xi}\right)\eta_{\mu\nu}\Box h \tag{C.50}$$

where, as for the Yang-Mills case, the new terms originating from the Lautrup-Nakanishi field or, alternatively, from the gauge-fixing to which it corresponds via its equation, contribute to as to form a non-degenerate kinetic kernel. One may simplify (C.51) by taking its trace,

$$\Box h = \left(\frac{\xi+2}{\xi+1}\right)\partial^\rho\partial^\sigma h_{\rho\sigma} \tag{C.51}$$

and substituting it back in: this has the effect of showing that the last term cancels exactly the third[1] so as to get

$$\Box h_{\mu\nu} - \left(\frac{\xi+2}{\xi}\right)\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h\right) = 0. \tag{C.52}$$

Next, recall the general ansatz for the graviton,

$$\begin{aligned} h_{\mu\nu} = A_{(\mu} \circ \tilde{A}_{\nu)} &+ \frac{\partial_\mu\partial_\nu}{\Box}\left(a_1 A^\rho \circ \tilde{A}_\rho + a_2 c^\alpha \circ \tilde{c}_\alpha\right) + \frac{a_3}{\Box}\left(\partial A \circ \partial_{(\mu}\tilde{A}_{\nu)} + \partial_{(\mu}A_{\nu)} \circ \partial\tilde{A}\right) \\ &+ \frac{a_4}{\Box^2}\partial_\mu\partial_\nu\partial A \circ \partial\tilde{A} + \eta_{\mu\nu}\left(b_1 A^\rho \circ \tilde{A}_\rho + b_2 c^\alpha \circ \tilde{c}_\alpha + \frac{b_3}{\Box}\partial A \circ \partial\tilde{A}\right) \end{aligned} \tag{C.53}$$

Hitting this with the various differential operators appearing in the equations of motion, one obtains the results of Table C.3. As it is apparent in the table, the $\eta_{\mu\nu}$ pieces, despite playing a completely separate role in the calculation of the BRST transformations, are allowed to mix with the other terms under the action of the Yang-Mills equations of motion: considering the last three rows of Table C.3, observe that the entries in the last two columns "lose" their $\eta_{\mu\nu}$ structure, owing to the index contractions imposed by the form of equations of motion. At this point, we may introduce the effective source for the graviton, so that we now have to reproduce

$$\Box h_{\mu\nu} - \left(\frac{\xi+2}{\xi}\right)\left(2\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \partial_\mu\partial_\nu h\right) - j_{\mu\nu}(h) = 0. \tag{C.54}$$

[1]It does not matter in which dimension $D \neq 2$ we work, and thus we take the trace: the factor of D cancels out. In two dimensions, it turns out that $\Box h$ is not related to $\partial^\rho\partial^\sigma h_{\rho\sigma}$ at all.

where the ansatz for the source is given by

$$j_{\mu\nu}(h) = \hat{a}_0 \frac{1}{\Box} j_{(\mu} \circ \tilde{j}_{\nu)} + \frac{\partial_\mu \partial_\nu}{\Box^2} \left(\hat{a}_1 j^\rho \circ \tilde{j}_\rho + \hat{a}_2 j^\alpha \circ \tilde{j}_\alpha \right) + \eta_{\mu\nu} \left(\frac{\hat{b}_1}{\Box} j^\rho \circ \tilde{j}_\rho + \frac{\hat{b}_2}{\Box} j^\alpha \circ \tilde{j}_\alpha \right) \tag{C.55}$$

It turns out that minimal "bases" for this calculation are 8-dimensional. We choose that consisting of the tensor structures

$$B = \left\{ \frac{\partial_\mu \partial_\nu}{\Box} \partial A \circ \partial \tilde{A},\ \frac{1}{\Box} \left(\partial A \circ \partial_{(\mu} j_{\nu)} + \partial_{(\mu} j_{\nu)} \circ \partial \tilde{A} \right),\ \frac{\partial_\mu \partial_\nu}{\Box^2} j^\rho \circ \tilde{j}_\rho, \right.$$
$$\left. \frac{\partial_\mu \partial_\nu}{\Box^2} j^\alpha \circ \tilde{j}_\alpha,\ \eta_{\mu\nu} \partial A \circ \partial \tilde{A},\ \frac{1}{\Box} j_{(\mu} \circ \tilde{j}_{\nu)},\ \frac{\eta_{\mu\nu}}{\Box} j^\rho \circ \tilde{j}_\rho,\ \frac{\eta_{\mu\nu}}{\Box} j^\alpha \circ \tilde{j}_\alpha \right\} \tag{C.56}$$

even though any 8-dimensional basis related to this one by the action of the Yang-Mills equations would do. Setting the coefficients of each of these "basis elements" to vanish yields a set of constraints on the parameters which, when combined with the remanining constraints coming from imposing BRST/antiBRST transformations, produces the results of Section 6.3.3.

$h_{\mu\nu}$	$\Box h_{\mu\nu}$	$-2\xi'_{(h)}\partial^\rho\partial_{(\mu}h_{\nu)\rho}$	$\xi'_{(h)}\partial_\mu\partial_\nu h$
$A_{(\mu}\circ\tilde{A}_{\nu)}$	$\Box\left(A_{(\mu}\circ\tilde{A}_{\nu)}\right)$	$-\xi'_{(h)}\left(\partial_{(\mu}A_{\nu)}\circ\partial\tilde{A}+\partial A\circ\partial_{(\mu}\tilde{A}_{\nu)}\right)$	$\xi'_{(h)}\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$
$a_1\frac{\partial_\mu\partial_\nu}{\Box}\left(A^\rho\circ\tilde{A}_\rho\right)$	$a_1\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$	$-2\xi'_{(h)}a_1\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$	$\xi'_{(h)}a_1\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$
$a_2\frac{\partial_\mu\partial_\nu}{\Box}(c^\alpha\circ\tilde{c}_\alpha)$	$a_2\partial_\mu\partial_\nu(c^\alpha\circ\tilde{c}_\alpha)$	$-2\xi'_{(h)}a_2\partial_\mu\partial_\nu(c^\alpha\circ\tilde{c}_\alpha)$	$\xi'_{(h)}a_2\partial_\mu\partial_\nu(c^\alpha\circ\tilde{c}_\alpha)$
$a_3\frac{1}{\Box}\left(\partial A\circ\partial_{(\mu}\tilde{A}_{\nu)}+\partial_{(\mu}A_{\nu)}\circ\partial\tilde{A}\right)$	$a_3\left(\partial A\circ\partial_{(\mu}\tilde{A}_{\nu)}+\partial_{(\mu}A_{\nu)}\circ\partial\tilde{A}\right)$	$-2\xi'_{(h)}a_3\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}-a_3\xi'_{(h)}\left(\partial A\circ\partial_{(\mu}\tilde{A}_{\nu)}+\partial_{(\mu}A_{\nu)}\circ\partial\tilde{A}\right)$	$2\xi'_{(h)}a_3\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$
$a_4\frac{\partial_\mu\partial_\nu}{\Box^2}\partial A\circ\partial\tilde{A}$	$a_4\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$	$-2\xi'_{(h)}a_4\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$	$\xi'_{(h)}a_4\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$
$b_1\eta_{\mu\nu}\left(A^\rho\circ\tilde{A}_\rho\right)$	$b_1\eta_{\mu\nu}\Box\left(A^\rho\circ\tilde{A}_\rho\right)$	$-2\xi'_{(h)}b_1\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$	$D\xi'_{(h)}b_1\partial_\mu\partial_\nu(A^\rho\circ\tilde{A}_\rho)$
$b_2\eta_{\mu\nu}(c^\alpha\circ\tilde{c}_\alpha)$	$b_2\eta_{\mu\nu}\Box(c^\alpha\circ\tilde{c}_\alpha)$	$-2\xi'_{(h)}b_2\partial_\mu\partial_\nu(c^\alpha\circ\tilde{c}_\alpha)$	$D\xi'_{(h)}b_2\partial_\mu\partial_\nu(c^\alpha\circ\tilde{c}_\alpha)$
$b_3\frac{\eta_{\mu\nu}}{\Box}\partial A\circ\partial\tilde{A}$	$b_3\eta_{\mu\nu}\partial A\circ\partial\tilde{A}$	$-2\xi'_{(h)}b_3\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$	$D\xi'_{(h)}b_3\frac{\partial_\mu\partial_\nu}{\Box}\partial A\circ\partial\tilde{A}$

Table C.3: Yang-Mills factorisation of the equations of motion for $h_{\mu\nu}$.

Bibliography

[1] J. M. Maldacena, Adv. Theor. Math. Phys **2**, 231 (1997), arXiv:9711200 [hep-th] .

[2] E. Witten, Adv. Theor. Math. Phys **2**, 253 (1998), arXiv:9802150 [hep-th] .

[3] H. Nastase, (2007), arXiv:0712.0689 .

[4] H. Kawai, D. C. Lewellen, and S.-H. H. Tye, Nuclear Physics **269**, 1 (1986).

[5] W. Siegel, Physics Letters B **211** (1988).

[6] W. Siegel, Physical Review D **53** (1995).

[7] Z. Bern, T. Dennen, Y.-T. Huang, and M. Kiermaier, Physical Review D **82** (2010), arXiv:1004.0693 .

[8] Z. Bern, J. Carrasco, and H. Johansson, Phys. Rev. Letters **105** (2010).

[9] Z. Bern, J. J. M. Carrasco, and H. Johansson, Physical Review D **78** (2008), arXiv:0805.3993v2 .

[10] H. Elvang and Y.-T. Huang, (2014), arXiv:1308.1697v2 .

[11] Z. Bern, L. Dixon, D. C. Dunbar, M. Perelstein, and J. S. Rozowsky, Nuclear Physics B **530** (1998), arXiv:9802162 .

[12] Z. Bern, L. J. Dixon, and R. Roiban, Physics Letters B **644** (2006), arXiv:0611086 .

[13] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, Phys. Rev. Letters **103** (2009).

[14] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov, and V. A. Smirnov, Physical Review D **90** (2013), arXiv:1309.2498v2 .

[15] Z. Bern, S. Davies, and T. Dennen, Physical Review D **90** (2014), arXiv:1409.3089v1 .

[16] Z. Bern, C. Cheung, H.-H. Chi, S. Davies, L. Dixon, and J. Nohle, Phys. Rev. Letters **115** (2015), arXiv:1507.06118v1 .

[17] J. Carrasco, R. Kallosh, R. Roiban, and A. Tseytlin, J. High Energ. Phys. **29** (2013), arXiv:1303.6219v2 .

[18] Z. Bern, M. Enciso, J. Parra-Martinez, and M. Zeng, J. High Energ. Phys. **137** (2017), arXiv:1703.08927v1 .

[19] Z. Bern, J. J. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban, and M. Zeng, (2018), arXiv:1804.09311v1 .

[20] J. J. Carrasco, M. Chiodaroli, M. Günaydin, and R. Roiban, J. High Energ. Phys. **56** (2013), arXiv:1212.1146v2 .

[21] P. H. Damgaard, R. Huang, T. Søndergaard, and Y. Zhang, J. High Energ. Phys. **101** (2012), arXiv:1206.1577v2 .

[22] Y.-t. Huang and H. Johansson, Phys. Rev. Letters **110** (2013), arXiv:1210.2255 .

[23] T. Bargheer, S. He, and T. McLoughlin, Phys. Rev. Letters **108** (2012), arXiv:1203.0562v1 .

[24] H. Johansson and A. Ochirov, J. High Energ. Phys. **46** (2015), arXiv:1407.4772v2 .

[25] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, J. High Energ. Phys. **81** (2015), arXiv:1408.0764v2 .

[26] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, J. High Energ. Phys. **74** (2017), arXiv:1511.01740v1 .

[27] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, Phys. Rev. Letters **117** (2015), arXiv:1512.09130v1 .

[28] M. Chiodaroli, (2016), arXiv:1607.04129v1 .

[29] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, J. High Energ. Phys. **2** (2017), arXiv:1703.00421v1 .

[30] M. Chiodaroli, M. Günaydin, H. Johansson, and R. Roiban, Phys. Rev. Letters **120** (2018), arXiv:1710.08796v1 .

[31] F. Cachazo, S. He, and E. Y. Yuan, J. High Energ. Phys. **33** (2014), arXiv:1309.0885v1 .

[32] L. Dolan and P. Goddard, J. High Energ. Phys. **10** (2014), arXiv:1311.5200 .

[33] S. G. Naculich, J. High Energ. Phys. **29** (2014), arXiv:1407.7836v2 .

[34] S. G. Naculich, J. High Energ. Phys. **50** (2015), arXiv:1501.03500v2 .

[35] F. Cachazo, S. He, and E. Y. Yuan, J. High Energ. Phys. **149** (2015), arXiv:1412.3479v2 .

[36] Y.-J. Du and C.-H. Fu, J. High Energ. Phys. **174** (2016), arXiv:1606.05846v2 .

[37] Y.-J. Du, F. Teng, and Y.-S. Wu, J. High Energ. Phys. **86** (2016), arXiv:1603.08158v3 .

[38] D. Nandan, J. Plefka, O. Schlotterer, and C. Wen, J. High Energ. Phys. **70** (2016), arXiv:1607.05701v1 .

[39] N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard, and B. Feng, J. High Energ. Phys. **94** (2016), arXiv:1608.00006v1 .

[40] L. Mason and D. Skinner, J. High Energ. Phys. **48** (2014), arXiv:1311.2564v1 .

[41] Y. Geyer, A. E. Lipstein, and L. Mason, Phys. Rev. Letters **113** (2014), arXiv:1404.6219v2 .

[42] E. Casali, Y. Geyer, L. Mason, R. Monteiro, and K. A. Roehrig, J. High Energ. Phys. **38** (2015), arXiv:1506.08771v2 .

[43] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Phys. Rev. Letters **115** (2015), arXiv:1507.00321 .

[44] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, J. High Energ. Phys. **114** (2016), arXiv:1511.06315v3 .

[45] Y. Geyer, L. Mason, R. Monteiro, and P. Tourkine, Physical Review D **94**, 2016 (2016), arXiv:1607.08887 .

[46] N. E. J. Bjerrum-Bohr, P. H. Damgaard, and P. Vanhove, Phys. Rev. Letters **103** (2009), arXiv:0907.1425v2 .

[47] S. Stieberger, (2010), arXiv:0907.2211v2 .

[48] C. R. Mafra, O. Schlotterer, and S. Stieberger, J. High Energ. Phys. **92** (2011), arXiv:1104.5224v1 .

[49] J. Broedel and L. J. Dixon, J. High Energ. Phys. **91** (2012), arXiv:1208.0876v2 .

[50] J. Broedel, O. Schlotterer, and S. Stieberger, Fortschritte der Physik **61**, 812 (2013), arXiv:1304.7267v2 .

[51] C. R. Mafra and O. Schlotterer, Fortschritte der Physik **63** (2015), arXiv:1410.0668v2 .

[52] C. R. Mafra and O. Schlotterer, J. High Energ. Phys. **97** (2016), arXiv:1510.08846v2 .

[53] C. R. Mafra and O. Schlotterer, J. High Energ. Phys. **124** (2015), arXiv:1505.02746v2 .

[54] S. He, R. Monteiro, and O. Schlotterer, J. High Energ. Phys. **171** (2016), arXiv:1507.06288v2 .

[55] O. Schlotterer, J. High Energ. Phys. **74** (2016), arXiv:1608.00130v2 .

[56] R. Monteiro and D. O'Connell, J. High Energ. Phys. **7** (2011), arXiv:1105.2565v2 .

[57] N. E. J. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro, and D. O'Connell, J. High Energ. Phys. **61** (2012), arXiv:1203.0944v2 .

[58] R. Monteiro and D. O'Connell, J. High Energ. Phys. **110** (2014), arXiv:1311.1151v1 .

[59] R. H. Boels, R. Sven Isermann, and R. Monteiro, J. High Energ. Phys. **107** (2013), arXiv:1301.4165v2 .

[60] C.-H. Fu and K. Krasnov, J. High Energ. Phys. **75** (2017), arXiv:1603.02033v2 .

[61] R. Monteiro, D. O 'Connell, and C. D. White, J. High Energ. Phys. **56** (2014), arXiv:1410.0239v2 .

[62] A. Luna, R. Monteiro, D. O'Connell, and C. D. White, Physics Letters B **750**, 272 (2015), arXiv:1507.01869v2 .

[63] A. Luna, R. Monteiro, I. Nicholson, D. O'Connell, and C. D. White, J. High Energ. Phys. **23** (2016), arXiv:1603.05737v1 .

[64] C. D. White, Physics Letters B **763**, 365 (2016), arXiv:1606.04724v2 .

[65] P.-J. De Smet and C. D. White, Physics Letters B **775**, 163 (2017), arXiv:1708.01103 .

[66] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O 'connell, N. Westerberg, and C. D. White, J. High Energ. Phys. **69** (2017), arXiv:1611.07508v2 .

[67] A. Luna, I. Nicholson, D. O 'Connell, and C. D. White, J. High Energ. Phys. **44** (2018), arXiv:1711.03901v1 .

[68] M. C. González, R. Penco, and M. Trodden, J. High Energ. Phys. **28** (2018), arXiv:1711.01296v1 .

[69] D. S. Berman, E. Chacón, A. Luna, and C. D. White, (2018), arXiv:1809.04063v1 .

[70] G. L. Cardoso, S. Nagy, and S. Nampuri, J. High Energ. Phys. **127** (2016), arXiv:1609.05022v2 .

[71] G. L. Cardoso, S. Nagy, and S. Nampuri, J. High Energ. Phys. **37** (2017), arXiv:1611.04409v1 .

[72] G. L. Cardoso, G. Inverso, S. Nagy, and S. Nampuri, in *PoS CORFU2017* (2018) arXiv:1803.07670v1 .

[73] C. N. Yang and R. L. Mills, Physical Review **96**, 191 (1954).

[74] M. F. Sohnius, Physics Reports **128** (1985).

[75] A. V. Proeyen, (1999), arXiv:9910030 .

[76] F. Quevedo, S. Krippendorf, and O. Schlotterer, (2010), arXiv:1011.1491v1 .

[77] J. STRATHDEE, International Journal of Modern Physics A **02**, 273 (1987).

[78] P. van Nieuwenhuizen, Physics Reports **68**, 189 (1980).

[79] B. De Wit, LesHouches Lect.Notes **76** (2003), arXiv:0212245v1 .

[80] D. Z. Freedman and A. V. Proeyen, *Supergravity* (Cambridge University Press, 2012) p. 607.

[81] Y. Tanii, *Introduction to Supergravities in Diverse Dimensions*, Tech. Rep. (1998) arXiv:9802138v1 .

[82] H. Samtleben, Classical and Quantum Gravity (2008), arXiv:0808.4076v1 .

[83] S. Ferrara and A. Marrani, Contemp. Math **490** (2009), arXiv:0808.3567v1 .

[84] C. M. Hull and P. K. Townsend, Nuclear Physics B **438**, 109 (1995), arXiv:9410167 .

[85] L. Borsten, M. Duff, L. Hughes, and S. Nagy, Phys. Rev. Letters **112** (2014), arXiv:1301.4176 .

[86] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, J. High Energ. Phys. **80** (2014), arXiv:1309.0546v3 .

[87] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, J. High Energ. Phys. **178** (2014), arXiv:1312.6523 .

[88] A. Anastasiou, L. Borsten, M. J. Hughes, and S. Nagy, J. High Energ. Phys. **148** (2016), arXiv:1502.05359 .

[89] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy, and M. Zoccali, in *Mile High Conference on Nonassociative Mathematics* (2017) arXiv:1711.08476v2 .

[90] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, Phys. Rev. Letters **113** (2014), arXiv:1408.4434v2 .

[91] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy, and M. Zoccali, Nuclear Physics B **934**, 606 (2018), arXiv:1707.03234v1 .

[92] D. Roest and H. Samtleben, Classical and Quantum Gravity **26**, 155001 (2009), arXiv:0904.1344 .

[93] M. J. Duff and S. Ferrara, Classical and Quantum Gravity **28** (2010), 10.1088/0264-9381/28/6/065005, arXiv:1009.4439 .

[94] G. Inverso, J. High Energ. Phys. **138** (2016), arXiv:1512.04500v1 .

[95] A. Anastasiou, L. Borsten, M. J. Duff, M. J. Hughes, A. Marrani, S. Nagy, and M. Zoccali, Physical Review D **96** (2017), arXiv:1610.07192 .

[96] C. Becchi, A. Rouet, and R. Stora, Annals of Physics **98**, 287 (1976).

[97] I. V. Tyutin, Preprint of the Lebedev Institute (1975), arXiv:0812.0580v2 .

[98] T. Kugo and I. Ojima, PHYSICS LETTERS B **73**, 459 (1978).

[99] T. Kugo and I. Ojima, Progress of Theoretical Physics **60**, 1869 (1978).

[100] T. Kugo and I. Ojima, Progress of Theoretical Physics **61**, 294 (1979).

[101] T. Kugo and I. Ojima, Supplement of the Progress of Theoretical Physics (1979).

[102] C. M. Becchi, Pramana **78**, 837 (2012), arXiv:1107.1070v2 .

[103] L. Faddeev and V. Popov, PHYSICS LETTERS B **25**, 29 (1967).

[104] L. Faddeev and V. Popov, *Perturbation theory for gauge invariant fields*, Tech. Rep. (1972).

[105] G. Curci and R. Ferrari, Physics Letters B **63**, 91 (1976).

[106] I. Ojima, Progress of Theoretical Physics **64** (1980).

[107] N. Nakanishi and I. Ojima, Zeitschrift für Physik C Particles and Fields **6**, 155 (1980).

[108] L. Bonora and M. Tonin, PHYSICS LETTERS B **98**, 48 (1981).

[109] L. Bonora, P. Pasti, and M. Tonin, Il Nuovo Cimento **63A**, 353 (1981).

[110] R. Delbourgo and P. D. Jarvis, J. Phys. A: Math. Gen. J. Phys. A: Math. Gen **15**, 611 (1982).

[111] L. Alvarez-Gaumé and L. Baulieu, Nuclear Physics **212**, 255 (1983).

[112] V. P. Spiridonov, Nuclear Physics **308**, 527 (1988).

[113] M. Kalb and P. Ramond, Phys. Rev. D Phys. Rev. D **7**, 2386 (1974).

[114] P. K. Townsend, PHYSICS LETTERS B **88**, 97 (1979).

[115] D. Z. Freedman and P. K. Townsend, Nuclear Physics **B177**, 282 (1981).

[116] T. O. Kimura, Prog. Theor. Phys. **64** (1980).

[117] T. Kimura, Progress of Theoretical Physics **65** (1981).

[118] S. P. De Alwis, M. T. Grisaru, and L. Mezincescu, PHYSICS LETTERS B **190**, 122 (1987).

[119] T. E. Clark, C.-H. Lee, and S. T. Love, Nuclear Physics B308 , 379 (1988).

[120] C. Batlle and J. Gomis, Physical Review D **38**, 1169 (1988).

[121] J. Thierry-Mieg, Nuclear Physics B **335**, 334 (1990).

[122] R. Ferraro, M. Henneaux, and M. Puchin, J. Math. Phys. **34**, 2757 (1993), arXiv:9210070 .

[123] G. Barnich, R. Constantinescu, and P. Gregoire, Physics Letters B **293**, 353 (1992), arXiv:9209007 .

[124] L. Bonora and R. P. Malik, Physics Letters B **655**, 75 (2007), arXiv:0707.3922v4 .

[125] L. Bonora and R. P. Malik, J. Phys. A **43** (2010), arXiv:0911.4919v2 .

[126] I. A. Batalin and G. A. Vilkovisky, PHYSICS LETTERS B **102**, 27 (1981).

[127] I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, Journal of Mathematical Physics **31**, 6 (1990).

[128] I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, Journal of Mathematical Physics **31**, 1487 (1990).

[129] I. A. Batalin, P. M. Lavrov, and I. V. Tyutin, Journal of Mathematical Physics **32**, 532 (1991).

[130] C. Hull, Modern Physics Letters A **5**, 1871 (1990).

[131] J. Gomis and J. Roca, Nuclear Physics **343** (1990).

[132] M. Henneaux, Nuclear Physics B **A**, 47 (1990).

[133] P. Gregoire and M. Henneaux, J. Phys. A: Math. Gen **26**, 6073 (1993).

[134] J. Gomis, J. París, and S. Samuel, Phys. Rept. **259**, 1 (1995), arXiv:9412228v1 .

[135] M. Henneaux and C. Teitelboim, *Quantization of gauge systems* (1991).

[136] J. Wess and B. Zumino, Nuclear Physics B **70**, 39 (1974).

[137] C. Pope, Lectures notes .

[138] S. Mizera, J. High Energ. Phys. **84** (2017), arXiv:1610.04230v1 .

[139] A. Anastasiou, *Yang-Mills origin of gravitational symmetries*, Ph.D. thesis (2015).

[140] M. Hughes, *Octonions and supergravity*, Ph.D. thesis (2015).

[141] S. Nagy, *A New Gauge / Gravity Dictionary via the Division Algebras*, Ph.D. thesis (2015).

[142] M. Günaydin, G. Sierra, and P. K. Townsend, Nuclear Physics **242** (1984).

[143] M. Gunaydin, G. Sierra, and P. K. Townsend, Physics Letters B **133**, 72 (1983).

[144] S. Gupta, in *Proceedings of the Physical Society* (1950).

[145] K. Bleuler, Helvetica Physica Acta **23** (1950).

[146] N. Nakanishi, Prog. Theor. Phys. **51**, 952 (1974).

[147] B. Lautrup, Mat. Fys. Medd. Dan. VId. Selsk. **35** (1967).

[148] K. Lichtenegger, *Aspects of Confinement in a Functional Approach to Coulomb Gauge QCD*, Ph.D. thesis (2010).

[149] L. Bonora and P. Cotta-Ramusino, Commun. Math. Phys. Mathematical Physics **87**, 589 (1983).

[150] P. West, *Introduction to strings and branes* (Cambridge University Press, 2012).