Symplectic Structure of Constrained Systems: Gribov Ambiguity and Classical Duals for 3D Gravity

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Abstract

The present thesis is divided into two parts. Part I is devoted to the study of Gribov ambiguity in gauge systems and its relation with the appearance of degeneracies in the symplectic structure of the corresponding reduced phase space after gauge fixation. Part II is concerned with classical dual field theories for three-dimensional Einstein gravity and the symplectic structure on coadjoint orbits of the corresponding asymptotic symmetry group.

In Part I, the Gribov problem is studied in the context of finite temperature QCD and the structure of the gluon propagator is analyzed. The standard confined scenario is found for low temperatures, while for high enough temperatures deconfinement takes place and a free gluon propagator is obtained. Subsequently, the relation between Gribov ambiguity and degeneracies in the symplectic structure of gauge systems is analyzed. It is shown that, in finite-dimensional systems, the presence of Gribov ambiguities in regular constrained systems always leads to a degenerate symplectic form upon Dirac reduction. The implications for the Gribov-Zwanziger approach to QCD and the symplectic structure of the theory are discussed.

In Part II, geometrical actions for three-dimensional Einstein gravity are constructed by studying the symplectic structure on coadjoint orbits of the asymptotic symmetry group. The geometrical action coming from the Kirillov-Kostant symplectic form on coadjoint orbits is analyzed thought Dirac's algorithm for constrained systems. By studying the case of centrally extended groups and semi-direct products, the symplectic structure on coadjoint orbits of the Virasoro and the BMS_3 group are analyzed. This allows one to associate separate geometric actions to each coadjoint orbit of the solution space, leading to two-dimensional dual field theories for asymptotically AdS and asymptotically flat three-dimensional gravity respectively.

The original results presented in this thesis led to the following publications:

- "Gribov gap equation at finite temperature",
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Preface

One of the cornerstones in modern physics is the gauge principle, which allows one to understand three of the four known interactions in nature as quantum Yang-Mills theories. Within this construction, the Standard Model of particle physics incorporates electromagnetism together with the weak nuclear interaction in the Glashow-Weinberg-Salam electroweak theory, and the strong nuclear interaction through quantum chromodynamics (QCD). However, despite the fact the Standard Model provides a very good description of many phenomena observed by experiments, it is still incomplete and many important questions remain unanswered. In the context of QCD, confinement is one of the main open problems in theoretical physics. This corresponds to the experimental fact that isolated color charged particles are not observed in nature, which has not yet received a satisfactory theoretical explanation, as it corresponds to a non-perturbative phenomenon. Due to asymptotic freedom [1], the standard perturbative analysis for QCD works only for high energies, but a characterization of the physical spectrum of the theories at low energy scales coming from first principles is still an outstanding problem. Over the last decades, however, a number of results signal a relation between the infrared two point function of QCD and *gauge fixing ambiguities* appearing the quantization scheme of Yang-Mills theories, first found by Gribov [2], which seem to be likely to provide some clues to the underlying structure of the theory that are responsible for confinement.

Another problem of the Standard Model is that it does not include gravity, the fourth fundamental interaction in the picture, described by General Relativity. So far, gravity can not be put in the same footing as the other forces described by the Standard Model, as it does not admit a gauge description in the Yang-Mills sense. Besides, the theory is not renorrmalizable and a quantum description of it is still lacking. Probably, one of the main breakthroughs in the pursuit of a quantum description of gravity is the holographic principle [3, 4], which postulates that gravity in a d-dimensional space-time can be described at the quantum level by a quantum field theory defined in a lower dimension. Within this context, Maldacena's conjecture [5] has played a central role and led to a connection between conformal field theories (CFT's) and supergravity theories in Anti-de Sitter (AdS) space-times. Evidence of AdS/CFT duality in physics can however be traced back even before Maldacena's conjecture, to the mid-80's when Brown and Henneaux showed that the asymptotic symmetry algebra of three dimensional gravity with negative cosmological constant turns out to be a central extension of the conformal algebra in two dimensions [6]. Based on this result, it can be shown that the Bekenstein–Hawking entropy of a three-dimensional black hole is in precise agreement with the entropy obtained by CFT reasonings using Cardy formula [7] and that the partition function for three-dimensional gravity can be understood as a character of some representation of the Virasoro algebra [8], showing that quantum aspects of gravity might be encoded on a dual CFT description.

The present thesis is divided into two parts, and deals with particular geometrical aspects of Yang-Mills theories and three-dimensional gravity.

Part I is concerned with the Gribov problem and gauge fixing ambiguities in Yang-Mills theories. This problem appears because, in spite of the geometrical beauty of gauge symmetry, it must be eliminated in order to quantize Yang-Mills theory. Due to gauge symmetry, when using the operator formalism it is not possible to define propagators for gauge fields and, if path integral quantization procedure is adopted, functional integrals diverge for they involve a sum over the infinite gauge transformations existing for every physical configuration. A way to overcome this issue is *gauge fixation*, which corresponds to a restriction in configuration space to the "fundamental modular region" where only one representative for each physical configuration exists. However, gauge theories present an obstruction to achieve a proper gauge fixing globally. This problem is called *Gribov ambiguity*. Given a gauge field configuration, it is possible to find an equivalent configuration or copy satisfying the same gauge conditions. The presence of Gribov copies is related with the existence of zero modes of the Faddeev-Popov operator and leads to an identically vanishing functional integral for the theory. A solution to eliminate Gribov copies connected with the identity is to restrict the domain of path integrals to the *Gribov region* which is the domain in the functional space of gauge potentials where the Faddeev-Popov operator is positive definite. This restriction has remarkable effects modifying the theory in the infrared and, in the case of QCD, might have a crucial role in the understanding of confinement.

In this manuscript, the relationship between the presence of Gribov ambiguity for gauge conditions and degeneracies in the symplectic structure of the resulting reduced theory will be studied. First, in the context of finite temperature Yang-Mills theory, the structure of the gluon propagator will be analyzed within the semiclassical Gribov approach. It will be shown that the theory displays a confined and a free regime, compatible with the confinement scenario. Subsequently, the existence of Gribov ambiguity will be studied in finite dimensional systems in the Hamiltonian framework. Using Dirac's theory for constrained Hamiltonian systems, it will be shown that Gribov ambiguity leads to a degenerate symplectic form for the reduced phase space. Finally, the implications for the Gribov formulation of QCD will be discussed

Part II of the thesis is focused on dual field theories of three-dimensional gravity. One of the few well-established examples of the AdS/CFT correspondence is three-dimensional gravity with negative cosmological constant. In this scenario, the first indication of duality comes from the notion of asymptotically AdS space-time, where the asymptotic symmetry algebra for the theory turns out to be given by two copies of the Witt algebra, which corresponds to the conformal algebra in two dimensions. The algebra of surface charges is given by a central extension of the asymptotic symmetry algebra and consists of two copies of the Virasoro algebra. The presence of the infinite-dimensional conformal group as asymptotic symmetry group suggests that the asymptotic dynamics of the theory is described by a two-dimensional CFT living at the boundary of the manifold over which the gravity theory is defined. This procedure can be carried out in the Hamiltonian framework by formulating Einstein gravity in three dimension with negative cosmological constant as two copies of the $SL(2,R)$ Chern-Simons theory [9], which can be redued to Liouville theory at the boundary by properly implementig the Brown-Henneaux boundary conditions [10]. This remarkable result has been generalized to the case of asymptotically flat three-dimensional gravity, finding a BMS_3 invariant two-dimensional field theory at the boundary [11, 12]. Even though a CFT can be obtained from the gravity action by Hamiltonian reduction, this duality is valid only at the classical level and the identification of Liouville theory as the dual theory to three-dimensional quantum gravity presents technical problems [13]. This analysis, however, does not take non trivial topology and the associated holonomies into account. When this is done, one expects modified Liouville type actions.

In this part of the thesis, dual theories for three dimensional gravity will be constructed by studying the symplectic structure on coadjoint orbits of the asymptotic symmetry group. Coadjoint orbits posses a natural symplectic form, from which geometrical actions can be constructed[14, 15]. It will be shown that this geometrical action match the dual field theory action for gravity obtained from the Chern-Simons formulation. Extra terms will be obtained, which label the orbit over which the action is defined and their physical interpretation will be discussed.

Part I

Gribov Ambiguity

Chapter 1

Introduction to Part I

In his seminal paper, Gribov showed that a standard gauge condition, such as the Coulomb or the Landau choices, fail to provide proper gauge fixings in Yang-Mills theories [2]. This so-called Gribov problem, that affects non-abelian gauge theories, means that a generic gauge fixing intersects the same gauge orbit more than once (Gribov copies) and may fail to intersect others. Algebraic gauge conditions free of Gribov ambiguities are possible, but those choices are affected by severe technical problems as, for instance, incompatibility with the boundary conditions that must be imposed on the gauge fields in order to properly define the configuration space for the theory [16]. Additionally, Singer [17] showed that Gribov ambiguities occur for all gauge fixing conditions involving derivatives (see also [18]), and moreover, the presence of the Gribov problem breaks BRST symmetry at a non-perturbative level [19].

The Gribov problem occurs because it is generically impossible to ensure positive definiteness of the Faddeev-Popov (FP) determinant everywhere in functional space, which make the path integral ill defined. Even when perturbation theory around vacuum is not affected by Gribov ambiguity when Yang-Mills theory is defined over a flat space-time with trivial topology [20], Gribov copies have to be taken into account when considering more general cases [21–25]. The configurations for which the FP operator develops a nontrivial zero mode are those where the gauge condition becomes 'tangent' to the gauge orbits and it therefore fails to intersect them. The Gribov horizon, where this happens, marks the boundary beyond which the gauge condition intersects the gauge orbits more than once (Gribov copies). The appearance of Gribov copies invalidates the usual approach to the path integral and one way to avoid overcounting is to restrict the sum over field configurations to the so-called Gribov region around $A_\mu = 0$, where the FP operator is positive definite [2, 26–30].

The most effective method to eliminate Gribov copies, proposed by Gribov himself in [2] and refined in [28, 27], corresponds to restricting the path integral to the so-called Gribov region, which is the region in the functional space of gauge potentials over which the Faddeev-Popov operator is positive definite. In [28] Dell'Antonio and Zwanziger showed that all the orbits of the theory intersect the Gribov region, indicating that no physical information is lost when implementing this restriction, which takes into account the infrared effects related to the partial elimination of the Gribov copies, in the sense that it only guarantees the exclusion of those copies obtained by gauge transformations connected with the identity [27, 31, 32] (copies with non-trivial winding number are still present [30]). Remarkably enough, the partial elimination of Gribov copies in perturbation theory is related to the non-perturbative infrared physics. When one takes into account the presence of suitable condensates [33–37] the agreement with lattice data is excellent [38, 39]. The non-perturbative input in the modified path-integral is the restriction to the Gribov region, which leads to a gluon propagator with imaginary poles, indicating that gluons are not in the physical spectrum of the theory. This remarkable result has opened a possible way to understand color confinement in QCD from a new perspective [40, 41]. Even though it is an experimental fact that quarks and gluons are confined and color charged states are unobservable as asymptotic states at low temperatures, it is expected that at high temperatures ($T_c \sim 150-200$ MeV) they become free [42, 43]. Such a phase transition from confinement to quark-gluon plasma (QGP) should be described within the framework of finite-temperature field theory allowing a better understanding of natural scenarios as the early universe or compact star physics [43–45].

In Dirac's formalism for constrained systems [46] gauge-invariant mechanical systems are characterized by the presence of first class constraints. Gauge fixing in those systems is achieved by the introduction of extra constraints, such that the whole set of constraints become second class. In this context, the Gribov problem is the statement that the second class nature of this set cannot hold globally: the Dirac matrix defined by their Poisson brackets is not invertible everywhere in phase space, it is *degenerate*.

Degenerate Hamiltonian systems on the other hand, are those whose symplectic form is not invertible in a subset of phase space [47]. In classical degenerate systems the evolution takes place over non-overlapping causally disconnected subregions of the phase space separated by degenerate surfaces. This means that if a system is prepared in one subregion, never evolves to a state in a different subregion. This still holds in the quantum domain for some simple degenerate systems [48]. Degenerate systems are ubiquitous in many areas of physics, from fluid dynamics [49] to gravity theories in higher dimensions [50, 51], in the strong electromagnetic fields of quasars [52], and in systems such as massive bi-gravity theory [53], which has been shown to possess degenerate sectors where the degrees of freedom change from one region of phase space to another [54].

In Chapter 2 Dirac's formalism for constrained systems is presented and the idea of gauge fixing in a gauge system is introduced. In Chapter 3 the quantization of Yang-Mills theories will be quickly reviewed focusing on the gauge fixing procedure and the Gribov problem will be introduced. In Chapter 4, the semiclassical Gribov approach to restrict the path integral to the Gribov horizon is introduced and the Gribov gap equation at finite temperature is analyzed. The solutions of the gap equation (which depend explicitly on the temperature) determine the structure of the gluon propagator within the semi-classical Gribov approach. The results found are consistent with the standard confinement scenario for low temperatures, while for high enough temperatures, deconfinement takes place and a free gluon propagator is obtained. An intermediate regime in between the confined and free phases can be read off from the resulting gluon propagator, which appears to be closely related to partial deconfinement.

In Chapter 5, the relation between Gribov ambiguity and degeneracies in the symplectic structure of physical systems is analyzed by studying toy models with finite number of degrees of freedom. It will be shown that Gribov ambiguity and the existence of degeneracies are related problems, and that the Gribov horizon can be identified as a surface of degeneracy in the reduced phase space. In finite-dimensional systems, the presence of Gribov ambiguities in regular constrained systems always leads to a degenerate symplectic structure upon Dirac reduction. This means that the system would be naturally confined to a region surrounded by a horizon, exactly as proposed by Zwanziger [26]. This interpretation of the Gribov horizon as surface of degeneracy that acts as a boundary beyond which the evolution cannot reach, makes the restriction in the sum over histories a natural prescription and not ad-hoc one. Finally, the implications for the Gribov-Zwanziger approach to QCD are discussed.

Chapter 2

Constrained Systems

Systems possessing local symmetries, such as the ones described by Yang-Mills theories or Einstein gravity, correspond to a class of systems called *constrained systems*, which present some degree of arbitrariness in their description. This is due to the fact the equations of motion include constraints between coordinates and velocities, which make the pass to the Hamiltonian formalism non-trivial. In this chapter, the main features of constrained systems are studied through Dirac's formalism.

2.1 Singular Lagrangians

Let us consider a system with a finite number N of degrees of freedom, whose dynamics is governed by a Lagrangian $L(q,\dot{q})$, where q^A and $\dot{q}^A$ correspond to the coordinates and velocities respectively and $A=1,\cdots,N$. The corresponding Euler-Lagrange equations for the system are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}-\frac{\partial L}{\partial q^A}=0 \ , \ A=1,\cdots,N. \tag{2.1}$$

Developing the time derivative, the equations of motion can be put in the form

$$W_{AB}(q,\dot{q})\ddot{q}^B=\frac{\partial L}{\partial q^A}-\frac{\partial^2 L}{\partial \dot{q}^A\partial q^B}\dot{q}^B, \tag{2.2}$$

where $W(q,\dot{q})$ is the Hessian matrix

$$W_{AB}(q,\dot{q})=\frac{\partial^2 L}{\partial \dot{q}^A\partial \dot{q}^B}. \tag{2.3}$$

If $\det W\neq 0$, it is possible to solve (2.2) and express the accelerations in terms of q^A and $\dot{q}^B$. If $\det W=0$, on the other hand, W can not be inverted. In this case the system is called *singular*.

2.2 Primary Constraints

In order to pass to the Hamiltonian formulation of the model, it is necessary to introduce the generalized momenta p_A conjugated to the coordinates q^A,

$$p_A = \frac{\partial L}{\partial \dot{q}^A}. \tag{2.4}$$

satisfying the canonical Poison brackets

$$\left[q^A, p_B\right] = \delta^A_B$$

In the case of non-singular systems, this definition allows to express the velocities in terms of the canonical variables

$$\dot{q}^A = f^A(q,p). \tag{2.5}$$

Subsequently, the canonical Hamiltonian is obtained by means of the Legendre transformation

$$H_0(q,p) = p_A f^A(q,p) - L(q, f(q,p)). \tag{2.6}$$

And the Hamilton equations are first order in the time derivatives. Let us note, however, that in order to get rid of the velocities, the matrix W should be invertible. In fact,

$$\begin{aligned} dp_A &= \frac{\partial p_A}{\partial \dot{q}^B} d\dot{q}^B + \frac{\partial p_A}{\partial q^B} dq^B \\ \Longrightarrow \frac{\partial p_A}{\partial \dot{q}^B} d\dot{q}^B &= dp_A - \frac{\partial p_A}{\partial q^B} dq^B, \end{aligned}$$

which to be solved for $d\dot{q}^J$ requires

$$\det\left(\frac{\partial p_A}{\partial \dot{q}^B}\right) = \det W \neq 0.$$

In the singular case, where $\det W = 0$, it is not possible to express all the velocities in terms of the coordinates and their associated canonical momenta. This means that the momenta are not independent, but there are relations among them. In order to see this, let W_{ab} $(a,b = 1,\cdots,R_W)$ be the submatrix with maximum rank R_W of W_{AB}, where a convenient arrangement of its components has been made. This allows to solve (2.4) for R_W velocities $\dot{q}^a$ in terms of the associated momenta p_a, the coordinates q^A and the remaining velocities, which will be labeled as $\dot{q}^\alpha$ with $\alpha = R_W + 1,\cdots,N$. Therefore, it is possible to write

$$\dot{q}^a = f^a\left(q, \{p_b\}, \{\dot{q}^\alpha\}\right) \tag{2.7}$$

Replacing these relations back into (2.4) leads to

$$p_A = h_A\left(q, \{p_a\}, \{\dot{q}^\alpha\}\right).$$

For $A = a$ this must reduce to an identity, while for $A = \alpha$ we get

$$p_\alpha = h_\alpha\left(q, \{p_a\}, \left\{\dot{q}^\beta\right\}\right).$$

However, the right hand side of this equation can not depend on the velocities $\dot{q}^\beta$, as in that case it would be possible to express more velocities in terms of the coordinates, the momenta, and the remaining velocities, which is not possible if W_{ab} has maximum rank. What we obtain in this case are relations of the form

$$\Phi_\alpha(q,p) = p_\alpha - h_\alpha\left(q, \{p_i\}\right) \approx 0, \tag{2.8}$$

which are called primary constraints, as they come uniquely from the definition of the canonical momenta, without making use of the equations of motion. These constraints define a subspace of dimension $2N - (N - R_w) = N + R_w$ in phase space, which will be called $\Gamma^{(0)}$. We have also introduced the *weak equality* symbol $\approx$ to denote equality in the constraint surface $\Gamma^{(0)}$. Two functions f and g of phase space are *weakly equal* if and only if they are equal when restricted to the surface defined by the constraints (2.8),

$$f \approx g \iff f\,|_{\Gamma^{(0)}} = g\,|_{\Gamma^{(0)}}. \tag{2.9}$$

In general cases it is be technically complicated to work with constraints in the explicit form (2.8). For instance, if the constraints could be written in a covariant way, it would be better not to destroy covariance to bring them to the form (2.8). As a general case, we will consider implicit constraints

$$\phi_m(q,p) \approx 0, \qquad m = 1, \cdots, M. \tag{2.10}$$

whose solutions for the momenta p_α are given by (2.10). There is, however, an ambiguity when considering implicit constraints. For instance, $\phi_m = 0$ and $\phi_m^2 = 0$ seem to be equally valid in $\Gamma^{(0)}$. In order to avoid this problem it is necessary to impose the following *regularity conditions* for the constraints ϕ_m, so that they properly represent $\Gamma^{(0)}$. The constraint surface $\phi_m = 0$ can be covered with open sets in which the constraints can be locally split into independent constraints $\phi_\alpha \approx 0$, $\alpha = 1, \cdots, N - R_w$, for which the Jacobian matrix $\partial(\phi_\alpha)/\partial(q^I, p_J)$ has maximum rank $N - R_w$ in $\Gamma^{(0)}$, and dependent constraints $\phi_{\bar{m}} \approx 0$, $\bar{m} = n - R_w + 1, \cdots, M$, which can be obtained form the independent ones.

In the subspace $\Gamma^{(0)}$, the canonical Hamiltonian (2.6) is only function of the coordinates q^A and the independent momenta p_A, but does not depend on the remaining coordinates $\dot{q}^\alpha$. In fact, the variation δH_0 induced by arbitrary variations of the positions and the momenta reads

$$\delta H_0 = \dot{q}^A \delta p_A + \left(p_A - \frac{\partial L}{\partial \dot{q}^A} \right) \delta \dot{q}^A - \delta q^A \frac{\partial L}{\partial q_A} = \dot{q}^A \delta p_A - \delta q^A \frac{\partial L}{\partial q^A}. \tag{2.11}$$

which means that $\partial H_0 / \partial \dot{q}^\alpha = 0$ and therefore

$$H_0 = H_0(q^A, p_a). \tag{2.12}$$

Equation (2.11) can be rewritten as

$$\left(\frac{\partial H_0}{\partial q^A} + \frac{\partial L}{\partial q^A} \right) \delta q^A + \left(\frac{\partial H_0}{\partial p_A} - \dot{q}^A \right) \delta p_A = 0. \tag{2.13}$$

The regularity conditions imply that the gradients $\partial \phi_m / \partial q^A$ and $\partial \phi_m / \partial p_A$ are linearly independent. This means that for any set of functions λ_A and μ^A satisfying $\lambda_A \delta q^A + \mu^A \delta p_A = 0$, where δq^A and δp_A are tangent to the constraint surface, it is always possible to write

$$\begin{aligned} \lambda_A &= u^\alpha \frac{\partial \phi_\alpha}{\partial q^A} \\ \mu^A &= u^\alpha \frac{\partial \phi_\alpha}{\partial p_A} \end{aligned} \tag{2.14}$$

where u^α are some functions of the coordinates and the momenta. Applying this result to (2.13), and using $\dot{p}_A = \partial L / \partial q^A$, leads to the Hamilton equations for a constrained system

$$\begin{aligned} \dot{q}^A &\approx \frac{\partial H_0}{\partial p_A} + u^\alpha \frac{\partial \phi_\alpha}{\partial p_A} \\ \dot{p}_A &\approx -\frac{\partial H_0}{\partial q^A} - u^\alpha \frac{\partial \phi_\alpha}{\partial q^A}. \end{aligned} \tag{2.15}$$

Defining the total Hamiltonian,

$$H_T = H_0 + u^\alpha \phi_\alpha, \tag{2.16}$$

Hamilton equations can be written as

$$\begin{aligned} \dot{q}^A &\approx [q_A, H_T] \approx \frac{\partial H_T}{\partial p_A}, \\ \dot{p}^A &\approx [p_A, H_T] \approx -\frac{\partial H_T}{\partial q^A}. \end{aligned} \tag{2.17}$$

2.3 Secondary Constraints

As a consistency requirement, the constraints must be preserved during the time evolution of the system, which means

$$\dot{\phi}_\alpha \approx [\phi_\alpha, H_0] + u^\beta \left[\phi_\alpha, \phi_\beta\right] \approx 0. \tag{2.18}$$

These relations may either fix the Lagrange multipliers u^β or can lead the new relations between the canonical variables, independent of the u's. In the case the new relations are independent of the primary constraints, they correspond to new constraints of the theory, which are called *secondary constraints,*

$$X_{\alpha'}(q,p) \approx 0 \tag{2.19}$$

These constraints restrict the dynamics of the system to a subspace $\Gamma^{(1)}$ of $\Gamma^{(0)}$, which weak equality should be referred to thereafter. This procedure must be repeated for the new constraints, which can lead to more restrictions to the Lagrange multipliers or more constraints. The procedure will end after a finite number of iterations, leading to the full set of constraints for the system, which will be denoted as

$$\phi_M \approx 0, \quad M = 1, \cdots, N_\phi, \tag{2.20}$$

and define a constraint surface Γ consistent with the dynamics.

2.4 First and Second Class Constraints

Given a set of constraints, they can be classified in the following way:

- First class constraints ψ_I, $I = 1, \ldots, N_\psi$, which have weakly vanishing Poisson bracket with all the constraints of the system.

$$[\psi_I, \phi_M] \approx 0. \tag{2.21}$$

- Second class constraints χ_Ω, $\Omega = 1, \ldots, N_\chi$, for which the *Dirac matrix* $C_{\Omega\Lambda} = [\chi_\Omega, \chi_\Lambda]$ has maximum rank in Γ, i.e.,

$$\det C \neq 0.$$

Once the whole set of constraints of the system have been obtained, they can be split into first and second class constraints $\phi_M = \{\psi_I, \chi_\Omega\}$ so that $N_\psi + N_\chi = N_\phi$. It is possible, however, that some linear combination of the second class constraints is first class. This will happen if the determinant of the Dirac matrix vanishes. In fact, if $\det C = 0$ there will be zero modes ξ^Ω satisfying $\xi^\Omega C_{\Omega\Lambda} = 0$, which can be written as

$$\left[\xi^\Omega \chi_\Omega, \chi_\Lambda\right] \approx 0,$$

implying that the combination $\xi^{\Omega}\chi_{\Omega}$ is a first class constraint.

After the second class constraints of the system have been obtained, they can be eliminated from the theory by defining a new Poisson structure for the system given by

$$[A,B]^* = [A,B] - [A,\chi_\Lambda]\, C^{\Lambda\Omega}\, [\chi_\Omega, B]\,, \tag{2.22}$$

which is called the *Dirac bracket*. This bracket satisfies the same properties as the Poisson bracket, namely

$$\begin{aligned} [A,B]^* &= -\,[B,A]^* \\ [A,BC]^* &= [A,B]^*\,C + B\,[A,C]^* \\ [AB,C]^* &= A\,[B,C]^* + [A,C]^*\,B \\ \left[[A,B]^*,C\right]^* + \left[[C,A]^*,B\right]^* &+ \left[[B,C]^*,A\right]^* = 0. \end{aligned} \tag{2.23}$$

For first class functions it is weakly equivalent to the Poisson bracket. In fact, for B and C first class functions and A arbitrary,

$$\begin{aligned} [A,B]^* &\approx [A,B] \\ \left[A,[B,C]^*\right]^* &\approx [A,[B,C]]\,. \end{aligned} \tag{2.24}$$

An important property of the Dirac bracket is that for any function of the canonical variables

$$[\chi_\Omega, F]^* = 0, \tag{2.25}$$

which allows to set the second class constraints strongly to zero and eliminate them from the formalism.

2.5 Gauge Transformations

First class constraints are related to the local symmetries of the system. In order to see this, let us consider the time evolution of a function $f(q(t),p(t))$ generated by the total Hamiltonian.

$$f(t_0+\delta t) = f(t_0) + \dot{f}(t_0)\delta t \approx f(t_0) + [f(t_0),H_T]\,\delta t \tag{2.26}$$

Let us also assume that the primary constraints ϕ_α entering in the total Hamiltonian (2.16) have been split into first class primary constraints ψ_i, $i = 1,\ldots,n_\gamma$, and second class primary constraints χ_ω, $\omega = 1,\ldots,n_\chi$, so that

$$H_T = H' + u^i \gamma_i \ \ , \ \ H' = H_0 + u^\omega \chi_\omega\ . \tag{2.27}$$

Equation (2.26) then takes the form

$$f(t_0+\delta t) \approx f(t_0) + \left[f(t_0), H'\right] \delta t + u^i \left[f(t_0), \psi_i\right] \delta t. \tag{2.28}$$

The Lagrange multipliers u^ω for the second class primary constraints are fixed by preservation in time of the constraints. The consistency conditions (2.18) for the set χ_ω read

$$\dot{\chi}_\omega = [\chi_\omega, H_0] + u^\sigma C_{\omega\sigma} \approx 0$$

and, as the constraints χ_ω are second class, the Dirac matrix $C_{\omega\sigma} = [\chi_\omega, \chi_\sigma]$ can be inverted, allowing to obtain u^σ.

On the other hand, the Lagrange multipliers u^i for the first class primary constraints are completely arbitrary and for another choice $u^i \longrightarrow v^i$, we get

$$f(t_0+\delta t) \approx f(t_0) + \left[f(t_0), H'\right] \delta t + v^i \left[f(t_0), \psi_i\right] \delta t. \tag{2.29}$$

The difference in δf between (2.28) and (2.29) is given to order δt by

$$\delta f \approx \left(v_a - v'_a\right) \delta t \left[f(t_0), \psi_i\right]. \tag{2.30}$$

As (2.28) and (2.29) describe the same physical configuration. This implies that local transformations of the form

$$\delta f \approx \varepsilon^i \left[f, \psi_i\right], \tag{2.31}$$

where $\varepsilon^i = u^i - v^i \delta t$ is the infinitesimal parameter of the transformation, are gauge transformations, i.e., they do not change the physical state and connect pairs $\left(q^A, p_B\right)$ describing the same physical configuration. First class primary constraints are then recognized as generators of infinitesimal gauge transformations.

The Poisson bracket preserves the first class nature of functions in phase space. This fact implies that, in general, the Poisson bracket of two first class primary constrains is a linear combination of the full set of first class constraints of the system ψ_I, i.e.,

$$\left[\psi_i, \psi_j\right] = C^I_{ij} \psi_I \tag{2.32}$$

Indicating that we can expect all the first class constraints to be generators of gauge transformations of the form

$$\delta f \approx \varepsilon^I \left[F, \psi_I\right] \tag{2.33}$$

This is known as *Dirac's conjecture* and turns out to be true for all the physical systems of interest. Thus, on the canonical variables, the most general infinitesimal gauge transformation has the form

$$\delta q^A(t) = \varepsilon^I(t)\left[q^A(t), \psi_I\right] \tag{2.34}$$
$$\delta p_B(t) = \varepsilon^I(t)\left[p_B(t), \psi_I\right]$$

where the function parameters ε^I can depend in time both implicitly through the canonical variables or explicitly.

The dynamics of the systems should allow any gauge transformation to be performed during time evolution. Therefore, following Dirac's conjecture, compatibility with the equations of motion requires to add all the first class secondary constraints to (2.27). This leads to the first class function

$$H_E = H' + u^I \psi_C \tag{2.35}$$

which is called the *extended Hamiltonian* and accounts for all the gauge freedom of the system.

2.6 Gauge Conditions

We have seen that the presence of first class constraints implies a gauge freedom in a constrained system, which is reflected in the arbitrary functions appearing in the equations of motion (2.15). Two sets of variables (q^A, p_A) and $(q^A + \delta q^A, p_A + \delta p_A)$, where δq^A and δp_A are given by (2.34) describe the same physical configuration. In order to quantize the system, however, it is practical to eliminate this arbitrariness and isolate the physical degrees of freedom. This can be done by implementing extra constraints by hand, called *gauge conditions,* to restrict phase space even more in such a way that there exist a one to one correspondence between physical configurations and values for the canonical variables in a reduced phase space Γ. The role of these extra conditions is to select one single representative of each set of canonical variables related by gauge transformations (2.34). These sets correspond to orbits in phase space, along which the action as well as the observables of the system take the same value.

A set of gauge conditions will be denoted by

$$G_I(q, p) = 0 \tag{2.36}$$

and in order to select one single representative of each orbit in phase space, must satisfy the following conditions:

- Accessibility: Given any set of canonical variables (q^A, p_A), there must be a gauge transformation obtained by iterations of infinitesimal transformations (2.34), taking (q^A, p_A) to some other set (q'^A, p'_A) satisfying the gauge conditions (2.36).

- Complete gauge fixation: Given a set of canonical variables (q^A, p_A) satisfying (2.36), there must not be any other set of variables connected with (q^A, p_A) by gauge transformations that also satisfies (2.36).

The second condition implies that there must be no gauge transformation other than the identity that preserves the gauge conditions. This means the the system of equations obtained after applying a gauge transformation to the gauge constraints,

$$\delta G_I(q,p) = \varepsilon^J [G_J, \gamma_J] \approx 0, \tag{2.37}$$

must lead to

$$\varepsilon^J = 0. \tag{2.38}$$

This requires $[G_I, \gamma_J]$ to be an invertible matrix and therefore the number of gauge conditions must be the same as the number of first class constraints in the system. The fact that

$$\det [G_I, \gamma_J] \neq 0$$

implies that, once the gauge is fixed, all the constraints are second class. Then, defining a Dirac bracket for the whole set of constraints

$$\gamma_N = \{G_I, \psi_J, \chi_\Omega\} \tag{2.39}$$

allows to set all the constraints strongly to zero and only the physical degrees of freedom of the theory are left, whose number is given by

$$\left(\begin{array}{c} N \text{ physical} \\ \text{degrees of freedom} \end{array} \right) = \frac{1}{2} \left[\left(\begin{array}{c} N \text{ canonical} \\ \text{variables} \end{array} \right) - \left(\begin{array}{c} N \text{ second class} \\ \text{constraints } \chi_\Omega \end{array} \right) \right] - \left(\begin{array}{c} N \text{ first class} \\ \text{constraints } \gamma_A \end{array} \right). \tag{2.40}$$

The relation (2.40) can be understood in the following way. Second class constraints come always in pairs, as the Dirac matrix for them is eve-dimensional. This comes from the fact that the Dirac matrix must be non-degenerate in order to define a true second. class set and odd-dimensional skew-symmetric matrices have always vanishing determinant. Thus, every pair of second class constraints eliminates a pair of canonical variables and therefore one degree of freedom of the theory. First class constraints, on the other hand, are eliminated once the gauge constraints (2.36) are introduced by hand. Together with the gauge conditions, first class constraints form pairs to eliminate pairs of canonical variables. This means that every first class constraint eminitanes one non-physical degree of freedom the theory. Putting all together, (2.40) is obtained.

Chapter 3

The Gribov Problem

In this chapter, the semi-classical procedure to restrict the path integral formulation of Yang-Mills theory to the Gribov region is discussed following the lines of [2, 20].

3.1 Gauge Fixing

The action functional for the $SU(N)$ Yang-Mills theory is given by

$$I_{YM}[A] = -\frac{1}{4g_0^2}\int d^4x F^a_{\mu\nu}F_a^{\mu\nu} \ , \tag{3.1}$$

where g_0 is the coupling constant, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc}A^b_\mu A^c_\nu$ is the field strength tensor associated to the gauge connection $A_\mu = A^a_\mu e_a$ and $\{e_a\}$ are the generators of the $\mathfrak{su}(N)$ algebra

$$[e_a, e_b] = f^c{}_{ab} e_c \ ,$$

with $f^a{}_{bc}$ the structure constants. The action (3.1) is invariant under $SU(N)$ gauge transformations

$$A_\mu \to A^g_\mu = g^{-1}\left(A_\mu + \partial_\mu\right) g \ \ , \ \ g \in SU(N) \ , \tag{3.2}$$

which defines naturally an equivalence relation for gauge potentials. Due to invariance under gauge transformations, the configuration space $\mathscr{A}$ for a gauge theory is divided in equivalence classes, each one corresponding to a different physical state. For Yang-Mills theory, two gauge potentials are *equivalent* if they are related by a gauge transformation (3.2)

$$A_\mu \sim A^g_\mu \ .$$

Thus, for a fixed A_μ, all the fields A'_μ that can be obtained from A_μ form a *gauge orbit* in configuration space, which corresponds to an equivalence class for the equivalence relation $\sim$.

Consequently, two gauge fields belonging to different orbits can not obtained from each other by gauge transformations.

Since the action functional I_{YM} is gauge-invariant, it takes a the same value at every point of a gauge orbit and physics does not change as we move along it.

The *physical* configuration space, denoted by $\mathscr{A}_{phy}$, is given by the quotient space

$$\mathscr{A}_{phy} = \mathscr{A}/G \ .$$

The prescription to obtain the physical configuration space from $\mathscr{A}$ is called *gauge fixing* and consist in imposing some constraints or *gauge conditions*

$$G^a\left[A_\mu\right] = 0 \ , \tag{3.3}$$

which must satisfy the following conditions[1]:

- The gauge conditions must be accessible: given a gauge potential A_μ, there must exist some $g \in G$ such that the transformed field A_μ^g satisfies the gauge conditions. In other words, the surface defined by G^a must intersect every orbit.

- The gauge conditions must fix the gauge completely: given a gauge field A_μ satisfying the gauge conditions, no other field related with A_μ by a gauge transformation can satisfy them. In other words, the surface defined by (3.3) must intersect every orbit only once.

When the gauge conditions satisfy this properties, they select one representative of every orbit and define a region in configuration space isomorphic to $\mathscr{A}_{phy}$, called "fundamental modular region".

The quantization of Yang-Mills theory is carried out defining the functional integral

$$Z = \mathscr{N} \int \mathscr{D}A \exp\left(-I_{EYM}\right) \ , \tag{3.4}$$

where I_{EYM} is action functional for $SU(N)$ Euclidean Yang-Mills theory

$$I_{EYM}[A] = \frac{1}{4g_0^2} \int d^4x F^a_{\mu\nu} F_a^{\mu\nu} \ , \tag{3.5}$$

$\mathscr{N}$ is a normalization and $\int \mathscr{D}A$ denotes the sum over all possible configurations. The action I_{EYM} is gauge invariant, but the integration runs over all gauge potentials A_μ including those which are related by gauge transformations. In other words, (3.4) includes the volume of the

[1] These conditions are the field theory version of the conditions defined in Section 2.6 for the gauge constraints (2.36). In the Hamiltonian formulation of Yang-Mills theory, the gauge conditions (3.3) correspond to constraints necessary, in principle, to render the first class constraints of the theory second class [46].

gauge group G, once per each space-time point. Because of this reason the integral in Z is divergent, as it sums over each physical configuration infinitely many times.

A way to overcome this problem is to restrict the sum over path integrals to one representative from each gauge orbit in such a way that the integration domain in Z is given by the fundamental modular region. In order to sum only over inequivalent configurations, a gauge fixing condition must be implemented via the Faddeev-Popov's trick. In the Landau gauge $\partial^\mu A_\mu = 0$, the gauge fixed path integral has the standard form [55],

$$Z = \mathscr{N} \int DA\delta\left(\partial^\mu A_\mu\right) \det\mathscr{M} \exp\left(-S_{EYM}\right) \ , \tag{3.6}$$

where $\mathscr{M}$ is the Faddeev-Popov operator for the Landau gauge:

$$\mathscr{M}^a{}_b = -\partial^\mu \left(D_\mu\right)^a{}_b \ , \tag{3.7}$$

and $\left(D_\mu\right)^a{}_b = \delta^a_b \partial_\mu - f^a{}_{bc} A^c_\mu$ is the covariant derivative in the adjoint representation.

In order for this procedure to be well-defined, the gauge fixing conditions must satisfy the properties previously explained, namely, they must be accessible and fix the gauge completely. Gribov showed, however, that the Coulomb or the Landau gauge does not fix the gauge completely [2]. In other words, the surface defined by the coulomb gauge in configuration space intersects some orbits more that once. After that, Singer showed that all gauge fixing conditions involving derivatives of the gauge fields have this problem [17]. This obstruction to achieve a proper gauge fixing is called *Gribov ambiguity*.

3.2 Gribov Region

Let's consider a gauge field configuration $A_\mu(x)$ satisfying some gauge condition

$$G^a\left[A_\mu\right] = 0 \ . \tag{3.8}$$

If this condition fixes the gauge completely, there can not be gauge fields related with $A_\mu(x)$ by a gauge transformation satisfying the same gauge. This means that the equation

$$G^a\left[A^g_\mu\right] = G^a\left[g^{-1}A_\mu g + g^{-1}\partial_\mu g\right] = 0 \tag{3.9}$$

must have the identity as the unique solution. In practice, this requirement is not fulfilled by the Landau or Coulomb gauge and nontrivial solutions for this equation, which are called *Gribov copies*, can be found [20–24].

In the case of infinitesimal gauge transformations, $\delta A_\mu = D_\mu \alpha$, the equation for copies (3.9) reduces to

$$G^a\left[A^g_\mu\right] = G^a\left[\left(A_\mu + D_\mu \alpha\right)\right] = 0 \ ,$$

$$\Longrightarrow \int d^4z \frac{\delta G^a \left[A_\mu (x)\right]}{\delta A^b_\mu (z)} \left(D^{(z)}_\mu\right)^b{}_c \alpha^c (z) = 0 \ , \tag{3.10}$$

and writing

$$\left(D^{(z)}_\mu\right)^a{}_b \alpha^b (z) = \int d^4y \frac{\delta A^{ag}_\mu (z)}{\delta \alpha^b (y)} \alpha^b (y) \ ,$$

the condition for the existence of copies takes the form

$$\int d^4y d^4z \frac{\delta G^a \left[A^g_\mu (x)\right]}{\delta A^{bg}_\mu (z)} \frac{\delta A^{bg}_\mu (z)}{\delta \alpha^c (y)} \alpha^c (y) = \int d^4y \frac{\delta G^a \left[A^g_\mu (x)\right]}{\delta \alpha^b (y)} \alpha^b (y) = 0 \ ,$$

$$\Longrightarrow \int d^4y \mathscr{M}^a{}_b(x,y) \alpha^b (y) = 0 \ . \tag{3.11}$$

Hence, Gribov copies connected with the identity correspond to zero modes of the Faddeev Popov operator (3.7). The presence of this copies imply that the gauge fixing procedure has failed and the functional integral Z is ill-defined. Furthermore, if zero modes for the Faddeev-Popov operator exist, the determinant in (3.6) vanishes and so does the functional integral.

In the case of the Landau gauge, the Faddeev-Popov operator is given by (3.7) and this equation takes the form

$$\partial^\mu \left(D_\mu\right)^a{}_b \alpha^b = 0 \ . \tag{3.12}$$

Let us consider now the eigenvalue equation for the Faddeev-Popov operator in the Landau gauge, i.e.,

$$-\partial^\mu \left(D_\mu\right)^a{}_b \alpha^b = \varepsilon \left(A_\mu\right) \alpha^a \ . \tag{3.13}$$

For vanishing gauge potentials this expression takes the form

$$-\partial^\mu \partial_\mu \alpha^a = \varepsilon \alpha^a \ ,$$

which has positive eigenvalues $\varepsilon = p^2$. This means that for small enough gauge fields A^a_μ the equation (3.13) has only positive eigenvalues. In particular, for euclidean abelian fields, the equation for the existence of Gribov copies (3.12) reduces to $\partial^\mu \partial_\mu \alpha = 0$ which has no smooth solutions apart from constant α's, which are proportional to the identity, implying that there are no Gribov copies in this abelian case. Therefore, for nonabelian gauge fields which are small everywhere, Gribov ambiguity does not affect perturbation theory. However, for sufficiently large gauge fields, a zero mode $\varepsilon_1 = 0$ appears, implying the presence of a Gribov copy (3.12), and for even larger values of A^a_μ the eigenvalue ε_1 becomes negative. For greater magnitudes of the gauge field, another eigenvalue ε_2 will vanish, becoming negative as the gauge field increases and so on. Because of this, the functional configuration space for the theory can be divided in regions C_n, over which the Faddeev-Popov operator has n negative eigenvalues separated by hypersurfaces δC_n where Gribov copies exist.

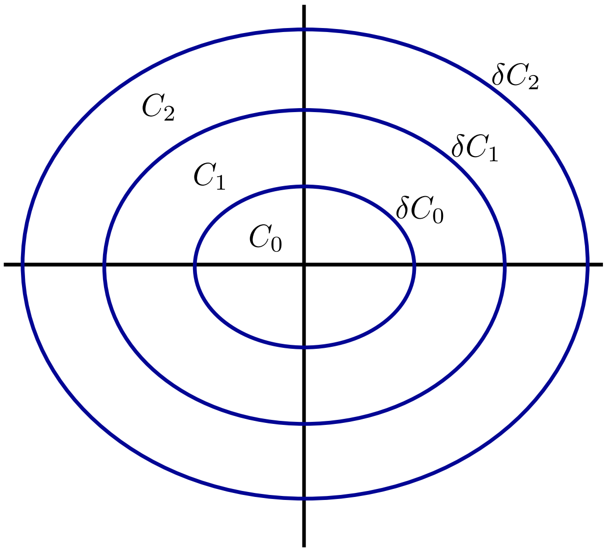

Fig. 3.1 Gribov horizons.

As a way to eliminate copies, Gribov proposed to restrict the integration domain in Z to the region C_0, in which the Faddeev-Popov operator is positive definite and doesn't reach any zero mode

$$C_0 \equiv \left\{ A_\mu, \partial^\mu A_\mu = 0 \middle| \det \mathscr{M} > 0 \right\} \ . \tag{3.14}$$

This is called the *Gribov region* and its boundary δC_0 is the *Gribov horizon*.

3.2.1 Alternative Definition

The Gribov region can also be defined as the set of local minima for the following functional

$$f[U] = \int d^4x tr \left[A^U_\mu A^{\mu U} \right] = \| A^U_\mu \|^2 \ . \tag{3.15}$$

In fact, the Euler-Lagrange equations obtained from with functional correspond to the equation for Gribov copies (3.9) in the Landau gauge**,**

$$\delta f = 0 \Longrightarrow \partial^\mu A^U_\mu = 0 \ .$$

The second variation of f leads to the Faddeev-Popov operator. In fact considering infinitesimal transformations,

$$\delta^2 f = \int d^4x tr \left[\alpha_0 \left(-\partial^\mu D_\mu A^U \right) \alpha_0 \right] \ ,$$

which must be positive if U is a minimum. Therefore, for every orbit there is at least one gauge field that is transversal and for which the Faddeev-Popov operator is positive definite.

G. Dell'Antonio and D. Zwanziger [28] proved that any gauge potential lying outside the Gribov region is a copy of a field inside the Gribov region. This means that every orbit intersect the Gribov region. This restriction eliminates copies connected with the identity. Copies with nontrivial winding number are not considered and thus, after restricting path integrals to the Gribov regions, integration over the fundamental modular region is still not achieved. However this is a remarkable improvement and has important effects in the theory. In the case of QCD, the theory is modified in the infrared opening a way to understand confinement of color [41].

3.3 Semi-classical Gribov Approach to QCD

We have seen in the previous section that, due to the presence of Gribov copies, the expression (3.6) is ill defined. To avoid zero modes of the Faddeev-Popov operator and eliminate copies, we will follow the strategy of restricting the relevant functional integrals to the so-called Gribov region (3.14), which corresponds to the region in the functional space of gauge potentials over which the Faddeev-Popov operator is positive definite. The restriction of (3.6) to the Gribov region can be implemented by redefining the generating functional as [20]

$$Z_G = \mathscr{N}\int DA\delta\left(\partial^\mu A_\mu\right)\det\left(\mathscr{M}\right)\exp\left(-S_{YM}\right)\mathscr{V}\left(C_0\right)\ , \tag{3.16}$$

where the factor $\mathscr{V}(C_0)$ ensures that the integration is performed only over C_0. In order to characterize $\mathscr{V}(C_0)$, we look at the connected two-point ghost function generated by (3.6):

$$\left\langle \bar{c}^a(x)\, c^b(y)\right\rangle = \mathscr{N}\int DA\delta\left(\partial^\mu A_\mu\right)\exp\left(-S_{YM}\right)\det\left(\mathscr{M}\right)\left(\mathscr{M}^{-1}(x,y)\right)^{ab}\ . \tag{3.17}$$

Singularities in (3.17) correspond to zero modes of the Faddeev-Popov operator, i.e. infinitesimal Gribov copies. In the momentum representation, singularities different from $k^2=0$ imply that $\mathscr{M}(x,y)$ can become negative definite, and therefore it is evaluated outside the Gribov horizon. The factor $\mathscr{V}(C_0)$ must be such that this kind of singularities is not present. This is known as the *no-pole condition*.

The standard connected ghost two-point function (3.17) can be put in the form

$$\left\langle \bar{c}^a(x)\, c^b(y)\right\rangle = \mathscr{N}\int DADcD\bar{c}\delta\left(\partial^\mu A_\mu\right)\exp\left(-S_{YM}\right)\langle \bar{c}^a(x)c^b(y)\rangle_A\ , \tag{3.18}$$

with $\langle \bar{c}^a(x)c^b(y)\rangle_A$ the connected ghost two-point function with A^a_μ playing the role of an external field. To second order in perturbation theory this can be written in momentum space as

$$\langle \bar{c}^a c_a\rangle_{k;A} = \frac{1}{k^2}\left(1+\sigma(k,A)\right) \approx \frac{1}{k^2}\frac{1}{\left(1-\sigma(k,A)\right)}, \tag{3.19}$$

where

$$\sigma(k,A) = \frac{Nk^\mu k^\nu}{3(N^2-1)k^2} \frac{1}{V} \sum_q \frac{A^{a\lambda}(-q)A_{a\lambda}(q)}{(k-q)^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) , \tag{3.20}$$

and V stands for the four-dimensional volume of the Euclidean space-time. Since $A_\mu^a(-q)A_{a\nu}(q)$ is a decreasing function of q^2, $\sigma(k,A)$ decreases as k^2 increases and the no-pole condition can be stated as

$$\sigma(0,A) = \frac{1}{4}\frac{N}{N^2-1}\frac{1}{V}\sum_q \frac{1}{q^2} A_\mu^a(-q)A_a^\mu(q) < 1 . \tag{3.21}$$

Hence, the factor $\mathcal{V}(C_0)$ needed in (3.16) to restrict path integrals to the Gribov horizon is given by $\mathcal{V}(C_0) = \Theta(1-\sigma(0,A))$, where $\Theta(x) = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\eta \frac{e^{\eta x}}{\eta}$ is the Heaviside step function. Implementing this factor in Z_G, the quadratic part of the path integral in the field A_μ can be put in the form

$$Z_G^{quad} = \mathcal{N}\int \frac{d\eta}{2\pi i} e^{f(\eta)} \ , \ f(\eta) = \eta - \ln\eta - \frac{3}{2}(N^2-1)\sum_q \ln\left(q^2 + \frac{\eta N g_0^2}{N^2-1}\frac{1}{2V}\frac{1}{q^2}\right) . \tag{3.22}$$

Using the steepest descent (saddle point) method, (3.22) can be approximated by $Z_G^{quad} \approx e^{f(\eta_0)}$, where η_0 satisfies the minimum condition $f'(\eta_0) = 0$. Defining the Gribov parameter $\gamma^4 = \frac{\eta_0 N g_0^2}{N^2-1}\frac{1}{2V}$, the minimum condition leads to the gap equation

$$1 - \frac{Ng_0^2}{\gamma^4(N^2-1)2V} - \frac{3Ng_0^2}{4V}\sum_q \frac{1}{q^4+\gamma^4} = 0 . \tag{3.23}$$

The solution of this equation in the infinite volume limit $V \to \infty$ is given by $\gamma^2 = \Lambda^2 e^{-\frac{64\pi^2}{3Ng_0^2}}$, where Λ is the ultraviolet cutoff, and it leads to a confining gauge propagator [20]

$$D_{\mu\nu}^{ab}(q) = \delta^{ab} g_0^2 \frac{q^2}{q^4+\gamma^4}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) . \tag{3.24}$$

For large q, (3.24) reduces to the standard perturbative result [55]. In the infrared, however, the gluon propagator is suppressed, as it displays imaginary poles. In other words, since $D_{\mu\nu}^{ab}(q)$ has a positivity violating Källén-Lehmann representation [40, 55], gluons cannot be considered as part of the physical spectrum and the propagator (3.24) is interpreted as confining. Replacing (3.24) in (3.20) leads to the following behavior for the ghost propagators (3.19), in the infrared limit:

$$\langle \bar{c}^a c_a \rangle_{q;A} \underset{q\to 0}{\longrightarrow} \frac{128\pi\gamma^2}{3Ng_0^2}\frac{1}{q^4} , \tag{3.25}$$

which means that the ghost propagator is not free-like, but enhanced for $q \to 0$.

Chapter 4

Semi-Classical Gribov Approach at Finite Temperature

Finite-temperature Yang-Mills theory can be studied using the imaginary time formalism [42, 56], which relates the corresponding quantum field theory generating functional with a quantum statistical partition function through a compactification of the temporal coordinate. In this formalism, the period of the compactified time is associated with the inverse of the temperature of a thermal bath, and the partition function can be written as

$$Z = \int DA \exp\left(\frac{1}{4g_0^2}\int_0^{\frac{1}{T}} d\tau \int d^3x F^a_{\mu\nu}F_a^{\mu\nu}\right) \ . \tag{4.1}$$

Since the temporal integration limits 0 and T^{-1} are identified, when passing to momentum space, temperature dependent fields are expanded in a Fourier series over discrete Matsubara frequencies ω_n.

$$\varphi(\tau,\mathbf{x}) = T^{\infty}_{n=-\infty}\int \frac{d^3q}{(2\pi)^3} e^{-i(\omega_n\tau+\mathbf{q}\cdot\mathbf{x})}\varphi(\omega_n,\mathbf{q}) \ , \quad \omega_n = 2\pi nT \ . \tag{4.2}$$

4.1 Dynamical Thermal Mass

When implementing the gauge fixing, the finite-temperature formalism must be applied to the generating functional (3.6), where the Euclidean action has to be written as a local functional for ghost and gauge fields and perturbation theory can be applied. For gluons, when considering one-loop corrections, the resumed gauge propagator in the Landau gauge takes the form [43]

$$D^{ab}_{\mu\nu}(q) = g^2\delta^{ab}\left(\frac{P^T_{\mu\nu}(q)}{q^2+\Pi_T(q)} + \frac{P^L_{\mu\nu}(q)}{q^2+\Pi_L(q)}\right) \ , \tag{4.3}$$

where g is the running coupling and

$$P^T_{\mu\nu}(q) = \delta^i_\mu \delta^j_\nu \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right) \,, \qquad (4.4)$$
$$P^L_{\mu\nu}(q) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - P^T_{\mu\nu}(q) \,,$$

are transverse projectors orthogonal to each other, ($P^T_{\mu\nu} q^\nu = P^L_{\mu\nu} q^\nu = 0$, $\delta^{\rho\sigma} P^T_{\mu\rho} P^L_{\sigma\nu} = 0$) and $\Pi_T(q)$, $\Pi_L(q)$ are the components of the self-energy $\Pi_{\mu\nu}$ along the projectors (4.4)

$$\Pi_{\mu\nu}(q) = P^T_{\mu\nu}(q)\,\Pi_T(q) + P^L_{\mu\nu}(q)\,\Pi_L(q) \,. \qquad (4.5)$$

In the plasma region, where $\omega_n >> |\mathbf{q}|$, the self-energy components $\Pi_T(q)$, $\Pi_L(q)$ are given, in the hard thermal loop approximation, by

$$\Pi_T(q) = \Pi_L(q) \approx \frac{Ng^2T^2}{9} \,, \qquad (4.6)$$

which means that, in a hot plasma, gauge fields acquire an effective thermal mass [43]

$$m^2_{pl} = \frac{Ng^2T^2}{9} \,. \qquad (4.7)$$

In this case the gauge propagator ((4.3)) takes the form

$$D^{ab}_{\mu\nu}(q) = \frac{g^2\delta^{ab}}{q^2 + m^2_{pl}} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \,. \qquad (4.8)$$

It is worth noting that ghost fields do not acquire a thermal mass [42], which implies that the no-pole condition (3.21) has no extra terms when one-loop corrections are considered. However, the expression for the gap equation will be modified by the presence of the effective thermal mass (4.7), as we will see below.

The effect of a dynamical mass m in the semi-classical Gribov approach can be obtained by adding a term of the form $m^2 A_\mu A^\mu$ to the quadratic action in (3.22). This approach was studied in [57] and modifies the gap equation (3.23) as

$$1 - \frac{3g^2}{\gamma^4 (N^2-1)\, 2V} - \frac{3Ng^2}{4V} \sum_q \frac{1}{q^4 + m^2 q^2 + \gamma^4} = 0 \,. \qquad (4.9)$$

The solution of this equation, if it exists, defines a massive (partially) confining gauge propagator

$$\bar{D}^{ab}_{\mu\nu}(q) = \delta^{ab} g^2 \frac{q^2}{q^4 + m^2 q^2 + \gamma^4} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \,. \qquad (4.10)$$

The confining character of this propagator relies on the presence of imaginary poles, which violates positivity of the spectral density function of the Källén-Lehmann representation [40, 55], indicating that it describes non-physical excitations. However, the presence of a dynamical mass m allows the possibility for the propagator (4.10) to acquire a physical degree of freedom. In fact, the poles of (4.10) are given by

$$z_{\pm} = \frac{1}{2}\left(-m^2 \pm \sqrt{m^4 - 4\gamma^4}\right) . \tag{4.11}$$

Hence, for $m^2 \geq 2\gamma^2$ the propagator $\bar{D}^{ab}_{\mu\nu}(q)$ can describe physical particles. Writing (4.10) in the form

$$\bar{D}^{ab}_{\mu\nu}(q) = \delta^{ab}\frac{g^2}{\sqrt{m^4 - 4\gamma^4}}\left[\frac{z_+}{(q^2 - z_+)} - \frac{z_-}{(q^2 - z_-)}\right]\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) , \tag{4.12}$$

we can see that the propagator splits into two terms with opposite residue sign, indicating that the gluon field A_μ has only one physical degree of freedom.

In general, if m is a function of some physical parameter, we can distinguish three scenarios for the behavior of the propagator.

- For $m^2 < 2\gamma^2$ both poles of (4.10) are complex, indicating that there are no propagating gluonic degrees of freedom (confined phase).
- For $m^2 \geq 2\gamma^2$ only one of the two gluonic degrees of freedom is physical (partially deconfined phase). Hence, if this regime appears (as will be shown in the following, it does) it shows qualitative characteristics both of the confined phase and of the deconfined phase.
- If there is no solution for the gap equation, the only consistent choice for the Gribov mass parameter is $\gamma = 0$, leading to a free gluon propagator (deconfined phase).

In the present case, the effect of the one-loop thermal mass (4.7) on the Gribov restriction will be considered by setting $m = m_{pl}(T)$, and it will be shown that there exist critical temperatures corresponding to the above three different regimes. It is worth noting that the inclusion of such a one-loop mass is fundamental in order to obtain these different phases.

4.2 Thermal Gap Equation

Two important requirements for the consistency of the analysis are the following. Firstly, the finite-temperature gap equation should have, when the temperature is low enough, solutions close to the zero-temperature one, describing confined gluons. Secondly, when the temperature is high enough, the gap equation should have no solution, which describes propagating gluons.

As is well known, these conditions are not easy to satisfy [58, 59]. In the present analysis, we will include the one-loop perturbative corrections both in the running coupling and in the field propagators (since the crucial role of the one-loop mass is well known: see [60] and references therein). In order to write down the gap equation for the finite-temperature case, we apply the prescription (4.2) to (4.9) and take the infinite spatial volume limit

$$\frac{1}{V}\sum_{q} \to T\sum_{n}\int \frac{d^3q}{(2\pi)^3} \; . \tag{4.13}$$

Finally, replacing the thermal gluon mass (4.7), we obtain the following thermal gap equation:

$$\frac{3Ng^2T}{8\pi^2}\sum_{n}\int_0^{\Lambda} \frac{r^2dr}{(r^2+\omega_n^2)^2+\frac{Ng^2T^2}{9}(r^2+\omega_n^2)+\gamma^4} = 1 \; , \tag{4.14}$$

where we have adopted polar coordinates, integrated over angular variables, and we defined a radial integration limit Λ, which corresponds to an ultraviolet cutoff. Let us note that we have neglected the second term of (4.9), as it goes to zero for an infinite spatial volume. Defining the dimensionless variables

$$\begin{aligned} R &= \frac{r}{\Lambda} \quad , \quad \lambda = \frac{2\pi T}{\Lambda} \; , \\ \theta_n &= \frac{\omega_n}{\Lambda} = n\lambda \quad , \quad \Gamma = \frac{\gamma}{\Lambda} \; , \end{aligned} \tag{4.15}$$

the thermal gap equation can be rewritten as

$$\frac{3Ng^2\lambda}{16\pi^3}\sum_{n}\int_0^{1} \frac{R^2dR}{(R^2+\theta_n^2)^2+\frac{Ng^2\lambda^2}{36\pi^2}(R^2+\theta_n^2)+\Gamma^4} = 1 \; . \tag{4.16}$$

The sum over all dimensionless Matsubara frequencies θ_n can be carried out analytically (see Appendix A), leading to

$$\begin{aligned} S(R,\lambda,\Gamma) &= \sum_{n} \frac{1}{(R^2+\theta_n^2)^2+\frac{Ng^2\lambda^2}{36\pi^2}(R^2+\theta_n^2)+\Gamma^4} \\ &= \frac{\pi}{2\lambda\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}} \left(\frac{\coth\left(\frac{\pi}{\lambda}\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}-\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}\right)}{\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}-\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}} - \frac{\coth\left(\frac{\pi}{\lambda}\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}+\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}\right)}{\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}+\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}} \right) , \end{aligned} \tag{4.17}$$

Then, the gap equation takes the form

$$\frac{3Ng^2\lambda}{16\pi^3}\int_0^1 dRR^2S(R,\lambda,\Gamma) = 1 \; , \tag{4.18}$$

which defines γ as a function of λ

$$\gamma = \Lambda\Gamma(\lambda) \ . \tag{4.19}$$

4.3 The Three Regimes

As we have shown in Section 4.1, the effective gluon propagator (4.10) can lead to three different regimes for gluons depending on the value of the thermal mass $m_{pl}(T)$, which in turn depends on the temperature T. These three regimes can be associated to two transition temperatures. In this section we present the numerical analysis of the gap equation (4.18) for QCD ($N = 3$) in the high-temperature regime and subsequently we study a possible infrared continuation.

4.3.1 High Temperature Running Coupling

Let us consider the thermal gap equation in the limit of high temperatures $T >> 1$. In finite-temperature QCD, the one-loop running coupling depends on the temperature T (or, in our case, on λ) as [58, 61]

$$g^2(\lambda) = \frac{8\pi^2}{11\ln\left(\frac{2\pi T}{\Lambda_{QCD}}\right)} = \frac{8\pi^2}{11\ln(\alpha\lambda)} \ , \tag{4.20}$$

where we have defined the ratio between the cutoff Λ and the energy scale Λ_{QCD} as

$$\alpha \equiv \frac{\Lambda}{\Lambda_{QCD}} \ . \tag{4.21}$$

For the left hand side of (4.18), we define the function

$$F(\lambda,\Gamma) = \frac{9g^2\lambda}{16\pi^3}\int_0^1 dRR^2 S(R,\lambda,\Gamma) \ . \tag{4.22}$$

Then the solution for the gap equation corresponds to the intersection of the curves $Y = F(\lambda,\Gamma)$ with $Y = 1$. In order to obtain the qualitative behavior for the solutions, we will consider $\alpha = 1$ in the analysis below (as it will be explained later on, the qualitative behavior of the gluon propagator does not depend on the value of α). From Figure 4.1, we see that the existence of solution depends on the temperature. In fact, the intersection occurs for λ's below a critical value $\lambda_c^{(1)} = 1.4$, see Figure 4.2. This corresponds to a phase transition at temperature

$$\frac{T_c^{(1)}}{\Lambda_{QCD}} = 0.22 \ . \tag{4.23}$$

For $T > T_c^{(1)}$ there is no solution for the gap equation (4.18). In this case the only consistent choice for the Gribov parameter is $\gamma = 0$, indicating that this regime represents the free phase.

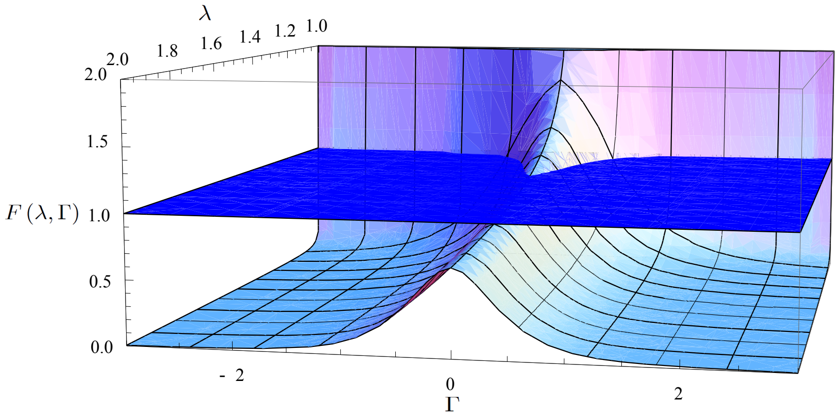

Fig. 4.1 Plot of the surface F for different values of λ and Γ. The intersection with the plane $Y = 1$ occurs for λ below the critical value $\lambda_c^{(1)} = 1.4$.

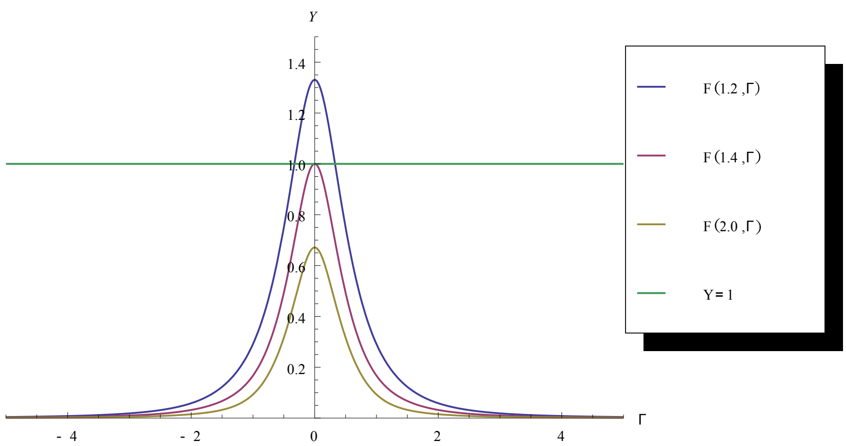

Fig. 4.2 Plot of F as a function of Γ, for $\lambda = 1.2$, 1.4 and 2.0.

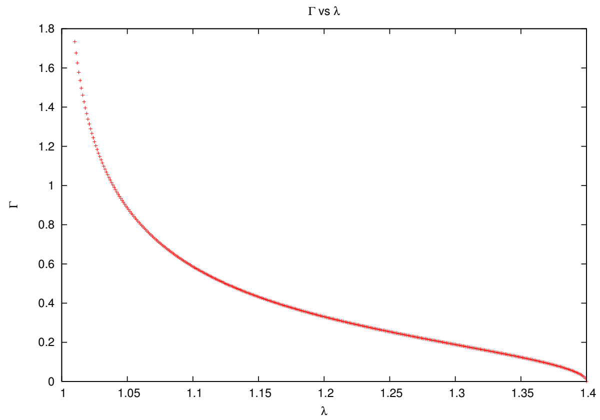

Fig. 4.3 Plot Γ vs. λ. At $\lambda^{(1)} \sim 1.4$ there exists a phase transition from a deconfined phase to a semi-confined one, which corresponds $\frac{T_c^1}{\Lambda} = 0.22$.

On the other hand, for $T < T_c^{(1)}$, there is a solution for the gap equation, which define the Gribov parameter γ. Therefore, as is shown in Figure 4.3, $\Gamma = \gamma/\Lambda$ decreases as λ increases and vanishes for $\lambda^{(1)} = 1.4$. Even though for $\lambda < 1.4$ there is a solution for the gap equation, the propagator is still not completely confining. As we saw in Section 4.1, depending on the sign of the discriminant in (4.11), a partial or total confinement can take place. In this case, the change of sign in (4.11) occurs for $\lambda_c^{(2)} = 1.08$ (see Figure 4.4), which corresponds to

$$\frac{T_c^{(2)}}{\Lambda_{QCD}} = 0.17 \ . \tag{4.24}$$

Hence, two phase transitions are found as the temperature decreases: a deconfined/partially deconfined phase transition at $T_c^{(1)}$ and a partially deconfined/confined phase transition at $T_c^{(2)}$. In the intermediate phase, only one degree of freedom of the gluon field is physical.

4.3.2 Infrared Continuation

In order to extend the analysis of the previous subsection to the low-temperature regime, we need a prescription to extend the definition (4.20) for $\lambda < 1$. A way to extend the running coupling to the infrared regime in zero-temperature QCD has been developed in [62] in the framework of quark-antiquark potentials by adding a non-perturbative contribution to the Wilson loop. In the finite-temperature case, the analog extension reads

$$g^2(g_0, \lambda) = \frac{g_0^2}{1 + \frac{11}{16\pi^2} g_0^2 \ln(1 + \alpha^2 \lambda^2)} \ . \tag{4.25}$$

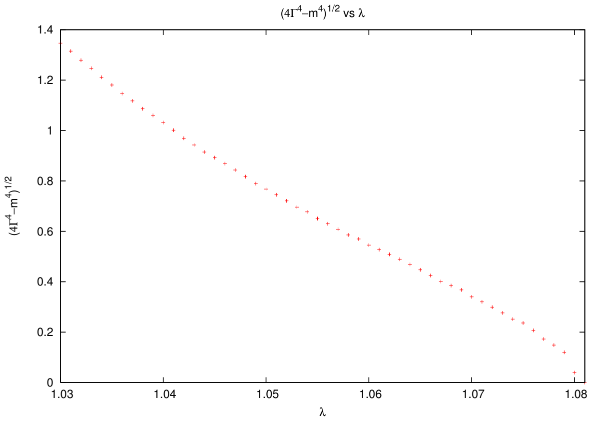

Fig. 4.4 Plot $\sqrt{4\Gamma^4 - m^4}$ vs. λ. At $\lambda^{(2)} \sim 1.08$ there exists a phase transition from a semi-confined phase to a confined one, which corresponds $\frac{T_c^2}{\Lambda} = 0.17$.

This expression reduces to (4.20) for large λ but, in the limit $\lambda \to 0$ the running coupling reduces to the bare coupling constant g_0

$$g^2 \underset{\lambda \to 0}{\longrightarrow} g_0^2 \ .$$

This choice is also consistent with the fact that the thermal gluon mass (4.7) must vanish as T goes to zero

$$m_{pl}^2 \underset{T \to 0}{\longrightarrow} 0 \ .$$

which is a necessary requirement to reduce (4.9) to (3.23) in this limit and to connect consistently with the standard $T = 0$ results [20]. Let us note that for large g_0 the behavior of $g(g_0, \lambda)$ becomes insensible to small variations of g_0 itself; see Figure 4.5. This is also consistent with the fact that in quantum field theory bare quantities are infinite but unobservable and they need to be renormalized. Replacing the expression (4.25) (with $\alpha = 1$) in the gap equation (4.18), the left hand side takes the form

$$G(g_0, \lambda, \Gamma) = \frac{9g^2\lambda}{16\pi^3} \int_0^1 dR R^2 S(R, g_0, \lambda, \Gamma) \ , \tag{4.26}$$

where $S(R, g_0, \lambda, \Gamma)$ is obtained replacing (4.25) in (4.17). Then the solution for the gap equation again corresponds to the intersection of the curves $Y = G(g_0, \lambda, \Gamma)$ and $Y = 1$, whose existence depends on λ (see Figure 4.6). Similarly to the previous subsection, we find two phase transitions. Choosing $g_0 = 1000$, the deconfined/partially deconfined phase transition occurs for the critical value $\lambda_c^{(1)} = 1.17$ (see Figures 4.7 and 4.8), which corresponds to

$$\frac{T_c^{(1)}}{\Lambda_{QCD}} = 0.19 \ , \tag{4.27}$$

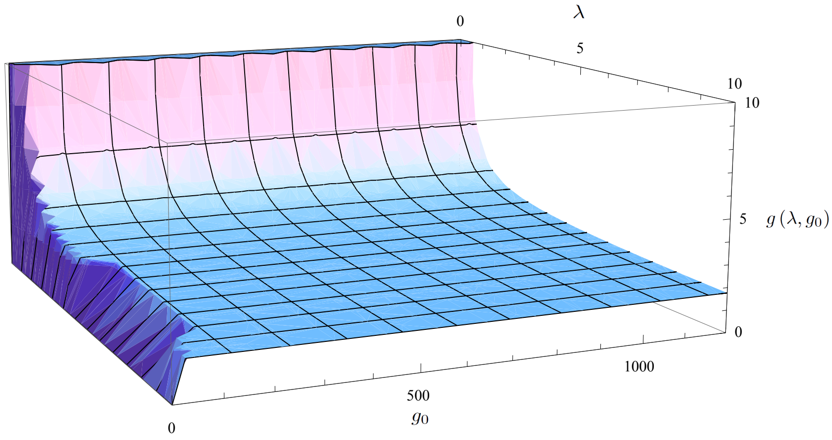

Fig. 4.5 Plot of the running coupling g as a function of g_0 and λ. For g_0 large, g becomes almost insensible to small variations of g_0.

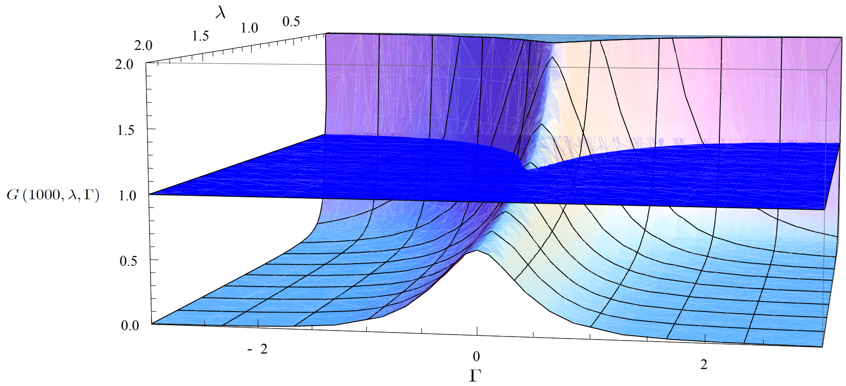

Fig. 4.6 Plot of the surface F for different values of λ and Γ. The intersection with the plane $Y = 1$ occurs for λ below a critical value $\lambda_c^{(1)} = 1.17$.

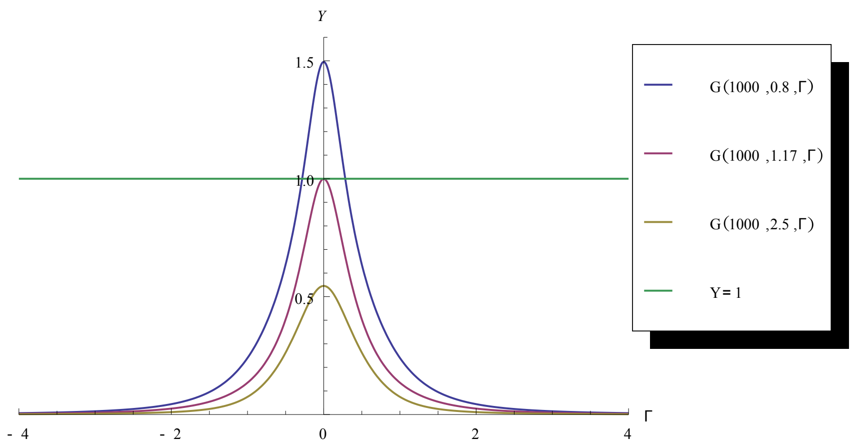

Fig. 4.7 Plot of F as a function of Γ, for $\lambda = 0.8$, 1.17 and 2.5.

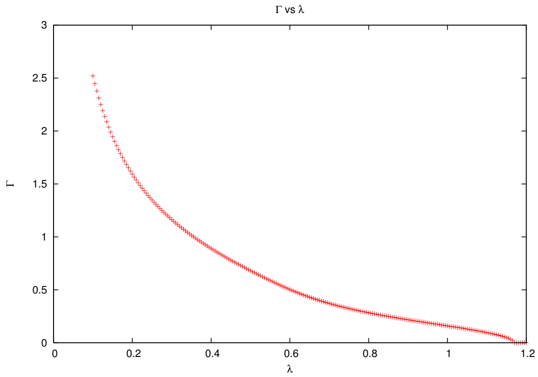

Fig. 4.8 Plot Γ vs. λ. At $\lambda^{(1)} \sim 1.17$ there exists a phase transition from a deconfined phase to a semi-confined one, which corresponds to $\frac{T_c^{(1)}}{\Lambda} = 0.19$.

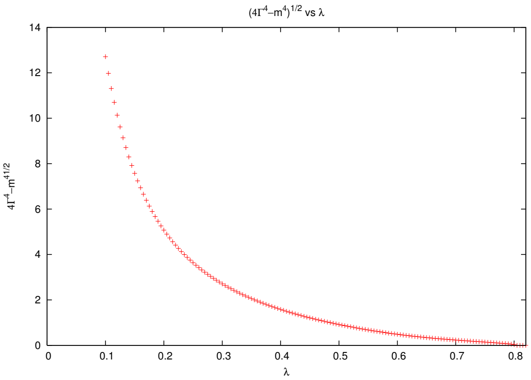

Fig. 4.9 Plot $\sqrt{4\Gamma^4 - m^4}$ vs. λ. At $\lambda^{(2)} \sim 0.81$ there exists a phase transition from a semi-confined phase to a confined one, which corresponds to $\frac{T_c^{(2)}}{\Lambda} = 0.13$.

while the partially deconfined/confined phase transition now occurs for $\lambda_c^{(2)} = 0.81$ (see Figure 4.9), i.e.,

$$\frac{T_c^{(2)}}{\Lambda_{QCD}} = 0.13 \; . \tag{4.28}$$

The results obtained with the prescription (4.25) are very similar to the ones obtained in the previous subsection. It is important to note that the qualitative behavior of the solution of the gap equation and the gluon propagator does not depend on the value of α in the definition (4.21). As we can see in Table 4.1, the greater the value of α that we consider in the analysis (i.e. the greater the cutoff Λ compared with QCD scale Λ_{QCD}), the greater will be the numerical values for the critical temperatures for the phase transitions. Hence, the fact that the integration cutoff Λ is much higher than the QCD scale Λ_{QCD} implies that the critical temperatures obtained with this method, when considering a more realistic ratio between this quantities, will be greater than the values obtained in this section.

	g_{HT}		g_{IC}	
α	$\frac{T_c^{(1)}}{\Lambda_{QCD}}$	$\frac{T_c^{(2)}}{\Lambda_{QCD}}$	$\frac{T_c^{(1)}}{\Lambda_{QCD}}$	$\frac{T_c^{(2)}}{\Lambda_{QCD}}$
1	0.223	0.172	0.186	0.128
10	0.437	0.331	0.414	0.301
100	0.758	0.558	0.752	0.540

Table 4.1 Critical temperatures $\frac{T_c^{(1)}}{\Lambda_{QCD}}$ and $\frac{T_c^{(2)}}{\Lambda_{QCD}}$ for different values of α. Here, g_{HT} and g_{IC} correspond to the running coupling at high temperature (4.20) and its infrared continuation (4.25), respectively.

On the other hand, in our analysis we have considered only gluon dynamics (without quarks). In [63, 64] it has been found that the value for the energy scale Λ_{QCD} that must be considered depends on the numbers of flavors that are included in the analysis and there have been found different values for T_c/Λ_{QCD} depending on these considerations.

Chapter 5

Gribov Ambiguity and Degenerate Systems

Degenerate Hamiltonian systems are those whose symplectic form is not invertible in a subset of phase space. They are ubiquitous in physics, as degeneracies of this kind appear in fluid dynamics [49], alternative theories of gravity [50, 51, 53, 54] and in the strong electromagnetic fields of quasars [52]. In this chapter we show that degenerate symplectic forms are also linked to the presence of Gribov ambiguities in gauge systems and study finite dimensional models in which degeneracies can be analyzed in an explicit way.

5.1 Degenerate Systems

We now briefly review classical [47] and quantum [48] degenerate systems. In order to fix ideas, let's consider a system described by the first order action,

$$I[u] = \int \mathrm{d}t \left(X_A(u)\dot{u}^A - H(u) \right) \text{, with } A = 1, \ldots, N \ . \tag{5.1}$$

This action can be interpreted in two not exactly equivalent ways:

- The u^A's are N generalized coordinates and $L(u,\dot{u}) = X(u)_A\dot{u}^A - H(u)$ is the Lagrangian [1].

- The u^A's are non-canonical coordinates in a N-dimensional phase space Γ, where N is necessarily even and (5.1) gives the action in Hamiltonian form.

[1] The notation here is different than in Chaper 2 . u^A denothes both the coordiantes and the momenta of the system and, in that case, N in (5.1) corrresponds to $2N$ in (2.1).

In the first approach, for each u there is a canonically conjugate momentum at the $2N$-dimensional canonical phase space $\tilde{\Gamma}$ given by

$$p_A = \frac{\partial L}{\partial \dot{u}^A} \ . \tag{5.2}$$

In this case, this definition gives a set of primary constraints,

$$\Phi_A = p_A - X_A(u) \approx 0 \ , \tag{5.3}$$

whose (canonical) Poisson brackets define the antisymmetric matrix

$$[\Phi_A, \Phi_B] = \partial_A X_B - \partial_B X_A \equiv \Omega_{AB}(u) \,. \tag{5.4}$$

If Ω_{AB} is invertible –which requires N to be even–, the constraints $\Phi_A \approx 0$ are second class and $\Omega_{AB}(u)$ gives the Dirac bracket (2.22) necessary to eliminate them. Elimination of these second class constraints in the $2N$-dimensional canonical phase space $\tilde{\Gamma} = \{u^A, p_A\}$ corresponds to choosing half of the u's as coordinates and the rest as momenta, and $\Omega_{AB}(u)$ will be identified as the (not necessarily canonical) pre-symplectic form in the reduced N-dimensional phase space Γ. In fact, in the Hamiltonian approach the pre-symplectic form can be read from the equations of motion for the action (5.1),

$$\Omega_{AB}(u)\, \dot{u}^A + E_A(u) = 0 \,, \tag{5.5}$$

where

$$\Omega_{AB} \equiv \partial_A X_B(u) - \partial_B X_A(u) \,, \quad \text{and} \quad E_A \equiv \partial_A H(u) \,. \tag{5.6}$$

Since Ω_{AB} is a curl, it satisfies the identity $\partial_A \Omega_{BC} + \partial_B \Omega_{CA} + \partial_C \Omega_{AB} = 0$, which shows that the presymplectic form is closed,

$$\Omega = dX \Longrightarrow d\Omega \equiv 0.$$

This reasoning shows that in the open sets where Ω_{AB} is invertible, the Lagrangian and Hamiltonian versions of this system are equivalent. In this case the inverse symplectic form, Ω^{AB}, defines the Poisson bracket for the theory in (not necessarily canonical) coordinates

$$\Omega^{AB} = [u^A, u^B] \ . \tag{5.7}$$

In what follows, we will refer to Γ as the phase space where u are the coordinates.

The pre-symplectic form $\Omega_{AB}(u)$ is a function of the phase space coordinates u^A and its determinant can vanish on some subset $\Sigma \subset \Gamma$ of measure zero. Degenerate systems are characterized by having a pre-symplectic form whose rank is not constant throughout phase space. Moreover, in its evolution a degenerate system can reach a degenerate surface Σ where $\det[\Omega_{AB}] = 0$ in a finite time,

$$\Sigma = \{u \in \Gamma \,|\, \Upsilon(u) = 0\} \,, \tag{5.8}$$

where $\Upsilon(u) = \varepsilon^{A_1 A_2 \cdots A_N} \Omega_{A_1 A_2} \cdots \Omega_{A_{N-1} A_N}$ is the Pfaffian of Ω_{AB}, and $\det[\Omega_{AB}] = (\Upsilon)^2$.

Generically, a degenerate surface represents a co-dimension one submanifold in phase space and, as shown in [47], the classical evolution cannot take the system across Σ. The equations of motion (5.5) can be solved for $\dot{u}^A$ provided Ω_{AB} can be inverted. Moreover, the velocity diverges in the vicinity of Σ, and if Ω has a simple zero, the velocity changes sign across Σ. Therefore an initial state on one side of Σ could never reach the other: there is no causal connection between configurations on opposite sides of Σ. This degeneracy surface Σ acts as a source or sink of orbits, splitting the phase space into causally disconnected, non overlapping regions.

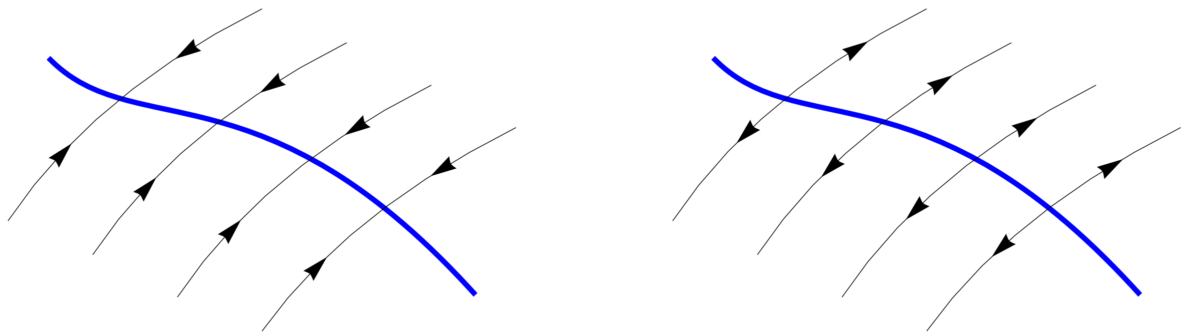

Fig. 5.1 Qualitative flow of the orbits near the degeneracy surface (blue lines), which can act as a sink or as a source.

In the quantum case, the degeneracy of the symplectic form becomes the singular set of the Hamiltonian and the corresponding Hilbert space $\mathscr{H}$ is endowed with a weighted scalar product,

$$< \Psi_1, \Psi_2 >= \int \mathrm{d}V \Psi_1^* w \Psi_2 \ , \tag{5.9}$$

where $\mathrm{d}V = \sqrt{g}\,\mathrm{d}^n u$ is the volume form and the weight $w(u)$ is the Pfaffian $\Upsilon(u)$ of the symplectic form Ω_{AB}

$$w(u) := \sqrt{\det[\Omega_{AB}(u)]} = \Upsilon(u) \, , \tag{5.10}$$

defined in order for the Hamiltonian to be symmetric and for the norm in $\mathscr{H}$ to be positive definite.

Since singular points must be excluded from the domain of the Hamiltonian operator, for consistency they should also be excluded from the domain of the wave functions. This means that the Hilbert space includes wave functions that can be discontinuous at the degenerate surfaces. Allowing discontinuous wave functions implies that the solutions can have support restricted to a single region bounded by Σ. Therefore the Hilbert space is a direct sum of orthogonal subspaces of functions defined on each side of the degenerate surface and, in complete analogy with the classical picture, there is no quantum tunneling across Σ.

5.2 Gauge Fixing and Gribov Ambiguity

The quantum description of a gauge-invariant system can be achieved by first fixing the gauge and then applying the quantization prescription to the remaining classical degrees of freedom. Let Γ be a phase space described by generalized coordinates u^A $(A = 1,2,\cdots,N)$, endowed with a symplectic form $\Omega_{AB}(u)$ everywhere invertible. Consider now an open patch of the phase space Γ where a system has local symmetries generated by a set of first class constraints $\psi_i(u) \approx 0$, $(i = 1,\ldots,n < N/2)$. Following Dirac's procedure, for a system with n first class constraints, an equal number of gauge fixing conditions,

$$G_i(u) \approx 0\,, \qquad i = 1,\ldots,n\,, \tag{5.11}$$

must be included so that the whole set of constraints

$$\{\gamma_I\} = \{G_i, \psi_j\}\,, \qquad I = 1,\ldots,2n < N\,, \tag{5.12}$$

is second class (see [46]). In order to define a proper gauge fixing, two conditions must be fulfilled: every orbit must intersect the surface defined by the set $\{G_i\}$ in Γ (accessibility), and orbits can't intersect the surface defined by $\{G_i\}$ more than once (complete gauge fixation). In other words, the surface in phase space defined by the gauge conditions (5.11) must intersect every orbit once and only once.

The submanifold defined by setting the constraints $\{\gamma_I\}$ strongly equal to zero, corresponds to the reduced gauge-fixed phase space of the system, which will be denoted by Γ_0

$$\Gamma_0 := \left\{u^A \in \Gamma\,|\,\gamma_I(u) = 0, I = 1,\ldots,2n\right\}. \tag{5.13}$$

In Γ_0 a new Poisson structure is introduced by the Dirac bracket (see (2.22))

$$[M,N]^* = [M,N] - [M,\gamma_I]C^{IJ}[\gamma_J,N]\,, \tag{5.14}$$

where C^{IJ} is the inverse of the Dirac matrix constructed from the second class constraints $\{\gamma_I\}$,

$$C_{IJ} = [\gamma_I,\gamma_J] = \Omega^{AB}\partial_A\gamma_I\partial_B\gamma_J\,. \tag{5.15}$$

The symplectic form for the gauge fixed system in the reduced phase space defines the Dirac bracket (5.14). Suppose now that the set of gauge conditions $\{G_i\}$ fails to fix completely the gauge in a region of phase space, leading to a Gribov ambiguity. This means that if a configuration u^A satisfies the gauge conditions $G_i(u) \approx 0$, there exists a gauge-transformed configuration $u^A + \delta u^A$ that also satisfies it, namely

$$\delta G_i(q,p) \approx \partial_A G_i \delta u^A = 0\,. \tag{5.16}$$

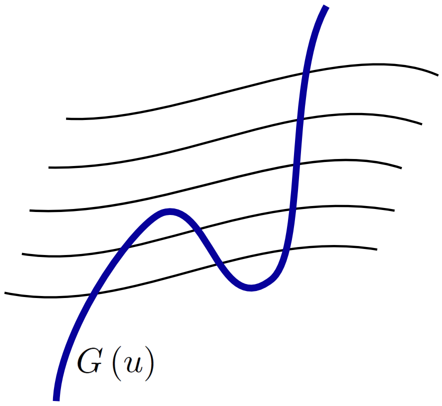

Fig. 5.2 The gauge condition $G_i(u) \approx 0$ (thick line) intersects the gauge orbits (thin lines) more than once provided there exist points where the orbits run tangent to the gauge condition.

Since gauge transformations are generated by first class constraints,

$$\delta u^A = \varepsilon^j \left[u^A, \psi_j\right] = \varepsilon^j \Omega^{AB} \partial_B \psi_j \,, \tag{5.17}$$

where ε^j are infinitesimal parameters, the condition for the existence of Gribov copies (5.16) takes the form

$$\varepsilon^j \Omega^{AB} \partial_A G_i \partial_B \psi_j = \varepsilon^j \left[G_i, \psi_j\right] = 0 \,, \tag{5.18}$$

which has nontrivial solutions ($\varepsilon^i \neq 0$) provided

$$\det\left[G_i, \psi_j\right] = 0 \,.$$

The matrix $\left[G_i, \psi_j\right]$ corresponds to the FP operator in gauge field theory, whose definition is

$$\mathscr{M}_{ij} = \left[G_i, \psi_j\right] = \Omega^{AB} \partial_A G_i \partial_B \psi_j \,. \tag{5.19}$$

Gribov ambiguity occurs if the determinant of the FP operator $\mathscr{M}_{ij}$ vanishes. The Gribov copies continuously connected to a given configuration are related by the corresponding zero modes. The Gribov horizon is defined to be the subset Ξ of phase space Γ where the FP determinant vanishes,

$$\Xi := \left\{ u^A \in \Gamma \,|\, \det[\mathscr{M}_{ij}] = 0 \right\} . \tag{5.20}$$

Now let's observe that the Dirac matrix (5.15) for the set of constraints $\{\gamma_I\}$ contains $\mathscr{M}_{ij}$ as a submatrix

$$C_{IJ} = [\gamma_I, \gamma_J] = \begin{pmatrix} \Omega^{AB} \partial_A G_i \partial_B G_j & \mathscr{M}_{ij} \\ -\mathscr{M}_{ij} & \Omega^{AB} \partial_A \psi_i \partial_B \psi_j \end{pmatrix} . \tag{5.21}$$

Hence, as the set $\{\psi_i\}$ is first class, the determinant of the Dirac matrix is given weakly by the square of the FP determinant

$$\det[C_{IJ}] \approx \left(\det[\mathscr{M}_{ij}]\right)^2 \ . \tag{5.22}$$

In an open set where $\mathscr{M}_{ij}$ is invertible, the Dirac bracket (5.14) can be safely defined. On the other hand, since at the Gribov horizon $\det[\mathscr{M}_{ij}]$ vanishes, the determinant of the Dirac matrix vanishes as well and the Dirac bracket becomes ill-defined there. Moreover, in the next section, we will see that a Gribov ambiguity implies a degeneracy of the symplectic form for the gauge-fixed system at the Gribov horizon.

5.3 Gribov Horizon and Degenerate Surfaces

In general, the gauge generators $\psi_i \approx 0$, together with the gauge fixing conditions $G_i \approx 0$ form a set of $2n$ second class constraints. However, this is not globally true in the presence of a Gribov ambiguity, which can have non-trivial consequences in the symplectic structure of the reduced phase space. This can be seen considering an open set where the Dirac matrix C_{IJ} is invertible, $C^{IJ}C_{JK} = \delta^I_K$. Setting the constraints strongly to zero defines the reduced gauge-fixed (*physical*) phase space, which is generically a co-dimension $2n$ surface Γ_0 embedded in phase space Γ.

Even though we started the analysis with a globally invertible symplectic form, implementing the gauge fixing changes the Poisson structure and a new symplectic form for the reduced phase space must be found. In order to explicitly write the symplectic form in the reduced phase space it is useful to take adapted coordinates $\{U^A\} = \{u^{*a}, v^I\}$, where $\{u^{*a}\}$ are *first class* coordinates (in the sense that they have vanishing brackets with all second class constraints, see [46])

$$u^{*a} = u^a - [u^a, \gamma_I]C^{IJ}\gamma_J \ \text{ with } \ a = 1,...,N-2n \ , \tag{5.23}$$

while $\{v^I\}$ is chosen as the set of second class constraints (5.12)

$$v^I = \gamma_I \ \text{ with } \ I = 1,...,2n \ . \tag{5.24}$$

Consequently $\{u^{*a}\}$ and $\{v^I\}$ are canonically independent coordinates, i.e.,

$$[u^{*a}, v^I] = 0 \, . \tag{5.25}$$

The conditions $v^I = 0$ define the reduced phase space, and the u^*'s fix the position within the reduced phase space Γ_0. The matrix of their Poisson brackets given by

$$\hat{\Omega}^{AB} = \left[U^A, U^B\right] = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & C_{IJ} \end{pmatrix} , \tag{5.26}$$

where

$$\omega^{ab} = [u^{*a}, u^{*b}] \approx [u^a, u^b]^* \ , \tag{5.27}$$

is the inverse of the symplectic form in the reduced phase space ω_{ab}.
The passage from the generic coordinates $\{u^A\}$ to the adapted ones, $\{U^A\} = \{u^{*a}, \gamma_I\}$, must be well defined. Then, the Jacobian for the transformation,

$$\mathscr{J}^A{}_B = \left(\frac{\partial U^A}{\partial u^B}\right) = \begin{pmatrix} \partial_B u^{*a} \\ \partial_B \gamma^I \end{pmatrix} , \tag{5.28}$$

is invertible. Assuming the original Poisson structure (5.7) to be well defined, i.e., $\det[\Omega^{AB}] = \Omega(u) \neq 0$, the new Poisson bracket in the adapted coordinates satisfies

$$\det[\hat{\Omega}^{AB}] = \left(\det[\mathscr{J}^A{}_B]\right)^2 \Omega \ . \tag{5.29}$$

Now we will see that for a system with Gribov ambiguity, the symplectic form on the reduced phase space, ω_{ab}, necessarily degenerates at the Gribov horizon, In fact, since the coordinates U^A are globally well defined, the determinant of the Jacobian (5.28) is finite everywhere. In particular, it must approach a finite value $\mathscr{J}(\bar{u})$ on the Gribov horizon,

$$\det[\mathscr{J}^A{}_B] \underset{u \to \bar{u}}{\longrightarrow} \mathscr{J}(\bar{u}) \neq 0 \ , \tag{5.30}$$

where $\bar{u}$ stands for the values of the coordinates at the Gribov horizon (5.20). From (5.26) this means that

$$\det[\hat{\Omega}^{AB}] = \det[\omega^{ab}] \det[C_{IJ}] \underset{u \to \bar{u}}{\longrightarrow} \mathscr{J}(\bar{u})^2 \Omega(\bar{u}) . \tag{5.31}$$

On the other hand, from (5.22) we know that the determinant of the Dirac matrix vanishes at the Gribov horizon, and therefore the determinant of the Poisson structure on the reduced phase space must be singular,

$$\det[\omega^{ab}] \underset{u \to \bar{u}}{\longrightarrow} \infty .$$

Consequently, the reduced phase space symplectic form necessarily degenerates at the Gribov horizon,

$$\det[\omega_{ab}] \underset{u \to \bar{u}}{\longrightarrow} 0 . \tag{5.32}$$

A well-defined Poisson structure ω^{ab} at the Gribov horizon ($\det[\omega^{ab}(\bar{u})]$ finite) requires $\det[\hat{\Omega}^{AB}] \underset{u \to \bar{u}}{\longrightarrow} 0$ and, consequently, the coordinates $\{U^A\}$ should be ill-defined there. This might happen if the constraints (9.27) are not functionally independent at the Gribov horizon, that is, if the constraints fail to be *regular*. If this problem is not produced by an erroneous choice of gauge fixing, it can only be due to an irregularity in the first class constraints at the Gribov horizon. Irregularity in dynamical systems is an independent problem from degeneracy and requires special handling to define the system in a consistent manner [65]. An example of a system with Gribov ambiguity where the reduced symplectic form is non-degenerate due to irregularities will be analyzed in Section 5.5.

The importance of this result is that when the global coordinates are well defined, the induced symplectic form of the gauge-fixed theory degenerates at the Gribov horizon. Consequently, as shown in [47] and [48], the dynamics is restricted to the regions of phase space bounded by the degeneracy surface. This argument puts the Gribov-Zwanziger restriction on a firm basis: the previous analysis (which strictly speaking only holds for finite dimensional systems) suggests that the system cannot cross the Gribov horizon (since it is a degenerate surface for the corresponding Hamiltonian system) and, therefore, the Gribov-Zwanziger restriction would be naturally respected by the dynamics.

5.4 The FLPR Model

In this section we illustrate the previous discussion with a solvable model proposed by Friedberg, Lee, Pang and Ren (FLPR), which presents a Gribov ambiguity for Coulomb-like gauge conditions [66]. This model has been extensively studied trying understand how the Gribov ambiguity could be circumvented [19, 67, 68]. We will show that, in this gauge, the symplectic form for the gauge-fixed system becomes degenerate at the Gribov horizon. Closely related models, for which Dirac quantization is non-trivial, have been analyzed in [69].

The Lagrangian for the FLPR model is

$$L = \frac{1}{2}\left((\dot{x}+\alpha y q)^2 + (\dot{y}-\alpha x q)^2 + (\dot{z}-q)^2\right) - V(\rho)\,, \tag{5.33}$$

where $\{x, y, z, q\}$ are Cartesian coordinates, $\rho = \sqrt{x^2+y^2}$, and $\alpha > 0$ is a coupling constant. The velocity $\dot{q}$ is absent and therefore the coordinate q plays the role of an auxiliary field or Lagrange multiplier. The associated canonical momenta are given by

$$\begin{array}{ll} p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}+\alpha y q\,, & p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y}-\alpha x q\,, \\ p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z}-q\,, & p_q = \frac{\partial L}{\partial \dot{q}} = 0\,. \end{array} \tag{5.34}$$

Following Dirac's procedure, we find one primary constraint

$$\varphi = p_q \approx 0\,. \tag{5.35}$$

The total Hamiltonian is given by

$$H_T = \frac{1}{2}(p_x^2+p_y^2+p_z^2) + [\alpha(xp_y - yp_x)+p_z]q + \xi\varphi + V(\rho)\,, \tag{5.36}$$

where ξ is a Lagrange multiplier. Time preservation of the constraint φ leads to the secondary constraint

$$\phi = p_z + \alpha\,(x p_y - y p_x) \approx 0\,, \tag{5.37}$$

which leads to no new constraints for the system. Since φ and ϕ have vanishing Poisson bracket, they form a first class set, reflecting the fact that they generate the local[2] gauge symmetries. The constraint φ generates arbitrary translations in q,

$$\delta_\varphi(x,y,z,q) = (0,0,0,\varepsilon(t))\,,\;\; \delta_\varphi(p_x,p_y,p_z,p_q) = 0\,, \tag{5.38}$$

while ϕ generates helicoidal motions,

$$\delta_\phi(x,y,z,q) = \varepsilon(t)(-\alpha y,\alpha x,1,0)\,,\;\; \delta_\phi(p_x,p_y,p_z,p_q) = \alpha\varepsilon(t)(-p_y,p_x,0,0)\,, \tag{5.39}$$

as it is shown in Figure 5.3. Both transformations leave invariant the Hamiltonian (5.36) for arbitrary $\varepsilon(t)$ and $\varepsilon(t)$. Note that the system is invariant under rotations in the $x-y$ plane, translations in z and time translations, but these are global symmetries that lead to conservation of the z-components of the angular and the linear momenta, and the energy. Symmetries (5.38, 5.39), instead, are not rigid but local.

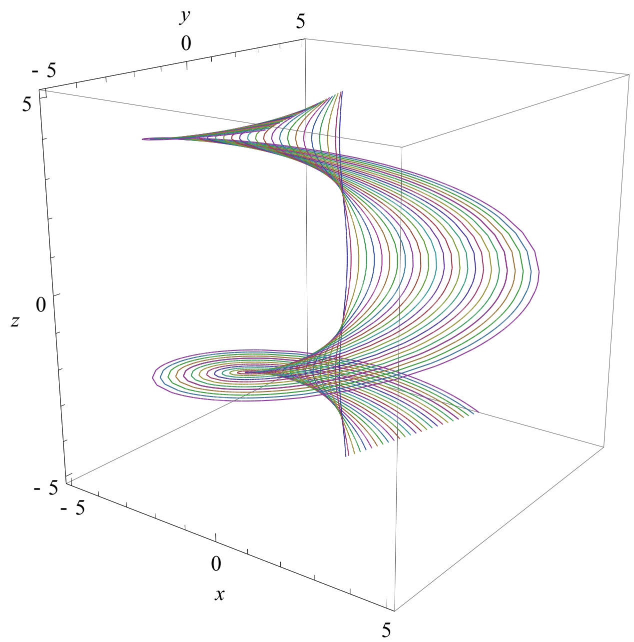

Fig. 5.3 The orbits generated by gauge transformations in the FLPR model are helicoids of the form $(x,y,z) = (\rho\cos[\alpha\varepsilon(t)],\rho\sin[\alpha\varepsilon(t)],\varepsilon(t))$.

[2] Locality here refers to time.

The gauge freedom generated by φ, can be fixed by the gauge condition

$$\mathscr{G} = q \approx 0\,, \tag{5.40}$$

which is analogous to the temporal gauge $A_0 = 0$ in Maxwell theory. Thus, the coordinate q and its conjugate momentum p_q, can be eliminated from phase space by an algebraic gauge choice, as it happens with A_0 in electrodynamics, which also enters as a Lagrange multiplier. This partial gauge fixing eliminates the term $\xi\varphi$ from Hamiltonian (5.36) and identifies q as a Lagrange multiplier. The result is a Hamiltonian system in the 6-dimensional phase space Γ with coordinates $\{u^A\} = \{x, p_x, y, p_y, z, p_z\}$ and a single (necessarily first class) constraint $\phi \approx 0$. The Poisson bracket in this phase space is given by

$$[M,N]_\Gamma = \Omega^{AB}\partial_A M \partial_A N\,, \tag{5.41}$$

where Ω^{AB} is the canonical Poisson bracket, and the canonical symplectic form is

$$\Omega_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}\,. \tag{5.42}$$

Following [66], the gauge freedom generated by ϕ is to be eliminated by a gauge condition $G(x,y,z) \approx 0$, where G is a linear homogeneous function, which is in some sense analogous to the Coulomb gauge. Since the system is invariant under rotations in the $x-y$ plane, we can choose the gauge condition to be independent of y. Hence, we take

$$G = z - \lambda x \approx 0\,, \tag{5.43}$$

which is called λ*-gauge.* As it can be seen, for $\lambda \neq 0$ the condition (5.43) does not fix the gauge globally (see Figure 5.4). In the same way as the Coulomb gauge does in Yang-Mills theory, it has a Gribov ambiguity at $y = -(\alpha\lambda)^{-1}$. In fact, the non trivial Poisson bracket,

$$\mathscr{M} = [G,\phi] = 1 + \alpha\lambda y\,, \tag{5.44}$$

which corresponds to the Faddeev-Popov determinant, indicates that these are second class constraints everywhere in Γ_0, except at $y = -(\alpha\lambda)^{-1}$. Consequently, $\det C = \mathscr{M}^2$ vanishes where the condition $G \approx 0$ fails to fix the gauge, that is on the Gribov horizon

$$\Xi = \{(x, p_x, y, p_y, z, p_z) \in \Gamma \,|\, \mathscr{M} = 0\}\,. \tag{5.45}$$

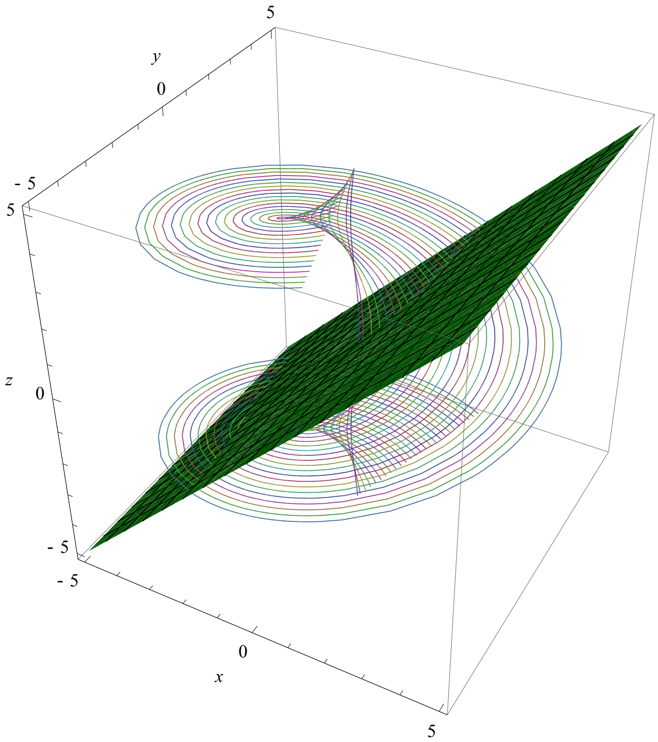

Fig. 5.4 The surface defined by the λ-gauge condition $G = z - \lambda x = 0$ is a plane (here plotted for $\lambda = 1$). The Gribov ambiguity in the FLPR model is reflected by the fact that this plane intersects some gauge orbits more than once.

When the second class constraints (5.37, 5.43) are set strongly equal to zero, z and p_z can be eliminated from the phase space. The four-dimensional reduced phase space Γ_0, parametrized with coordinates (x, p_x, y, p_y), acquires a non-canonical Poisson structure given by the Dirac bracket (5.14) , where γ_I are the second class constraints $\{G, \phi\}$

$$\gamma_I : \; \gamma_1 = G = z - \lambda x \,, \qquad \gamma_2 = \phi = p_z + \alpha(x p_y - y p_x) \,, \tag{5.46}$$

and C^{IJ} is the inverse of the Dirac matrix $C_{IJ} \equiv [\gamma_I, \gamma_J]$. In this case, the Dirac matrix is given by

$$C_{IJ} = [\gamma_I, \gamma_J]_\Gamma = \begin{pmatrix} 0 & \mathscr{M} \\ -\mathscr{M} & 0 \end{pmatrix} , \tag{5.47}$$

and the Dirac brackets are given by

$$\begin{aligned} &[x, p_x]^* = \frac{1}{\mathscr{M}} \;\;,\;\; [x, y]^* = 0 \;\;,\;\; [x, p_y]^* = 0 \,, \\ &[y, p_y]^* = 1 \;\;,\;\; [y, p_x]^* = \frac{\alpha \lambda x}{\mathscr{M}} \;\;,\;\; [p_x, p_y]^* = -\frac{\alpha \lambda p_x}{\mathscr{M}} \,. \end{aligned} \tag{5.48}$$

In the reduced phase space, the Poisson matrix (5.27) takes the form

$$\omega^{ab} = \begin{pmatrix} 0 & \frac{1}{\mathscr{M}} & 0 & 0 \\ -\frac{1}{\mathscr{M}} & 0 & -\frac{\alpha\lambda x}{\mathscr{M}} & -\frac{\alpha\lambda p_x}{\mathscr{M}} \\ 0 & \frac{\alpha\lambda x}{\mathscr{M}} & 0 & 1 \\ 0 & \frac{\alpha\lambda p_x}{\mathscr{M}} & -1 & 0 \end{pmatrix} , \tag{5.49}$$

and the corresponding symplectic form is

$$\omega_{ab} = \begin{pmatrix} 0 & -\mathscr{M} & -\alpha\lambda p_x & \alpha\lambda x \\ \mathscr{M} & 0 & 0 & 0 \\ \alpha\lambda p_x & 0 & 0 & -1 \\ -\alpha\lambda x & 0 & 1 & 0 \end{pmatrix} . \tag{5.50}$$

It can be checked that this symplectic form is closed, $\partial_a\omega_{bc} + \partial_b\omega_{ca} + \partial_c\omega_{ab} = 0$ and therefore in a local chart it can be expressed as the exterior derivative of a one-form, $\omega_{ab} = \partial_a X_b - \partial_b X_a$ (or $\omega = dX$), which can be integrated as

$$X(x, p_x, y, p_y) = (p_x + \alpha\lambda[yp_x - xp_y])dx + p_y dy \, . \tag{5.51}$$

The determinant of the symplectic form in the reduced phase space can be read off from (5.50), and is given by

$$\det[\omega_{ab}] = \mathscr{M}^2 \, . \tag{5.52}$$

Clearly, ω_{ab} degenerates precisely at the Gribov (5.45) restricted to the constraint surface and the degeneracy surface (5.8) is given by

$$\Sigma = \{(x, p_x, y, p_y) \in \Gamma_0 | \Upsilon(u) \equiv \mathscr{M} = 0\} \, . \tag{5.53}$$

This corresponds to a particular realization of the behavior (5.32). In fact, defining $\sigma^2 = 1 + \alpha^2\lambda^2\rho^2 > 0$, the eigenvalues of the reduced symplectic form are given by $\{\pm i\omega_+, \pm i\omega_-\}$, where

$$\omega_\pm = \frac{1}{\sqrt{2}} \left[\sigma^2 + \mathscr{M}^2 \pm \sqrt{(\sigma^2 + \mathscr{M}^2)^2 - 4\mathscr{M}^2} \right]^{1/2} \, . \tag{5.54}$$

Near the degeneracy ω_+ and ω_- can be expanded in powers of $\mathscr{M}$, leading to

$$\omega_+ \approx \sigma \quad , \quad \omega_- \approx \frac{\mathscr{M}}{\sigma} \, . \tag{5.55}$$

Hence, as the system approaches to the degeneracy ω_+ goes linearly to zero while ω_- never vanishes, which means that the symplectic form ω_{ab} has a simple zero in the degeneracy surface and this system corresponds to the class of degenerate systems discussed in [47] and [48].

It is reassuring to confirm that the degeneracy is not an artifact introduced by the change of coordinates $\{U^A\} \to \{u^{*a}, v^I\}$ defined in (5.23), which in this case is given by

$$
\begin{aligned}
x^* &= \frac{x + \alpha yz}{\mathscr{M}} , & p_x^* &= \frac{p_x + \alpha p_y z + \alpha\lambda p_z}{\mathscr{M}} , \\
y^* &= y - \frac{\alpha x(z - \lambda x)}{\mathscr{M}} , & p_y^* &= p_y - \frac{\alpha p_x(z - \lambda x)}{\mathscr{M}} , \\
v^1 &= \gamma_1 = z - \lambda x , & v^2 &= \gamma_2 p_z + \alpha\,(x p_y - y p_x) .
\end{aligned} \quad , \tag{5.56}
$$

In fact, the Jacobian (5.28) is given in this case by

$$
\mathscr{J}^A{}_B = \begin{pmatrix}
\frac{1}{\mathscr{M}} & 0 & 0 & 0 & \frac{\alpha y}{\mathscr{M}} & 0 \\
0 & \frac{1}{\mathscr{M}} & -\frac{\alpha\lambda p_x}{\mathscr{M}} & \frac{\alpha\lambda x}{\mathscr{M}} & \frac{\alpha\lambda p_y}{\mathscr{M}} & \frac{\lambda}{\mathscr{M}} \\
\frac{\alpha\lambda x}{\mathscr{M}} & 0 & 1 & 0 & -\frac{\alpha x}{\mathscr{M}} & 0 \\
\frac{\alpha\lambda p_x}{\mathscr{M}} & 0 & 0 & 1 & -\frac{\alpha\lambda p_x}{\mathscr{M}} & 0 \\
-\lambda & 0 & 0 & 0 & 1 & 0 \\
\alpha p_y & -\alpha y & -\alpha p_x & \alpha x & 0 & 1
\end{pmatrix} , \tag{5.57}
$$

which, in spite of the the apparent singularities in its entries, has unit determinant everywhere in phase space, $(\det \mathscr{J})|_\Gamma \equiv 1$.

5.4.1 Effective Lagrangian for the Gauge-fixed System

The gauge-fixed system is a degenerate one described by a first order Hamiltonian action, as presented in (5.1),

$$
I_{gf}[u] = \int \mathrm{d}t \, [\dot{u}^a X_a(u) - H_{gf}(u)] \ , \tag{5.58}
$$

where X_a is given by (5.51), H_{gf} is the gauge-fixed Hamiltonian,

$$
\begin{aligned}
H_{gf} &= \frac{1}{2}(1 + \alpha^2 y^2) p_x^2 + \frac{1}{2}(1 + \alpha^2 x^2) p_y^2 - \alpha^2 xy p_x p_y + V(x^2 + y^2) \\
&= \frac{1}{2} g^{ij} p_i p_j + V(x^2 + y^2) .
\end{aligned} \tag{5.59}
$$

Here the matrix

$$
g^{ij} := \begin{bmatrix} (1 + \alpha^2 y^2) & -\alpha^2 xy \\ -\alpha^2 xy & (1 + \alpha^2 x^2) \end{bmatrix} \tag{5.60}
$$

is the inverse of the metric

$$
g_{ij} := \frac{1}{1 + \alpha^2 \rho^2} \begin{bmatrix} (1 + \alpha^2 x^2) & \alpha^2 xy \\ \alpha^2 xy & (1 + \alpha^2 y^2) \end{bmatrix} . \tag{5.61}
$$

5.4.2 Gauge Orbits and Phase-Space

Gribov ambiguity results from the fact that the surface defined by a gauge condition does not intersect every gauge orbit once and only once. As it was mentioned in Section 5.2, this is a requirement to achieve a proper gauge fixing [46]. In the case of the FLPR model this clearly happens because the plane defined by (5.43) intersects some gauge orbits many times for $\lambda > 0$, as it can be seen in Figure 5.4. The $G = 0$ plane intersects more than once any orbit such that $x^2 + y^2 > (\alpha\lambda)^{-2}$. The only way that this doesn't happen is if $\lambda = 0$.

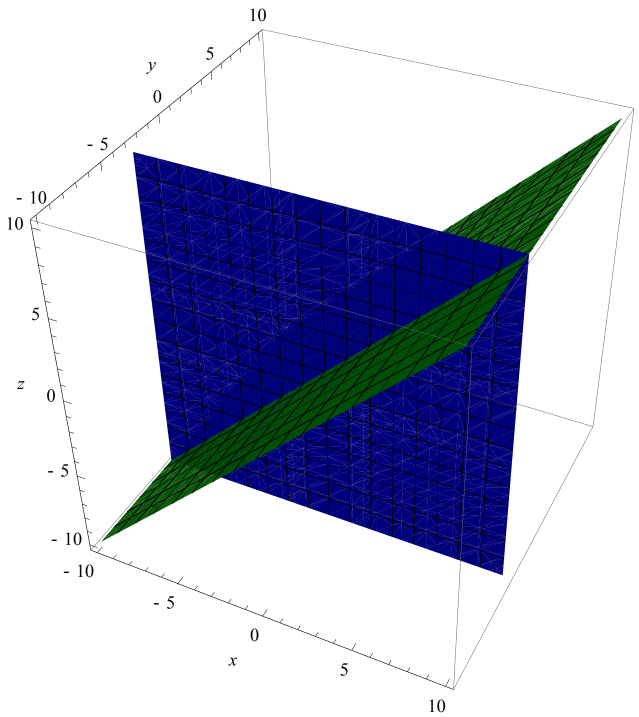

Fig. 5.5 In the case of the FLPR model, the Gribov horizon (blue plane), $y = -(\alpha\lambda)^{-1}$, and the constraint surface (green plane), $G = 0$, are plotted for $\lambda = 1$ and $\alpha = 1/3$. The GH divides the constraint surface in two dynamically disconnected regions.

Degenerate surfaces divide phase space into dynamically disconnected regions. In this case the presence of the Gribov horizon defines two regions in physical gauge fixed space (see Figure 5.5)

$$C_+ := \{(x,y,z)\,|\,z - \lambda x = 0,\, 1 + \alpha\lambda y > 0\} \ , \tag{5.62}$$

$$C_- := \{(x,y,z)\,|\,z - \lambda x = 0,\, 1 + \alpha\lambda y < 0\} \ . \tag{5.63}$$

These two regions are not equivalent since only C_+ contains at least one representative of every gauge orbit, while not all gauge orbits pass through C_-. To restrict the analysis of the system to one region or the other is consistent in the sense that all states whose initial condition is in one region will remain there always (see [48]). In Yang-Mills theories the Gribov region corresponds to the neighborhood of $A_\mu = 0$ in the functional space of connections where the FP operator is positive definite [16] and *small copies* (namely, points infinitesimally close which belong to the same gauge orbit) are absent. In the Yang-Mills case all the gauge orbits cross the Gribov region at least once [29]. Similarly to what happens in the Yang-Mills case, within the region C_+ (which contains at least one representative of every gauge orbit) there are still large copies [30].

5.4.3 Quantization

In order to define the quantum theory, the Hilbert space for the system must be equipped with an inner product that provides a scalar product and a norm

$$\|\Psi(u)\| = \left(\int \mathrm{d}^2u\,\sqrt{g}\,w(u)\,|\Psi(u)|^2\right)^{1/2} . \tag{5.64}$$

In the FLPR model, $g = (1+\alpha^2\rho^2)^{-1}$ is the determinant of the metric (5.61) and the weight $w(u)$ is such that the Hamiltonian is symmetric, that is,

$$\int \mathrm{d}^2u\,\sqrt{g}\,w(u)\Psi_1^*(u)\left(\hat{H}\Psi_2(u)\right) = \int \mathrm{d}^2u\,\sqrt{g}\,w(u)\left(\hat{H}\Psi_1(u)\right)^*\Psi_2(u) . \tag{5.65}$$

As discussed in Section 5.1, the proper choice for the measure $w(u)$ corresponds to the Pfaffian of the symplectic form ω_{ab} (5.50), given in this case by (5.53),

$$w(u) = \Upsilon = \mathscr{M} = 1+\alpha\lambda y , \tag{5.66}$$

whose zeros define the degeneracy surface (5.8). In order to see this, let's define new variables $\{\pi_i\}$ canonically conjugate to the u's, so that

$$[u^i,\pi_j]^* = \delta^i_j, \quad \{u^i\} = \{x,y\} . \tag{5.67}$$

A simple calculation using (5.48) leads to the following expression of the momenta in terms of the π's

$$p_x = \frac{1}{1+\alpha\lambda y}(\pi_x + \alpha\lambda x\pi_y) , \quad p_y = \pi_y . \tag{5.68}$$

The quantum operators are then obtained via the prescription

$$\begin{aligned} u^i &\longrightarrow \hat{u}^i = u^i , \\ \pi_i &\longrightarrow \hat{\pi}_i = -i\hbar\partial_i , \\ [\ ,\ ]^* &\longrightarrow \frac{1}{i\hbar}[\ ,\ ]\,(\text{Commutator}) . \end{aligned} \tag{5.69}$$

Using (5.68), the classical Hamiltonian (5.59) can be rewritten as

$$H = \frac{1}{2} h^{ij} \pi_i \pi_j + V \ , \tag{5.70}$$

where h^{ij} is the inverse of the metric

$$h_{ij} = \frac{1}{1+\alpha^2\rho^2} \begin{pmatrix} (1+\alpha\lambda y)^2 + \alpha^2 \left(1+\lambda^2\right) x^2 & \alpha^2 xy - \alpha\lambda x \\ \alpha^2 xy - \alpha\lambda x & 1+\alpha^2 y^2 \end{pmatrix} . \tag{5.71}$$

At the quantum level, the correct ordering for the quantum operators (5.69) that renders the Hamiltonian symmetric –and invariant under general coordinate transformations– is the one for which $\hat{H}$ is a Laplacian for the metric h_{ij} [70], i.e.

$$\hat{H} = -\frac{\hbar^2}{2} \frac{1}{\sqrt{|h|}} \partial_i \left(\sqrt{|h|} h^{ij} \partial_j \right) + V(\rho) \ , \tag{5.72}$$

where h is the determinant of (5.71) and where integration measure in (5.64) is $\int d^2u\sqrt{h}$. A straightforward computation leads to

$$\sqrt{h} = \frac{(1+\alpha\lambda y)}{\sqrt{1+\alpha^2\rho^2}} = \sqrt{g}\Upsilon \ , \tag{5.73}$$

which confirms (5.66). Hence, the measure of the Hilbert space vanishes exactly where the symplectic form does. Then, according to the results in [48] this permits to interpret the corresponding Hilbert space as a collection of causally disconnected subspaces: there is no tunneling from one side of the degenerate surface to the other. In turn, this confirms the dynamical correctness of imposing the restriction to the interior of the Gribov region, at least for first quantization.

5.5 Irregular Case

As mentioned in Section 4, there is an exceptional case in which the reduced symplectic form is non-degenerate at the Gribov horizon. As it will be shown in the following, this could happen if the constraints fail to be functionally independent, i.e., if they are irregular [46, 65].

A set of constraints is regular if they are functionally independent on the constraints surface. For a set of constraints (5.12) this is ensured by demanding that the Jacobian

$$\mathscr{K}^I{}_B = \left. \frac{\partial \gamma_I}{\partial u^B} \right|_{\Gamma_0} \tag{5.74}$$

has maximal rank on the constraint surface. In particular, for a set of two constraints $\{G, \phi\}$, this means

$$dG \wedge d\phi|_{\Gamma_0} \neq 0 \Longrightarrow \partial_{[A} G \partial_{B]} \phi|_{\Gamma_0} \neq 0 \ , \tag{5.75}$$

while the Dirac matrix (5.21) takes the form

$$C_{IJ} = \begin{pmatrix} 0 & \mathscr{M} \\ -\mathscr{M} & 0 \end{pmatrix} \ , \tag{5.76}$$

where $\mathscr{M} = [G, \phi]$ the FP determinant. Hence, using (9.29) and (5.21), the reduced phase space symplectic form (5.27) can be expressed weakly as

$$\omega^{ab} \approx [u^a, u^b]^* = \Omega^{ab} + \mathscr{M}^{-1} \Omega^{aC} \Omega^{Db} \partial_{[C} G \partial_{D]} \phi \ . \tag{5.77}$$

This suggests that, if the constraints fail the regularity test (5.75) at the Gribov horizon, the singularity in the inverse of the FP determinant $\mathscr{M}^{-1}$ can be cancelled by the vanishing quantity $\partial_{[C} G \partial_{D]} \phi$ and no degeneracies would appear even in the presence of Gribov ambiguity.

Another way to see this picture for a general set of constraints (5.12), $\{\gamma_I\} = \{G_i, \phi_j\}$, is by noting that, as the original symplectic structure (5.7) is considered to be well defined ($\det[\Omega^{AB}] = \Omega$), the determinant of the Poisson bracket in the new coordinates $U^A = [u^{*a}, v^I]$, defined by (5.23) and (5.24) is given by (5.29), which can be evaluated on the constraint surface Γ_0,

$$\det[\hat{\Omega}^{AB}]\Big|_{\Gamma_0} = \left(\det\left[\mathscr{J}^A{}_B\right]\right)^2 \Omega\Big|_{\Gamma_0} \ . \tag{5.78}$$

On the other hand, the Jacobian (5.28) evaluated on $\gamma_I = 0$ can be written in terms of (5.74) as

$$\mathscr{J}^A{}_B\Big|_{\Gamma_0} = \begin{pmatrix} \partial_B u^{*a} \\ \mathscr{K}^I{}_B \end{pmatrix} \ . \tag{5.79}$$

Hence, if the constraints (9.27) are irregular at the Gribov horizon, both $\mathscr{K}^I{}_B$ and $\mathscr{J}^A{}_B|_\Gamma$ have non-maximal rank, implying that the determinant $\det\left[\mathscr{J}^A{}_B\right]$ vanishes at the intersection of the Gribov horizon and Γ_0. Therefore,

$$\det[\hat{\Omega}^{AB}]\Big|_{\Gamma_0} \underset{u \to \tilde{u}}{\longrightarrow} 0 \ . \tag{5.80}$$

Then, looking again at (5.26), we see that in this case the vanishing of $\det[C_{IJ}]$ at the Gribov horizon does not imply that the reduced phase space Poisson structure should blow up and degeneracies in the symplectic structure of the gauge fixed system can be overcome. However, this situation is even more pathological than the degenerate one, as the gauge-fixed system doesn't describe the dynamics of the original system. In the following, an explicit example of this situation will be presented.

5.5.1 Example: Christ-Lee Model

The Lagrangian for the Christ-Lee model [71] is given by

$$L = \frac{1}{2}(\dot{x} + \alpha y q)^2 + (\dot{y} - \alpha x q)^2 - V(x^2 + y^2)\,,$$

where $\alpha > 0$ is a coupling constant. The canonical momenta of the system are given by

$$p_x = \dot{x} + \alpha y q\,, \quad p_y = \dot{y} - \alpha x q\,, \quad p_q = 0\,. \tag{5.81}$$

Dirac's method leads to the following first class constraints

$$\varphi = p_q \approx 0\,, \qquad \phi = x p_y - y p_x \approx 0\ , \tag{5.82}$$

which generate arbitrary translations in q and rotations in the $x-y$ plane respectively. The total Hamiltonian is given by

$$H_T = \frac{1}{2}(p_x^2 + p_y^2) + \alpha(x p_y - y p_x)q + \xi\varphi + V(\rho)\ , \tag{5.83}$$

where ξ is a Lagrange multiplier As before, the constraint φ can be trivially eliminated by the introduction of a gauge condition $\mathscr{G} = q \approx 0$. The Dirac bracket associated to this pair of constraints is just the Poisson bracket in the coordinates $\{x, p_x, y, p_y\}$, and using this we can set φ and $\mathscr{G}$ strongly to zero. Now we will focus on the constraint ϕ, whose action on the coordinates generates circular orbits in phase space.

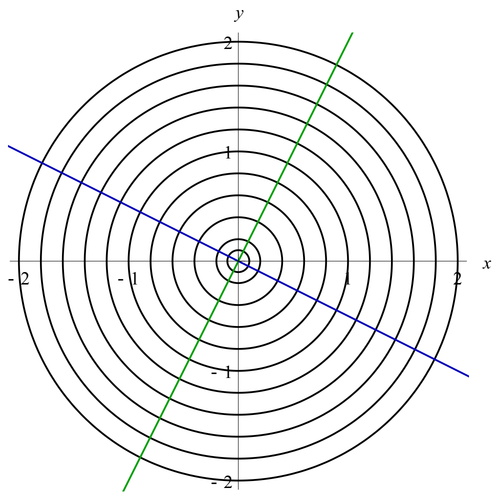

Fig. 5.6 Orbits for the Christ-Lee model are given by circles centered at the origin. The GH (blue line) and the surface $G = 0$ (green line) are plotted for $\mu = 2$. The GH restricted to the constraint surface corresponds to the point $x = y = 0$.

As we are interested in Gribov ambiguity, we will pick the following gauge condition [67]

$$G = y - \mu x \approx 0 \ , \tag{5.84}$$

with μ a constant. The Dirac matrix for this set of constraints $\gamma_I = \{G, \phi\}$ with $I = 1,2$ is given by (5.76) where

$$\mathscr{M} = [G, \phi] = x + \mu y \ ,$$

and there exists a Gribov horizon (5.20) defined by

$$\Xi := \left\{ (x, p_x, y, p_y) \in \Gamma \,|\, \mathscr{M} = 0 \right\} . \tag{5.85}$$

The Poisson structure of the space is given via the Dirac bracket (5.14), where γ_I are the second class constraints $\{G, \phi\}$. This leads to

$$\begin{aligned} [x, p_x]^* &= \frac{x}{\mathscr{M}} \ , \ [x, y]^* = 0 \ , \ [x, p_y]^* = \frac{y}{\mathscr{M}} \ , \\ [y, p_y]^* &= \frac{\mu y}{\mathscr{M}} \ , \ [y, p_x]^* = \frac{-\mu y}{\mathscr{M}} \ , \ [p_x, p_y]^* = \frac{\mu p_x p_y}{\mathscr{M}^2} \ . \end{aligned} \tag{5.86}$$

Once the second class constraints G and ϕ are strongly equal to zero

$$y = \mu x \ , \qquad p_y = \mu p_x \ , \tag{5.87}$$

we are left with only one degree of freedom corresponding to the variable x. The Gribov horizon restricted to the constraint surface $G = 0$ is given by $x = 0$ (see Figure 5.6). Then, the reduced phase space symplectic form (5.27) turns out to be non-degenerate

$$\omega_{ab} = \begin{pmatrix} 0 & -\left(1 + \mu^2\right) \\ 1 + \mu^2 & 0 \end{pmatrix} \ , \ \det[\omega_{ab}] = \left(1 + \mu^2\right)^2 \ . \tag{5.88}$$

However, in this case, the constraints $\{G, \phi\}$ are not functionally independent at the Gribov horizon. To see this consider the sub-block (5.74) of (5.79) whose rank determines the functional independence of the constraints $\{G, \phi\}$,

$$\mathscr{K}^I{}_B = \left. \frac{\partial \gamma_I}{\partial u^B} \right|_{\Gamma_0} = \begin{pmatrix} -\mu & 0 & 1 & 0 \\ \mu p_x & -\mu x & -p_x & x \end{pmatrix} . \tag{5.89}$$

This matrix has non-maximal rank on the Gribov horizon restricted to the constraint surface ($x = 0$), then the constraints are not regular there because their gradients are proportional.

The gauge-fixed Lagrangian now reads

$$L = \frac{1}{2}\left(1 + \mu^2\right) \dot{x}^2 - V\left((1 + \mu^2) x^2\right) \ , \tag{5.90}$$

which seems to be free of degeneracy at the Gribov horizon. However this is an illusion because the absence of degeneracy results from the fact that the constraints are no longer functionally independent, so that the system, on the Gribov horizon, fails to be regular.

Chapter 6

Conclusions and Future Directions

In the first part of the thesis, the relation between Gribov ambiguity and degeneracy in Hamiltonian systems has been studied. In this context, the Gribov restriction for QCD can be seen as a prescription consistent with the fact that it is respected by the dynamics, both classical and quantum mechanically, at least in finite dimensional Hamiltonian systems.

In gauge systems with finite number of degrees of freedom, the existence of Gribov ambiguity in the gauge fixing conditions leads to a degenerate symplectic structure for the reduced system: the degenerate surface in the reduced phase space is the Gribov horizon restricted to the constraint surface. It is important to observe that, although in the FLPR model the Gribov ambiguity can be circumvented by choosing $\lambda = 0$ (leading to the analog of the axial gauge in field theory), an analogous choice is not possible for Yang-Mills theories. In fact, as shown in [17], in order to include relevant non-trivial configurations –like instantons– in the function space of the theory, certain boundary conditions must be imposed on the fields, which rule out algebraic gauge conditions (see also [16]). In this sense, a consistent analog of the limit $\lambda \to 0$ for field theories does not exist, the Gribov ambiguity is unavoidable for gauge theories, and degeneracies should be expected in the gauge-fixed system. As we have shown, when the requirement of regularity is not imposed, a non-degenerate gauge-fixed systems can be obtained. However this is not a solution to the problem. Regularity is a key requirement for a set of constraints to be well defined, as irregularities lead to a Lagrangian that does not describe the real dynamics of the original system.

Even if the generalization of our results to field theories is conceptually straightforward, an interesting future direction for this work is to look for explicit degeneracies in the gauge-fixed symplectic form of Yang-Mills type theories.

On the other hand, it has been shown that the semi-classical Gribov approach applied to finite-temperature Yang-Mills theory is consistent with the presence of a confined/deconfined phase transition. This is reflected in the fact that the existence of solutions of the Gribov the gap equation depends on the temperature. A key ingredient for the consistent description of these different regimes is the inclusion of a mass term in the gluon propagator, which comes from the one-loop corrections to the theory, and is consistent with the fact that the thermal mass (4.7) causes gluon deconfinement, as explained in [60]. This result suggest that, when temperature is included in the theory, the degeneracy in the reduced phase space produced by the Gribov horizon disappears at some critical temperature.

In order to be able to study the low-temperature limit, we have introduced a modified running coupling g, which interpolates between the standard perturbative result in the ultraviolet regime and a constant (in principle infinite but unobservable) for the infrared regime. It is worth to note that this modification has been considered only for consistency, as it allows the gluon thermal mass to go to zero for low temperature, but the presence of these phase transitions does not depend on this fact. Indeed, the same qualitative behavior for the gluon propagator was obtained when considering the standard one-loop running coupling (4.20) and, furthermore, it can be shown that phase transitions are also present if only a constant coupling is considered in the whole analysis.

For the analysis of the Gribov gap equation at finite temperature, the scaling solution has been considered, in which the gluon propagator (3.24) vanishes and the ghost propagator (3.25) blows up as $1/q^4$ in the infrared limit $q \to 0$. On the other hand, it is clear by now that the decoupling solution (where the gluon propagator goes to a constant in the infrared limit while the ghost propagator has a free-like behavior) is the relevant one [72–74]. The decoupling solution has a strong lattice support [75–78] and can be obtained analytically within the refined Gribov-Zwanziger theory by including some condensates [79, 35, 37]. It would certainly be of interest to study the refined Gribov-Zwanziger approach at finite temperature.

It is important to note that even though the analysis made for finite dimensional Hamiltonian systems extends in a straightforward way the case of Yang-Mills theory in which there are infinite Gribov horizons [80], the problem involves additional important technical difficulties, as for instance, the definition of the reduced phase space when non-algebraic gauge conditions are adopted. In particular, when set strongly to zero, this kind of gauge conditions does not allow to express one field as local functions of the remaining ones, and a local action for the physical degrees of freedom with the reduced symplectic form is not available. These difficulties in the standard Hamiltonian formulation for Yang-Mills theories make the path integral formalism better suited. However, an interesting novel Hamiltonian approach to QCD, where Dirac reduction is considered, has been recently developed in [81], which could be worth to study within this context.

On the other hand, the question of whether the degeneracy surface can act as a sink or as a source in Yang-Mills theories is as interesting as extremely difficult and deserves further investigation. The difficulty stems from the fact that in Yang-Mills the Gribov horizon is an infinite-dimensional hyper surface with quite a complicated topology. Hence, in order to determine whether the degeneracy is a sink or a source, one should have a complete characterization of the geometry in the vicinity of the horizon.

An interesting result in the analysis of finite temperature QCD is the appearance of an intermediate regime in between the confined and free regimes, in which only one of the two gluonic degrees of freedom is physical, while the other one does not belong to the physical spectrum. In this sense, this new regime captures traces of both confined and deconfined regimes. Hence, this scenario could be interpreted as a partial deconfinement or a semi-QGP phase, which has been studied in [82–84]. Another future direction of this thesis is the understanding of this behavior at the Hamiltonian level in the context of degeneracies in the symplectic structure of gauge fixed systems.

The fact that the Gribov horizon is a degeneracy surface for the gauge fixed system, which persists at the quantum level, strongly supports the consistency of the Gribov restriction for QCD, as the degeneracy divides phase space into causally disconnected regions. Even though the Gribov-Zwanziger idea is heuristic and supported by the fact that every orbit intersects the Gribov region [28] (which means that no physical information is lost if the restriction is applied), the results it yields have gained acceptance by their match with the lattice data. Our results provide a novel point of view for the problem in support of the Gribov-Zwanziger proposal that makes it worth to be studied deeper within the Hamiltonian framework.

Part II

Three-dimensional Gravity

Chapter 7

Introduction to Part II

A prime example of duality between a three-dimensional and a two-dimensional theory is the relation between a Chern-Simons theory in the presence of a boundary and the associated chiral Wess-Zumino-Witten (WZW) model: on the classical level for instance, the variational principles are strictly equivalent as the latter is obtained from the former by solving the constraints in the action [85–87].

In the case of the Chern-Simons formulation of 3d gravity [9, 88], the role of the boundary is played by non trivial fall-off conditions. For anti-de Sitter or flat asymptotics, a Gibbons-Hawking like boundary term is required to make solutions with the prescribed asymptotics true extrema of the variational principle. Furthermore, the fall-off conditions lead to additional constraints that correspond to fixing a subset of the conserved currents of the WZW model [10, 89]. The associated reduced phase space description is Liouville theory in the AdS case and a suitable limit thereof in the flat case [11, 12]. As already noted in [89] and also more recently in [90, 91], this analysis does not take non trivial topology and the associated holonomies into account. When this is done, one expects modified chiral boson or Liouville type actions actions.

In this part of the thesis we follow a different approach to construct dual two-dimensional action principles for the gauge fixed solution space of three-dimensional gravity. Indeed, both in the asymptotically AdS and flat cases, this space is known to coincide with the centrally extended coadjoint representation, at fixed values of the central charges, of the asymptotic symmetry groups, via two copies of the Virasoro group [6, 92–97] and the centrally extended $\widehat{\mathrm{BMS}}_3$ group [98–100, 11, 101, 102], respectively (see also [103, 104] for recent related considerations). As a consequence, the gravitational solution space admits a partition into coadjoint orbits. For any group G, the individual orbits are symplectic spaces (see e.g. [105, 106, 14, 15] and original references therein), to which are associated in a canonical way to *geometrical actions*, which admit G as a global symmetry group [107].

When applied to three-dimensional gravity, we will get in this way finer actions than those of [10, 89, 11, 12], precisely adapted to the individual orbits. In the anti-de Sitter case for instance, one finds an intriguing connection between $3d$ and $2d$ gravity [108–110]. In the flat case, this approach allows us to construct generalized $\widehat{\mathrm{BMS}}_3$ invariant actions.

Another interesting aspect of this approach to two-dimensional conformal or BMS_3 invariant actions is that, exactly like in the case of loop groups and the associated WZW models, they can also be interpreted as one-dimensional particle-type actions for infinite-dimensional groups. The spatial dimension is hidden or emergent, depending on whether one uses a Fourier expansion for the Lie algebra generators and their duals with associated infinite mode sums or an inner product with an explicit integration over the circle.

In Chapter 8, the main aspects of three dimensional gravity are exposed and its Chern-Simons formulation is reviewed. It will be also shown that 2+1 dimensional anti-de Sitter space-times are Lorentz-flat and the implications for the dual theories for gravity will be discussed. In Chapter 9, the symplectic structure on coadjoint orbits will be discussed and the Hamiltonian analysis of the Kirillov-Kostant action will be performed. Chapter 10 will be devoted to the construction of dual field theories for three-dimensional gravity as geometrical action on coadjoint orbits of the associated asymptotic symmetry groups.

Chapter 8

Classical Duals of Three-dimensional Gravity

8.1 Einstein Gravity in Three-dimensions

The dynamics of the gravitational field, described by the metric $g_{\mu\nu}$, is governed by Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 , \tag{8.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and Λ is the cosmological constant. This equations can be obtained from the d-dimensional Einstein Hilbert action

$$I_{EH} \equiv \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^d x , \tag{8.2}$$

where we have considered $c = 1$, G is the $d-$dimensional Newton constant, g is the determinant of the metric.

In dimension $d \geq 4$ General Relativity can not be written as a gauge theory for the Poincare or $(A)dS$ group, making its quantization an intractable problem [111]. In three dimensions, however, Einstein Gravity can be formulated as true gauge theory. In fact, for $d = 3$ the Einstein Hilbert action can be written as a first order theory using the language of differential forms

$$I_{EH} = -\frac{1}{16\pi G} \int \varepsilon_{abc} \left(R^{ab} e^c - \frac{\Lambda}{3} e^a e^b e^c \right) , \tag{8.3}$$

where $R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$ is the curvature $2-$form, $\omega^{ab} = \omega^{ab}_\mu dx^\mu$ is the Lorentz spin-connection and $e^a = e^a_\mu dx^\mu$ correspond to the dreibein forms, whose components define the soldering form between open sets in the manifold M and the corresponding tangent spaces and satisfy $g_{\mu\nu} \equiv e^a_\mu e^b_\nu \eta_{ab}$, with $\eta_{ab} = diag(-++)$ the Minkowski metric

8.1.1 Chern-Simons Formulation

The action (8.3) can be formulated as Chern-Simons theory [88]. In fact, defining the gauge connection

$$A = e^a P_a + \omega^a J_a \ ,$$

where is the dual spin connection $\omega^a = \frac{1}{2}\varepsilon^a{}_{bc}\omega^{bc}$, the Einstein-Hilbert action can be written as

$$I_{EH} = -\frac{k}{4\pi}\int \left\langle AdA + \frac{2}{3}A^3 \right\rangle \ ,$$

where $k = \frac{1}{16\pi G}$ and the gauge generators $\{J_a, P_a\}$ satisfy the algebra

$$\begin{aligned} [J_a, J_b] &= \varepsilon^c{}_{ab} J_c \ , \\ [J_{ab}, P_c] &= \varepsilon^c{}_{ab} P_c \ , \\ [P_a, P_b] &= \pm\frac{1}{l^2}\varepsilon^c{}_{ab} J_c \ , \end{aligned}$$

where $l^2 = \mp 1/\Lambda$, which corresponds to the $SO(2,2)$ ($SO(3,1)$) group in the case of negative (positive) cosmological constant and to the $ISO(2,1)$ group for the case of vanishing cosmological constant $\Lambda = 0$.

8.1.2 Lorentz Flat Geometries

One of the most remarkable properties of three-dimensional Einstein gravity with negative cosmological constant is that it admits the BTZ black hole solution [112], whose line element is given by

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 (N_\phi dt + d\phi)^2 \ , \tag{8.4}$$

where $f^2 = -M + r^2/\ell^2 + J^2/(2r)^2$, and $N_\phi = -J/(2r^2)$. Here M is the mass, J is the angular momentum, and the coordinates have the standard ranges, $-\infty < t < \infty$, $0 < r < \infty$, $0 \leq \phi \leq 2\pi$. The BTZ black hole shares all the features of the more realistic $3+1$ counterparts, such as the existence of an event horizon that surrounds the central singularity, its formation by collapsing matter, the emission of Hawking radiation consistent with thermodynamics, and the relation between entropy and the area of the horizon, among others. On the other hand, the enormous simplification resulting from the absence of propagating degrees of freedom in $2+1$ dimensions makes it an ideal laboratory to test gravitation theory in a lighter setting [113].

Another exceptional feature of the BTZ black hole is that in any simply connected patch U of the geometry, it is parallelizable with respect to a Lorentz connection [114]. This means that U can be covered with a family of locally inertial frames (freely falling observers) so that they can all be obtained by parallel transport from a given one U_0, independently of the path taken to connect them. The notion of parallelism here is the one relevant to the Lorentz group, characterized by the connection one-form $\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu$. This connection defines the covariant derivative of a Lorentz vector v^a with respect to the Lorentz group as

$$Dv^a = dv^a + \omega^a{}_b v^b \ , \tag{8.5}$$

and the corresponding Lorentz curvature is $R^a{}_b := d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$.

The geometry of the BTZ solution is the quotient of the AdS_3 space-time by an isometry that identifies points along a Killing vector [115],

$$\mathscr{M}_{\mathrm{BTZ}} = \mathrm{AdS}_3/\Gamma_K \, . \tag{8.6}$$

Varying the action with respect to the metric, the field equations describe a manifold of constant negative Riemann curvature,

$$\mathscr{R}^{\alpha\beta}{}_{\mu\nu} = -\ell^{-2}\left(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu\right) . \tag{8.7}$$

As is well known, parallel transport of a vector (or a frame) in a closed loop produces a rotated vector (or frame) by a magnitude that depends on the total curvature enclosed by the loop. Hence, the possibility of covering the region U with a family of parallel-transported frames independently of the path in a consistent manner, requires the corresponding curvature to vanish,

$$R^a{}_b(x) = 0 \, , \quad \forall x \in U \ . \tag{8.8}$$

Since the Lorentz curvature does not make any reference to the metric $g_{\mu\nu}(x)$, a natural question to ask would be, what is the most general metric consistent with a Lorentz-flat geometry? In other words, does $R^{ab} = 0$ determine, or impose some constraints, on the metric of the manifold? In order to answer this question, we can start by defining the metric in terms of the local frame one-forms (vielbeins), $e^a(x) = e^a{}_\mu(x)dx^\mu$,

$$g_{\mu\nu}(x) = \eta_{ab}e^a{}_\mu e^b{}_\nu \ . \tag{8.9}$$

The vielbeins are vectors under the Lorentz group acting in the tangent space, and their covariant derivative defines the torsion two-form,

$$T^a = De^a = de^a + \omega^a{}_b e^b \ , \tag{8.10}$$

which is also independent of the metric. However, the covariant derivative of the torsion must vanish, because $DT^a = R^a{}_b e^b$. So, we conclude that a Lorentz-flat space-time $R^a{}_b = 0$ can only admit a covariantly constant torsion,

$$DT^a = 0 \ . \tag{8.11}$$

Splitting the Lorentz connection into a torsion-free part $\bar{\omega}^a{}_b$ and the contorsion, $\kappa^a{}_b = \omega^a{}_b - \bar{\omega}^a{}_b$, we obtain

$$T^a = \kappa^a{}_b e^b \ . \tag{8.12}$$

The Lorentz curvature can also be split into a purely metric part and torsion-dependent terms,

$$R^a{}_b = \mathscr{R}^a{}_b + \bar{D}\kappa^a{}_b + \kappa^a{}_c \kappa^c{}_b \ , \tag{8.13}$$

where $\mathscr{R}^a{}_b$ is the curvature for the torsion-free part of the connection, given by the Riemann tensor as

$$\mathscr{R}^{ab} = \frac{1}{2} e^a{}_\alpha e^b{}_\beta \mathscr{R}^{\alpha\beta}{}_{\mu\nu} dx^\mu dx^\nu \ . \tag{8.14}$$

In $2+1$ dimensions, the condition $DT^a = 0$ can be integrated to

$$T^a = \tau \varepsilon^a{}_{bc} e^b e^c \ , \tag{8.15}$$

where $\varepsilon^a{}_{bc} = \eta^{ad}\varepsilon_{dbc}$ is the Levi-Civita anti-symmetric invariant symbol. In (8.15) τ is a free integration parameter. From this last expression, the contorsion can be identified as $\kappa^a{}_b = -\tau\varepsilon^a{}_{bc} e^c$. Using this expression in (8.13) yields

$$R^{ab} = \mathscr{R}^{ab} + \tau^2 e^a e^b \ . \tag{8.16}$$

In other words, a space-time with vanishing Lorentz curvature corresponds to an anti-de Sitter ($\tau \neq 0$) or flat ($\tau = 0$) Riemannian geometry, where $\ell = 1/\tau$ is the radius of curvature. The fact that (8.4) defines a Lorentz flat geometry with covariantly constant torsion can also be explicitly verifiedby defining the dreibein forms

$$e^0 = f dt \ , \tag{8.17}$$

$$e^1 = f^{-1} dr \ , \tag{8.18}$$

$$e^2 = r(d\varphi + N^\varphi dt) \ . \tag{8.19}$$

The vanishing torsion condition, $de^a + \bar{\omega}^a{}_b e^b = 0$, can be solved for the connection and leads to

$$\bar{\omega}^0{}_1 = \frac{r}{l^2} dt - \frac{J}{2r} d\varphi \ , \tag{8.20}$$

$$\bar{\omega}^1{}_2 = -f d\varphi \ , \tag{8.21}$$

$$\bar{\omega}^2{}_0 = -\frac{J}{2fr^2} dr \ . \tag{8.22}$$

The corresponding Riemannian two-form has constant, negative (or zero) curvature, $\bar{R}^{ab} = -l^{-2}e^a e^b$. On the other hand, the full Lorentz connection $\omega^a{}_b$ reads

$$\omega^0{}_1 = \left(\frac{r}{l} - \varepsilon\frac{J}{2r}\right)\left[\frac{1}{l}dt + \varepsilon d\varphi\right] \,, \tag{8.23}$$

$$\omega^1{}_2 = -f\left[\frac{\varepsilon}{l}dt + d\varphi\right] \,, \tag{8.24}$$

$$\omega^0{}_2 = -\frac{1}{lf}\left(\frac{Jl}{2r^2} + \varepsilon\right) dr \,, \tag{8.25}$$

which can b explicitly checked to be flat, $R^{ab} = 0$.

Now, since 2+1 black holes for any M and J are obtained by an identification of AdS_3, all of them are locally Lorentz-flat. In fact, this feature can also be extended to other locally AdS_3 solutions, like the naked singularities obtained by identifications that produce a conical singularity [116].

8.2 Classical Dual Field Theories

In this section we will review the procedure to obtain classical two-dimensional dual field theories for Einstein gravity both for the Asymptotically AdS and asymptotically flat case following [10, 12]. The theories are constructed in the Chern-Simons formulation by first solving the constraints in the action and subsequently by imposing the corresponding boundary conditions at the level of the currents of the theory.

8.2.1 AdS_3 Case

The notion of asymptotically AdS space-times in three dimensions, which at spatial infinity have the form

$$ds^2 \underset{r\to\infty}{\longrightarrow} -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\phi^2 \,,$$

can be made precise by defining the following set of boundary conditions due to Brown and Henneaux [6].

$$\mathscr{L}_\xi g_{rr} \sim O\left(r^{-4}\right) \,,\; \mathscr{L}_\xi g_{ra} \sim O\left(r^{-3}\right) \,,\; \mathscr{L}_\xi g_{ab} \sim O(1) \,, \tag{8.26}$$

where $a, b = t, \phi$. These boundary conditions determine the form of the asymptotic Killing vectors to be

$$\begin{aligned}
\xi^t &= \ell\left(T^+ + T^- + \tfrac{\ell^2}{r^2}\partial_+^2 T^+ + \tfrac{\ell^2}{r^2}\partial_-^2 T^-\right) + O\left(r^{-4}\right) \,, \\
\xi^\phi &= T^+ - T^- - \tfrac{\ell^2}{2r^2}\partial_+^2 T^+ + \tfrac{\ell^2}{2r^2}\partial_-^2 T^- + O\left(r^{-4}\right) \,, \\
\xi^r &= -\left(\partial_+ T^+ + \partial_- T^-\right) r + O\left(r^{-1}\right) \,,
\end{aligned}$$

where $T^{\pm} = T^{\pm}(t/\ell \pm \phi)$. Decomposing the vectors in T^{+} and T^{-} dependent parts as $\xi = \lambda^{+}[T^{+}] + \lambda^{-}[T^{-}]$. Expanding in Fourier modes by defining the basis $\lambda_n^{\pm} = \lambda^{\pm}\left[e^{in(t/\ell\pm\phi)}\right]$, the asymptotic symmetry algebra is found to be given by two copies of the Witt algebra, which corresponds to the conformal algebra in two dimensions:

$$i\left[\lambda_m^{\pm}, \lambda_n^{\pm}\right] = (m-n)\,\lambda_{m+n}^{\pm} \,. \tag{8.27}$$

Thus, the asymptotic symmetry algebra corresponding to the boundary conditions (8.26) is an infinite-dimensional extension of the isometry algebra.

The algebra of surface charges for the theory can be obtained by supplementing the generators of the asymptotic symmetries with surface terms necessary to render their variations well defined [117] and is isomorphic to the algebra of asymptotic deformations up to a central extension. The central extension turns out to be $c = \frac{3l}{2G}$ [6] and the algebra of surface charges is given by two copies of the Virasoro algebra:

$$i\left[L_m^{\pm}, L_n^{\pm}\right] = (m-n)\,L_{m+n}^{\pm} + \frac{c}{12} m\left(m^2-1\right)\delta_{m+n,0} \,, \tag{8.28}$$

where $L_m^{\pm}$ correspond to the surface charge that generates $\lambda_m^{\pm}$.

The presence of the infinite-dimensional conformal group as asymptotic symmetry group suggests that the asymptotic dynamics of the theory is described by a two-dimensional CFT living at the boundary of the manifold over which the gravity theory is defined.

This procedure can be carried out in the Hamiltonian framework by noting that, due to the isomorphism $SO(2,2) \simeq SL(2,R) \times SL(2,R)$, the three-dimensional Einstein-Hilbert action in with negative cosmological constant can be rewritten as two copies of the $SL(2,R)$ Chern-Simons theory [9].

$$I[e,\omega] = I_{CS}\left[A^{+}\right] - I_{CS}\left[A^{-}\right] - \frac{k}{4\pi}\int_{\partial\Sigma} dtd\phi Tr\left[\left(A_{\phi}^{+}\right)^2 + \left(A_{\phi}^{-}\right)^2\right] \,, \tag{8.29}$$

where boundary terms necessary to render the variational problem well-defined have been added and the Chern-Simons action in Hamiltonian form is given by

$$I_{CS}[A] = -\frac{kl}{4\pi}\int dtdrd\phi Tr\left[A_{\phi}\dot{A}_r - A_r\dot{A}_{\phi}\right] - \frac{kl}{4\pi}\int dtd\phi\varepsilon^{ij} Tr\left[A_0 F_{ij}\right] \,. \tag{8.30}$$

In this context, the Brown-Henneaux boundary conditions for the metric are translated in the following boundary conditions for the $SL(2,R)$ connections [10]:

$$A \sim \begin{pmatrix} \frac{1}{2r}dr & \frac{l}{r}\Xi_{++}dx^{+} \\ \frac{r}{l}dx^{+} & -\frac{1}{2r}dr \end{pmatrix} \,, \quad \bar{A} \sim \begin{pmatrix} \frac{1}{2r}dr & \frac{r}{l}dx^{-} \\ \frac{l}{r}\Xi_{--}dx^{-} & -\frac{1}{2r}dr \end{pmatrix} \,, \tag{8.31}$$

where $x^{\pm}$ are light-cone coordinates.

Varying the action with respect to the Lagrange multipliers $A_0^{\pm}$ gives the constraints $F_{ij}^{\mp}=0$, whose general solutions are locally given by pure gauge configuration of the form $A_i^{\pm}=G_{\pm}^{-1}\partial_i G_{\pm}$. Imposing the gauge fixing conditions $\partial_\phi A_r^{\pm}=0$, the general solution factorizes as

$$G_{\pm}=g_{\pm}(t,\phi)\,h_{\pm}(r,t)\ , \tag{8.32}$$

which, when replaced back in the action leads to the sum of two chiral WZW models,

$$I[e,\omega]=I_{+}[g_{+}]+I_{-}[g_{-}]\ ,$$

where

$$I_{\pm}[g_{\pm}]=\pm\frac{k}{2\pi}\int dtd\phi Tr\left[\left(g_{\pm}^{-1}\partial_{+}g_{\pm}-g_{\pm}^{-1}\partial_{-}g_{\pm}\right)g_{\pm}^{-1}\partial_{\mp}g_{\pm}\right]\pm\frac{kl}{2\pi}\Gamma[G_{\pm}]\ , \tag{8.33}$$

and

$$\Gamma[G]=\frac{1}{3!}\int Tr\left(G^{-1}dG\right)^{3}\ . \tag{8.34}$$

The action (8.33) can be rewritten as one single non-chiral WZW model in Hamiltonian form by applying the transformation $G=G_{+}^{-1}G_{-}$, $g=g_{+}^{-1}g_{-}$, $\pi=-g_{-}^{-1}g_{+}'g_{+}^{-1}g_{-}-g_{-}^{-1}g_{-}'$, leading to

$$I[e,\omega]=\frac{kl}{2\pi}\int dtd\phi Tr\left[\frac{1}{2}\pi g^{-1}\dot{g}-\frac{1}{4l}\left(\pi^{2}+\left(g^{-1}g'\right)^{2}\right)\right]-\frac{kl}{2\pi}\Gamma[G]\ , \tag{8.35}$$

which, after eliminating the momenta, can be put in the standard form

$$I[g]=-\frac{kl^{2}}{8\pi}\int dud\phi Tr\left[\eta^{\mu\nu}g^{-1}\partial_{\mu}gg^{-1}\partial_{\nu}g\right]-\frac{kl}{2\pi}\Gamma[G]\ . \tag{8.36}$$

The non-chiral WZW model is invariant under the transformations $g\longrightarrow\Theta_{+}(x^{+})g\Theta_{-}^{-1}(x^{-})$. The infinitesimal form of these transformation have the form $\delta g=\theta_{+}g-g\theta_{-}$ and lead to the Noether currents whose time components are given by

$$J_{\pm}^{0}=2Tr[\theta_{\pm}I_{\pm}],\qquad I_{+}=\pm\frac{kl}{4\pi}\partial_{+}gg^{-1},\qquad I_{-}=-\frac{kl}{4\pi}g^{-1}\partial_{-}g\ , \tag{8.37}$$

and, in the Hamiltonian formulation, satisfy a Poisson algebra isomorphic to two commuting copies of the $\mathfrak{sl}(2,\mathbb{R})$ Kac-Moody algebra

$$\begin{aligned}\left\{I_a^{\pm}(\phi),I_b^{\pm}(\phi')\right\} &= \varepsilon^{c}{}_{ab}I_c^{\pm}(\phi)\delta\left(\phi-\phi'\right)\pm\frac{kl}{4\pi}\eta_{ab}\partial_\phi\delta\left(\phi-\phi'\right)\ ,\\ \left\{I_a^{+}(\phi),I_b^{-}(\phi')\right\} &= 0\ .\end{aligned} \tag{8.38}$$

The conformal algebra can be obtained from (8.38) by means of the Sugawara construction. The energy-momentum tensor for the theory can be written as quadratic combinations of the currents

(8.37) and its non-vanishing component, which in light-cone coordinates are given weakly by

$$T^{+}_{++} \approx \frac{2\pi}{k} I^{+}_{a} I^{a}_{+} \; , \qquad (8.39)$$
$$T^{-}_{--} \approx \delta^{\pm}_{-} \frac{2\pi}{k} I^{-}_{a} I^{a}_{-} \; ,$$

satisfy the Poisson algebra

$$\left\{T^{\pm}_{\pm\pm}(\phi), T^{\pm}_{\pm\pm}(\phi')\right\}^{*} = \pm l \left(T^{\pm}_{\pm\pm}(\phi) + T^{\pm}_{\pm\pm}(\phi')\right) \partial_{\phi} \delta \left(\phi - \phi'\right) \; , \qquad (8.40)$$

which after expanding in modes has the form (8.27). The theory can be further reduced by implementing the boundary conditions (8.26), which in terms of the currents (8.37) can be written as $\left(g_{\pm}^{-1} \partial_{\phi} g_{\pm}\right)^{\mp} = 1$. In terms of the parametrization

$$g_{+} = \begin{pmatrix} 1 & 0 \\ \sigma_{+} & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\varphi_{+}} & 0 \\ 0 & e^{\frac{1}{2}\varphi_{+}} \end{pmatrix} \begin{pmatrix} 1 & \tau_{+} \\ 0 & 1 \end{pmatrix} , \qquad (8.41)$$

$$g_{-} = \begin{pmatrix} 1 & \sigma_{-} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\varphi_{-}} & 0 \\ 0 & e^{-\frac{1}{2}\varphi_{-}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau_{-} & 1 \end{pmatrix} . \qquad (8.42)$$

The reduced action for Einstein gravity with negative cosmological constant takes the form

$$I^{red}\left[e, \omega\right] = I^{red}_{+} + I^{red}_{-} \; ,$$

where $I^{red}_{\pm}$ correspond, up to boundary terms, to the chiral boson actions

$$I^{red}_{\pm} = \pm \frac{k}{4\pi} \int dt d\phi \left[\varphi'_{\pm} \partial_{\mp} \varphi_{\pm}\right] \; . \qquad (8.43)$$

The sum of the chiral boson actions can be written as a Liouville theory by applying a suitable transformation to the fields [10]. The change of variables is not well defined for the zero modes and implies that the equivalence of the sum of the two chiral models with the non-chiral theory does not hold in this sector. The reduction of the chiral models to Liouville theory can be understood as a Hamiltonian reduction where the reduced phase space is endowed with the symplectic structure coming from the corresponding Dirac bracket. In that context, the first class energy-momentum tensor can be obtained from (8.39) $\tilde{T}^{\pm} \approx T^{\pm} \pm \frac{1}{l} \partial_{\phi} I^{\pm}_{2}$, whose Dirac brackets are given by

$$\left\{\tilde{T}^{\pm}(\phi), \tilde{T}^{\pm}(\phi')\right\}^{*} = \pm \frac{1}{l} \left(\tilde{T}^{\pm}(\phi) + \tilde{T}^{\pm}(\phi')\right) \partial_{\phi} \delta \left(\phi - \phi'\right) \mp \frac{k}{4\pi l} \partial^{3}_{\phi} \delta \left(\phi - \phi'\right) \; . \qquad (8.44)$$

In terms of modes $L_m^{\pm} = \frac{1}{l}\int_0^{2\pi} d\phi e^{\pm im\phi}\tilde{T}_{\pm\pm}^{\pm}$, this gives a Dirac bracket algebra isomorphic to the Virasoro algebra (8.28) with the Brown-Henneaux central extension

$$i\left\{L_m^{\pm},L_n^{\pm}\right\}^* = (m-n)L_{m+n}^{\pm} + \frac{c}{12}m^3\delta_{m+n}^0, \;\; c = 6kl = \frac{3l}{2G} \; . \tag{8.45}$$

8.2.2 Flat Case

In order to analyze the limit of vanishing cosmological constant, it is convenient to express the Einstein-Hilbert (8.3) action in Hamiltonian form [12]

$$I[e,\omega] = -\frac{k}{2\pi}\int dtdrd\phi\varepsilon^{ij}\left(\omega_{ai}\dot{e}_j^e - \mathscr{H}\right) - \frac{k}{4\pi}\int_{\partial\Sigma} dtd\phi\left[\omega_\phi^a\omega_{a\phi} + \frac{1}{l^2}e_\phi^a e_{a\phi}\right] \; , \tag{8.46}$$

where the Hamiltonian is given by $\mathscr{H} = e_{ta}\gamma_\omega^a + \omega_{ta}\gamma_e^a$ with

$$\begin{aligned}
\gamma_\omega^a &= \frac{1}{2}\varepsilon^{ij}\left(\partial_i e_j^a - \partial_j e_i^a + \varepsilon^{abc}\left[\omega_{ib}e_{jc} - \omega_{jb}e_{ic}\right]\right) \; , \\
\gamma_\omega^a &= \frac{1}{2}\varepsilon^{ij}\left(\partial_i\omega_j^a - \partial_j\omega_i^a + \varepsilon^{abc}\left[\omega_{ib}\omega_{jc} + \frac{1}{l^2}e_{ib}e_c\right]\right) \; ,
\end{aligned}$$

and the boundary terms have been included to render the variational problem well defined. By adopting the gauge conditions $\partial_\phi\omega_r = 0 = \partial_\phi e_r$, the general solution to the constraints $\gamma_\omega = 0$ and $\gamma_e = 0$ is given by

$$\begin{aligned}
&\omega_i = \Lambda^{-1}\partial_i\Lambda \;\; , \;\; \Lambda = \lambda(t,\phi)\mu(r,t) \; , \\
&\tilde{e}_i = \Lambda^{-1}\partial_i\left(\alpha + \lambda\beta\lambda^{-1}\right)\Lambda \; .
\end{aligned} \tag{8.47}$$

Inserting this solution into the action (8.46) and taking the limit $l \longrightarrow \infty$, which correspond to the vanishing cosmological constant case $\Lambda \longrightarrow 0$, leads to the flat WZW model action

$$I[\lambda,\alpha] = \frac{k}{\pi}\int dtd\phi Tr\left[\dot{\lambda}\lambda^{-1}\alpha' - \frac{1}{2}\left(\lambda'\lambda^{-1}\right)^2\right] \; . \tag{8.48}$$

The action is invariant under the transformation $\lambda \longrightarrow \lambda\Theta^{-1}(\phi), \alpha \longrightarrow \alpha - t\lambda\Theta^{-1}\Theta'\lambda$ and $\lambda \longrightarrow \lambda$, $\alpha \longrightarrow \lambda^{-1}\Sigma(\phi)\lambda$, whose infinitesimal forms are given respectively by $\delta_\theta\lambda = -\lambda\theta$, $\delta_\theta\alpha = -t\lambda\theta'\lambda^{-1}$, and $\delta_\sigma\lambda = 0$, $\delta_\sigma\alpha = \lambda\sigma\lambda^{-1}$. The time components of the associated Noether currents are given by

$$J_\theta^0 = 2Tr[\theta J] \, , \quad J = -\frac{k}{2\pi}\left[\lambda^{-1}\alpha'\lambda - t\left(\lambda^{-1}\lambda'\right)'\right] \; , \tag{8.49}$$

$$P_\sigma^0 = 2Tr[\sigma P] \, , \quad P = \frac{k}{2\pi}\lambda^{-1}\lambda' \; , \tag{8.50}$$

which, in the Hamiltonian formulation, satisfy a Dirac brackets algebra isomorphic to the $\mathfrak{iso}(2,1)$ Kac-Moody algebra

$$\begin{aligned}
\left\{P_a(\phi),P_b(\phi')\right\}^* &= 0\,, \\
\left\{J_a(\phi),P_b(\phi')\right\}^* &= \varepsilon^c{}_{ab}P_c(\phi)\delta\left(\phi-\phi'\right)-\frac{k}{2\pi}\eta_{ab}\partial_\phi\delta\left(\phi-\phi'\right)\,, \\
\left\{J_a(\phi),J_b(\phi')\right\}^* &= \varepsilon^c{}_{ab}J_c(\phi)\delta\left(\phi-\phi'\right)\,.
\end{aligned} \tag{8.51}$$

The theory can be further reduced by considering the parametrization

$$\lambda=\begin{pmatrix}1 & 0\\ \frac{\sigma}{\sqrt{2}} & 1\end{pmatrix}\begin{pmatrix}e^{-\frac{1}{2}\varphi} & 0\\ 0 & e^{\frac{1}{2}\varphi}\end{pmatrix}\begin{pmatrix}1 & \sqrt{2}\tau\\ 0 & 1\end{pmatrix},\ \alpha=\begin{pmatrix}1+\frac{\theta}{2} & \frac{\zeta}{2}\\ \frac{\eta}{\sqrt{2}} & 1-\frac{\theta}{2}\end{pmatrix}, \tag{8.52}$$

and imposing the boundary conditions corresponding to asymptotically flat space-times. In the Chern Simons formulation for gravity, the conditions for asymptotically flat space-times can be obtained by writing the chiral connections entering in (8.29) in the BMS gauge, which after implementing (8.47) lead to $\left(\lambda^{-1}\lambda'\right)^-\approx 1/\sqrt{2},\left(\lambda^{-1}\alpha'\lambda\right)^-\approx 0$ and in terms of the parametrization (8.52) take the form

$$\sigma'e^{-\varphi}\approx 1,\ \ \eta'\sigma^2+2\theta'\sigma-2\zeta'\approx 0\,. \tag{8.53}$$

By inserting (8.52) and (8.53) in the action and neglecting all boundary terms, the reduced action can be written as the flat Liouville theory [12]

$$I^{red}=\frac{k}{2\pi}\int dtd\phi\left[\zeta'\dot{\varphi}-\varphi'^2\right]\,. \tag{8.54}$$

In the Hamiltonian formalism, the Hamiltonian and momentum densities for the Flat WZW model (8.48) are given weakly as quadratic combinations of the currents as $\mathscr{H}\approx\frac{\pi}{k}P^aP_a$ and $\mathscr{P}\approx-\frac{2\pi}{k}J^aP_a$. When implementing the reduction to the flat Liouville theory, the generators that commute with the first class reduction constraints are given by $\tilde{\mathscr{H}}=\mathscr{H}+\partial_\phi P_2$ and $\tilde{\mathscr{P}}=\mathscr{P}+\partial_\phi j_2$, whose Dirac bracket algebra is given by

$$\begin{aligned}
\left\{\tilde{\mathscr{H}}(\phi),\tilde{\mathscr{H}}(\phi')\right\}^* &= 0\,, \\
\left\{\tilde{\mathscr{H}}(\phi),\tilde{\mathscr{P}}(\phi')\right\}^* &= \left(\tilde{\mathscr{H}}(\phi)+\tilde{\mathscr{H}}(\phi')\right)\partial_\phi\delta\left(\phi-\phi'\right)-\frac{k}{2\pi}\partial_\phi^3\delta\left(\phi-\phi'\right)\,, \\
\left\{\tilde{\mathscr{P}}(\phi),\tilde{\mathscr{P}}(\phi')\right\}^* &= \left(\tilde{\mathscr{P}}(\phi)+\tilde{\mathscr{P}}(\phi')\right)\partial_\phi\delta\left(\phi-\phi'\right)\,,
\end{aligned} \tag{8.55}$$

and corresponds the centrally extended $\widehat{\mathfrak{bms}}_3$ algebra, which in terms of modes, $P_m=\int_0^{2\pi}d\phi e^{im\phi}\tilde{\mathscr{H}}$, $J_m=\int_0^{2\pi}d\phi e^{im\phi}\tilde{\mathscr{P}}$, takes the form

$$
\begin{aligned}
i\{P_m,P_n\}^* &= 0\,, \\
i\{J_m,P_n\}^* &= (m-n)\,P_{m+n}+\frac{c_2}{12}m^3\delta^0_{m+n},\;\; c_2=12k=\frac{3}{G}\,, \\
i\{J_m,J_n\}^* &= (m-n)\,J_{m+n}+\frac{c_1}{12}m^3\delta^0_{m+n},\;\; c_1=0\,.
\end{aligned}
\tag{8.56}
$$

8.2.3 Lorentz-flat-Geometries

An action containing the Lorentz-flat geometries as solutions is given by a Chern-Simons theory for the $SL(2,R)$ group

$$I=\int\left\langle \omega d\omega+\frac{2}{3}\omega^3\right\rangle,$$

whose field equations are $R=0$. By considering geometries whose torsion is covariantly constant, this theory can include all the asymptotically AdS space-times defined by the Brown-Henneaux boundary conditions as asymptotically Lorentz-flat geometries. The boundary conditions for the $SL(2,R)$ connection describing asymptotically Lorentz-flat geometries and read

$$
\begin{aligned}
&\omega^0\sim\varepsilon\tfrac{r}{l}\left(\tfrac{dt}{l}+\varepsilon d\phi\right)+O\left(r^{-1}\right)dt+O\left(r^{-1}\right)d\phi+O\left(r^{-4}\right)dr\,, \\
&\omega^1\sim\varepsilon\left(\tfrac{1}{r}+O\left(r^{-3}\right)\right)dr+O\left(r^{-2}\right)dt+O\left(r^{-2}\right)d\phi\,, \\
&\omega^2\sim\tfrac{r}{l}\left(\tfrac{dt}{l}+\varepsilon d\phi\right)+O\left(r^{-1}\right)dt+O\left(r^{-1}\right)d\phi+O\left(r^{-4}\right)dr\,.
\end{aligned}
\tag{8.57}
$$

Here $\langle\;\rangle$ denotes the symmetrized trace. and where M is an orientable 3-dimensional manifold over which the connection ω is defined. Now suppose M has the topology $\Sigma\times\mathbb{R}$. Considering $\mathbb{R}$ as the temporal line and Σ as a spatial section, we can split the gauge field in time and space components $\omega_\mu dx^\mu=\omega_t dt+\omega_i dx^i$, $(i=1,2)$. The action then takes the form

$$I=\kappa\int_M d^3x\varepsilon^{ij}\left\langle\dot{\omega}_i\omega_j+\omega_t R_{ij}\right\rangle-\frac{\varepsilon\kappa}{l}\int_B dt d\phi\left\langle\omega_\phi^2\right\rangle\,. \tag{8.58}$$

Here ω_t is interpreted as a Lagrangian multiplier which enforces the constraint

$$R_{ij}=0\,. \tag{8.59}$$

Solving the constraint (5.21) leads to the local the form of the spatial connection

$$\omega_i=G^{-1}\partial_i G\,, \tag{8.60}$$

where $G\in SO(2,1)$.

By means of the gauge condition $\partial_\phi\omega_r=0$, (8.60) can be solved and gives

$$G=g(t,\phi)\,h(r)\,, \tag{8.61}$$

where g and h are elements of $SO(2,1)$. Replacing (8.61) in (8.58) yields

$$I_{cWZW} = \kappa \int_B dt d\phi \left\langle g^{-1}\dot{g}g^{-1}g' - \frac{\varepsilon}{l}\left(g^{-1}g'\right)^2 \right\rangle + \Gamma[G] \qquad (8.62)$$

and correspond to the action of a chiral WZW model, whose chirality depends on the sign of ε. At this point, the reduction is only partial as we have not still imposed the boundary conditions (8.57). Let's write the chiral WZW action as

$$I_{cWZW} = I_\varepsilon[g] + \Gamma[G] \ , \qquad (8.63)$$

where we have defined

$$I_\varepsilon[g] = \kappa \int_B dt d\phi \left\langle g^{-1}\dot{g}g^{-1}g' - \frac{\varepsilon}{l}\left(g^{-1}g'\right)^2 \right\rangle \ . \qquad (8.64)$$

The variation of $I_\varepsilon[g]$ is given by

$$\delta I_\varepsilon = -\kappa \int_B dt d\phi \left\langle g^{-1}\delta g \partial_\phi\left(g^{-1}\dot{g}\right) + g^{-1}\delta g \partial_t\left(g^{-1}g'\right) - \frac{2\varepsilon}{l} g^{-1}\delta g \partial_\phi\left(g^{-1}g'\right) \right\rangle + \delta\Gamma[G] \ , \qquad (8.65)$$

while the variation of Γ is

$$\delta\Gamma[G] = \kappa \int_B dt d\phi \varepsilon^{\alpha\beta} \left\langle g^{-1}\delta g \partial_\alpha\left(g^{-1}\partial_\beta g\right)\right\rangle = \kappa \int_B dt d\phi \left\langle -g^{-1}\delta g \partial_t\left(g^{-1}g'\right) + g^{-1}\delta g \partial_\phi\left(g^{-1}\dot{g}\right)\right\rangle \ , \qquad (8.66)$$

where we have considered $\varepsilon^{t\phi} = -1$. Therefore, putting (8.65) and (8.66) together, we obtain the form of the variation of the chiral WZW action

$$\delta I_{cWZW} = -2\kappa \int_B dt d\phi \left\langle g^{-1}\delta g \left(\partial_t - \frac{\varepsilon}{l}\partial_\phi\right)\left(g^{-1}g'\right) \right\rangle \ .$$

The field equations for the chiral WZW are then given by

$$\left(\partial_t - \frac{\varepsilon}{l}\partial_\phi\right)\left(g^{-1}g'\right) = 0 \ ,$$

and express the conservation of the current

$$J = g^{-1}g\ ^{\cdot\prime} \qquad (8.67)$$

In therms this current, the remaining boundary conditions (8.57) read

$$\begin{aligned} &J^{(2)} \sim 0 \ , \\ &J^{(+)} \sim \tfrac{1-\varepsilon}{2}\left(\tfrac{r}{l}\right)^{1+\varepsilon} + O\left(r^{-(1-\varepsilon)}\right) \ , \\ &J^{(-)} \sim \tfrac{1+\varepsilon}{2}\left(\tfrac{r}{l}\right)^{1-\varepsilon} + O\left(r^{-(1+\varepsilon)}\right) \ . \end{aligned} \qquad (8.68)$$

Let's consider now the following Cartan decomposition for the group element

$$g = ABC \ , \tag{8.69}$$

where

$$A = \begin{pmatrix} 1 & \frac{1-\varepsilon}{2}x \\ -\frac{1+\varepsilon}{2}y & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} e^{-\frac{\varepsilon}{2}\phi} & 0 \\ 0 & e^{\frac{\varepsilon}{2}\phi} \end{pmatrix} \ , \ C = \begin{pmatrix} 1 & -\frac{1+\varepsilon}{2}x \\ \frac{1-\varepsilon}{2}y & 1 \end{pmatrix} \ . \tag{8.70}$$

Then, the boundary conditions (8.68) imply that on the boundary

$$\begin{array}{r} \frac{1-\varepsilon}{2}x' e^{\varepsilon\phi} = \frac{1-\varepsilon}{2}\left(\frac{r}{l}\right)^{1+\varepsilon} \\ -\frac{1+\varepsilon}{2}y' e^{-\varepsilon\phi} = \frac{1+\varepsilon}{2}\left(\frac{r}{l}\right)^{1-\varepsilon} \end{array} \ . \tag{8.71}$$

Replacing (8.70) and (8.71) in (8.63) leads, up to boundary terms, to the following reduced action

$$I_{red} = \frac{\kappa}{2}\int_B dt d\phi \left(\dot{\phi}\phi' - \frac{\varepsilon}{l}\left(\phi'\right)^2\right) \ , \tag{8.72}$$

which corresponds to a chiral boson action, whose chirality is determined by the sign of ε.

Chapter 9

Actions on Coadjoint orbits

In this chapter we describe the general construction of geometrical actions using coadjoint orbits of a Lie group G. In Section 9.1 we review the notions of coadjoint orbits and the Kirillov-Kostant symplectic form, which allows to construct a geometrical action. In Section 9.2 we show how this action is restricted to a coadjoint orbit, over which the Kirillov-Kostant symplectic form is explicitly non-degenerate and can be inverted. In Section 9.3 we analyze the geometrical action as a constrained system by means of Dirac formalism to find the associated Poisson bracket on each orbit.

9.1 Coadjoint Orbits

A Lie group G can act on itself by conjugation, i.e., any element $g \in G$ defines an group automorphism given by

$$\begin{aligned} \sigma_g : \quad & G \longrightarrow G \\ & g' \longmapsto \sigma_g(g') = gg'g^{-1} \ . \end{aligned} \tag{9.1}$$

Given a curve in the group manifold $c(t)$ with $t \in \mathbb{R}$, the action of the group on the points of the curve is given by $\sigma_g(ct) = gc(t)g^{-1}$. The Lie algebra $\mathfrak{g}$ associated to G is isomorphic to the tangent space T_eG to G at the identity e and the adjoint action of G on $\mathfrak{g}$ is defined as the differential of the group automorphism σ_g at the identity

$$Ad_gX = \frac{d}{dt}\left(gc(t)g^{-1}\right)\Big|_{t=0} \ , \tag{9.2}$$

where $X \in \mathfrak{g}$ corresponds to the tangent vector to the curve $c(t)$. Then, the map

$$\begin{aligned} Ad : \quad & Ad : G \longrightarrow Aut(\mathfrak{g}) \\ & g \longmapsto Ad_g \ , \end{aligned} \tag{9.3}$$

defines the adjoint representation of the group G.

Let $\mathfrak{g}^*$ be the dual space of the Lie algebra $\mathfrak{g}$ and $\langle , \rangle$ the paring between them. The coadjoint representation of G on $\mathfrak{g}^*$ is then defined by the map

$$\begin{aligned} Ad^* : \quad G &\longrightarrow Aut(\mathfrak{g}^*) \\ g &\longmapsto Ad^*_g \ , \end{aligned} \tag{9.4}$$

where, for every element $U \in \mathfrak{g}^*$ and $g \in G$, the coadjoint action of G on $\mathfrak{g}^*$ satisfies

$$\left\langle Ad^*_g B, X \right\rangle = \left\langle B, Ad_{g^{-1}} X \right\rangle \ . \tag{9.5}$$

Let us consider now the adjoint action of elements of the curve $c(t)$ on some Lie algebra element $Y \in \mathfrak{g}$, $Ad_{c(t)}Y$. The differential of the map Ad at the identity defines the adjoint action of $\mathfrak{g}$ on itself by

$$ad_X Y = \frac{d}{dt}\left(Ad_{c(t)}Y\right)\bigg|_{t=0} = [X, Y] \ , \tag{9.6}$$

and the map

$$\begin{aligned} ad : \quad \mathfrak{g} &\longrightarrow End(\mathfrak{g}) \\ X &\longmapsto ad_X \ , \end{aligned} \tag{9.7}$$

corresponds to the adjoint representation of the Lie algebra $\mathfrak{g}$. Similarly, the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ is defined by

$$\begin{aligned} ad^* : \quad \mathfrak{g} &\longrightarrow End(\mathfrak{g}^*) \\ X &\longmapsto ad^*_X \ , \end{aligned} \tag{9.8}$$

where the adjoint action ad^*_X of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by the differential of $Ad^*_{c(t)}$ at the identity, for which (9.5) leads to

$$\langle ad^*_X B, Y \rangle = -\langle B, ad_X Y \rangle \ . \tag{9.9}$$

The coadjoint orbit O_U of a Lie group G through a certain point $U^{(0)} \in \mathfrak{g}^*$ is defined as

$$O_U = \left\{ U = Ad^*_g U^{(0)} \mid g \in G \right\} \ . \tag{9.10}$$

The isotropy group H of the orbit is given by the elements in G leaving $U^{(0)}$ stationary

$$H = \left\{ h \in G \mid Ad^*_h U^{(0)} = U^{(0)} \right\} \ , \tag{9.11}$$

and therefore, coadjoint orbits are group manifolds of the form $O_U \cong G/H$.

9.2 Kirillov-Kostant Form and Geometrical Actions

Let us consider the case in which G is a matrix group, where the adjoint action is simply given by the matrix conjugation,

$$Ad_g X = gXg^{-1} \ . \tag{9.12}$$

In that case the coadjoint action is also given by matrix conjugation and elements of the orbit O_U

are expressed in terms of a fixed representative as

$$U = Ad^*_g X = gU^{(0)}g^{-1} \ .$$

One interesting feature of coadjoint orbits is that they are symplectic manifolds, which are naturally endowed by a canonical G-invariant symplectic structure. In other words, for each coadjoint orbit O_U it is possible to define a closed and non-degenerate symplectic form given by the Kirillov-Kostant form

$$\Omega\left(v_X, v_Y\right) = \langle U, [X,Y]\rangle \ , \tag{9.13}$$

where v_X is the vector field on O_U induced by X through the coadjoint action of G on the orbit.

In the following a different definition of the Kirillov-Kostant form will be used. This construction is better suited for our purposes and we will prove in the next section that it matches the standard definition. Let us define the Kirillov-Kostant symplectic $2-$form on O_U as

$$\Omega = \left\langle U, \kappa^2 \right\rangle \ , \tag{9.14}$$

where κ corresponds to the right invariant Maurer-Cartan form (B.6).

As the symplectic form (9.14) is closed, it is locally exact. In fact, it is easy to see that it can be written as

$$\Omega = -d\left\langle U, \kappa\right\rangle \ ,$$

which allows to define a *geometric action* by means of the $1-$form $\langle U, \kappa\rangle$

$$I = -\int \langle U, \kappa\rangle \ . \tag{9.15}$$

In order to see that the Kirillov-Kostant form is non-degenerate on O_U, let us split the generators of G as $\{e_A, e_\alpha\}$ with e_α are the generators of H. Then, we can consider following parametrization for the group elements:

$$g = \hat{g}h \ , \tag{9.16}$$

where

$$\hat{g} = e^{\zeta^A e_A} \ , \ h = e^{\zeta^\alpha e_\alpha} \in H \ . \tag{9.17}$$

Using this parametrization, the right invariant Maurer-Cartan form κ takes the form $\kappa = d\hat{g}\hat{g}^{-1} + \hat{g}dhh^{-1}\hat{g}^{-1}$. A simple computation shows that the Kirillov-Kostant symplectic form (9.14) has no component along H , i.e.

$$\Omega = -d\left\langle U, d\hat{g}\hat{g}^{-1}\right\rangle \ , \tag{9.18}$$

implying that the geometrical action (9.15) can be put in the form

$$I = -\int \left\langle U, d\hat{g}\hat{g}^{-1}\right\rangle \ . \tag{9.19}$$

In order to see that Ω is explicitly non-degenerate let's note that (9.18) can be put in the form

$$\Omega = \left\langle U, \left(d\hat{g}\hat{g}^{-1}\right)^2\right\rangle = \left\langle U^{(0)}, \left(\hat{g}^{-1}d\hat{g}\right)^2\right\rangle = -\frac{1}{2} f^c_{AB} U^{(0)}_c \hat{M}^A{}_C \hat{M}^B{}_D d\zeta^C \wedge d\zeta^D \ , \tag{9.20}$$

where $\hat{M}^A{}_C = M^A{}_C\big|_{\zeta^\alpha=0}$. The Kirillov-Kostant symplectic form can then be written as

$$\Omega = \frac{1}{2}\Omega_{AB} d\zeta^A \wedge d\zeta^B \ , \tag{9.21}$$

where

$$\Omega_{AB} = C_{CD}\hat{M}^C{}_A \hat{M}^D{}_B \ \ , \ \ C_{AB} = -f^c_{AB} U^{(0)}_c \ . \tag{9.22}$$

The matrix C_{AB} is invertible, as all the zeros coming from the adjoint action of the little group H (9.11) have been removed, and the matrix $\hat{M}^C{}_A$ is invertible, as $M^a{}_b$ is a triangular matrix when expressed as a 2×2 block matrix. Therefore, the matrix Ω_{AB} is invertible, and its inverse defines a Poisson bracket for the coordinates of the orbit G/H given by

$$\left\{\zeta^A, \zeta^B\right\} = \Omega^{AB} = C^{CD}\left(\hat{M}^{-1}\right)^A{}_C \left(\hat{M}^{-1}\right)^B{}_D \ . \tag{9.23}$$

9.3 Hamiltonian Formulation

In order to perform the Hamiltonian analysis of the geometrical actions on coadjoint orbits, lets consider local coordinates $\{\zeta^a\}$ for G. Then, using (B.14) and (B.5), the geometrical action (9.15) can be written as

$$I = -\int \left\langle U^{(0)}, \theta\right\rangle = -\int dt U^{(0)}_a M^a{}_b \dot{\zeta}^b \ .$$

The canonical momenta are then given by

$$\eta_a = \frac{\partial \mathscr{L}}{\partial \dot{\zeta}^a} = -U^{(0)}_b M^b{}_a \ ,$$

and lead to the constraints

$$\phi_a = \eta_a + U_b^{(0)} M^b{}_a \ . \tag{9.24}$$

The matrix of Poisson brackets of the set of constrains (9.24) define the pre-symplectic form of the phase space of the system and is given by

$$\Omega_{ab} = \{\phi_a, \phi_b\} = U_c^{(0)} f_{de}^c M^d{}_a M^e{}_b \ .$$

This matrix is not invertible, as it has zero modes given by $L_\beta^a = \left(M^{-1}\right)_\beta^a$, with β running on the little group of the orbit defined by $U^{(0)}$, of dimension n. Using (B.13) the infinitesimal form of the definition (9.11) for the little group can be expressed as $f_{\alpha\beta}^c U_c^{(0)} = 0$, implying

$$\Omega_{ab} \left(M^{-1}\right)^b{}_\beta = U_c^{(0)} f_{d\beta}^c M^d{}_a = 0 \ .$$

This means that there are n zero modes

$$(v^a)_\beta = \left(M^{-1}\right)^a{}_\beta \ ,$$

which define n first class constraints hidden in the set ϕ_a, given by

$$\psi_\beta = \phi_a \left(v^a\right)_\beta = \phi_a \left(M^{-1}\right)^a{}_\beta = \eta_a \left(M^{-1}\right)^a{}_\beta + U_\beta^{(0)} \ , \tag{9.25}$$

the transformations generated by these constraints are given by

$$\delta\zeta^a = \varepsilon^\beta \left\{\zeta^a, \psi_\beta\right\} = \varepsilon^\beta \left(M^{-1}\right)^a{}_\beta \ . \tag{9.26}$$

These transformations correspond to the action of the little group H on G. In fact the transformation $g \longrightarrow g' = gh = e^{\zeta^a e_a} e^{\varepsilon^\beta e_\alpha} = e^{\zeta'^a e_a}$ produce a change in the coordinates given by the Baker-Campbell- Hausdorff formula,

$$\zeta'^a = \zeta^a + \delta_\beta^a \varepsilon^\beta + \frac{1}{2}\zeta^b \varepsilon^\beta f_{b\beta}^a + \frac{1}{12}\zeta^b \zeta^c \varepsilon^\beta f_{bd}^a f_{c\beta}^d + \dots \ ,$$

which can be written as

$$\zeta'^a = \zeta^a + \varepsilon^\beta \left(M^{-1}\right)^a{}_\beta \ ,$$

and reproduces (9.26).

Having isolated the set of first class constraints (9.25), the set of second class constraints can be taken as $\chi_A = \eta_c \left(M^{-1}\right)^c{}_A + U_A^{(0)}$, whose Poisson brackets define the matrix

$$C_{AB} = \{\chi_A, \chi_B\} \approx U_c^{(0)} f_{AB}^c \ ,$$

which is invertible, as its entries can be written as $C_{AB} \approx -\left\langle ad^*_{e_A} U^{(0)}, e_B \right\rangle$, showing that all the zeros coming from the adjoint action of the little group on $U^{(0)}$ are not present.

In order to carry out the Hamiltonian reduction, the gauge must be fixed, which corresponds to the implementation of new constraints which make (9.25) second class. As gauge conditions we consider

$$G^\alpha = \zeta^\alpha - \left(M^{-1}\right)^\alpha{}_A C^{AB} \chi_B \approx 0 \,,$$

whose Poisson brackets with the constraints read

$$\begin{aligned} \left\{G^\alpha, \psi_\beta\right\} &= \left(M^{-1}\right)^\alpha{}_\beta - \left(M^{-1}\right)^\alpha{}_A C^{AB} \left\{\chi_B, \psi_\beta\right\} \approx \left(M^{-1}\right)^\alpha{}_\beta \,, \\ \left\{G^\alpha, \chi_A\right\} &= \left(M^{-1}\right)^\alpha{}_A - \left(M^{-1}\right)^\alpha{}_B C^{BC} C_{CA} = 0 \,, \\ \left\{G^\alpha, G^\beta\right\} &\approx \left(M^{-1}\right)^\alpha{}_C C^{CD} \left(M^{-1}\right)^\beta{}_D \,. \end{aligned}$$

Defining the set of constraints

$$\gamma_\Lambda = \left\{\chi_A, G^\alpha, \psi_{\alpha'}\right\} \,, \tag{9.27}$$

where capital greek indices run as $\Lambda = \{A, \alpha, \alpha'\}$; $\Sigma = \{B, \beta, \beta'\}$; $\Omega = \{C, \gamma, \gamma'\}$,etc. The Dirac matrix for the set (9.27) is given by

$$S_{\Lambda\Sigma} \approx \begin{pmatrix} C_{AB} & 0 & 0 \\ 0 & \left(M^{-1}\right)^\alpha{}_C C^{CD} \left(M^{-1}\right)^\beta{}_D & \left(M^{-1}\right)^\alpha{}_{\beta'} \\ 0 & -\left(M^{-1}\right)^\beta{}_{\alpha'} & 0 \end{pmatrix} \,. \tag{9.28}$$

The constraints (9.27) are second class provided the block $\left(M^{-1}\right)^\alpha{}_\beta$ of the matrix $\left(M^{-1}\right)^a{}_b$ is invertible. Denoting de inverse of $\left(M^{-1}\right)^\alpha{}_\beta$ by $\tilde{M}^\alpha{}_\beta$, the inverse of (9.28) is given by

$$S^{\Lambda\Sigma} = \begin{pmatrix} C^{AB} & 0 & 0 \\ 0 & 0 & -\tilde{M}^{\beta'}{}_\alpha \\ 0 & \tilde{M}^{\alpha'}{}_\beta & \tilde{M}^{\alpha'}{}_\gamma \left(M^{-1}\right)^\gamma{}_C C^{CD} \left(M^{-1}\right)^\delta{}_D \tilde{M}^{\beta'}{}_\delta \end{pmatrix} \,,$$

where C^{AB} is the inverse of the matrix.

We define now the Dirac bracket

$$\{F, G\}^* = \{F, G\} - \{F, \gamma_\Lambda\} S^{\Lambda\Sigma} \{\gamma_\Sigma, G\} \,. \tag{9.29}$$

After setting the constraints (9.27) strongly to zero, the reduced phase space variables are ζ^A, whose bracket can be computed to give

$$\left\{\zeta^A, \zeta^B\right\}^* = \left[\left(M^{-1}\right)^A{}_C - \left(M^{-1}\right)^A{}_\alpha \tilde{M}^\alpha{}_\gamma \left(M^{-1}\right)^\gamma{}_C\right] C^{CD} \left[\left(M^{-1}\right)^B{}_D - \left(M^{-1}\right)^B{}_\beta \tilde{M}^\beta{}_\delta \left(M^{-1}\right)^\delta{}_D\right] \,. \tag{9.30}$$

Writing the matrix $\left(M^{-1}\right)^{a}{}_{b}$ in a 2×2 block form

$$\left(M^{-1}\right)^{a}{}_{b} = \begin{pmatrix} \left(M^{-1}\right)^{A}{}_{B} & \left(M^{-1}\right)^{A}{}_{\beta} \\ \left(M^{-1}\right)^{\alpha}{}_{B} & \left(M^{-1}\right)^{\alpha}{}_{\beta} \end{pmatrix} , \tag{9.31}$$

we see that when computing the inverse, as for any 2×2 block matrix, the fact that $\left(M^{-1}\right)^{a}{}_{b}$ and $\left(M^{-1}\right)^{\alpha}{}_{\beta}$ are invertible imply that $\left(M^{-1}\right)^{A}{}_{B} - \left(M^{-1}\right)^{A}{}_{\alpha}\tilde{M}^{\alpha}{}_{\beta}\left(M^{-1}\right)^{\beta}{}_{B}$ is also a non-degenerate matrix. In fact, its inverse is given by $M^{A}{}_{B}$ and therefore (9.30) can be inverted to find the symplectic structure for the reduced phase space of the systems $O_U = G/H$ with coordinates $\left\{\zeta^{A}\right\}$, which reads

$$\omega_{AB} = U_a^{(0)} f^{a}_{CD} M^{C}{}_{A} M^{D}{}_{B} , \tag{9.32}$$

and leads to the following reduced Hamiltonian action.

$$I_{red} = -\int dt U_a^{(0)} M^{a}{}_{A} \dot{\zeta}^{A} .$$

Let us consider now the coordinates for $\mathfrak{g}^*$ (B.19), which can be written as functions of the coordinates ζ^a as

$$U_a = U_b^{(0)} \left(K^{-1}\right)^{b}{}_{a} .$$

The properties (5.68) and (B.17), after some algebra the bracket for the U_a's is found to be

$$\{U_a, U_b\}^* = \frac{\partial U_a}{\partial \zeta^A}\frac{\partial U_b}{\partial \zeta^B}\left\{\zeta^A, \zeta^B\right\}^* = -f^{c}_{ab} U_c . \tag{9.33}$$

This implies that the Hamiltonian vector fields with respect to the Poisson bracket (9.33). are given by the vector fields defined by the adjoint action (B.20). In fact

$$v_a = \{U_a, U_b\}^* \frac{\partial}{\partial U_b} = -f^{c}_{ab} U_c \frac{\partial}{\partial U_b} . \tag{9.34}$$

On the other hand, the vector fields (B.21) can be regarded as right invariant vector fields on G/H. In fact, starting from (B.3) and using (B.16) we find

$$R_a = R^{c}{}_{a}\frac{\partial}{\partial \zeta^c} = \left(N^{-1}\right)^{c}{}_{a}\frac{\partial U_b}{\partial \zeta^c}\frac{\partial}{\partial U_b} = -U_c f^{c}_{ab}\frac{\partial}{\partial U_b} ,$$

which means they are dual to the right invariant Maurer-Cartan forms κ^a. Letting (9.14) act on two vector field gives

$$\Omega(v_a, v_b) = \frac{1}{2}\langle U, [e_c, e_d]\rangle \kappa^c(R_a)\wedge\kappa^d(R_b)$$
$$\Longrightarrow \Omega(v_a, v_b) = \langle U, [e_c, e_d]\rangle ,$$

allowing to recover the standard definition (9.13).

Chapter 10

Geometrical Actions for 3D Gravity

In the previous chapter we have seen how to construct a geometrical action on coadjoint orbit of a given Lie group from its symplectic structure. In the context of three-dimensional gravity this method is particularly interesting as the asymptotic structure Einstein gravity in three-dimensions, both in the asymptotically AdS and the asymptotically flat case, is described by infinite dimensional groups whose coadjoint representation is identified with the corresponding solution space of the theory. In fact the general solution for the Einstein equations in three-dimensions with negative cosmological constant and with Brown-Hennaux boundary conditions is given by

$$ds^2 = \frac{l^2}{r^2}dr^2 - r^2\left(dx^+ - \frac{8\pi Gl}{r^2}L^- dx^-\right)\left(dx^- - \frac{8\pi Gl}{r^2}L^+ dx^+\right),$$

where $L^\pm = L^\pm\left(x^\pm\right)$ arbitrary smooth periodic functions. Under the action of the conformal group at infinity these functions transform as

$$\delta L^\pm = \varepsilon\left(L^\pm\right)' + 2\varepsilon' L^\pm - \frac{c}{12}\varepsilon''',$$

which means that $L^\pm$ are in the coadjoint representation of the Virasoro group. Therefore, the solution space of asymptotically AdS three-dimensional gravity can be identified with two copies of the dual space of the Virasoro algebra, $\widehat{Vec}^*\left(S^1\right)$.

In the case $\Lambda = 0$, the the solution for Einstein gravity with asymptotically flat boundary conditions is given by [100]

$$ds^2 = \Theta du^2 - 2dudr + \left(2\Xi + u\Theta'\right)dud\phi + r^2 d\phi^2,$$

where in this case, under the action of the BMS_3 group the functions Θ and Ξ transform as

$$\delta\Theta = Y\Theta' + 2Y'\Theta,$$
$$\delta\Xi = Y\Xi' + 2Y'\Xi + T\Theta' + T'\Theta.$$

Implying the in the case the solution space for asymptotically flat three-dimensional Einstein gravity is identified with the coadjoint representation of the BMS_3 group [102].

Therefore, it is interesting to see what kind of geometrical actions can be obtained for these infinite dimensional groups and how they are related with the known classical duals found in the literature and presented in Chapter. With this aim, this section is devoted to the construction of the Kirillov-Kostant symplectic form and the associated geometrical action for the case of centrally extended groups following [108]. The results are applied for the cases of the Kac-Moody and the Virasoro group, obtaining WZW models former case and chiral boson action in the later case. New terms appear in the geometrical actions, which label the coadjoint orbit over which the action is defined. Finally, in order to analyze the flat case, the procedure is generalized to the case of semi-direct product groups and their central extensions.

10.1 Central Extensions

Let us consider a Lie group G and its central extension $\widehat{G} = G \times \mathbb{R}$, whose elements will be denoted as (g,m). The group operation in $\widehat{G}$ is given by

$$(g,m)\left(g',n\right) = \left(gg', m+n+\xi\left(g,g'\right)\right) ,$$

where $\xi : G \times G \to \mathbb{R}$ is a $2-$cocycle satisfying the group cocycle identity

$$\xi\left(g'',g'\right) + \xi\left(g''g',g\right) = \xi\left(g'',g'g\right) + \xi\left(g',g\right) ,$$

together with the normalization $\xi\left(g,e\right) = \xi\left(e,g\right) = 0$ and $\xi\left(g,g^{-1}\right) = \xi\left(g^{-1},g\right)$, where e is the identity in G. The identity in $\widehat{G}$ is given by $(e,0)$, while of the inverse of (g,m) has the form

$$(g,m)^{-1} = \left(g^{-1}, -m-\xi\left(g,g^{-1}\right)\right) .$$

The adjoint action of $\widehat{G}$ on its Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$, whose elements are denoted by (X,m), (Y,n), *etc*, is then obtained from definition (9.2), and yields

$$Ad_{(g,m)}\left(Y,n\right) = \left(Ad_g Y, n + \frac{d}{dt}\left[\xi\left(g,g_t g^{-1}\right) + \xi\left(g_t,g^{-1}\right)\right]\Big|_{t=0}\right) , \tag{10.1}$$

where $\frac{d}{dt}g_t\big|_{t=0} = Y$ and the term $\frac{d}{dt}\left[\xi\left(g, g_t' g^{-1}\right) + \xi\left(g_t', g^{-1}\right)\right]\big|_{t=0}$ is a linear function of Y on $\mathbb{R}$ and therefore can be written as

$$\frac{d}{dt}\left[\xi\left(g, g_t g^{-1}\right) + \xi\left(g_t, g^{-1}\right)\right]\bigg|_{t=0} = \lambda\left\langle S\left(g^{-1}\right), Y\right\rangle_0 \,, \tag{10.2}$$

where $S : G \to \mathfrak{g}^*$ is a $1-$cocycle on G, λ is a constant and $\langle,\rangle_0 : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the pairing between $\mathfrak{g}$ and its dual. Then, the adjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}$ takes the form

$$Ad_{(g,m)}(Y,n) = \left(Ad_g Y, n + \lambda\left\langle S\left(g^{-1}\right), Y\right\rangle_0\right) \,. \tag{10.3}$$

The function $S(g)$ is restricted by the requirement that $Ad_{(g,m)}$ must be a representation, which means that the composition law of the group must be respected, $Ad_{(g,m)(g',m')}(Y,n) = Ad_{(g,m)}Ad_{(g',m')}(Y,n)$. Working out both sides of this expressions yields to the condition

$$S\left(gg'\right) = S(g) + Ad_g^* S\left(g'\right) \,. \tag{10.4}$$

On the other hand, the adjoint representation of the identity must satisfy $Ad_{(e,0)}(X,m) = (X,m)$ which means that $S(e) = 0$. Finally, the existence of inverse requires $Ad_{(g,m)}Ad_{(g,m)^{-1}}(Y,n) = (Y,n)$,which leads to

$$S(g) = -Ad_g^* S\left(g^{-1}\right) \,. \tag{10.5}$$

Denoting the elements of the dual $\widehat{\mathfrak{g}}^*$ as (U,c), we can define the pairing between the central extended algebra $\widehat{\mathfrak{g}}$ and its dual $\langle,\rangle : \widehat{\mathfrak{g}}^* \times \widehat{\mathfrak{g}} \to \mathbb{R}$ as

$$\langle (U,c), (X,m)\rangle = \langle U, X\rangle_0 + cm \,. \tag{10.6}$$

The coadjoint action of $\widehat{G}$ on $\widehat{\mathfrak{g}}^*$ is then defined by (9.9) which in the central extended case has the form

$$Ad^*_{(g,m)}(U,c) = \left(Ad_g^* U + c\lambda S(g), c\right) \,, \tag{10.7}$$

where $Ad_g^* U$ is the coadjoint representation of G.

The commutation relations for $\widehat{\mathfrak{g}}$ are given by the infinitesimal form of the adjoint action (10.3). The explicit form for the bracket can be obtained by differentiating (10.3) with respect to some parameter t which parametrizes $g = g(t)$, and evaluating the result at $t = 0$. The infinitesimal version of $Ad_g Y$ is given by (9.6), while the infinitesimal form of S can be written as

$$\begin{aligned} &s : \mathfrak{g} \to \mathfrak{g}^* \\ &s(X) = \frac{dS(g(t))}{dt}\bigg|_{t=0} \,. \end{aligned}$$

From (10.2) we see that $s(X)$ is linear in X and therefore

$$\left.\frac{dS\left(g^{-1}\right)}{dt}\right|_{t=0} = -s(X) \ .$$

The infinitesimal form of (10.3) is then be given by

$$ad_{(X,m)}(Y,n) \equiv [(X,m),(Y,n)] = \left([X,Y], -\lambda \langle s(X), Y\rangle_0\right) \ . \tag{10.8}$$

The coadjoint representation of the algebra $\widehat{\mathfrak{g}}$ is defined by (9.9), which in this case leads to

$$ad^*_{(X,m)}(U,c) = \left(ad^*_X U + c\lambda s(X), 0\right) \ , \tag{10.9}$$

and corresponds to the infinitesimal version of (10.7), as expected.

10.2 Geometrical Actions for Centrally Extended Groups

The procedure to construct geometrical actions on coadjoint orbits of centrally extended groups can be straightforwardly generalized from the discussion exposed in the previous sections. Given a representative $\left(U^{(0)}, c\right)$ of some orbit $O_{(U,c)}$ of a centrally extended group $\hat{G}$, any point of $O_{(U,c)}$ can be reached by the coadjoint action on $\left(U^{(0)}, c\right)$ with some group element (g,m),

$$(U,c) = Ad^*_{(g,m)}\left(U^{(0)}, c\right) = \left(Ad^*_g U^{(0)} + c\lambda S(g), c\right) \ . \tag{10.10}$$

Then, the Kirillov-Kostant symplectic structure on $O_{(U,c)}$ can be generalized form (9.14) as

$$\Omega = \left\langle (U,c), (\kappa, m_\kappa)^2 \right\rangle \ , \tag{10.11}$$

where the pairing $\langle\ ,\rangle$ has been defined in (10.6) and (κ, m_κ) is the right invariant Maurer-Cartan forms on $\hat{G}$, which satisfies

$$d(\kappa, m_\kappa) - \frac{1}{2}[(\kappa, m_\kappa), (\kappa, m_\kappa)] = 0 \ , \tag{10.12}$$

Using (10.12), the symplectic form can be written as

$$\Omega = -da \ \ , \ \ a = -\langle (U,c), (\kappa, m_\kappa)\rangle \ ,$$

which leads to the associated geometrical action analogous to (9.15)

$$I = \int a = -\int \langle (U,c), (\kappa, m_\kappa)\rangle \ . \tag{10.13}$$

Finally using (10.6), we find

$$I = -\int \left\langle Ad^*_g U^{(0)}, \kappa \right\rangle_0 + c\left(m_\kappa + \lambda \left\langle S(g), \kappa \right\rangle_0\right) . \tag{10.14}$$

10.2.1 Kac-Moody Group

The Loop LG group associated to a semi-simple Lie group G corresponds to the group of continuous maps from the unit circle to G

$$\begin{aligned} g: \quad & S^1 \longrightarrow G \\ & \phi \longmapsto g(\phi) \ , \end{aligned}$$

where $\phi \in [0, 2\pi)$, with the multiplication law $g(\phi)\, g'(\phi) = gg'(\phi)$. In the same way, the loop algebra $L\mathfrak{g}$ is given by the continuous maps from S^1 to $\mathfrak{g}$, the Lie algebra associated to G, and its elements will be denoted by $x(\phi)$. The elements of the dual space $L\mathfrak{g}^*$ will be denoted by $u(\phi)$. Here x and u are continuous functions on S^1 taking values in $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively.

The pairing $\langle\ ,\ \rangle_0$ between $L\mathfrak{g}$ and $L\mathfrak{g}^*$ is given by

$$\langle u, x\rangle_0 = \mathrm{Tr} \int_0^{2\pi} d\phi\, xu \ , \tag{10.15}$$

where the integration is taken over S^1.

The Kac-Moody group $\widehat{LG}$ is given by the central extension of the Loop group of G. The elements of the Kac-Moody algebra $\widehat{L\mathfrak{g}}$ will be denoted by pairs $(x(\phi), m), (y(\phi), n), etc$, while the dual space elements will be denoted by $(u(\phi), c)$. The paring between $\widehat{L\mathfrak{g}}$ and $\widehat{L\mathfrak{g}^*}$ is the given by (10.6), which in this case takes the form

$$\langle (x(\phi), m), (u(\phi), c)\rangle_0 = \mathrm{Tr} \int_0^{2\pi} d\phi\, xu + cm \ . \tag{10.16}$$

The central extension of the loop group is given by means of the 1$-$cocycle on the loop group

$$S(g) = \partial_\phi g g^{-1} \ .$$

Together with the normalization $\lambda = -1/2\pi$, S specifies the adjoint and coadjoint representations for the Kac-Moody group, which can be read off from (10.3) and (10.7) respectively

$$Ad_{(g,m)}(y, n) = \left(gyg^{-1}, n + \frac{1}{2\pi} \mathrm{Tr} \int_0^{2\pi} d\phi\, g^{-1} \partial_\phi g y \right) , \tag{10.17}$$

$$Ad^*_{(g,m)}(u, c) = \left(gug^{-1} - \frac{c}{2\pi} \partial_\phi g g^{-1}, c \right) , \tag{10.18}$$

where we have considered a matrix representation for LG so that $Ad_g y = g\, y\, g^{-1}$ and $Ad_g^* u = gug^{-1}$. The infinitesimal form of $S(g)$ in this case is given by

$$s(y) = \partial_\phi y \,, \tag{10.19}$$

while the infinitesimal adjoint and coadjoint action of $L\mathfrak{g}$ are given by $ad_x y = [x,y]$ and $ad_x^* u = [x,u]$ respectively. Therefore, the infinitesimal adjoint and coadjoint actions of the Kac-Moody algebra $\widehat{L\mathfrak{g}}$ are obtained from (10.8) and (10.9) by replacing (10.19)

$$ad_{(x,m)}(y,n) = [(x,m),(y,n)] = \left([x,y], \frac{1}{2\pi}\mathrm{Tr}\int_0^{2\pi} d\phi\, \partial_\phi x y\right) \,, \tag{10.20}$$

$$ad^*_{(x,m)}(u,c) = \left([x,u] - \frac{c}{2\pi}\partial_\phi x \,, 0\right) \,. \tag{10.21}$$

In order to compute the geometrical action for the Kac-Moody group, we need to find the right invariant Maurer-Cartan form which satisfy the equation (10.12)

$$d(\kappa, m_\kappa) - \frac{1}{2}[(\kappa,m_\kappa),(\kappa,m_\kappa)] = 0 \,,$$

whose solution has been computed in Appendix C.1 and reads

$$(\kappa, m_\kappa) = \left(dgg^{-1}, \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\left(dgg^{-1}\right)^2 \partial_\phi gg^{-1}\right)\right]\right) \,.$$

Let's consider a coadjoint orbit with representative (u_0, c) so that all the elements of the orbit can be reached from the representative by the coadjoint action of the Kac-Moody group, i.e.,

$$(u,c) = Ad_g^*(u_0,c) = \left(gu_0 g^{-1} - \frac{c}{2\pi}\partial_\phi gg^{-1}, c\right) \,. \tag{10.22}$$

With all these relations, the symplectic form for an orbit $O_{(u,c)}$ can be obtained from (10.11) and after some algebra gives

$$\begin{aligned}\Omega_{(u,c)} &= \left\langle (u,c), \frac{1}{2}[(\kappa,m_\kappa),(\kappa,m_\kappa)]\right\rangle \\ &= \mathrm{Tr}\int_0^{2\pi} d\phi \left[u_0\left(g^{-1}dg\right)^2 - \frac{c}{4\pi}\partial_\phi gg^{-1}\left(dgg^{-1}\right)^2 + \frac{c}{4\pi} d\left(\partial_\phi g\right) g^{-1}dgg^{-1}\right] \,,\end{aligned}$$

while the corresponding geometrical action can be directly read off from (10.13) and gives

$$I = -\int \langle (u,c),(\kappa,m_\kappa)\rangle \tag{10.23}$$

$$= -\int\int_0^{2\pi} d\phi \mathrm{Tr}\left(gu_0 g^{-1} dgg^{-1}\right) \tag{10.24}$$

$$-c\left(\frac{1}{4\pi}\int_0^{2\pi} d\phi \mathrm{Tr}\left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\left(dgg^{-1}\right)^2\partial_\phi gg^{-1}\right)\right] - \frac{1}{2\pi}\left\langle \partial_\phi gg^{-1}, dgg^{-1}\right\rangle_0\right) \tag{10.25}$$

After some algebra, the action takes the form

$$I = -\int\int_0^{2\pi} d\phi \mathrm{Tr}\left(u_0 g^{-1} dg\right) - \frac{c}{4\pi}\int\int_0^{2\pi} d\phi \mathrm{Tr}\left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\partial_\phi gg^{-1}dgg^{-1}dgg^{-1}\right)\right] , \tag{10.26}$$

which corresponds to the kinetic term of the chiral WZW model [10] plus an extra term of the form $u_0 g^{-1}dg$, which represents the coupling to point sources and the inclusion of holonomies corresponding angular defects or event horizons in the case of black holes [87].

10.2.2 Virasoro Group

Let us denote by $Diff\left(S^1\right)$ the group of all orientation-preserving diffeomorphisms of the circle and let $Vec\left(S^1\right)$ be its Lie algebra, whose elements will be denoted by $X = X(\phi)\partial_\phi$. The elements of the dual space $Vec\left(S^1\right)^*$ correspond to quadratic differentials on S^1 and will be denoted by $p = p(\phi)(d\phi)^2$. The paring is then given by

$$\langle p, X\rangle_0 = \int_0^{2\pi} d\phi X(\phi) p(\phi) \ .$$

The adjoint action of $Diff\left(S^1\right)$ corresponds a reparametrization of the form $\phi \longmapsto F(\phi)$, $F \in Diff\ \ S^1$. Acting on X it has the form

$$X{=}X(\phi)\partial_\phi = X(F(\phi))\frac{\partial}{\partial F(\phi)} = X(F(\phi))\frac{1}{F'(\phi)}\partial_\phi \ .$$

Hence, we obtain

$$Ad_F X = \frac{1}{F'(\phi)} X(F(\phi))\partial_\phi \ . \tag{10.27}$$

In the same way for a element of the dual space, the coadjoint action can then be found using (9.5) which in this case yields

$$Ad_F^* p = F^{'}(\phi)^2 p(F(\phi))(d\phi)^2 \ ,$$

which corresponds to a diffeomorphism acting on the quadratic differential. In fact

$$p = p(\phi)(d\phi)^2 = p(F(\phi))(dF(\phi))^2 \tag{10.28}$$

$$= p(F(\phi))F^{'}(\phi)^2(d\phi)^2 \ . \tag{10.29}$$

The infinitesimal adjoint action of $Vec\left(S^1\right)$ is given by the commutator of two vector fields on the circle, i.e.

$$ad_X Y = [X,Y] = \left[X(\phi)\partial_\phi, Y(\phi)\partial_\phi\right] = \left(X_1(\phi)X_2'(\phi) - X_2(\phi)X_1'(\phi)\right)\partial_\phi \ . \tag{10.30}$$

The infinitesimal coadjoint action can be obtained using (9.9) and in this case leads to

$$ad^*_X p = \left(p'(\phi)X(\phi) + 2p(\phi)X'(\phi)\right) d\phi^2 \,. \tag{10.31}$$

The Virasoro group corresponds to a central extension of $Diff\left(S^1\right)$ and will be denoted by $\widehat{Diff}\left(S^1\right)$. The elements of the Virasoro algebra $\widehat{Vec}\left(S^1\right)$ will be denoted by $(X,-ia)$, with X a vector field on S^1, while the elements of the dual will be denoted by (p,ic), where p is an element in the dual of $Vec\left(S^1\right)$. The pairing between $\widehat{Vec}\left(S^1\right)$ and its dual is then given by

$$\langle (X,-ia),(p,ic)\rangle = \int_0^{2\pi} d\phi\, p(\phi) X(\phi) + ca \,. \tag{10.32}$$

The $1-$cocyle for the conformal group is given by the Schwarzian derivative

$$S(F) = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 , \tag{10.33}$$

which, together with the normalization $\lambda = -1/24\pi$, determines the adjoint and coadjoint action of $\widehat{Diff}\left(S^1\right)$ according to (10.3) and (10.7) giving

$$Ad_F(X(\phi)\partial_\phi, -ia) = \left(\frac{1}{F'}X(F(\phi))\partial_\phi, -i\left(a - \frac{1}{24\pi}\int_0^{2\pi} d\phi X(\phi) S(1/F)\right)\right), \tag{10.34}$$

$$Ad^*_F(p(\phi)d\phi^2, ic) = \left(\left(F'(\phi)^2 p(F(\phi)) - \frac{c}{24\pi}S(F)\right)d\phi^2, ic\right) . \tag{10.35}$$

In order to determine the infinitesimal adjoint and coadjoint action of the Virasoro algebra we need the infinitesimal version of (10.33). Considering an infinitesimal diffeomorphism $F(\phi) = \phi + X(\phi)$ we find

$$S(1+X) = \frac{X'''}{1+X'} + \frac{3}{2}\left(\frac{X''}{1+X'}\right)^2 \approx X''' , \tag{10.36}$$

and therefore

$$s(X) = X(\phi)''' .$$

The infinitesimal adjoint and coadjoint action of can then be read off from (10.8) and (10.9)

$$ad_X(Y(\phi), -ia) = \left(\left(X(\phi)Y'(\phi) - Y(\phi)X'(\phi)\right)\partial_\phi, -\frac{i}{24\pi}\int_0^{2\pi} d\phi X'''(\phi) Y(\phi)\right), \tag{10.37}$$

$$ad^*_X(p(\phi), ic) = \left(\left(p'(\phi)X(\phi) + 2p(\phi)X'(\phi)\right)d\phi^2 - \frac{c}{24\pi}X(\phi)''', 0\right) . \tag{10.38}$$

Finally, in order to construct the associated geometrical action, we need to solve and find the right invariant Maurer-Cartan form, which solves (10.12)

$$d\left(\kappa,-im_\kappa\right)=\left(\kappa^2,-\frac{i}{48\pi}\int_0^{2\pi}d\phi\,\kappa'''(\phi)\kappa(\phi)\right)\;.$$

The solution is given by (see Appendix C.2)

$$\left(\kappa,-im_\kappa\right)=\left(\frac{dF}{F'}\partial_\phi,-\frac{i}{48\pi}\int_0^{2\pi}d\phi\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'}\right]\right)\;.$$

Given a coadjoint orbit $O_{(p,ic)}$ with representative $\left(p_0 d\phi^2, ic\right)$, its elements can be put in the form

$$\left(p\left(\phi\right)d\phi^2,ic\right)=\left(\left(F'\left(\phi\right)^2 p_0\left(F\left(\phi\right)\right)-\frac{c}{24\pi}S(F)\right)d\phi^2,ic\right)\;,$$

which corresponds to the coadjoint action (10.35) on the representative. The geometrical action coming from the Kirillov-Kostant symplectic form is then given by

$$\begin{aligned} I &= -\int\left\langle\left(p\left(\phi\right)d\phi^2,ic\right),\left(\frac{dF}{F'}\partial_\phi,-\frac{i}{48\pi}\int_0^{2\pi}d\phi\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'}\right]\right)\right\rangle \\ &= \int d\phi\left[-p_0\left(F\right)F'dF+\frac{c}{48\pi}\frac{dF}{F'}\left(\frac{F'''}{F'}-2\left(\frac{F''}{F'}\right)^2\right)\right]\;. \end{aligned} \tag{10.39}$$

Furthermore, defining $\varphi=\log F'$ we find

$$d\varphi\varphi'=\left(\frac{dFF''}{F'^2}\right)'-\frac{dF}{F'}\left(\frac{F'''}{F'}+\left(\frac{F''}{F'}\right)^2\right)\;, \tag{10.40}$$

and using the notation $dF=\dot{F}dt$ we can write the action, up to boundary terms, as

$$I=\int d\phi dt\left[-p_0(F)F'\dot{F}-\frac{c}{48\pi}\dot{\varphi}\varphi'\right]\;, \tag{10.41}$$

which, apart from the representative of the orbit, corresponds to the kinetic term of the chiral boson action [12].

10.3 Semi-direct Products

Let $\mathscr{A}$ be an abelian vector space, whose elements are denoted by α,α',etc, and let G be a Lie group with elements g,g',etc. Then, the action of G on $\mathscr{A}$ will be denoted by $\sigma_g\alpha$, with σ a representation of G on $\mathscr{A}$. The semi-direct product $\mathscr{S}=G\ltimes_\sigma\mathscr{A}$ is a group with elements of the form (g,α), whose group operation is given by

$$\left(g',\alpha'\right)\left(g,\alpha\right)=\left(g'g,\alpha'+\sigma_{g'}\alpha\right)\;. \tag{10.42}$$

Consider now the corresponding Lie algebra $\mathfrak{s} = \mathfrak{g} \oplus \mathscr{A}$, with elements (X, α), (Y, β), *etc*, and its dual $\mathfrak{g}^*$ with elements (j, p), (j', p'), *etc*. The adjoint action of G on $\mathfrak{g}$ is defined as

$$Ad_{(g,\alpha)}(Y, \beta) = \frac{d}{dt}(g, \alpha)\left(g'_t, \alpha'_t\right)(g, \alpha)^{-1}\bigg|_{t=0} ,$$

where $\frac{d}{dt}(g'_t, \alpha'_t)\big|_{t=0} = (Y, \beta)$ and leads to

$$Ad_{(g,\alpha)}(Y, \beta) = \left(Ad_g Y, \sigma_g \beta - \Sigma_{Ad_g X} \alpha\right) . \tag{10.43}$$

Here Ad_h stands for the adjoint representation of $\mathscr{S}$ and Σ is the representation of $\mathfrak{h}$ on $\mathscr{A}$ corresponding to the infinitesimal form of σ. The bilinear form for $\mathfrak{s}$ can then be defined as

$$\langle (j, p), (X, \alpha) \rangle_0 = \langle j, X \rangle_{\mathfrak{g}} + \langle p, \alpha \rangle_{\mathscr{A}} , \tag{10.44}$$

where $\langle , \rangle_{\mathfrak{g}}$ and $\langle , \rangle_{\mathscr{A}}$ are the natural pairings in $\mathfrak{g}$ and $\mathscr{A}$ respectively. The coadjoint action of $\mathscr{S}$ on $\mathfrak{s}^*$ is then defined as

$$\left\langle Ad^*_{(g,\alpha)}(j, p), (X, \beta) \right\rangle_0 = \left\langle (j, p), Ad_{(g,\alpha)^{-1}}(X, \beta) \right\rangle_0 ,$$

which leads to

$$Ad^*_{(g,\alpha)}(j, p) = \left(Ad^*_h j + \sigma^*_g p \odot \alpha, \sigma^*_g p\right) , \tag{10.45}$$

where $\mathscr{P} \odot \alpha$ is defined as

$$\langle p \odot \alpha, X \rangle_{\mathfrak{g}} = \langle p, \Sigma_X \alpha \rangle_{\mathscr{A}} = -\langle \Sigma^*_X p, \alpha \rangle_{\mathscr{A}} , \tag{10.46}$$

and σ^*, Σ^* are the dual maps of σ, Σ respectively (with respect to the pairing $\langle , \rangle_{\mathscr{A}}$). The commutation relations for $\mathfrak{s}$ are determined by the infinitesimal form of (10.43), i.e.,

$$[(X, \alpha), (Y, \beta)] = ad_{(X,\alpha)}(Y, \beta) = ([X, Y], \Sigma_X \beta - \Sigma_Y \alpha) . \tag{10.47}$$

10.3.1 Adjoint Representation

Consider now the semi-direct product in which σ corresponds to the adjoint representation and $\mathscr{A}$ is given by the Lie algebra of G seen as the abelian additive group of a vector space, which will be denoted by $\mathfrak{g}_{aj}$. The adjoint and action of $\mathscr{S}$ takes the form

$$Ad_{(g,\alpha)}(Y, \beta) = \left(Ad_g Y, Ad_g \beta - ad_{Ad_g Y} \alpha\right) , \tag{10.48}$$

while for the coadjoint action we get

$$Ad^*_{(g,\alpha)}(j,p) = \left(Ad^*_g j + Ad^*_g \left[ad^*_{Ad_{g^{-1}}\alpha} p\right], Ad^*_g p\right) \ , \tag{10.49}$$

which, using the identity $Ad^*_g \circ ad^*_{Ad_{g^{-1}}\alpha} \circ Ad^*_{g^{-1}} = ad^*_\alpha$ can equivalently written as

$$Ad^*_{(g,\alpha)}(j,p) = \left(Ad^*_g j + ad^*_\alpha Ad^*_g p, Ad^*_g p\right) \ . \tag{10.50}$$

The infinitesimal versions of (10.48) and (10.50) lead to the adjoint and coadjoint representations of the corresponding algebra $\mathfrak{s} = \mathfrak{g} \rtimes \mathfrak{g}_{ab}$

$$\begin{aligned} ad_{(X,\alpha)}(Y,\beta) &= [(X,\alpha),(Y,\beta)] = ([X,Y],[X,\beta]-[Y,\alpha]) \ , && (10.51) \\ ad^*_{(X,\alpha)}(j,p) &= (ad^*_X j + ad^*_\alpha p, ad^*_X p) \ . && (10.52) \end{aligned}$$

10.4 Centrally Extended Semi-direct Products

Let us consider a now the central extended group $\widehat{\mathscr{S}}$, i.e. a central extension of a semi-direct product group, whose elements will be denoted by (g,m_1,α,m_2). The elements of the algebra $\widehat{\mathfrak{s}}$ will be denoted by $(X,m_1,\alpha,m_2),(Y,n_1,\beta,n_2)$, etc, while the elements of the dual as (j,c_1,p,c_2). The adjoint and the coadjoint action of $\widehat{\mathscr{S}}$ can be constructed by starting with (10.48) and (10.50), which in this case take form

$$Ad_{(g,m_1;\alpha,m_2)}(Y,n_1;\beta,n_2) = \left(Ad_{(g,m_1)}(Y,n_1); Ad_{(g,m_1)}(\beta,n_2) - ad_{Ad_{(g,m_1)}(Y,n_1)}(\alpha,m_2)\right) \ , \tag{10.53}$$

$$Ad^*_{(g,m_1;\alpha,m_2)}(j,c_1;p,c_2) = \left(Ad^*_{(g,m_1)}(j,c_1) + ad^*_{(\alpha,m_2)} Ad^*_{(g,m_1)}(p,c_2); Ad^*_{(g,m_1)}(p,c_2)\right) \ , \tag{10.54}$$

and then replacing (10.3), (10.7), (10.8) and (10.9), leading to

$$\begin{aligned} & Ad_{(g,m_1;\alpha,m_2)}(Y,n_1;\beta,n_2) \\ = \ & \left(Ad_g Y, n_1 + \lambda \left\langle S\left(g^{-1}\right), Y\right\rangle_0; Ad_g\beta - ad_{Ad_g Y}\alpha, n_2 + \lambda\left[\left\langle S\left(g^{-1}\right),\beta\right\rangle_0 + \left\langle s\left(Ad_g Y\right),\alpha\right\rangle_0\right]\right) && (10.55) \\ & Ad^*_{(g,m_1;\alpha,m_2)}(j,c_1;p,c_2) \\ = \ & \left(Ad^*_g\left[j + ad^*_\alpha p + c_2\lambda s(\alpha)\right] + c_1\lambda S(g), c_1; Ad^*_g p + c_2\lambda S(g), c_2\right) \ . && (10.56) \end{aligned}$$

For later purposes, the definition (10.50) will be also useful, which in this case has the form

$$Ad^*_{(g,m_1;Ad_g\alpha,m_2)}(j,c_1;p,c_2) = \left(Ad^*_{(g,m_1)}(j,c_1) + Ad^*_{(g,m_1)}\left[ad^*_{(\alpha,m_2)}(p,c_2)\right]; Ad^*_{(g,m_1)}(p,c_2)\right) \ . \tag{10.57}$$

The infinitesimal adjoint and coadjoint action can similarly be read off from (10.51) and (10.52), which in this case have the form

$$
\begin{aligned}
ad_{(X,m_1;\alpha,m_2)}(Y,n_1;\beta,n_2) &= [(X,m_1;\alpha,m_2),(Y,n_1;\beta,n_2)] \\
&= ([(X,m_1),(Y,n_1)];[(X,m_1),(\beta,n_2)]-[(Y,n_1),(\alpha,m_2)]) \ ,
\end{aligned} \tag{10.58}
$$

$$
ad^*_{(X,m_1;\alpha,m_2)}(j,c_1;p,c_2) = \left(ad^*_{(X,m_1)}(j,c_1)+ad^*_{(\alpha,m_2)}(p,c_2),ad^*_{(X,m_1)}(p,c_2)\right) \ . \tag{10.59}
$$

Replacing (10.8) and (10.9) we find

$$
ad_{(X,m_1;\alpha,m_2)}(Y,n_1;\beta,n_2) = ([X,Y],-\lambda\langle s(X),Y\rangle_0;[X,\beta]-[Y,\alpha],-[\lambda\langle s(X),\beta\rangle_0-\langle s(Y),\alpha\rangle_0]) \ , \tag{10.60}
$$

$$
ad^*_{(X,m_1;\alpha,m_2)}(j,c_1;p,c_2) = (ad^*_X j+ad^*_\alpha p+c_1\lambda s(X)+c_2\lambda s(\alpha),0;ad^*_X p+c_2\lambda s(X),0) \ . \tag{10.61}
$$

In order to construct the geometric action, the right invariant Maurer-Cartan form of $\widehat{\mathscr{S}}$ must be determined, which will be denoted by $(\kappa,m_\kappa,\alpha_\kappa,n_k)$ and satisfies the Maurer-Cartan equation (B.8); which in this case has the form

$$
d(\kappa,m_\kappa;\alpha_\kappa,n_k)-\frac{1}{2}[(\kappa,m_\kappa;\alpha_\kappa,n_k)(\kappa,m_\kappa;\alpha_\kappa,n_\kappa)]=0 \ .
$$

Given an orbit $O_{(j,c_1;p,c_2)}$ its elements are constructed by letting the whole group act on a representative (j_0,c_1,p_0,c_2) by the coadjoint action, which for convenience will be taken in the form (10.57) rather than (10.56). This means

$$
\begin{aligned}
(j,c_1;p,c_2) &= Ad_{(g,m_1,Ad_g\alpha,m_2)}(j_0,c_1,p_0,c_2) \\
&= \left(Ad^*_g[j_0+ad^*_\alpha p_0+c_2\lambda s(\alpha)]+c_1\lambda S(g),c_1;Ad^*_g p_0+c_2\lambda S(g),c_2\right) \ .
\end{aligned}
$$

Then, the geometrical action (9.15) takes the form

$$
\begin{aligned}
I &= \int\langle(j,c_1;p,c_2)(\kappa,m_\kappa;\alpha_\kappa,n_\kappa)\rangle \\
&= \int\left\langle\left(Ad^*_g[j_0+ad^*_\alpha p_0+c_2\lambda s(\alpha)]+c_1\lambda S(g)\right)\kappa\right\rangle \\
&\quad+\int\left\langle\left(Ad^*_g p_0+c_2\lambda S(g)\right)\alpha_\kappa\right\rangle+c_1 m_\kappa+c_2 n_\kappa \ .
\end{aligned}
$$

10.4.1 Kac-Moody Algebra of $G \ltimes \mathfrak{g}$

Let us consider now the group $\widehat{LG}\ltimes\widehat{L\mathfrak{g}}_{ab}$, where G a compact semi-simple Lie group, which corresponds to a central extension of the loop algebra of a semi-direct product. The adjoint and the coadjoint action for such a group can be constructed using (10.55) and (10.56) and replacing the results found for the Kac-Moody group (10.17) and (10.18). In order to write the geometrical action for a coadjoint orbit, the expression (10.57) will be useful later, which in this case takes form

$$Ad^{*}_{(g;g\alpha g^{-1})}(j,c_1;p,c_2) = \left(g\left[j+[\alpha,p]-\frac{c_2}{2\pi}\alpha'\right]g^{-1}-\frac{c_1}{2\pi}g'g^{-1},c_1;gpg^{-1}-\frac{c_2}{2\pi}g'g^{-1},c_2\right) . \tag{10.62}$$

Let us turn now our attention to the Maurer-Cartan one-form $(\kappa,m_\kappa;\beta_\kappa,n_\kappa)$ taking values in the Kac-Moody algebra. Using (10.60) the Maurer-Cartan equation (B.8) takes the form

$$\begin{aligned} d(\kappa,m_\kappa;\beta_\kappa,n_\kappa) &= \frac{1}{2}\left[(\kappa,m_\kappa;\beta_\kappa,n_\kappa),(\kappa,m_\kappa;\beta_\kappa,n_\kappa)\right] \\ &= \left(\frac{1}{2}[\kappa,\kappa],\frac{1}{4\pi}\int d\theta \mathrm{Tr}\left(\kappa'\kappa\right);[\kappa,\beta_\kappa],\frac{1}{2\pi}\int d\theta\mathrm{Tr}\left(\kappa'\beta_\kappa\right)\right) , \end{aligned} \tag{10.63}$$

whose solution to (10.63) is given by (see Appendix C.3)

$$(\kappa,m_\kappa;\beta_\kappa,n_\kappa) = \left(dgg^{-1},\frac{1}{4\pi}\int d\theta\mathrm{Tr}\begin{pmatrix} g'g^{-1}dgg^{-1} \\ +d^{-1}\left[g'g^{-1}dgg^{-1}dgg^{-1}\right]\end{pmatrix};gd\alpha g^{-1},\frac{1}{2\pi}\int d\theta\mathrm{Tr}\left(g'd\alpha g^{-1}\right)\right) .$$

Now, using (10.62) to parametrize the coadjoint orbits for the loop algebra

$$(j,c_1;p,c_2) = Ad^{*}_{(g;g\alpha g^{-1})}(j_0,c_1;p_0,c_2) ,$$

the action for an orbit with representative $(j_0,c_1;p_0,c_2)$ can be constructed using (9.15)

$$I = -\int_{(j_0;p_0)}\left\langle Ad^{*}_{(g;g\alpha g^{-1})}(j_0,c_1;p_0,c_2),(\kappa,m_\kappa;\beta_\kappa,n_\kappa)\right\rangle ,$$

which after some algebra leads to the action

$$\begin{aligned} I &= -\int d\theta\mathrm{Tr}\left(j_0 g^{-1}dg+p_0\left(d\alpha+\left[g^{-1}dg,\alpha\right]\right)\right) \\ &\quad +\int d\theta\mathrm{Tr}\left(\frac{c_2}{2\pi}\alpha' g^{-1}dg+\frac{c_1}{4\pi}\left(g'g^{-1}dgg^{-1}-d^{-1}\left[g'g^{-1}dgg^{-1}dgg^{-1}\right]\right)\right) . \end{aligned} \tag{10.64}$$

The second integral corresponds to the kinetic term of the flat WZW model [12].

10.4.2 BMS$_3$

Let us consider now the $\widehat{\mathrm{BMS}}_3$ group, which corresponds to the semi-direct product of the Virasoro group and its algebra (as an abelian additive group) under the adjoint action

$$\widehat{\mathrm{BMS}}_3 = \widehat{Diff}\left(S^1\right)\ltimes Vec\left(S^1\right)_{ab} ,$$

the elements of the $\widehat{\mathfrak{bms}_3}$ algebra will be denoted by $(X,-im_1,\alpha,-im_2)$, $(Y,-in_1,\beta,-in_2)$, *etc*. The adjoint and coadjoint action of the Virasoro group are given by (10.55) and (10.56). Replacing that the relations obtained for the Virasoro group (10.34) and (10.35), we obtain

$$
\begin{aligned}
& Ad_{(F,\alpha)}(Y,-in_1,\beta,-in_2) \\
&= \left(\begin{array}{c} \frac{Y(F)}{F'},-i\left(n_1-\frac{1}{24\pi}\int d\theta S\left(F^{-1}\right)Y\right) \\ ;\frac{\beta(F)-\alpha'Y(F)}{F'}+2\alpha\left(\frac{Y(F)}{F'}\right)',-i\left(n_2-\frac{1}{24\pi}\int d\theta\left[S\left(F^{-1}\right)\beta+\left(\frac{Y(F)}{F'}\right)'''\alpha\right]\right)\end{array}\right) \quad (10.65) \\
& Ad^*_{(F,(F')^{-1}\alpha(F))}(j,ic_1;p,ic_2) \quad (10.66) \\
&= \left(\left(F'\right)^2\left[j+p'\alpha+2p\alpha'-\frac{c_2}{24\pi}\alpha'''\right]\circ F-\frac{c_1}{24\pi}S(F),ic_1;\left(F'\right)^2p(F)-\frac{c_2}{24\pi}S(F),ic_2\right).
\end{aligned}
$$

Similarly, their infinitesimal versions can be obtained from (10.60) and (10.61) and read

$$
\begin{aligned}
& ad_{(X,\alpha)}(Y,-in_1,\beta,-in_2)=[(X,-im_1;\alpha,-im_2),(Y,-in_1;\beta,-in_2)] \\
&=\left([X,Y],-\frac{i}{24\pi}\int d\theta X'''Y;X\beta'-\beta X'-Y\alpha'+\alpha Y',-\frac{i}{24\pi}\int d\theta\left[X'''\beta-Y'''\alpha\right]\right) \quad (10.67) \\
& ad^*_{(X,\alpha)}(j,ic_1;p,ic_2) \\
&=\left(j'X+2jX'-\frac{c_1}{24\pi}X'''+p'\alpha+2p\alpha'-\frac{c_1}{24\pi}\alpha''',0;p'X+2pX'-\frac{c_2}{24\pi}X''',0\right). \quad (10.68)
\end{aligned}
$$

The Maurer-Cartan form $(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa)$ in this case satisfies the equation (B.8), which using (10.60) takes the form

$$
\begin{aligned}
d(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa) &= \frac{1}{2}\left[(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa),(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa)\right] \\
&= \left(\frac{1}{2}\left[(\kappa,-im_\kappa),(\kappa,-im_\kappa)\right];\left[(\kappa,-im_\kappa),(\beta_\kappa,-in_\kappa)\right]\right) \\
&= \left(\kappa\kappa',-\frac{i}{48\pi}\int d\theta\kappa'''\kappa;\kappa\beta_\kappa'+\beta_\kappa\kappa',-\frac{i}{24\pi}\int d\theta\kappa'''\beta_\kappa\right).
\end{aligned}
$$

The Maurer-Cartan form has been computed in Appendix C.4 is then given by

$$
(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa)=\left(\begin{array}{c}\frac{dF}{F'},-\frac{i}{48\pi}\int d\theta\left(\frac{F'''}{F'}-\frac{F''^2}{F'^2}\right)\frac{dF}{F'} \\ ;\frac{F}{F'}\left(d\alpha-\frac{dF\alpha'}{F'}\right),-\frac{i}{24\pi}\int d\theta S(F)\frac{F}{F'}\left(d\alpha-\frac{dF\alpha'}{F'}\right)\end{array}\right).
$$

Now we use (9.15) to construct the geometrical action for an orbit $O_{(j,ic_1;p,ic_2)}$ with elements parametrized as

$$
\begin{aligned}
(j,ic_1;p,ic_2) &= Ad^*_{\left(F,-im_1,(F')^{-1}\alpha(F),im_1\right)}(j_0,ic_1;p_0,ic_2) \quad (10.69) \\
&= \left(\begin{array}{c}\left(F'\right)^2\left[j_0+p_0'\alpha+2p_0\alpha'-\frac{c_2}{24\pi}\alpha'''\right]\circ F-\frac{c_1}{24\pi}S(F),ic_1 \\ ;\left(F'\right)^2p_0(F)-\frac{c_2}{24\pi}S(F),ic_2\end{array}\right).
\end{aligned}
$$

The action is given by

$$I=-\int\left\langle Ad^{*}_{\left(F,-im_1,(F')^{-1}\alpha(F),im_1\right)}\left(j_0,ic_1;p_0,ic_2\right),\left(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa\right)\right\rangle \, .$$

In the notation $d=dt\left(\dot{}\right)$ the action takes the form

$$I=-\int d\theta dt\left[\begin{array}{c} F'\dot{F}\left[j_0+p_0'\alpha+2p_0\alpha'\right]\circ F+F\left(F'\dot{\alpha}-\dot{F}\alpha'\right)p_0\left(F\right) \\ -\frac{c_2}{24\pi}F'\dot{F}\alpha'''\left(F\right)-\frac{c_1}{48\pi}\frac{\dot{F}}{F'}\left(\frac{F'''}{F'}-2\frac{F''^2}{F'^2}\right)\end{array}\right] .$$

Defining the variables $\phi=\log\left(F'\right)$ $\xi=\alpha'\left(F\right)$ we can write

$$\begin{aligned}
\dot{\phi}\phi' &= \frac{\dot{F}'F''}{F'^2}=\left(\frac{\dot{F}F''}{F'^2}\right)'-\frac{\dot{F}}{F'}\left(\frac{F'''}{F'}-2\frac{F''^2}{F'^2}\right), \\
\dot{\phi}\xi' &= \dot{F}'\alpha''\left(F\right)=\left(\dot{F}\alpha''\left(F\right)\right)'-\dot{F}F'\alpha'''\left(F\right) \, .
\end{aligned}$$

Therefore, the action on a orbit of the $\widehat{\mathfrak{bms}}_3$ algebra is

$$I=-\int d\theta dt\left[F'\dot{F}\left[j_0+p_0'\alpha+2p_0\alpha'\right]\circ F+F\left(F'\dot{\alpha}-\dot{F}\alpha'\right)p_0\left(F\right)+\frac{c_2}{24\pi}\dot{\phi}\xi'+\frac{c_1}{48\pi}\dot{\phi}\phi'\right] , \tag{10.70}$$

where the terms proportional to the central charges are recognized as the classical dual for asymptotically flat Einstein gravity in three dimensions [12].

Chapter 11

Conclusions and Future Developments

In the second part of the thesis, classical dual field theories for three-dimensional gravity with negative and vanishing cosmological constant have been studied as geometrical actions on coadjoint orbits of the corresponding asymptotic symmetry group. The coadjoint orbits of the Virasoro and BMS_3 group have been studied, the Kirillov-Kostant symplectic form has been analyzed from the Hamiltonian point of view and two-dimensional action principles for the gauge fixed solution space of three-dimensional gravity have been constructed. These actions can also be interpreted as one-dimensional particle-type actions for infinite-dimensional groups. In the case of the Kac-Moody group, the geometrical action corresponds to a WZW model plus extra contributions, which label the orbit on which the theory is defined and from the physical point of view they represent the coupling to point sources or the inclusion of holonomies representing angular defects or the presence of an event horizon in the case of black holes [87]. In the case of the Virasoro group, a chiral boson is obtained allowing to reconstruct the dual field theory for asymptotically AdS three-dimensional Einstein gravity when two copies of the Virasoro group are considered. When studying semi-direct products groups, these results can be extended to obtain generalized $\widehat{\mathrm{BMS}}_3$ invariant actions, which include the known classical dual for asymptotically flat three-dimensional Einstein gravity [12].

Another result presented in this manuscript is the fact that locally AdS geometries can be also understood as Lorentz flat geometries in the presence of covariantly constant torsion. In the euclidean case, the Adams-Hopf theorem [118] states that the three-sphere is parallelizable, namely, it can be endowed with a globally trivial $SO(3)$ connection. Equivalently, the statement that AdS_3 is Lorentz-flat is just the continuation to Lorentzian signature of the Adams-Hopf result. Since the Adams-Hopf theorem establishes that S^7 is parallelizable, one should expect that some interesting covariantly constant torsion geometries would also exist in AdS_7. Other local Lorentz flat black hole solutions can be constructed in the presence a locally flat but globally nontrivial gauge connection. This is the case, for instance in the vacuum sector of some supersymmetric Chern-Simons theories that include the $U(1)$ or $SU(2)$ connections [119]. Those solutions, for particular values of the parameters, are configurations admitting globally defined Killing spinors and therefore define stable BPS ground state. This result is interesting, as it can be continued to the asymptotic analysis to formulate asymptotically AdS spaces as Asymptotically Lorentz-flat geometries. The construction of a Chern-Simons action invariant under the Lorentz group containing these solutions would lead to a single chiral boson as the classical dual theory at the boundary. The implications of this will be explored in a future project.

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Appendix A

Sum Over Matsubara Frequencies

Let's consider the gap equation (4.16)

$$\frac{3Ng^2\lambda}{16\pi^3}\sum_n \int_0^1 \frac{R^2 dR}{(R^2+\theta_n^2)^2+\frac{Ng^2\lambda^2}{36\pi^2}(R^2+\theta_n^2)+\Gamma^4}=1\ ,$$

and let's compute the following sum over the dimensionless Matsubara frequencies θ_n

$$\sum_n \frac{1}{(R^2+\theta_n^2)^2+\frac{Ng^2\lambda^2}{36\pi^2}(R^2+\theta_n^2)+\Gamma^4}=\sum_n \frac{1}{P(n^2)}\ , \tag{A.1}$$

where

$$P(x)=\lambda^4 (x+a_-)(x+a_+)\,, \tag{A.2}$$

$$a_\pm=\frac{R^2}{\lambda^2}+\frac{Ng^2}{72\pi^2}\pm\sqrt{\frac{N^2g^4}{72^2\pi^4}-\frac{\Gamma^4}{\lambda^4}}\ . \tag{A.3}$$

Using algebraic manipulations, we can write (A.1) as

$$\sum_n \frac{1}{P(n^2)}=\frac{1}{\lambda^4}\frac{1}{a_+-a_-}\sum_n\left(\frac{1}{n^2+a_-}-\frac{1}{n^2+a_+}\right)\ . \tag{A.4}$$

Then, using the residue theorem applied sum series

$$\sum_{n=-\infty}^{\infty} f(z)=-\sum res\left[\pi\cot(\pi z) f(z)\right]\ ,$$

we obtain for (A.4)

$$\sum_n \frac{1}{P(n^2)}=\frac{1}{\lambda^4}\frac{1}{a_+-a_-}\left(\frac{\pi\coth\left(\pi\sqrt{a_-}\right)}{\sqrt{a_-}}-\frac{\pi\coth\left(\pi\sqrt{a_+}\right)}{\sqrt{a_+}}\right)\ . \tag{A.5}$$

Defining $S(R,\lambda,\Gamma)=\sum_n \frac{1}{P\left(n^2\right)}$ and using (A.3) we obtain (4.17)

$$S(R,\lambda,\Gamma)=\frac{\pi}{2\lambda\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}\left(\frac{\coth\left(\frac{\pi}{\lambda}\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}-\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}\right)}{\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}-\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}}-\frac{\coth\left(\frac{\pi}{\lambda}\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}+\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}\right)}{\sqrt{R^2+\frac{Ng^2\lambda^2}{72\pi^2}+\sqrt{\frac{N^2g^4\lambda^4}{72^2\pi^4}-\Gamma^4}}}\right), \tag{A.6}$$

and the thermal gap equation (4.16) takes the form

$$\frac{3Ng^2\lambda}{16\pi^3}\int_0^1 dRR^2S(R,\lambda,\Gamma)=1\ .$$

Appendix B

Generalities on Lie Groups

Consider local coordinates ζ^a on a Lie group G such that $\zeta^a = 0$ corresponds to the identity e and $e_a = \frac{\partial g}{\partial \zeta^a}|_e$ is a basis of $\mathfrak{g}$, the tangent space at e, and satisfy

$$[e_a, e_b] = f^c_{ab} e_c \ , \tag{B.1}$$

where f^c_{ab} are the structure constants. Abusing notation, we write the left and right invariant vector fields on G arising from the differential of the left and right action of G on itself as

$$L_a(g) = g e_a = L^b{}_a(\zeta) \frac{\partial g}{\partial \zeta^b} \ , \tag{B.2}$$

$$R_a(g) = e_a g = R^b{}_a(\zeta) \frac{\partial g}{\partial \zeta^b} \ , \tag{B.3}$$

with Lie brackets

$$[L_a, L_b] = f^c_{ab} L_c, \quad [R_a, R_b] = -f^c_{ab} R_c, \quad [L_a, R_b] = 0 \ . \tag{B.4}$$

The associated dual left and right invariant one-forms are $\theta^a(g) = e^a g^{-1}$ and $\kappa^a = g^{-1} e^a$ and the left and right invariant Maurer-Cartan forms are given by

$$\theta = g^{-1} dg = \theta^a e_a = M^a{}_b d\zeta^b e_a \ , \tag{B.5}$$

$$\kappa = dg g^{-1} = e_a \kappa^a = N^a{}_b e_a d\zeta^b \ , \tag{B.6}$$

where $M^a{}_b = \left(L^{-1}\right)^a{}_b$ and $N^a{}_b = \left(R^{-1}\right)^a{}_b$, which satisfy the Maurer-Cartan equations

$$d\theta + \frac{1}{2}[\theta, \theta] = 0 \tag{B.7}$$

$$d\kappa - \frac{1}{2}[\kappa, \kappa] = 0 \ . \tag{B.8}$$

By the same abuse of notation, we write the adjoint action of G on the basis of $\mathfrak{g}$ by replacing $X = e_a$ in (9.12), which gives

$$Ad_g e_a = g e_a g^{-1} = K^b{}_a(\zeta) e_b \ . \tag{B.9}$$

Let us consider now a basis e^a for $\mathfrak{g}^*$, dual to e_a with pairing

$$\langle e^a, e_b \rangle = \delta^a_b \ . \tag{B.10}$$

The coadjoint action of G on the basis of $\mathfrak{g}^*$ can be obtained by replacing $U = e^a$ in (9.5) and leads to

$$Ad^*_g e^a = g e^a g^{-1} = (K^{-1})^a{}_b e^b \ . \tag{B.11}$$

The infinitesimal form of (B.9) and (B.11) are given by

$$ad_{e_a} e_b = [e_a, e_b] = f^c_{ab} e_c \ , \tag{B.12}$$

$$ad^*_{e_a} e^b = [e_a, e^b] = -f^b_{ac} e^c \ . \tag{B.13}$$

Note that due to the property

$$\kappa = Ad_g \theta = g \theta g^{-1} \ , \tag{B.14}$$

it follows that

$$N^a{}_b = K^a{}_c M^c{}_b \ . \tag{B.15}$$

Another useful properties of the matrix $K^a{}_c$ are

$$\frac{\partial K^a{}_b}{\partial \zeta^c} = f^d_{be} N^e{}_c K^a{}_d \ , \tag{B.16}$$

$$f^c_{ab} K^d{}_c = f^d_{ce} K^c{}_a K^e{}_b \ . \tag{B.17}$$

Let v denote a vector field induced by the coadjoint action of G, which is given by (9.5). In fact,

$$v(U) = \frac{d}{dt} \left(Ad^*_{c(t)} U \right) \Big|_{t=0} = ad^*_X U \ , \ X = \frac{dc}{dt} \Big|_{t=0} \ . \tag{B.18}$$

Let us consider now a set of local coordinates U_a in $\mathfrak{g}^*$ such that every element $U \in \mathfrak{g}^*$ can be written as

$$U = U_a e^a \ . \tag{B.19}$$

Then, using (B.13), (B.18) takes the form

$$v(U) = X^a U_c ad^*_{e_a} e^c = -X^a U_c f^c_{ab} e^b = X^a v_a \ ,$$

where we have defined the vector fields

$$v_a(U) = -U_b f^c_{ab} e^b \ , \tag{B.20}$$

which span $T_U \mathfrak{g}^*$. The vector field v_a can then be written the form

$$v_a = -U_c f^c_{ab} \frac{\partial}{\partial U_b} \ . \tag{B.21}$$

Appendix C

Details on Maurer-Cartan Forms

In this Appendix we consider some examples of how to construct the right invariant Maurer-Cartan form for some infinite dimensional groups that will be useful in Chapter 10. Left invariant forms can be constructed along the same lines.

C.1 Kac-Moody Group

Let (κ, m_κ) be the Maurer-Cartan form for the Kac-Moody group, the Maurer-Cartan equation (B.8) can be written explicitly using (10.20)

$$d(\kappa, m_\kappa) = [(\kappa, m_\kappa), (\kappa, m_\kappa)] = \frac{1}{2}\left([\kappa, \kappa], \frac{1}{2\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \partial_\phi \kappa\kappa\right) .$$

Therefore, the equations to solve are given by

$$d\kappa = \frac{1}{2}[\kappa, \kappa] \quad , \quad dm_\kappa = \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \partial_\phi \kappa\kappa . \tag{C.1}$$

Here κ corresponds to the right invariant Maurer-Cartan form for LG, which is given by (B.6),

$$\kappa = dgg^{-1}, \tag{C.2}$$

Replacing (C.2) in (C.1), the equation for m_κ takes the form

$$\begin{aligned} dm_\kappa &= \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \partial_\phi \left(dgg^{-1}\right) dgg^{-1} = \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \left(d\partial_\phi gg^{-1} + dg\partial_\phi g^{-1}\right) dgg^{-1} \\ &= \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi d\left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\left(dgg^{-1}\right)^2 \partial_\phi gg^{-1}\right)\right] . \end{aligned}$$

Therefore we find

$$m_\kappa = \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\left(dgg^{-1}\right)^2\partial_\phi gg^{-1}\right)\right] , \tag{C.3}$$

and the right invariant Maurer-Cartan forms is given by

$$(\kappa, m_\kappa) = \left(dgg^{-1}, \frac{1}{4\pi}\mathrm{Tr}\int_0^{2\pi} d\phi \left[dgg^{-1}\partial_\phi gg^{-1} - d^{-1}\left(\left(dgg^{-1}\right)^2\partial_\phi gg^{-1}\right)\right]\right) .$$

C.2 Virasoro Group

The Maurer-Cartan equation (B.8) for the Virasoro group is given by

$$\begin{aligned} d(\kappa, -im_\kappa) &= \left(\kappa^2, -\frac{i}{48\pi}\int_0^{2\pi} d\phi\, \kappa'''(\phi)\kappa(\phi)\right) \\ &\Longrightarrow d\kappa = \kappa^2 \quad , \quad dm_\kappa = \frac{1}{48\pi}\int_0^{2\pi} d\phi\, \kappa'''\kappa , \end{aligned} \tag{C.4}$$

were we have used (10.37). The solution for κ is given by

$$\kappa = \frac{dF}{F'}\partial_\phi . \tag{C.5}$$

In fact

$$\kappa^2 = \frac{dF}{F'}\partial_\phi \frac{dF}{F'}\partial_\phi = \frac{dF dF'}{F'^2}\partial_\phi = d\left(\frac{dF}{F'}\partial_\phi\right) = d\kappa . \tag{C.6}$$

Let us now turn our attention to the equation for m_κ

$$dm_\kappa = \frac{1}{48\pi}\int_0^{2\pi} d\phi\, \kappa'''\kappa . \tag{C.7}$$

Let's compute first $\kappa'''(\phi)$

$$\begin{aligned} &\kappa'(\phi) = \tfrac{dF'}{F'} - \tfrac{dF}{F'^2}F'' , \\ &\kappa''(\phi) = \tfrac{dF''}{F'} - 2\tfrac{dF'}{F'^2}F'' + 2\tfrac{dF}{F'^3}F''^2 - \tfrac{dF}{F'^2}F''' , \\ &\kappa'''(\phi) == \tfrac{dF'''}{F'} - 3\tfrac{dF''}{F'^2}F'' + 6\tfrac{dF'}{F'^3}F''^2 - 3\tfrac{dF'}{F'^2}F''' - 6\tfrac{dF}{F'^4}F''^3 + 6\tfrac{dF}{F'^3}F''F''' - \tfrac{dF}{F'^2}F'''' . \end{aligned}$$

Therefore

$$\kappa'''(\phi)\kappa(\phi) = \frac{dF'''}{F'^2}dF - 3\frac{F''}{F'^3}dF''dF + 6\frac{F''^2}{F'^4}dF'dF - 3\frac{F''}{F'^3}dF'dF . \tag{C.8}$$

Now we will show that the term $\kappa'''(\phi)\kappa(\phi)$ is equivalent, up to a total derivative in ϕ, to the expression

$$d\left[\left(\frac{F'''}{F'} - \left(\frac{F''}{F'}\right)^2\right)\kappa\right] . \tag{C.9}$$

In fact

$$\begin{aligned} d\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'}\right] &= \left(\frac{dF'''}{F'}-\frac{F'''}{F'^2}dF'-2\frac{F''}{F'}\left(\frac{dF''}{F'}-\frac{F''}{F'^2}dF'\right)\right)\frac{dF}{F'} \\ &\quad +\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'^2}dF' \\ &= \frac{dF'''}{F'^2}dF-\frac{F'''}{F'^3}dF'dF-2\frac{F''}{F'^3}dF''dF+2\frac{F''^2}{F'^4}dF'dF \\ &\quad +\frac{F'''}{F'^3}dFdF'-\frac{F''^2}{F'^3}dFdF' \\ &= \frac{dF'''}{F'^2}dF-2\frac{F''}{F'^3}dF''dF+3\frac{F''^2}{F'^4}dF'dF-2\frac{F'''}{F'^3}dFdF' \end{aligned} \tag{C.10}$$

and (C.10) differs from (C.8) by

$$\frac{F''}{F'^3}dF''dF-3\frac{F''^2}{F'^4}dF'dF+\frac{F'''}{F'^3}dFdF'=\partial_\phi\left(\frac{F''}{F'^3}dF'dF\right)\ . \tag{C.11}$$

Therefore, the integrand in (C.7) is given by (C.9) up to boundary and we can write

$$dm_\kappa=\frac{1}{48\pi}\int_0^{2\pi}d\phi d\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\kappa\right] \tag{C.12}$$

$$\Longrightarrow m_\kappa=\frac{1}{48\pi}\int_0^{2\pi}d\phi\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'}\right]\ , \tag{C.13}$$

and the right invariant Maurer-Cartan form is

$$(\kappa,-im_\kappa)=\left(\frac{dF}{F'}\partial_\phi,-\frac{i}{48\pi}\int_0^{2\pi}d\phi\left[\left(\frac{F'''}{F'}-\left(\frac{F''}{F'}\right)^2\right)\frac{dF}{F'}\right]\right)\ .$$

C.3 Kac-Moody Group of $G \ltimes \mathfrak{g}$

Using (10.8), the Maurer-Cartan equation (C.1) yields

$$(d\kappa,dm_\kappa;d\beta_\kappa,dn_\kappa)=\left(\frac{1}{2}[\kappa,\kappa],\frac{1}{4\pi}\int d\theta\mathrm{Tr}\left(\kappa'\kappa\right);[\kappa,\beta_\kappa],\frac{1}{2\pi}\int d\theta\mathrm{Tr}\left(\kappa'\beta_\kappa\right)\right)\ , \tag{C.14}$$

where the first and the third equations can be solved straightforwardly, leading to

$$\kappa=dgg^{-1}\ \ ,\ \ \beta_\kappa=gd\alpha g^{-1}\ ,$$

while the second equation was solved in and the solution is given by

$$m_\kappa = \frac{1}{4\pi}\int d\theta \mathrm{Tr}\left(g'g^{-1}dgg^{-1} + d^{-1}\left[g'g^{-1}dgg^{-1}dgg^{-1}\right]\right) .$$

We are then left with

$$\begin{aligned} dn_\kappa &= \frac{1}{2\pi}\int d\theta \mathrm{Tr}\left(\left(dgg^{-1}\right)' g d\alpha g^{-1}\right) \\ &= \frac{1}{2\pi}\int d\theta \mathrm{Tr}\left(d\left(g'd\alpha g^{-1}\right) + g'd\alpha dg^{-1} - dgg^{-1}g'g^{-1}gd\alpha g^{-1}\right) \\ &= \frac{1}{2\pi}\int d\theta \mathrm{Tr} d\left(g'd\alpha g^{-1}\right) , \end{aligned}$$

$$\Longrightarrow n_\kappa = \frac{1}{2\pi}\int d\theta \mathrm{Tr}\left(g'd\alpha g^{-1}\right) .$$

Hence, the solution to (C.14) is given by

$$(\kappa, m_\kappa; \beta_\kappa, n_\kappa) = \left(dgg^{-1}, \frac{1}{4\pi}\int d\theta \mathrm{Tr}\begin{pmatrix} g'g^{-1}dgg^{-1} \\ +d^{-1}\left[g'g^{-1}dgg^{-1}dgg^{-1}\right] \end{pmatrix}; gd\alpha g^{-1}, \frac{1}{2\pi}\int d\theta \mathrm{Tr}\left(g'd\alpha g^{-1}\right)\right) .$$

C.4 BMS$_3$

The Maurer-Cartan equation in the case of the $\widehat{\mathrm{BMS}}_3$ group has the form

$$\begin{aligned} d(\kappa, -im_\kappa; \beta_\kappa, -in_\kappa) &= \frac{1}{2}\left[(\kappa, -im_\kappa; \beta_\kappa, -in_\kappa), (\kappa, -im_\kappa; \beta_\kappa, -in_\kappa)\right] \\ &= \left(\frac{1}{2}\left[(\kappa, -im_\kappa), (\kappa, -im_\kappa)\right]; \left[(\kappa, -im_\kappa), (\beta_\kappa, -in_\kappa)\right]\right) \\ &= \left(\kappa\kappa', -\frac{i}{48\pi}\int d\theta \kappa'''\kappa; \kappa\beta_\kappa' + \beta_\kappa\kappa', -\frac{i}{24\pi}\int d\theta \kappa'''\beta_\kappa\right) , \end{aligned}$$

where we have used (10.67) . Hence the equations to solve are

$$d\kappa = \kappa\kappa' \tag{C.15}$$

$$d\beta_\kappa = \kappa\beta_\kappa' + \beta_\kappa\kappa' \tag{C.16}$$

$$dm_\kappa = \frac{1}{48\pi}\int d\theta \kappa'''\kappa \tag{C.17}$$

$$dn_\kappa = \frac{1}{24\pi}\int d\theta \kappa'''\beta_\kappa . \tag{C.18}$$

The solution for the equations (C.15) and (C.16) are

$$\kappa = \frac{dF}{F'} \tag{C.19}$$

$$\beta_\kappa = \frac{F}{F'}\left(d\alpha - \kappa\alpha'\right) \ , \tag{C.20}$$

while the solution for (C.17) was computed in Appendix C.1 and reads

$$m_\kappa = \frac{1}{48\pi}\int d\theta \left(\frac{F'''}{F'} - \frac{F''^2}{F'^2}\right)\kappa \ .$$

For equation (C.18), we compute $\kappa'''\beta_\kappa$ using (C.8)

$$\begin{aligned}
\kappa'''\beta_\kappa &= \begin{pmatrix} \frac{dF'''}{F'} - 3\frac{dF''F''}{F'^2} - 3\frac{dF'F'''}{F'^2} + 6\frac{dF'F''^2}{F'^3} \\ -\frac{dFF''''}{F'^2} + 6\frac{dFF''F'''}{F'^3} - 6\frac{dFF''^3}{F'^4} \end{pmatrix}\left(\frac{Fd\alpha}{F'} - \frac{FdF\alpha'}{F'^2}\right) \\
&= \begin{pmatrix} \frac{dF'''Fd\alpha}{F'^2} - 3\frac{dF''F''Fd\alpha}{F'^3} - 3\frac{dF'F'''Fd\alpha}{F'^3} \\ +6\frac{dF'F''^2Fd\alpha}{F'^4} - \frac{dFF''''Fd\alpha}{F'^3} + 6\frac{dFF''F'''Fd\alpha}{F'^4} - 6\frac{dFF''^3Fd\alpha}{F'^5} \end{pmatrix} \\
&\quad -\left(\frac{dF'''FdF\alpha'}{F'^3} - 3\frac{dF''F''FdF\alpha'}{F'^4} - 3\frac{dF'F'''FdF\alpha'}{F'^4} + 6\frac{dF'F''^2FdF\alpha'}{F'^5}\right) \ .
\end{aligned}$$

Let us work out the terms in the first bracket

$$\begin{aligned}
&\frac{dF'''Fd\alpha}{F'^2} - \frac{3}{2}\frac{d\left(F''^2\right)Fd\alpha}{F'^3} - 3\frac{dF'F'''Fd\alpha}{F'^3} + 6\frac{dF'F''^2Fd\alpha}{F'^4} \\
&-\frac{dFF''''Fd\alpha}{F'^3} + 6\frac{dFF''F'''Fd\alpha}{F'^4} - 6\frac{dFF''^3Fd\alpha}{F'^5} && \text{(C.21)} \\
= \ & d\left(\frac{F'''Fd\alpha}{F'^2} - \frac{3}{2}\frac{F''^2Fd\alpha}{F'^3}\right) - \frac{F'''dFd\alpha}{F'^2} + \frac{3}{2}\frac{F''^2dFd\alpha}{F'^3} - \frac{dF'F'''Fd\alpha}{F'^3} && \text{(C.22)} \\
&+\frac{3}{2}\frac{dF'F''^2Fd\alpha}{F'^4} - \frac{dFF''''Fd\alpha}{F'^3} + 6\frac{dFF''F'''Fd\alpha}{F'^4} - 6\frac{dFF''^3Fd\alpha}{F'^5} \ .
\end{aligned}$$

For the second bracket we have

$$\begin{aligned}
&\frac{dF'''FdF\alpha'}{F'^3} - 3\frac{dF''F''FdF\alpha'}{F'^4} - 3\frac{dF'F'''FdF\alpha'}{F'^4} + 6\frac{dF'F''^2FdF\alpha'}{F'^5} \\
= \ & d\left(\frac{F'''FdF\alpha'}{F'^3} - \frac{3}{2}\frac{F''^2FdF\alpha'}{F'^4}\right) + \left(\frac{F'''FdFd\alpha}{F'^3}\right)' - \frac{3}{2}\left(\frac{F''^2FdFd\alpha}{F'^4}\right)' && \text{(C.23)} \\
&-\frac{F''''FdFd\alpha}{F'^3} - \frac{F'''dFd\alpha}{F'^2} - \frac{F'''FdF'd\alpha}{F'^3} + 6\frac{F'''FdFF''d\alpha}{F'^4} + \frac{3}{2}\frac{F''^2dFd\alpha}{F'^3} \\
&+\frac{3}{2}\frac{F''^2FdF'd\alpha}{F'^4} - 6\frac{F''^2FdFF''d\alpha}{F'^5} \ . && \text{(C.24)}
\end{aligned}$$

Subtracting (C.22) and (C.23) we find

$$\kappa'''\beta_\kappa = d\left(\left(\frac{F'''}{F'}-\frac{3}{2}\frac{F''^2}{F'^2}\right)\left(\frac{Fd\alpha}{F'}-\frac{FdF\alpha'}{F'^2}\right)\right)-\left(\left(\frac{F'''}{F'}-\frac{3}{2}\frac{F''^2}{F'^2}\right)\frac{F'''FdFd\alpha}{F'^3}\right)' ,$$

and equation (2.3) takes the form

$$dn_\kappa = \frac{1}{24\pi}\int d\theta d\left(\left(\frac{F'''}{F'}-\frac{3}{2}\frac{F''^2}{F'^2}\right)\left(\frac{FdA}{F'}-\frac{FdF\alpha'}{F'^2}\right)\right) = \frac{1}{24\pi}\int d\theta d\left(S(F)\beta_\kappa\right)$$

$$\Longrightarrow n_\kappa = \frac{1}{24\pi}\int d\theta S(F)\beta_\kappa .$$

The Maurer-Cartan form in this case is then given by

$$(\kappa,-im_\kappa;\beta_\kappa,-in_\kappa) = \begin{pmatrix} \frac{dF}{F'},-\frac{i}{48\pi}\int d\theta\left(\frac{F'''}{F'}-\frac{F''^2}{F'^2}\right)\frac{dF}{F'} \\ ;\frac{F}{F'}\left(d\alpha-\frac{dF\alpha'}{F'}\right),-\frac{i}{24\pi}\int d\theta S(F)\frac{F}{F'}\left(d\alpha-\frac{dF\alpha'}{F'}\right) \end{pmatrix} .$$