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RESEARCH ARTICLE

Nima Khosravi · **Hamid Reza Sepangi** · **Babak Vakili**

A cosmological viewpoint on the correspondence between deformed phase-space and canonical quantization

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Abstract We employ the familiar canonical quantization procedure in a given cosmological setting to argue that it is equivalent to and results in the same physical picture if one considers the deformation of the phase-space instead. To show this we use a probabilistic evolutionary process to make the solutions of these different approaches comparable. Specific model theories are used to show that the independent solutions of the resulting Wheeler–DeWitt equation are equivalent to solutions of the deformation method with different signs for the deformation parameter. We also argued that since the Wheeler–DeWitt equation is a direct consequence of diffeomorphism invariance, this equivalence is only true provided that the deformation of phase-space does not break such an invariance.

Keyword Quantum cosmology

1 Introduction

Standard cosmological models based on classical general relativity have no convincing and precise answer for the presence of the so-called "Big-Bang" singularity. Any hope of dealing with such singularities would be in vein unless a reliable quantum theory of gravity can be constructed. In the absence of a full theory of quantum gravity, it would be useful to describe the quantum states of the universe within the context of quantum cosmology, introduced in the works of DeWitt [1] and later Misner [2]. In this formalism which is based on the canonical quantization procedure, one first freezes a large number of degrees of freedom and then quantizes the remaining ones. The quantum state of the universe so obtained is

N. Khosravi School of Physics Institute for Research in Fundamental Sciences (IPM) P. O. Box 19395-5531, Tehran, Iran nima@ipm.ir; n-khosravi@sbu.ac.ir · N. Khosravi H. R. Sepangi Department of Physics Shahid Beheshti University G. C., Evin 19839 Tehran, Iran hr-sepangi@sbu.ac.ir · B. Vakili Department of Physics Azad University of Chalous P. O. Box 46615-397, Chalous, Iran b-vakili@cc.sbu.ac.ir; bvakili45@gmail.com

then described by a wave function in the mini-superspace, a function of the 3-geometry of the model and matter fields presented in the theory, satisfying the Wheeler–DeWitt (WD) equation. In more recent times, the research in this area has been quite active with different approaches [3; 4; 5; 6; 7; 8; 9; 10; 11], see [12] for a review. Interesting applications can be found in [13; 14] where canonical quantization is applied to many models with different matter fields as the source of gravity.

An important ingredient in any model theory related to the quantization of a cosmological model is the choice of the quantization procedure used to quantized the system. As mentioned above, the most widely used method has traditionally been the canonical quantization method based on the WD equation which is nothing but the application of the Hamiltonian constraint to the wave function of the universe. However, one may solve the constraint before using it in the theory and in particular, before quantizing the system. If we do so, we are led to a Schrödinger type equation where a time reparameterization in terms of various dynamical variables can be done before quantization [15; 16; 17]. A particularly interesting but rarely used approach to study quantum effects is to introduce a deformation in the phase-space of the system. It is believed that such a deformation of phase-space is an equivalent path to quantization, in particular to canonical and path integral quantizations [18].

An important question then naturally arises in applying the various quantization methods to cosmological models, that is, if they are equivalent. To at least partially answer the above question, we propose to quantize some simple cosmological models, namely the de Sitter, dusty FRW, FRW with radiative matter and Bianchi type I, using the methods described above. First we introduce a deformation to the Poisson algebra of the corresponding phase-space of the models. This will lead us to a deformed Hamiltonian from which the equations of motion can be constructed. We will show that the presence of the deformation parameter in the solutions can be interpreted as a quantum effect. This is done by comparing the resulting solutions with that of the other quantization method which is nothing but the usual canonical WD approach. Indeed, we will show that both of these quantization methods have the same physical interpretation for our chosen models.

A remark on the WD approach is that, as is well known, in canonical quantization the evolution of states disappear since in all diffeomorphism invariant theories the Hamiltonian becomes a constraint. Indeed, the wave function in the WD equation is independent of time, i.e., the universe has a static picture in this scenario. On the other hand in the deformed phase-space method, time does appear and so the
evolution becomes meaningful. Therefore, the problem of the evolution of states which is a major problem in quantum cosmology may also be addressed in this approach. To propose a possible solution to this problem we will use what we call the “Probabilistic Evolutionary Process” which was introduced in [19; 20]. This mechanism is briefly studied in the next section because of its crucial role in this work. In Sect. 3 we review the quantization of the de Sitter and dusty FRW cosmological models inspired by the deformed special relativity (DSR), discussed in [21; 22]. The canonical quantization for both models will be studied in Sect. 4. We will close the paper with a discussion and comparison of the results.

2 Probabilistic evolutionary process (PEP)

To quantize a classical model the following procedure is usually followed. The classical Hamiltonian is written in its corresponding operator form and the resulting Schrödinger equation obtained after quantization, i.e., $i\hbar\frac{\partial}{\partial t}\Psi = \mathscr{H}\Psi$, becomes the relevant equation to describe time evolution of the quantum states. However, in diffeomorphism invariant models the Hamiltonian becomes a constraint, $\mathscr{H} = 0$, and therefore does not provide for the evolution of the corresponding states. This means that in such models all the states are stationary. One of the common examples of this situation is general relativity which is employed to investigate the evolution of the cosmos. In quantum cosmology, the Schrödinger equation becomes the Wheeler–DeWitt equation, $\mathscr{H}\Psi = 0$. As is well known, quantum cosmology suffers from a number of problems, namely the construction of the Hilbert space for defining a positive definite inner product of the solutions of the WD equation, the operator ordering problem and most importantly, the problem of time; the wave function in the WD equation is independent of time, i.e., the universe has a static picture in this scenario. This problem was first addressed in [1] by DeWitt himself. However, he argued that the problem of time should not be considered as a hinderance in the sense that the theory itself must include a suitable well-defined time in terms of its geometry or matter fields. In this scheme time is identified with one of the characters of the geometry, usually the scale factor and is referred to as the intrinsic time, or with the momentum conjugate to the scale factor or even with a scalar character of the matter fields coupled to gravity in any specific model, known as the extrinsic time.

In general, the crucial problem in canonical quantum gravity is the presence of constraints in the gravitational field equations. Identification of time with one of the dynamical variables depends on the method we use to deal with theses constraints. Different approaches arising from these methods have been investigated in detail in [15]. As discussed in [15], time may be identified before or after quantization has been done. There are approaches, on the other hand, in which time has no fundamental role. The problem is how one can describe the evolution of the universe since observations show that the universe is not presently in a stationary state. In a previous work [19; 20], we introduced a mechanism which we have called the probabilistic evolutionary process (PEP), based on the probabilistic structure of quantum systems, to provide a sense of the evolution embedded in the wave function of the universe. This is based on the fact that in quantum systems the square of a state defines the probability, $\mathscr{P}_a = |\Psi(a)|^2$. The mechanism introduced as PEP says that the state Ψ_a makes a transition to the state Ψ_{a+da} if their distance, da, is infinitesimal and continuous, and that the higher the value of the transition probability[1] the larger the value of $\mathscr{P}_{a+da} - \mathscr{P}_a$.[2] The mechanism for transition from one state to another is through a small external perturbation.[3]

[1] This transition probability can play the role of the speed of transition, i.e., the higher the probability of transition the larger the speed of transition.

[2] Since there is no constraint on the positivity of $\mathscr{P}_{a+da} - \mathscr{P}_a$ then PEP can describe a tunneling process too.

[3] Certainly, in quantum cosmology, the universe is considered as one whole [23; 24] and the introduction of an external force is irrelevant. However, because of the lack of a full theory to describe the universe, these small external forces are merely used to afford a better understanding of the discussions presented here.

Fig. 1 Figure used to explain the idea of PEP

To make the discussion above more clear, we take an example which we shall encounter later on but will present the result in the form of the following figure. In what follows, we shall focus on the probabilistic description of our quantum states provided by Fig. 1 without worrying about the details of the model of which the figure is a result.

Let us take a specific initial condition, say $a = 2.5$, corresponding to the point P. Then, PEP states that the system (here specified by the scale factor a) moves continuously to a state with higher probability and thus P moves to the right to reach the point Q, a local maximum. Here, since Q locally has the maximum probability the system stays at Q. This means that the scale factor becomes constant as the time passes.[4] We show this transition by $P \stackrel{PEP}{\longrightarrow} Q$. Now let the initial condition be the point R. Then we have $R \stackrel{PEP}{\longrightarrow} Q$, and so on. Note that $R \stackrel{PEP}{\longrightarrow} S$ is possible but it has much smaller probability compared to the transitions $R \stackrel{PEP}{\longrightarrow} Q$. We note that the transitions $R \stackrel{PEP}{\longrightarrow} S$ and $S \stackrel{PEP}{\longrightarrow} T$ may be interpreted as tunnelling precesses in ordinary quantum mechanics in the sense that the probability of being at S is zero. It means that PEP can reproduce tunnelling processes but with a very small probability.

In the following we will use PEP to describe the evolution of quantum cosmological states in the canonical method of quantization. We insist that PEP plays a crucial role in the interpretation of the states in canonical quantization and allows them to be compared to the states resulting from quantization by deformation of the phase-space structure.

3 Phase-space deformation: a procedure for quantization

It has long been argued that a deformation in phase-space can be seen as an alternative path to quantization, based on Wigner quasi-distribution function and Weyl correspondence between quantum-mechanical operators in Hilbert space and ordinary c-number functions in phase-space, see for example [18] and the references therein. The deformation in the usual phase-space structure is introduced by Moyal brackets which are based on the Moyal product [25; 26; 27; 28; 29]. However, to introduce such deformations it is more convenient to work with Poisson brackets rather than Moyal brackets.

From a cosmological point of view, models are built in a minisuper-(phase)-space. It is therefore safe to say that studying such a space in the presence of deformations mentioned above can be interpreted as studying the quantum effects on cosmological solutions. One should note that in gravity the effects of quantization are woven into the existence of a fundamental length [30]. The question then arises as to what form of deformations in phase-space is appropriate for studying quantum effects in a cosmological model? Studies in noncommutative geometry [31; 32; 33] and generalized uncertainty principle (GUP) [34] have been a source of inspiration for those who have been seeking an answer to the above question.

[4] A perturbation around the local maximum is acceptable as described before.

More precisely, introduction of modifications to the structure of geometry in the way of noncommutativity has become the basis from which similar modifications in the phase-space have been inspired. In this approach, the fields and their conjugate momenta play the role of coordinate basis in noncommutative geometry [35; 36]. In doing so an effective model is constructed whose validity will depend on its power of prediction. For example, if in a model field theory the fields are taken as noncommutative, as has been done in [35; 36], the resulting effective theory predicts the same Lorentz violation as a field theory in which the coordinates are considered as noncommutative [37; 38; 39]. As a further example, it is well known that string theory can be used to suggest a modification to the bracket structure of coordinates, also known as GUP [34] which is used to modify the phase-space structure [40; 41; 42; 43; 44; 45]. Over the years, a large number of works on noncommutative fields [25; 26; 27; 28; 29] have been inspired by noncommutative geometry model theories [31; 32; 33].

In this paper we study the effects of the existence of a fundamental length in a cosmological scenario by constructing a model based on the noncommutative structure of the Deformed (Doubly) Special Relativity [46; 47; 48; 49; 50; 51; 52] which is related to what is known as the κ-deformation [53]. This way of introducing noncommutativity is interesting because of its compatibility with Lorentz symmetry, as is commonly believed [53; 54]. The κ-deformation is introduced and studied in [55; 56]. The κ-Minkowski space [57] arises naturally from the κ-Poincare algebra [46; 47; 48; 49; 50; 51; 52] such that the ordinary Poisson brackets between the coordinates are replaced by

$$\{x_0, x_i\} = \frac{1}{\kappa} x_i, \tag{1}$$

where κ is the deformation (noncommutative) parameter which has the dimension of mass $\kappa = \varepsilon \ell^{-1}$ when $c = \hbar = 1$, and $\varepsilon = \pm 1$ [58] such that κ and ℓ are interpreted as dimensional parameters which are identified with the fundamental energy and length, respectively. As mentioned above, one can change the structure of the phase-space based on Eq. (1). Here we will examine a new kind of modification in the phase-space structure inspired by relation (1), much the same as has been done in [25; 26; 27; 28; 29; 40; 41; 42; 43; 44; 45; 59]. In what follows we introduce noncommutativity based on κ-Minkowskian space and study its consequences on the de Sitter and dusty FRW cosmologies.

Let us start by briefly studying the ordinary, spatially flat FRW model where the metric is given by

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{2}$$

with $N(t)$ being the lapse function. The Einstein-Hilbert Lagrangian with a general energy density $V(a)$ becomes

$$\begin{aligned} \mathscr{L} &= \sqrt{-g}(R[g] - V(a)) \\ &= -6N^{-1}a\dot{a}^2 - Na^3V(a), \end{aligned} \tag{3}$$

where $R[g]$ is the Ricci scalar and in the second line the total derivative term has been ignored. The corresponding Hamiltonian up to a sign becomes

$$\mathscr{H}_0 = \frac{1}{24} N a^{-1} p_a^2 - N a^3 V(a). \tag{4}$$

Here, we note that since the momentum conjugate to $N(t), \pi = \frac{\partial \mathscr{L}}{\partial \dot{N}}$ vanishes, the term $\lambda \pi$ must be added as a constraint to Hamiltonian (4). The Dirac Hamiltonian then becomes

$$\mathscr{H} = \frac{1}{24} N a^{-1} p_a^2 - N a^3 V(a) + \lambda \pi. \tag{5}$$

To introduce noncommutativity one can start with

$$\{N'(t), a'(t)\} = \ell a'(t). \tag{6}$$

This is similar to Eq. (1) since one can interpret $N(t)$ and $a(t)$, appearing as the coefficients of dt and d, in the same manner as x_0 and x_i appearing in (1) respectively. For this reason we shall call it the κ-Minkowskian-minisuper-phase-space. In this case the Hamiltonian becomes

$$\mathscr{H}_0' = \frac{1}{24} N' a'^{-1} p'^2_a - N' a'^3 V(a'), \tag{7}$$

where the ordinary Poisson brackets are satisfied except in (6). To move along, one introduces the following variables [21; 22; 60]

$$\begin{cases} N'(t) = N(t) - \ell a(t) p_a(t), \\ a'(t) = a(t), \\ p'_a(t) = p_a(t). \end{cases} \tag{8}$$

It can be easily checked that the above variables will satisfy (6) if the unprimed variables satisfy the ordinary Poisson brackets. The term $-\ell a(t) p_a(t)$ may be looked upon as a direct consequence of a phase-space deformation of relation (6) which, as has been suggested, could originate from string theory, noncommutative geometry and so on, see [18; 59; 61]. With the above transformations, Hamiltonian (7) takes the form

$$\mathscr{H}_0^{nc} = \frac{1}{24} N a^{-1} p_a^2 - N a^3 V(a) - \frac{1}{24} \ell p_a^3 + \ell a^4 V(a) p_a. \tag{9}$$

It is clear that the momentum conjugate to $N(t)$ does not appear in (9), i.e., $\pi = 0$ is a primary constraint. It can be checked by using Legendre transformations that the conjugate momentum corresponding to $N(t), \pi = \frac{\partial \mathscr{L}}{\partial \dot{N}}$, vanishes. It is therefore necessary to add the term $\lambda \pi$ to Hamiltonian (9) to obtain the Dirac Hamiltonian

$$\mathscr{H}^{nc} = \frac{1}{24} N a^{-1} p_a^2 - N a^3 V(a) - \frac{1}{24} \ell p_a^3 + \ell a^4 V(a) p_a + \lambda \pi. \tag{10}$$

The equations of motion resulting from Hamiltonian (10) are

$$\begin{aligned} \dot{a} &= \{a, \mathscr{H}^{nc}\} = \frac{1}{12} N a^{-1} p_a - \frac{1}{8} \ell p_a^2 + \ell a^4 V(a), \\ \dot{p}_a &= \{p_a, \mathscr{H}^{nc}\} = \frac{1}{24} N a^{-2} p_a^2 + 3 N a^2 V(a) + N a^3 V'(a) \\ &\quad - 4 \ell a^3 V(a) p_a - \ell a^4 V'(a) p_a, \\ \dot{N} &= \{N, \mathscr{H}^{nc}\} = \lambda, \\ \dot{\pi} &= \{\pi, \mathscr{H}^{nc}\} = -\frac{1}{24} a^{-1} p_a^2 + a^3 V(a), \end{aligned} \tag{11}$$

Fig. 2 The scale factor for $\ell = 0.0001$ in the de Sitter model. The initial conditions are $C_1 = 1$ and $\Lambda = 6$ for the *left graph* and $C_1 = -0.0001$ for the *right one*

where a prime denotes differentiation with respect to the argument. The requirement that the primary constraints should hold during the evolution of the system means that $\dot{\pi} = \{\pi, \mathscr{H}^{nc}\} = 0$. If p_a is now calculated from the secondary constraint, $\dot{\pi} = 0$, and the result is substituted in the first equation in (11) one obtains

$$\dot{a} + 2\ell a^4 V(a) = N\sqrt{\frac{1}{6}a^2 V(a)}. \tag{12}$$

It is easy to check that the above equation is consistent with the second equation in (11) as well.

3.1 The de Sitter model

As is well known, in the de Sitter model $V(a) = 2\Lambda$ and Eq. (12) becomes

$$\dot{a} + 4\ell\Lambda a^4 = N\sqrt{\frac{1}{3}\Lambda a^2}, \tag{13}$$

which is compatible with other equations. The solution of the above equation, the scale factor, for a constant N can be written as[5]

$$a(t) = 3^{\frac{1}{6}}\left(\frac{N e^{\sqrt{3\Lambda}Nt}}{C_1 + 12\ell\sqrt{\Lambda}e^{\sqrt{3\Lambda}Nt}}\right)^{\frac{1}{3}}, \tag{14}$$

where C_1 is a constant of integration. It is seen that the usual de Sitter solution (without any deformations) is recovered in the limit $\ell \to 0$. The scale factor calculated in (14) and its behavior are shown in Fig. 2 for $\ell > 0$ and in Fig. 3 for $\ell < 0$. In what follows, we discuss the different figures separately. This may seem irrelevant here since the behavior of the scale factor for different choices are similar to each other. However, in the next case when we study the dust model, the solutions are essentially different, demanding separate discussions on their behavior. The same is also true when we study the relation of the deformed phase-space solutions to that of the canonical method at the end of next section.

The left plot in Fig. 2 shows that the scale factor has started from an initial value and becomes constant in the end. During this passage the behavior of the scale factor is monotonically increasing. The right plot in Fig. 2 on the other hand, shows a conversing behavior and predicts a monotonically decreasing scale factor. Note that in this case the scale factor becomes constant in its final state too. The scale factor represented by the left plot in Fig. 3 has the same behavior as that in the left plot in Fig. 2. In Fig. 3 one must restrict the discussion to the second part of the right plot only since the denominator of the scale factor in (14) cannot be zero for physical reasons. This means, for example,

[5] This means that we restrict ourselves to a certain class of gauges, namely $N = const.$, which is equivalent to the choice $\lambda = 0$ in the equation of motion (11).

Fig. 3 The scale factor for $\ell = -0.0001$ in the de Sitter model. The initial conditions are $C_1 = -0.0001$ and $\Lambda = 6$ for the *left graph* and $C_1 = 1$ for the *right one*

Fig. 4 The scale factor for $\ell = 0.01$ in the dust model. The initial conditions are $C_2 = 1$ and $\rho_0 = 1$ for the *left graph* and $C_2 = -1$ for the *right one*

that the universe has come into being at $t = 1.5$ in our case. Now the behavior of the scale factor is the same as that of the scale factor in the right plot of Fig. 2. Note that in all the figures we have taken $N = 1$ which only changes the values of the initial and final scale factor without any change in the behavior.[6] It should be mentioned here that these different initial and final values are crucial to our discussion on the equivalence between these two different approaches.

3.2 The dust model

In this case, substituting $V(a) = \rho_0 a^{-3}$ in Eq. (12) results in

$$\dot{a} + 2\ell\rho_0 a = N\sqrt{\frac{\rho_0}{6a}}. \tag{15}$$

The solution for $N = const.$ is given by

$$a(t) = \frac{N^{\frac{2}{3}}}{2\times 3^{\frac{1}{3}}}\left[\frac{1 - C_2 e^{-3\ell\rho_0 t}}{\ell\sqrt{\rho_0}}\right]^{\frac{2}{3}}, \tag{16}$$

for $\ell > 0$ and

$$a(t) = \frac{N^{\frac{2}{3}}}{2\times 3^{\frac{1}{3}}}\left[\frac{-1 + C_2 e^{3|\ell|\rho_0 t}}{|\ell|\sqrt{\rho_0}}\right]^{\frac{2}{3}}, \tag{17}$$

for $\ell < 0$, where C_2 is an integration constant which, for the second solution (17), must satisfy $C_2 e^{3|\ell|\rho_0 t} \geq 1$, indicating that $C_2 \geq 1$. Figures 4 and 5 show the behavior of the scale factors for $\ell > 0$ and $\ell < 0$ respectively. The scale factor presented by the left plot in Fig. 4 suggests that the universe starts from an initial state and reaches a constant final state due to a monotonically increasing behavior. As can be seen from the right plot in Fig. 4, the converse is also true. It is worth mentioning that only for $C_2 = 1$ the limit $\ell \rightarrow 0$ results in exactly the same behavior as that in the absence of any deformations.[7]

As mentioned in the previous section, the FRW dust model represents a completely different behavior for $\ell < 0$, which is represented in Fig. 5. In this case the scale factor monotonically increases without reaching a final constant value.

[6] This is compatible with the spirit of gauge invariance, that is, there is no physical difference between multiple gauge fixing. The measured value of the system parameters could be different for different gauges but as mentioned before, they have the same behavior.

[7] This does not make other choices for C_2 irrelevant since the limiting process is as yet questionable. Specifically, the solution represented on the right in Fig. 4, does not have a corresponding example in the non-deformed phase-space and it is a prediction of quantum cosmology only.

Fig. 5 The scale factor for $\ell=-0.01$ in the dust model. The initial conditions are $C_2=1$ and $\Lambda=6$. Note that in this case, for all the valid values of C_2 ($C_2\geq 1$) we have the same behavior

Fig. 6 The scale factor for $\ell=1, \rho_0=1$ and $C_3=1$ in the radiation model is shown in the *left figure*. In the *right figure* the scale factor for $C_3=-1$ is shown

Fig. 7 *Left*, the scale factor for $\ell=-1, \rho_0=1$ and $C_3=1$ in the radiation model and *right*, the scale factor for $C_3=1$. Note that since $a^2(t)$ appears in metric (2), the absolute value of the scale factor (20) is meaningful

3.3 The radiation model

This case is more interesting since it aims to describe the early phases of the universe, when quantum geometrical properties are expected to dominate. Mathematically, for the radiation era $V(a)=\rho_0 a^{-4}$ which makes Eq. (12)

$$\dot{a}+2\ell\rho_0=\sqrt{\frac{\rho_0}{6}}Na^{-1}. \tag{18}$$

The solution of the above equation reads

$$a(t)=\frac{\sqrt{6}N}{12\ell\sqrt{\rho_0}}\left[1+\mathscr{W}\left(\frac{C_3 e^{(-1-4\sqrt{6}N^{-1}\ell^2\rho_0^{\frac{3}{2}}t)}}{\sqrt{6}N}\right)\right], \tag{19}$$

for $\ell>0$ and

$$a(t)=\frac{\sqrt{6}N}{12\ell\sqrt{\rho_0}}\left[1+\mathscr{W}\left(-\frac{C_3 e^{(-1-4\sqrt{6}N^{-1}\ell^2\rho_0^{\frac{3}{2}}t)}}{\sqrt{6}N}\right)\right], \tag{20}$$

and for $\ell<0$ where C_3 is an integration constant, $\mathscr{W}(z)$ is the Lambert W function and gives the principal solution for w in $z=we^w$. Figures 6 and 7 show the behavior of scale factor for $\ell>0$ and $\ell<0$, respectively.

4 Canonical quantization: the WD equation

We now focus attention on the study of the quantum cosmology of the models described above. For this purpose we quantize the dynamical variables with the use of the WD equation, that is, $\mathscr{H}\Psi=0$, where $\mathscr{H}$ is the operator form of the Hamiltonian given by Eq. (4), and Ψ is the wave function of the universe, a function of the scale factor and the matter fields, if they exist.

Fig. 8 The probability density for the $J_{\frac{1}{3}}$ term with $\Lambda = 0.0001$

4.1 The de Sitter model

Let us set $V(a) = 2\Lambda$. Then the corresponding WD equation becomes

$$\mathscr{H}\Psi(a) = \left[\frac{1}{24}a^{-1}p_a^2 - 2\Lambda a^3\right]\Psi(a) = 0. \tag{21}$$

Choice of the ordering $a^{-1}p_a^2 = p_a a^{-1} p_a$ to make the Hamiltonian Hermitian and use of $[a, p_a] = i\hbar$ and $p_a = -i\hbar\partial_a$ results in

$$\partial_a^2\Psi(a) - a^{-1}\partial_a\Psi(a) + \frac{48\Lambda}{\hbar^2}a^4\Psi(a) = 0, \tag{22}$$

with solutions

$$\Psi(a) = c_1 a J_{\frac{1}{3}}\left(\frac{4a^3\sqrt{\Lambda}}{\hbar\sqrt{3}}\right) + c_2 a J_{-\frac{1}{3}}\left(\frac{4a^3\sqrt{\Lambda}}{\hbar\sqrt{3}}\right), \tag{23}$$

where J is the Bessel function and c_1 and c_2 are arbitrary constants. The constants must be so chosen as to satisfy the initial conditions. However, in what follows, our discussions are independent of the values of these constants. Here we show and emphasize the different behavior of both solutions. To do this we need to calculate the probability density $|\Psi(a)|^2$. In Figs. 8 and 9 the probability density for $c_2 = 0$ and $c_1 = 0$ are shown respectively.

Now, we start from a chosen initial condition, say, $a = 1$. The PEP procedure may now be employed to describe the physical interpretation of the resulting states. In Fig. 8, let the initial state be at point P. One then expects, keeping in mind the discussion presented previously, the transition $P \xrightarrow{PEP} Q$ to appear, with the following physical interpretation. The point P shows an initial state and the point Q shows the final state. The universe begins its evolution by a monotonically increasing scale factor to reach the point Q. Point Q is a state with a greater scale factor and is finite. Note that since the point Q is a local maximum, PEP predicts $Q \xrightarrow{PEP} Q$ which means that the scale factor remains in this final state. The transition $P \xrightarrow{PEP} Q$ then means starting from an initial state, increasing monotonically, and finally arriving at a constant final state. This behavior is completely similar to the behavior of the scale factor in the previous section, presented in the left plot in Fig. 2.

Since for the other cases the details are similar, in the following we mention only the correspondence between the solutions without any detail. In Fig. 8, the transition $R \xrightarrow{PEP} Q$ is similar to the right plot in Fig. 2. On the other hand in the same figure, transition $T \xrightarrow{PEP} U$ is similar to that of the left plot in Fig. 2 with a different initial condition, that is, with an appropriate C_1. The transition $Q \xrightarrow{PEP} Q$ represents the scale factor (14) with $C_1 = 0$. In Fig. 9, similar discussions as above apply and the transitions $Q \xrightarrow{PEP} P$ and $T \xrightarrow{PEP} S$ have the behavior and interpretation

Fig. 9 The probability density for the $J_{-\frac{1}{3}}$ term with $\Lambda = 0.0001$

compatible with the right plot in Fig. 3 with different initial conditions. Again, $R \xrightarrow{PEP} S$ is similar to the left plot in Fig. 3.

Now, an interesting question arises as to the meaning of $P \xrightarrow{PEP} P$ in this case. One possible answer would be that only superpositions of the solutions are appropriate for comparison with the previous section results, e.g., $c_1 = c_2 = 1$ and $c_1 = -c_2 = 1$.

It is appropriate at this point to discuss different values of the final scale factor predicted by the WD equation. Obviously, since the maximum probabilities in Figs. 8 and 9 occur at different values then the final scale factor assumes different values too. This prediction is compatible with the deformed phase-space method due to the appearance of the lapse function, N, as a constant parameter in relation (14) controlling the behavior of the scale factor. It is worth mentioning that in both approaches there are two parameters that control the behavior of the scale factor; C_1 in the deformed phase-space method and also N which is fixed by the maximum value of the scale factor and its initial values in each region of the WD solution. Note that each region is an interval between two minima that contains one maximum. For example in Fig. 6 the region containing points P and Q is one region and that containing T and U is another region and the same is true for other figures.

4.2 The dust model

In this case we set $V(a) = \rho_0 a^{-3}$ in Hamiltonian (4). The corresponding WD equation becomes

$$\mathscr{H}\Psi(a) = \left[\frac{1}{24}a^{-1}p_a^2 - \rho_0\right]\Psi(a) = 0. \tag{24}$$

With ordering mentioned above, the WD equation is

$$\partial_a^2\Psi(a) - a^{-1}\partial_a\Psi(a) + \frac{24\rho_0}{\hbar^2}a\Psi(a) = 0, \tag{25}$$

with solution

$$\Psi(a) = c_1'\mathrm{Ai}'\left[2\left(\frac{-3\rho_0}{\hbar^2}\right)^{1/3}a\right] + c_2'\mathrm{Bi}'\left[2\left(\frac{-3\rho_0}{\hbar^2}\right)^{1/3}a\right], \tag{26}$$

where c_1' and c_2' are integration constants and Ai' and Bi' are derivatives of the Airy functions Ai and Bi with respect to a respectively. In this case the corresponding probability densities $|\Psi(a)|^2$ are shown in Figs. 10 and 11 where $c_1' = 0$ and $c_2' = 0$ respectively. In the dust model two different solutions show completely different behaviors. One, Fig. 10, has local maxima but the other, Fig. 11, is monotonically increasing. In Fig. 10, the transition $Q \xrightarrow{PEP} R$ is compatible with the left plot in

Fig. 10 The probability density for the Bi′ term with $\rho_0 = 1$

Fig. 11 The probability density for the Ai′ term with $\rho_0 = 1$

Fig. 12 The probability density for the J-Bessel term with $\rho_0 = 1$ and $\hbar = 1$

Fig. 13 The probability density for the Y-Bessel term with $\rho_0 = 1$ and $\hbar = 1$

Fig. 4. Transition $S \xrightarrow{PEP} R$ however, is compatible with the right plot in Fig. 4 with transitions $P \xrightarrow{PEP} P$ and $R \xrightarrow{PEP} R$ being compatible with scale factor (16) with different initial conditions and constants.

It is worth noting that the transition $P \xrightarrow{PEP} Q$ in Fig. 11 means that no matter what the initial state is, the scale factor moves to a larger value and continues to grow since in this case no local maximum exists at all. This behavior is completely in agreement with the behavior of the scale factor represented in Fig. 5. Note that a similar discussion like that of the last paragraph of the previous subsection goes for the dust case. Note also that for Fig. 11 there is only one region.

4.3 The radiation model

In the radiation case the Hamiltonian (4) with $V(a) = \rho_0 a^{-4}$ results in the following WD equation

$$\mathscr{H}\Psi(a) = \left[\frac{1}{24}a^{-1}p_a^2 - \rho_0 a^{-1}\right]\Psi(a) = 0. \tag{27}$$

Using the above mentioned ordering, it reduces to a differential equation as follows

$$\partial_a^2\Psi(a) - a^{-1}\partial_a\Psi(a) + \frac{24\rho_0}{\hbar^2}a\Psi(a) = 0, \tag{28}$$

with solution

$$\Psi(a) = c_1'' a J_1\left(2\sqrt{6}\hbar a\right) + c_2'' a Y_1\left(2\sqrt{6}\hbar a\right), \tag{29}$$

where c_1'' and c_2'' are integration constants and J and Y are Bessel functions. The behavior of these functions are represented in Figs. 12 and 13. The discussions on the comparison between quantum cosmological solutions and their corresponding form from the deformed phase-space formalism, i.e., Fig. 6, are the same as previous
models, namely the cosmological constant and dust models. Similar discussion as above would be applicable to this case as well.

5 Bianchi type I model

In this section we consider a more complicated model, i.e., the Bianchi model with different matter fields and compare its quantum behavior to that modeled by a kind of phase-space deformation and also canonical quantization. We separate Bianchi models from previous ones since there is more than one variable which would make our conjecture more reasonable.

5.1 Phase-space deformation

Let us consider a cosmological model in which the spacetime is assumed to be of Bianchi type I whose metric can be written as

$$ds^2 = -N^2(t)dt^2 + e^{2u(t)}e^{2\beta_{ij}(t)}dx^i dx^j, \tag{30}$$

where $N(t)$ is the lapse function, $e^{u(t)}$ is the scale factor of the universe and $\beta_{ij}(t)$ determine the anisotropic parameters $v(t)$ and $w(t)$ as follows

$$\beta_{ij} = \mathrm{diag}\left(v + \sqrt{3}w, v - \sqrt{3}w, -2v\right). \tag{31}$$

To simplify the model we take $w = 0$, which is equivalent to a universe with two scale factors in the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2) + c^2(t)dz^2. \tag{32}$$

The anisotropy in the above metric can be achieved by introducing a large scale homogeneous magnetic field in a flat FRW spacetime. Such a magnetic field results in a preferred direction in space along the direction of the field. If we introduce a magnetic field which has only a z component, the resulting metric can be written in the form (32) where there are equal scale factors in the transverse directions x and y and a different one, $c(t)$, in the longitudinal direction z. The Hamiltonian for gravity coupled to a perfect fluid with equation of state $p = \gamma\rho$ is

$$\mathscr{H} = \frac{1}{24}Ne^{-3u}\left(-p_u^2 + p_v^2\right) + NM_\gamma e^{-3\gamma u} + \lambda\pi, \tag{33}$$

where, M_γ is a model dependent constant. To introduce noncommutativity one can start with

$$\left\{N'(t), e^{u'+v'}\right\} = \ell e^{u'+v'}, \quad \left\{N'(t), e^{u'-2v'}\right\} = \ell e^{u'-2v'}, \tag{34}$$

which are the two dimensional generalization of the relation (6). The Hamiltonian of this model becomes

$$\mathscr{H}'_0 = \frac{1}{24}N'e^{-3u'}\left(-p'^2_u + p'^2_v\right) + N'M_\gamma e^{-3\gamma u'}. \tag{35}$$

Now we introduce as before

$$\begin{cases} N'(t) = N(t) - \alpha\ell p_u(t) - \beta\ell p_v(t), \\ u'(t) = u(t), \quad v'(t) = v(t), \\ p'_u(t) = p_u(t), \quad p'_v(t) = p_v(t) \end{cases} \tag{36}$$

Fig. 14 *Left plot* is shown for $\gamma = -1$ (de Sitter) and *right one* for $\gamma = 0$ (dust). We take $\alpha = \beta = 1/2$, $p_{0v} = 1, N = 1, \ell = 1, M_\gamma = 1$ and initial conditions $u(t=0) = v(t=0) = 1, p_u(t=0) = 0$. The *dashed lines* represent the u-behavior and the *solid line* the v-behavior in both figures

Fig. 15 These figures are sketched only for dust, $\gamma = 0$. We take $\alpha = \beta = 1/2, p_{0v} = -1, N = 1, \ell = 1, M_\gamma = 1$ and initial conditions $u(t=0) = v(t=0) = 1, p_u(t=0) = 0$. *Left*, the behavior of u and *right*, the behavior of v

where $\alpha + \beta = 1$. The deformed Hamiltonian by imposing the above transformations takes the form

$$\mathscr{H}^{nc} = [N(t) - \alpha\ell p_u(t) - \beta\ell p_v(t)]\left[\frac{1}{24}e^{-3u}\left(-p_u^2 + p_v^2\right) + M_\gamma e^{-3\gamma u}\right] + \lambda\pi. \tag{37}$$

The classical dynamics is governed by the Hamilton equations, that is

$$\begin{cases} \dot{u} = \{u, \mathscr{H}^{nc}\} = -\frac{1}{12}e^{-3u}p_u\,[N(t) - \alpha\ell p_u(t) - \beta\ell p_v(t)] - \alpha\ell\left[\frac{1}{24}e^{-3u}\left(-p_u^2 + p_v^2\right) + M_\gamma e^{-3\gamma u}\right], \\ \dot{v} = \{v, \mathscr{H}^{nc}\} = \frac{1}{12}e^{-3u}p_v[N(t) - \alpha\ell p_u(t) - \beta\ell p_v(t)] - \beta\ell\left[\frac{1}{24}e^{-3u}\left(-p_u^2 + p_v^2\right) + M_\gamma e^{-3\gamma u}\right], \\ \dot{p_u} = \{p_u, \mathscr{H}^{nc}\} = [N(t) - \alpha\ell p_u(t) - \beta\ell p_v(t)]\left[-\frac{1}{8}e^{-3u}\left(-p_u^2 + p_v^2\right) - 3\gamma M_\gamma e^{-3\gamma u}\right], \\ \dot{p_v} = \{p_v, \mathscr{H}^{nc}\} = 0, \\ \dot{N} = \{N, \mathscr{H}^{nc}\} = \lambda, \\ \dot{\pi} = \{\pi, \mathscr{H}^{nc}\} = \frac{1}{24}e^{-3u}\left(-p_u^2 + p_v^2\right) + M_\gamma e^{-3\gamma u}. \end{cases} \tag{38}$$

The requirement that the primary constraints should hold during the evolution of the system means that $\dot{\pi} = \{\pi, \mathscr{H}^{nc}\} = 0$. Applying this condition to the above equations we find

$$\begin{cases} \dot{u} = -\frac{1}{12}e^{-3u}p_u\,[N - \alpha\ell p_u(t) - \beta\ell p_{0v}], \\ \dot{v} = \frac{1}{12}e^{-3u}p_v\,[N - \alpha\ell p_u(t) - \beta\ell p_{0v}], \\ \dot{p_u} = -3(\gamma - 1)M_\gamma e^{-3\gamma u}\,[N - \alpha\ell p_u(t) - \beta\ell p_{0v}]. \end{cases} \tag{39}$$

Since no analytic solution exists for the above equations, their behavior is represented in Figs. 14 and 15, employing numerical methods. In Fig. 14 for u, there is a decreasing era before the steady behavior, in contrast to the v-behavior. However, other choices are possible too. For example, in Fig. 15, both u and v behave similarly in a decreasing manner at early times. These figures are plotted for typical numerical values of the parameters and initial conditions. Use of other initial conditions results in an almost repeated behavior.

5.2 Canonical quantization

The WD equation for our model reads

$$\mathscr{H}\Psi(u,v) = 0 \Rightarrow \left[\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} + 24M_\gamma e^{3(1-\gamma)u}\right]\Psi(u,v) = 0. \tag{40}$$

The solutions to the above equation are

$$\Psi_\nu(u,v) = e^{\pm i\nu v} J_{\pm i\nu/3}\left(4\sqrt{\frac{M_\gamma}{6}}e^{3u}\right), \tag{41}$$

for $\gamma = -1$ and

$$\Psi_\nu(u,v) = e^{\pm i\nu v} J_{\pm 2i\nu/3}\left(4\sqrt{\frac{2M_\gamma}{3}}e^{3u/2}\right), \tag{42}$$

for $\gamma = 0$, where ν is a separation constant. We have chosen oscillatory functions $e^{\pm i\nu v}$ since the real exponents would lead to exponentially increasingwave func-

Fig. 16 *Right*, the plot of $|\Psi(u,v)|^2$ and *left*, its corresponding contour plot for de Sitter case with $M_\gamma = 20$. Only the first two terms of relation (43) with a constant weight function have been taken

Fig. 17 *Right*, the plot of $|\Psi(u,v)|^2$ and *left*, its corresponding contour plot for dust with $M_\gamma = 20$. Only the first two terms of relation (43) with a constant weight function have been taken

tions for $v \to \pm\infty$ which would not show physical behavior. Therefore, we may now write the general solution of the WD equation as a superposition of the above eigenfunctions

$$\Psi(u,v) = \int_{-\infty}^{+\infty} C(\nu)\Psi_\nu(u,v)d\nu, \tag{43}$$

where $C(\nu)$ can be chosen as a shifted Gaussian weight function $e^{-a(\nu-b)^2}$. The square of the wave functions for different cases are shown in Figs. 16 and 17 for de Sitter and dust cases respectively. In Fig. 16, the point Q is located at a local maximum probability so that PEP predicts a transition from point P to Q, i.e., $P \stackrel{PEP}{\longrightarrow} Q$. Physically, this transition causes u moving down and v moving up and after $P \stackrel{PEP}{\longrightarrow} Q$, since Q is located at a maximum, then $Q \stackrel{PEP}{\longrightarrow} Q$ which means the system is steady at Q. This behavior is exactly similar to the previous result which is represented in the left plot in Fig. 14. The same is true for the left plot in Fig. 14 and $P \stackrel{PEP}{\longrightarrow} Q$ transition in Fig. 17. Since point S is located at a maximum local probability, so that PEP predicts the transitions $R \stackrel{PEP}{\longrightarrow} S$ and then $S \stackrel{PEP}{\longrightarrow} S$. This means that there is the possibility that both u and v move down. This latter result is completely in agreement with the previous results, i.e., the case represented in Fig. 15.

There is a notable difference between the Bianchi model and models presented previously in that there exists more than one direction in the configuration space for the Bianchi model. This makes some ambiguities in the path of transition to more probable states in the PEP paradigm. But as has been discussed in [19], the path in these examples is on the gradient of $|\Psi|^2$ (hyper-)surface which has the steepest slope. This proposal results in the definition of a unique transition path in PEP for more complicated configuration spaces.

6 Discussion

In this paper we have argued, using familiar examples, that one can draw a completely equivalent physical interpretation for two different approaches to quantization, namely the usual canonical quantization method and the phase-space deformation method, at least within the framework of the examples used in this work. A by-product is that when one studies the deformation of phase-space, one should realize that this is effectively a quantization procedure and should refrain from using any other quantization method simultaneously. However, since in the phase-space deformation method for quantization the evolutionary parameter, t, does appear, in contrast to the canonical method, a mechanism for retrieving the dynamical information of the system, when canonically quantized, is at hand and can be used. This had been done in a previous work [19; 20]. The results show that if the equivalence of different methods of quantization discussed above are assumed, then the probabilistic evolutionary mechanism works consistently and therefore can be used to address the problem of evolution in diffeomorphism invariant theories.

However, such an equivalence cannot hold true in examples where the deformation in phase-space is introduced in a Lorentz non-invariant manner. This is not hard to predict since the WD equation is a direct result of diffeomorphism invariance and so if a deformation in phase-space breaks such an invariance then the results of different quantization methods should be different. In this connection we note that κ-Minkowski deformation and GUP, for example, preserve Lorentz invariance [54].

Another feature which is of interest is that the WD equation is usually a second order differential equation and has therefore two independent solutions. However, in the deformed phase-space quantum models, the equations of motion are first order, in contrast to the former. The disparity lies in the fact that, as discussed above, the parameter of deformation, ℓ, is arbitrary, and hence can be chosen to be either positive or negative. It has been shown that different signs of ℓ relate to different solutions of the WD equation.

It is worth mentioning that this paper can be read as a companion to our previous one [19]. In [19] we have discussed the PEP paradigm in detail. However, a brief mention of the key points of the theory would be helpful here. The PEP proposal has some roots in the second law of thermodynamics in that a system in an initial state evolves and approaches another state with a higher entropy, i.e., $S_f - S_i = \triangle S \geq 0$. A vital problem in PEP is the velocity of transition from an initial state to a more probable state. There is as yet no concrete answer to this question since the theory is still in its infancy. However, in a model discussed in [19] a relation for the velocity of transition is presented which is obtained by comparison of PEP's results with the Causal Dynamical Triangulation method as a model of quantum gravity.

Finally, it is important to note that if a proposal is shown to be correct in a few examples it does not necessarily guarantee its universal correctness and this is true in what we have presented here. However, we have shown that there is a correspondence between different methods of quantization. We suggest that this correspondence should be considered until a more complete understanding of the universe is achieved.[8] In the absence of a full theory, e.g., quantum gravity, all the meaningful approaches should be considered in the hope that a comprehensive theory eventually emerges. In this paper we have shown that two different methods of quantization would lead to the same results, at least in the specific examples presented, and that the simultaneous use of these methods must be avoided to prevent double quantization. This could lead to a new understanding of the quantization methods in quantum cosmology and eventually be instrumental in achieving a satisfactory theory of quantum gravity.

[8] Specially, in our case, a theory of quantum gravity or quantum cosmology.

References

1. B.S. DeWitt (1967) *Phys. Rev.* **160** 1113
2. C.W. Misner (1969) *Phys. Rev.* **186** 1319
3. A. Vilenkin (1982) *Phys. Lett. B* **117** 25
4. A. Vilenkin (1983) *Phys. Rev. D* **27** 2848
5. A. Vilenkin (1986) *Phys. Rev. D* **33** 3560
6. A. Vilenkin (1988) *Phys. Rev. D* **37** 888
7. A. Vilenkin (1994) *Phys. Rev. D* **50** 2581
8. J.B. Hartle S.W. Hawking (1983) *Phys. Rev. D* **28** 2960
9. S.W. Hawking (1984) *Nucl. Phys. B* **239** 257
10. J.J. Halliwell S.W. Hawking (1985) *Phys. Rev. D* **31** 1777
11. G.W. Gibbons J.B. Hartle (1990) *Phys. Rev. D* **42** 2458
12. Wiltshire, D.L.: An Introduction to Quantum Cosmology. arXiv: gr-qc/0101003
13. M.P. Ryan (1972) Hamiltonian Cosmology Springer Berlin
14. M.P. Ryan L.C. Shepley (1975) Homogeneous Relativistic Cosmologies Princeton University Press Princeton
15. Isham, C.J.: Canonical quantum gravity and the problem of time. In: Lectures Presented at NATO Advanced Study Institute, Salamanca (1992). arXiv: gr-qc/9210011
16. W.F. Blyth C.J. Isham (1975) *Phys. Rev. D* **11** 768
17. Vakili, B., Sepangi, H.R.: Ann. Phys. **323** (2008) 548. arXiv:0709.2988[gr-qc]
18. C.K. Zachos D.B. Fairlie T.L. Curtright eds (2005) Quantum Mechanics in Phase Space World Scientific Publishing Co. Singapore
19. Khosravi, N., Sepangi, H.R.: Phys. Lett. B **673**, 297 (2009). arXiv: 0903.1914
20. Khosravi, N., Jalalzadeh, S., Sepangi, H.R.: Gen. Relativ. Gravit. **39**, 899 (2007). arXiv: gr-qc/0702067
21. Khosravi, N., Sepangi, H.R.: J. Cosmol. Astropart. Phys. **JCAP04**, 011 (2008). arXiv: 0803.1714[gr-qc]
22. Khosravi, N., Sepangi, H.R.: Phys. Lett. A **372**, 3356 (2008). arXiv: 0802.0767 [gr-qc]
23. Halliwell, J.J.: The Interpretation of Quantum Cosmology and the Problem of Time. arXiv: gr-qc/0208018
24. Halliwell, J.J.: The Interpretation of Quantum Cosmological Models. arXiv: gr-qc/9208001
25. Garcia-Compean, H., Obregon, O., Ramirez, C.: Phys. Rev. Lett. **88**, 161301 (2002). arXiv: hep-th/0107250
26. Lopez-Dominguez, J.C., Obregon, O., Ramirez, C., Sabido, M.: Phys. Rev. D **74**, 084024 (2006). arXiv: hep-th/0607002
27. Barbosa, G.D., Pinto-Neto, N.: Phys. Rev. D **70**, 103512 (2004). arXiv: hep-th/0407111
28. Khosravi, N., Jalalzadeh, S., Sepangi, H.R.: J. High Energy Phys. **JHEP 01**, 134 (2006). arXiv: hep-th/0601116
29. Vakili, B., Khosravi, N., Sepangi, H.R.: Class. Quantum Grav. **24**, 931

(2007). arXiv: gr-qc/0701075
30. Garay, L.J.: Int. J. Mod. Phys. A **10**, 145 (1995). arXiv: gr-qc/9403008
31. A. Connes (1994) Noncommutative Geometry Academic Press New York
32. Connes, A.: J. Math. Phys. **41**, 3832 (2000). arXiv: hep-th/0003006
33. Majid, S.: Lect. Notes Phys. **541**, 227 (2000). arXiv: hep-th/0006166
34. Kempf, A., Mangano, G., Mann, R.B.: Phys. Rev. D **52**, 1108 (1995). arXiv: hep-th/9412167
35. Carmona, J.M., Cortes, J.L., Gamboa, J., Mendez, F.: J. High Energy Phys. **JHEP 03**, 058 (2003). arXiv: hep-th/0301248
36. Carmona, J.M., Cortes, J.L., Gamboa, J., Mendez, F.: Phys. Lett. B **565**, 222 (2003). arXiv: hep-th/0207158
37. Carroll, S.M., Harvey, J.A., Kostelesky, V.A., Lane, C.D., Okamoto, T.: Phys. Rev. Lett. **87**, 141601 (2001). arXiv: hep-th/0105082
38. Carlson, C.E., Carone, C.D., Lebed, R.F.: Phys. Lett. B **518**, 201 (2001). arXiv: hep-ph/0107291
39. Carlson, C.E., Carone, C.D., Lebed, R.F.: Phys. Lett. B **549**, 337 (2002). arXiv: hep-ph/0209077
40. Vakili, B., Sepangi, H.R.: Phys. Lett. B **651**, 79 (2007). arXiv: 0706.0273 [gr-qc]
41. Vakili, B.: Phys. Rev. D **77**, 044023 (2008). arXiv: 0801.2438[gr-qc]
42. Battisti, M.V., Montani, G.: Phys. Lett. B **656**, 96 (2007). arXiv: gr-qc/0703025
43. Battisti, M.V., Montani, G.: Phys. Rev. D **77**, 023518 (2008). 0707.2726 [gr-qc]
44. Sepangi, H.R., Shakerin, B., Vakili, B.: Class. Quantum Grav. **26**, 065003 (2009). arXiv: 0901.3829 [gr-qc]
45. Vakili, B.: Int. J. Mod. Phys. D **18**, 1059 (2009). arXiv: 0811.3481 [gr-qc]
46. Amelino-Camelia, G.: Int. J. Mod. Phys. D **11**, 1643 (2002). arXiv: gr-qc/0210063
47. Kowalski-Glikman, J.: Introduction to doubly special relativity. Lect. Notes Phys. **669**, 131 (2005). arXiv: hep-th/0405273
48. Amelino-Camelia, G.: Int. J. Mod. Phys. D **11**, 35 (2002). arXiv: gr-qc/0012051
49. Amelino-Camelia, G.: Phys. Lett. B **510**, 255 (2001). arXiv: hep-th/0012238
50. Magueijo, J., Smolin, L.: Phys. Rev. Lett. **88**, 190403 (2002). arXiv: hep-th/0112090
51. Kowalski-Glikman, J., Nowak, S.: Phys. Lett. B **539**, 126 (2002). arXiv: hep-th/0203040
52. J. Lukierski A. Nowicki H. Ruegg V.N. Tolstoy (1991) *Phys. Lett. B* **268** 331
53. Kowalski-Glikman, J.: Phys. Lett. A **299**, 454 (2002). arXiv: hep-th/0111110
54. Romero, J.M., Vergara, J.D., Santiago, J.A.: Phys. Rev. D **75**, 065008 (2007). arXiv: hep-th/0702113
55. J. Lukierski A. Nowicki H. Ruegg (1992) *Phys. Lett. B* **293** 344
56. Majid, S., Ruegg, H.: Phys. Lett. B **334**, 348 (1994). arXiv: hep-th/9405107
57. Freidel, L., Kowalski-Glikman, J., Nowak, S.: Phys. Lett. B **648**, 70 (2007).

arXiv: hep-th/0612170
58. Bruno, N.R., Amelino-Camelia, G., Kowalski-Glikman, J.: Phys. Lett. B **522**, 133 (2001). arXiv: hep-th/0107039
59. Khosravi, N., Sepangi, H.R., Sheikh-Jabbari, M.M.: Phys. Lett. B **647**, 219 (2007). arXiv: hep-th/0611236
60. Chaichian, M., Sheikh-Jabbari, M.M., Tureanu, A.: Phys. Rev. Lett. **86**, 2716 (2001). arXiv: hep-th/0010175
61. Nascimento, J.R., Petrov A.Y., Ribeiro, R.F.: Europhys. Lett. **77**, 51001 (2007). arXiv: hep-th/0601077