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Generalized Entanglement Entropy and Holography

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Abstract.

A nonextensive statistical mechanics entropy that depends only on the probability distribution is proposed in the framework of superstatistics. It is based on a $\Gamma(\chi^2)$ distribution that depends on β and also on p_l . The corresponding modified von Neumann entropy is constructed; it is shown that it can also be obtained from a generalized Replica trick. We address the question whether the generalized entanglement entropy can play a role in the gauge/gravity duality. We pay attention to 2dCFT and their gravity duals. The correction terms to the von Neumann entropy result more relevant than the usual UV (for c = 1) ones and also than those due to the area dependent AdS_3 entropy which result comparable to the UV ones. Then the correction terms due to the new entropy and the AdS entropy in a different manner than the UV ones or than the corrections to the AdS_3 area dependent entropy.

1. Introduction

Information entropies to quantify predictability in a process have been proposed in the literature [1–5]. The most notable of them is due to the seminal work of Shannon [3]. By maximizing the appropriate information measures [1] the associated probability distributions can be calculated. For some of these generalized information and entropy measures their potential physical applications have been discussed elsewhere [6].

By considering nonequilibrium systems with a long-term stationary state that possesses a spatio-temporal fluctuating intensive quantity more general statistics have been formulated known as superstatistics [7]. Selecting the temperature as the fluctuating quantity among other available intensive variables, the microscopic system is considered as made up of smaller cells that are temporarily in local equilibrium. Within each cell β , the inverse temperature, is approximately constant. Each cell is large enough to obey standard statistical mechanics but has a different temperature assigned to it, in accordance to a general distribution $f(\beta)$, from this distribution one can get an effective Boltzmann factor

$$B(E) = \int_0^\infty d\beta f(\beta) e^{-\beta E},\tag{1}$$

where E is the energy of a microstate associated with each of the considered cells. A distribution $f(\beta) = \delta(\beta - \beta_0)$ gives the ordinary Boltzmann factor. Other distributions, the $\Gamma(\chi^2)$, log-normal and F-distribution have been considered leading to their associated Boltzmann factors.

To deduce the entropies corresponding to these Boltzmann factors B(E) a formalism was introduced [8]. Following this procedure the Boltzmann-Gibbs entropy and the non-extensive

statistical mechanics entropy were obtained. The entropy is expressed as $S = k \sum_{k=1}^{\Omega} s(p_l)$ in terms of a generic s(x) that can be calculated, x is at this stage a parameter. By maximizing the appropriate information measure this parameter is identified with the probability distribution. For other distributions, among them, the log-normal and F-distribution it is not possible to get closed analytical expressions for their associated entropies and the calculations were performed numerically [8] by means of the corresponding B(E).

In [9] a $f(\beta)$, $\Gamma(\chi^2)$ distribution was proposed depending on a parameter p_l , from it the associated Boltzmann factor B(E) and entropy were calculated. The parameter p_l is then identified with the probability distribution by maximizing the appropriate information measure. The resulting generalized information entropy depends only on the probability distribution. This new entropy was further studied in [10]. In this paper, we will review some of our previous results in [9, 10]. We will also present some new results which correspond to work in progress with several collaborators and that will be further developed and presented elsewhere. First, in Section 2 several generalized, p_l dependent distributions and their associated Boltzmann factors are presented. It is shown that they coincide up to the first correction term in their expansions. The one corresponding to the $\Gamma(\chi^2)$ distribution can be obtained in an analytically closed form. In Section 3 we will obtain the entropy for the $\Gamma(\chi^2)$ distribution. We will show several aspects of this non-extensive generalized entropy that depends only on the probability.

We turn to quantum aspects in Section 4, and construct the corresponding generalization of the von Neumann entropy and exhibit an extended Replica trick from which this same entropy also arises. We address then the question whether an alternative entanglement entropy can play a role in the gauge/gravity duality. In particular, based on our generalized entropy we obtain correction terms to the usual 2dCFT entanglement entropy. These are exponentially suppressed, but are larger than the standard ultraviolet terms, that are also exponentially suppressed. We point out to the possible implications of this entropy in connection to the Ryu-Takayanagi identification between the CFT entanglement entropy and their proposed AdS_3 area dependent entropy.

2. Generalized distributions and their Boltzmann factors

We begin by assuming a Γ (or χ^2) distribution on the inverse temperature β , depending on a parameter p_l to be identified with the probability associated with the microscopic configuration of the system by means of maximizing the associated entropy. We may write this parameter p_l Γ distribution as

$$f_{p_l}(\beta) = \frac{1}{\beta_0 p_l \Gamma\left(\frac{1}{p_l}\right)} \left(\frac{\beta}{\beta_0} \frac{1}{p_l}\right)^{\frac{1-p_l}{p_l}} e^{-\beta/\beta_0 p_l},\tag{2}$$

where β_0 is the average inverse temperature. Integration over β yields the generalized Boltzmann factor

$$B_{p_l}(E) = (1 + p_l \beta_0 E)^{-\frac{1}{p_l}},\tag{3}$$

as shown in [10], this kind of expression can be expanded for small $p_l \beta_0 E$, to get

$$B_{p_l}(E) = e^{-\beta_0 E} \left[1 + \frac{1}{2} p_l \beta_0^2 E^2 - \frac{1}{3} p_l^2 \beta_0^3 E^3 + \dots \right].$$
(4)

We follow now the same procedure for the log-normal distribution, this can be written in terms of p_l as

$$f_{p_l}(\beta) = \frac{1}{\sqrt{2\pi}\beta[\ln(p_l+1)]^{1/2}} \exp\left\{-\frac{\left[\ln\frac{\beta(p_l+1)^{1/2}}{\beta_0}\right]^2}{2\ln(p_l+1)}\right\}.$$
(5)

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The generalized Boltzmann factor can be obtained in leading order, for small variance of the inverse temperature fluctuations,

$$B_{p_l}(E) = e^{-\beta_0 E} \left[1 + \frac{1}{2} p_l \beta_0^2 E^2 - \frac{1}{6} p_l^2 (p_l + 3) \beta_0^3 E^3 + \cdots \right].$$
(6)

In general, the *F*-distribution has two free constant parameters. We consider, the particular case in which one of these constant parameters is chosen as v = 4. For this value of this constant parameter we define a *F*-distribution in function of the inverse of the temperature and p_l as

$$f_{p_l}(\beta) = \frac{\Gamma\left(\frac{8p_l-1}{2p_l-1}\right)}{\Gamma\left(\left(\frac{4p_l+1}{2p_l-1}\right)} \frac{1}{\beta_0^2} \left(\frac{2p_l-1}{p_l+1}\right)^2 \frac{\beta}{\left(1+\frac{\beta}{\beta_0}\frac{2p_l-1}{p_l+1}\right)^{\left(\frac{8p_l-1}{2p_l-1}\right)}}.$$
(7)

Once more the associated Boltzmann factor can not be evaluated in a closed form, but for a small variation of the fluctuations we obtain the series expansion

$$B_{p_l}(E) = e^{-\beta_0 E} \left[1 + \frac{1}{2} p_l \beta_0^2 E^2 + \frac{1}{3} p_l \frac{(5p_l - 1)}{p_l - 2} \beta_0^3 E^3 + \dots \right].$$
(8)

As we will demonstrate and already shown in [9, 10] one can obtain in a closed form the entropy corresponding to Eqs. (2) and (3) resulting in

$$S = k \sum_{l=1}^{\Omega} (1 - p_l^{p_l}), \tag{9}$$

where k is the conventional constant and $\sum_{l=1}^{\Omega} p_l = 1$. The expansion of Eq. (9) gives

$$-\frac{S}{k} = \sum_{l=1}^{\Omega} p_l \ln p_l + \frac{(p_l \ln p_l)^2}{2!} + \frac{(p_l \ln p_l)^3}{3!} + \cdots .$$
(10)

Given that the Boltzmann factor coincides up to the second term for the $\Gamma(\chi^2)$, log-normal and *F*-distributions, for enough small $p_l \beta_0 E$ the entropy Eq. (10) corresponds to all these distributions up to the first term that modifies the usual Shannon entropy. We expect at least this modification to the entropy for several possible $f(\beta)$ distributions. So, for an stationary system with some temperature distribution in the mentioned sense we could describe the actual β -distributions by some of the ones considered here. As up to the first correction term to the standard Boltzmann factor is the same for several distributions, as shown, also the first correction term to the various corresponding entropies will be the same, namely the second term in Eq.(10).

3. Entropy from the Boltzmann factor

We begin by assuming the Γ (or χ^2) inverse temperature β distribution depending on a parameter p_l Eq. (2), to be identified with the probability associated with the microscopic configuration of the system. As shown, integration over β yields the generalized Boltzmann factor which can be expanded as in Eq. (4).

The examples studied in [7] have been nicely addressed in [8] in order to deduce the entropies from their corresponding Boltzmann factors. Another possible way to reconstruct entropies

depending on constant parameters has been proposed in [11–13], this approach provides other expressions. In [12,13] it has been shown that there exists a duality between these two procedures. We will restrict to the first proposal, by means of which the Boltzmann-Gibbs entropy and the non extensive statistical mechanics entropy can be obtained in a closed analytic form [8]. However, the entropies corresponding to the Boltzmann factors associated to the log-normal or the Fdistributions can not be obtained analytically and were calculated numerically. Following [7,8], we present the procedure to obtain the entropy corresponding to the $f(\beta)$ distribution Eq. (2) and to its associated generalized Boltzmann factor Eq. (3). We begin by defining the entropy $S = k \sum_{l=1}^{\Omega} s(p_l)$ in terms of a generic $s(p_l)$, where p_l can be considered at this stage an arbitrary parameter; for $f(\beta) = \delta(\beta - \beta_0) s(x) = -x \ln x$ the Shannon entropy is recovered. As shown in [8] it is possible to express s(x) and a generic internal energy u(x) in terms of integrals on a function E(y) that is obtained from the Boltzmann factor B(E) of interest. By these means s(x)and u(x) can be written as

$$s(x) = \int_0^x dy \frac{\delta + E(y)}{1 - E(y)/E^*} \quad , \tag{11}$$

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and

$$u(x) = (1 + \delta/E^*) \int_0^x \frac{dy}{1 - E(y)/E^*} , \qquad (12)$$

where E(y) is to be identified with the inverse function of $B_{p_l}(E) / \int_0^\infty dE' B_{p_l}(E')$. One selects first the $f(\beta)$ of interest, then B(E) is calculated and the integral $\int_0^\infty B(E') dE'$ is performed. Inverting the axes of the variables, E(y) for several superstatistics can be found [8], and from it E^* . In our case, the starting points are the distribution Eq. (2) and the Boltzmann factor Eq. (3). E(y) and E^* are given by

$$E(y) = \frac{y^{-x} - 1}{x} , \qquad (13)$$

$$E^* = -\frac{1}{x}.\tag{14}$$

A straightforward calculation gives for u(x) and s(x)

$$u(x) = x^{x+1},$$
(15)

$$s(x) = 1 - x^x. (16)$$

where δ has been determined by means of the condition u(1) = 1.

Expressions Eqs.(15) and (16) fulfill the expected conditions for the entropy and the energy s(0) = 0, u(0) = 0 and u(1) = 1, s(1) = 0. By these means, the entropy results in Eq.(9) and its expansion in Eq.(10).

Using these results, the corresponding functional including restrictions is given by

$$\Phi = \frac{S}{k} - \gamma \sum_{l=1}^{\Omega} p_l - \beta \sum_{l=1}^{\Omega} p_l^{p_{l+1}} E_l,$$
(17)

where the second restriction concerns the average value of the energy and γ and β are Lagrange parameters, and then by maximizing Φ , p_l is obtained as

$$1 + \ln p_l + \beta E_l (1 + p_l + p_l \ln p_l) = p_l^{-p_l}.$$
(18)

The dominant term in this expression corresponds to the Gibbs-Boltzmann prediction, $p_l = e^{-\beta_o E_l}$. In general, however, we cannot analytically express p_l as function of βE_l . In Fig. 1, p_l is given as a function of the reduced energy βE_l . We notice that for relative large values of βE_l the usual values for p_l coincide with the ones given by Eq. (18). As expected, they coincide also for $p_l \sim 1$.



Figure 1. (Color online) Comparison of the two probabilities. The blue dotted line corresponds to $p_l = f(\beta E_l)$, Eq. (18), and red dashed line to the standard one $p_l = e^{-\beta E_l}$

As we have shown, by choosing $f_{p_l}(\beta)$ Eq.(2), then $B_{p_l}(E)$ Eq. (3) is obtained by integrating over β ; by inverting the axes of the variable the inverse function E(y) Eq. (13) and E^* Eq. (14) can be found. This procedure has allowed us to calculate u(x) Eq. (15) and s(x) Eq. (16) and consequently the entropy Eqs.(9) and (16). If we assume in $f_{p_l}(\beta)$ Eq. (2) the equiprobable condition, $p_l = \frac{1}{\Omega}$, then the corresponding distribution is given by

$$f_{\Omega}(\beta) = \frac{\Omega}{\beta_0 \Gamma(\Omega)} \left(\frac{\beta \Omega}{\beta_0}\right)^{\Omega-1} e^{-\frac{\beta \Omega}{\beta_o}},\tag{19}$$

where the Boltzmann factor and the entropy are now

$$B_{\Omega}(E) = (1 + \beta_0 E/\Omega)^{-\Omega}, \qquad (20)$$

$$S = k\Omega \left[1 - \frac{1}{\Omega^{\frac{1}{\Omega}}} \right], \tag{21}$$

or, in terms of Boltzmann's entropy, $S_B = k \ln \Omega$,

$$\frac{S}{k} = \frac{S_B}{k} - \frac{1}{2!} e^{-S_B/k} \left(\frac{S_B}{k}\right)^2 + \frac{1}{3!} e^{-\frac{2S_B}{k}} \left(\frac{S_B}{k}\right)^3 \cdots .$$
(22)



Figure 2. (Color online) The entropies as function of Ω (small)the blue dashed and red dotted lines correspond to $\frac{S}{k}$, Eq.(21) and, $\frac{S_{\beta}}{k}$, respectively ($p_l = 1/\Omega$ equipartition)



Figure 3. (Color online) The entropies as function of Ω (large) the blue dashed and red dotted lines correspond to $\frac{S}{k}$, Eq. (21) and $\frac{S_{\beta}}{k}$, respectively ($p_l = 1/\Omega$ equipartition)

Figs. 2 and 3 show the Boltzmann entropy $\frac{S_B}{k}$ and the entropy $\frac{S}{k}$ given by expression Eq. (21). As mentioned in the Introduction, it was shown in [9], in relation with the entropy of a black hole, that if we associate its entropy, depending linearly on its area, with $\frac{S_B}{k}$ the standard entropy, then the entropy $\frac{S}{k}$ will be given as a function of the area by means of Eqs. (21) and (22). This would imply a modification to Newton's law and to general relativity according with the possibility that gravity could be thought as an equation of state [14]; explained as an entropic force [15–17]. However, for most gravitational systems one expects a large Ω , namely large $\frac{S_B}{k}$ and will not greatly differ with $\frac{S}{k}$, Eq. (21) and Fig. 3.

We notice that in the range of low values of Ω the entropies $\frac{S_{\beta}}{k}$ and $\frac{S}{k}$ differ. Instead, for large Ω the two entropies essentially coincide. Since Ω is basically a measure of the phase space volume, what we are finding is that for systems with reduced number of microstates the entropies become different, whereas for the opposite case they will be essentially identical. Then, we could conclude from here that the model could be useful when we have a clear indication of restriction of available states, like by strong confinement of fluids or low temperatures.

The entropy derived in [9,10] and whose main properties we review and extended here, Eqs.(9) and (21), has as an important feature to be independent of any arbitrary constant parameter,

and to depend only on the probability distribution p_l Eq.(9), associated with the microscopic configuration of the system. Its expansion provides as a first term Shannon's entropy Eq.(10) and correspondingly Boltzmann's entropy Eq.(22). This entropy corresponds to the Γ distribution Eqs.(2) and (3).

4. Holography and the generalized entanglement entropy

A quantum many body system in one dimension and at criticality corresponding to a 2dCFT has an entanglement entropy [18] given by

$$S_A = \frac{c}{3} \ln\left(\frac{L}{\pi a} \sin\frac{\pi l}{L}\right),\tag{23}$$

where l, is the length of the subsystem, A and L the length of the total system AB, both ends of AUB are periodically identified, a is a UV cutoff, the lattice spacing, is the central charge c of the 2dCFT. Away from criticality the previous expression [18] is replaced by

$$S_A = \frac{1}{3} \ln \frac{l}{a},\tag{24}$$

where for simplicity we have putted c = 1. The UV correction terms to (24) have been calculated [19] and can be written as

$$S = \frac{1}{3} \ln \frac{l}{a} - 2\left(1 + \ln\left(\frac{a}{l}\right)\frac{\partial}{\partial\ln\left(\frac{a}{l}\right)}\right) \sum_{k=1}^{\infty} \ln\left(1 - \left(\frac{a}{l}\right)^{2k}\right),\tag{25}$$

in terms of S_A and for $\frac{a}{I} \ll 1$,

$$S \sim S_A - 12S_A e^{-6S_A} + 2e^{-6S_A} + \cdots,$$
 (26)

the correction terms are exponentially suppressed. The entropy S_A corresponds to the most relevant term, the von Neumann entropy for the 2dCFT under consideration.

The entropy in AdS_3 is proportional to a minimal surface [20]. In this case the length of an static geodesic determines its value. A geodesic going from radius ρ_0 and $\theta = 0$ to radius ρ_0 and $\theta = \frac{\pi l}{L}$ has a length given by

$$L_{\gamma_A} = Rarccosh \left[1 - sin^2 \left(\frac{l\pi}{L} \right) + \frac{1}{2} e^{2\rho_0} sin^2 \left(\frac{l\pi}{L} \right) + \frac{1}{2} e^{-2\rho_0} sin^2 \left(\frac{l\pi}{L} \right) \right].$$
(27)

For enough large ρ_0 and identifying

$$e^{\rho_0} \equiv \frac{L}{\pi a}, \qquad \frac{R}{4G} \equiv \frac{c}{6}.$$
 (28)

It was shown that the length Eq.(27) provides an area entropy dependent that coincides with Eq.(24). Further terms can be calculated from the length Eq.(27), these result in the correction terms to the AdS_3 entropy. It can be shown that they are proportional to the same negative exponentials than the 2*dCFT* entropy *UV* correction terms Eq.(26).

We turn now to the generalization of the von Neumann entropy that corresponds to the proposed generalized Shannon entropy Eq.(9), namely

$$S_{+} = T_{r}(1 - \rho^{\rho}), \tag{29}$$

in which ρ is the density matrix. This same entropy arises also from the following generalized Replica trick that we propose [21,22]

$$S_{+} = -\sum_{k\geq 1} \frac{1}{k!} \lim_{n \to k} \frac{\partial^{k}}{\partial n^{k}} Tr \rho_{A}^{n}.$$
(30)

Making n-copies of the system is possible to compute $Tr\rho_4^n$ for n a positive integer. The partition function of the system with this n-sheeted structure reads [18] $Z_n(A)$ with

$$Tr\rho_A^n = \frac{Z_n(A)}{Z^n}.$$
(31)

For all the examples of 2dCFT using the transformation property of the two-point functions when the lattice spacing is very small one has a generic expression

$$Tr\rho_A^n \sim b^{(l/n-n)/6}.$$
(32)

We show here the simplest case, for $b = \frac{l}{a}$ and then the standard Replica trick, namely

$$S_A = -\frac{\partial}{\partial n} Tr \rho_A^n |_{n=1} \tag{33}$$

reproduces the entropy S_A Eq.(24).

Further terms in the entropy Eq.(29), are given by the second and third derivatives in Eq.(30) and the S_{\pm} entropy results in

$$S_{+} = S_{A} + \frac{1}{16}e^{-\frac{3}{4}S_{0}}S_{0}\left(1 + \frac{25}{8}S_{0}\right) - \frac{1}{6}S_{0}e^{-4S_{0}/3}\left(\frac{1}{27} + \frac{5}{181}S_{0} + \frac{125}{729}S_{0}^{3}\right) + \cdots$$
(34)

We notice that the correction terms to S_A due to the proposed generalized entanglement entropy Eq.(29) are also exponentially suppressed. They are however larger than the usual ultraviolet corrections to the 2dCFT entanglement entropy Eq.(26) and, as mentioned, consequently also larger than the correction entropy terms to the AdS_3 space. The proposed entanglement entropy Eq.(29) has its origin in other considerations Eqs.(9) and (30). It seems of interest to search for an entropy for the AdS_3 space that would correspond to the entropy S_+ Eq.(29). This is not a straightforward task one would need to find a well justified and appropriate generalization of the AdS_3 space entropy based on a possibly different relation with the geodesical length Eq.(27) and probably then another theory of gravitation would be needed. These matters require further study and are beyond the scope of this work.

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