Distinguishing general relativity and modified theories of gravity using quasi-normal modes

A thesis submitted for the degree of

Doctor of Philosophy

in the

School of Physics

by

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July 2019

AUTHOR'S DECLARATION

I hereby declare that the work reported in this thesis is original and has been carried out by me during my tenure as a PhD student at **School of Physics, Indian Institute of Science Education and Research Thiruvananthapuram (IISER-TVM)** under the supervision of **Dr. Sreedhar Dutta**. This work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution/ University. Material obtained from other sources has been duly acknowledged in the thesis. This work has been checked for plagiarism by the Central Library, IISER Thiruvananthapuram.

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“Don’t panic!”

— Douglas Adams, The Hitchhiker’s Guide to the Galaxy

CERTIFICATE

This is to certify that the work contained in this thesis entitled **"Distinguishing general relativity and modified theories of gravity using quasi-normal modes"** submitted by **Soham Bhattacharyya** (**Roll No: IPHD13013**) to **School of Physics, Indian Institute of Science Education and Research Thiruvananthapuram** towards the requirement of **Doctor of Philosophy** in **Physics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree. This work has been checked for plagiarism by the Central Library, IISER Thiruvananthapuram.

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Date: December 30, 2019 — Doctoral Advisor

Dedicated to the memory of all the giants whose collective shoulders I stand on.

ACKNOWLEDGMENTS

I would like to thank the Ministry of Human Resource Development (MHRD), Government of India for the fellowship which supported this work. I would like to thank the School of Physics, IISER Thiruvananthapuram for providing me with the research facilities. I would like to thank all the faculty members, students, and staff of the School of Physics, IISER Thiruvananthapuram for their continual support throughout the course of this journey. I would also like to thank P Ajith's group at ICTS Bangalore, Bruce Allen and Badri Krishnan's group at AEI Hannover, and S Shankaranarayanan's group at Department of Physics, IIT Bombay for hospitality. I have used Kasper Peeter's symbolic algebra system Cadabra [1] extensively for calculations. Finally, I would like to thank my thesis advisor Dr. Sreedhar Dutta and the members of my doctoral committee for their support.

ABSTRACT

Quasi-Normal Modes, or the mathematical description of gravitational waves emitted during the ring-down of a perturbed black hole, provide critical information about the structure of these compact objects. Since regions around Black Holes have some of the strongest gravitational fields in the Universe, hence, Quasi-Normal Modes can be a tool for strong field tests of General Relativity and possible deviations from it. In the case of General Relativity, it is known for a long time that a relation between two types of Black Hole perturbations: even parity (Zerilli) and odd parity (Regge-Wheeler), leads to an equality of reflection coefficients for both parities. With the direct detection of Gravitational waves, it is now natural to ask: whether the same relation (between even and odd parity perturbations) holds for modified gravity theories? If not, whether one can use this as a way to probe deviations from General Relativity. As a first step, this thesis shows explicitly that the above relation between Regge-Wheeler and Zerilli potentials break down for modifications to gravity, and hence the two perturbations do not share equality of reflection coefficients. This thesis also discusses the implication of this inequality on the gravitational wave observations.

Contents

List of Figures

Synopsis of the thesis

The General theory of Relativity (GR) predicts the motion and dynamics of astronomical bodies to accuracies detectable by current experimental standards — from the perihelion precession of Mercury, to the detection of gravitational waves, and direct imaging of black holes. Tests of GR, as well as estimating physically relevant parameters about the source objects from gravitational waves involve matching of the detected signal to a template bank of numerically simulated waveforms. However, since the direct detection of gravitational waves, the question of whether nature follows GR or a more generalized, modified theory of gravity, whose low energy limit is GR, has gained interest. Various methods have been employed in the extraction of information from the detected signals, which involve searching for dispersion of the waves in vacuum, and the mass of the graviton. However, many of these methods are not model independent. Hence, finding model independent methods to constrain deviation from GR is the need of the hour. In this thesis, a model independent parameter is found, which can act as a quantifier to distinguish GR from a modified gravity theory.

It has been shown in the thesis that based on what can be obtained through gravitational wave observations, modified theories of gravity can be classified into two types—parity conserving and parity violating. The above two types of modifications leave unique signatures to emitted gravitational waves by perturbed black holes. Hence, the current thesis take two modified gravity theories, $f(R)$ and Chern-Simons modifications to GR , which are parity conserving and parity violating theories, respectively. Perturbation studies about spherically symmetric black hole solutions of said theories are performed and the specific signatures which they impart to emitted gravitational waves are extracted. A strategy to detect the above mentioned signatures from gravitational wave observation by earth or space based detectors is also outlined.

Certain theoretical problems faced by GR in the form of non-renormalizability and singularity formation at the heart of black holes have inspired various authors to propose generalizations to GR. These generalizations are usually motivated from string theories and loop quantum gravity. Hence, they are low energy modification to high energy theories which give GR in the weak field limit, or in the limit of currently possible observational precision. Some generalizations contain all solutions of GR and some exclude certain GR solutions. In Chapter 1, various problems plagued by GR is discussed and attempts at possible solutions outlined which involve description of two different types of generalizations to GR, namely $f(R)$ gravity and Chern-Simons modification to GR.

In general relativity perturbation about a background solution leads to excitation of the system which relaxes by emitting gravitational waves, whose (complex) frequencies do not depend on the details of the process which caused the perturbation, but only on the parameters of the background black hole space-time — mass, charge, and angular momentum. The mathematical description of the gravitational waves, which get radiated due a perturbed black hole relaxing, are given by Quasi-normal modes, which has been reviewed in Chapter 2. It is known in the literature that perturbation about a spherically symmetric black hole in GR consists of two decoupled parts, corresponding to two opposite parities — even and odd. It was shown by Chandrasekhar that given the differential equations of the two kinds of perturbations follow the same boundary conditions, purely ingoing at the horizon and outgoing at infinity, both parities shall have the same set of quasi-normal frequencies. Both parities also share equality of the fraction of the initial perturbation energy that gets radiated to asymptotic infinity. This particular relationship between the two decoupled modes is a fragile balance known as *isospectral* equality in literature.

Chapter 2 also deals with the review of perturbation techniques of charged black hole space-times in GR. It was shown by Gunter (1980) that a purely gravitational perturbation of a charged black hole space-time leads to the emission of both gravitational and electromagnetic waves. It has also been shown that the equality of the fraction of the initial perturbation energy that travels to asymptotic infinity through the odd and even parity is broken in the presence of charge, while the set of quasi-normal frequencies remaining the same for both parities. Based on the published results of the author of this thesis, a dimensionless parameter was defined which will have a specific value in the absence of charge (which can be obtained from numerical relativity simulations) and a different value in the presence of it. Observed value of the dimensionless parameter can confirm or infirm the presence of charge in astrophysical black holes.

$f(R)$ theories of gravity contain all stable black hole solutions of GR. However, for the same background solution, the perturbed space-time in GR and $f(R)$ gravity behaves differently. Owing to a massive scalar degree of freedom intrinsic to the graviton, and its preferential coupling to one of the parities, leaving the other untouched, a distinct signature is imparted on the gravitational waves emitted during ring-down, breaking isospectrality, as shown in Chapter 3 for a Schwarzschild solution in $f(R)$ gravity. The breaking of the isospectral equality leads to quasi-normal frequencies of the two parities to differ, as well as modifying the fraction of the initial perturbation energy that gets radiated in the form of gravitational waves to asymptotic infinity. These phenomena leave distinct *signatures* on the emitted gravitational waves which is possible to be observed by future generations of gravitational wave detectors with better sensitivity in the ring-down regime. A dimensionless parameter corresponding to the relative radiated intensities through the odd and even parity perturbations is defined, whose observed value will be different, if there exists a modification to gravity, from its corresponding value in GR (obtained from numerical simulations).

Chapter 4 deals with a perturbed Reissner-Nördstrom (also a solution of $f(R)$ gravity) background space-time. It is shown that the iso-spectral equality is broken and a further decrease in the fraction of the perturbation energy that is radiated to asymptotic infinity takes place, This leads to a change in the observed value of the dimensionless parameter (if nature follows a modified theory of gravity) from the corresponding GR value.

Modifying gravity in a parity violating manner by adding a pseudo-scalar field in the action coupling non-minimally with the background curvature was done by Jackiw and Pi (2003), known as the canonical Chern-Simons modification to GR. Certain theoretical problems are faced by the canonical model with respect to black hole perturbations. However, the problems can be avoided if a generalization of the canonical to the dynamical Chern-Simons modification to GR is done. A comparison of the two models with respect to perturbations about a Schwarzschild background was done in Chapter 5, and it was shown that the dynamical theory is more physical than the canonical one. Based on published results by the author of the current thesis, it is shown that the perturbed parity violating scalar field preferentially couples to one of the parities, which is different from $f(R)$, and modifies its dynamics, leaving the other parity untouched. Similar to the analysis of previous chapters, a dimensionless parameter, quantifying the relative radiated gravitational energies through the odd and even channels, was defined. The dimensionless parameter, in dynamical Chern-Simons modified gravity, shows a change from its corresponding GR value that is different from the change as shown by $f(R)$ gravity. A method to calculate the dimensionless parameter in terms of observables at gravitational wave detectors was outlined.

Chapter 6 summarizes all the results from the preceding chapters and motivates an argument to treat the dimensionless parameter defined in previous chapters to be a model independent one. The dimensionless parameter is also shown to classify a modification to gravity into parity conserving and parity violating, and hence, any deviation of the observed value of the parameter, from its GR value, will indicate whether any observed effective modification to GR violates or conserve parity.

List of publications

1. S. Bhattacharyya and S. Shankaranarayanan, "'Quasinormal modes as a distinguisher between general relativity and f(R) gravity", *Phys. Rev. D* **96** (2017) 06044 [arXiv:1704.07044].

2. S. Bhattacharyya and S. Shankaranarayanan, "Quasinormal modes as a distinguisher between general relativity and f(R) gravity: Charged black-holes", *Eur. Phys. J. C 78 (9), 737 (2018)* **78 (9)** (2018) 737 [arXiv:1803.07576].

3. S. Bhattacharyya and S. Shankaranarayanan, "Distinguishing general relativity from Chern-Simons gravity using gravitational wave polarizations", [arXiv:1812.00187] *Accepted for publication in Phys. Rev. D*.

Chapter 1

Introduction

Predicting the motion of heavenly bodies have always been a goal for the human civilization. The remarkable periodicity that is shown by such bodies have helped us to keep track of time and seasons. But a mathematical description of the motion of these bodies was lacking until 1686. Newton's gravity was the first mathematical framework to observe and predict the motions of planets around the sun. A tantalizing hint of a baffling object (that forms the base of this thesis) comes from one of the predictions of Newton's gravity itself; a dark star, by a geologist called John Michell in 1783. A star so dense that not even light cannot escape its gravity. This was centuries before Einstein came up with his General theory of Relativity (GR) [2] and subsequent works [3–7] formulated the 'black hole' (BH) solutions that we are familiar with today.

Such dark stars or black holes were indirectly observed while looking at the trajectories of stellar bodies around the center of the milky way galaxy [8]. From the motion of these objects, an estimate of the central potential was made, and consequently, the mass of the central object as well as the size of the region in which it is confined could be estimated. It was found that an enormously dense object sits at the heart of the milky way galaxy, which according to GR could only be a black hole. This was followed by a relatively recent direct observation and imaging of a black hole [9]. These BHs can be used as natural laboratories to test the degree of validity of GR and also constrain deviations from it.

In this chapter a brief introduction to GR is given and its predictions & their corresponding observational successes are listed. This is followed by some of the theoretical and observational problems that GR faces, as well as motivations for developing modified theories of gravity. Classification of various modified theories of gravity are given, out of which two different theories coming from different motivations are introduced that are relevant for the rest of the thesis. In this thesis, $c = G = 1$ (Geometric units) is set, and the metric signature $(-, +, +, +)$ is used.

1.1 General theory of relativity

1.1.1 Brief introduction to GR

Newton treats gravity as an instantaneous force. General relativity (GR) builds on the concept of causality from special theory of relativity and treats gravity, like electrodynamics, as a field with a finite speed of propagation [10, 11]. In GR, the concept of a three dimensional space is replaced with a four dimensional space-time whose local geometry quantifies the gravitational 'force'. In the absence of matter, the geometry of space-time, and consequently the gravitational strength, is characterized by tensor quantity known as the space-time metric $g_{\mu\nu}$ [12]. This tensor quantifies length between two infinitesimally separated points in space as well as the rates at which time flows in a region of space. Specifically, one constructs an infinitesimal scalar invariant 'measure' in 4D space-time known as the infinitesimal line element ds

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

where dx^μ is an infinitesimal space-time interval — a vector whose components are infinitesimal time and distances respectively. dx^μ is defined around a space-time point (or event) x^μ, which is used as the coordinate for the space-time. GR is a coordinate independent theory, and hence an infinitesimal change of coordinate

$$x^\mu \quad\to\quad x'^\mu = x^\mu + \xi^\mu \tag{1.2}$$

keeps the line element (1.1) invariant.

$g_{\mu\nu}$ in general is a function of both space and time and follows a set of second order quasi-linear differential equations known as field equations which can be derived from the celebrated action principle [13]:

$$\mathcal{S} \;=\; \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R \tag{1.3}$$

where R is referred as the Ricci scalar which is a function of $g_{\mu\nu}$ and its (second) derivatives. Extremizing the action leads to the following equations of motion for $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\; 0 \tag{1.4}$$

where $R_{\mu\nu}$ is a tensor which is a function of $g_{\mu\nu}$ and its (second) derivatives and $R = g^{\mu\nu}R_{\mu\nu}$. Both the action and the equations of motion remain invariant under coordinate transformations, although the individual components of the metric tensors change - a feature that will be utilized in Chapter 2.

The set of differential equations (1.4) can be solved for the components of $g_{\mu\nu}$ under certain physically simplifying assumptions like spherical or axial symmetry. Since BHs are the simplest objects in GR, under the assumptions of spherical symmetry, the Schwarzschild solution was the first exact solution that was obtained [3]. It is given by the infinitesimal line element

$$ds^2 \;=\; -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega^2 \tag{1.5}$$

where M is the quasi-local mass of the black hole space-time and $d\Omega$ is the infinitesimal solid angle. (1.5) is not the solution for the entire space-time, but only from $r = 2M$ to ∞. This is because of a special surface at $r = 2M = r_H$ (also known as the event horizon or simply horizon) where the inner causal boundary of the space-time lies, the outer being at ∞. It is a surface of infinite redshift for light and a surface of no return for particles. Curiously, the outer vacuum space-time of any star can be approximated by (1.5). Although the inner solution ($r < r_H$) can technically be solved for, it is plagued by certain pathologies like space-time singularities, which will be discussed later in this chapter.

Motion of particles and light rays in different space-times can be obtained by minimizing the action whose lagrangian density is the line element [14]

$$\mathcal{S} = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda \tag{1.6}$$

where λ is an affine parameter which parametrizes the geodesic, λ can also be used as the proper time as measured by the photon or the particle along the geodesic. Sign of the line element determines whether one is considering light or particles. For particles, $ds < 0$ (also known as time-like geodesics); whereas for light $ds = 0$ (also known as light-like or null geodesic). The equations of motion of various geodesics, after minimizing (1.6) becomes

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{1.7}$$

where $\Gamma^\mu_{\alpha\beta}$ is known as the Christoffel symbol. Another surface of interest can be found by solving (1.7) for photons, and is known as the photon sphere [14]. Orbits on the photon sphere lie at the maxima of a potential, and hence, are in an unstable equilibrium. Any geodesic (or a freely falling particle) crossing the photon sphere will inevitably cross the horizon, and hence there are no stable orbit, light or massive, on or inside the photon radius r_P. For Schwarzschild, $r_P = \frac{3r_H}{2}$. This particular radial distance from the black hole will be of consequence in the later chapters of the thesis, even more important than the horizon itself.

Axisymmetric vacuum solutions like Kerr are the most abundant BHs in the universe. However, to obtain the difference in effects of modified gravity theories from GR, this thesis only considers spherically symmetric BHs in this thesis.

In the presence of matter, the space-time geometry around the matter distribution curves, and test particles around the distribution follow the path of least distance on a curved hypersurface, instead of straight line. However, the presence of matter is not the only condition for space-time to curve. The region outside a static black hole, being devoid of matter, has a curved geometry. Other instances of a curved vacuum space-time is a Universe with a non-zero background curvature.

In the presence of matter, the action (1.3) is generalized to [14]

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} + \mathcal{L}_m\right) \tag{1.8}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu} \tag{1.9}$$

$$\kappa^2 = \frac{8\pi G}{c^4} \tag{1.10}$$

where $\mathcal{L}_m$ and $T_{\mu\nu}$ are the matter Lagrangian density and the matter energy-momentum tensor respectively. Solutions of (1.9) describes the dynamics of the space-time and the matter is subject to the constraint

$$\nabla^\mu T_{\mu\nu} = 0 \tag{1.11}$$

where ∇_μ is the covariant derivative. The simplest non-vacuum solution is the charged BH, known as the Reissner-Nördstrom space-time, where an electromagnetic field exists in the space-time. The action is given by [4–6, 14]

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\right) \tag{1.12}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and μ_0 is the magnetic permeability of vacuum. For a BH with central mass (M) and charge (Q), the line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega^2. \tag{1.13}$$

Although the long standing assumption is that charged BHs quickly lose their charge through accretion of the opposite charge, the assumption or the rate at which the BHs lose charge can be tested and will be discussed in Chapter 2.

1.1.2 Predictions and observational successes GR

GR, from the beginning of its conception has been a phenomenal success in its power of prediction and standing the test of time. It is a theory that is based on certain principles and makes certain predictions, and tests have been done since its inception to probe the validity of both the principles and the predictions. In the following table a (non-exhaustive) list of the tests of the principles and predictions are mentioned [15]

Principles/Predictions	Observational tests
The Weak Equivalence Principle: All freely falling test particles fall at the same rate irrespective of their internal composition.	[16], [17]
Local Lorentz invariance (principle): Non-gravitational physical laws are independent of the velocity of the freely falling frame in which they are described.	[18], [19], [20]
Local position invariance (principle): Non-gravitational physical laws are independent of the position in space or time of the freely falling frame in which they are described.	[21], [22–24], [25]

Gravitational deflection of light (prediction): A ray of light passing by a massive object will bend towards the object, changing the apparent position of the source from which it was emitted.	[26], [27, 28]
Perihelion precession of Mercury (prediction): Unexplainable by Newtonian gravity, the solar contribution to the slow perihelion shift of mercury at 43 arcseconds per century was correctly predicted by GR [29].	Observation preceded prediction when Le Verrier in 1859 found the perihelion shift discrepancy even after factoring in perturbing effects from other planets. Most recent observation include Messenger spacecraft observations [30].
Lense-Thirring effect (prediction): A rotating massive body will drag inertial frames around its vicinity along the direction of rotation, leading to a precession in a gyroscope's spin if it is not parallel to the angular momentum of the rotating body.	[31] [32], [33], [34].
Gravitational redshift (prediction): An observer located at a distance from a massive luminous body will see the light from the body to be redshifted compared to an observer located closer to the surface of the body.	[35], [36], [37]

Gravitational waves (prediction): Mass distributions with time varying Quadrupole moments and higher will shed orbital/rotational energy by radiating gravitational waves.	Indirect observation through loss of orbital energy: [38, 39]. Direct detection in binary BH and binary neutron star collisions [40, 41]
Existence of BHs (predicted): A profound prediction of GR is the existence of a BH, and consequently, the presence of a causal boundary - horizon of a BH, a central dark region surrounded by a light ring.	Indirect confirmation through the frequency evolution data of a detected inspiral-merger process. The sharp drop in the chirp signal [40] after a certain frequency is reached is indicative of the fact that the final state can be very well described by a static black hole solution of GR. Direct imaging through the Event Horizon telescope of the central BH of neighboring galaxy M87 [42–47]

1.2 Problems with GR

In spite of all its successes GR is plagued by some fundamental problems, the most important of which, in the context of this thesis, is the existence of space-time singularities at the center of BHs. As mentioned earlier, an isolated BH in GR is characterized by a metric (like (1.5), consisting of an *outer* and *inner* space-time, separated by a one way surface known as the event horizon. The inner space-time of a black hole is causally disconnected from the outer space-time. A black hole is formed when the gravitational effects become larger than the fermionic pressure of a matter distribution, i.e. when the density of some localized matter distribution is high enough - stars burn out their fuel and collapse upon themselves. When the matter density is high enough, a trapped surface forms which causally disconnects the dense matter from the outer space-time - a surface which in the (infinite) future turns out to be the event horizon. GR predicts that the dense matter inside the trapped surface will keep on collapsing on itself, eventually forming a singularity, with infinite matter density. Other quantities like the space-time curvature, characterizing the 'gravitational pull', also become infinite. This can be seen in the behavior of the Kretschmann scalar, constructed out of the self contraction of the Riemann tensor ($R_{\mu\nu\rho\sigma}$). In a Schwarzschild space-time, for example,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \tag{1.14}$$

which indicates the rate at which the curvature shoots up to infinity as one approaches the center of a black hole. However, from the history of theorizing about the natural world, we know that singularities or infinities in a theory are pathologies which indicates that the theory stops being valid in the regions where such singularities form. This means that GR cannot be a complete description of space-time in and around large and dense objects in nature.

The aforementioned problem is one of the predictions that can never be directly observed, given the causal disconnect between the inner and outer regions. But a possible resolution of the above can have potential observable signatures. From current trends in observations, it is expected that a deviation from general relativity can occur in regions of intense gravitational fields - regions of high spatial curvatures.

A closely related problem is the non-renormalizability of gravity, i.e. the GR action (1.8) is non-renormalizable [48, 49]. Mathematically it implies that the procedure of eliminating infinities in field theoretic calculations while trying to calculate physical quantities break down, which physically imply that the theory shows ultraviolet divergences. In other words, non-renormalizability leads to the theory predicting singular short ranged behavior, which is not the case that observations tell us. Hence, short ranged limit of GR does not lead to another well established theory of short ranges - quantum field theory. This problem of reconciling the theory of large distances with the small lead to string theory and loop quantum gravity. Low energy limits of both type of theories lead to corrections to the Einstein-Hilbert action, which are expected to show up in regions of strong gravitational fields that have either not been probed, or current observational sensitivities are not enough to pick up deviations from the predicted behavior.

1.3 Modified theories of gravity

1.3.1 Classification

Recent attempts at modifying GR to address its fundamental shortcomings can be broadly of two types:

- Local theories of gravity
- Non-local theories of gravity

Broadly, non-local theories of gravity [50] tries to unify the non-local nature of quantum fluctuations with the local-ness of GR using a quantum effective action [51, 52], and are aimed at curing the bad behavior of ultraviolet (short-distance) limit of GR, as well as its infrared (long-distance) effects in cosmology like problems of dark energy [53], flattening of galaxy rotation curves [54, 55], etc. Non-local theories put forth certain possible resolution of these fundamental problems at the compromise of causality and the principle of equivalence. However, such theories are beyond the scope of the present thesis and will be ignored henceforth.

Another possible classification is

- Four dimensional theories
- Higher dimensional theories

Both local and non-local theories have been formulated in both of these regimes. The simplest higher dimensional modification is the higher dimensional GR ($n > 4$), where a radial and time-like direction are embedded in an $(n-2)$-sphere [56–58]. However, gravitational waves in such a theory will have a falloff that is faster than r^{-2} as well as other signatures that have not been seen in observations [59]. The current thesis will only deal with space-times of four dimensions.

Four dimensional local modifications to GR can be listed into the following non-exhaustive list

- Scalar-tensor [60], vector-tensor [61], scalar-vector-tensor [62] theories.
- Higher (> 2) derivative gravity theories: Lovelock [63], [64].
- Relativistic Modified Newtonian Dynamics (MOND) [65].
- Theories with more than one tensor: [66–68].
- Parity violating theories of gravity: Chern-Simons modification to general relativity [69, 70].

1.3.2 Motivations for modifications

Linearization and gauge fixing of (1.9) reveals that the gravitational radiation contains only two massless spin-2 degrees of freedom. However, in order that gravity be renormalizable, the action must contain terms comprised of higher powers of curvature objects [71–73], leading to equations of motion that contains more than two derivatives of the metric tensor. These extended theories have a low frequency GR limit, but at high frequency can be radically different from GR. The inclusion of such terms lead to a softening of the divergence of quantities near the singularity, while an infinite series comprising of higher powers lead to renormalizability. However, since the current context deals with observations, in the low energy limit, the modified action can be written as [74], after excluding $\mathcal{O}\left(R^3\right)$ terms

$$\mathcal{S}_{mod} = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa^2}\left(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + \mathcal{L}_M\right] \tag{1.15}$$

whose field equations are quasi-linear but now has four derivatives of the metric tensor. References [71, 72] find that the linearized radiation corresponding to (1.15) has, in addition to massless spin-2 degrees of freedom, massive spin-2 and spin-0 degrees of freedom. A curious feature/pathology of (1.15) is that the linearized energy of the massive spin-2 field is unbounded from below, implying that the mode is unstable. Such a feature is referred to as a ghost in literature. As a low energy limit of high energy modifications, a simplifying assumption can get rid of the ghost, and forms an important part of this thesis.

1.3.3 $f(R)$ theories

If one restricts the modification to gravity in the low energy limit just to the αR^2 term, one does away with the ghost degree of freedom. Hence

$$\mathcal{S}_{mod} = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa^2} \left(R + \alpha R^2 \right) + \mathcal{L}_M \right] \tag{1.16}$$

which has, in addition to the two massless spin-2 degrees of freedom, one additional massive scalar degree of freedom. The presence of the extra massive degree of freedom can be demonstrated intuitively by performing a particular conformal transformation on the metric.

$$\tilde{g}_{\mu\nu} = (1 + 2\alpha R)^2 g_{\mu\nu} \tag{1.17}$$

which leads to the action (1.16), in the absence of matter, becoming

$$\mathcal{S}_{mod} = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa^2} \tilde{R} + \frac{1}{2} \left(\partial\phi \right)^2 - V\left(\phi\right) \right] \tag{1.18}$$

$$\phi = c \ln \left(1 + 2\alpha R \right) \tag{1.19}$$

$$V\left(\phi\right) = -\frac{\left(e^{c\phi} - 1 \right)^2}{4\alpha} e^{-2c\phi} \tag{1.20}$$

where $c = \sqrt{\frac{3c^4}{16\pi G}}$. Eq. (1.18) is the Einstein-Hilbert action in the presence of a scalar field with a non-trivial potential. Small oscillations of ϕ about the minima of the potential gives the mass of the scalar field as $m_{eff} = \frac{1}{\sqrt{6\alpha}}$. Eq. (1.16) is a special case of $f(R)$

theories of gravity, where the GR Lagrangian R is replaced by an arbitrary function of the Ricci scalar. $f(R)$ theories have no unstable (ghost) degrees of freedom [75]. The field equations of $f(R)$ theories are as follows

$$f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' - \frac{f}{2}g_{\mu\nu} = 0 \tag{1.21}$$

where $f' = \frac{dF}{dR}$. According to the *no-hair* theorem [76], a scalar field cannot exist as a stable configuration around any stable black hole solutions of GR. Since $f(R)$ theories can be written as GR + scalar, all black hole solutions in vacuum and trace free matter of GR are also valid solutions in $f(R)$ theories.

Another way of obtaining the extra massive scalar degree of freedom in $f(R)$ theories is to linearize its field equations about flat space-time. This approach helps us understand exactly what combinations of higher derivative terms in the field equations can be represented as the massive scalar. Linearizing about a flat background,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.22}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1.23}$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} \tag{1.24}$$

leads to (1.21) becoming [77]

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} - 2\alpha\partial_\mu\partial_\nu R^{(1)} + 2\alpha\eta_{\mu\nu}\Box R^{(1)} = 0 \tag{1.25}$$

whose trace gives the following

$$\Box R^{(1)} - \frac{1}{6\alpha}R^{(1)} = 0 \tag{1.26}$$

which is the Klein-Gordon equation with mass $\frac{1}{\sqrt{6\alpha}}$. From Eq. (1.26) it is clear that the Ricci scalar acts as the massive spin-0 degree of freedom. Hence, one can extract out the massive scalar from the massless tensor part of the wave by a redefinition of the perturbation variable and using gauge conditions

$$\psi_{\mu\nu} = h_{\mu\nu} - \left(2\alpha R^{(1)} + \frac{h}{2}\right)\eta_{\mu\nu} \tag{1.27}$$

$$\partial^\mu \psi_{\mu\nu} = 0 \tag{1.28}$$

$$\psi = 0 \tag{1.29}$$

Using these in Eq. (1.25), one obtains

$$\Box \psi_{\mu\nu} = 0 \tag{1.30}$$

These correspond to the wave equation of a massless spin-2 particle with two ($+$ and $\times$) polarizations.

GR treats space-time as a geometrical object where presence of mass-energy curves the *fabric* of space-time, forcing test particles to follow the geometry of the *fabric*. However, for modified gravity theories like $f(R)$, the classical geometric nature of space-time is not immediately obvious. The inability to grasp the geometric nature of modified theories however does not stop one from trying to observe the effects of such modifications in observational tests, in gravitational waves for example. In order to circumvent the mathematical complexity of trying to understand modification to gravity from the geometric point of view, one can treat modifications to gravity (higher derivatives, extra fields) as an effective fluid, appearing on the RHS of the Einstein equations as an effective energy-momentum tensor. For example, (1.21) can be written as follows

$$G_{\mu\nu} = \kappa^2 T^{eff}_{\mu\nu} \tag{1.31}$$

$$T^{eff}_{\mu\nu} = \frac{1}{\kappa^2 f'}\left[\nabla_\mu \nabla_\nu f' - g_{\mu\nu}\Box f' + \frac{1}{2}\left(Rf' - f\right) g_{\mu\nu}\right] \tag{1.32}$$

Any realistic theory of gravity should be able to reproduce r^{-1} behavior of the gravitational potential in the Newtonian limit (like GR). Modified gravity theories (like $f(R)$) can have corrections to the r^{-1} term at high energies, i.e.

$$\Phi_{Newtonian} = -\frac{GM}{r}\left(1 + \frac{e^{-r\, m_{eff}}}{3}\right) \tag{1.33}$$

$$m^2_{eff} = \frac{1}{3f''(R)} \tag{1.34}$$

The modification of $\Phi_{Newtonian}$ in $f(R)$ theories can lead to deviations of the perihelion precession rates of planets. Such solar system tests put bounds on the value of m_{eff}, and correspondingly leads to $\alpha \leq 10^{12}\, km^2$ [78]. However, the strength of gravitational fields in the solar system is weak, and hence tighter bounds on α can be put from signals emanating from regions of intense gravitational fields, like gravitational waves (GW) from perturbed black holes. Since the strength of the gravitational field around a black hole goes as the inverse of black hole mass squared, smaller black holes would provide excellent test beds for tests of GR and for putting stringent limits on the value of α.

1.3.4 Chern-Simons modification to GR

Another way of modifying GR is to add scalar fields to the action. For vacuum spacetimes, as seen previously, *GR + scalar field* can be replaced by an $f(R)$ theory. Hence, one other option is to add a pseudo-scalar field (ϑ) to the theory which couples non-minimally with the background curvature. A pseudo-scalar is an object that changes sign under a parity transformation (reflection about the origin), unlike a scalar which does not change sign. In spherical symmetry,

$$\vartheta\left(r, \pi-\theta, \phi+\pi\right) = -\vartheta\left(r, \theta, \phi\right) \tag{1.35}$$

Such a theory dynamically violates local Lorentz symmetry, as well as parity, and can be used as a toy model for baryogenesis. Chern-Simons (CS) modifications to GR [69] were inspired from a modification to electrodynamics of the same name, it adds a parity violating pseudo-scalar which couples with another parity violating curvature invariant - contraction of the Riemann tensor with its own dual ${}^*R^{\tau}{}_{\sigma\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta}$. CS modifications to gravity come as the low energy limit of string theories and loop quantum gravity [79]. The action is of the form

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{2\kappa^2} + \frac{\alpha}{4}\vartheta^* RR - \frac{\beta}{2}\left(\nabla\vartheta\right)^2 - \frac{\beta}{2}V\left(\vartheta\right)\right] \tag{1.36}$$

where ϑ is a constant or a dynamical pseudo-scalar field. ϑ is chosen to be dimensionless, which leads to $[\alpha] = L^2$ and $[\beta]$ is dimensionless and ${}^*RR$ is

$${}^*RR = \frac{1}{2}R_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\alpha\beta}R^{\rho\sigma}{}_{\alpha\beta} \tag{1.37}$$

referred to as Pontryagin density quantifying the extent to which local Lorentz invariance is violated. CS theories are broadly of two types

- One in which ϑ takes a constant and a special value, with no kinetic and potential term - canonical CS theory [69].
- One in which ϑ is a fully dynamical field - dynamical CS theories [70].

For spherically symmetric space-times, the Pontryagin density vanishes, and one is left with *GR + scalar field with potential*. Owing to the no-hair theorem [65], for a spherically symmetric background, the only stable solution possible is that of the Schwarzschild. Hence, Schwarzschild space-time is a solution of both kinds of CS theories [69]. However, axisymmetric solutions are non-trivial to construct in CS theories owing to the fact that the Pontryagin density does not vanish for such space-times. However, it is possible to construct axisymmetric solutions from spherically symmetric solutions perturbatively in spin [80]. So far in the literature, no fast spinning Kerr-like solution exists in either kind of CS theories.

There are no extra intrinsic degrees of freedom of the gravitational field in this modification, except the two usual massless spin-2 degrees of freedom. In the literature, one usually assumes $V(\vartheta) = 0$, then, the field equations of (1.36) lead to

$$R_{\mu\nu} = -2\kappa^2\alpha C_{\mu\nu} + \kappa^2\beta\vartheta_{;\mu}\vartheta_{;\nu} \tag{1.38}$$

$$\Box\vartheta = -\frac{\alpha}{4\beta}{}^*RR \tag{1.39}$$

$$C_{\mu\nu} = \frac{1}{2}\left[\vartheta_{;\sigma}\left(\epsilon^{\sigma}{}_{\mu}{}^{\alpha\beta}R_{\nu\beta;\alpha} + \epsilon^{\sigma}{}_{\nu}{}^{\alpha\beta}R_{\mu\beta;\alpha}\right) + \vartheta_{;\tau\sigma}\left({}^*R^{\tau}{}_{\mu}{}^{\sigma}{}_{\nu} + {}^*R^{\tau}{}_{\nu}{}^{\sigma}{}_{\mu}\right)\right] \tag{1.40}$$

where $C_{\mu\nu}$ is known as the Cotton tensor in literature [69]. Linearizing (1.38) about a Minkowski background, and choosing the transverse-traceless gauge, one obtains the radiative part of the perturbed metric as

$$h_{ab}(t,r) = \frac{1}{2\pi}\int_p \begin{pmatrix} h_+ - ip\dot{\vartheta}h_\times & h_\times + ip\dot{\vartheta}h_+ \\ h_\times + ip\dot{\vartheta}h_+ & -h_+ + ip\dot{\vartheta}h_\times \end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{r}}dp \tag{1.41}$$

Hence, CS modifications lead to the plus and cross carrying different intensities, as well as imparting a circular polarization to the linearly polarized modes of GR.

Another effect of CS modification is the phenomenon of *birefringence of vacuum*. Although the circularly polarized gravitational radiation in CS theories travel at the speed of light, one of the consequences of parity violation is that different frequency components travel at different velocities in vacuum [79] - something that can technically be observed by GW detectors.

1.4 Structure of the thesis

In the rest of the thesis spherically symmetric BH space-times are perturbed and properties of the perturbations are looked at for $f(R)$ and CS modified gravity, and ways to distinguish GWs (during ring-down) between GR and the modified theories are shown.

In Chapter 2, the theory of black hole perturbations in GR will be outlined, salient features of the solutions of the perturbed equations of motion will be outlined for both Schwarzschild and Reissner-Nördstrom space-times.

In Chapter 3, analysis of a perturbation about a Schwarzschild background in $f(R)$ gravity will be performed and differences in its dynamics with the perturbation dynamics of GR Schwarzschild will be discussed. Effects of the difference in dynamics on observations and method to detect said difference to constrain deviation from GR will be shown.

In Chapter 4, a charged BH space-time in $f(R)$ gravity will be perturbed and the effects of the modified perturbation dynamics on observations will be discussed.

In Chapter 5, comparison between two types of Chern Simons modifications to GR (canonical and dynamical), and their effects on the perturbation dynamics in Schwarzschild space-times will be compared. The effects of dynamical CS modification to gravity on BH ring-down, and on the emitted gravitational waves will be shown and methods to detect such effects will be discussed.

Chapter 6 summarizes the results from previous chapter and draws some conclusions about signatures imparted to gravitational waves from modified gravitational theories in general. Possible future directions the analysis of the current thesis can take are also discussed.

In Appendix A, the form of the parametrized Pöschl-Teller potential which replaces the odd/even parity potentials in the scattering problem and the form of the complex Reflection coefficient as a function of angular frequency ω are shown.

In Appendix B, the higher derivative modifications to GR are expressed in terms of the parameters of an effective fluid. Exact form of an effective source term showing preferential coupling, modifying the dynamics of the perturbations about a Schwarzschild background in $f(R)$ gravity is derived.

In Appendix C, an energy-momentum pseudo-tensor corresponding to the radiated gravitational radiation by a perturbed BH in $f(R)$ gravity is constructed.

In Appendix D, the preferentially coupled effective source terms modifying the dynamics of a perturbation about a charged BH background in $f(R)$ gravity is calculated.

In Appendix E, an energy-momentum pseudo-tensor corresponding to the radiated gravitational radiation by a perturbed charged BH in $f(R)$ gravity is constructed.

In Appendix F, the linearly perturbed Pontryagin density is derived in terms of one of the gravitational parity modes. A preferentially coupling effective source term, modifying the dynamics of a perturbation about a Schwarzschild BH in dynamical Chern-Simons modification to GR is calculated.

In Appendix G, an energy-momentum pseudo-tensor corresponding to the radiated gravitational radiation by a perturbed BH in a dynamical CS modification to GR is constructed.

In Appendix H, the constancy of a relative intensity factor between odd and even parity perturbations in GR, and its time dependence in dynamical CS modified gravity, as defined in Chapter 5, is shown, enabling one to test for deviations from GR from gravitational wave observations.

In Appendix I, a relation between the polarization amplitudes of perturbation in Minkowski space-times, and the same at spatially asymptotic BH space-times in dynamical CS modified gravity was made.

Chapter 2

Theory of black hole perturbations in General Relativity

Astronomical processes like particle in-fall/orbit around a black hole (BH) and binary BH collisions make the external space-time of a remnant BH undergo damped ringing while emitting gravitational waves. A ringing external space-time of a BH will asymptote towards spherical (Schwarzschild) or axial (Kerr) symmetry — which are stable, static, and unique [81] solutions of GR. A single distorted BH can be represented mathematically by a small perturbation about a stable BH external space-time metric that the perturbed system will eventually evolve to. Studies on the dynamics and properties of perturbations about the Schwarzschild solution were first done in [82], [83], and then in [84]. From Sec. (2.1) to Sec. (2.7.2), a review of various aspects of black hole perturbation theory will be done, whereas in Sec. (2.7.3), some new results regarding perturbed space-times of charged BHs will be shown.

The conventions of Chapter 1 will be followed. Greek alphabets have been used for indices 0-3, lower Latin for 0-1, and upper Latin for 2-3. The various physical quantities with the over-line refer to the values evaluated for the spherically symmetric background, whereas superscript (n) represents the n-th order perturbed quantity. ∇_μ and $\Box$ represents covariant derivative and Laplace-Beltrami operator for the full space-time,

$\tilde{D}_a$ and $\tilde{\Box}$ represents covariant derivative and Laplace-Beltrami operator for 0-1 indices, while $\hat{D}_a$ and $\hat{\Box}$ represents covariant derivative and Laplace-Beltrami for 2-3 indices respectively. $\tilde{D}_a r$ and $\tilde{D}_a \tilde{D}_b r$ have been represented as r_a and r_{ab} respectively throughout the thesis.

2.1 Formalism and definitions

A perturbation about the Schwarzschild solution is represented as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu} \tag{2.1}$$

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} \tag{2.2}$$

where $\bar{g}_{\mu\nu}$ is the metric tensor corresponding to the line element (1.5) describing lengths and clock rates in the external vacuum space-time of a spherically symmetric BH space-time. ϵ is so chosen that $\frac{\epsilon|h_{\mu\nu}|}{|\bar{g}_{\mu\nu}|} << 1$, i.e. the effect of the perturbation on the background space-time should be sufficiently small. The relaxation of $g_{\mu\nu}$ to $\bar{g}_{\mu\nu}$ can be described by solving for $h_{\mu\nu}$ up to $\mathcal{O}(\epsilon)$ in the field equations satisfied by $g_{\mu\nu}$:

$$R_{\mu\nu}(g_{\mu\nu}) = R_{\mu\nu}(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) \tag{2.3}$$

$$\simeq \bar{R}_{\mu\nu} + \epsilon R^{(1)}_{\mu\nu} \tag{2.4}$$

$$\Rightarrow R^{(1)}_{\mu\nu} = 0 \tag{2.5}$$

where $\bar{R}_{\mu\nu} = 0$ is the background solution corresponding to $\bar{g}_{\mu\nu}$, and $R^{(1)}_{\mu\nu}$ is the first order perturbed Riemann tensor describing the dynamics of $h_{\mu\nu}$. Retaining only $\mathcal{O}(\epsilon)$ terms, one finds

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h + \nabla_\mu\nabla_\sigma h^\sigma_\nu + \nabla_\nu\nabla_\sigma h^\sigma_\mu - 2\bar{R}_{\mu\sigma\nu\rho}h^{\sigma\rho}\right) = 0 \tag{2.6}$$

The background space-time represented by the line element (1.5), owing to a spherical symmetry in a two dimensional subspace of the four dimensional manifold, can be split into two 2-D space-times in the following way

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.7}$$
$$= g_{ab} dy^a dy^b + r^2 \bar{g}_{AB} dz^A dz^B \tag{2.8}$$

where $x_\mu \equiv (y_a, z_A)$ is the coordinate 4-vector, while $y_a \equiv (t, r)$ and $z_A \equiv (\theta, \phi)$ are coordinate 2-vectors corresponding to an orbit space with metric g_{ab} and a 2-sphere with metric $\bar{g}_{AB}$ respectively. Hence, the whole space-time is foliated by a concentric spheres with radii ranging from $2M$ to ∞.

2.2 Coordinate transformations and spherical harmonics

Under a coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu , \tag{2.9}$$

where ξ^μ is a 4-vector, the perturbed tensor transforms as follows

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} = h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \tag{2.10}$$

Since GR is a coordinate independent theory, transformation (2.9) leaves (2.6) invariant. This is akin to the gauge freedom of electrodynamics where the 4-vector potential A^μ cannot be uniquely determined. The invariance of (2.6) under transformation (2.10) thus imply that the solution $h_{\mu\nu}$ of (2.6) is uncertain up to a 4-vector ξ^μ.

A special coordinate transformation in the space-times under study is rotation, which is a symmetry of the Schwarzschild solution. The form of the 4-vector ξ^μ that generates rotation in the space-time is of the form

$$\xi^\mu \equiv \begin{pmatrix} \mathbf{0} \\ \xi^A \end{pmatrix} \tag{2.11}$$

Under this restricted transformation, different components of the metric perturbation $h_{\mu\nu}$ transform in different manner [82, 85]

$$h_{ab} \to \tilde{h}_{ab} = h_{ab} \tag{2.12}$$

$$h_{aA} \to \tilde{h}_{aA} = h_{aA} - r^2 \tilde{D}_a \left(\frac{\xi_A}{r^2}\right) \tag{2.13}$$

$$h_{AB} \to \tilde{h}_{AB} = h_{AB} - \hat{D}_A \xi_B - \hat{D}_B \xi_A \tag{2.14}$$

implying that h_{ab} transforms as a scalar, h_{aA} as a 2-vector, and h_{AB} as a tensor under the 2-D rotation. Symmetry in the subspace implies that the angular dependence of a general perturbation can be expressed as an infinite series of spherical harmonic functions. Thus the angular dependence of the scalar components of $h_{\mu\nu}$ under rotation can be expressed as

$$h_{ab} = \sum_{\ell,m} f_{ab}^{\ell,m}\left(y^c\right) \mathbf{S}^{\ell,m}\left(z^A\right) \tag{2.15}$$

where (ℓ, m) are multipole indices, y^a and z^A are as defined in Eq. (2.8), and $\mathbf{S}^{\ell,m}$ is the scalar spherical harmonic function and proportional to the associated Legendre polynomial [86]. It follows the following property

$$\hat{\Box}\mathbf{S}^{\ell,m} = -k^2\ \mathbf{S}^{\ell,m} \tag{2.16}$$

$$k = \sqrt{\ell\left(\ell+1\right)}. \tag{2.17}$$

Similarly, the vector transforming part of $h_{\mu\nu}$, h_{aA} can have an angular dependence of two types - a curl-less vector $\mathbf{S}_A^{\ell,m}$ (parity conserving or even under a parity transformation $\theta \to \pi - \theta$, $\phi \to \phi + \pi$) , and a divergence-less vector $V_A^{\ell,m}$ (parity violating or odd under parity transformation). Both can be obtained from the scalar spherical harmonic function in the following manner

$$\mathbf{S}_A^{\ell,m} = \hat{D}_A \mathbf{S}^{\ell,m} \tag{2.18}$$

$$\mathbf{V}_A^{\ell,m} = \epsilon_{AB} \hat{D}^B \mathbf{S}^{\ell,m} \tag{2.19}$$

which are gradient and pseudo-gradient of the scalar spherical harmonic function respectively. Therefore, h_{aA} can be expressed as

$$h_{aA} = \sum_{\ell,m} \left[f_a^{E,\ell m}\left(y^b\right) \mathbf{S}_A^{\ell,m}\left(z^B\right) + f_a^{O,\ell m}\left(y^b\right) \mathbf{V}_A^{\ell,m}\left(z^B\right) \right] \tag{2.20}$$

where the indices E and O stands for even parity and odd parity respectively. Tensor transforming components of $h_{\mu\nu}$ under rotation can similarly be expressed as an infinite series of three types of tensor spherical harmonics, two of even parity and one of odd parity. Out of which, the simplest even parity tensor spherical harmonic is $\bar{g}_{AB}\mathbf{S}^{\ell,m}$ where $\bar{g}_{AB}$ is the metric of a 2-sphere. Two other spherical harmonic tensors can be defined

$$\mathbf{S}^{\ell,m}_{AB} = \hat{D}_A\hat{D}_B\mathbf{S}^{\ell,m} + \frac{1}{2}k^2\bar{g}_{AB}\mathbf{S}^{\ell,m} \tag{2.21}$$

$$\mathbf{V}^{\ell,m}_{AB} = -\frac{1}{2}\left(\hat{D}_A\mathbf{V}^{\ell,m}_B + \hat{D}_B\mathbf{V}^{\ell,m}_A\right) \tag{2.22}$$

such that $\mathbf{S}^A_A = \mathbf{V}^A_A = 0$, where the ℓ, m indices have been omitted. The components h_{AB} can now be written as

$$h_{AB} = \sum_{\ell,m}\left[r^2H^{\ell,m}_L(y^a)\,\bar{g}_{AB}\mathbf{S}^{\ell,m} + r^2H^{E,\ell m}_T(y^a)\,\mathbf{S}^{\ell,m}_{AB} + H^{O,\ell m}_T(y^a)\,\mathbf{V}^{\ell,m}_{AB}\right] \tag{2.23}$$

where the indices L and T stand for longitudinal and transverse respectively. Thus given these expansions, significant simplification can be done on (2.5), reducing quasi-linear second order differential equations of four variables into quasilinear second order differential equations of two variables.

The scalar harmonic functions, being proportional to the associated Legendre polynomials, form an orthogonal set. The vector and tensor harmonic functions also follow orthogonality properties [86]

$$\int \mathbf{S}^{A,\ell m}\mathbf{S}^{\ell' m'}_A d\Omega = \ell(\ell+1)\,\delta_{\ell\ell'}\delta_{mm'} \tag{2.24}$$

$$\int \mathbf{V}^{A,\ell m}\mathbf{V}^{\ell' m'}_A d\Omega = \ell(\ell+1)\,\delta_{\ell\ell'}\delta_{mm'} \tag{2.25}$$

$$\int \mathbf{S}^{A,\ell m}\mathbf{V}^{\ell' m'}_A d\Omega = 0 \tag{2.26}$$

$$\int \mathbf{S}^{AB,\ell m}\mathbf{S}^{\ell' m'}_{AB} d\Omega = \frac{1}{2}(\ell-1)\,\ell\,(\ell+1)\,(\ell+2)\,\delta_{\ell\ell'}\delta_{mm'} \tag{2.27}$$

$$\int \mathbf{V}^{AB,\ell m}\mathbf{V}^{\ell' m'}_{AB} d\Omega = \frac{1}{2}(\ell-1)\,\ell\,(\ell+1)\,(\ell+2)\,\delta_{\ell\ell'}\delta_{mm'} \tag{2.28}$$

$$\int \mathbf{S}^{AB,\ell m}\mathbf{V}^{\ell' m'}_{AB} d\Omega = 0 \tag{2.29}$$

Unlike a tensor, components of a 4-vector transform only as a scalar or a vector under rotation. Specifically, for a 4-vector ξ^μ, the components ξ^a transform as scalars, and ξ^A transforms as a 2-vector. Therefore, a vector that generates a general coordinate transformation can be expanded in terms of spherical harmonic scalars and vectors

$$\xi_a = \sum_{\ell,m} L_a^{\ell,m}\mathbf{S}^{\ell,m} \tag{2.30}$$

$$\xi_A = \sum_{\ell,m}\left(L_E^{\ell,m}\mathbf{S}_A^{\ell,m} + L_O^{\ell,m}\mathbf{V}_A^{\ell,m}\right) \tag{2.31}$$

Like the metric perturbation $h_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ can also be similarly split into R_{ab}, R_{aA}, and R_{AB}, each of which can be split into even and odd parity parts [85], which form two decoupled sets of equations owing to the orthogonality relations (2.24)-(2.29) and the spherical symmetry of the background.

The various components of an energy-momentum tensor $T_{\mu\nu}$ in a spherically symmetric background transforms in the same way as the components of $h_{\mu\nu}$, and hence can be written as a sum over spherical harmonic scalars (T_{ab}), vectors (T_{aA}), and tensors (T_{AB}). The coefficient multiplying a spherical harmonic object (scalar/vector/tensor) of each ℓ, m in the spherical harmonic expansion of various components of $T_{\mu\nu}$ can be obtained in the same manner as the coefficient set $\left(f_{ab}, f_a^{E/O}, H_L, H_T^{E/O}\right)$ of $h_{\mu\nu}$.

For even parity, coefficients of expansion can be found using the orthogonal properties (2.24)-(2.29)

$$\tau^{ab}_{\ell,m} = 8\pi\int T^{ab}\mathbf{S}^{\ell,m}d\Omega \tag{2.32}$$

$$\tau^{a}_{E,\ell m} = \frac{16\pi r^2}{k^2}\int T^{aA}\mathbf{S}_A^{\ell,m} \tag{2.33}$$

$$P_{\ell,m} = 8\pi r^2\int T^{AB}\gamma_{AB}\mathbf{S}^{\ell,m} \tag{2.34}$$

$$\tau_T^{E,\ell m} = \frac{32\pi r^4}{\mu}\int T^{AB}\mathbf{S}_{AB}^{\ell,m} \tag{2.35}$$

$$\mu = (\ell-1)\,\ell\,(\ell+1)\,(\ell+2) \tag{2.36}$$

while for odd parity on obtains

$$\tau^{a}_{O,\ell m} = \frac{16\pi r^2}{k^2}\int T^{aA}\mathbf{V}_A^{\ell,m} \tag{2.37}$$

$$\tau_T^{O,\ell m} = \frac{16\pi r^4}{\mu}\int T^{AB}\mathbf{V}_{AB}^{\ell,m} \tag{2.38}$$

From this point onward, the indices ℓ, m and the summations over them will be implicitly assumed.

2.3 The gauge invariant formalism

Early works of BH perturbation involved utilizing of the gauge freedom of the metric perturbation to set certain components of $h_{\mu\nu}$ to zero, yielding a single scalar (or master function) for each parity [82, 83]. That one scalar per parity encompassed the entire dynamics of the perturbed external space-time for that parity. However, it was not immediately obvious why the Regge-Wheeler gauge choice could be termed as the 'physical gauge'. A gauge invariant study done by Moncrief [87] using Newman-Penrose formalism [88] showed that the gauge was indeed physical and that the Regge-Wheeler and Zerilli definition of the master functions were also gauge invariant. Using the metric perturbation approach, Gerlach and Sengupta [89] developed a gauge invariant approach to the perturbation theory, which was developed later by Martel-Poisson [86] and extended to higher dimensions by Kodama-Ishibashi [58, 85, 90].

In the gauge invariant approach, one tries to find how the coefficients of the spherical harmonic expansion in (2.15), (2.20), and (2.23) transform under (2.9). Using the expansions (2.30) and (2.31) in (2.12) and (2.14), one obtains for the even parity components

$$f_{ab} \to \tilde{f}_{ab} = f_{ab} - \tilde{D}_a L_b - \tilde{D}_b L_a \tag{2.39}$$

$$f_a^E \to \tilde{f}_a^E = f_a^E - L_a - \tilde{D}_a L_E + \frac{2}{r} r_a L_E \tag{2.40}$$

$$H_L \to \tilde{H}_L = H_L + \frac{k^2}{r^2} L_E - \frac{2}{r} r^a L_a \tag{2.41}$$

$$H_T^E \to \tilde{H}_T^E = H_T^E - \frac{2}{r^2} L_E \tag{2.42}$$

Similarly, the odd parity components transform as

$$f_a^O \to \tilde{f}_a^O = f_a^O - \tilde{D}_a L_O + \frac{2}{r} r_a L_O \tag{2.43}$$

$$H_T^O \to \tilde{H}_T^O = H_T^O - 2L_O \tag{2.44}$$

Given the relations (2.39)-(2.44), one can take linear combinations of the various coefficients to define other quantities which remain invariant under a coordinate transformation. If one can obtain such quantities, then they can be used as observables. One finds the following gauge invariant quantities for the even parity sector [85, 86]

$$F_{ab} = f_{ab} - \tilde{D}_a j_b - \tilde{D}_b j_a \tag{2.45}$$

$$F = H_L + \frac{k^2}{2} H_T^E - \frac{2}{r} r^a j_a \tag{2.46}$$

where $j_a = f_a^E - \frac{r^2}{2} H_T^E$. Similarly, for the odd sector one has [85, 86]

$$F_a = f_a^O - \frac{1}{2} \tilde{D}_a H_T^O + \frac{1}{r} r_a H_T^O \tag{2.47}$$

2.4 Master equations for perturbation and boundary conditions

Using the gauge invariant variables to define two master variables ($\Psi_{E/O}$), one for each parity, (2.5) reduces to two decoupled 2-D equations of the form

$$\tilde{\Box}\Phi_{E/O} - \frac{V_{E/O}}{g}\Phi_{E/O} = S_{E/O} \tag{2.48}$$

where $g \equiv g(r) = 1 - \frac{2M}{r}$ and as found by [86],

$$\Phi_E = \frac{2r}{(\ell-1)(\ell+2)+2}\left[F + \frac{2g}{(\ell-1)(\ell+2)+\frac{6M}{r}}\left(gF_{rr} - r\partial_r F\right)\right] \tag{2.49}$$

$$\Phi_O = \frac{2rk^2}{\mu}\left(\partial_r F_t - \partial_t F_r - \frac{2}{r}F_t\right) \tag{2.50}$$

$$\tag{2.51}$$

where the quantities F_{ab}, F, and F_a are as defined in (2.45), (2.46), and (2.47) respectively.

(2.48) have the form of the equation of motion of a wave moving in a background curvature induced effective potential $V_{E/O}$ and have a simpler 1-D form when the quantities $\Phi_{E/O}$ are expanded using plane wave fronts, i.e. giving $\Phi_{E/O}$ a $e^{i\omega t}$ time dependence. Then, for a plane wave $\Phi_{E/O}$ of frequency ω incident on the BH space-time, one finds

$$\frac{d^2\Phi_{E/O}}{dr_*^2} + \left(\omega^2 - V_{E/O}\right)\Phi_{E/O} = 0 \tag{2.52}$$

where a radial coordinate transformation from r to r_* was done, which are related to each other through

$$r_* = r + 2M\ln\left|\frac{r}{2M} - 1\right| \tag{2.53}$$

The exact forms of $V_{E/O}$ will be given later, however, both the potentials asymptote to zero at the boundaries and have a single maxima at $r = r_P$, which was defined in Sec. (1.1.1). Thus, the problem of the relaxation of a spherically symmetric BH on perturbation can be replaced by a wave scattering problem in 1-D.

Being a 2nd order DE, one needs two initial conditions, which can either be

- Values of the variables $\Phi_{E/O}$ at the boundaries: horizon and asymptotic infinity.
- Values and the first derivative values at either of the two boundaries.

A scattering problem like (2.52) in the BH context require a specific type of boundary conditions. Owing to the fact that the horizon is a one way surface, these classical waves should be purely ingoing at the horizon, while at ∞ there can be both ingoing and outgoing radiation corresponding to incident and reflected parts of the wave respectively. Hence, the boundary conditions on (2.52) will be of the form

$$\Phi_{E/O} \sim T_{E/O}(\omega)\, e^{i\omega(t+r_*)} \quad r_* \to -\infty \tag{2.54}$$

$$\sim e^{i\omega(t+r_*)} + R_{E/O}(\omega)\, e^{i\omega(t-r_*)} \quad r_* \to \infty \tag{2.55}$$

where the sign before r_* is a convention that differs in literature. Fig. 2.1 provides a pictorial representation of the scattering process and the boundary conditions. In this thesis, positive sign before r_* will imply an ingoing wave, and a negative sign will imply outgoing wave. Since one of the boundaries in this problem has a purely ingoing boundary condition, it is a dissipative system - implying the asymptotic solution will be decaying in time. This is natural because the perturbations have to decay away so that a stable solution is reached eventually, otherwise, BHs would be very unstable objects. Because of this, all the values ω's can take will be complex with a positive imaginary part such that solutions can have an $e^{-\omega_I t}$ form ($\omega_I \equiv \Im(\omega)$). The $T_{E/O}(\omega)$ stands for complex transmission amplitude, and $R_{E/O}(\omega)$ stands for complex reflection amplitude. Hence, $|T_{E/O}|^2$ and $|R_{E/O}|^2$ are the transmission and reflection coefficients respectively - quantifying the fraction of the incident gravitational energy (or the initial energy of perturbation) that gets absorbed by the BH or is radiated away to asymptotic infinity respectively. They follow the constraint

$$|T_{E/O}|^2 + |R_{E/O}|^2 = 1 \tag{2.56}$$

implying conservation of energy. In the case where $T_{\mu\nu} = 0$, the profile of the potentials $V_{E/O}$ determine the complex amplitudes $R_{E/O}(\omega)$ and $T_{E/O}(\omega)$ [91, 92].

2.5 Isospectrality

The exact forms of the potentials $V_{E/O}$ are given by [82–84, 86]

$$\begin{aligned} V_E &= \frac{1}{\Lambda^2}\left(1-\frac{2M}{r}\right)\left[(\ell-1)^2(\ell+2)^2\left(\frac{(\ell-1)(\ell+2)+2}{r^2}+\frac{6M}{r^3}\right)\right. \\ &\quad \left.+\frac{36M^2}{r^4}\left((\ell-1)(\ell+2)+\frac{2M}{r}\right)\right] \end{aligned} \tag{2.57}$$

$$V_O = \left(1-\frac{2M}{r}\right)\left(\frac{k^2}{r^2}-\frac{6M}{r^3}\right) \tag{2.58}$$

where $\Lambda = (\ell-1)(\ell+2) + \frac{6M}{r}$. Although (2.57) and (2.58) look functionally very different, a curious relationship between the potentials $V_{E/O}$ was found by Chandrasekhar [84], where it was shown that both the potentials can be obtained from the same func-

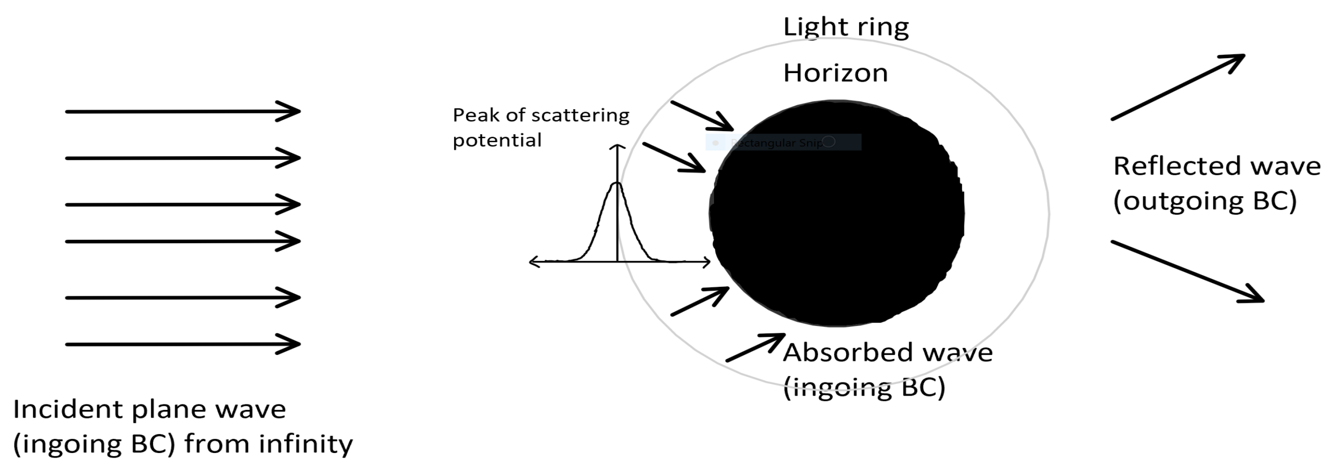


Figure- 2.1: Schematic diagram of the scattering process and the boundary conditions (2.54)

tion $W(r)$

$$gV_{E/O} = W(r)^2 \mp gW'(r) - \frac{\left(k^2-2\right)^2 k^4}{36 r_H^2} \tag{2.59}$$

$$W(r) = \frac{3r_H\left(r_H - r\right)}{r^2\left[3r_H + \left(k^2-2\right) r\right]} - \frac{k^2\left(k^2-2\right)}{6r_H} \tag{2.60}$$

where $g = 1 - \frac{2M}{r}$, the negative sign in the RHS of (2.59) is for the even parity potential and the positive sign is for the odd parity potential, and the prime denotes radial derivative. Using the fact that both even and odd parities follow the same boundary conditions (2.54), it was shown by [93] that the range of allowed values of ω for both even and odd parity sectors are the same - or that the two parities are iso-spectral.

Another consequence of isospectrality is the following

$$|T_E|^2 = |T_O|^2 \tag{2.61}$$

$$|R_E|^2 = |R_O|^2 \tag{2.62}$$

implying that the fraction of radiated gravitational energy to the incident gravitational energy (perturbation energy or initial conditions) is the same for both odd and even parity perturbations - a feature that has been shown to be unique to GR [94, 95]. Isospectrality also breaks in the case of a non-vanishing cosmological constant [94].

2.6 Gravitational radiation at asymptotic infinity

2.6.1 Matching of a perturbed Schwarzschild metric with asymptotic Minkowski

In Refs. [86, 96, 97] it was shown that only the traceless part of h_{AB} contributes to the radiation escaping to asymptotic infinity, and a connection was found between polarizations $h_{+/\times}$ and the gauge invariant perturbation variables of a Schwarzschild space-time

In Ref. [96] it was shown using the tetrad formalism developed in [88] that at asymptotic infinity, h_{AB} can be written in the locally flat coordinate system of an observer as

$$h_{\hat{A}\hat{B}} = \mathbf{e}^A_{\hat{A}} \mathbf{e}^B_{\hat{B}} h_{AB} \tag{2.63}$$

where $\mathbf{e}^A_{\hat{A}} = diag\left[r^{-1}, (r\sin\theta)^{-1}\right]$ is the observer's local tetrad and $\hat{A}$ is the tetrad index. The traceless part of h_{AB} has the form [85], where an implicit summation over multipole index ℓ and projection index m was assumed

$$h_{AB} = r^2 \left(H_T^E \mathbf{S}_{AB} + H_T^O \mathbf{V}_{AB}\right) \tag{2.64}$$

Using (2.63) in (2.64) one obtains the traceless part of the perturbation in $\mathcal{S}^2$ at a large distance from the black-hole in a locally flat space-time as

$$h_{\hat{A}\hat{B}} \equiv H_T^E \begin{pmatrix} \mathbf{S}_{\theta\theta} & \mathbf{S}_{\theta\phi} \\ \frac{\mathbf{S}_{\theta\phi}}{\sin\theta} & \frac{\mathbf{S}_{\phi\phi}}{\sin\theta} \end{pmatrix} + H_T^O \begin{pmatrix} \mathbf{V}_{\theta\theta} & \mathbf{V}_{\theta\phi} \\ \frac{\mathbf{V}_{\theta\phi}}{\sin\theta} & \frac{\mathbf{V}_{\phi\phi}}{\sin\theta} \end{pmatrix} \tag{2.65}$$

$$= \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}. \tag{2.66}$$

from which polarization amplitudes $h_{+/\times}$ can be read off from (2.65) as

$$h_+ = \left(H_T^E \mathbf{S}_{\theta\theta} + H_T^O \mathbf{V}_{\theta\theta}\right) \tag{2.67}$$

$$h_\times = \frac{1}{\sin\theta}\left(H_T^E \mathbf{S}_{\theta\phi} + H_T^O \mathbf{V}_{\theta\phi}\right). \tag{2.68}$$

The asymptotic relationship between the perturbation variables and the scalar/vector master variables were found from [86, 96], using which (2.67) and (2.68) becomes

$$h_+ \simeq \frac{1}{r}\left(\Phi_E \mathbf{S}_{\theta\theta} + \Phi_O \mathbf{V}_{\theta\theta}\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{2.69}$$

$$h_\times = \frac{1}{r\sin\theta}\left(\Phi_E \mathbf{S}_{\theta\phi} + \Phi_O \mathbf{V}_{\theta\phi}\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{2.70}$$

which can be inverted (after truncating $\mathcal{O}\left(\frac{1}{r^2}\right)$ onward terms)

$$\frac{\Phi_E}{r} = \int h_+ \mathbf{S}_{\theta\theta} d\Omega + \int \sin\theta h_\times \mathbf{S}_{\theta\phi} d\Omega \tag{2.71}$$

$$\frac{\Phi_O}{r} = \int h_+ \mathbf{V}_{\theta\theta} d\Omega + \int \sin\theta h_\times \mathbf{V}_{\theta\phi} d\Omega \tag{2.72}$$

In the observational point of view, the above relations are impractical since it involves observing the polarization amplitudes at each point of the surface of a sphere and integrating over it — which is unlikely, unless in future the technological challenge of encompassing the entirety of a black hole with detectors can be overcome. Earth bound detectors can only observe gravitational waves on a small patch of the sphere, given which, it is useful to find the quantity in the LHS of (2.71) and (2.72) per unit solid angle, for which one obtains

$$\frac{d\Psi_E}{d\Omega} = h_+ \mathbf{S}_{\theta\theta} + \sin\theta h_\times \mathbf{S}_{\theta\phi} \tag{2.73}$$

$$\frac{d\Psi_O}{d\Omega} = h_+ \mathbf{V}_{\theta\theta} + \sin\theta h_\times \mathbf{V}_{\theta\phi} \tag{2.74}$$

where $\Psi_{E/O} = \frac{\Phi_{E/O}}{r}$.

The tensor spherical harmonics are related to the spin weighted spherical harmonics $({}_{-2}Y_{\ell m})$ used for the description of gravitational radiation using null tetrads instead of the metric tensor [86, 96, 98], and hence (2.67) and (2.68) can be rearranged into

$$h_+ - ih_\times \simeq \frac{1}{r}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\Phi_E + i\Phi_O\right)\,{}_{-2}Y_{\ell m} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{2.75}$$

LHS is the doubly integrated Weyl scalar Ψ_4 at asymptotic infinity (also known as the gravitational strain), and can be expanded in spin-weighted spherical harmonics as [96, 99] (summation over ℓ, m have been implicitly assumed).

$$h_+ - ih_\times = h_{\ell m} \, {}_{-2}Y_{\ell m}. \tag{2.76}$$

Comparing (2.75) and (2.76) one has,

$$\Re\left(h_{\ell m}\right) \simeq \frac{1}{r}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\Phi_E \tag{2.77}$$

$$\Im\left(h_{\ell m}\right) \simeq \frac{1}{r}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\Phi_O \tag{2.78}$$

2.6.2 Energy-momentum pseudotensor

Perturbations involve supplying extra energy to a system, which if stable to perturbations have various ways of relaxing to a lower energy stable state. Systems where the end-state is a vacuum BH (formed from merger of a binary BH or an neutron star-BH system) relax to a stable state by absorbing and radiating extra energy which slightly changes the symmetry of the background. This can be seen as a wave scattering problem (as found in Sec. (2.4)) where the BH absorbs a portion of some incident radiation, radiating away the rest. Presence of radiation modifies the energy-momentum content of the background space-time, therefore, for a perturbation (2.1), one can quantify the associated radiation by expanding the field tensor in powers of ϵ

$$\bar{R}_{\mu\nu} + \epsilon R^{(1)}_{\mu\nu} + \epsilon^2 R^{(2)}_{\mu\nu} = 0 \tag{2.79}$$

where the third term in the above expression is the *back-reaction* due to perturbation. Since at $\mathcal{O}\left(\epsilon\right)$ one solves for $R^{(1)}_{\mu\nu} = 0$, the radiation due to perturbation becomes a second order effect, and acts as an energy-momentum (EM) pseudo-tensor for the perturbation [100, 101]

$$\bar{R}_{\mu\nu} = \kappa^2 t_{\mu\nu} \tag{2.80}$$

$$= \quad -\epsilon^2 \left\langle R^{(2)}_{\mu\nu} \right\rangle \tag{2.81}$$

where $< ... >$ is a symbolic representation of averaging over wavelengths that are much smaller than the length scales over which the background space-time changes appreciably - yielding a gauge invariant measure of the energy-momentum content of perturbation [100]. Operationally, the averaging procedure allows one to do the following algebra in GR

- Total derivative terms of the form $\nabla_\mu A_{\alpha\beta\ldots}$ can be put to zero.
- $\nabla_\mu \mathbf{A} \nabla_\nu \mathbf{B} = -\mathbf{A} \nabla_\mu \nabla_\nu \mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are tensor objects.

Thus, calculating $R^{(2)}_{\mu\nu}$ due to (2.1), one obtains

$$\begin{aligned} R^{(2)}_{\mu\nu} \quad = \quad & -\frac{1}{2}\left[\frac{1}{2} h^{\rho\tau}_{;\beta} h_{\rho\tau;\alpha} + h^{\rho\tau}\left(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\tau\rho} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\right) + h^{\tau;\rho}_{\beta}\left(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}\right)\right. \\ & \left. - \left(h^{\rho\tau}_{;\rho} - \frac{1}{2} h^{;\tau}\right)\left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau}\right)\right] \end{aligned} \tag{2.82}$$

Using the the transverse-traceless gauge

$$\psi_{\mu\nu} \quad = \quad h_{\mu\nu} - \frac{h}{2}\bar{g}_{\mu\nu} \tag{2.83}$$

$$\nabla_\mu \psi^\mu_\nu \quad = \quad 0 \tag{2.84}$$

$$\psi \quad = \quad 0 \tag{2.85}$$

where h and ψ are the trace of $h_{\mu\nu}$ and $\psi_{\mu\nu}$ respectively, one finds $t_{\mu\nu}$ to be

$$t_{\mu\nu} \quad = \quad \frac{\epsilon^2}{4\kappa^2}\left\langle \nabla_\mu \psi^{\alpha\beta} \nabla_\nu \psi_{\alpha\beta} \right\rangle \tag{2.86}$$

Gravitational Energy radiated per unit time per unit solid angle ($d\Omega$) in the outgoing radial direction hence will be of the form [102, 103]

$$\frac{d^2 E}{d\Omega dt} \quad = \quad r^2 \; \hat{r} \; t_{tr} \tag{2.87}$$

where $\hat{r}$ is a unit vector defining the outward radial null direction. On using the form of $h_{\mu\nu}$ in the transverse-traceless gauge at asymptotic infinity, one obtains

$$\frac{d^2 E}{d\Omega dt} \quad = \quad \frac{\epsilon^2 r^2}{4\kappa^2}\left\langle \left|\dot{h}_+\right|^2 + \left|\dot{h}_\times\right|^2 \right\rangle \tag{2.88}$$

which on converting to the odd/even master functions using (2.75) become

$$\frac{d^2E}{d\Omega dt} = \frac{\epsilon^2\mu}{4\kappa^2}\left\langle\left|\dot{\Phi}_E\right|^2 + \left|\dot{\Phi}_O\right|^2\right\rangle \tag{2.89}$$

where the summation over ℓ, m have been implicitly assumed.

2.7 Charged BHs in GR

The line element for a charged BH external space-time is given by (1.13). It corresponds to the background electromagnetic vector potential

$$\bar{A}_\mu \equiv \left(\frac{\sqrt{2}Q}{\kappa r}, 0, 0, 0\right) \tag{2.90}$$

A perturbation of the electromagnetic field about the background (2.90), is represented by

$$A_\mu = \bar{A}_\mu + A_\mu^{(1)} \tag{2.91}$$

where $A_\mu^{(1)}$ is a 4-vector whose components $A_a^{(1)}$ transform as a scalar, and $A_A^{(1)}$ transforms as a vector under rotation (2.11). Hence, $A_\mu^{(1)}$ can be similarly expanded using scalar and vector harmonics as

$$A_\mu^{(1)} = (\mathcal{A}_a\mathbf{S}, \mathcal{A}_E\mathbf{S}_A + \mathcal{A}_O\mathbf{V}_A) \tag{2.92}$$

Thus for odd parity, $\mathcal{A}_O$ is the only variable, and hence can be used as a master function for perturbation dynamics. For even parity three scalars characterize the perturbation, which can be combined using the perturbed Maxwell equations [104] to yield a gauge invariant variable which will be denoted by $\mathcal{A}_E$. Combining the perturbed field equations and the Maxwell equations, one finds the following coupled DEs [104]:

2.7.1 Odd parity perturbations

$$r^2 D_a\left(\frac{1}{r^2}D^a\Omega\right) - \frac{k^2-1}{r^2}\Omega = \frac{2\sqrt{2}\kappa Q}{r^2}\mathcal{A}_O \tag{2.93}$$

$$\tilde{\Box}\mathcal{A}_O - \frac{1}{r^2}\left(k^2+1+\frac{4Q^2}{r^2}\right)\mathcal{A}_O = \frac{\sqrt{2}Q\,(k^2-1)}{\kappa r^4}\Omega \tag{2.94}$$

where $\Phi_V^0 = \frac{\Omega}{r}$. which can be decoupled using the following substitution

$$\Phi_\pm^O = a_\pm^O \Phi_O^0 + b_\pm^O \mathcal{A}_O \tag{2.95}$$

$$\left(a_+^O, b_+^O\right) \equiv \left(\frac{Q(k^2-1)}{3M+\Delta}, \frac{\kappa}{\sqrt{2}}\right) \tag{2.96}$$

$$\left(a_-^O, b_-^O\right) \equiv \left(1, \frac{-2\sqrt{2}\kappa Q}{3M+\Delta}\right) \tag{2.97}$$

yielding

$$\frac{d^2\Phi_\pm^O}{dx^2} + \left(\tilde{\omega}^2 - V_\pm^O\right)\Phi_\pm^O = 0 \tag{2.98}$$

where

$$V_\pm^O = \frac{g}{r^2}\left(k^2+1+\frac{4Q^2}{r^2}+\frac{-3M\pm\Delta}{r}\right) \tag{2.99}$$

$$\Delta = \sqrt{9M^2+4(k^2-1)Q^2} \tag{2.100}$$

2.7.2 Even parity perturbations

$$g\frac{d}{dr}\left(g\frac{d\Phi_E^0}{dr}\right) + \left(\omega^2 - V_E\right)\Phi_E^0 = 0 \tag{2.101}$$

$$\tilde{\Box}\mathcal{A}_E - \frac{1}{r^2}\left(k^2+\frac{8Q^2 g}{r^2 H}\right)\mathcal{A}_E = \frac{\sqrt{2}Q}{\kappa r^3}\left(\frac{2H^2-P_Z}{4H}\Phi_E^0 + gr\frac{d\Phi_E^0}{dr}\right) \tag{2.102}$$

where $H = k^2 - 2 + \frac{6M}{r} - \frac{4Q^2}{r^2}$, and $P_Z = \frac{8M^2}{r^2} + \frac{2M}{r}\left(-\frac{4Q^2}{r^2} + 6\,k^2 - 6\right) - \frac{8\left(k^2+1\right)Q^2}{r^2} - 4\,k^2 + 8$, and can similarly be decoupled using the following substitution

$$\Phi_\pm^E = a_\pm^E \Phi_E^0 + b_\pm^E \mathcal{A}_E \tag{2.103}$$

$$\left(a_+^E, b_+^E\right) \equiv \left(\frac{Q(k^2 r + 3\,M + 3\,\nu - 2\,r)}{2r}, \frac{3\,(M+\nu)\,\kappa}{\sqrt{2}}\right)$$

(2.104)

$$\left(a_-^E, b_-^E\right) \equiv \left(3(M+\nu) - \frac{4Q^2}{r}, -\frac{4\sqrt{2}Q}{\kappa}\right) \tag{2.105}$$

where $\nu = \sqrt{M^2 + \frac{4}{9}(k^2-2)Q^2}$, leading to the decoupled DEs

$$\frac{d^2\Phi_\pm^E}{dx^2} + \left(\tilde{\omega}^2 - V_\pm^E\right)\Phi_\pm^E = 0 \tag{2.106}$$

where $V_\pm^E$ are scattering potentials similar to $V_\pm^O$.

2.7.3 Gravitational energy suppression in charged BH space-times

Similar to the Schwarzschild, isospectral relations between $V_\pm^E$ and $V_\pm^O$ exists, establishing equality of spectrum, as well as transmission and reflection coefficients. However, the isospectral relationship exists in the decoupled form, which is the dynamics of a variable that is a combination of gravitational and electro-magnetic perturbations.

At the boundaries $x \to \pm\infty$, $\Phi_\pm^{E/O}$ is given by

$$\Phi_\pm^{E/O} \sim e^{i\tilde{\omega}x} + \sqrt{R_\pm^{E/O}}e^{i\left(\delta_{\pm,(r)}^{E/O} - \tilde{\omega}x\right)} \qquad x \to \infty \tag{2.107}$$

$$\Phi_\pm^{E/O} \sim \sqrt{T_\pm^{E/O}}e^{i\left(\delta_{\pm,(t)}^{E/O} + \tilde{\omega}x\right)} \qquad x \to -\infty \tag{2.108}$$

where $R_\pm^{E/O}$ & $T_\pm^{E/O}$ are the reflection and transmission coefficients associated with the potentials $V_\pm^{E/O}$, while $\delta_{\pm,(r/t)}^{E/O}$ are the changes in phase of the incoming wave due to reflection/transmission off of the potential barriers $V_\pm^{E/O}$.

A purely gravitational wave incident on the black-hole space-time from $x \to \infty$ is given by

$$\Phi_{E/O,(i)}^0 \neq 0 \tag{2.109}$$

$$\mathcal{A}_{E/O,(i)} = 0 \tag{2.110}$$

where the reflected waves [105], compared to the incident wave is given by

$$\left|\Phi_{E/O,(r)}^0\right| = \left|\Phi_{E/O,(i)}^0\right| \left[R_+^{E/O}\sin^2\epsilon + R_-^{E/O}\cos^2\epsilon\right.$$

$$+2\sqrt{R_{+}^{E/O}R_{-}^{E/O}}\cos\left(\delta_{+,(r)}^{O/E}-\delta_{-,(r)}^{E/O}\right)\Big]^{\frac{1}{2}}\sin\epsilon\cos\epsilon \tag{2.111}$$

$$\left|\mathcal{A}_{E/O,(r)}\right| = \left|\Phi^{0}_{E/O,(i)}\right| C_{E/O} \tag{2.112}$$

$$C_{E/O} = \Big[R_{+}^{E/O}+R_{-}^{V/S} -2\sqrt{R_{+}^{E/O}R_{-}^{E/O}}\cos\left(\delta_{+,(r)}^{O/E}-\delta_{-,(r)}^{E/O}\right)\Big]^{\frac{1}{2}}\sin\epsilon\cos\epsilon \tag{2.113}$$

$$\sin 2\epsilon = \mp 2\sqrt{\frac{-q_1 q_2}{\left(q_1-q_2\right)^2}} \qquad S(-),V(+) \tag{2.114}$$

$$q_i = 3M+\left(-1\right)^{i-1}\sqrt{9M^2+4\left(k^2-2\right)Q^2} \qquad i=1,2 \tag{2.115}$$

indicating that a purely gravitational wave on scattering off of the curvature of a Reissner-Nördstrom space-time will have a minor electromagnetic component in the net scattered radiation. The factor multiplying $\left|\Phi^{0}_{E/O,(i)}\right|$ in (2.112) is called the conversion factor ($C_{E/O}$) and was first calculated in [105], and shown that $C_E \geq C_O$ - implying a higher fraction of converted electromagnetic energy for the even parity compared to the odd parity [105]. This feature can be used to confirm or infirm the consensus in the scientific community that astrophysical BHs are charge neutral.

An estimate of relative radiated gravitational energy difference between odd and even parity modes as a function of the BH charge and initial conditions of the ring-down regime can be estimated using (2.113)

$$\Delta^{\ell m}_{GR}\left(q,I_E,I_O\right) \equiv \frac{\left(1-C_O\right)I_O-\left(1-C_E\right)I_E}{\left(1-C_O\right)I_O+\left(1-C_E\right)I_E} \tag{2.116}$$

where I_E and I_O are the initial intensity of the gravitational perturbations, and $q=\frac{Q}{M}$ is the BH charge scaled with respect to the mass. Solutions of 2.52 at asymptotic infinity as a function of time is given by

$$\Psi_{E/O} = A_{E/O}e^{-\omega_I t}e^{i\omega t} \tag{2.117}$$

which on using in (2.116) and the definition (2.89) for flux per (even/odd) mode becomes

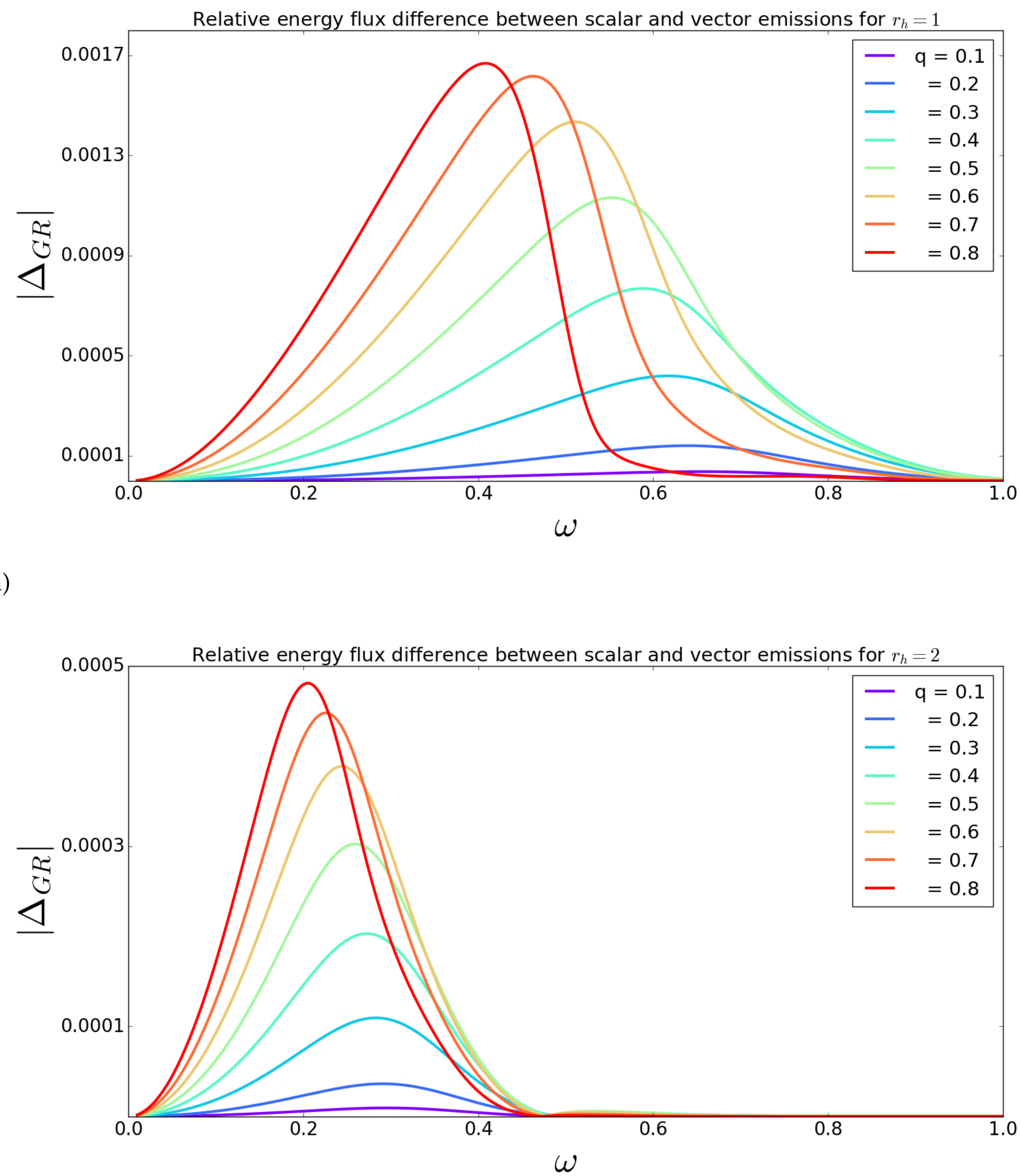
$$\Delta_{GR}^{\ell m}\left(q, A_E, A_O\right) \equiv \frac{\left(1-C_O\right) A_O^2-\left(1-C_E\right) A_E^2}{\left(1-C_O\right) A_O^2+\left(1-C_E\right) A_E^2} \tag{2.118}$$

where $A_{E/O}$ are initial conditions for solving (2.52) and are the initial amplitudes of perturbation, which in turn depend on the specifics of the process that caused the perturbation - like mass ratio and spin ratio of a binary BH merger [106]. All such parameters can be estimated using the properties of the waveform of the gravitational wave signal just before the system transitions into the ring-down regime. Numerical relativity simulations can tell us the value of $\Delta_{GR}^{\ell m}$ for uncharged BHs of the same initial parameters, whereas any presence of charge in BHs found in nature will have a value of $\Delta_{GR}^{\ell m}$ that is lesser than the value found from simulations, owing to the fact that even parity emits less gravitational energy than odd parity. For a particular case where odd and and even modes are equally excited $(A_O = A_E)$, (2.118) can be replaced with a definition that is significantly simpler and can be expressed simply as a function of the mass scaled charge q [107]

$$\left.\Delta_{GR}^{\ell m}\right|_{A_E=A_O} = \frac{C_E-C_O}{1-C_O}, \tag{2.119}$$

in which case the above quantity simply becomes the relative difference between the radiated odd and even gravitational fluxes and have been plotted in Fig. 2.2 for various values of q and different values of the horizon radius r_h [107]. In order to obtain the reflection coefficients $R_\pm$ and the phases $\delta_\pm^{E/O}$, the potentials $V_\pm^{E/O}$ were replaced with a properly parametrized Pöschl-Teller potential, details of which are given in Appendix A.

In the next chapter, changes in the isospectral relations due to modified gravity will be discussed.

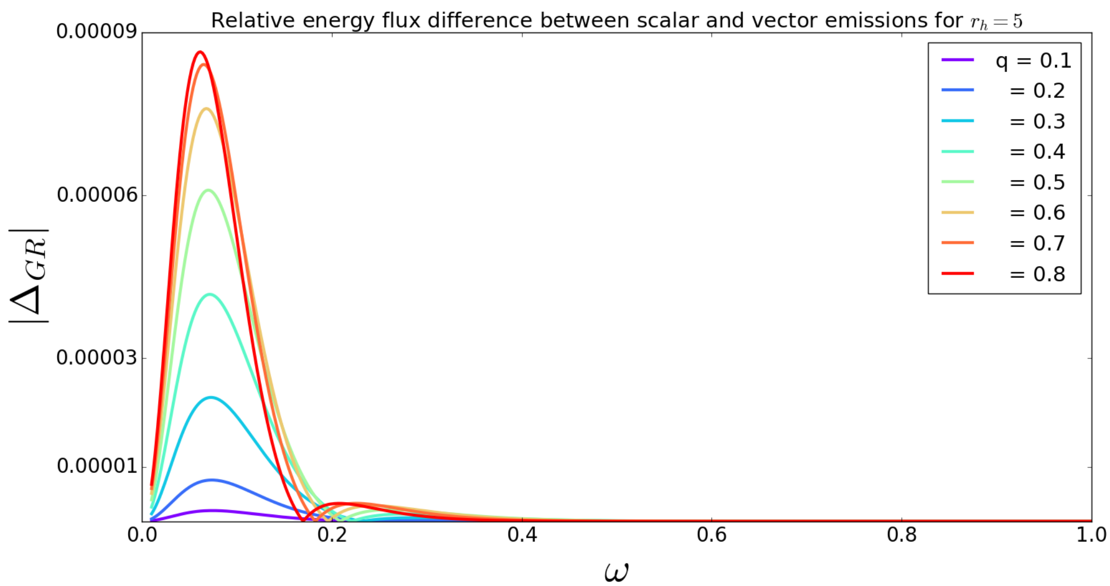


Relative energy flux difference between scalar and vector emissions for $r_h = 1$
0.0017
0.0013
0.0009
0.0005
0.0001
$|\Delta_{GR}|$
0.0
0.2
0.4
0.6
0.8
1.0
ω
q = 0.1
= 0.2
= 0.3
= 0.4
= 0.5
= 0.6
= 0.7
= 0.8

(a)

Relative energy flux difference between scalar and vector emissions for $r_h = 2$
0.0005
0.0003
0.0001
$|\Delta_{GR}|$
0.0
0.2
0.4
0.6
0.8
1.0
ω
q = 0.1
= 0.2
= 0.3
= 0.4
= 0.5
= 0.6
= 0.7
= 0.8

(b)



(c)

Figure- 2.2: Relative energy flux difference $|\Delta^{22}_{GR}|$ for three different horizon radii. Increasing BH size leads to larger characteristic length scales for the space-time leading to the shift of the profiles towards low frequencies, and leads to a decrease in the quantity (2.119).

Chapter 3

Signatures of $f(R)$ gravity from perturbed Schwarzschild black holes

In this chapter, perturbation studies of a Schwarzschild BH, which is also a solution of $f(R)$ theories of gravity (as seen in Sec. (1.3.3)), will be performed. Using a gauge invariant analysis following [86], it will be shown that signatures of modifications to gravity, like $f(R)$, will alter the way gravitational energy is radiated through odd and even parities. As a consequence it will be shown that in gravitational wave observations, the alteration in energy radiation will leave a specific signature, which can be utilized to detect or constrain deviations from GR. This chapter is based on the published works [95, 107].

The notations of Chapter 1 and 2 will be continued throughout the current chapter.

3.1 Modification to gravity as GR + effective fluid

As seen in Sec. (1.3.3), higher derivative correction terms to the Einstein field equations can be represented as an effective fluid which modifies the background or the perturbed space-time. Since the effective fluid energy-momentum tensor (1.31) explicitly depends on the Ricci scalar (R), for Ricci flat space-times (i.e. $R = 0$), the energy-momentum tensor vanishes, leaving behind GR, satisfying

$$\bar{R}_{\mu\nu} = 0 \tag{3.1}$$

Consequently, the background remains a solution of GR, and in this chapter a background Schwarzschild space-time will be perturbed. A perturbed Schwarzschild space-time in $f(R)$ gravity is quite different from a perturbed Schwarzschild space-time in GR, owing to the presence of first order perturbed effective fluid of the higher derivative terms. When $f(R)$, for simplicity, takes a specific form of

$$f(R) = R + \alpha R^2, \tag{3.2}$$

the perturbed equations of motion corresponding to the metric perturbation (2.1), about a Schwarzschild background $\bar{g}_{\mu\nu}$, are given by

$$R^{(1)}_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} = T^{eff}_{\mu\nu} \tag{3.3}$$

$$T^{eff}_{\mu\nu} = 2\alpha\left(\nabla_\mu\nabla_\nu R^{(1)} - \bar{g}_{\mu\nu}\Box R^{(1)}\right) \tag{3.4}$$

as well as the trace of the field equations (3.3), given by [95, 108]

$$\Box R^{(1)} - \frac{1}{6\alpha}R^{(1)} = 0 \tag{3.5}$$

which, similar to Sec. (1.3.3), is the Klein-Gordon equation, describing the propagation of a massive scalar field (with mass parameter $\sqrt{\frac{1}{6\alpha}}$) on a curved background space-time. As was found in Eq. (1.30), Eq. (3.5) also indicates the presence of an extra massive scalar degree of freedom for the graviton, in addition to the two massless tensor degrees of freedom. Whereas the two massless tensor degrees of freedom are transverse, the massive scalar is a combination of longitudinal and breathing modes [77, 109, 110].

3.1.1 Dynamics of extra scalar degree of freedom

The dynamics of the massive scalar field and its properties are the easiest to calculate. Since the first order Ricci $\left(R^{(1)}\right)$ is a scalar, its angular dependence can be expanded in terms of an infinite sum of spherical harmonic functions. Expanding $R^{(1)}$ in terms of spherical harmonics and performing a substitution on the spherical harmonic coefficients as follows

$$R^{(1)} = \sum_{\ell,m} \frac{\Phi_{\ell m}}{r} \mathbf{S}^{\ell,m} \tag{3.6}$$

and using the orthogonality properties of the spherical harmonic functions, Eq. (3.5) for each ℓ, m becomes

$$\Box\Phi - \frac{\tilde{V}_{RW}}{g}\Phi = 0 \tag{3.7}$$

where the multipolar indices ℓ, m have been suppressed. The above, expanding in plane wave fronts, and transforming to *tortoise* coordinates, the above expression for each ω becomes

$$\frac{d^2\Phi}{dr_*^2} + \left(\omega^2 - \tilde{V}_{RW}\right)\Phi = 0. \tag{3.8}$$

where the effective potential due to background curvature is unlike the potentials $V_{E/O}$ of (2.52). Its form in a Schwarzschild space-time of central mass M and multipole ℓ is given by

$$\tilde{V}_{RW} = \left(1 - \frac{2M}{r}\right)\left(\frac{k^2}{r^2} + \frac{2M}{r^3} + \frac{1}{6\alpha}\right) \tag{3.9}$$

where k has been defined in Eq. (2.17). As found from Sec. (2.5), $V_{E/O}$ asymptotes to zero as $r \to \infty$. However, Eq. (3.9) does not vanish at asymptotic infinity, but asymptotes to the value $\frac{1}{6\alpha}$. But like the potentials $V_{E/O}$, $\tilde{V}_{RW}$ also vanishes at the horizon. From recent limits placed on deviations from GR from gravitational wave observations [111, 112], it is evident that a deviation from GR, if any, will be smaller than the current experimental accuracy. This implies that the deviation parameter α in (3.2) is a small number, such that for the strong field space-times from where gravitational waves have been detected so far, the first term of the action (3.2) dominates over the second term, i.e.

$$\alpha R << 1. \tag{3.10}$$

An immediate consequence of α being small reflects on the asymptotic value of $\tilde{V}_{RW}$, which becomes a large number. From earth based fifth force experiments like Eöt-Wash [77, 113, 114] a conservative constraint of

$$\alpha \leq 10^{-9} m^2 \tag{3.11}$$

can be placed. Given (3.11), the profile of $\tilde{V}_{RW}$ can be plotted for various values of the mass M, as has been done in Fig. 3.1 as a function of the horizon radius scaled radial coordinate. Fig. 3.1 shows that an incident massive scalar mode for astrophysical

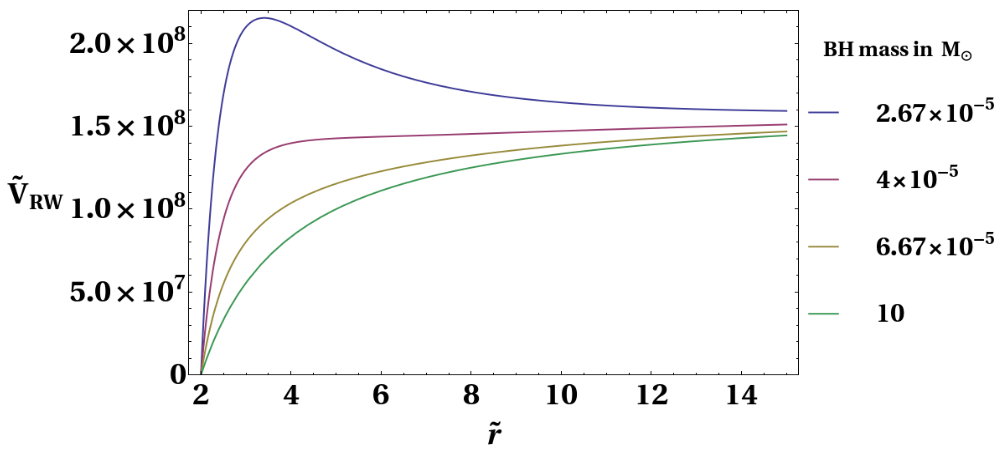


Figure- 3.1: Plot of Eq. (3.9) for a range of BH masses. For small M ($\sim 10^{-5} M_\odot$), the maxima at the light ring reappears, making it similar to the scattering potentials $V_{E/O}$. For such small masses the massive scalar radiation can propagate to ∞ for $\omega^2 \geq (6\alpha)^{-1}$, and can have a larger share of the net emitted gravitational radiation. For comparison, the green curve shows the potential encountered by Φ for a $10M_\odot$ BH.

sized BHs face no potential barrier at the light ring. Given the fully ingoing boundary condition at the horizon holds, all of an incident scalar wave will be absorbed. Hence, any excitation of the massive field due to perturbation near the light ring will also be absorbed by the BH, making it a non-propagating degree of freedom, compared to the two propagating tensor degrees of freedom. A massive scalar excitation will lead to a transient halo of scalar field existing outside the horizon which decays rapidly with increasing radial distance given the large potential barrier it faces with increasing distance from the BH. Hence, the massive scalar field cannot exist as a stable configuration around the BH - indicating that the *no scalar hair* theorem holds for spherically symmetric vacuum BH solution of $f(R)$ theories of gravity. However, it is possible that an ultralight scalar field can exist for a long while (possibly comparable to the age of the Universe) around a black hole, as shown in [115, 116]. But as seen from Eq. (3.9), the mass term for the scalar mode of $R+\alpha R^2$ gravity is $\sqrt{\frac{1}{6\alpha}}$. Since, from observations it is apparent that nature follows GR to the best of our knowledge, α has to be a small number, and correspondingly, the mass of the field is large. Massive fields, owing to their small decay times, cannot exist as a long-term configuration around a black hole.

3.1.2 Effective fluid description of the massive scalar field perturbation

The effective energy-momentum tensors' (3.3) individual components like T^{eff}_{ab}, T^{eff}_{aA}, and T^{eff}_{AB} transform like scalars, vectors, and tensors respectively, similar to $h_{\mu\nu}$, and hence can be expanded in spherical harmonic functions as shown in (2.32)-(2.38). The coefficients of expansion in as a function of the variable Φ and its derivatives are given by (after ignoring the multipole indices) [95]

$$\tau_{ab} = 2\alpha\left[D_a D_b - g_{ab}\left(\tilde{\Box} + \frac{2}{r}D^c r D_c - \frac{k^2}{r^2}\right)\right]\left(\frac{\Phi}{r}\right) \tag{3.12}$$

$$\tau^E_a = -\frac{4\alpha r}{k^2} D_a\left(\frac{\Phi}{r^2}\right) \tag{3.13}$$

$$P = 2\alpha\left(\frac{k^2}{2r^2} - \tilde{\Box} - \frac{2}{r}D^a r D_a\right)\left(\frac{\Phi}{r}\right) \tag{3.14}$$

$$\tau^E_T = \frac{4\alpha\Phi}{r} \tag{3.15}$$

$$\tau^O_a = 0 \tag{3.16}$$

$$\tau^O_T = 0 \tag{3.17}$$

where P can be seen as pressure of an effective fluid along the radial direction, whereas τ_T^E is the anisotropic stress. Derivation of the above can be found in Appendix B.1 From (3.12)-(3.17) it is clear that the modification to GR in $R + \alpha R^2$ gravity results in an effective fluid being present only in the even parity sector, leaving the odd parity untouched (which has a dynamics same as GR).

The effective fluid of modifications to gravity can be expressed as an effective source term which couples to the odd parity dynamics. After expanding both odd parity variable Φ_O and the massive scalar Φ in plane wave fronts

$$\Phi_O(t,r) = \int_{-\infty}^{\infty} e^{i\omega t}\Phi_O(\omega,r)\,d\omega \tag{3.18}$$

$$\Phi(t,r) = \int_{-\infty}^{\infty} e^{i\sigma t}\Phi(\sigma,r)\,d\sigma, \tag{3.19}$$

for a specific set of quasinormal frequencies (ω,σ) (corresponding to the odd parity mode and the massive scalar respectively), the odd and even parity dynamics, respectively become [95]

$$\frac{d^2\Phi_O}{dr_*^2} + \left(\omega^2 - V_O\right)\Phi_O = 0 \tag{3.20}$$

$$\frac{d^2\Phi_E}{dr_*^2} + \left(\omega^2 - V_E\right)\Phi_E = S_{eff} \tag{3.21}$$

where (derivation in Appendix B.2)

$$S_{eff} = \left[C_1(\sigma,\omega,r) + C_2(\sigma,\omega,r)\frac{d}{dr_*}\right]\frac{4\alpha\Phi}{k^2 + \frac{6M}{r} - 2} \tag{3.22}$$

$$\begin{aligned} C_1(\sigma,\omega,r) = {} & \sigma^2\left(1+\frac{\sigma}{\omega}\right) - \frac{Mg}{r^3}\left(1+\frac{18M}{rH}\right) - \left(\frac{\sigma}{\omega}\right)\frac{g}{r^2}\left[\frac{54M^2}{r^2H} - \frac{72gM^2}{r^2H^2} - \frac{18M}{rH}\right. \\ & \left. + \frac{1}{2}\frac{P_1}{H} - \frac{3M}{r} + \frac{\tilde{V}_{RW}}{g}\right] \end{aligned} \tag{3.23}$$

$$C_2(\sigma,\omega,r) = \frac{3M}{r^2} - \left(\frac{\sigma}{\omega}\right)\left[\frac{12Mg}{r^2H} - \frac{M}{r^2}\right] \tag{3.24}$$

$$P_1 = -\frac{48M^2}{r^2} + \frac{8M}{r}\left(8-k^2\right) - 2k^2(k^2-2) \tag{3.25}$$

Presence of a source term in Eq. (3.21) and the absence of the same in (3.20) implies breaking of isospectrality, with the even parity quasinormal frequencies and reflection coefficient being modified, while the odd parity quasinormal frequencies and reflection coefficient remains the same as in GR. This feature can leave a signature on the radiated gravitational energy ratio of the two parities that can technically be detected by detectors at asymptotic infinity.

3.2 Energy-momentum pseudo-tensor of perturbation in $f(R)$ theories

3.2.1 About Minkowski background

The energy-momentum content of a linear gravitational perturbation, Eq. (2.1), about some background can be expressed in the form of a pseudo-tensor derived from the averaged field equations at $\mathcal{O}\left(\epsilon^2\right)$, as was shown in Sec. (2.6.2). It is quite illuminating to look at the energy-momentum pseudo-tensor of perturbation in Minkowski spacetime. The linearly perturbed field equations about a Minkowski background and its dynamics was shown in Sec. (1.3.3). Ref. [77] finds the energy momentum pseudo-tensor of radiation about a Minkowski background at $\mathcal{O}\left(\epsilon^2\right)$ to be

$$t_{\mu\nu} \;=\; \frac{\epsilon^2}{4\kappa^2}\left\langle -\partial_\mu\psi_{\alpha\beta}\partial_\nu\psi^{\alpha\beta} + 24\alpha^2\partial_\mu R^{(1)}\partial_\nu R^{(1)}\right\rangle . \tag{3.26}$$

where $\psi_{\mu\nu}$ is the transverse-traceless metric perturbation as defined in Eq. (1.27). For details on the derivation of (3.26), see Appendix C. In contrast to Eq. (2.86), the EM pseudo-tensor now has a kinetic term corresponding to the extra massive scalar $R^{(1)}$, which is of the order α^2. Eq. (1.26), for small values of α, will have exponentially decaying solutions for frequencies less than $\frac{c}{2\pi}\sqrt{\frac{1}{6\alpha}}$ — implying that only very high frequency gravitational radiation will be able to excite the massive mode. Based on earth based laboratory tests of short distances [77], the upper bound on α was found to be

$$\alpha \;<\; 10^{-9}m^2 \tag{3.27}$$

implying that the minimum frequency required to excite the massive mode in flat space is

$$f_{min} \gtrapprox 10^{11}\ Hz \tag{3.28}$$

which is a rather conservative estimate. However, earth based tests are not in the high curvature regime unlike black hole external space-times, where the αR^2 term may have a significant contribution, and the upper bound on α can be significantly higher.

This leads to the conclusion that the direct detection of a massive scalar mode intrinsic to gravity, if any, will not be possible for present or future generations of detectors.

3.2.2 About a general (curved) background space-time

Even though (1.26) and (3.26) points toward an apparent impossibility of direct detection of a massive intrinsic scalar degree of freedom, there are indirect way with which one can probe the presence of non-tensorial degrees of freedom. This is possible by looking at how non-tensorial degrees of freedom in modified gravity theories interact with the usual tensor modes. In flat space-times, (as seen from (1.26) and (3.26)) the dynamics of the massless tensor modes are completely decoupled from the massive scalar mode. But it is not so in curved space-times, as is seen from the modified odd parity dynamics (3.20) close to BHs. Hence, it is important to extend the energy-momentum pseudotensor method to perturbations in curved space-times. For general curved space-times, the perturbed field equations of $R + \alpha R^2$ gravity at $\mathcal{O}(\epsilon)$ in terms of the redefined variable $\psi_{\mu\nu}$ are given by [107]

$$\Box\psi_{\mu\nu} + 2\bar{R}_{\alpha\mu\beta\nu}\psi^{\alpha\beta} = 0 \tag{3.29}$$

where $\bar{R}_{\alpha\mu\beta\nu}$ is the background Riemann tensor. Hence, Eq. (3.29) and Eq. (1.26) determine the dynamics of both the massless and massive modes in curved space-times. In order to find the energy-momentum pseudotensor, it is necessary to calculate the perturbed field equations at $\mathcal{O}(\epsilon^2)$ as well, which is given by [107]

$$\mathcal{G}^{(2)}_{\mu\nu} \equiv R^{(2)}_{\mu\nu} - h_{\mu\nu}R^{(1)} + \bar{g}_{\mu\nu}h^{\alpha\beta}R^{(1)}_{\alpha\beta} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}R^{(2)}_{\alpha\beta} - \alpha\left[4\nabla^{(1)}_{\mu}\nabla_{\nu}R^{(1)} - 2h_{\mu\nu}\Box R^{(1)}\right.$$

$$+2\bar{g}_{\mu\nu}h^{\alpha\beta}\nabla_\alpha\nabla_\beta R^{(1)} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\nabla^{(1)}_\alpha\nabla_\beta R^{(1)} + \bar{g}_{\mu\nu}\left(R^{(1)}\right)^2 - 4R^{(1)}R^{(1)}_{\mu\nu}\Big] \tag{3.30}$$

where $\mathcal{G}_{\mu\nu}$ represents the modified field tensor, superscript 2 indicates perturbation at $\mathcal{O}(\epsilon^2)$, and $\nabla^{(1)}_\mu$ represents the first order perturbed covariant derivative. Changing variables from $h_{\mu\nu}$ to $\psi_{\mu\nu}$ and employing the averaging-over-short-wavelengths procedure from Sec. (2.6.2), one obtains [107]

$$t_{\mu\nu} = \frac{1}{\kappa^2}\left\langle -\frac{1}{4}\nabla_\mu\psi_{\alpha\beta}\nabla_\nu\psi^{\alpha\beta} + \frac{\alpha}{6}\left(R^{(1)}\right)^2 + 18\alpha^2\nabla_\mu R^{(1)}\nabla_\nu R^{(1)}\right\rangle \tag{3.31}$$

which leads to the appearance of a potential energy like term of $\mathcal{O}(\alpha)$ in curved space-times, in addition to the kinetic terms of the massive scalar of order $\mathcal{O}(\alpha^2)$. Hence, in curved space-times, the effect of the massive scalar field is significantly stronger, and any interactions with the tensorial modes in these regions would leave a signature of $\mathcal{O}(\alpha)$ on the tensorial modes that travel to gravitational wave detectors. The preferential coupling of the massive scalar with the odd parity perturbation would imply that the even parity sector exchanges energy with the massive scalar of order α, leaving the odd sector untouched. Hence, like Eq. (2.89), energy radiated in the outward radial direction per unit time per unit solid angle can be obtained from Eq. (3.31) as

$$\frac{d^2E}{dtd\Omega} = \frac{\epsilon^2}{\kappa^2}\left\langle \frac{1}{4}\left(\left|\dot{\tilde{\Phi}}_E\right|^2 + \left|\dot{\Phi}_O\right|^2\right) + \frac{\alpha}{6}\left|\Phi\right|^2 + 18\alpha^2\left|\dot{\Phi}\right|^2\right\rangle . \tag{3.32}$$

Given the same initial energy of perturbation, the following inequality holds

$$\left|\dot{\tilde{\Phi}}_E\right|^2 < \left|\dot{\Phi}_E\right|^2 \tag{3.33}$$

from which a dimensionless parameter like (2.116) can be defined in the context of $f(R)$ theories of gravity

$$\Delta_{\ell m} = \frac{\left|\dot{\tilde{\Phi}}_O\right|^2 - \left|\dot{\Phi}_E\right|^2}{\left|\dot{\tilde{\Phi}}_O\right|^2 + \left|\dot{\Phi}_E\right|^2} \tag{3.34}$$

whose value will be more in $f(R)$ compared to GR, and will be time varying (compared to a constant in GR).

Given the relations (2.77) and (2.78), value of the above quantity, as well as the quasi-normal frequencies of the two parities, can be obtained from observations. Both of these observations, taken together, can help in detecting and constraining deviations from GR.

3.2.3 Generalization to polynomial $f(R)$ theories

In this chapter, a specific model of $f(R)$ gravity was chosen, i.e. $R + \alpha R^2$. However, for any general $f(R)$ theory which can be expanded in a power series of the Ricci scalar

$$f(R) = \sum_{n=1}^{\infty} a_n R^n, \tag{3.35}$$

a linear perturbation about a Ricci flat space-time will reduce the linearized equations of motion identically to that of the linearized equations of $R + \alpha R^2$ gravity. To see this consider the effective stress tensor for a general $f(R)$ theory, as was given by (1.31), which when linearized about a Ricci flat background yields

$$T^{(1),eff}_{\mu\nu} = \frac{1}{\kappa^2}\left[\nabla_\mu \nabla_\nu \delta f' - \bar{g}_{\mu\nu}\Box\delta f'\right] \tag{3.36}$$

where $f' = \frac{df}{dR}$ and $\delta f'$ is the first order perturbed f', which for a Ricci flat background space-time and a form (3.35) becomes

$$\delta f' = 2\alpha R^{(1)}. \tag{3.37}$$

Substituting the above in (3.36) gives back (3.3).

In the next chapter the current treatment will be extended to charged BH solutions of $f(R)$ theories.

Chapter 4

Signatures of $f(R)$ gravity from perturbed charged black holes

For charged BH solutions in GR, the quantity $\Delta_{\ell m}$ was obtained as a function of BH charge and initial conditions of the perturbation, as was seen in Section 2.7. Simultaneous gravitational and electromagnetic perturbation, corresponding to (2.1) and (2.91) respectively, about a Reissner-Nördstrom solution in $R + \alpha R^2$ gravity will, like the Schwarzschild solution, contain an effective fluid corresponding to the higher derivative modifications to GR, and will leave signatures of modification, in addition to charge, in gravitational waves being emitted by ringing BHs, and correspondingly in a quantity like $\Delta_{\ell m}$. The above mentioned effects, alongwith effects of the same in observations will be discussed in the present chapter.

This chapter is based on the published work [107]

4.1 Perturbation dynamics in a charged BH in $f(R)$ gravity

4.1.1 The background

Electromagnetic fields can be included in an $f(R)$ theory via the following action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa^2} - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\right] \tag{4.1}$$

whose equations of motion are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T^{EM}_{\mu\nu} + T^{eff}_{\mu\nu} \tag{4.2}$$

where $T^{eff}_{\mu\nu}$ was defined in Eq. (1.31), and

$$T^{EM}_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{4.3}$$

is the energy-momentum tensor corresponding to the electromagnetic field. The background solution $\bar{F}_{\mu\nu}$ can be found by solving for the background electromagnetic 4-potential A^{μ}, whose equations of motion are given by

$$\Box A_{\nu} - R_{\mu\nu}A^{\mu} = 0 \tag{4.4}$$

which can be solved in the Lorentz gauge and the solution was given in Eq. (2.90). The background EM tensor then has the form

$$\bar{T}^{EM}_{\mu\nu} = \left(\begin{array}{c|c} -P\delta^a_b & 0 \\ --- & --- \\ 0 & P\delta^A_B \end{array}\right) \tag{4.5}$$

$$P = \frac{Q^2}{\kappa^2 y^4}\,. \tag{4.6}$$

where the dimensionless radial distance $y = \frac{r}{r_H}$ is scaled with respect to the BH horizon radius r_H, which in turn is given by

$$r_H = M + \sqrt{M^2 - Q^2}. \tag{4.7}$$

A dimensionless charge $q = \frac{Q}{r_H}$ can also be defined for convenience, due to which, the mass can be written as a function of the horizon radius and q

$$M = \frac{r_H\left(1+q^2\right)}{2}. \tag{4.8}$$

It is to be noted that $\bar{g}^{\mu\nu}\bar{T}^{EM}_{\mu\nu} = 0$, hence the background space-time is Ricci flat, and $f(R)$ can be replaced by $R + \alpha R^2$ for linear perturbations about the background, as found in Sec. (3.2.3).

4.1.2 Effective fluid of higher derivative modifications in a charged BH background

On simultaneously perturbing the metric and the 4-potential about a Reissner-Nördstrom background and the background 4-potential (2.90), respectively, the perturbed electromagnetic field tensor $F^{(1)}_{\mu\nu}$ can, in the same manner as any other tensor in a spherically symmetric space-time, be split into odd and even parity parts, where components of one parity remains decoupled with the components of the opposite parity in the differential equations they satisfy, i.e.

$$F^{(1)}_{\mu\nu} = F^{(1),E}_{\mu\nu} + F^{(1),O}_{\mu\nu}, \tag{4.9}$$

which follows the following linearized equations

$$\nabla^{\mu} F^{(1)}_{\mu\nu} = 0 \tag{4.10}$$

$$\nabla_{[\lambda} F^{(1)}_{\mu\nu]} \tag{4.11}$$

whereas the gravitational field equations are given by

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} = \kappa^2 T^{(1),EM}_{\mu\nu} + T^{(1),eff}_{\mu\nu} \tag{4.12}$$

$$T^{(1),eff}_{\alpha\beta} = 2\alpha\left(R^{(1)}_{\mu\nu} - \bar{g}_{\mu\nu}\Box R^{(1)} - 2R^{(1)}\bar{R}_{\mu\nu}\right) \tag{4.13}$$

$$T^{(1),EM}_{\alpha\beta} = F^{(1)}_{\alpha\mu}\bar{F}^{\mu}_{\beta} + \bar{F}_{\alpha\mu}F^{(1)\mu}_{\beta} - h^{\mu\nu}\bar{F}_{\alpha\mu}\bar{F}_{\beta\nu} - \frac{1}{4}\left[h_{\alpha\beta}\bar{F}^2 - \bar{g}_{\alpha\beta}h^{\rho\mu}\bar{F}_{\mu\nu}\bar{F}^{\nu}_{\rho} - \bar{g}_{\alpha\beta}h^{\rho\nu}\bar{F}_{\mu\nu}\bar{F}^{\mu}_{\rho} + 2\bar{g}_{\alpha\beta}\bar{F}.F^{(1)}\right]$$

(4.14)

The set of equations (4.10)-(4.12) can be split into two sectors, corresponding to each parity. The odd parity dynamics of both the gravitational and the electromagnetic perturbations are not modified due to preferential coupling of the massive scalar/effective fluid with the even parity sector. Hence, the odd parity equations of motion remain the same as Eq. (2.98). Similarly, the set of quasinormal frequencies that satisfy Eq. (2.98) and the boundary conditions (2.107)-(2.108), also remain the same as that of GR.

The even parity perturbation dynamics gets modified by the presence of the effective fluid of higher derivative modifications. The coefficients of spherical harmonic expansion of the first order effective energy-momentum tensor $T_{\mu\nu}^{(1),eff}$ were given in Eq. (3.12)-(3.15), using which two source terms $\left(S_{\pm}^{eff}\right)$, corresponding to the variables $\Phi_{\pm}^{E}$ respectively, can be defined, which appears in the odd parity dynamics as follows

$$\frac{d^2\Phi_{\pm}^{E}}{dr_*^2}\left(\omega^2 - V_{\pm}^{E}\right) \quad = \quad S_{\pm}^{eff} \tag{4.15}$$

with the following form of the source term

$$S_{\pm}^{eff} \quad = \quad c_{\pm}\tilde{\Phi} + d_{\pm}\frac{d\tilde{\Phi}}{dx} \tag{4.16}$$

where

$$x = y - \frac{\ln\left(y-1\right)}{q^2-1} + \frac{q^4\ln\left(y-q^2 right\right)}{q^2-1} \tag{4.17}$$

is the horizon radius scaled, generalized tortoise coordinate for charged space-times, $\tilde{\Phi} = \frac{4\alpha\Phi}{H}$, $H\left(r\right) \equiv H = k^2 - 2 + \frac{3\left(1+q^2\right)}{y} - \frac{4q^2}{y^2}$. The exact forms of $(c_{\pm}, d_{\pm})$ are complicated and only relevant around the horizon. Hence, $(c_{\pm}, d_{\pm})$ can be expanded about the horizon in powers of g (defined in Chapter 1) as follows, and the leading order contribution comes from $\mathcal{O}\left(\frac{1}{g}\right)$, as shown in the following

$$c_{+} \quad = \quad -\frac{\left(\frac{3}{2}\,q^2 + k^2y - 2\,y + \frac{1}{2}\,\sqrt{9+9\,q^4+\left(-14+16\,k^2\right)q^2} + \frac{3}{2}\right)q\left(q^2y - 16\,q^2 + y\right)\left(q^2y - 2\,q^2 + y\right)}{4gy^7H} \tag{4.18}$$

$$d_+ = \frac{\left(\frac{3}{2}\,q^2 + k^2 y - 2\,y + \frac{1}{2}\,\sqrt{9+9\,q^4+\left(-14+16\,k^2\right)q^2} + \frac{3}{2}\right) q \left(q^2 y - 2\,q^2 + y\right)}{2y^4 g} \tag{4.19}$$

$$c_- = -\frac{3\left(q^2 y - \frac{8}{3}\,q^2 + \frac{1}{3}\,\sqrt{9+9\,q^4+\left(-14+16\,k^2\right)q^2 y + y}\right)\left(q^2 y - 2\,q^2 + y\right)\left(q^2 y - 16\,q^2 + y\right)}{4gy^7 H} \tag{4.20}$$

$$d_- = \frac{3\left(q^2 y - \frac{8}{3}\,q^2 + \frac{1}{3}\,\sqrt{9+9\,q^4+\left(-14+16\,k^2\right)q^2 y + y}\right)\left(q^2 y - 2\,q^2 + y\right)}{2y^4 g} \tag{4.21}$$

see Appendix D for derivation.

4.1.3 Dynamics of the massive scalar field in Reissner-Nördstrom space-times and breaking of isospectrality

The perturbed Ricci scalar dynamics (3.5), after the substitution (3.6), and the transformation to generalized tortoise coordinate (4.17), becomes

$$\frac{d^2\Phi}{dx^2} + \left(\omega^2 - \tilde{V}_{RW}\right)\Phi = 0 \tag{4.22}$$

where the effective potential $\tilde{V}_{RW}$ as experienced by the massive scalar in the charged space-time is given by

$$\tilde{V}_{RW} = \left(1 - \frac{1+q^2}{y} + \frac{q^2}{y^2}\right)\left(\frac{k^2}{y^2} + \frac{1+q^2}{y^3} - \frac{q^2}{y^4} + \frac{1}{6\tilde{\alpha}}\right), \tag{4.23}$$

where $\tilde{\alpha} = \frac{\alpha}{r_H^2}$. At large distances from the black hole $(y \to \infty)$ $\tilde{V}_{RW} \to \frac{1}{6\tilde{\alpha}}$, which gives back the flat space limit of the minimum frequency required to excite the massive field. However, for very small α, the potential can be approximated as

$$\tilde{V}_{RW} \approx \left(1 - \frac{1+q^2}{y} + \frac{q^2}{y^2}\right)\frac{r_H^2}{6\alpha}. \tag{4.24}$$

Thus near small black-holes the minimum frequency (and hence energy) required to excite the massive field is lesser than that of flat space.

Presence of a source term in the RHS of (4.15) and the absence of any in the odd parity equations implies isospectrality, which was broken between the odd and even parity in the uncharged space-time, is broken for the charged BH in $f(R)$ theories as well. Hence, the quasinormal spectrum will be different for odd and even parities, as well as a reduction in the fraction of the incident energy that is radiated away to asymptotic infinity by the even parity mode. A simultaneous measurement of the ratio of radiated odd and even intensities (of the dominant mode) detected at asymptotic infinity and the (dominant) quasinormal frequencies will thus detect or constrain deviations from GR. An estimate of the change in the intensity ratio can be found from the energy-momentum pseudo-tensor of perturbation.

4.2 Energy-momentum pseudo-tensor of perturbation

Ringing BHs relax by radiating energy through gravitational waves to asymptotic infinity. As seen from Sec. (2.6.2), the (small wavelength averaged) perturbed field equations at $\mathcal{O}\left(\epsilon^2\right)$ acts as the back-reaction for the perturbation and quantifies the energy-momentum content of the gravitational waves. Hence, the EM pseudo-tensor of perturbation on a Reissner-Nördstrom space-time, in terms of the redefined metric perturbation $\psi_{\mu\nu}$ (1.27) is given by [107]

$$\begin{aligned} t_{\mu\nu} &= -\frac{1}{4\kappa^2}\left\langle \psi^{\alpha\beta}_{;\mu}\psi_{\alpha\beta;\nu}\right\rangle + \frac{\alpha}{6\kappa^2}\bar{g}_{\mu\nu}\left\langle \left(R^{(1)}\right)^2\right\rangle + \frac{18\alpha^2}{\kappa^2}\left\langle R^{(1)}_{;\mu}R^{(1)}_{;\nu}\right\rangle \\ &\quad +2\kappa^2\left\langle A^{(1)\alpha}_{;\mu}A^{(1)}_{\alpha;\nu}\right\rangle + \kappa^2\langle\mathcal{P}_{\mu\nu}\rangle \end{aligned} \tag{4.25}$$

where the on-shell conditions ($\mathcal{O}\left(\epsilon\right)$ field + Maxwell equations) were used to obtain the above, $\mathcal{O}\left(\kappa^4\right)$ terms were ignored (see Appendix E), and where

$$\begin{aligned} \langle\mathcal{P}_{\alpha\beta}\rangle &= -\frac{1}{2}\bar{F}^{\epsilon}_{\beta}\bar{F}^{\rho}_{\epsilon}\langle\psi_{\alpha}{}^{\tau}\psi_{\rho\tau}\rangle - \frac{3}{2}\bar{F}^{\epsilon}_{\alpha}\bar{F}^{\rho}_{\epsilon}\langle\psi_{\beta}{}^{\tau}\psi_{\rho\tau}\rangle \\ &\quad -\frac{1}{8}\bar{F}^{\epsilon\rho}\bar{F}_{\epsilon\rho}\langle\psi_{\beta}{}^{\tau}\psi_{\alpha\tau}\rangle + \frac{1}{2}\bar{F}^{\epsilon\rho}\bar{F}^{\tau}_{\epsilon}\langle\psi_{\alpha\rho}\psi_{\beta\tau}\rangle \\ &\quad +2\bar{F}^{\epsilon}_{\alpha}\bar{F}^{\rho\tau}\langle\psi_{\beta\rho}\psi_{\epsilon\tau}\rangle - \bar{F}^{\epsilon\rho}\bar{F}^{\tau}_{\epsilon}\langle\psi_{\alpha\beta}\psi_{\rho\tau}\rangle \\ &\quad +\bar{g}_{\alpha\beta}\left[\frac{3}{2}\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle - \bar{F}^{\rho}_{\epsilon}\bar{F}^{\tau}_{\rho}\langle\psi^{\epsilon\mu}\psi_{\mu\tau}\rangle \right. \\ &\quad \left. +\frac{1}{8}\left(\bar{F}\right)^2\langle\psi^2\rangle\right] \end{aligned} \tag{4.26}$$

Similar to what was found in Sec. (3.2.2), near the BH, the contribution of the modification to gravity towards the EM pseudo-tensor is $(\mathcal{O}(\alpha))$, compared to $\mathcal{O}(\alpha^2)$ in flat space-times. Hence, a quantifying measure of the change in relative intensities between the odd and even parity radiated energy due to modifications to gravity can be defined from Eq. (4.25) as

$$\Delta_{\ell m} \equiv \frac{2}{3}\left(1+\frac{(1-C_O)\,A_O^2}{(1-C_E)\,\tilde{A}_E^2}\right)\frac{\alpha}{\tilde{\omega} r_H^2} \tag{4.27}$$

where $\tilde{\omega} = \omega r_H$ is a dimensionless (horizon radius scaled) frequency. From the above, it can be seen that the dimensionless parameter $\Delta^q_{\ell m}$ is a function of the charge, horizon radius, initial amplitudes of the odd and even perturbations, and is directly proportional to the parameter of deviation α. Hence, for a charged BH space-time, the net relative radiated intensity $\Delta^{net}_{\ell m}$ between the odd and even parities is given by

$$\Delta^{net}_{\ell m} = \Delta^{\ell m}_{GR} + \Delta_{\ell m} \tag{4.28}$$

where $\Delta^{\ell m}_{GR}$ was defined in (2.118). Thus, the extra suppression of radiated even parity energy can be quantified through the second term of the above. The quantity $\Delta_{\ell m}$ will be zero for GR and will be greater than the corresponding GR value in $f(R)$ theories. For a special case where the gravitational odd and even parities are equally excited due to a perturbation, Δ_{22} was plotted in Fig. 4.2 as a function of the dimensionless charge q for three values of the horizon radii. A 3-D plot of Δ_{22} was plotted in Fig. 4.1 as a function of the dimensionless charge q and frequency $\tilde{\omega}$, which shows that its behavior is dominated by the $\frac{1}{\tilde{\omega}^2}$ dependence on frequency, and has a very weak dependence on the BH charge — implying that the modification to the EM pseudo-tensor due to $f(R)$ gravity is independent of the BH charge. Observation wise, the quantity $\Delta^{net}_{\ell m}$ will have the form (3.34), with the asymptotic forms of $\tilde{\Phi}_E$ and Φ_O given by Eq. (2.77) and (2.78) respectively. Obtaining the value of the parameter (3.34), and then comparing it with its GR value from numerical simulations of charged BHs can technically be used to detect or constrain deviations from GR. Similarly, the dominant mode quasinormal frequencies of the odd and even parities can also be compared to check if isospectrality

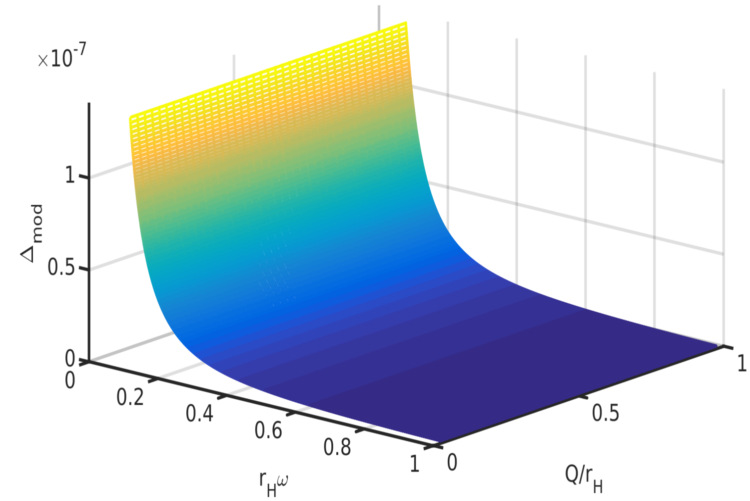
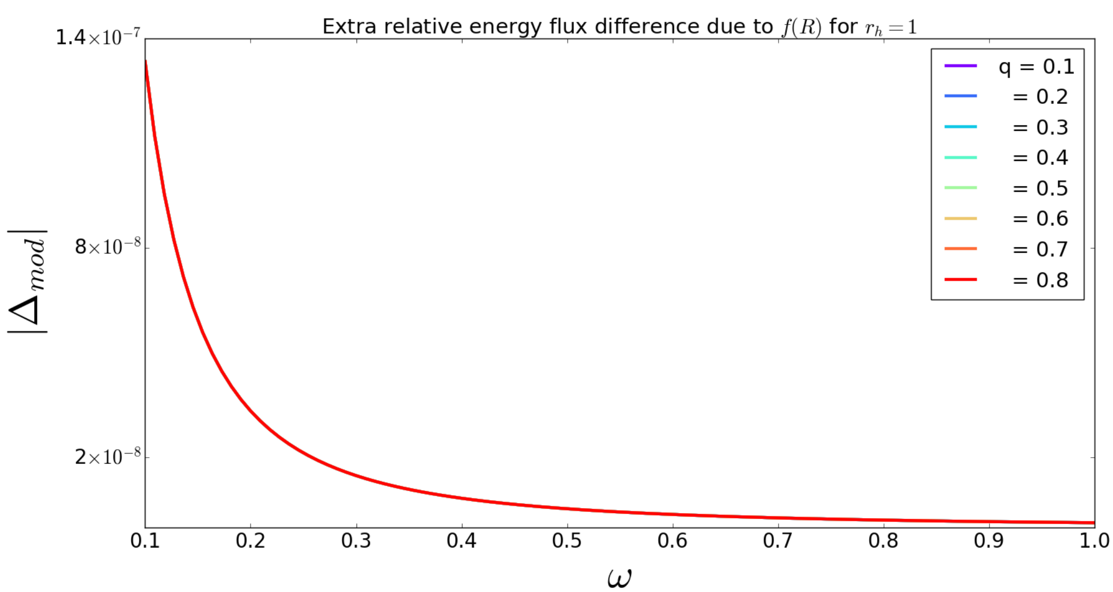


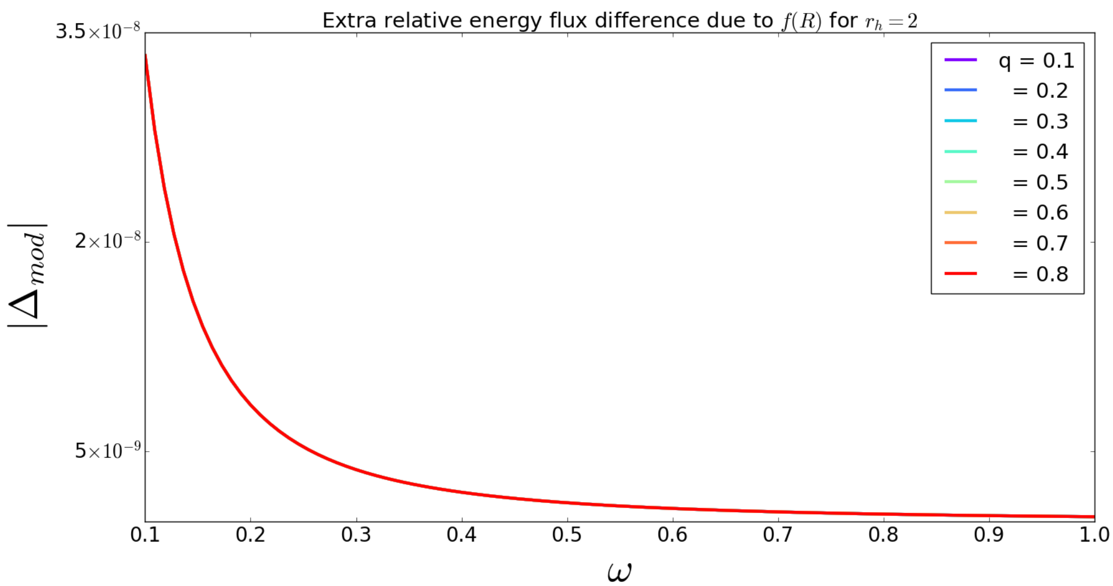
Figure- 4.1: Δ_{22} as a function of the scaled charge q and dimensionless frequency $\tilde{\omega}$ for $r_H = 1$.

holds or breaks. Both of these observations taken together can help ascertain if nature follows GR or a modified theory which limits to GR for weak gravitational fields.

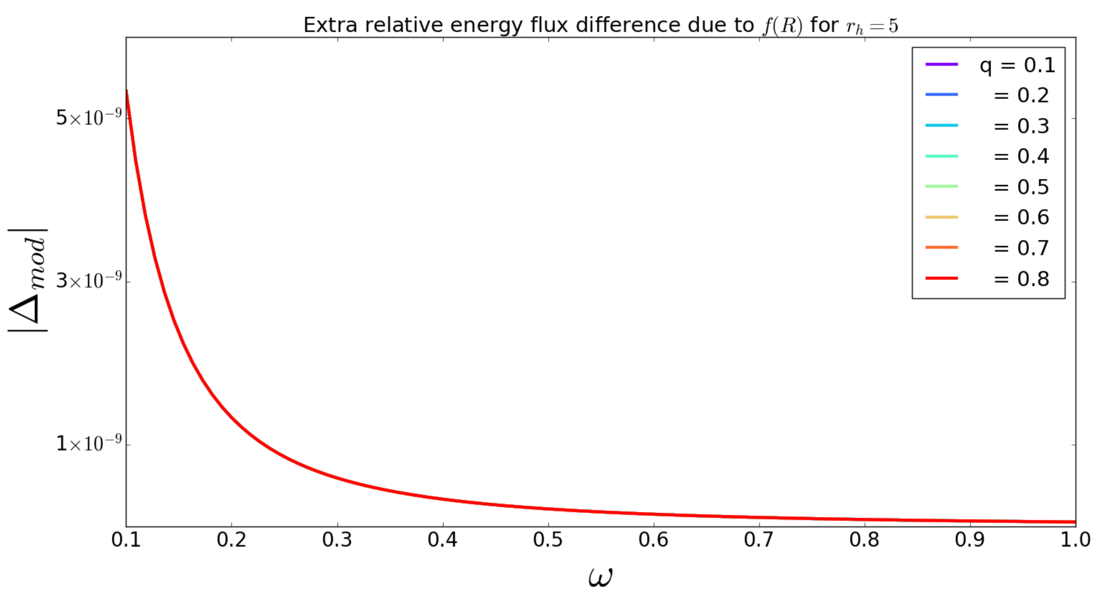
In the next chapter, the effects of Chern Simons modification to gravity on the odd/even parity modes and their observational implications shall be discussed.



(a)



(b)



(c)

Figure- 4.2: Relative energy flux difference Δ_{22} for three different horizon radii ($r_H = 1, 2, 5$). Increasing BH size leads to a reduction in Δ_{22}, as seen from its definition (4.27).

Chapter 5

Observational signatures of Chern Simons gravity from BH ring-down

In this chapter, Chern Simons (CS) modifications to gravity and its signatures on gravitational waves from ringing BHs will be studied, along with their observational consequences. Broadly, as was discussed in Sec. (1.3.4), two types of Chern Simons modifications to gravity exists, canonical CS and dynamical CS theory. CS theories add a pseudo-scalar field of either a fixed form (leading to a preferred direction in a 4-D manifold) or a dynamical one. First, canonical CS and perturbation studies of spherically symmetric BH solutions of canonical CS shall be reviewed, followed by a discussion on problems with the model regarding certain observational predictions, and how dynamical CS gets rid of the problems of canonical CS. The formalism of Chapter 2 will be continued.

This chapter is based on the published results of [117]

5.1 Canonical Chern Simons modification to GR

5.1.1 Action and background

The canonical CS modification to GR was first shown in [69] with the action

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \frac{\alpha}{4} \vartheta {}^*RR \right] \tag{5.1}$$

with the specific form of the pseudo-scalar ϑ, such that its derivative is a time-like constant vector, indicating a preferred direction in the space-time

$$\nabla_\mu \vartheta = v_\mu \equiv \left(\frac{1}{\mu}, \mathbf{0} \right), \tag{5.2}$$

leading to broken diffeomorphism invariance. The amount to which diffeomorphism invariance is violated can be quantified by the Pontryagin scalar ${}^*RR$ defined in Eq. (1.37). The equations of motion force Eq. (1.37) to vanish dynamically, which for Ricci flat space-times are given by

$$R_{\mu\nu} = -2\kappa^2 \alpha C_{\mu\nu} \tag{5.3}$$

where $C_{\mu\nu}$ was defined in Eq. (1.40). For the given form of Eq. (5.2), it was shown by [69] that the Cotton tensor $C_{\mu\nu}$ vanishes identically in spherically symmetric space-times leading to Eq. (5.3) supporting the Schwarzschild solution [69].

5.1.2 Gravitational perturbation about a Schwarzschild background

It was shown in [118] that perturbing about a Schwarzschild background solution in canonical CS gravity lead to mixing of two opposite parities of perturbations. Specifically, it was shown that odd (even) parity equations of motion had even (odd) parity metric components and their derivatives as source terms, leading to a coupled dynamics between even and odd parities. Furthermore, it was shown that the linearized Pontryagin scalar relates to the odd parity master function as follows

$${}^*RR^{(1)} = -\frac{24M}{r^6} \frac{(\ell+2)!}{(\ell-2)!} \Phi_O \mathbf{S} \tag{5.4}$$

where **S** was defined in Sec. (2.2). Since the quantity ${}^*RR$ is dynamically suppressed by the equations of motion, the above relation indicates that the odd parity master function is also suppressed dynamically and gravitational energy radiated through the odd parity mode is severely reduced. However, as found by [118], the even parity is dynamically coupled to the odd parity through the linearized equations of motion, leading to energy exchange between the two modes, and thereby drastically reducing/freezing the system to emission of gravitational waves altogether — an effect that should lead to no gravitational waves being detected at asymptotic infinity. This becomes a pathology for the canonical CS gravity, and can be solved by modifying the pseudo-scalar of canonical CS to have full dynamic dependence, namely dynamical CS modification to gravity.

5.2 Dynamical Chern Simons modification to GR

5.2.1 Background

Dynamical CS solves the problem of freezing of ring-down emission in canonical CS by making the pseudo-scalar into a dynamical variable, i.e. $\vartheta \equiv \vartheta\left(x^{\mu}\right)$. This leads to kinetic terms of the pseudo-scalar appearing on the action, as well as a possible potential term, as was shown in Eq. (1.36). For spherically symmetric backgrounds, the Cotton tensor vanishes, leaving only the second term in the RHS of Eq. (1.38). This amounts to adding a scalar field to a spherically symmetric space-time in GR. But, in [76], it was shown that a scalar field cannot exist in a stable configuration in a spherically symmetric BH space-time. Hence, the Schwarzschild solution remains a solution of the dynamical CS modification to GR as well, given which, perturbations about the background can be expanded in spherical harmonic tensors.

5.2.2 Perturbations about a Schwarzschild background

Linearizing Eq. (1.38) and (1.39) about a Schwarzschild background leads to the following

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} = -2\kappa^2\alpha C^{(1)}_{\mu\nu} \tag{5.5}$$

$$\Box\vartheta = -\frac{\alpha}{4\beta}{}^{*}RR^{(1)}, \tag{5.6}$$

where $\vartheta^{(1)} \equiv \vartheta$ was used, implying the CS pseudo-scalar appears only at first order of perturbation. Expanding the metric perturbation $h_{\mu\nu}$ in tensor spherical harmonics, ϑ in scalar spherical harmonics in the same way as Eq. (3.6), expanding in plane waves, and transforming to tortoise coordinates one obtains two coupled equations of motion for the odd parity perturbations and the CS pseudo-scalar [117, 119]

$$\frac{d^2\Phi_O}{dr_*^2} + \left(\omega^2 - V_O\right)\Phi_O = S^{eff} \tag{5.7}$$

$$\frac{d^2\varphi}{dr_*^2} + \left(\omega^2 - V_\varphi\right)\varphi = \frac{6\alpha\mu M f}{\beta r^5}\Phi_O \tag{5.8}$$

$$\mu = (\ell-1)\,\ell\,(\ell+1)\,(\ell+2) \tag{5.9}$$

φ is related to ϑ as [119, 120]

$$\vartheta\left(t, r, \Omega\right) = \frac{\varphi\left(r\right)}{r}\mathbf{S}(\Omega)e^{i\omega t} \tag{5.10}$$

and $\mathbf{S}(\Omega)$ is a scalar spherical harmonic function. V_O and V_φ are the odd parity effective potential and the effective potential for a massless spin-0 field (corresponding to ϑ), respectively. S^{eff} is given by

$$S^{eff} = \frac{\kappa^2\alpha}{(\ell-1)\,(\ell+2)}\left[\frac{6M}{r}\partial^2_{r_*}\varphi - \frac{12M}{r^2}\partial_{r_*}\varphi + \frac{6\omega^2 M}{r}\varphi\right] \tag{5.11}$$

details of which have been given in Appendix F. In stark contrast to the case of $f(R)$ theories, it is the odd parity that the pseudo-scalar couples with, leaving the even parity perturbations untouched, which follow the same dynamics as in GR. This parity preferential coupling leads to energy exchange between the odd parity sector and the pseudo-scalar, with the ringing BH being allowed an extra channel through which to radiate gravitational energy and relax. Hence, the amount of gravitational radiation that is emitted through the odd parity channel is reduced, a feature that can act as a distinguishing tool between GR and CS modified gravity in the context of observations [117].

An estimate of the relative intensities of the odd-to-even parity emissions can be found by calculating the energy-momentum tensor of perturbation, which will be shown in the following section.

5.3 Relative energetic difference between odd and even parities

5.3.1 First order perturbation of a general space-time in dynamical CS gravity

For a general vacuum space-time (external space-times of BHs) in dynamical CS gravity, the linearized field equations in terms of the transverse-traceless variable $\psi_{\mu\nu}$, defined in Eq. (2.83), is given by [117]

$$\Box\psi_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\psi^{\alpha\beta} = 2\kappa^2\alpha\vartheta_{;\tau\sigma}\left({}^*\bar{R}^{\tau}{}_{\mu}{}^{\sigma}{}_{\nu} + {}^*\bar{R}^{\tau}{}_{\nu}{}^{\sigma}{}_{\mu}\right) \tag{5.12}$$

$$\begin{aligned}\Box\vartheta = -\frac{\alpha}{4\beta}\Big[&2\psi^{\mu\nu;\beta\alpha}\left({}^*\bar{R}_{\mu\alpha\nu\beta} + {}^*\bar{R}_{\mu\beta\nu\alpha}\right)\\ &+\bar{R}^{\alpha\beta\gamma\mu}\left({}^*\bar{R}_{\alpha}{}^{\nu}{}_{\gamma\mu}\psi_{\beta\nu} + {}^*\bar{R}_{\alpha\beta\sigma\mu}\right)\psi^{\sigma}_{\gamma}\Big]\end{aligned} \tag{5.13}$$

It is important to note that in the asymptotic limit Eqs. (5.12) and (5.13) decouple, and both $\psi_{\mu\nu}$ and ϑ is a light field, which is in contrast to $f(R)$ gravity where the extra (intrinsic) degree of freedom was a massive one. The above equations form the *on-shell* conditions that will be used to obtain the energy-momentum pseudo-tensor of perturbation.

5.3.2 Energy-momentum pseudo-tensor of perturbation

The perturbed field equations of dynamical CS at $\mathcal{O}\left(\epsilon^2\right)$ due to a linear metric perturbation and pseudo-scalar perturbation is given by

$$\mathfrak{G}_{\mu\nu}^{(2)} = -2\epsilon^2\kappa^2\alpha C_{\mu\nu}^{(2)} - \epsilon^2 G_{\mu\nu}^{(2)} + \epsilon^2\kappa^2\beta\vartheta_{;\mu}\vartheta_{;\nu} \tag{5.14}$$

where $\mathfrak{G}_{\mu\nu}$ is the modified field tensor for dynamical CS gravity and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein field tensor . It should be noted that the term in the equations of motion arising out of the kinetic term of the pseudo-scalar in the action appears only at second order of perturbation. After averaging over small wavelengths, as in Sec. (2.6.2) , transforming from $h_{\mu\nu}$ to $\psi_{\mu\nu}$, and using the on-shell conditions (5.12) and (5.13), one obtains the energy-momentum pseudo-tensor of perturbation as [117]

$$t_{\mu\nu} = \frac{1}{4}\left\langle\psi^{\rho\tau}_{;\mu}\psi_{\rho\tau;\nu}\right\rangle + \frac{\kappa^2\alpha^2}{2\beta}\left\langle\mathcal{P}_{\mu\nu}\right\rangle - \kappa^2\beta\left\langle\vartheta_{;\mu}\vartheta_{;\nu}\right\rangle \tag{5.15}$$

$$\begin{aligned}\left\langle\mathcal{P}_{\mu\nu}\right\rangle = {} & -2\left\langle\psi^{\beta\gamma;\delta\lambda}\psi_{\nu\alpha;\sigma}{}^{;\rho}\right\rangle\epsilon_{\mu\rho}{}^{\sigma\alpha}\left({}^*\bar{R}_{\lambda\beta\delta\gamma} + {}^*\bar{R}_{\delta\beta\lambda\gamma}\right) - 2\left\langle\psi^{\beta\gamma;\delta\lambda}\psi_{\mu\alpha;\sigma}{}^{;\rho}\right\rangle\epsilon_{\nu\rho}{}^{\sigma\alpha}\left({}^*\bar{R}_{\lambda\beta\delta\gamma}\right. \\ & \left.+{}^*\bar{R}_{\delta\beta\lambda\gamma}\right) - 2\left\langle\psi^{\rho\sigma;\delta}\psi^{\alpha\beta;\gamma}\right\rangle\left[{}^*\bar{R}_{\gamma\alpha\delta\beta}\left({}^*\bar{R}_{\mu\sigma\nu\rho} + {}^*\bar{R}_{\nu\sigma\mu\rho}\right) + {}^*\bar{R}_{\delta\alpha\gamma\beta}\left({}^*\bar{R}_{\mu\sigma\nu\rho} + {}^*\bar{R}_{\nu\sigma\mu\rho}\right)\right] \\ & +\bar{R}^{\rho\sigma\alpha\beta}\left[\epsilon_{\mu\gamma}{}^{\delta\lambda}\left(\left\langle\psi^{\eta;\gamma}_{\sigma}\psi_{\nu\lambda;\delta}\right\rangle{}^*\bar{R}_{\rho\eta\alpha\beta} + \left\langle\psi^{\eta;\gamma}_{\alpha}\psi_{\nu\lambda;\delta}\right\rangle{}^*\bar{R}_{\rho\sigma\eta\beta}\right) + \epsilon_{\nu\lambda}{}^{\delta\gamma}\left(\left\langle\psi^{\eta;\gamma}_{\sigma}\psi_{\mu\lambda;\delta}\right\rangle{}^*\bar{R}_{\rho\eta\alpha\beta}\right.\right. \\ & \left.\left. + \left\langle\psi^{\eta;\gamma}_{\alpha}\psi_{\mu\lambda;\delta}\right\rangle{}^*\bar{R}_{\rho\sigma\eta\beta}\right)\right] + \bar{R}^{\rho\sigma\alpha\beta}\left[\left\langle\psi^{\gamma\delta}\psi^{\lambda}_{\alpha}\right\rangle{}^*\bar{R}_{\rho\sigma\lambda\beta}\left({}^*\bar{R}_{\mu\delta\nu\gamma} + {}^*\bar{R}_{\nu\delta\mu\gamma}\right)\right. \\ & \left. + \left\langle\psi^{\gamma\delta}\psi^{\lambda}_{\sigma}\right\rangle{}^*\bar{R}_{\rho\lambda\alpha\beta}\left({}^*\bar{R}_{\mu\delta\nu\gamma} + {}^*\bar{R}_{\nu\delta\mu\gamma}\right)\right]\end{aligned} \tag{5.16}$$

The derivation of the above has been given in Appendix G. A number of things need to be noted about Eq. (5.15). Firstly, the first term in the RHS contains the energy-momentum of the gravitational waves, while the second and third appear due to dynamical CS modification to GR, similar to as one sees for $f(R)$ in Eq. (3.31). Secondly, unlike the massive scalar of $f(R)$, only first derivatives of the CS pseudo-scalar appear in Eq. (5.15). Thirdly, instead of the potential term, second term in the RHS of Eq. (3.31), a graviton-graviton coupling term appears whose form is given in Eq. (5.16) — a term whose contributions drops off quickly (due to the presence of the background Riemann tensor as a product) as one goes away from the BH. While the dominant modified term at large distances could be the third term of Eq. (5.15), corresponding to the kinetic term of the pseudo-scalar; nearer to a BH, the second term can have a stronger impact owing to the bounds on the constants that were found from [119, 121], i.e., $\beta > \frac{\alpha^2}{\beta}$. The second term can have a markedly stronger effect close to small BHs, owing to the fact that smaller BHs have stronger Riemann curvatures.

In order to get an estimate of the energy density due to the CS modification, one tries to obtain the approximate form of the various fields as a function of the radial distance and some background parameter L, which will be discussed below. From the form of (5.16), it is clear that it does not correspond to the energy-momentum content of a propagating field, but gets absorbed by the BH. An estimate of its energy density can be made by considering its 00 component, which at leading order of r has the form

$$\langle \mathcal{P}_{00} \rangle \sim \frac{\kappa^2 \alpha^2}{\beta} \frac{1}{L r^5}, \tag{5.17}$$

given the metric perturbation, and the background Riemann has the form

$$\psi_{\mu\nu} \sim \frac{L}{r} \tag{5.18}$$

$$\bar{R}_{\mu\nu\rho\sigma} \sim \frac{L}{r^3}. \tag{5.19}$$

while the first term in the RHS of Eq. (5.15) can be represented as

$$\left\langle \psi^{\rho\tau}_{;0} \psi_{\rho\tau;0} \right\rangle \sim \frac{1}{r^2}. \tag{5.20}$$

Using the above approximations, RHS of Eq. (5.15) can be written in terms of dimensionless variables as

$$\begin{aligned} t_{\mu\nu} &= -\frac{\epsilon^2}{4L^2} \left\langle \widetilde{(\nabla\psi)^2} \right\rangle_{\mu\nu} - \frac{\epsilon^2\kappa^2\alpha^2}{4\beta L^6} \left[\widetilde{\mathbf{R}} \left\langle \widetilde{\nabla^2\psi\nabla^2\psi} \right\rangle_{\mu\nu} + \widetilde{\mathbf{RR}} \left\langle \widetilde{\nabla\psi\nabla\psi} \right\rangle_{\mu\nu} + \widetilde{\mathbf{RRR}} \widetilde{\langle \psi\psi \rangle}_{\mu\nu} \right] \\ &+ \frac{\epsilon^2\kappa^2\beta}{L^2} \left\langle \widetilde{\nabla\vartheta\nabla\vartheta} \right\rangle_{\mu\nu} \end{aligned} \tag{5.21}$$

where $\psi_{\mu\nu}$, ϑ and their derivatives have been scaled with respect to a characteristic length scale L of the background space-time (which in this case is of the order of the size of the photon sphere around a black-hole, which was discussed in Sec. (1.1.1)) such that quantities inside the angular brackets are dimensionless. $\widetilde{\mathbf{R}}$ is the shorthand notation for dimensionless Riemann tensor. Hence, a ratio can be defined between the second and the first term, indicating the relative energy density of the graviton-graviton coupling term with respect to the emitted gravitational waves, as [117]

$$\Delta_{CS} = \frac{\kappa^2\alpha^2}{\beta}\frac{1}{Lr^3} \tag{5.22}$$

Scaling the radial variable with respect to the background characteristic length scale as $r = yL$,

$$\Delta_{CS} = \frac{\kappa^2\alpha^2}{\beta L^4}\frac{1}{y^3} \tag{5.23}$$

the $\frac{1}{y^3}$ dimensionless factor can be integrated out to give a factor which will not change the approximate order of Δ_{CS}, hence one obtains,

$$\Delta_{CS} = \frac{\kappa^2\alpha^2}{\beta L^4}. \tag{5.24}$$

The above indicates the relative energy density content of the second term in the RHS of (5.15), whose form is given in Eq. (5.16), with respect to the radiated gravitational wave energy density will be more for smaller BHs.

The energy density of the kinetic term of the pseudo-scalar is a non-trivial problem since it involves the simultaneous solving of Eq. (5.7) and Eq.(5.8) for the form of ϑ. The author is not aware of the existence of any such technique in the literature.

5.3.3 Observational signatures

An effect of CS modification to GR is imparting elliptical polarization to the linear polarization tensor of GR, as seen from Eq. (I.3). Similar to the calculation of Sec. (2.6.1), a connection between the elliptical polarization and the master functions can be established in the following manner

$$\tilde{h}_+ - i\tilde{h}_\times \simeq \frac{1}{r}\sum_{\ell,m}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\Phi_E + i\Phi_O\right)\, {}_{-2}Y_{\ell m} \tag{5.25}$$

whose derivation and definitions of $\tilde{h}_{+/\pm}$ have been given in Appendix I.

The rate at which gravitational radiation escapes to asymptotic infinity in GR, given in terms of the odd and even master functions, was defined in Eq.(2.89). For dynamical CS gravity, the rate at which radiation (both gravitational and scalar) escapes to asymptotic infinity can be obtained terms of the odd/even parity master functions and CS pseudo-scalar as [117]

$$\left\langle \dot{E} \right\rangle\Big|_{CS} = \frac{1}{64\pi}\sum_{\ell m}\mu\left\langle \left|\dot{\Phi}_E\right|^2 + \left|\dot{\tilde{\Phi}}_O\right|^2 + \kappa^2\beta\left|\dot{\varphi}\right|^2 \right\rangle \tag{5.26}$$

where φ was defined in Eq. (5.10). There is also energy loss $\left\langle \dot{E}_{coup,CS} \right\rangle$ in the form of the graviton-graviton coupling near the BH region, whose form was given in Eq. (5.16), does not travel to asymptotic infinity, thereby effectively reducing the odd parity reflection coefficient, or the fraction of the odd parity initial excitation that gets scattered off to asymptotic infinity, compared to GR. Considering the same initial perturbation energy for a Schwarzschild solution in GR and dynamical CS, the latter shall then radiate lesser gravitational flux, with the difference in energy coming from both the graviton-graviton coupling (which absorbed by the BH), and the kinetic term of the pseudoscalar field. Thus, one can write the following

$$\left\langle \dot{E} \right\rangle\Big|_{CS} + \left\langle \dot{E} \right\rangle\Big|_{coup,CS} = \left\langle \dot{E} \right\rangle\Big|_{GR} \tag{5.27}$$

from which one obtains the following inequality

$$\left|\dot{\Phi}_O\right|^2 > \left|\dot{\tilde{\Phi}}_O\right|^2 \tag{5.28}$$

at all times. A suitable ansatz for the modified odd parity wavefunction for CS gravity can be

$$\tilde{\Phi}_O = \tilde{A}_O e^{-\tilde{\kappa}_O t} e^{i\tilde{\omega}_O t} \tag{5.29}$$

where $\tilde{A}_O < A_O$, the real and imaginary parts of the odd parity QNM frequency are modified due to the coupling with the CS field in the form of an inhomogeneous term in the RHS of the differential equation (9) in the main text. This leads to the following

$$\frac{\left|\dot{\tilde{\Phi}}_O\right|^2}{\left|\dot{\Phi}_E\right|^2} = \frac{\tilde{A}_O^2\left(\tilde{\kappa}_O^2 + \tilde{\omega}_O^2\right)}{A_E^2\left(\kappa_E^2 + \omega_E^2\right)} e^{-2(\tilde{\kappa}_O - \kappa_E)t} \tag{5.30}$$

which is less than the corresponding GR value at all times courtesy inequality (5.28), with a growth/decay rate proportional to $e^{-2(\tilde{\kappa}_O - \kappa_E)t}$ (depending on whether the imaginary part of the odd parity dominant mode frequency is enhanced or suppressed due to CS modification). However, for the same initial energy of perturbation, the odd parity mode can now relax to a stable Schwarzschild faster, because of the presence of further channels (pseudo-scalar and graviton-graviton coupling) to take away the initial perturbation energy. This leads to a shorter modified decay time for the odd parity mode compared to the even parity, , i.e. $\tilde{\kappa}_O > \kappa_E$ — implying that the quantity (5.30), and correspondingly

$$\Delta_{\ell m} = \frac{\left|\dot{\Phi}_O\right|^2 - \left|\dot{\Phi}_E\right|^2}{\left|\dot{\Phi}_O\right|^2 + \left|\dot{\Phi}_E\right|^2}, \tag{5.31}$$

will be a decreasing function of time in CS gravity [117], the derivation of which can be found in Appendix H. This feature can be directly tested from observations, given one has the values of the functions $\Phi_{E/O}$, calculated from the dominant multipole complex coefficient of the spherically decomposed gravitational strain, as was shown in Eq. (2.77) and (2.78). Any non-constant value of the observed $\Delta_{\ell m}$, thus, may signal deviation from GR.

Chapter 6

Conclusions and future outlook

6.1 Conclusions

In the current thesis, it was shown that various modifications to gravity leave telltale signatures in the gravitational wave emitted by ringing BH solutions of the theory. Depending on parity conservation or violation of the modified theory, a dimensionless parameter can be defined which behaves in two different manner, making it a quantifier for deviations from GR. The current thesis dealt with two such theories, one which conserves, like $f(R)$ gravity, and the other violates parity, dynamical Chern Simons modifications to GR.

Ringing BHs in GR relax by emitting gravitational waves in the form of damped sinusoids, whose wave description involves complex frequencies. A review of BH perturbation literature was done in Chapter 2, where it was shown that a perturbed spherically symmetric BH can relax by emitting gravitational waves through two channels of opposite parities — odd and even. It is known in the literature that for GR the frequency and damping times (corresponding to the real and imaginary parts of the complex frequency) of both the parities are equal, as well as the fraction of the initial energy of perturbation that travel to asymptotic infinity (the reflection coefficient of a wave scattering process). The "fragile" equality of the frequency and reflection coefficients of the two opposite parity channels hence serve as a feature that can be used to test GR at strong gravity regimes.

Based on works of [105], it was also shown in Chapter 2 that perturbation of charged BHs in GR lead to a suppression of the radiated gravitational energy in general, with the even parity radiating lesser than the odd parity. This is because the net scattered energy off of a charged BH from a purely gravitational initial perturbation will contain electromagnetic waves, with the conversion from gravitational to electromagnetic energy being higher for the even parity compared to the odd. The difference in relative suppression depends on the central charge of the BH, a feature using which one can define a dimensionless parameter quantifying the relative suppression, which in turn can technically show up in gravitational wave observations. Even though the energy radiation is different, perturbation about a Reissner-Nördtrom space-time in GR guarantee the same frequency and damping times for both parities, preserving isospectrality.

In Chapter 3, a specific form of $f(R) = R + \alpha R^2$ was taken and a perturbation about a Schwarzschild background solution of the theory was performed. $f(R)$ being a theory with an extra intrinsic massive scalar degree of freedom (other than the massless tensor modes), it was shown that the perturbed Ricci scalar acts as the extra massive mode in Chapter 1. While in Chapter 3 it was shown that the massive mode corresponding to the Ricci scalar has an independent dynamics, unaffected by the massless tensor modes, seen in Eq. (3.8). The massive mode faces an increasing potential wall, Eq. (3.9), whose value saturates to $\frac{1}{6\alpha}$ at asymptotic infinity, which for small α corresponds to a large potential barrier, and a non-scattering one (unlike the potentials faced by the two tensor modes). Similarly, the dynamics of the odd parity mode also remains unaffected by the introduction of the massive scalar, as seen in Eq. (3.20). However, the even parity mode gets affected, with the massive scalar acting as a source term to the even parity dynamics, as seen from Eq. (3.21). This leads to an energy share between the massive scalar and the even parity perturbation tensor, suppressing the reflection coefficient of the even parity, as well as modifying its quasinormal frequencies.

It was also shown in Sec. (3.2) of Chapter 3 that the effect of higher derivative modifications to gravity shows up in the energy-momentum content of the radiated gravitational wave at $\mathcal{O}(\alpha^2)$ in Minkowski space-times. However, the same shows up at $\mathcal{O}(\alpha)$ close to BHs or in general curved space-times, with the energy-momentum content corresponding to the modification depending inversely on the BH size, implying that at the observation level, smaller BHs can act as better test bed for constraining deviations from GR. A dimensionless parameter $\Delta_{\ell m}$ was defined in Eq. (3.34) (which remains constant in GR) which is time dependent for $f(R)$ theories, and is greater than the corresponding GR value. Observed values of $\Delta_{\ell m}$ can help put strict constraints on the parameter α. It was also shown that any general $f(R)$ theory that can be written in a polynomial form has signatures in gravitational waves that are identical to that imparted by $R + \alpha R^2$ theory.

In Chapter 4, charged BH solutions in $f(R)$ gravity were considered and their perturbations studied. Preferential coupling of the massive scalar degree of freedom to the even parity wavefunction (which was shown to be a linear combination of gravitational and electromagnetic master functions), make sure that the even parity effective reflection coefficient is reduced. It was shown that this leads to further suppression of energy through the gravitational even parity channel, modifying the relative radiated intensity factor $\Delta_{\ell m}$, defined in (5.31), between the odd and even parities. Ring-down of charged BH solutions in GR can be numerically simulated for various charges and initial perturbation amplitudes ($A_{O/E}$) and $\Delta_{\ell m}$ can be calculated for various multipole numbers. Comparing simulated values with the observed values of $\Delta_{\ell m}$ may indicate deviation from GR. Isospectrality, or the equality of the real and imaginary parts of the quasinormal frequencies of the odd and even parities for all multipoles, is broken in charged BH, as was shown in Sec. (4.1.2), with the odd parity quasinormal frequencies staying the same as GR. Simultaneous measurement of the relative amplitude factor $\Delta_{\ell m}$ and the frequency shift can put strong constraints on the parameter α.

In Chapter 5, perturbation studies of Schwarzschild BHs in canonical and dynamical Chern Simons modification to GR were compared. Certain pathologies of canonical CS was shown to be absent in dynamical CS gravity. It was shown that the parity violating CS scalar field couples only with the odd parity gravitational degree of freedom leaving the even parity unaffected. This leads to an exchange of energy between the odd parity perturbation and the CS pseudo-scalar, reducing the radiated odd parity gravitational energy, hence modifying the relative intensity factor $\Delta_{\ell m}$. The difference in energy was found to be radiated partly through a kinetic term of the CS pseudo-scalar, and part of it is absorbed by the BH. It was shown that the difference in the relative intensity factor of GR and dynamical CS gravity is higher for smaller BHs. An effect of CS modification to GR is to impart elliptical polarization to the linear polarization of GR, as was seen in Eq. (I.3). An asymptotic connection was established between the even/odd parity master functions and the elliptical polarization amplitudes, from which observed $\Delta_{\ell m}$ can put strong constraints on the parameter $\frac{\alpha^2}{\beta}$.

The two kinds of modifications to gravity that were discussed in this thesis have certain similarities as well as differences. While isospectrality breaking is the similar feature for both types of modifications, the relative intensity factor $\Delta_{\ell m}$ will be more than the corresponding GR value for $f(R)$ modification to gravity, whereas for dynamical CS modification to GR, it is less than the corresponding GR value. Based on whether the observed value is more or less than the GR value obtained from simulations, the question of whether nature follows a parity conserving or violating modification to gravity can be answered. One scenario is possible where the action has both $f(R)$ and CS terms and by fine tuning of the coupling parameters, $\Delta_{\ell m}$ can be made to have the corresponding GR value. But a situation like that is very special and is unlikely to be true in nature.

An obvious generalization of the analysis of the current thesis is to rotating spacetimes. Kerr black holes, being Ricci flat, are solutions of $f(R)$ theories. However, owing to less symmetry in the system, separating angular and radial functions using spheroidal harmonics is a non-trivial problem even in GR and the author is not aware of any work that has successfully separated the same in the metric perturbation approach. However, using the NP formalism [88], Teukolsky [122] managed to obtain two decoupled master equations corresponding to the radial and the angular parts in GR. Recently, in Ref. [123], using the NP formalism, the authors have obtained two decoupled equations for a Kerr solution in $f(R)$ gravity. Since, the analysis of the energy-momentum pseudo-tensor of perturbation is background independent, the master functions of [123] can be calculated at asymptotic infinity in terms of the polarization amplitudes $h_{+/\times}$, which can help in putting bounds on the parameter α from the most abundant astrophysical source for ring-down gravitational waves, a spinning BH.

A fast spinning BH solution in CS modified gravity is yet to be found, since the background Pontryagin density is non-vanishing, hence putting a constraint on how fast a spinning BH in CS modified gravity can rotate. However, as was shown in Ref. [124], slowly rotating BH solutions can be obtained, and in general, faster rotating solutions can be obtained perturbatively in powers of the spin parameter a as shown in [125]. As was shown by [126], even though odd and even parity dynamics get coupled from $\mathcal{O}\left(a^2\right)$ onward, the mixing of modes do not take place for the same multipole numbers of opposite parities. For example, the $(2,2)$ even parity mode does not couple with the odd parity mode of the same multipole index, ensuring no exchange of energy takes place between the two modes and the relative intensity ratio will still indicate possible deviation from GR.

The detectability of the effects of parity (conserving/violating) hinges on the ability to detect QNMs. Since the Signal-to-Noise-ratio in the ring-down regime is small for current generation of detectors, one can, at most, estimate the fundamental mode $(n = 1, \ell = 2, m = 2)$ from the ring-down data. [127] outlines a method to obtain higher overtones and modes by a coherent stacking of the data of multiple events. One possible future work can be to extract the odd/even signal from the detected gravitational strain (preferably using an event with a small final spin) in the ring-down regime, following the concept outlined in Sec. 5.3.3, and then fitting them to a damped sinusoid to estimate the amplitude, frequency, and damping times of each parity. Estimates on the three parameters of each parity can help us quantify the detected $\Delta_{\ell,m}$, which can be compared to GR numerical relativity studies of systems with the same parameters (like mass and spin of individual BHs as estimated from the inspiral regime).

Isospectrality breaks in GR due to environmental contaminants (accretion disks, dark matter halo) around BHs [128]. [129, 130] show that one can account for contaminants by considering their effects as a perturbation on the effective potentials, thereby shifting the QNM spectra. However, the studies only consider the Regge-Wheeler (odd parity) potential and possible perturbative modifications to the same due to contaminants. It is possible to compute shifts in the quasinormal spectra for a 'dirty BH' in the $f(R)$ or Chern-Simons case for both parities as a function of two parameters, each corresponding to the perturbative change in the effective potentials in the respective parities. But broken isospectrality (due to the combined effect of modification to gravity and contaminants), in general, will lead to the odd and even parity potentials modifying differently, thereby resulting in different shifts of the quasinormal spectra (compared to a clean BH in a modified theory) for the odd and even parities. While it is technically possible in multi-messenger astronomy to observe accretion disks that are luminous through electro-magnetic follow ups; thin and sparse accretion disks, and dark matter halo will not be accounted for from such follow ups or simultaneous observations. [131] show that certain models of contaminants can significantly affect the possibility of testing GR. However, environmental contaminants around a BH would vary with each detection and would show up as different $\Delta_{\ell,m}$ and different shifts in the odd/even spectra in different cases. However, a possible modification to GR is an universal effect, and its contribution to the value of $\Delta_{\ell,m}$ must remain constant (even if overshadowed by factors like environmental contaminants) in all observations.

6.2 Future outlook

With subsequent upgradation of the technology used to detect gravitational waves, like aLIGO [132], LISA [133], and Cosmic Explorer [134], it will be possible to obtain *signal-to-noise ratio* of the ring-down regime up to 50 [135]. Better energy and time resolution of the ring-down regime will help in calculating the factor $\Delta_{\ell m}$ to more accuracy, and consequently, constrain deviations from GR.

Numerical simulations of the ring-down of charged BH in GR can help in expanding the parameter space for creation of waveforms, which are ultimately matched with the detected signal. Even though astrophysical BHs are assumed to be charge neutral, it is possible that the rate at which gravitational energy escapes the ringing system is faster than the rate at which charge neutralization by the infall of accreting opposite charge takes place, which may show up in observations as the rate of detection goes up in the future.

The chances of a direct detection of extra degrees of freedom can be increased by making detectors more sensitive towards scalar or vector degrees of freedom. Since the CS pseudo-scalar is a long range light field, it is possible that it can be detected.

Better theoretical understanding of the Kerr perturbation of various modified theories of gravity is required, which can help in connecting the two propagating modes from perturbed Kerr BHs with the plus and cross amplitudes at asymptotic infinity. If a preferential coupling phenomenon due to conserving/violating of parity exists in the perturbed equations, the phenomenon could be used to calculate the NP observables and their relative ratios.

Since the presence of quasi-stable accreting disk of matter outside a ringing BH modifies dynamics and breaks isospectrality, with the increase in the number of detectors, and availability of better imaging techniques like Event Horizon Telescope, it may be possible to directly image the accreting matter, and compare the electromagnetic observation obtained properties of the accreting mass with the properties obtained from gravitational wave observations affected by the degree of isospectrality breaking. Differences in the properties obtained from two different kinds of observations can help in constraining deviation from GR and/or probing the limits of currently available and future BH imaging techniques.

Appendix A

The Pöschl-Teller method

To calculate $C_{V/S}$ one needs to calculate the absolute ($\sqrt{R_{\pm}^{V/S}}$) and phase ($\delta_{\pm,(r)}^{V/S}$) parts of the reflection amplitude of the form $\sqrt{R_{\pm}^{V/S}}e^{i\delta_{\pm,(r)}^{V/S}}$ - which can be found by utilizing the method used in [91]. In this scheme the potentials $V_{\pm}^{V/S}$ are replaced by a properly parametrized Pöschl-Teller potential which is of the form

$$U_{PT}(x) = \frac{U_0}{\cosh^2\left[\beta(x - x_0)\right]} \tag{A.1}$$

where $U_0 = U_{PT}(x_0)$ is the maximum value of the potential and $\beta = -\sqrt{\frac{1}{2U_0}\left.\frac{d^2U_{PT}}{dx^2}\right|_{x=x_0}}$ is the curvature about the maximum. Reflection amplitude for this potential was found from [91]

$$R(\omega) = \frac{\Gamma\left(\frac{-i\omega}{\beta}\right)\Gamma\left(1+\chi+\frac{i\omega}{\beta}\right)\Gamma\left(-\chi+\frac{i\omega}{\beta}\right)}{\Gamma\left(\frac{i\omega}{\beta}\right)\Gamma\left(1+\chi\right)\Gamma\left(-\chi\right)} \tag{A.2}$$

where $\Gamma(a)$ is the Gamma function and $\chi = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{U_0}{\beta}}$. Absolute and phase parts of (A.2) give the reflection coefficient and the phase change on scattering respectively.

Appendix B

Effective fluid and source term of $f(R)$ gravity

B.1 Relating higher derivative terms with the perturbed Energy Momentum tensor

Christoffel symbols of the background metric written in 2+2 form becomes

$$\Gamma^a_{bc} =^2 \Gamma^a_{bc} \tag{B.1}$$

$$\Gamma^a_{BC} = -rD^a r\gamma_{BC} \tag{B.2}$$

$$\Gamma^A_{aB} = \frac{D_a r}{r}\delta^A_B \tag{B.3}$$

$$\Gamma^A_{BC} = \hat{\Gamma}^A_{BC} \tag{B.4}$$

Where ${}^2\Gamma^a_{bc}$ and $\hat{\Gamma}^A_{BC}$ are Christoffel symbols on (t, r) space and the 2-sphere respectively. Using (B.1)-(B.4), various double covariant derivatives were calculated as follows

$$\Box F = \tilde{\Box} F + \frac{1}{r^2}\hat{\Box} F + \frac{2}{r} D^a r D_a F \tag{B.5}$$

$$\nabla_a \nabla_b F = D_a D_b F \tag{B.6}$$

$$\nabla_a \nabla_B F = rD_a \left(\frac{1}{r} D_B F \right) \tag{B.7}$$

$$\nabla_A \nabla_B F = D_A D_B F + rD^a r D_a F \gamma_{AB} \tag{B.8}$$

for some scalar function $F(y^\mu)$. The higher derivative terms of the modified field equation, bundled as an effective Energy Momentum tensor has a perturbation around $\bar{R} = 0$ was given by Eq. (3.4), while $R^{(1)}$ was separated using Eq. 3.6. Using (B.5)-(B.8) in (3.4), one obtains

$$T^{eff}_{ab} = \frac{2\alpha}{\kappa^2}\left[D_a D_b \Omega - g_{ab}\left(\tilde{\Box}\Omega + \frac{2}{r}D^c r D_c \Omega - \frac{k^2}{r^2}\Omega\right)\right]\mathbf{S} \tag{B.9}$$

$$T^{eff}_{aB} = -\frac{2\alpha}{\kappa^2} k r D_a \left(\frac{\Omega}{r}\right)\mathbf{S}_B \tag{B.10}$$

$$T^{eff}_{AB} = \frac{2\alpha}{\kappa^2}\left[k^2 \Omega \mathbf{S}_{AB} - r^2 \gamma_{AB}\left(\tilde{\Box}\Omega + \frac{2}{r}D^a r D_a \Omega - \frac{k^2}{2r^2}\Omega\right)\mathbf{S}\right] \tag{B.11}$$

where $\Omega = \frac{\Phi}{r}$, and τ_{ab}, $\tau_a^{(E/O)}$, P, and $\tau_T^{(E/O)}$ were found by comparing the above relations with the definitions (2.32)-(2.38)

$$\tau_{ab} = 2\alpha\left[D_a D_b - g_{ab}\left(\tilde{\Box} + \frac{2}{r}D^c r D_c - \frac{k^2}{r^2}\right)\right]\left(\frac{\Phi}{r}\right) \tag{B.12}$$

$$\tau_a^E = -\frac{4\alpha r}{k^2} D_a \left(\frac{\Phi}{r^2}\right) \tag{B.13}$$

$$P = 2\alpha\left(\frac{k^2}{2r^2} - \tilde{\Box} - \frac{2}{r}D^a r D_a\right)\left(\frac{\Phi}{r}\right) \tag{B.14}$$

$$\tau_T^E = \frac{4\alpha\Phi}{r} \tag{B.15}$$

$$\tau_a^O = 0 \tag{B.16}$$

$$\tau_T^O = 0 \tag{B.17}$$

where Φ is the extra scalar mode.

B.2 The effective source term

The inhomogeneous source term for a general matter perturbation in a Schwarzschild background was obtained for $m + n$ spacetimes with electromagnetic presence in [104]. In this thesis, a restricted version of the above is used by putting the background and perturbed electromagnetic sources to zero and $m = n = 2$. One obtains,

$$S_S^{eff} = \frac{g}{rH}\left[-HS_T - \frac{P_1}{H}\frac{S_t}{i\omega} - 4g\frac{r\left(S_t\right)'}{i\omega} - 4rgS_r + \frac{P_2}{H}\frac{rS_t^r}{i\omega} + 2r^2\frac{\left(S_t^r\right)'}{i\omega} + 2r^2 S_r^r\right], \quad \text{(B.18)}$$

where the prime denotes radial derivative and

$$S_b^a = \kappa^2 \tau_b^a; \quad S_a = \frac{r\kappa^2}{k}\tau_a^{(S)}; \quad S_T = \frac{2r^2\kappa^2}{k^2}\tau_T^{(S)}, \quad \text{(B.19)}$$

$$P_1 = -\frac{48M^2}{r^2} + \frac{8M}{r}\left(8 - k^2\right) - 2k^2(k^2 - 2) \quad \text{(B.20)}$$

$$P_2 = \frac{24M}{r}; \quad H = k^2 + \frac{6M}{r} - 2 \quad \text{(B.21)}$$

(B.19) was calculated using (B.12), (B.13), and (B.15) and substituted to (B.18). Time dependence of Φ was separated out using

$$\Phi\left(r,t\right) \equiv \Phi\left(r\right)e^{i\sigma t} \quad \text{(B.22)}$$

Double radial derivatives were reduced by using the equation of motion of Φ

$$\frac{d^2\Phi}{dr_*^2} + \left(\sigma^2 - \tilde{V}_{RW}\right)\Phi = 0 \quad \text{(B.23)}$$

from which the effective source term was obtained as

$$S_S^{eff} = \left[C_1(\sigma,\omega,r) + C_2(\sigma,\omega,r)\frac{d}{dr_*}\right]\tilde{\Phi}_S \quad \text{(B.24)}$$

where $\tilde{\Phi}_E = \frac{4\alpha\Phi}{H}$, $H\left(r\right) \equiv H = k^2 + \frac{6M}{r} + 2$ and the coefficients were obtained as

$$\begin{aligned} C_1\left(\sigma,\omega,r\right) &= \sigma^2\left(1 + \frac{\sigma}{\omega}\right) - \frac{Mg}{r^3}\left(1 + \frac{18M}{rH}\right) - \left(\frac{\sigma}{\omega}\right)\frac{g}{r^2}\left[\frac{54M^2}{r^2H} - \frac{72gM^2}{r^2H^2}\right. \\ &\quad \left. - \frac{18M}{rH} + \frac{1}{2}\frac{P_1}{H} - \frac{3M}{r} + \frac{\tilde{V}_{RW}}{g}\right] \quad \text{(B.25)} \\ C_2\left(\sigma,\omega,r\right) &= \frac{3M}{r^2} - \left(\frac{\sigma}{\omega}\right)\left[\frac{12Mg}{r^2H} - \frac{M}{r^2}\right] \quad \text{(B.26)} \\ P_1 &= -\frac{48M^2}{r^2} + \frac{8M}{r}\left(8 - k^2\right) - 2k^2(k^2 - 2) \quad \text{(B.27)} \end{aligned}$$

Appendix C

Energy momentum pseudo-tensor of perturbation for a ringing Schwarzschild space-time in $f(R)$ gravity

A first order perturbed modified field equations for $f(R)$ gravity for a general background is given by

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(1)} - 2\alpha\left(\nabla_\mu\nabla_\nu R^{(1)} - \bar{g}_{\mu\nu}\Box R^{(1)}\right) = 0, \tag{C.1}$$

which on redefinition $h_{\mu\nu} \to \psi_{\mu\nu}$, defined in Eq. (1.27), and using the transverse-traceless gauge condition becomes

$$\Box\psi_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\psi^{\alpha\beta} = 0 \tag{C.2}$$

alongwith the dynamics for the massive scalar mode, $R^{(1)}$, given by (1.26).

Detailed calculations of the terms at $\mathcal{O}\left(\epsilon^2\right)$ follows. Cadabra [1] was used to calculate the second order quantities.

C.1 Convention

$$
\begin{aligned}
f(R) &= R + \alpha R^2 && \text{(C.3)}\\
g_{\alpha\beta} &= \bar{g}_{\alpha\beta} + \epsilon h_{\alpha\beta} && \text{(C.4)}\\
g^{\alpha\beta} &= \bar{g}^{\alpha\beta} - \epsilon h^{\alpha\beta} + \mathcal{O}\left(\epsilon^2\right) && \text{(C.5)}\\
R_{\alpha\beta} &= \bar{R}_{\alpha\beta} + \epsilon R^{(1)}_{\alpha\beta} + \epsilon^2 R^{(2)}_{\alpha\beta} && \text{(C.6)}\\
\bar{R}_{\alpha\beta} &= 0 && \text{(C.7)}\\
\nabla_\alpha A_\beta &= A_{\beta;\alpha} && \text{(C.8)}\\
\bar{g}^{\mu\nu}\nabla_\mu\nabla_\nu A &= \Box A && \text{(C.9)}\\
\nabla^{(1)}_\mu &\equiv \delta\nabla_\mu && \text{(C.10)}
\end{aligned}
$$

C.2 The Einstein tensor

$$
\begin{aligned}
G^{(2)}_{\alpha\beta} &= R^{(2)}_{\alpha\beta} - \left(h_{\alpha\beta}R^{(1)} + \frac{1}{2}\bar{g}_{\alpha\beta}R^{(2)}\right) && \text{(C.11)}\\
R^{(2)} &= -2h^{\mu\nu}R^{(1)}_{\mu\nu} + \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} && \text{(C.12)}\\
G^{(2)}_{\alpha\beta} &= R^{(2)}_{\alpha\beta} - h_{\alpha\beta}R^{(1)} + \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} && \text{(C.13)}
\end{aligned}
$$

C.3 The effective energy momentum tensor of $f(R)$

$$
\begin{aligned}
T^{eff}_{\alpha\beta} &= \alpha\left(2\nabla_\alpha\nabla_\beta R - 2g_{\alpha\beta}\Box R + \frac{1}{2}g_{\alpha\beta}R^2 - 2RR_{\alpha\beta}\right) && \text{(C.14)}\\
\delta^2\left(\nabla_\alpha\nabla_\beta R\right) &= 2\delta\nabla_\alpha\left(\nabla_\beta R\right) + \underbrace{\cancel{2\nabla_\alpha\nabla_\beta R^{(2)}}}_{\to\, 0,\ \text{of the form} S^{\rho}_{\mu\nu;\rho}} && \text{(C.15)}\\
\delta^2\left(g_{\alpha\beta}\Box R\right) &= h_{\alpha\beta}\Box R^{(1)} - \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} + \bar{g}_{\alpha\beta}\delta\nabla^\mu\left(\nabla_\mu R^{(1)}\right) + \underbrace{\cancel{\bar{g}_{\alpha\beta}\Box R^{(2)}}}_{\to\, 0,\ \text{of the form} S^{\rho}_{\mu\nu;\rho}} && \text{(C.16)}\\
\delta^2\left(g_{\alpha\beta}R^2\right) &= 2\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 && \text{(C.17)}\\
\delta^2\left(RR_{\alpha\beta}\right) &= 2R^{(1)}R^{(1)}_{\alpha\beta} && \text{(C.18)}
\end{aligned}
$$

$$
\begin{aligned}
T^{(2),eff}_{\alpha\beta} &= \alpha\left[4\delta\nabla_\alpha R^{(1)}_{;\beta} - 2h_{\alpha\beta}\Box R^{(1)} + 2\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} - \bar{g}_{\alpha\beta}\delta\nabla^\mu R^{(1)}_{;\mu} + \bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 \right. \\
&\left. -4R^{(1)}R^{(1)}_{\alpha\beta}\right] \qquad \text{(C.19)}
\end{aligned}
$$

C.4 Individual terms

C.4.1 Covariant derivative variation

$$
\delta\nabla_\mu A_\nu = \frac{1}{2}\left(-h_{\mu\nu}^{;\sigma} + h^\sigma_{\nu;\mu} + h^\sigma_{\mu;\nu}\right)A_\sigma \qquad \text{(C.20)}
$$

C.4.2 Terms in $h_{\alpha\beta}$ and $R^{(1)}$ form

$$
\begin{aligned}
T1: \quad R^{(2)}_{\alpha\beta} &= -\frac{1}{2}\left[\frac{1}{2}h^{\rho\tau}_{;\beta}h_{\rho\tau;\alpha} + h^{\rho\tau}\left(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\tau\rho} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\right)\right. \\
&\left. + h_\beta^{\tau;\rho}\left(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}\right) - \left(h^{\rho\tau}_{;\rho} - \frac{1}{2}h^{;\tau}\right)\left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau}\right)\right] \qquad \text{(C.21)}
\end{aligned}
$$

$$
T2: \quad -h_{\alpha\beta}R^{(1)} = -h_{\alpha\beta}\left(\Box h - h^{\mu\nu}_{;\mu\nu}\right) \qquad \text{(C.22)}
$$

$$
T3: \quad \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}\bar{g}_{\alpha\beta}h^{\mu\nu}\left(h_{;\mu\nu} + \Box h_{\mu\nu} - h^\rho_{\alpha;\beta\rho} - h^\rho_{\beta;\alpha\rho}\right) \qquad \text{(C.23)}
$$

$$
\begin{aligned}
T4: \quad -\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} &= -\bar{g}_{\alpha\beta}\left(-\frac{3}{4}\nabla^\mu h^{\nu\rho}\nabla_\mu h_{\nu\rho} - \frac{1}{2}h^{\mu\nu}\nabla^\rho\nabla_\rho h_{\mu\nu} - \frac{1}{2}h^{\mu\nu}\nabla_\mu\nabla_\nu h\right. \\
&+ h^{\mu\nu}\nabla^\rho\nabla_\mu h_{\nu\rho} + \frac{1}{2}\nabla^\mu h^{\nu\rho}\nabla_\nu h_{\mu\rho} + \nabla^\mu h_\mu{}^\nu\nabla^\rho h_{\nu\rho} \\
&\left. -\frac{1}{2}\nabla^\mu h_\mu{}^\nu\nabla_\nu h - \frac{1}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{1}{4}\nabla^\mu h\nabla_\mu h\right) \qquad \text{(C.24)}
\end{aligned}
$$

$$
T5: \quad -4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta} = -\alpha\left(-h^{;\sigma}_{\alpha\beta} + h^\sigma_{\beta;\alpha} + h^\sigma_{\alpha;\beta}\right)R^{(1)}_{;\sigma} \qquad \text{(C.25)}
$$

$$
T6: \quad 2\alpha h_{\alpha\beta}\Box R^{(1)} \qquad \text{(C.26)}
$$

$$
T7: \quad -2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} \qquad \text{(C.27)}
$$

$$
T8: \quad \alpha\bar{g}_{\alpha\beta}\delta\nabla^\mu R^{(1)}_{;\mu} = \frac{\alpha}{2}\bar{g}_{\alpha\beta}\left(-h^{;\sigma} + 2h^{\mu\sigma}_{;\mu}\right)R^{(1)}_{;\sigma} \qquad \text{(C.28)}
$$

$$
T9: \quad -\alpha\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 \qquad \text{(C.29)}
$$

$$T10: \quad 4\alpha R^{(1)}R^{(1)}_{\alpha\beta} \quad = \quad 2\alpha\left(h_{;\alpha\beta} + \Box h_{\alpha\beta} - h^{\rho}_{\alpha;\rho\beta} - h^{\rho}_{\beta;\rho\alpha}\right)R^{(1)} \tag{C.30}$$

C.5 Gauge fixing and averaging

C.5.1 Field redefinition and gauge

$$\psi_{\alpha\beta} \quad = \quad h_{\alpha\beta} - \bar{g}_{\alpha\beta}\left(\frac{h}{2} + 2\alpha R^{(1)}\right) \tag{C.31}$$

$$\psi^{\alpha\beta}_{;\beta} \quad = \quad 0 \tag{C.32}$$

C.5.2 Averaging procedure guidelines

- Total derivative terms of the form $\langle A_{\alpha..\beta;\mu}\rangle = 0$.
- $\langle A_{;\alpha}B_{;\beta}\rangle = -\langle A_{;\alpha\beta}B\rangle$, where A and B are indexed tensor objects.

C.6 Terms with redefined variables

C.6.1 T1

Cadabra output:

$$\begin{aligned}
R^{(2)}_{\alpha\beta} \quad = \quad & -\frac{1}{4}\nabla_{\alpha}\psi^{\rho\tau}\nabla_{\beta}\psi_{\rho\tau} - \frac{1}{2}\psi^{\rho\tau}\nabla_{\alpha}\nabla_{\beta}\psi_{\rho\tau} - \frac{1}{4}\psi\nabla_{\alpha}\nabla_{\beta}\psi - \frac{1}{2}\psi^{\rho\tau}\nabla_{\rho}\nabla_{\tau}\psi_{\alpha\beta} + \frac{1}{4}\psi^{\rho\tau}g_{\alpha\beta}\nabla_{\rho}\nabla_{\tau}\psi \\
& +\frac{1}{2}\psi^{\rho\tau}\nabla_{\beta}\nabla_{\rho}\psi_{\alpha\tau} - \frac{1}{4}\psi_{\alpha}{}^{\rho}\nabla_{\beta}\nabla_{\rho}\psi + \frac{1}{2}\psi^{\rho\tau}\nabla_{\alpha}\nabla_{\rho}\psi_{\beta\tau} - \frac{1}{4}\psi_{\beta}{}^{\rho}\nabla_{\alpha}\nabla_{\rho}\psi + \frac{3}{8}\nabla_{\alpha}\nabla_{\beta}\psi\psi \\
& +\frac{1}{4}\nabla^{\rho}\nabla_{\rho}\psi_{\alpha\beta}\psi - \frac{1}{8}\nabla^{\rho}\nabla_{\rho}\psi\psi g_{\alpha\beta} - \frac{1}{4}\nabla_{\beta}\nabla^{\rho}\psi_{\alpha\rho}\psi + \frac{1}{8}\nabla_{\beta}\nabla_{\alpha}\psi\psi - \frac{1}{4}\nabla_{\alpha}\nabla^{\rho}\psi_{\beta\rho}\psi \\
& -\frac{1}{2}\nabla^{\rho}\psi_{\alpha}{}^{\tau}\nabla_{\rho}\psi_{\beta\tau} + \frac{1}{4}\nabla^{\rho}\psi_{\alpha\beta}\nabla_{\rho}\psi + \frac{1}{2}\nabla^{\rho}\psi_{\alpha}{}^{\tau}\nabla_{\tau}\psi_{\beta\rho} - \frac{1}{4}\nabla_{\alpha}\psi_{\beta}{}^{\rho}\nabla_{\rho}\psi + \frac{1}{4}\nabla^{\rho}\psi\nabla_{\rho}\psi_{\alpha\beta} \\
& -\frac{1}{8}g_{\alpha\beta}\nabla^{\rho}\psi\nabla_{\rho}\psi + \frac{1}{8}\nabla_{\alpha}\psi\nabla_{\beta}\psi + \frac{1}{2}\nabla^{\rho}\psi_{\rho}{}^{\tau}\nabla_{\beta}\psi_{\alpha\tau} - \frac{1}{4}\nabla^{\rho}\psi_{\alpha\rho}\nabla_{\beta}\psi + \frac{1}{2}\nabla^{\rho}\psi_{\rho}{}^{\tau}\nabla_{\alpha}\psi_{\beta\tau} \\
& -\frac{1}{4}\nabla^{\rho}\psi_{\beta\rho}\nabla_{\alpha}\psi - \frac{1}{2}\nabla^{\rho}\psi_{\rho}{}^{\tau}\nabla_{\tau}\psi_{\alpha\beta} + \frac{1}{4}\nabla^{\rho}\psi_{\rho}{}^{\tau}g_{\alpha\beta}\nabla_{\tau}\psi - \frac{1}{4}\nabla^{\rho}\psi\nabla_{\beta}\psi_{\alpha\rho} - \frac{1}{8}\nabla^{\rho}\psi\nabla_{\rho}\psi g_{\alpha\beta} \\
& +\frac{1}{8}\nabla^{\rho}\psi g_{\alpha\beta}\nabla_{\rho}\psi + \alpha\left(-\nabla_{\beta}R\nabla_{\alpha}\psi - \psi\nabla_{\alpha}\nabla_{\beta}R + \psi^{\rho\tau}g_{\alpha\beta}\nabla_{\rho}\nabla_{\tau}R - \psi_{\alpha}{}^{\rho}\nabla_{\beta}\nabla_{\rho}R - \psi_{\beta}{}^{\rho}\nabla_{\alpha}\nabla_{\rho}R\right.
\end{aligned}$$

$$-\frac{1}{2}\nabla^{\rho}\nabla_{\rho}R\psi g_{\alpha\beta}+\frac{1}{2}\nabla_{\beta}\nabla_{\alpha}R\psi+\frac{1}{2}\nabla_{\alpha}\nabla_{\beta}R\psi+\frac{3}{2}\nabla_{\alpha}\nabla_{\beta}\psi R-2R\nabla_{\alpha}\nabla_{\beta}\psi+\nabla^{\rho}\nabla_{\rho}\psi_{\alpha\beta}R$$
$$-\frac{1}{2}\nabla^{\rho}\nabla_{\rho}\psi R g_{\alpha\beta}-\nabla_{\beta}\nabla^{\rho}\psi_{\alpha\rho}R+\frac{1}{2}\nabla_{\beta}\nabla_{\alpha}\psi R-\nabla_{\alpha}\nabla^{\rho}\psi_{\beta\rho}R+\nabla^{\rho}\psi_{\alpha\beta}\nabla_{\rho}R-\nabla_{\alpha}\psi_{\beta}{}^{\rho}\nabla_{\rho}R$$
$$-\frac{1}{2}g_{\alpha\beta}\nabla^{\rho}\psi\nabla_{\rho}R+\frac{1}{2}\nabla_{\alpha}\psi\nabla_{\beta}R-\frac{1}{2}g_{\alpha\beta}\nabla^{\rho}R\nabla_{\rho}\psi-\nabla^{\rho}\psi_{\alpha\rho}\nabla_{\beta}R-\nabla^{\rho}\psi_{\beta\rho}\nabla_{\alpha}R$$
$$+\nabla^{\rho}\psi_{\rho}{}^{\tau}g_{\alpha\beta}\nabla_{\tau}R-\frac{1}{2}\nabla_{\alpha}R\nabla_{\beta}\psi+\nabla^{\rho}R\nabla_{\alpha}\psi_{\beta\rho}-\frac{1}{2}\nabla^{\rho}\psi\nabla_{\rho}R g_{\alpha\beta}+\nabla^{\rho}R g_{\alpha\beta}\nabla_{\rho}\psi\Big)$$
$$+\alpha^{2}\left(-6\nabla_{\alpha}R\nabla_{\beta}R-8R\nabla_{\alpha}\nabla_{\beta}R-2\nabla^{\rho}\nabla_{\rho}RRg_{\alpha\beta}+2\nabla_{\beta}\nabla_{\alpha}RR+2\nabla_{\alpha}\nabla_{\beta}RR-2g_{\alpha\beta}\nabla^{\rho}R\nabla_{\rho}R\right.$$
$$\left.-2\nabla^{\rho}R\nabla_{\rho}Rg_{\alpha\beta}+4\nabla^{\rho}Rg_{\alpha\beta}\nabla_{\rho}R\right) \tag{C.33}$$

After following averaging guidelines:

- Coefficient of α^0: $\langle\frac{1}{4}\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}-\frac{1}{8}\psi_{;\alpha}\psi_{;\beta}\rangle$
- Coefficient of α: $\langle-\psi_{;\alpha}R^{(1)}_{;\beta}\rangle$
- Coefficient of α^2: $\langle-2R^{(1)}_{;\alpha}R^{(1)}_{;\beta}-2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle$

$$\langle R^{(2)}_{\alpha\beta}\rangle = \langle\frac{1}{4}\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}-\frac{1}{8}\psi_{;\alpha}\psi_{;\beta}-\alpha\psi_{;\alpha}R^{(1)}_{;\beta}-\alpha^{2}\left(2R^{(1)}_{;\alpha}R^{(1)}_{;\beta}+2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\right)\rangle \tag{C.34}$$

C.6.2 T2

Cadabra output

$$-h_{\alpha\beta}R^{(1)} = -\alpha\left(-6\psi_{\alpha\beta}\nabla^{\rho}\nabla_{\rho}R+3g_{\alpha\beta}\psi\nabla^{\rho}\nabla_{\rho}R\right)-12\alpha^{2}g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}R \tag{C.35}$$

After averaging

$$\langle-h_{\alpha\beta}R^{(1)}\rangle = \langle-6\alpha\Box\psi_{\alpha\beta}R^{(1)}-12\alpha^{2}\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{C.36}$$

C.6.3 T3

Cadabra output:

$$\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}g_{\alpha\beta}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\rho}\psi_{\tau\epsilon} - g_{\alpha\beta}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\tau}\psi_{\epsilon\rho} + \frac{1}{2}g_{\alpha\beta}\psi\nabla^{\rho}\nabla^{\tau}\psi_{\tau\rho} + \alpha\left(-2g_{\alpha\beta}\psi^{\rho\tau}\nabla_{\rho}\nabla_{\tau}R + 2g_{\alpha\beta}\psi\nabla^{\rho}\nabla_{\rho}R + g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}\psi + 2g_{\alpha\beta}R\nabla^{\rho}\nabla^{\tau}\psi_{\tau\rho}\right) + 12\alpha^{2}g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}R \quad \text{(C.37)}$$

After averaging and using equation of motion

$$\langle\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu}\rangle = \langle 12\alpha^{2}\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \quad \text{(C.38)}$$

C.6.4 T4

Cadabra output:

$$-\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = -\bar{g}_{\alpha\beta}\left(-\frac{3}{8}\nabla^{\rho}\psi^{\tau\epsilon}\nabla_{\rho}\psi_{\tau\epsilon} + \frac{1}{16}\nabla^{\rho}\psi\nabla_{\rho}\psi - \frac{1}{4}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\rho}\psi_{\tau\epsilon} + \frac{1}{2}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\tau}\psi_{\epsilon\rho} + \frac{1}{4}\nabla^{\rho}\psi^{\tau\epsilon}\nabla_{\tau}\psi_{\rho\epsilon}\right) - \alpha\bar{g}_{\alpha\beta}\left(-\frac{1}{2}\nabla^{\rho}\psi\nabla_{\rho}R - \frac{1}{2}\nabla^{\rho}R\nabla_{\rho}\psi - \psi\nabla^{\rho}\nabla_{\rho}R + \psi^{\rho\tau}\nabla_{\rho}\nabla_{\tau}R + \nabla^{\rho}R\nabla^{\tau}\psi_{\rho\tau}\right) - \bar{g}_{\alpha\beta}\alpha^{2}(-3\nabla^{\rho}R\nabla_{\rho}R - 6R\nabla^{\rho}\nabla_{\rho}R) \quad \text{(C.39)}$$

After averaging and using equation of motion

- Coefficient of $\alpha^{0} = 0$
- Coefficient of $\alpha = 0$
- Coefficient of $\alpha^{2} = 3\alpha^{2}\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}$

$$\langle\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\rangle = \langle 3\alpha^{2}\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \quad \text{(C.40)}$$

C.6.5 T5

Cadabra output:

$$\begin{aligned} -4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta} &= \alpha\Big[\nabla^\rho\psi_{\alpha\beta}\nabla_\rho R - \frac{1}{2}g_{\alpha\beta}\nabla^\rho\psi\nabla_\rho R - \nabla_\alpha\psi_\beta{}^\rho\nabla_\rho R + \frac{1}{2}\nabla_\alpha\psi\nabla_\beta R - \nabla_\beta\psi_\alpha{}^\rho\nabla_\rho R \\ &\quad + \frac{1}{2}\nabla_\beta\psi\nabla_\alpha R - \alpha\left(2g_{\alpha\beta}\nabla^\rho R\nabla_\rho R + 4\nabla_\alpha R\nabla_\beta R\right)\Big] \end{aligned} \tag{C.41}$$

After averaging and using equations of motion

$$\langle -4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta}\rangle = \langle \alpha\psi_{;\alpha}R^{(1)}_{;\beta} + 2\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)} + 4\alpha^2 R^{(1)}_{;\alpha}R^{(1)}_{;\beta}\rangle \tag{C.42}$$

C.6.6 T6

Cadabra output

$$2\alpha h_{\alpha\beta}\Box R^{(1)} = 2\alpha\psi_{\alpha\beta}\nabla^\rho\nabla_\rho R - \alpha g_{\alpha\beta}\psi\nabla^\rho\nabla_\rho R - 4\alpha^2 g_{\alpha\beta}R\nabla^\rho\nabla_\rho R \tag{C.43}$$

After averaging and using equations of motion

$$\langle 2\alpha h_{\alpha\beta}\Box R^{(1)}\rangle = \langle 2\alpha\Box\psi_{\alpha\beta}R^{(1)} - 4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{C.44}$$

C.6.7 T7

Cadabra output:

$$-2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} = -\alpha\bar{g}_{\alpha\beta}\left(2\psi^{\mu\nu}\nabla_\mu\nabla_\nu R - \psi\nabla^\mu\nabla_\mu R - 4\alpha R\nabla^\mu\nabla_\mu R\right) \tag{C.45}$$

After averaging and using equations of motion

$$\langle -2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu}\rangle = \langle 4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{C.46}$$

C.6.8 T8

Cadabra output:

$$\alpha\bar{g}_{\alpha\beta}\delta\nabla^{\mu}R^{(1)}_{;\mu} = 2\alpha^2 g_{\alpha\beta}\nabla^{\rho}R\nabla_{\rho}R \tag{C.47}$$

After averaging

$$\langle\alpha\bar{g}_{\alpha\beta}\delta\nabla^{\mu}R^{(1)}_{;\mu}\rangle = \langle-2\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{C.48}$$

C.6.9 T10

From calculations of T3 and equations of motion:

$$4\alpha R^{(1)}R^{(1)}_{\alpha\beta} = 4\alpha R^{(1)}\left[-\frac{1}{2}\Box\psi_{\alpha\beta}+\frac{1}{4}\bar{g}_{\alpha\beta}\Box\psi - R^{(0)}_{\mu\alpha\nu\beta}\psi^{\mu\nu}+\alpha\left(2R^{(1)}_{;\alpha\beta}+\bar{g}_{\alpha\beta}\Box R^{(1)}\right)\right] \tag{C.49}$$

Using equations of motion and averaging which becomes

$$\langle 4\alpha R^{(1)}R^{(1)}_{\alpha\beta}\rangle = \langle-8\alpha^2 R^{(1)}_{;\alpha}R^{(1)}_{;\beta}+4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{C.50}$$

C.7 Second order perturbed $f(R)$ equations of motion

$$\langle G^{(2)}_{\alpha\beta}-T^{(2),eff}_{\alpha\beta}\rangle = \langle\sum_{i=1}^{10}Ti\rangle \tag{C.51}$$

$$= \langle\frac{1}{4}\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}-\frac{1}{8}\psi_{;\alpha}\psi_{;\beta}-\alpha\left[4\Box\psi_{\alpha\beta}R^{(1)}+\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2\right] +\alpha^2\left(-6R^{(1)}_{;\alpha}R^{(1)}_{;\beta}+5\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\right)\rangle \tag{C.52}$$

$$= \langle\frac{1}{4}\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}-\frac{1}{8}\psi_{;\alpha}\psi_{;\beta}+\alpha\left[R^{(0)}_{\mu\alpha\nu\beta}\psi^{\mu\nu}R^{(1)}-\frac{1}{6}\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2\right]-6\alpha^2R^{(1)}_{;\alpha}R^{(1)}_{;\beta}\rangle \tag{C.53}$$

Two points

(i). It is possible to, by a gauge transformation, to put $\psi = 0$ (Gair 2011).

(ii). Redefinition of fields, from $h_{\alpha\beta} \rightarrow \psi_{\alpha\beta}$ takes away the extra degree of freedom $R^{(1)}$ and leaves one with $\psi_{\alpha\beta}$, which is the pure spin-2 field. The first order Ricci scalar $R^{(1)}$, in an $f(R)$ theory is a scalar field that is an independent degree of freedom. Hence, a term like $\langle \psi^{\mu\nu} R^{(1)} \rangle$, which is a correlation between two independent fields, must be zero.

Hence, one obtains

$$\langle G^{(2)}_{\alpha\beta} - T^{(2),eff}_{\alpha\beta} \rangle \;=\; \langle \underbrace{\frac{1}{4}\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}}_{\mathbf{I}} - \underbrace{\frac{\alpha}{6}\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2}_{\mathbf{II}} - \underbrace{6\alpha^2 R^{(1)}_{;\alpha} R^{(1)}_{;\beta}}_{\mathbf{III}} \rangle = t_{\mu\nu} \tag{C.54}$$

where $t_{\mu\nu}$ is the energy momentum tensor for gravitational waves. The energy density in the waves then is given by t_{00}.

It is to be noted that

$$t^{\mathbf{II}}_{00} \;\rightarrow\; 0, r \rightarrow r_H \quad (\bar{g}_{00} \rightarrow 0) \tag{C.55}$$

$$\rightarrow\; 0, r \rightarrow \infty \quad \left(R^{(1)} \rightarrow 0\right) \tag{C.56}$$

Thus $t^{\mathbf{II}}_{00}$ must have a maximum in $r_H < r < \infty$. Since there is no scattering of waves beyond the maxima of the scattering potentials of the even/odd potentials, that maxima can thus be taken as a point of maximum energetic difference between the massless and the massive mode that can be detected at ∞.

Appendix D

Details of the effective source term of charged BHs in $f(R)$ gravity

For a scalar perturbed energy-momentum tensor given by

$$T^E_{\mu\nu} \equiv \left(\begin{array}{c|c} \tau_{ab}\mathbf{S} & r\tau^E_a \mathbf{S}_B \\ --- & ----------- \\ r\tau^E_a \mathbf{S}_B & r^2 P \bar{g}_{AB}\mathbf{S} + r^2 \tau^E_T \mathbf{S}_{AB} \end{array} \right), \tag{D.1}$$

from [104] the source term for the scalar perturbation of a charged black-hole was found to be

$$S^{eff}_{\pm} = a^S_{\pm} S_\Phi + b^S_{\pm} S_{\mathcal{A}} \tag{D.2}$$

$$S_\Phi = \frac{g}{rH}\left[-HS_T - \frac{P_1}{H}\frac{S_t}{i\omega} - 4g\frac{r\left(S_t\right)'}{i\omega} - 4rgS_r + \frac{P_2}{H}\frac{rS^r_t}{i\omega} + 2r^2\frac{\left(S^r_t\right)'}{i\omega} + 2r^2 S^r_r\right] \tag{D.3}$$

$$S_{\mathcal{A}} = \frac{2\sqrt{2}Qg}{i\omega r^2 H}\left(2gS_t - rS^r_t\right) \tag{D.4}$$

where the prime denotes radial derivatives and

$$P_1 = -\frac{32Q^4}{r^4} + \frac{48Q^2}{r^2}\left(\frac{2M}{r} - 1\right) - \frac{48M^2}{r^2} + \frac{4M}{r}\left(8 - k^2\right) - 2k^2\left(k^2 - 2\right) \tag{D.5}$$

$$P_2 = -\frac{32Q^2}{r^2} + \frac{24M}{r} \tag{D.6}$$

$$S^a_b = \kappa^2 \tau^a_b \qquad S_a = \frac{r\kappa^2}{k}\tau^E_a \qquad S_T = \frac{2r^2\kappa^2}{k^2}\tau^E_T \tag{D.7}$$

Equating $T^E_{\mu\nu} = T^{eff}_{\mu\nu}$ the components of $T^E_{\mu\nu}$ were found from [97] in terms of the massive field Φ, using which components of S^a_b, S_a, and S_T were found for (D.3) and (D.4). $S^{eff}_{\pm}$ is only relevant around the horizon of the black hole where the presence of Φ is at the largest. Hence, the coefficients $c_\pm$ and $d_\pm$ of Eq. (4.16) were calculated around the horizon in a power series of $g(y) \equiv g$ around the horizon and only contribution from $\frac{1}{g}$ is relevant, which turn out to be

$$c_+ = -\frac{\left(\frac{3}{2}\,q^2 + k^2 y - 2\,y + \frac{1}{2}\,\sqrt{9 + 9\,q^4 + (-14 + 16\,k^2)\,q^2} + \frac{3}{2}\right) q\left(q^2 y - 16\,q^2 + y\right)\left(q^2 y - 2\,q^2 + y\right)}{4gy^7 H} \tag{D.8}$$

$$d_+ = \frac{\left(\frac{3}{2}\,q^2 + k^2 y - 2\,y + \frac{1}{2}\,\sqrt{9 + 9\,q^4 + (-14 + 16\,k^2)\,q^2} + \frac{3}{2}\right) q\left(q^2 y - 2\,q^2 + y\right)}{2y^4 g} \tag{D.9}$$

$$c_- = -\frac{3\left(q^2 y - \frac{8}{3}\,q^2 + \frac{1}{3}\,\sqrt{9 + 9\,q^4 + (-14 + 16\,k^2)\,q^2}y + y\right)\left(q^2 y - 2\,q^2 + y\right)\left(q^2 y - 16\,q^2 + y\right)}{4gy^7 H} \tag{D.10}$$

$$d_- = \frac{3\left(q^2 y - \frac{8}{3}\,q^2 + \frac{1}{3}\,\sqrt{9 + 9\,q^4 + (-14 + 16\,k^2)\,q^2}y + y\right)\left(q^2 y - 2\,q^2 + y\right)}{2y^4 g} \tag{D.11}$$

Appendix E

Energy momentum pseudo-tensor of perturbation for a ringing Reissner-Nördstrom space-time in $f(R)$ gravity

E.1 Convention and background

$$f(R) = R + \alpha R^2 \tag{E.1}$$

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \epsilon h_{\alpha\beta} \tag{E.2}$$

$$g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \epsilon h^{\alpha\beta} + \mathcal{O}\left(\epsilon^2\right) \tag{E.3}$$

$$F_{\alpha\beta} = \bar{F}_{\alpha\beta} + \epsilon F^{(1)}_{\alpha\beta} \tag{E.4}$$

$$R_{\alpha\beta} = \bar{R}_{\alpha\beta} + \epsilon R^{(1)}_{\alpha\beta} + \epsilon^2 R^{(2)}_{\alpha\beta} \tag{E.5}$$

$$\bar{R}_{\alpha\beta} = \kappa^2 T_{\alpha\beta} \tag{E.6}$$

$$R = \bar{R} + \epsilon R^{(1)} \tag{E.7}$$

$$\bar{R} = 0 \tag{E.8}$$

$$\bar{F}^{;\mu}_{\mu\nu} = 0 \tag{E.9}$$

$$\nabla_\alpha A_\beta = A_{\beta;\alpha} \tag{E.10}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu A = \Box A \tag{E.11}$$

E.2 The Einstein tensor

$$\delta^2 G_{\alpha\beta} = \delta^2\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) \tag{E.12}$$

$$= R^{(2)}_{\alpha\beta} - \left(h_{\alpha\beta}R^{(1)} + \frac{1}{2}\bar{g}_{\alpha\beta}R^{(2)}\right) \tag{E.13}$$

$$R^{(2)} = \delta^2\left(g^{\mu\nu}R_{\mu\nu}\right) = -2h^{\mu\nu}R^{(1)}_{\mu\nu} + \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} \tag{E.14}$$

$$G^{(2)}_{\alpha\beta} = R^{(2)}_{\alpha\beta} - h_{\alpha\beta}R^{(1)} + \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} \tag{E.15}$$

E.3 Energy momentum tensor due to electromagnetic field

$$T_{\alpha\beta} = F_{\alpha\mu}F^{\mu}_{\beta} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{E.16}$$

$$T^{(1)}_{\alpha\beta} = F^{(1)}_{\alpha\mu}\bar{F}^{\mu}_{\beta} + \bar{F}_{\alpha\mu}F^{(1)\mu}_{\beta} - h^{\mu\nu}\bar{F}_{\alpha\mu}\bar{F}_{\beta\nu} - \frac{1}{4}\left[h_{\alpha\beta}(\bar{F})^2 - \bar{g}_{\alpha\beta}h^{\rho\mu}\bar{F}_{\mu\nu}\bar{F}^{\nu}_{\rho} - \bar{g}_{\alpha\beta}h^{\rho\nu}\bar{F}_{\mu\nu}\bar{F}^{\mu}_{\rho} + 2\bar{g}_{\alpha\beta}\bar{F}.F^{(1)}\right] \tag{E.17}$$

$$\langle T^{(2)}_{\alpha\beta}\rangle = \langle 2F^{(1)}_{\alpha\mu}F^{(1)\mu}_{\beta} - \frac{1}{4}\left(2\bar{g}_{\alpha\beta}F^{(1)}.F^{(1)} - h_{\alpha\beta}h^{\rho\mu}\bar{F}_{\mu\nu}\bar{F}^{\nu}_{\rho} - h_{\alpha\beta}h^{\rho\nu}\bar{F}_{\mu\nu}\bar{F}^{\mu}_{\rho}\right)\rangle \tag{E.18}$$

E.4 The effective energy momentum tensor of $f(R)$

$$T^{eff}_{\alpha\beta} = \alpha\left(2\nabla_\alpha\nabla_\beta R - 2g_{\alpha\beta}\Box R + \frac{1}{2}g_{\alpha\beta}R^2 - 2RR_{\alpha\beta}\right) \tag{E.19}$$

$$\delta^2\left(\nabla_\alpha\nabla_\beta R\right) = 2\delta\nabla_\alpha\left(\nabla_\beta R\right) + \underbrace{2\nabla_\alpha\nabla_\beta R^{(2)}}_{0,\text{ of the form} S^{\rho}_{\mu\nu;\rho}} \tag{E.20}$$

$$\delta^2\left(g_{\alpha\beta}\Box R\right) = h_{\alpha\beta}\Box R^{(1)} - \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} + \bar{g}_{\alpha\beta}\delta\nabla^{\mu}\left(\nabla_\mu R^{(1)}\right) + \underbrace{\bar{g}_{\alpha\beta}\Box R^{(2)}}_{0,\text{ of the form} S^{\rho}_{\mu\nu;\rho}} \tag{E.21}$$

$$\delta^2\left(g_{\alpha\beta}R^2\right) = 2\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 \quad (E.22)$$

$$\delta^2\left(RR_{\alpha\beta}\right) = 2R^{(1)}R^{(1)}_{\alpha\beta} \quad (E.23)$$

$$T^{(2),eff}_{\alpha\beta} = \alpha\left[4\delta\nabla_\alpha R^{(1)}_{;\beta} - 2h_{\alpha\beta}\Box R^{(1)} + 2\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} - \bar{g}_{\alpha\beta}\delta\nabla^\mu R^{(1)}_{;\mu} + \bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 - 4R^{(1)}R^{(1)}_{\alpha\beta}\right] \quad (E.24)$$

E.5 Field redefinition and gauge

$$\psi_{\alpha\beta} = h_{\alpha\beta} - \bar{g}_{\alpha\beta}\left(\frac{h}{2} + 2\alpha R^{(1)}\right) \quad (E.25)$$

$$\psi^{\alpha\beta}_{;\beta} = 0 \quad (E.26)$$

$$\psi = 0 \quad (E.27)$$

$$A^{(1);\mu}_{\mu} = 0 \quad (E.28)$$

E.5.1 $f(R)$ equations of motion in curved space

$$\Box\psi_{\alpha\beta} + 2\bar{R}_{\mu\alpha\nu\beta}\psi^{\mu\nu} = \kappa^2\left(\mathcal{U}_{\alpha\beta} + \mathcal{T}_{\alpha\beta}\right) \quad \text{: spin 2} \quad (E.29)$$

$$\Box R^{(1)} - \frac{1}{6\alpha}R^{(1)} = 0 \quad \text{: spin 0} \quad (E.30)$$

$$\Box A^{(1)}_\nu = \mathcal{V}_\nu + 2\kappa^2 T_{\mu\nu}A^{(1)\mu} \quad \text{: spin 1} \quad (E.31)$$

where

$$\mathcal{U}_{\alpha\beta} = 2\psi^{\mu\nu}\bar{F}_{\alpha\mu}\bar{F}_{\beta\nu} - \bar{g}_{\alpha\beta}\psi^{\mu\nu}\bar{F}_{\nu\rho}\bar{F}^{\rho}_{\mu} - 2\bar{F}^{\nu}_{\mu}\bar{F}^{\mu}_{(\alpha}\psi_{\beta)\nu} \quad (E.32)$$

$$\mathcal{T}_{\alpha\beta} = -2F^{(1)}_{\alpha\mu}\bar{F}^{\mu}_{\beta} - 2\bar{F}_{\alpha\mu}F^{(1)\mu}_{\beta} + \bar{g}_{\alpha\beta}\bar{F}.F^{(1)} \quad (E.33)$$

$$\mathcal{V}_\nu = 2\psi^{\alpha\beta}\bar{F}_{\alpha\nu;\beta} + \psi^{\beta;\alpha}_{\nu}\bar{F}_{\alpha\beta} \quad (E.34)$$

E.6 Individual terms

E.6.1 Covariant derivative variation

$$\delta\nabla_\mu A_\nu = \frac{1}{2}\left(h^{;\sigma}_{\mu\nu} - h^{\sigma}_{\nu;\mu} - h^{\sigma}_{\mu;\nu}\right) A_\sigma \tag{E.35}$$

E.6.2 Terms in $h_{\alpha\beta}$ and $R^{(1)}$ form

$$T1: \quad R^{(2)}_{\alpha\beta} = -\frac{1}{2}\left[\frac{1}{2}h^{\rho\tau}_{;\beta}h_{\rho\tau;\alpha} + h^{\rho\tau}\left(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\tau\rho} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\right) + h^{\tau;\rho}_{\beta}\left(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}\right) - \left(h^{\rho\tau}_{;\rho} - \frac{1}{2}h^{;\tau}\right)\left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau}\right)\right] \tag{E.36}$$

$$T2: \quad -h_{\alpha\beta}R^{(1)} = -h_{\alpha\beta}\left(\Box h - h^{\mu\nu}_{;\mu\nu}\right) \tag{E.37}$$

$$T3: \quad \bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}\bar{g}_{\alpha\beta}h^{\mu\nu}\left(h_{;\mu\nu} + \Box h_{\mu\nu} - h^{\rho}_{\alpha;\beta\rho} - h^{\rho}_{\beta;\alpha\rho}\right) \tag{E.38}$$

$$T4: \quad -\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} = -\bar{g}_{\alpha\beta}\left(-\frac{3}{4}\nabla^\mu h^{\nu\rho}\nabla_\mu h_{\nu\rho} - \frac{1}{2}h^{\mu\nu}\nabla^\rho\nabla_\rho h_{\mu\nu} - \frac{1}{2}h^{\mu\nu}\nabla_\mu\nabla_\nu h + h^{\mu\nu}\nabla^\rho\nabla_\mu h_{\nu\rho} + \frac{1}{2}\nabla^\mu h^{\nu\rho}\nabla_\nu h_{\mu\rho} + \nabla^\mu h_\mu{}^\nu\nabla^\rho h_{\nu\rho} - \frac{1}{2}\nabla^\mu h_\mu{}^\nu\nabla_\nu h - \frac{1}{2}\nabla^\mu h\nabla^\nu h_{\mu\nu} + \frac{1}{4}\nabla^\mu h\nabla_\mu h\right) \tag{E.39}$$

$$T5: \quad -4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta} = \alpha\left(-h^{;\sigma}_{\alpha\beta} + h^{\sigma}_{\beta;\alpha} + h^{\sigma}_{\alpha;\beta}\right)R^{(1)}_{;\sigma} \tag{E.40}$$

$$T6: \quad 2\alpha h_{\alpha\beta}\Box R^{(1)} \tag{E.41}$$

$$T7: \quad -2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} \tag{E.42}$$

$$T8: \quad 2\alpha\bar{g}_{\alpha\beta}\delta\nabla^\mu R^{(1)}_{;\mu} = \alpha\bar{g}_{\alpha\beta}\left(h^{;\sigma} - 2h^{\mu\sigma}_{;\mu}\right)R^{(1)}_{;\sigma} \tag{E.43}$$

$$T9: \quad -\alpha\bar{g}_{\alpha\beta}\left(R^{(1)}\right)^2 \tag{E.44}$$

$$T10: \quad 4\alpha R^{(1)}R^{(1)}_{\alpha\beta} = 2\alpha\left(h_{;\alpha\beta} + \Box h_{\alpha\beta} - h^{\rho}_{\alpha;\rho\beta} - h^{\rho}_{\beta;\rho\alpha}\right)R^{(1)} \tag{E.45}$$

$$T11: \quad -2\kappa^2 F^{(1)}_{\alpha\mu}F^{(1)\mu}_{\beta} \tag{E.46}$$

$$T12: \quad \frac{\kappa^2}{2}\bar{g}_{\alpha\beta}F^{(1)}.F^{(1)} \tag{E.47}$$

$$T13: \quad -\frac{\kappa^2 h_{\alpha\beta}}{4}\left(h^{\rho\mu}\bar{F}_{\mu\nu}\bar{F}^{\nu}_{\rho} + h^{\rho\nu}\bar{F}_{\mu\nu}\bar{F}^{\mu}_{\rho}\right) \tag{E.48}$$

E.7 Averaging

E.7.1 Averaging procedure guidelines

- Total derivative terms of the form $\langle A_{\alpha..\beta;\mu}\rangle = 0$.
- $\langle A_{;\alpha}B_{;\beta}\rangle = -\langle A_{;\alpha\beta}B\rangle$, where A and B are indexed tensor objects.
- Averages of a product of independent fields are zero.

E.8 Terms with redefined variables

E.8.1 T1

Cadabra output:

$$
\begin{aligned}
R^{(2)}_{\alpha\beta} \;=\;& -\frac{1}{4}\nabla_\beta\psi^{\rho\tau}\nabla_\alpha\psi_{\rho\tau} - \frac{1}{2}\psi^{\rho\tau}\nabla_\alpha\nabla_\beta\psi_{\rho\tau} - \frac{1}{2}\psi^{\rho\tau}\nabla_\rho\nabla_\tau\psi_{\alpha\beta} + \frac{1}{2}\psi^{\rho\tau}\nabla_\beta\nabla_\rho\psi_{\alpha\tau} + \frac{1}{2}\psi^{\rho\tau}\nabla_\alpha\nabla_\rho\psi_{\beta\tau}\\
&-\frac{1}{2}\nabla^\rho\psi_\beta{}^\tau\nabla_\rho\psi_{\alpha\tau} + \frac{1}{2}\nabla^\rho\psi_\beta{}^\tau\nabla_\tau\psi_{\alpha\rho} + \alpha\left(\psi^{\rho\tau}g_{\alpha\beta}\nabla_\rho\nabla_\tau R - \psi^\rho{}_\alpha\nabla_\beta\nabla_\rho R - \psi^\rho{}_\beta\nabla_\alpha\nabla_\rho R\right.\\
&+\nabla^\rho\nabla_\rho\psi_{\alpha\beta}R + \nabla^\rho\psi_{\alpha\beta}\nabla_\rho R - \nabla_\alpha\psi_\beta{}^\rho\nabla_\rho R + \nabla^\rho R\nabla_\rho\psi_{\beta\alpha} + \nabla^\rho R\nabla_\beta\psi_{\alpha\rho} - \nabla_\rho R\nabla_\beta\psi^\rho{}_\alpha\\
&\left.-\nabla_\rho R\nabla_\alpha\psi^\rho{}_\beta + \nabla_\rho R\nabla^\rho\psi_{\alpha\beta} + 2\nabla^\rho R\nabla_\alpha\psi_{\beta\rho} - 2\nabla^\rho R\nabla_\rho\psi_{\alpha\beta}\right) + \alpha^2\left(-6\nabla_\beta R\nabla_\alpha R\right.\\
&-8R\nabla_\alpha\nabla_\beta R - 2\nabla^\rho\nabla_\rho R R g_{\alpha\beta} + 2\nabla_\beta\nabla_\alpha R R + 2\nabla_\alpha\nabla_\beta R R - 2g_{\alpha\beta}\nabla^\rho R\nabla_\rho R\\
&\left.-2\nabla^\rho R\nabla_\rho R g_{\alpha\beta} + 4\nabla^\rho R g_{\alpha\beta}\nabla_\rho R\right) \qquad \text{(E.49)}
\end{aligned}
$$

After following averaging guidelines:

- Coefficient of α^0:

$$
\begin{aligned}
&\frac{1}{4}\langle\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}\rangle + \kappa^2\Big\langle\Big(-\frac{1}{2}\bar{F}_\beta{}^\epsilon\bar{F}_\epsilon{}^\rho\psi_\alpha{}^\tau\psi_{\rho\tau} - \frac{1}{2}\bar{F}_\alpha{}^\epsilon\bar{F}_\epsilon{}^\rho\psi_\beta{}^\tau\psi_{\rho\tau} - \frac{1}{8}\bar{F}^{\epsilon\rho}\bar{F}_{\epsilon\rho}\psi_\beta{}^\tau\psi_{\alpha\tau}\\
&+\frac{1}{2}\bar{F}^{\epsilon\rho}\bar{F}_\epsilon{}^\tau\psi_{\alpha\rho}\psi_{\beta\tau} + 2\bar{F}_\alpha{}^\epsilon\bar{F}^{\rho\tau}\psi_{\beta\rho}\psi_{\epsilon\tau} - \bar{F}^{\epsilon\rho}\bar{F}_\epsilon{}^\tau\psi_{\alpha\beta}\psi_{\rho\tau} - \bar{F}_\alpha{}^\epsilon\bar{F}_\epsilon{}^\rho\psi_\rho{}^\tau\psi_{\beta\tau}\Big)\Big\rangle
\end{aligned}
$$

- Coefficient of α: 0
- Coefficient of α^2: $\langle -2R^{(1)}_{;\alpha}R^{(1)}_{;\beta} - 2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle$

$$\begin{aligned}\langle R^{(2)}_{\alpha\beta}\rangle &= \frac{1}{4}\langle\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}\rangle + \kappa^2\Bigg(-\frac{1}{2}\bar{F}_{\beta}{}^{\epsilon}\bar{F}_{\epsilon}{}^{\rho}\langle\psi_{\alpha}{}^{\tau}\psi_{\rho\tau}\rangle - \frac{1}{2}\bar{F}_{\alpha}{}^{\epsilon}\bar{F}_{\epsilon}{}^{\rho}\langle\psi_{\beta}{}^{\tau}\psi_{\rho\tau}\rangle\rangle \\ &-\frac{1}{8}\bar{F}^{\epsilon\rho}\bar{F}_{\epsilon\rho}\langle\psi_{\beta}{}^{\tau}\psi_{\alpha\tau} + \frac{1}{2}\bar{F}^{\epsilon\rho}\bar{F}_{\epsilon}{}^{\tau}\langle\psi_{\alpha\rho}\psi_{\beta\tau}\rangle + 2\bar{F}_{\alpha}{}^{\epsilon}\bar{F}^{\rho\tau}\langle\psi_{\beta\rho}\psi_{\epsilon\tau}\rangle - \bar{F}^{\epsilon\rho}\bar{F}_{\epsilon}{}^{\tau}\langle\psi_{\alpha\beta}\psi_{\rho\tau}\rangle \\ &-\bar{F}_{\alpha}{}^{\epsilon}\bar{F}_{\epsilon}{}^{\rho}\langle\psi_{\rho}{}^{\tau}\psi_{\beta\tau}\rangle\Bigg) - \alpha^2\left(2\langle R^{(1)}_{;\alpha}R^{(1)}_{;\beta}\rangle + 2\bar{g}_{\alpha\beta}\langle R^{(1)}\Box R^{(1)}\rangle\right)\end{aligned} \tag{E.50}$$

E.8.2 T2

Cadabra output

$$-h_{\alpha\beta}R^{(1)} = -\alpha\left(-6\psi_{\alpha\beta}\nabla^{\rho}\nabla_{\rho}R + 3g_{\alpha\beta}\psi\nabla^{\rho}\nabla_{\rho}R\right) - 12\alpha^2 g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}R \tag{E.51}$$

After averaging

$$\langle -h_{\alpha\beta}R^{(1)}\rangle = \langle -6\alpha\Box\psi_{\alpha\beta}R^{(1)} - 12\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{E.52}$$

E.8.3 T3

Cadabra output:

$$\begin{aligned}\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} &= \frac{1}{2}g_{\alpha\beta}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\rho}\psi_{\tau\epsilon} - g_{\alpha\beta}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\tau}\psi_{\epsilon\rho} + \frac{1}{2}g_{\alpha\beta}\psi\nabla^{\rho}\nabla^{\tau}\psi_{\tau\rho} + \alpha\left(-2g_{\alpha\beta}\psi^{\rho\tau}\nabla_{\rho}\nabla_{\tau}R\right. \\ &\left.+2g_{\alpha\beta}\psi\nabla^{\rho}\nabla_{\rho}R + g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}\psi + 2g_{\alpha\beta}R\nabla^{\rho}\nabla^{\tau}\psi_{\tau\rho}\right) + 12\alpha^2 g_{\alpha\beta}R\nabla^{\rho}\nabla_{\rho}R\end{aligned} \tag{E.53}$$

After averaging and using equation of motion

- Coefficient of α^0 : $\kappa^2\bar{g}_{\alpha\beta}\left[2\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle - \bar{F}^{\rho}_{\epsilon}\bar{F}^{\tau}_{\rho}\langle\psi^{\epsilon\mu}\psi_{\mu\tau}\rangle + \frac{1}{4}\left(\bar{F}\right)^2\langle\psi^2\rangle\right]$

- Coefficient of α: 0
- Coefficient of α^2: $12\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}$

$$\begin{aligned}\langle\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu}\rangle &= \kappa^2\bar{g}_{\alpha\beta}\left[2\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle - \bar{F}^{\rho}_{\epsilon}\bar{F}^{\tau}_{\rho}\langle\psi^{\epsilon\mu}\psi_{\mu\tau}\rangle + \frac{1}{4}\left(\bar{F}\right)^2\langle\psi^2\rangle\right] \\ &\quad +12\alpha^2\bar{g}_{\alpha\beta}\langle R^{(1)}\Box R^{(1)}\rangle \end{aligned} \tag{E.54}$$

E.8.4 T4

Cadabra output:

$$\begin{aligned}-\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} &= -\bar{g}_{\alpha\beta}\Bigg(-\frac{3}{8}\nabla^{\rho}\psi^{\tau\epsilon}\nabla_{\rho}\psi_{\tau\epsilon} + \frac{1}{16}\nabla^{\rho}\psi\nabla_{\rho}\psi - \frac{1}{4}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\rho}\psi_{\tau\epsilon} + \frac{1}{2}\psi^{\tau\epsilon}\nabla^{\rho}\nabla_{\tau}\psi_{\epsilon\rho} \\ &\quad +\frac{1}{4}\nabla^{\rho}\psi^{\tau\epsilon}\nabla_{\tau}\psi_{\rho\epsilon}\Bigg) - \alpha\bar{g}_{\alpha\beta}\Bigg(-\frac{1}{2}\nabla^{\rho}\psi\nabla_{\rho}R - \frac{1}{2}\nabla^{\rho}R\nabla_{\rho}\psi - \psi\nabla^{\rho}\nabla_{\rho}R \\ &\quad +\psi^{\rho\tau}\nabla_{\rho}\nabla_{\tau}R + \nabla^{\rho}R\nabla^{\tau}\psi_{\rho\tau}\Bigg) - \bar{g}_{\alpha\beta}\alpha^2(-3\nabla^{\rho}R\nabla_{\rho}R - 6R\nabla^{\rho}\nabla_{\rho}R) \end{aligned} \tag{E.55}$$

After averaging and using equation of motion

- Coefficient of α^0: $-\frac{\kappa^2\bar{g}_{\alpha\beta}}{2}\left[\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle + \frac{1}{4}\left(\bar{F}\right)^2\langle\psi^2\rangle\right]$
- Coefficient of $\alpha = 0$
- Coefficient of $\alpha^2 = 3\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}$

$$\langle\frac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\rangle = -\frac{\kappa^2}{2}\bar{g}_{\alpha\beta}\left[\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle + \frac{1}{4}\left(\bar{F}\right)^2\langle\psi^2\rangle\right] + 3\alpha^2\bar{g}_{\alpha\beta}\langle R^{(1)}\Box R^{(1)}\rangle \tag{E.56}$$

E.8.5 T5

Cadabra output:

$$-4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta} = \alpha\left(-2\nabla^\rho\psi_{\alpha\beta}\nabla_\rho R + 2\nabla_\alpha\psi_\beta{}^\rho\nabla_\rho R + 2\nabla_\beta\psi_\alpha{}^\rho\nabla_\rho R\right) + \alpha^2\left(4g_{\alpha\beta}\nabla^\rho R\nabla_\rho R - 8\nabla_\alpha R\nabla_\beta R\right) \tag{E.57}$$

After averaging and using equations of motion

$$\langle -4\alpha\delta\nabla_\alpha R^{(1)}_{;\beta}\rangle = \langle -4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)} - 8\alpha^2 R^{(1)}_{;\alpha}R^{(1)}_{;\beta}\rangle \tag{E.58}$$

E.8.6 T6

Cadabra output

$$2\alpha h_{\alpha\beta}\Box R^{(1)} = 2\alpha\psi_{\alpha\beta}\nabla^\rho\nabla_\rho R - \alpha g_{\alpha\beta}\psi\nabla^\rho\nabla_\rho R - 4\alpha^2 g_{\alpha\beta}R\nabla^\rho\nabla_\rho R \tag{E.59}$$

After averaging and using equations of motion

$$\langle 2\alpha h_{\alpha\beta}\Box R^{(1)}\rangle = \langle 2\alpha\Box\psi_{\alpha\beta}R^{(1)} - 4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{E.60}$$

E.8.7 T7

Cadabra output:

$$-2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{;\mu\nu} = -\alpha\bar{g}_{\alpha\beta}\left(2\psi^{\mu\nu}\nabla_\mu\nabla_\nu R - \psi\nabla^\mu\nabla_\mu R - 4\alpha R\nabla^\mu\nabla_\mu R\right) \tag{E.61}$$

After averaging and using equations of motion

$$\langle -2\alpha\bar{g}_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu}\rangle = \langle 4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{E.62}$$

E.8.8 T8

Cadabra output:

$$\alpha\bar{g}_{\alpha\beta}\delta\nabla^{\mu}R^{(1)}_{;\mu} \quad = \quad -4\alpha^2 g_{\alpha\beta}\nabla^{\rho}R\nabla_{\rho}R \tag{E.63}$$

After averaging

$$\langle\alpha\bar{g}_{\alpha\beta}\delta\nabla^{\mu}R^{(1)}_{;\mu}\rangle \quad = \quad \langle 4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{E.64}$$

E.8.9 T10

From calculations of T3 and equations of motion:

$$4\alpha R^{(1)}R^{(1)}_{\alpha\beta} \quad = \quad 4\alpha R^{(1)}\left[-\frac{1}{2}\Box\psi_{\alpha\beta}+\frac{1}{4}\bar{g}_{\alpha\beta}\Box\psi-\bar{R}_{\mu\alpha\nu\beta}\psi^{\mu\nu}+\alpha\left(2R^{(1)}_{;\alpha\beta}+\bar{g}_{\alpha\beta}\Box R^{(1)}\right)\right] \tag{E.65}$$

Using equations of motion and averaging which becomes

$$\langle 4\alpha R^{(1)}R^{(1)}_{\alpha\beta}\rangle \quad = \quad \langle -8\alpha^2 R^{(1)}_{;\alpha}R^{(1)}_{;\beta}+4\alpha^2\bar{g}_{\alpha\beta}R^{(1)}\Box R^{(1)}\rangle \tag{E.66}$$

E.8.10 T11

Using gauge condition and equations of motion

$$-2\kappa^2\langle F^{(1)}_{\alpha\mu}F^{(1)\mu}_{\beta}\rangle \quad = \quad -2\kappa^2\langle A^{(1)}_{\mu;\alpha}A^{(1)\mu}_{;\beta}\rangle+4\kappa^4\bar{T}_{\alpha\beta}\langle(A^{(1)})^2\rangle \tag{E.67}$$

E.8.11 T12

Using gauge condition and equations of motion

$$\langle\frac{\kappa^2}{2}\bar{g}_{\alpha\beta}F^{(1)}_{\mu\nu}F^{(1)\mu\nu}\rangle \quad = \quad 2\kappa^4\bar{g}_{\alpha\beta}\bar{T}_{\mu\nu}\langle A^{(1)\mu}A^{(1)\nu}\rangle \tag{E.68}$$

E.8.12 T13

Using gauge condition and equations of motion

$$\langle -\frac{\kappa^2 h_{\alpha\beta}}{4}\left(h^{\rho\mu}\bar{F}_{\mu\nu}\bar{F}^{\nu}_{\rho}+h^{\rho\nu}\bar{F}_{\mu\nu}\bar{F}^{\mu}_{\rho}\right)\rangle \;=\; 0 \tag{E.69}$$

E.9 Second order perturbed $f(R)$ equations of motion in RN space-times

Ignoring terms of $\mathcal{O}\left(\kappa^4\right)$

$$\langle G^{(2)}_{\alpha\beta}-T^{(2),eff}_{\alpha\beta}\rangle \;=\; \langle\sum_{i=1}^{13}Ti\rangle \tag{E.70}$$

$$= \frac{1}{4}\langle\psi^{\rho\tau}_{;\alpha}\psi_{\rho\tau;\beta}\rangle-\frac{1}{6}\alpha\bar{g}_{\alpha\beta}\langle\left(R^{(1)}\right)^2\rangle-18\alpha^2\langle R^{(1)}_{;\alpha}R^{(1)}_{;\beta}\rangle -2\kappa^2\langle A^{(1)}_{\mu;\alpha}A^{(1)\mu}_{;\beta}\rangle+\kappa^2\langle\mathcal{P}_{\alpha\beta}\rangle \tag{E.71}$$

where

$$\begin{aligned}\langle\mathcal{P}_{\alpha\beta}\rangle \;=\; & -\frac{1}{2}\bar{F}^{\epsilon}_{\beta}\bar{F}^{\rho}_{\epsilon}\langle\psi_{\alpha}{}^{\tau}\psi_{\rho\tau}\rangle-\frac{3}{2}\bar{F}^{\epsilon}_{\alpha}\bar{F}^{\rho}_{\epsilon}\langle\psi_{\beta}{}^{\tau}\psi_{\rho\tau}\rangle-\frac{1}{8}\bar{F}^{\epsilon\rho}\bar{F}_{\epsilon\rho}\langle\psi_{\beta}{}^{\tau}\psi_{\alpha\tau}\rangle\\ & +\frac{1}{2}\bar{F}^{\epsilon\rho}\bar{F}^{\tau}_{\epsilon}\langle\psi_{\alpha\rho}\psi_{\beta\tau}\rangle+2\bar{F}^{\epsilon}_{\alpha}\bar{F}^{\rho\tau}\langle\psi_{\beta\rho}\psi_{\epsilon\tau}\rangle-\bar{F}^{\epsilon\rho}\bar{F}^{\tau}_{\epsilon}\langle\psi_{\alpha\beta}\psi_{\rho\tau}\rangle\\ & +\bar{g}_{\alpha\beta}\left[\frac{3}{2}\bar{F}_{\epsilon\rho}\bar{F}_{\mu\tau}\langle\psi^{\epsilon\mu}\psi^{\rho\tau}\rangle-\bar{F}^{\rho}_{\epsilon}\bar{F}^{\tau}_{\rho}\langle\psi^{\epsilon\mu}\psi_{\mu\tau}\rangle+\frac{1}{8}\left(\bar{F}\right)^2\langle\psi^2\rangle\right]\end{aligned} \tag{E.72}$$

Appendix F

Linearly perturbed Pontryagin density and effective source term of odd parity dynamics

F.1 Perturbed Pontryagin density

The perturbed metric tensor can be 2+2 decomposed following Eq. (2.15), (2.20), and (2.23). A covariant Levi-Civita on a 2-sphere can be constructed by projecting out of a Levi-Civita in the full space-time as

$$\epsilon_{AB} = \frac{1}{\sqrt{2}r^2}\epsilon_{AaBb}\epsilon^{ab} \tag{F.1}$$

where ϵ_{ab} is the covariant Levi-Civita on the (t, r) space. (F.1) satisfies all the properties of the antisymmetric 2-form in the 2-sphere. Using Eq. (2.15), (2.20), (2.23), and (F.1), the perturbed Pontryagin density for a background Schwarzschild space-time in terms of the *Cunningham-Price-Moncrief* variable Φ_O (as defined in [86]) becomes

$$\delta\left({}^*RR\right) = \frac{24\left(\ell-1\right)\ell\left(\ell+1\right)\left(\ell+2\right)M}{r^6}\,\Phi_O\,\mathbf{S} \tag{F.2}$$

where $\mathbf{S} \equiv \mathbf{S}_{\ell m}$ is the scalar spherical harmonic as defined in (2.16) and it's seen that only the odd parity master function contributes to the perturbed Pontryagin density.

F.2 Perturbed Cotton tensor as an effective source.

The perturbed Cotton tensor can be written as an effective energy-momentum tensor in the following manner

$$
\begin{aligned}
R^{(1)}_{\mu\nu} &= \kappa^2 \mathcal{T}_{\mu\nu} && \text{(F.3)}\\
\mathcal{T}_{\mu\nu} &= -\alpha \Theta_{;\tau\sigma} \left({}^*\bar{R}^{\tau}{}_{\mu\nu}{}^{\sigma} + {}^*\bar{R}^{\tau}{}_{\nu\mu}{}^{\sigma} \right) && \text{(F.4)}
\end{aligned}
$$

A 2-vector and a scalar can be defined from $\mathcal{T}_{\mu\nu}$ in the following manner, following [86]

$$
\begin{aligned}
P^a &= \frac{\kappa^2 r^2}{k^2} \int \mathcal{T}^{aA} V_A \; d\Omega && \text{(F.5)}\\
P &= \frac{\kappa^2 r^4}{(\ell-1)\,\ell\,(\ell+1)\,(\ell+2)} \int \mathcal{T}^{AB} V_{AB} \; d\Omega && \text{(F.6)}
\end{aligned}
$$

Using φ as defined in Eq. (5.10), one obtains the following

$$
\begin{aligned}
P^t &= -\frac{6M}{r^2} \partial_r \varphi && \text{(F.7)}\\
P^r &= \frac{6i\omega M}{r^2} \varphi && \text{(F.8)}\\
P &= 0 && \text{(F.9)}
\end{aligned}
$$

The above components satisfy the conservation equation $\nabla^\mu \mathcal{T}_{\mu\nu} = 0$. Following [86], one finds

$$
\partial_t P^t + \partial_r P^r + \frac{2}{r} P^r = 0 \qquad \text{(F.10)}
$$

which serves as a consistency check for the obtained components. Again, following [86], the effective source term coupling with the odd parity gravitational perturbation was found to be

$$
S^{eff} = \frac{\kappa^2 \alpha}{(\ell-1)\,(\ell+2)} \left[\frac{6M}{r} \partial^2_{r_*} \varphi - \frac{12M}{r^2} \partial_{r_*} \varphi + \frac{6\omega^2 M}{r} \varphi \right] \qquad \text{(F.11)}
$$

Appendix G

Gravitational radiation in the shortwave limit of dynamical CS modified gravity

A vanishing background ϑ and transverse-traceless gauge was used. For a simultaneous metric and CS field perturbation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + eh_{\mu\nu} \tag{G.1}$$

$$\vartheta = e\vartheta, \tag{G.2}$$

the modified field tensor $\mathfrak{G}_{\mu\nu} = R_{\mu\nu} 2\kappa^2 \alpha C_{\mu\nu} - \kappa^2 \beta \vartheta_{;\mu} \vartheta_{;\nu}$ can be expanded in powers of ϵ as

$$\bar{\mathfrak{G}}_{\mu\nu} + e\mathfrak{G}^{(1)}_{\mu\nu} + e^2 \mathfrak{G}^{(2)}_{\mu\nu} = 0 \tag{G.3}$$

Solving for $\mathfrak{G}^{(1)}_{\mu\nu} = 0$ gives the dynamics of the perturbation. While the radiated energy and momentum flux due to perturbation can be found from an energy-momentum pseudotensor due to perturbation. From (G.3) one then obtains

$$\bar{\mathfrak{G}}_{\mu\nu} = \kappa^2 t_{\mu\nu} \tag{G.4}$$

$$= -e^2 \left\langle \mathfrak{G}^{(2)}_{\mu\nu} \right\rangle \tag{G.5}$$

$$t_{\mu\nu} = -\frac{e^2}{\kappa^2} \left\langle \mathfrak{G}^{(2)}_{\mu\nu} \right\rangle \tag{G.6}$$

$$\mathfrak{G}^{(2)}_{\mu\nu} = G^{(2)}_{\mu\nu} - 4\kappa^2 \alpha \left[\nabla^{(1)}_{\sigma} \nabla_{\tau} \vartheta \; {}^*\bar{R}^{\tau}{}_{(\mu}{}^{\sigma}{}_{\nu)} \right.$$

$$+\nabla_\sigma \nabla_\tau \vartheta \ {}^*R^{(1)\tau}{}_{(\mu}{}^{\sigma}{}_{\nu)}\Big] - \kappa^2 \beta \vartheta_{;\mu}\vartheta_{;\nu} \tag{G.7}$$

The operation $\langle ... \rangle$ was defined in [101] and consists of the following effective operations

- Total derivative terms are put to zero.
- $\langle A_{;\mu} B_{;\nu} \rangle = -\langle A_{;\mu\nu} B \rangle$
- Covariant derivatives commute.
- Average of a product of two different fields are put to zero, since for high frequencies they are Gaussian random variables.

From which $\left\langle \mathfrak{G}^{(2)}_{\mu\nu} \right\rangle$ was found to be

$$-\left\langle \mathfrak{G}^{(2)}_{\mu\nu} \right\rangle = \frac{1}{4}\left\langle \psi^{\rho\tau}_{;\mu}\psi_{\rho\tau;\nu}\right\rangle + \frac{\kappa^2\alpha^2}{2\beta}\left\langle \mathcal{P}_{\mu\nu}\right\rangle - \kappa^2\beta\left\langle \vartheta_{;\mu}\vartheta_{;\nu}\right\rangle \tag{G.8}$$

$$\begin{aligned}
\left\langle \mathcal{P}_{\mu\nu}\right\rangle = {} & -2\left\langle \psi^{\beta\gamma;\delta\lambda}\psi_{\nu\alpha;\sigma}{}^{;\rho}\right\rangle \epsilon_{\mu\rho}{}^{\sigma\alpha}\left({}^*\bar{R}_{\lambda\beta\delta\gamma} + {}^*\bar{R}_{\delta\beta\lambda\gamma}\right) \\
& -2\left\langle \psi^{\beta\gamma;\delta\lambda}\psi_{\mu\alpha;\sigma}{}^{;\rho}\right\rangle \epsilon_{\nu\rho}{}^{\sigma\alpha}\left({}^*\bar{R}_{\lambda\beta\delta\gamma} + {}^*\bar{R}_{\delta\beta\lambda\gamma}\right) \\
& -2\left\langle \psi^{\rho\sigma;\delta}\psi^{\alpha\beta;\gamma}\right\rangle \left[{}^*\bar{R}_{\gamma\alpha\delta\beta}\left({}^*\bar{R}_{\mu\sigma\nu\rho} + {}^*\bar{R}_{\nu\sigma\mu\rho}\right)\right. \\
& \left.+{}^*\bar{R}_{\delta\alpha\gamma\beta}\left({}^*\bar{R}_{\mu\sigma\nu\rho} + {}^*\bar{R}_{\nu\sigma\mu\rho}\right)\right] \\
& +\bar{R}^{\rho\sigma\alpha\beta}\left[\epsilon_{\mu\gamma}{}^{\delta\lambda}\left(\left\langle \psi^{\eta;\gamma}_{\sigma}\psi_{\nu\lambda;\delta}\right\rangle {}^*\bar{R}_{\rho\eta\alpha\beta}\right.\right. \\
& \left.+\left\langle \psi^{\eta;\gamma}_{\alpha}\psi_{\nu\lambda;\delta}\right\rangle {}^*\bar{R}_{\rho\sigma\eta\beta}\right) \\
& +\epsilon_{\nu\lambda}{}^{\delta\gamma}\left(\left\langle \psi^{\eta;\gamma}_{\sigma}\psi_{\mu\lambda;\delta}\right\rangle {}^*\bar{R}_{\rho\eta\alpha\beta}\right. \\
& \left.\left.+\left\langle \psi^{\eta;\gamma}_{\alpha}\psi_{\mu\lambda;\delta}\right\rangle {}^*\bar{R}_{\rho\sigma\eta\beta}\right)\right] \\
& +\bar{R}^{\rho\sigma\alpha\beta}\left[\left\langle \psi^{\gamma\delta}\psi^{\lambda}_{\alpha}\right\rangle {}^*\bar{R}_{\rho\sigma\lambda\beta}\left({}^*\bar{R}_{\mu\delta\nu\gamma} + {}^*\bar{R}_{\nu\delta\mu\gamma}\right)\right. \\
& \left.+\left\langle \psi^{\gamma\delta}\psi^{\lambda}_{\sigma}\right\rangle {}^*\bar{R}_{\rho\lambda\alpha\beta}\left({}^*\bar{R}_{\mu\delta\nu\gamma} + {}^*\bar{R}_{\nu\delta\mu\gamma}\right)\right]
\end{aligned} \tag{G.9}$$

Appendix H

Constancy of $\Delta_{\ell m}$ in GR and its time dependence in CS

H.1 Constancy in GR

The quantity $\Delta_{\ell m}$ in the main text given by

$$\Delta_{\ell,m} = \frac{\left|\dot{\Psi}_O^{\ell,m}\right|^2 - \left|\dot{\Psi}_E^{\ell,m}\right|^2}{\left|\dot{\Psi}_O^{\ell,m}\right|^2 + \left|\dot{\Psi}_E^{\ell,m}\right|^2} \tag{H.1}$$

can be written as

$$\Delta_{\ell,m} = \frac{\frac{\left|\dot{\Psi}_O^{\ell,m}\right|^2}{\left|\dot{\Psi}_E^{\ell,m}\right|^2} - 1}{\frac{\left|\dot{\Psi}_O^{\ell,m}\right|^2}{\left|\dot{\Psi}_E^{\ell,m}\right|^2} + 1} \tag{H.2}$$

In the wave zone, the odd/even modes are of the form

$$\Psi_{E/O} = A_{E/O} e^{-\kappa_{E/O} t} e^{i\omega_{E/O} t} \tag{H.3}$$

where $A_{E/O}$ is a constant amplitude that depends on the initial conditions of the perturbation process. Due to isospectrality relation for GR, $\kappa_E = \kappa_O = \kappa$ and

$\omega_E = \omega_O = \omega$. Substituting (H.3) in (H.2) one obtains

$$\Delta_{\ell,m} = \frac{\left|\frac{A_O}{A_E}\right|^2 - 1}{\left|\frac{A_O}{A_E}\right|^2 + 1} \tag{H.4}$$

which is a constant.

H.2 Time dependent $\Delta_{\ell,m}$ in CS gravity

Radiation rate escaping to asymptotic infinity for general relativity is given by [86]

$$\left\langle \dot{E} \right\rangle \Big|_{GR} = \frac{1}{64\pi} \sum_{\ell m} \mu \left\langle \left|\dot{\Psi}_E\right|^2 + \left|\dot{\Psi}_O\right|^2 \right\rangle \tag{H.5}$$

$$\mu = (\ell - 1)\,\ell\,(\ell + 1)\,(\ell + 2) \tag{H.6}$$

Similarly, for dynamical CS gravity the rate at which radiation (both gravitational and scalar) escapes to asymptotic infinity can be given by

$$\left\langle \dot{E} \right\rangle \Big|_{CS} = \frac{1}{64\pi} \sum_{\ell m} \mu \left\langle \left|\dot{\Psi}_E\right|^2 + \left|\dot{\tilde{\Psi}}_O\right|^2 + \kappa^2 \beta \left|\dot{\varphi}\right|^2 \right\rangle \tag{H.7}$$

There is also energy loss $\left\langle \dot{E}_{coup,CS} \right\rangle$ in the form of the graviton-graviton coupling near the BH region (G.9) which does not travel to asymptotic infinity, thereby effectively reducing the odd parity reflection coefficient, or the fraction of the odd parity initial excitation that gets scattered off to asymptotic infinity, compared to GR. Considering the same initial perturbation energy for a Schwarzschild solution in GR and dynamical CS, the latter shall then radiate lesser gravitational flux, with the difference in energy coming from both the graviton-graviton coupling (which absorbed by the BH), and the kinetic term of the pseudoscalar field. Thus, one can write the following

$$\left\langle \dot{E} \right\rangle \Big|_{CS} + \left\langle \dot{E} \right\rangle \Big|_{coup,CS} = \left\langle \dot{E} \right\rangle \Big|_{GR} \tag{H.8}$$

from which one obtains the following inequality

$$\left|\dot{\Psi}_O\right|^2 > \left|\dot{\tilde{\Psi}}_O\right|^2 \tag{H.9}$$

at all times. A suitable ansatz for the modified odd parity wavefunction for CS gravity can be

$$\tilde{\Psi}_O = \tilde{A}_O e^{-\tilde{\kappa}_O t} e^{i\tilde{\omega}_O t} \tag{H.10}$$

where $\tilde{A}_O < A_O$, the real and imaginary parts of the odd parity QNM frequency are modified due to the coupling with the CS field in the form of an inhomogeneous term in the RHS of the differential equation (9) in the main text. This leads to the following

$$\frac{\left|\dot{\tilde{\Psi}}_O\right|^2}{\left|\dot{\Psi}_E\right|^2} = \frac{\tilde{A}_O^2\left(\tilde{\kappa}_O^2 + \tilde{\omega}_O^2\right)}{A_E^2\left(\kappa_E^2 + \omega_E^2\right)} e^{-2(\tilde{\kappa}_O - \kappa_E)t} \tag{H.11}$$

which is less than the corresponding GR value at all times courtesy Eq. (H.9), with a growth/decay rate proportional to $e^{-2(\tilde{\kappa}_O - \kappa_E)t}$ (depending on whether the imaginary part of the odd parity dominant mode frequency is enhanced or suppressed due to CS modification). However, for the same initial energy of perturbation, the odd parity mode can now relax to a stable Schwarzschild faster, because of the presence of further channels (pseudo-scalar and graviton-graviton coupling) to take away the initial perturbation energy. This leads to a shorter modified decay time for the odd parity mode compared to the even parity [1], i.e. $\tilde{\kappa}_O > \kappa_E$ — implying that RHS of Eq. (H.11), and correspondingly $\Delta_{\ell,m}$, will be decreasing functions of time in CS gravity.

[1]A feature that is also seen in charged BHs in GR, with the damping time decreasing with increase in charge (See Table I of [136]). This is because, for a purely gravitational perturbation, extra energy in the system can now be radiated away through the electromagnetic waves as well (in addition to gravitational degrees of freedom), making the relaxation to a stable solution faster. Although the signatures imparted to gravitational waves due to the presence of charge is quite distinct from the signatures imparted due to modifications to gravity. Whereas isospectrality holds for perturbed charged GR BHs, the same does not hold for a perturbed Schwarzschild in CS modified gravity.

Appendix I

Matching elliptical amplitudes observed in detectors to odd and even parity master functions

In order to equate the parity polarizations with the plus and cross, the radiative part of the metric perturbation is projected on a tetrad of freely falling observers in the radiation zone. The radiative part is simply the perturbation about the background 2-sphere in the 2+2 decomposed metric. One has from [69, 79, 86, 96]

$$h_{\hat{A}\hat{B}} = e^A_{\hat{A}} e^B_{\hat{B}} h_{AB} \tag{I.1}$$

$$= \frac{\Phi_E}{r} \begin{pmatrix} \mathbf{S}_{\theta\theta} & \frac{\mathbf{S}_{\theta\phi}}{\sin\theta} \\ \frac{\mathbf{S}_{\theta\phi}}{\sin\theta} & \frac{\mathbf{S}_{\phi\phi}}{\sin^2\theta} \end{pmatrix} + \frac{\Phi_O}{r} \begin{pmatrix} \mathbf{V}_{\theta\theta} & \frac{\mathbf{V}_{\theta\phi}}{\sin\theta} \\ \frac{\mathbf{V}_{\theta\phi}}{\sin\theta} & \frac{\mathbf{V}_{\phi\phi}}{\sin^2\theta} \end{pmatrix} \tag{I.2}$$

$$= \begin{pmatrix} h_+ - ip\dot{\vartheta}h_\times & h_\times + ip\dot{\vartheta}h_+ \\ h_\times + ip\dot{\vartheta}h_+ & -h_+ + ip\dot{\vartheta}h_\times \end{pmatrix} \tag{I.3}$$

$$= \begin{pmatrix} \tilde{h}_+ & \tilde{h}_\times \\ \tilde{h}_\times & -\tilde{h}_+ \end{pmatrix} \tag{I.4}$$

where an implicit summation of ℓ, m was assumed and p is the wavenumber corresponding the plus/cross polarizations. Comparing (I.2) and (I.4) and using the relation between the tensor spherical harmonics and spin-weighted spherical harmonics [96] obtains

$$\tilde{h}_{+} - i\tilde{h}_{\times} \simeq \frac{1}{r}\sum_{\ell,m}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left(\Phi_E + i\Phi_O\right)\ {}_{-2}Y_{\ell m} \tag{I.5}$$

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