Faculty of Science

# Super-membranes, M(atrix) theory and scattering

## A (mostly) self contained introduction

**Christian Leppänen**

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**Department of Physics**
**University of Gothenburg**

UNIVERSITY OF GOTHENBURG

# SUPER-MEMBRANES, MATRIX THEORY AND SCATTERING

CHRISTIAN LEPPÄNEN

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Department of Fundamental Physics
Faculty of Science
Göteborgs Universitet
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To my family,
who always adds fuel to my curiosity.

# ABSTRACT

We analyze the bosonic and super-membrane in 11D Minkowski space-time. Passing on to light-cone coordinates, and gauge, to truncate the infinite dimensional Lie algebra on the membrane to arrive at M(atrix) theory. We give an introduction to M(atrix) theory, develop some tools needed to calculate the scattering of two D0-branes in M(atrix) theory to first loop order. We derive the M(atrix) theory from Yang-Mills theory, calculate the scattering of two D0-branes and compare this to the first term in super-gravity. We find that M(atrix) theory at this order indeed coincide with super-gravity.

*Non-Euclidean geometry and non-commutative algebra, which at one time were considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be necessary for the description of general facts of the physical world.* — Paul Dirac

# ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my supervisor Martin Cederwall who suggested such an interesting topic and who guided me on this journey through extra dimensions filled with charged branes as far as the eye can see. With out his supervision this thesis would never been written. I would also like to thank my friends for wanting to discuss this technical thesis in layman terms which deepened my understanding. Many thanks to my sister who spurred me through the writings of the last chapter. Thanks to the creator of classical thesis and the contributors[1].

Many thanks goes out for the people in the Hilbert-room[2], especially Farhad, where most of this thesis have been written, for helping in times of need, proofreading and valuable discussions.

1 http://code.google.com/p/classicthesis/

2 Kemivägen 9, Chalmers University of Technology, Gothenburg, Sweden

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Part I

# INTRODUCTION

We give an historical introduction and an outline of the thesis, followed by some conventions and notations.

# 1 INTRODUCTION

## 1.1 HISTORY

Dirac was the first to study relativistic membranes, in 1962, trying to model an electron with a vibrating spherical shell [1]. This theory did however not include spin and to implement it was hard, the theory was therefore abandoned.

String theory appeared in the late 1960s'. In string theory one considers strings, that is 1-dimensional objects instead of what previously had been considered, point particles, that is; zero dimensional particles. But if one starts to study 1- dimensional object a natural question arises; why not consider membranes, i.e. 2-dimensional objects or even higher dimensional objects? The first paper to investigate the Lagrangian and Hamiltonian formalism of the bosonic membrane was [2] published in 1976, where they also considered some different gauge choices, which arises naturally since one is bound to have a reparametrization invariance of the membranes world volume if one does not gauge fix.

Since bosonic membranes lack fermions it was natural to try to introduce fermions into the theory, this was done using supersymmetry. This can be done in 1 of 3 ways, one can introduce supersymmetry on the world volume of the membrane (called a "spinning membrane"), or embed the membrane in a super-space, or both at the same time (called super-embedding). There is however a paper which presents a no-go theorem for spinning membranes [3], which in turn made most people consider the target-space alternative, as shall we.

To match the bosonic degrees of freedom with the fermionic degrees of freedom one needs a fermionic symmetry, called kappa-symmetry ($\kappa$-symmetry). This symmetry was derived for general membranes in 1987 [4].

Hoppe and Goldstone found a clever way in 1982 [5] to regularize the field theory living on the spherical bosonic membrane's world volume. This was done by truncating the infinite Lie algebra to a finite Lie algebra, described by $su(n)$[1]. One then recovers the infinite Lie algebra in the $N \to \infty$ limit in some sense, more on this in chapter (4.4.4). This was later done for a supersymmetric membrane of arbitrary geometry [6, 7]. This truncation leads to a theory called M(atrix) theory, which the last part of this thesis will treat.

1 $su(n)$ is for the case where the center of mass motion has been removed, if one considers a theory where the center of mass motions has not been removed the Lie algebra will be $u(n)$.

To date no normalized ground state for the super-membrane is known.

## 1.2 OUTLINE

In the second part we establish the mathematical framework, we define vectors, dual vectors, manifolds and some other mathematical tools.

In the third part we describe the bosonic membrane. First in the Lagrangian formalism where we analyze the equations of motion and symmetries of the Lagrangian. Then we make the transition to the Hamiltonian formalism. Here we analyze the constraints due to the reparameterization invariance of the membranes world volume. We choose the so called light-cone coordinates and a gauge called the light-cone gauge. We analyze the infinite dimensional Lie algebra and truncate it to arrive at a finite dimensional M(atrix) theory.

We will try to make calculations explicit in the third part so that one can follow them if one stares at the equations for a reasonable time. In the fourth part the calculations often get to long to be written out explicitly, but one should have a better feeling for the equations from the bosonic part.

In the fourth part we will basically do the same thing as in the third part but with a supersymmetric membrane. For the super-membranes we need an additional symmetry called the $\kappa$-symmetry. We show how this symmetry works and elaborate on it's necessity. In the last section of this part we make the transition to supersymmetric M(atrix) theory.

In the fifth and final part we give an introduction to M(atrix) theory, connecting it to other theories (not only membranes), develop some tools and lastly give an explicit calculation to show that scattering of two D0-branes in M(atrix) theory, at one loop correction, gives the first term of super-gravity.

## 1.3 NOTATION AND CONVENTIONS

We will use units in which $\hbar = c = 1$. This means that we will measure everything in units of mass[2], i.e. [time] = [length] = $-1$ and [energy] = 1, where "1" means $\text{Mass}^1$ and "$-1$" means $\text{Mass}^{-1}$.

2 The relations can be seen from the standard well know equations, $E = mc^2$, $E = \hbar\omega$ and $ct = x$.

Partial derivatives are written as $\partial_i = \frac{\partial}{\partial \sigma^i}$ and $\partial_m = \frac{\partial}{\partial X^m}$. The indices are (for the bosonic and super-membrane chapters)

$$M, N, P \text{ coordinate super indices} \begin{cases} m, n, p & \text{coordinate vector indices} \\ \mu, \nu, \rho & \text{coordinate fermionic indices} \end{cases}$$

$$A, B, C \text{ inertial super indices} \begin{cases} a, b, c & \text{inertial vector indices} \\ \alpha, \beta, \gamma & \text{inertial fermionic indices} \end{cases}$$

and

$$i, j, k \text{ are world-volume indices}$$

with $i, j, k = 0, 1, 2$. $r, s$ will be range over 1 to 2. If we run out of indices we will prime the indices, e.g. $M, N, P, M', N', P', M'', \ldots$

In this thesis we consider 11-dimensional target space[3].

3 The treatment of the number of dimensions the membrane can live in and what this implies for the spinors and number of supersymmetries is investigated in the so called "brane scan". See e.g. [8] for such a review.

# Part II

# MATHEMATICAL PRELIMINARIES

In this part we define the manifold, the Lie derivative and we take a look at the Cartan formalism. The larger part of this is heavily based on [9]. We will define the terminology, show some properties and make some claims. For a deeper treatment, more rigorous and systematic introduction see [10] or [11].

# 2 MATHEMATICAL PRELIMINARIES

We will go over the mathematical preliminaries. This is not strictly necessary since there is a great deal one could understand about the membrane with ordinary calculus and some good guessing, but it is necessary in order to get a deeper and proper understanding of the theory. First we will try to get an intuition of what a manifold is, then we will define maps and special kinds of sets, after this we define the manifold. We will mention some important concepts as pullbacks and pushforwards. We will take a look at two types of vector representations (and their dual vectors, called forms) and look at different kinds of connections and relate these to each other. At the end we will take a look at some special kinds of tools for manifolds, that is: Covariant derivatives, Lie derivatives in different kinds of shapes and some other tools.

## 2.1 MANIFOLDS

A manifold (to be define later in this chapter) is a topological space[1] which is topologically equivalent[2] to $\mathbb{R}^m$ **locally**. Especially is $\mathbb{R}^m$ a manifold. Some other examples are

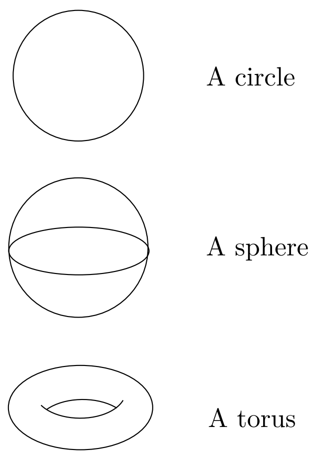


Figure 1: Examples of manifolds.

Examples of non-manifolds are

1 A set of points each with it's set of neighborhood points that satisfy a set of axioms relating points and neighborhoods.
2 One can think of "topologically equivalent to" as "looks like".

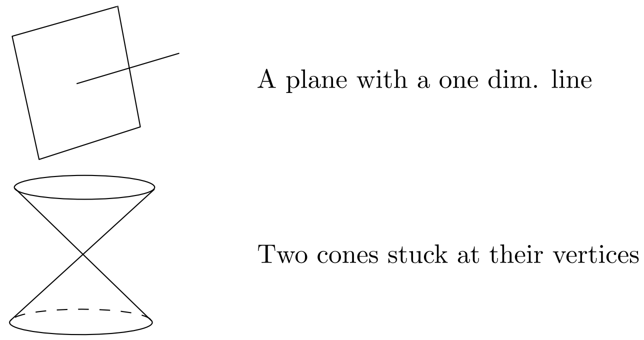


Figure 2: Examples of non-manifolds.

since they are not topologically equivalent to $\mathbb{R}^m$ locally.

We will introduce some terminology and definitions so that we can define a manifold a little more exactly.

**Definition 1. Map**

A map $\phi$ between two sets $M$ and $N$ is a relationship which assigns exactly one element in $N$ to every element in $M$. We write this as $\phi : M \to N$. ◇

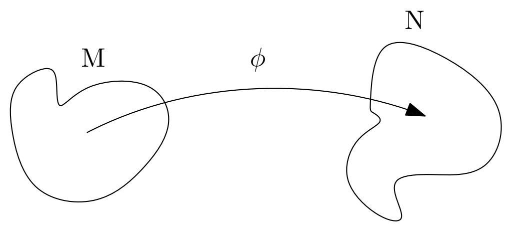


Figure 3: The map $\phi$ maps every element in the set $M$ to exactly one element in the set $N$.

Note that $\phi$ is not always invertible.

When we have many maps we can compose them. We define the composition:

**Definition 2. Composition**

Given two maps $\phi : L \to M$ and $\psi : M \to N$, we define the composition $\psi \circ \phi : L \to N$ by the operation $(\psi \circ \phi)(l) = \psi(\phi(l))$, with $l \in L$, $\phi(l) \in M$ and $\psi(\phi(l)) \in N$. ◇

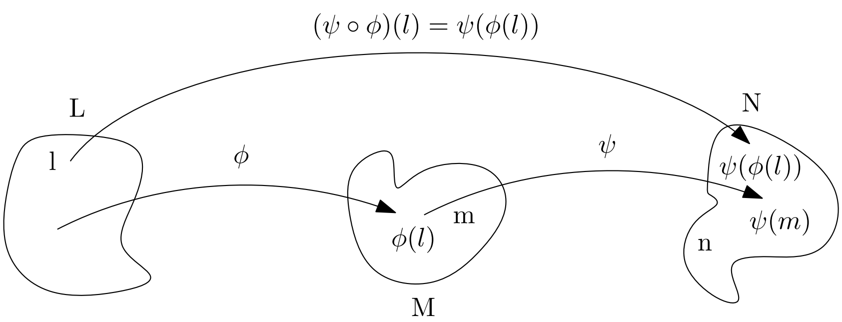


Figure 4: Composition.

A map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ takes an $m$-tuple $(x^1, x^2, \ldots, x^m)$ to an $n$-tuple $(y^1, y^2, \ldots, y^n)$. We can therefore think of the map as $n$ functions $y^i = \phi^i(x^1, x^2, \ldots, x^m)$ with $(i = 1, 2, \ldots, n)$. That is, $y^i$ are functions of $m$ variables.

Onward!

We define a very usable word, namely smooth:

**Definition 3. p times differentiable and Smooth**

If a function is $p$ times differentiable and continuous, we denote this by $C^p$. In particular $C^\infty$ maps are called "smooth". ◇

Looking ahead, it turns out that our Lagrangian will be invariant under reparametrization[3]. We say that the Lagrangian for the membrane has diffeomorphisms. The mathematical definition is:

**Definition 4. Diffeomorphic & diffeomorphism**

If there exists a smooth ($C^\infty$) map $\phi : M \to N$ with a smooth ($C^\infty$) inverse $\phi^{-1} : N \to M$, we call $\phi$ a diffeomorphism and we say that $M$ and $N$ are diffeomorphic. ◇

Smooth maps are differentiable by definition. We claim that there is a chain rule for compositions of maps:

*Claim* 1. **Chain rule**

If we have two maps $f : \mathbb{R}^l \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$, we will also have it's composition $(g \circ f) : \mathbb{R}^l \to \mathbb{R}^n$ with coordinates $x^a \in \mathbb{R}^l$, $y^b \in \mathbb{R}^m$ and $z^c \in \mathbb{R}^n$. The chain rule relates the partial derivatives of the composition to the partial derivatives on the individual maps

$$\frac{\partial (g \circ f)^c}{\partial x^a} = \sum_b \frac{\partial f^b}{\partial x^a} \frac{\partial g^c}{\partial y^b} \tag{1}$$

which we usually write as

$$\frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a} \frac{\partial}{\partial y^b}. \tag{2}$$

Any manifold will be topologically equivalent to $\mathbb{R}^m$ by definition so this will be the local chain rule of the manifold. ◇

Manifolds are built from atlases which are built from charts which are built from open sets which are built from open balls. We need to define these to define the manifold. Here we go:

**Definition 5. Open ball**

An open ball is the set of all points $x^i \in \mathbb{R}^m$ such that all points $x^i$ lies inside some fixed radius, that is $\left[\sum_{i=1}^m (x^i - y^i)^2\right]^{1/2} < r$ for some fixed $y \in \mathbb{R}^m$ and $r \in \mathbb{R}$ with $i = 1, \ldots, m$. ◇

*Note the strict inequality.*

3 No physical properties can depend on the choice of coordinates.

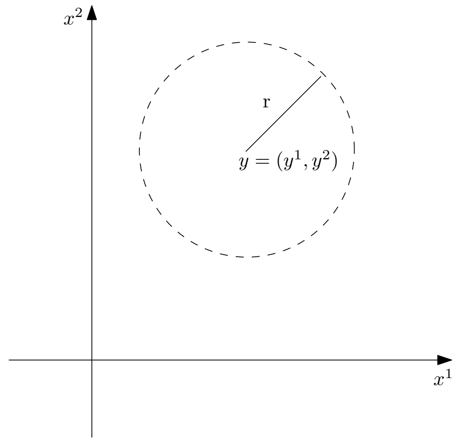


Figure 5: Open ball in $\mathbb{R}^2$.

The open ball allows us to define the open set.

**Definition 6. Open set**
An open set in $\mathbb{R}^m$ is a set constructed from an arbitrary union of open balls. ⋄

Which allows us to define a chart.

**Definition 7. Chart**
A chart consists of a subset $U \subseteq M$ along with a one-to-one continuous (and invertible obviously) map $\phi : U \to \mathbb{R}^m$ such that the set of points $\phi(U)$ is an open set in $\mathbb{R}^m$. That is, a chart is $(U, \phi)$. ⋄

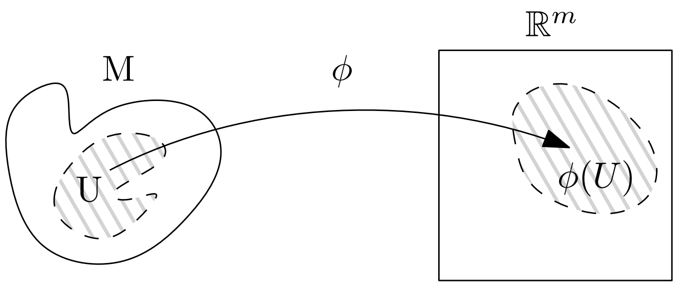


Figure 6: Chart.

We are now ready to define a special kind of atlas. The smooth ($C^\infty$) atlas.

**Definition 8. Smooth Atlas**
A smooth atlas is an indexed collection of charts $\{(U_\alpha, \phi_\alpha)\}$ where the union of the $U_\alpha$ covers all of $M$. The charts are also smoothly sewn together; that is, if $U_\alpha \cap U_\beta \neq \emptyset$ then $(\phi_\alpha \circ \phi_\beta^{-1}) : [\phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n] \to [\phi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n]$ (see fig 7) and we require these maps to be smooth everywhere they are defined. ⋄

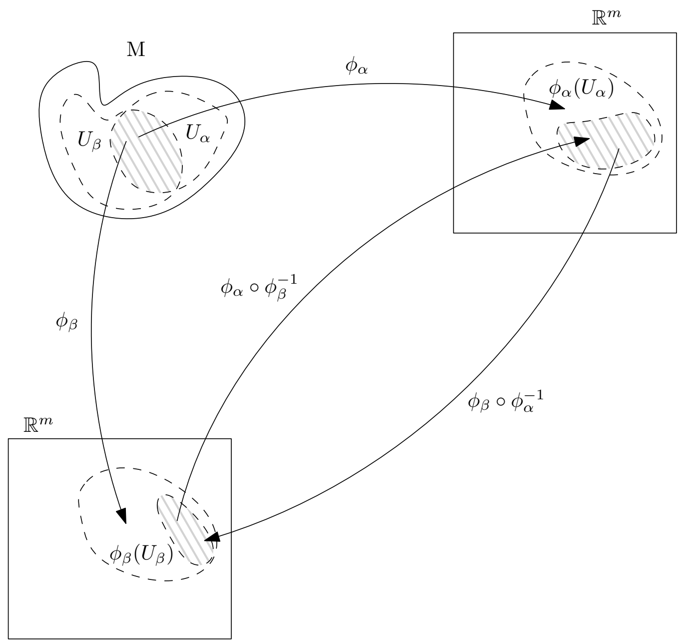


Figure 7: Atlas.

We can now define the manifold.

**Definition 9. Manifold**
A (smooth) manifold is a set $M$ that contains every possible (smooth) atlas built from every compatible chart. ◇

The meaning of the partial derivative on a manifold is (by the chain rule, see fig 8)

$$\frac{\partial f}{\partial x^\mu} \equiv \frac{\partial(\psi \circ f \circ \phi^{-1})(x^\mu)}{\partial x^\mu}, \tag{3}$$

where $x^\mu$ are the coordinates in $\mathbb{R}^m$.

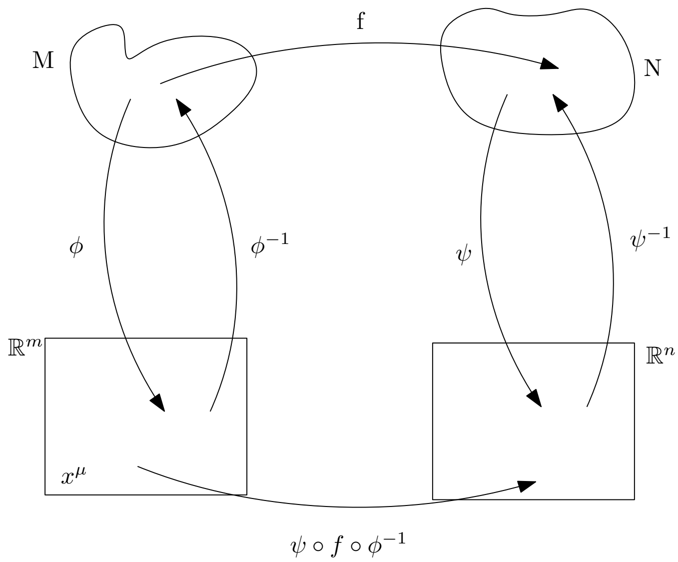


Figure 8: Derivatives on the manifold.

## 2.2 BASIS VECTORS

Vectors on the manifold are defined through curves on the manifold. Every point $p$ on the manifold will have its own vector space.

**Definition 10. Vectors**
Consider a curve on a manifold $M$ parametrized by a parameter $\lambda$ described by the equations $x^i = x^i(\lambda)$ with $i = 1, \ldots, m$, where $m$ is the dimension of the manifold M. Consider also a differentiable function $f(x^i)$ on $M$. There then has to be another function $g(\lambda)$ which gives the value $f$ of every point on the curve parametrized by $\lambda$. Differentiation and using the chain rule at a point $p$ on the manifold $M$ we get

$$\frac{\mathrm{d}g}{\mathrm{d}\lambda} = \sum_{i=1}^{m} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\partial f}{\partial x^i} \tag{4}$$

but this has to be true for any function $g$ and $f$ related in this way.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \sum_{i=1}^{m} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\partial}{\partial x^i}. \tag{5}$$

Now the object $\frac{\mathrm{d}}{\mathrm{d}\lambda}$ is a vector which lives in the tangent space of the manifold $M$ at point $p$. $\{\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\}$ are to be seen as the vector components and $\frac{\partial}{\partial x^i}$ are the basis vectors[4]. ◇

We can only compare vectors which lives in the same tangent space, that is; on the same point $p$ on the manifold. We will however develop tools which will enable us to move these vectors between the tangent spaces to compare them, but at this point we can't move the vectors between the vector spaces.

There are also dual vectors:

**Definition 11. Dual vectors (One-forms)**
The dual vectors (also called one-forms) $\omega$ are defined as linear, real-valued functions of vectors which takes the vectors to $\mathbb{R}$ (or $\mathbb{C}$ if the underlying field is complex.).

$$\omega : V \rightarrow \mathbb{R}. \tag{6}$$

◇

We will also define vector fields:

**Definition 12. Vector fields**
A vector field is given by some rule which defines a vector at each point of the manifold $M$. ◇

Lets take a look at the different basis for vectors and dual vectors (one-forms). There are two main choices of basis vectors, "the coordinate basis" and "the Cartan formalism".

4 Working in the $\{\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\}$ basis we often write the vector $V$ as $V = (\frac{\mathrm{d}x^1}{\mathrm{d}\lambda}, \frac{\mathrm{d}x^2}{\mathrm{d}\lambda}, \ldots, \frac{\mathrm{d}x^m}{\mathrm{d}\lambda})$.

### 2.2.1 *Coordinate basis*

**Definition 13. Coordinate basis**

We say that we have chosen the coordinate basis if we at point $p$ on the manifold $M$ choose the partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ as a basis, denoted by $\hat{e}_{(\mu)} = \partial_\mu$ and take the dual vector space basis to be $\hat{\theta}^{(\mu)} = dx^\mu$. Here the coordinates $x^\mu$ are the local coordinates for the neighborhood which are topologically equivalent to $\mathbb{R}^m$. ◇

It is easy to prove that these choices form a vector space, we will however not prove this here.

*Claim* 2. **Complete basis for tangent space $T_p$ and cotangent space $T_p^*$**

The coordinate basis $\{\partial_\mu\}$ at point $p$ form a complete basis for the tangent space $T_p$ at point $p$.

The basis $\{dx^\mu\}$ form a complete basis for the cotangent space $T_p^*$ at point $p$. $T_p^*$ is the set of linear maps which takes a vector $V$ in $T_p$ to $\mathbb{R}$ $\omega : T_p \to \mathbb{R}$. ◇

Lets see how these objects transforms. By the chain rule we get the transformation

$$\partial_{\mu'} \equiv \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \tag{7}$$

from our old coordinate system $x^\mu$ to the new coordinate system $x^{\mu'}$.

The transformation for a vector is then (by demanding scalars to be invariant)

$$\begin{aligned} V &= V^\mu \partial_\mu \\ &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \end{aligned} \tag{8}$$

so that

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu. \tag{9}$$

For a one-form $\omega$ we get (by demanding scalars to be invariant)

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu \tag{10}$$

since the basis transforms (by the chain rule)

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu. \tag{11}$$

Note that we have the following relation between the tangent space and cotangent space

$$\hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = dx^\mu \partial_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{12}$$

### 2.2.2 *Cartan formalism*

This is a set of basis vectors in the tangent space which is not derived from any coordinate system, and can't be. This basis is also referred to as "non-coordinate basis".

**Definition 14. Non-coordinate basis and veilbein**
We denote the basis vectors by $\hat{e}_{(a)}$ and choose them to be orthonormal, i.e the inner-product are $g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}$ where $\eta_{ab}$ is the metric of the tangent space. ◇

The relation between the non-coordinate basis and the coordinate basis are

$$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)} \tag{13}$$

where $e^a_\mu$ is an $n \times n$ invertible matrix with $n$ being the number of values $a$ or $\mu$ takes. We will often refer to $e^a_\mu$ as the vielbein's. The inverse obeys

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \qquad \text{and } e^a_\mu e^\mu_b = \delta^a_b \tag{14}$$

so that

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{15}$$

In an analogs way we introduce the same concept to the dual vector space. It turns out that the relation are the inverses

$$\hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)}. \tag{16}$$

The relation between the dual vector space and the vector space are

$$\hat{\theta}^{(a)} \hat{e}_{(b)} = \delta^a_b. \tag{17}$$

As an example a vector $V$ is written as $V^\mu \hat{e}_{(\mu)}$ in coordinate basis and written as $V^a \hat{e}_{(a)}$ in the non-coordinate basis. The components are related by $V^a = e^a_\mu V^\mu$.

## 2.3 PULLBACKS, PUSHFORWARDS AND INTEGRAL CURVES

Now that we have established the manifold and vectors we will define some important concepts.

**Definition 15. Pullback**
If we have two maps $\phi : M \to N$ and $f : N \to \mathbb{R}$ we can compose the map $(f \circ \phi) : M \to \mathbb{R}$. We define the "pullback" of $f$ by $\phi$

$$\phi_* f = (f \circ \phi) \tag{18}$$

since we think of $\phi_*$ as pulling back $f$ from $N$ to $M$, see fig 9. ◇

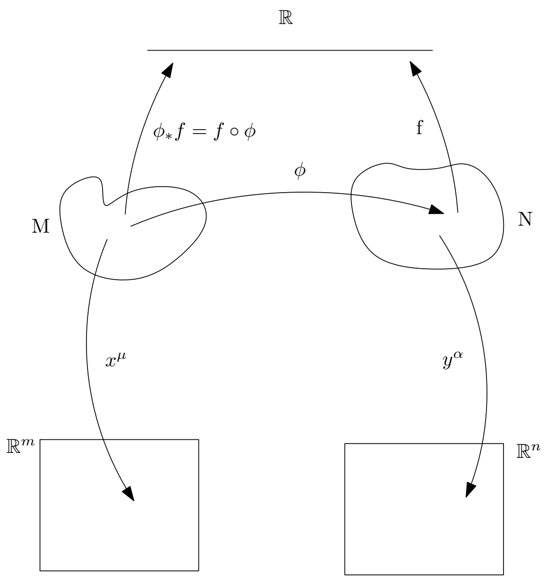


Figure 9: Pullback.

Note that $\phi$ do not need to have an inverse. So we can only pullback functions from $N$ to $M$. We can however pushforward vectors from $M$ to $N$ since vectors can be thought of as derivative operators that maps smooth functions to real numbers $V : f \to \mathbb{R}$.

**Definition 16. Pushforward of vectors**

Consider a vector at a point $p$ on $M$, $V(p)$. We define the pushforward vector $\phi^* V$ at point $\phi(p)$ on $N$ as

$$(\phi^* V)(f) = V(\phi_* f) \tag{19}$$

see fig 10. ◇

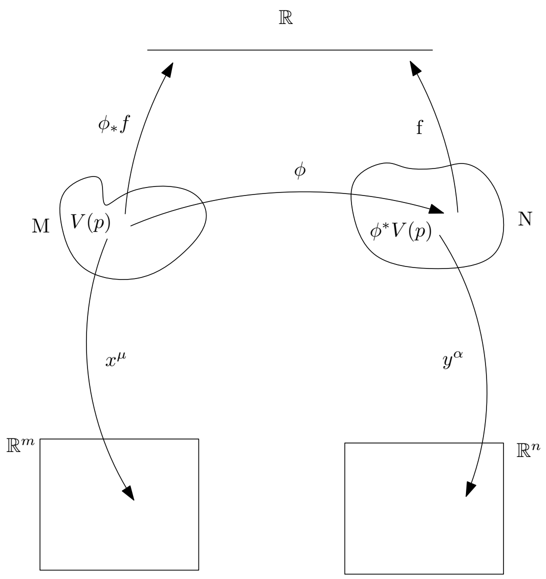


Figure 10: Pushforward of vectors.

So on $M$ the vector will be $V = V^\mu \partial_\mu$ and on $N$ $(\phi^* V) = (\phi^* V)^\alpha \partial_\alpha$. We use a test function $f$ and the chain-rule to find the relation between these two

$$\begin{aligned}(\phi^* V)^\alpha \partial_\alpha f =& V^\mu \partial_\mu (\phi_* f) \\ =& V^\mu \partial_\mu (f \circ \phi) \\ =& V^\mu \partial_\mu \big(f(\underbrace{\phi(x)}_{y})\big) \\ =& V^\mu \frac{\partial y^\alpha}{\partial x^\mu} (\partial_\alpha f)\end{aligned}$$

where we in the first equality used the definition of the pullback. We see that $(\phi^* V)^\alpha = (\phi^*)^\alpha{}_\mu V^\mu$ with the matrix expressed in coordinates $(\phi^*)^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu}$. Note that $\alpha$ and $\mu$ do not need to have the same allowed values and the matrix $(\phi^*)^\alpha{}_\mu$ need not to be invertible.

We define a pullback of forms, where a scalar function can be seen as a special case, a zero-form.

**Definition 17. Pullback of forms**

In analogy with the two previous definitions we define the pullback of a one-form $\omega$, for a map $\phi : M \to N$ and $f : N \to \mathbb{R}$ as

$$(\phi_* \omega)(V) = \omega(\phi^* V) \tag{20}$$

with $(\phi_*)_\mu{}^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}$, which again, might not be invertible. ◇

The reason why vectors can be pushed forward but not pulled back and that one-forms can be pulled back but not pushed forward if we have a map $\phi : M \to N$ is due to the fact that $\phi$ may not be invertible. If $\phi$ is invertible then we can pushforward and pullback arbitrary tensors (to be defined) however we choose to. This defines a diffeomorphism between $M$ and $N$ since they are the same abstract manifold. This is why diffeomorphisms are coordinate transformations. Diffeomorphisms are "active coordinate transformation" and traditional coordinate transformation are "passive". This is illustrated in fig 11.

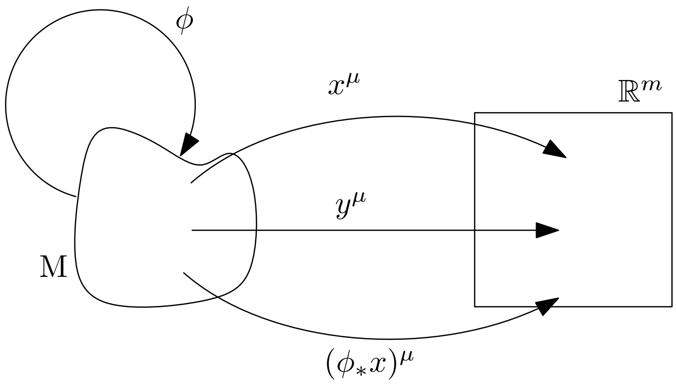


Figure 11: Diffeomorphisms and traditional coordinate transformations.

Here $x^\mu : M \to \mathbb{R}^n$ are coordinate maps on $M$, $y^\mu : M \to \mathbb{R}^n$ are the changes in the coordinate maps while holding the manifold $M$ fixed and the diffeomorphisms $\phi : M \to M$ moves the points on the manifold and the pullback $(\phi_* x)^\mu : M \to \mathbb{R}^n$ evaluates the new points.

The diffeomorphisms allows us to compare tensors on the manifold which lives in different tangent spaces on $M$. This suggests a different kind of derivative, something called the Lie derivative, which measures the rate of change of the tensors under diffeomorphisms (change of coordinates). For this we require a one parameter family of diffeomorphisms, say $\phi_t$ for a continuous parameter $t$. Which we will define now:

**Definition 18. Integral curves**

Given a vector field $V^\mu(x)$ we define the integral curves of the vector field to be the curves $x^\mu(t)$ which obeys the equations

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = V^\mu(x), \tag{21}$$

where $t$ is a continuous parameter. ◇

Our parameter $\phi_t$ is the flow down the integral curves. We say that the vector field is the generator of diffeomorphisms.

**Definition 19. Tensors**

Direct products of arbitrary many one-forms and vectors are called tensors. The tensors will have components $T^{\mu^1 \dots \mu^k}{}_{\nu^1 \dots \nu^l}$ with $k$ being the number of one-form arguments and $l$ being the number of vectors arguments the tensor will need to take to arrive at $\mathbb{R}$. ◇

## 2.4 MATHEMATICAL TOOLS

We will define the covariant derivative in the two basis, we will also define torsion and the Lie derivative.

### 2.4.1 *Covariant derivative*

The partial derivative $\partial_\mu$ in flat space in Cartesian coordinates is a map which takes $(k, l)$ tensors to $(k, l+1)$ tensors. Where $k$ is the vector space and $l$ is the dual vector space, i.e. $T^{\mu^1 \dots \mu^k}{}_{\nu^1 \dots \nu^l}$. However, this is not true for arbitrary spaces in arbitrary coordinate systems. Therefore we will construct a new derivative called the covariant derivative $\nabla$, where this map $\nabla : (k, l) \to (k, l+1)$ will be true in any coordinate system, in any space. It will have to obey some properties, namely:

1. Linearity: $\nabla(T + S) = \nabla T + \nabla S$.
2. Leibniz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$.

3. Commutes with contractions: $\nabla_\mu (T^\lambda_{\ \lambda\rho}) = (\nabla T)_{\mu\ \lambda\rho}^{\ \lambda}$.

4. Reduces to partial derivatives on scalars: $\nabla_\mu \phi = \partial_\mu \phi$.

It turns out that we seek[5]

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{22}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{23}$$

for some $n \times n$ matrix $\left(\Gamma_\mu\right)^\rho_{\ \sigma}$ for each $\mu$, where $n$ is the dimension of the manifold. These matrices are called affine connections. We do not assume that they are symmetric in any indices (as one usually do in an introduction to GR), that is, we do not assume that the space is torsion free (to be defined). The only thing we require from the connections $\Gamma$ are that they transform according to

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{24}$$

The first term is what we would expect if it transformed as an tensor, while the second term cancels the non-tensorial transformation term from the partial derivative. This makes the linear combination of the partial derivative and the connection to transform as a tensor. The special derivative $\nabla_\mu$ is called the covariant derivative.

**Definition 20. Covariant derivative (coordinate basis)**
The covariant derivative of a vector $V = V^\nu \partial_\nu$ in coordinate basis acts on it's components as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{25}$$

and for a one-form $\omega = \omega_\nu \mathrm{d}x^\nu$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda. \tag{26}$$

◇

We define the torsion tensor before continuing with the covariant derivative

**Definition 21. Torsion tensor**
The torsion tensor $T^\lambda_{\mu\nu}$ is defined as

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{27}$$

◇

5 We just state these results here. For a derivation see e.g. [12].

From the fact that the commutator of two vector fields $X$ and $Y$ gives a new vector field $[X,Y]^\mu = X^\nu\partial_\nu Y^\mu - Y^\nu\partial_\nu X^\mu$, we can write the torsion as a map from two vector fields to a third vector field

$$T(X,Y) = X^\mu\nabla_\mu Y - Y^\mu\nabla_\mu X - [X,Y]. \tag{28}$$

We will now define the covariant derivative in the non-coordinate basis and although going from the coordinate basis to non-coordinate basis is mostly putting vielbeins in the right places. This is not the case when we differentiate. When we differentiate in the non-coordinate basis we have to use another connection, called the spin connection $\omega_\mu{}^a{}_b$. ($\omega$ is the standard symbol for both the spin connection and for an arbitrary one-form, do not confuse them)

**Definition 22. Covariant derivative (non-coordinate basis)**
The covariant derivative of a vector $V = V^a\hat{e}_{(a)}$ in non-coordinate basis acts on it's components as

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b, \tag{29}$$

and for a one-form $\Omega = \Omega_a \hat{e}^{(a)}$

$$\nabla_\mu \Omega_a = \partial_\mu \Omega_a - \omega_\mu{}^b{}_a \Omega_b. \tag{30}$$

◇

Under general coordinate transformation the lower index $\mu$ transforms as a one-form and under local Lorentz transformations it transforms as

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda_{b'}{}^b \omega_\mu{}^a{}_b - \Lambda_{b'}{}^c \partial_\mu \Lambda^{a'}{}_c \tag{31}$$

which is exactly what we need for the covariant derivative to transform like a tensor, in analogy with $\Gamma^\nu_{\mu\lambda}$.

Since we have a relation between the both basis there should be a relation between the affine connection $\Gamma^\nu_{\mu\lambda}$ and the spin connection $\omega_\mu{}^b{}_a$.

*Claim* 3. **Spin connection and affine connection relation**
The relations for the spin connection and the affine connection are

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b, \tag{32}$$

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda. \tag{33}$$

◇

From this it follows that $\nabla_\mu e^a_\nu = 0$.

Using the Cartan formalism we are able to describe spinor fields on space-time and take their covariant derivative which is impossible in the coordinate basis. It also allows us to think of tensors as tensor valued differential forms, e.g.

$$\Omega_{\mu}^{\ a} \qquad \text{"vector valued one-form"}, \tag{34}$$

$$A_{\mu\nu\ b}^{\ \ a} \qquad \text{"(1,1)-tensor valued two-form"} \tag{35}$$

with $A_{\mu\nu\ b}^{\ \ a}$ being anti-symmetric in the indices $\mu$ and $\nu$.

### 2.4.2 *Lie derivative*

A very useful tool is the Lie derivative. It measures the change in a tensor due to diffeomorphisms (change of coordinates).

**Definition 23. Lie derivative**
Given a vector field $V^{\mu}(x)$ which generates a family of diffeomorphisms $\phi_t$, parametrized by $t$, we define the change of a tensor traveling down the integral curves as

$$\Delta_t T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(p) = \phi_{t*}\left(T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(\phi_t(p))\right) - T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(p) \tag{36}$$

and the Lie derivative $\mathfrak{L}_V$ along the vector field $V^{\mu}(x)$, at point $p$, as

$$\mathfrak{L}_V T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(p) = \lim_{t \to 0}\left(\frac{\Delta_t T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}(p)}{t}\right). \tag{37}$$

◇

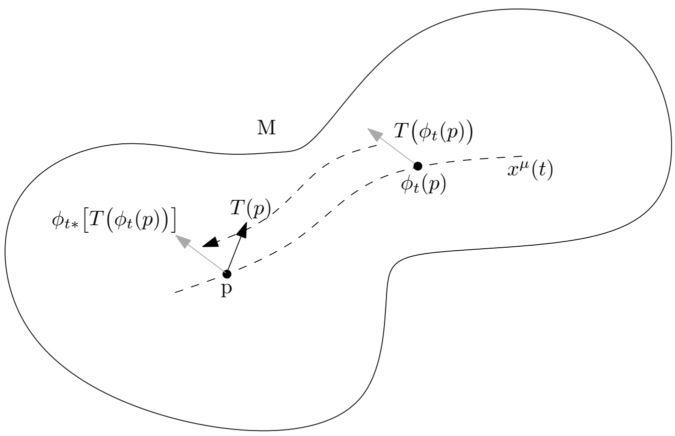


Figure 12: Lie derivative.

So we evaluate the tensor $T$ at two different points, $p$ and $\phi_t(p)$, to get the tensors $T(p)$ and $T(\phi_t(p))$. Then we pullback $T(\phi_t(p))$ to the point $p$ so that we can compare $T(p)$ and $T(\phi_t(p))$. We take the limit to get the instantaneous change at the point $p$ for $T(p)$. This is the Lie derivative.

Here follows some properties:

*Claim* 4. **Lie derivative properties**

With $a, b$ constants and $T, S$ tensors we have the following properties

1. Linearity: $\mathfrak{L}_V(aT + bS) = a\mathfrak{L}_V T + b\mathfrak{L}_V S$.
2. Leibniz rule: $\mathfrak{L}_V(T \otimes S) = (\mathfrak{L}_V T) \otimes S + T \otimes (\mathfrak{L}_V S)$.
3. Commutes with contractions: $\mathfrak{L}_V(T^{\lambda}{}_{\lambda\rho}) = (\mathfrak{L}_V T)^{\lambda}{}_{\lambda\rho}$.
4. Reduces to partial derivatives on scalars: $\mathfrak{L}_V f = V(f) = V^\mu \partial_\mu f$. ◇

There are some tricks to taking the Lie derivative.

*Claim* 5. **Lie derivative of a vector**

The Lie derivative along a vector field $V = V^\mu \partial_\mu$ of a vector $U = U^\mu \partial_\mu$ is given by the commutator

$$\mathfrak{L}_V U = \mathfrak{L}_V(U^\mu \partial_\mu) = [V, U]^\mu \, \partial_\mu = [V, U] = (V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu) \, \partial_\mu. \tag{38}$$

◇

From this follows that $\mathfrak{L}_V U = -\mathfrak{L}_U V$

*Claim* 6. **Lie derivative of a one-form**

The Lie derivative along a vector field $V$ of a one-form $\omega = \omega_\mu \mathrm{d}x^\mu$ is

$$\mathfrak{L}_V \omega = \mathfrak{L}_V(\omega_\mu \mathrm{d}x^\mu) = (V^\nu \partial_\nu \omega_\mu + (\partial_\mu V^\nu)\omega_\nu)\mathrm{d}x^\mu \tag{39}$$

where the first term can be thought of as a transport term and the second as a shift in coordinates (a pullback of $V$). ◇

Despite it's explicit appearance the Lie derivative is covariant. It takes $(k, l)$ tensors to $(k, l)$ tensors. A reason why the Lie derivative is so important are that; if we have a physical situation represented by a manifold $M$ with objects $T_i$ and a diffeomorphism $\phi : M \to M$ then $(M, T_i)$ represent the same physical situation as $(M, \phi_* T_i)$. This is important because we do not want to over count configurations and this gives us the tools to transform the object to spaces where they might be easier to work with.

By defining the exterior derivative and the interior product of one-forms we will get an alternative formulation for the Lie derivative of forms.

**Definition 24. Exterior derivative**

The exterior derivative d is a map which takes a $p$-form to a $(p+1)$-form. It's action on a $p$-form

$$\omega = \frac{1}{p!}\omega_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \tag{40}$$

is defined by

$$\mathrm{d}\omega = \frac{1}{p!}\left(\partial_\nu \omega_{\mu_1 \ldots \mu_p}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{41}$$

◇

*Claim* 7. **Exterior derivative properties**
The exterior derivative obeys

1. $\mathrm{d}(\omega + \Omega) = \mathrm{d}\omega + \mathrm{d}\Omega$.
2. $\mathrm{d}(\omega \wedge \chi) = \mathrm{d}\omega \wedge \chi + (-1)^p \omega \wedge \mathrm{d}\chi$.
3. $\mathrm{d}(\mathrm{d}\omega) = 0$.

Here $\omega, \Omega$ are $p$-forms and $\chi$ are a $q$-form. ◇

**Definition 25. Interior product**
The interior product $i_V$ (for a vector field $V$) is a map which takes a $p$-form $\omega = \frac{1}{p!}\omega_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$ to a $(p-1)$-form defined as

$$i_V \omega = \frac{1}{(p-1)!} V^\nu \omega_{\nu \mu_2 \ldots \mu_p} \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}. \tag{42}$$

◇

With these two definitions we can give an alternative formula of the Lie derivative of forms.

*Claim* 8. **Lie derivative of forms (Cartan's formula)**
The Lie derivative of forms is given by the anti-commutator of the exterior derivative and the interior product on a form

$$\mathfrak{L}_V \omega = \left(\mathrm{d}i_V + i_V \mathrm{d}\right) \omega = \mathrm{d}\left(i_V \omega\right) + i_V \left(\mathrm{d}\omega\right). \tag{43}$$

◇

### 2.4.3 *Integration*

We can also integrate forms. But this operation is only defined over a manifold $M$ if $M$ is "orientable", so we need to define orientation of a manifold first.

**Definition 26. Orientation**
Let M be a connected manifold covered by the charts $\{U_i\}$. The manifold is orientable if the Jacobian is greater then zero for any $U_j \cap U_i \neq \emptyset$, with local coordinates $\{x^\mu\}$ and $\{y^\alpha\}$ respectively, i.e $J = \det(\partial x^\mu / \partial y^\alpha) > 0$. ◇

For an $m$-dimensional manifold $M$, all $m$-forms at a point $p$ form an one-dimensional vector space[6]. This tells us that there exists some

6 This can easily be seen due to the fact that all forms are anti-symmetric.

function $f(x^1, \ldots, x^m)$ such that $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m$, where $\{x^i\}$ are the local coordinates at point $p$. To integrate we divide it up into small regions, the integral of the function $f$ is then approximately the sum of the product of the value of $f$ at each cell times the volume of the region. We then take the limit as these regions goes to zero.

**Definition 27. Integration of a form**
The integration of a $m$-form $\omega$ on an $m$-dimensional manifold $M$ is defined as

$$\begin{aligned}\int \omega &\equiv \int f(x^1, \ldots, x^m)\mathrm{d}x^1 \ldots \mathrm{d}x^m \\ &\equiv \int f(x^1, \ldots, x^m)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \end{aligned} \tag{44}$$

◇

Part III

# BOSONIC MEMBRANE

We consider a toy model of the membrane. That is, one that only contains bosons in space-time, known as the bosonic membrane. We look at two different formulations of the action, and their constraints, derive the general Hamiltonian and the Hamiltonian in the so called light-cone gauge. Then we make a transition from the membrane theory to M(atrix) theory. Going from the infinite dimensional group of area preserving diffeomorphisms to $SU(N)$ (neglecting the center of mass motion) or $U(N)$ (keeping the center of mass motion).

# 3 ACTION AND LAGRANGIAN

A membrane is a 2-dimensional object for a given time-slice in space-time. As it travels in time it sweeps out a 3-dimensional "world-volume" in space-time on which we want to construct a field theory. The coordinates for the 11-dimensional space-time will be denoted by $x^m$ ($m = 0, \dots, 10$) and the world-volume will be parametrized by $\sigma^i$ ($i = 0, 1, 2$). We will take $\sigma^0$ to be the time coordinate and $\sigma^1$ and $\sigma^2$ to parameterize the area of the membrane. We define a map from the world-volume $\Sigma_3$ to the 11-dimensional space-time $M_{11}$ called "target space"

$$X^m(\sigma^i) : \Sigma_3 \to M_{11}(g_{mn}). \tag{45}$$

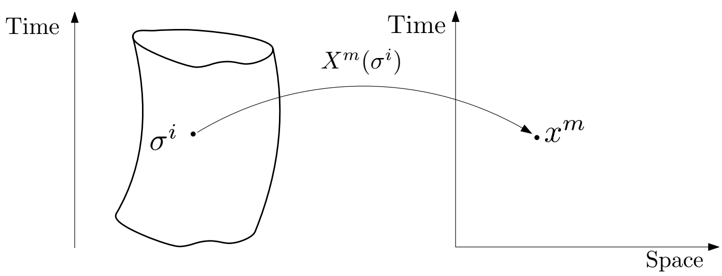


Figure 13: Mapping the world-volume to space-time.

We will look at two different Lagrangians. We will simply mention the first one (the Nambu-Goto action) since it's intuitive, but we will work with the second one (the Polyakov action) since this gets rid of the square root.

## 3.1 NAMBU-GOTO ACTION

The Nambu-Goto action is given by

*Note that the action is proportional to the world-volume.*

$$S = -T \int d^3\sigma \sqrt{-\det\left(\gamma_{ij}\right)} \tag{46}$$

where $\gamma_{ij}(\sigma)$ is the metric on the world-volume induced by the space-time metric

$$\gamma_{ij}(\sigma) = \partial_i X^m \partial_j X^n g_{mn}(X(\sigma)). \tag{47}$$

and the integral is over the world-volume. This action is invariant under reparameterization. That is; independent of our choice of $\sigma^i$, as it better be. It's proportional to the world-volume hence Lorentz invariant, as it also should be. $T$ is the constant of proportionality which has the dimension $[T] = 3$. In the following we will set $T = 1$.

## 3.2 POLYAKOV ACTION

We can get rid of the square-root at the cost of introducing an auxiliary field $h^{ij}$. The action can then be written as

$$S = \frac{1}{2} \int d^3\sigma \sqrt{-h}\left(1 - h^{ij}\partial_i X^m \partial_j X^n g_{mn}\right) \tag{48}$$

with $h = \det(h_{ij})$. This form of the action is called "The Polyakov action". It gives the same equations of motions as the Nambu-Goto action (46), i.e. the Polyakov action and the Nambu-Goto action are equivalent "on-shell" (which will be shown later). Both the world-volume metric $h^{ij}$ and the target space metric $g_{mn}$ have inverses that satisfy

*On-shell refers to when the equations of motion is are satisfied.*

$$g^{mn} g_{np} = \delta^m_p, \tag{49}$$

$$h^{ij} h_{jk} = \delta^i_k, \tag{50}$$

where $\delta^i_j$ is the Kronecker delta which is defined as

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{else} \end{cases}. \tag{51}$$

This allows us to raise and lower the indices $g_{mn}A^n = A_m$, $g^{mn}A_n = A^m$ and $h_{ij}B^j = B_i$, $h^{ij}B_j = B^i$.

### 3.2.1 *Equations of motion and boundary*

Varying the Polyakov action (48) with respect to[1] $X^m$ and integrating by parts we get the equations of motion according to the Lagrangian theory

$$\delta S = T \int d^3\sigma \left( \delta X^m \left[ \partial_i\left(\sqrt{-h} h^{ij} \partial_j X^n g_{mn}\right) - \frac{1}{2}\sqrt{-h} h^{ij} \partial_i X^p \partial_j X^n \left(\partial_m g_{pn}\right) \right] \right.$$
$$\left. - \partial_i\left(\sqrt{-h} h^{ij} \delta X^m \partial_j X^n g_{mn}\right) \right) = 0. \tag{52}$$

The equations of motion are given by the equation in the square brackets on the first row, which has to vanish independently of the second term by local causality,

$$\partial_i\left(\sqrt{-h} h^{ij} \partial_j X^n g_{mn}\right) - \frac{1}{2}\sqrt{-h} h^{ij} \partial_i X^p \partial_j X^n \left(\partial_m g_{pn}\right) = 0. \tag{53}$$

The second row gives us the constraint on the boundary, we expand this equation in the $i$-sum to get three terms. The first one

$$\int d\sigma^1 d\sigma^2 \left(\sqrt{-h} h^{0j} \delta X^m \partial_j X^n g_{mn}\right)\Big|_{\sigma^0=\sigma^0_i}^{\sigma^0=\sigma^0_f} \tag{54}$$

1 We treat $X^m$ and $h^{ij}$ as independent variables. The space-time metric $g_{mn}$ does however depend on space-time $g_{mn} = g_{mn}(x)$.

is trivial since we want to know how the membrane evolves from an arbitrary but fixed time $\sigma_i^0$ to an arbitrary but fixed time $\sigma_f^0$. If we allow these points to vary we do not know which times we are considering. We therefore demand that the variations vanish, i.e. $\delta X^m(\sigma_i^0, \sigma^1, \sigma^2) = \delta X^m(\sigma_f^0, \sigma^1, \sigma^2) = 0$, so that the above equation (54) vanishes. We are then left with two boundary conditions

$$\int d\sigma^0 \Bigg( d\sigma^2 \sqrt{-h} h^{1j} \delta X^m \partial_j X^n g_{mn} \Bigg|_{\sigma^1=\sigma_i^1}^{\sigma^1=\sigma_f^1} \tag{55}$$

$$+ d\sigma^1 \sqrt{-h} h^{2j} \delta X^m \partial_j X^n g_{mn} \Bigg|_{\sigma^2=\sigma_i^2}^{\sigma^2=\sigma_f^2} \Bigg) = 0. \tag{56}$$

These conditions will be satisfied if we have, with $r = 1, 2$,

- "Neumann boundary conditions"

  $$\partial_i X^m = 0 \tag{57}$$

  for $\sigma^r = \sigma_i^r$ and $\sigma^r = \sigma_f^r$.

- Or "Dirichlet boundary condition"

  $$\delta X^m = 0 \quad \Longleftrightarrow \quad X^m = \text{fixed} \quad \Longleftrightarrow \quad \partial_0 X^m = 0$$

  for $\sigma^r = \sigma_i^r$ and $\sigma^r = \sigma_f^r$.

However here we have to be careful, we can not have Dirichlet boundary condition in the time direction (if e.g $X^0$ is chosen as the time coordinate, it cannot satisfy a Dirichlet boundary condition) since the membrane has to be able to travel in time. We will however not consider boundaries any further in this thesis.

Varying the Polyakov action (48) with respect to $h^{ij}$ yields

*Note that $\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h} h_{ij} \delta h^{ij}$ by Jacobi's formula.*

$$\delta S = \frac{1}{2} \int d^3\sigma \sqrt{-h} \delta h^{ij} \Bigg[ -\frac{1}{2} h_{ij} (1 - h^{ki'} \partial_k X^m \partial_{i'} X^n g_{mn}) - \partial_i X^m \partial_j X^n g_{mn} \Bigg] = 0 \tag{58}$$

or, using the notation of the induced metric (47) for the integrand (which must vanish for the variation of the action to vanish)

$$-\frac{1}{2} h_{ij} (1 - h^{ki'} \gamma_{ki'}) - \gamma_{ij} = 0. \tag{59}$$

Taking the trace of this we find that $h^{ki'} \gamma_{ki'} = 3$. Using this in the equation above we find that $h_{ij}$ is simply the induced metric on the world-volume by the target space

$$h_{ij} = \partial_i X^m \partial_j X^n g_{mn}. \tag{60}$$

If we plug in (60) into (48) we recover (46), which shows the equality of the Polyakov action (48) and the Nambu-Goto action (46) and shows that $h_{ij}$ is only an auxiliary field.

### 3.2.2 *Symmetries*

There is two types of symmetries for the bosonic membrane in flat 11-dimensional Minkowski space.

- Global Poincaré symmetries which is due to the invariance of space-time

$$X^m \to \Lambda^m_n X^n + c^m$$

  where $\Lambda^m_n$ is a Lorentz transformation and $c^m$ is a translations in space-time.

- Local gauge invariance (reparameterization invariance or diffeomorphism) which is simply a change of parametrization coordinates

$$\sigma^i \to \tilde{\sigma}^i(\sigma^0, \sigma^1, \sigma^2).$$

# 4

# HAMILTONIAN FORMALISM

Since we are going to quantize the theory using the correspondence principle rather than the path integral formulation we will have to rewrite the Lagrangian theory in the Hamiltonian formalism. Here we will consider Minkowski space-time as the target space rather than curved space-time, so we change the curved space-time metric $g_{mn}$ to the flat Minkowski space-time metric $\eta_{mn}$ according to $g_{mn} \to \eta_{mn}$, with $\eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1)$.

If one is not familiar with Dirac's generalization of the Hamiltonian formalism there is a short overview in appendix A.

## 4.1 GENERAL HAMILTONIAN AND DIRAC BRACKETS

We define the conjugate momenta

$$\mathscr{P}_m \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 X^m)} = -\sqrt{-h}h^{0j}\partial_j X^n \eta_{mn}, \tag{61}$$

$$(\mathscr{P}_h)_{ij} \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 h^{ij})} = 0. \tag{62}$$

Multiplying (61) by $\partial_i X^m$ or $\mathscr{P}^m$ yields the "primary constraints"[1]. We are not allowed to use the primary constraints before we work out all the Poisson brackets of the theory since phase space variables may have non-vanishing Poisson brackets with the constrains. The primary constraints are

*Use $h^{00} = \det(h_{rs})/h$ (where $h_{rs}$ is the lower right part of $h_{ij}$, see (72)) for $\mathscr{P}^m$ case and (60).*

$$\begin{cases} \phi_1 \equiv & \mathscr{P}_m \partial_1 X^m \approx 0 \\ \phi_2 \equiv & \mathscr{P}_m \partial_2 X^m \approx 0 \\ \phi_3 \equiv & \mathscr{P}_m \mathscr{P}^m + \left(\partial_1 X^m \partial_1 X_m \partial_2 X^n \partial_2 X_n - \left(\partial_1 X^m \partial_2 X_m\right)^2\right) \approx 0 \end{cases} . \tag{63}$$

1 In fact the primary constraints using the Polyakov action is given by $(\mathscr{P}_h)_{ij} = 0$, but this is not very illuminating so we will use the equations of motion for $h_{ij}$ given by (60) to get the primary constraints in the Nambu-Goto formulation.

Where ($\approx$) stands for weakly, i.e. holds on-shell. The Legendre transformation gives us the Hamiltonian density

$$\begin{aligned}\mathscr{H} =& \mathscr{P}_m(\partial_0 X^m) - \mathscr{L} \\ =& -\sqrt{-h}h^{0j}\partial_j X^n \eta_{mn}(\partial_0 X^m) + \frac{1}{2}\sqrt{-h}\left(1 - h^{ij}\partial_i X^m \partial_j X^n \eta_{mn}\right) \\ =& -\sqrt{-h}\left(h^{0j}h_{0j} - 1\right) \\ =& 0 \end{aligned} \tag{64}$$

We build the new Hamiltonian density according to Dirac's theory

$$\mathscr{H}' = \mathscr{H} + u^L \phi_L = u^L \phi_L \tag{65}$$

with $L = 1, 2, 3$. We have to ensure that the primary constraints are preserved in time so that we don't get secondary constraints. It turns out that they are preserved [2], which means that the Poisson brackets which governs the time evolution are weakly zero. The Poisson brackets are

$$\dot{\phi}_L = \left\{\phi_L, \mathscr{H}'\right\}_p \approx 0, \quad \forall L. \tag{66}$$

So the total Hamiltonian is

$$H_T = \int \mathrm{d}^2\sigma \, \mathscr{H}' = \int \mathrm{d}^2\sigma \, u^L \phi_L \tag{67}$$

where the $u^L$ are to be determined by the gauge choices and $\mathrm{d}^2\sigma = \mathrm{d}\sigma^1 \mathrm{d}\sigma^2$.

### 4.1.1 *Getting a feel for the constraints*

If we make the gauge choice $X^0 = \sigma^0$, where $\sigma^0$ is our time coordinate, and consider the membrane in a fixed time we see that the first two constraints

$$\begin{aligned}\phi_1 \equiv \mathscr{P}_m \partial_1 X^m \approx 0 \\ \phi_2 \equiv \mathscr{P}_m \partial_2 X^m \approx 0\end{aligned}$$

tells us that the momenta are normal to the membrane while for the last constraint we make the additional choice $\partial_1 X^m \partial_2 X_m = 0$, which means that $\sigma^1$ is orthogonal to $\sigma^2$. We can then write the constraint as

$$\phi_3 \equiv \mathscr{P}_m \mathscr{P}^m + \left(\partial_1 X^m \partial_1 X_m \partial_2 X^n \partial_2 X_n\right) \approx 0.$$

We see that this constraint relates the magnitude of the momenta to the (area)$^2$ in the target-space associated with a unit $\mathrm{d}\sigma^1\mathrm{d}\sigma^2$ element in parameter space.

## 4.2 LIGHT-CONE COORDINATES

*This choice simplifies a lot of equations.*

We introduce the so called "light-cone coordinates", which are defined as

$$X^m = \left\{ X^\pm = \frac{1}{\sqrt{2}} \left( X^{10} \pm X^0 \right);\, X^I,\, I = 1, \ldots, 9 \right\}. \tag{68}$$

In these coordinates the length element is given by

$$ds^2 = dx^+ dx^- + dx^- dx^+ + \sum_{I=1}^{9} dx^I dx^I. \tag{69}$$

This means that the vectors will satisfy $A_+ = A^-$, $A_- = A^+$ and $A_I = A^I$.

The equations of motion for $h_{ij}$ (60) becomes

$$h_{ij} = \partial_i X^+ \partial_j X^- + \partial_i X^- \partial_j X^+ + \partial_i X^I \partial_j X^{I'} \eta_{II'}, \tag{70}$$

### 4.2.1 *Light-cone gauge*

We use the fact that the action (48) is parameterization invariant to choose coordinates so that[2]

$$X^+ = X^+(0) + \sigma^0 \quad \Longleftrightarrow \quad \partial_i X^+ = \delta_{i,0}. \tag{71}$$

This choice is called the "light-cone gauge" and it gauge fixes the time coordinate $\sigma^0$. So now we are left with only 2 parameterization choices, both of them spatial on the world-volume $\sigma^1, \sigma^2$. We have one requirement for the parameterization choices; they are not allowed to change the area of a time-slice of the world-volume.

## 4.3 HAMILTONIAN IN LIGHT-CONE COORD. AND GAUGE.

In the light-cone coordinates (68) and light-cone gauge (71) the equations of motion for $h_{ij}$ (70) become

$$h_{ij} = \left( \begin{array}{c|c} h_{00} & u_r \\ \hline u_r & h_{rs} \end{array} \right) \tag{72}$$

with

$$h_{00} = 2\partial_0 X^- + \partial_0 X^I \partial_0 X^{I'} \eta_{II'}, \tag{73}$$

$$h_{0r} \equiv u_r = \partial_r X^- + \partial_0 X^I \partial_r X^{I'} \eta_{II'}, \tag{74}$$

$$h_{rs} = \partial_r X^I \partial_s X^{I'} \eta_{II'}. \tag{75}$$

2 This choice removes the constraint $\phi_3$ from (63) since this becomes identically satisfied ($0 = 0$) for all choices of $\sigma^1$ and $\sigma^2$.

with $r, s = 1, 2$. Using this in the action (48) yields (by using Leibniz formula for determinants)

$$S = -\int d^3\sigma \sqrt{-h} \\ = -\int d^3\sigma \sqrt{h' U} \tag{76}$$

with $U = (-h_{00} + u_r h^{rs} u_s)$ and $h' = \det(h_{rs})$. The conjugate momenta are

$$\mathscr{P}^I \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 X_I)} = \sqrt{\frac{h'}{U}} \left( \partial_0 X^I - u_r h^{rs} \partial_s X^I \right), \tag{77}$$

$$\mathscr{P}^+ \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 X^-)} = \sqrt{\frac{h'}{U}}. \tag{78}$$

We use the Legendre transformation to get the Hamiltonian density

$$\mathscr{H}_{lc} = \mathscr{P}^I (\partial_0 X^{I'}) \eta_{II'} + \mathscr{P}^+ (\partial_0 X^-) - \mathscr{L} \\ = \frac{\mathscr{P}^I \mathscr{P}^{I'} \eta_{II'} + h'}{2\mathscr{P}^+} = -\mathscr{P}^- \tag{79}$$

where the last equality follows from the relation between the general Hamiltonian (64) and the light-cone Hamiltonian above $\mathscr{H} = 0 = \mathscr{P}^-(\partial_0 X^+) + \mathscr{H}_{lc}$.

We find the two primary constraints

$$\phi_r \equiv \mathscr{P}^I \partial_r X^{I'} \eta_{II'} + \mathscr{P}^+ \partial_r X^- \approx 0. \tag{80}$$

There are only two left since we gauge fixed one parameterization. We build the Hamiltonian density according to Dirac's theory

$$\mathscr{H}' = \mathscr{H}_{lc} + c^r \phi_r. \tag{81}$$

Checking for inconsistencies or secondary constraints we find [7]

$$\dot{\phi}_r = \left\{ \phi_r, \mathscr{H}' \right\}_p \approx 0 \tag{82}$$

We take the Lie derivative of $u_r$ to see how it transforms under coordinate transformations

$$\sigma^r \to \sigma^r + \xi^r(\sigma^i),$$

this yields

$$\mathfrak{L}_\xi u_r = (\partial_r \xi^s)\, u_s + \xi^s \partial_s u_r + \partial_0 \xi_r.$$

We choose such a parameterization that $\partial_0 \xi_r$ cancel the other terms, i.e [3] $u_r = 0$. Using the Hamiltonian equations

$$\partial_0 X^m = \frac{\partial \mathscr{H}_{lc}}{\partial \mathscr{P}_m} + c^r \frac{\partial \phi_r}{\partial \mathscr{P}_m}, \\ \partial_0 \mathscr{P}_m = -\frac{\partial \mathscr{H}_{lc}}{\partial X^m} - c^r \frac{\partial \phi_r}{\partial X^m} \tag{83}$$

3 This choice removes the two last constraints given by (80), since they are identically satisfied ($0 = 0$).

we find $\partial_0 X^I = \frac{\mathscr{P}^I}{\mathscr{P}^+} + c^r \partial_r X^I$ which rearranged, using (77), shows that $c^r = u^r = 0$. We get the total Hamiltonian density

$$\mathscr{H}_T = \mathscr{H}_{lc}. \tag{84}$$

From $u_r = 0$ in (74) it follows that

$$\partial_r X^- = -\partial_0 X^I \partial_r X^{I'} \eta_{II'} \tag{85}$$

multiplying by $\epsilon^{rs}\partial_s$ we get the constraint ($\epsilon$ is anti-symmetric and derivatives commute)

$$\epsilon^{rs}(\partial_s \partial_0 X^I)\partial_r X^{I'} \eta_{II'} = 0. \tag{86}$$

In phase space variables and with the brackets defined in (96) this condition is given by

$$\left\{ \frac{\mathscr{P}_I}{\mathscr{P}^+}, X^I \right\} = 0. \tag{87}$$

This constraint is called the Gauss constraint.

### 4.3.1 *Mass of membrane*

Using the Hamilton's equations (83) for $\mathscr{P}^+$ we find

$$\partial_0 \mathscr{P}^+ = 0. \tag{88}$$

Since $\mathscr{P}^+$ transforms as a density[4] we write it as $\mathscr{P}^+ = \lambda\sqrt{w(\sigma)}$, with $\lambda$ a scalar, $\int d^2\sigma \sqrt{w(\sigma)} = 1$ and $w(\sigma) = \det(w_{rs}(\sigma))$. $w_{rs}$ are to be viewed as a spatial Euclidean metric on the membrane at a given time-slice, while $h_{ij}$ are the Lorentzian metric on the world-volume. We have one restriction on $w_{rs}$, it is not allowed to be singular anywhere on the membrane.

The momentum are the integrals over the momentum densities

$$p^+ = \int d^2\sigma \mathscr{P}^+ = \lambda, \tag{89}$$

$$p^- = -\int d^2\sigma \mathscr{H}_{lc} = \frac{1}{2\lambda}\int \frac{d^2\sigma}{\sqrt{w(\sigma)}}\left(\mathscr{P}^I \mathscr{P}_I + h'\right), \tag{90}$$

$$p^I = \int d^2\sigma \mathscr{P}^I \tag{91}$$

and

$$h' = \frac{w(\sigma)}{2}\left\{X^I, X^{I'}\right\}^2 \tag{92}$$

with

$$\left\{X^I, X^{I'}\right\} \equiv \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_r X^I \partial_s X^{I'}. \tag{93}$$

4 Since $p^+$ is proportional to the square root of a determinant, see (78).

$p^I$ is just the center of mass motion by definition of center of mass. The mass becomes

$$\begin{aligned} M^2 = -p^\mu p_\mu =& 2p^+ p^- - p^I p_I \\ =& \int d^2\sigma \left( \frac{[\mathscr{P}^I \mathscr{P}_I]'}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2} \left\{ X^I, X^{I'} \right\}^2 \right) \end{aligned} \tag{94}$$

where the prime $[\,]'$ indicates that the center of mass momentum has been omitted. We see that this is the sum of the kinetic energy and potential energy $M^2 = T + V$. The Hamiltonian is given by

$$H = \int d^2\sigma \frac{1}{2\lambda} \left( \frac{\mathscr{P}^I \mathscr{P}_I}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2} \left\{ X^I, X^{I'} \right\}^2 \right). \tag{95}$$

## 4.4 FROM MEMBRANE TO M(ATRIX) THEORY

### 4.4.1 *Infinite dimensional Lie algebra*

The bracket

$$\{A, B\} \equiv \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_r A \partial_s B \tag{96}$$

is anti-symmetric $\{A, B\} = -\{B, A\}$, satisfies the Jacobi identity

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0 \tag{97}$$

and it is bi-linear

$$\{aA + bB, C\} = a\{A, C\} + b\{B, C\}, \tag{98}$$

for $a, b$ being scalars independent of $\sigma$ and $A = A(\sigma)$, $B = B(\sigma)$. So this bracket is a Lie bracket which together with the functions on the membrane forms an infinite dimensional Lie algebra.

### 4.4.2 *Area preserving diffeomorphisms (APD)*

We still have some residual gauge invariance called "area preserving diffeomorphisms". The APD are generated by $\xi^s$ defined through

$$\xi^s = \frac{\epsilon^{rs}}{\sqrt{w}} \partial_r \Lambda \tag{99}$$

where $\Lambda$ is a scalar function of $\sigma^s$. $\xi^s$ are divergence free vector-fields

$$\partial_s \left( \sqrt{w} \xi^s \right) = \epsilon^{rs} \partial_r \partial_s \Lambda = 0. \tag{100}$$

We will now show that this does not change the potential energy of the membrane.

The change in the coordinates are

$$\delta\sigma^s = \xi^s(\sigma^1, \sigma^2) \tag{101}$$

so that the change in $X^I$ becomes

$$\delta X^I = \epsilon^s \partial_s X^I = \left\{\Lambda, X^I\right\}. \tag{102}$$

The change in the potential energy is then

$$\begin{aligned}
\delta V = &\int d^2\sigma \sqrt{w} \left\{\left\{\Lambda, X^I\right\}, X^{I'}\right\} \{X_I, X_{I'}\} \\
=&\frac{1}{2} \int d^2\sigma \sqrt{w} \left\{\Lambda, \left\{X^I, X^{I'}\right\}\right\} \{X_I, X_{I'}\} \\
=&\frac{1}{4} \int d^2\sigma \sqrt{w} \left\{\Lambda, \left\{X^I, X^{I'}\right\} \{X_I, X_{I'}\}\right\} \\
=&\frac{1}{4} \int d^2\sigma \epsilon^{rs} \partial_r \Lambda \partial_s \left\{X^I, X^{I'}\right\} \{X_I, X_{I'}\} \\
=&-\frac{1}{4} \int d^2\sigma \epsilon^{rs} \partial_r \partial_s \Lambda \left\{X^I, X^{I'}\right\} \{X_I, X_{I'}\} + \text{Boundary} \\
=&0
\end{aligned} \tag{103}$$

where we used the Jacobi identity (97) and partial integration. So the change in the potential energy is the same, as promised, and hence for all purposes we are still describing the same membrane which proves the residual diffeomorphisms.

### 4.4.3 *Membrane topology and second quantization*

As we have seen the potential energy is given by

$$V = \frac{1}{4\lambda} \int d^2\sigma \sqrt{w(\sigma)} \left\{X^I, X^{I'}\right\}^2 \tag{104}$$

which means that the potential energy vanishes whenever $X^I$ only depend on either $\sigma^1$ or $\sigma^2$ but not both. This happens in the regions where the membrane grows infinitely thin tubes on it's world volume. Since the potential energy vanishes for such tubes there is no distinction between two free membranes and two membranes connected by a tube. This implies that the number of membranes are not conserved.

The spectrum of the classical bosonic membrane is continuous[13, 14], the quantum version is discrete[13, 14], the spectrum of the quantized super-membrane is continuous[15][5]. Therefore the quantized

5 "We give a rigorous proof that the quantum mechanical Hamiltonians of a class of supersymmetric matrix models have a continuous spectrum starting at zero. We are thus led to conclude that super-membranes (which can be regarded as a limit of such models) have a continuous mass spectrum and, in particular, no mass gap" from the abstract of [15]

super-membrane theory is a second quantized theory, a multi-"particle" theory where the number of "particles" can change.

Membranes can also use this fact to change shape. A spherical membrane can grow a tube which connects two points on the surface making it's topology that of a torus, even though the membranes are equivalent since the tubes carry no energy. So it is not quite clear if one are even able to talk about membranes with a fixed topology [16].

### 4.4.4 *General membrane regularization*

We expand $X^I$ in a complete set of orthonormal functions $Y_A$ (here $A, B, C$ are the basis function indices, not to be confused with the flat super indices)

$$X^I(\sigma) = X^{IA} Y_A(\sigma), \qquad A = 0, 1, 2, \ldots \tag{105}$$

with the orthogonality condition

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y^B(\sigma) = \delta_A^B. \tag{106}$$

We can then define a metric $\eta_{AB}$

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) = \eta_{AB}, \tag{107}$$

hence the metric lowers and raises the indices

$$\eta^{AB} Y_B(\sigma) = Y^A(\sigma) \tag{108}$$

and the inverse is defined

$$\eta^{AB} \eta_{BC} = \delta_C^A. \tag{109}$$

Multiplying (105) with $\sqrt{w(\sigma)} Y^B(\sigma)$ on both sides and integrating over $d^2\sigma$ we get (renaming indices and arguments)

$$X^{IA} = \int d^2\sigma' \sqrt{w(\sigma')} Y^A(\sigma') X^I(\sigma') \tag{110}$$

Using this in (105) we get

$$X^I(\sigma) = \int d^2\sigma' X^I(\sigma') Y_A(\sigma) \sqrt{w(\sigma')} Y^A(\sigma') \tag{111}$$

so that we get the completeness relation

$$Y_A(\sigma) Y^A(\sigma') = \frac{\delta(\sigma - \sigma')}{\sqrt{w(\sigma')}}. \tag{112}$$

The structure constants are defined through

$$\{Y_A(\sigma), Y_B(\sigma)\} = f_{AB}^{\;\;C} Y_C(\sigma) \tag{113}$$

with

$$f_{AB}^{\ C} = \int d^2\sigma' \epsilon^{rs} \partial_r Y_A(\sigma') \partial_s Y_B(\sigma') Y^C(\sigma'), \tag{114}$$

which can be seen by plugin in the explicit expression for the structure constant in the equation above and use the completeness relation (112). Now we introduce a cut-off $\Lambda$ on the number of modes $Y_A$, i.e. $A, B, C = 0, 1, \ldots, \Lambda$. Our infinite group of area preserving diffeomorphisms $G_{\text{APD}}$ will then be approximated by a finite Lie group $G_\Lambda$

$$\lim_{\Lambda\to\infty} f_{AB}^{\ C} \in G_\Lambda = f_{AB}^{\ C} \in G_{\text{APD}}. \tag{115}$$

The group $G_\Lambda$ will be the $SU(N)$ group, i.e. $G_\Lambda = SU(N)$ (if we include the center of mass motion the group will be $U(N)$), with $\Lambda = N^2 - 1$. This was proved for membranes of arbitrary genus [6]. We won't give the proof here since it's far to complicated. We will however prove this statement for the torus. But since the membrane can change genus through infinitely thin tubes this may be enough.

### 4.4.5 *Torus membrane regularization*

We choose the basis functions to be a double Fourier expansion

$$Y_A(\sigma^r) = e^{iA_r\sigma^r}, \qquad A = 0, 1, 2, \ldots \tag{116}$$

with $r = 1, 2$, and $0 \leq \sigma^r \leq 2\pi$. The definition of the metric (107) gives us

$$\int_0^{2\pi} d^2\sigma \sqrt{w(\sigma)} e^{i(A_r+B_r)\sigma^r} = \delta_{0,A_r+B_r} \tag{117}$$

if we choose $\sqrt{w(\sigma)} = \frac{1}{4\pi^2}$, which we do. Calculating the bracket

$$\{Y_A(\sigma^r), Y_B(\sigma^r)\} = -(2\pi)^2 \epsilon^{rs} A_r B_s Y_{A+B}(\sigma) \tag{118}$$

reveals that

$$f_{AB}^{\ C} = -(2\pi)^2 \epsilon^{rs} A_r B_s \delta_{A+B,C}. \tag{119}$$

A generalization of the Pauli matrices are the shift and clock matrices, defined

$$\Sigma_s = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \qquad \Sigma_c = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}, \tag{120}$$

where $N$ is defined from the group dimension, i.e from $SU(N)$, and $\omega = e^{i2\pi k/N}$, $k \in \mathbb{Z}$. They commute up to a phase factor

$$\Sigma_s \Sigma_c = \omega \Sigma_c \Sigma_s \tag{121}$$

This is very useful since any trace-less $N \times N$ matrix can be written as a linear combination of $\Sigma_s^{A_1}\Sigma_c^{A_2}$ [16]. The commutator is then

$$\left\{\Sigma_s^{A_1}\Sigma_c^{A_2}, \Sigma_s^{B_1}\Sigma_c^{B_2}\right\} = \left(\omega^{A_2 B_1} - \omega^{A_1 B_2}\right) \Sigma_s^{A_1+B_1}\Sigma_c^{A_2+B_2}. \tag{122}$$

Keeping $A$ and $B$ fixed and taking the limit as $N \to \infty$ we have

$$\lim_{N\to\infty} \omega^C = 1 + \frac{i2\pi kc}{N} + \mathscr{O}(1/N^2) \tag{123}$$

which gives

$$\lim_{N\to\infty} \left\{\Sigma_s^{A_1}\Sigma_c^{A_2}, \Sigma_s^{B_1}\Sigma_c^{B_2}\right\} = -\frac{i2\pi k}{N}\epsilon^{rs} A_r B_s \Sigma_s^{A_1+B_1}\Sigma_c^{A_2+B_2} + \mathscr{O}(1/N^2). \tag{124}$$

This is the same Lie-algebra as in (118) after re-scaling.

This shows that we successfully truncated the infinite dimensional Lie algebra (APD) to a finite dimensional Lie algebra of matrices.

## 4.5 SUMMARY

Our whole theory is now described by the Hamiltonian

$$H = \int d^2\sigma \frac{1}{2\lambda}\left(\frac{\mathscr{P}^I\mathscr{P}_I}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2}\left\{X^I, X^{I'}\right\}^2\right) \tag{125}$$

the Gauss-constraint[6] $\left\{\mathscr{P}_I, X^I\right\} = 0$ and the residual gauge invariance $\delta\sigma^r = \xi^r$, with $\partial_r(\sqrt{w}\xi^r) = 0$. Because of the residual gauge invariance we have 8 degrees of freedom for the bosonic membrane. Furthermore we have used our initial gauge freedom to fix $\partial_i X^+ = \delta_{i,0}$ and $u_r = 0$. Note that the canonical momenta are now defined as $\mathscr{P}^I = \mathscr{P}^+\partial_0 X^I$.

6 If we choose the spatial metric $w(\sigma)$ on the membrane to be constant, as we do for the torus case for example.

# Part IV

## SUPER-MEMBRANE

We describe the super-membrane. We introduce fermions in the target space-time. We take a look at the super-membrane Lagrangian. Since we relate bosons to fermions they need to have the same degrees of freedom. For this we need something called $\kappa$-symmetry (kappa-symmetry), which we also take a look at. Then we derive the Hamiltonian in the light-cone gauge and truncate the infinite dimensional algebra to arrive at super M(atrix) theory.

# 5 ACTION AND LAGRANGIAN

Since the bosonic membrane lacks fermions, which we know exist, we introduce supersymmetry. We introduce fermionic coordinates in target space which are denoted by $\theta$ (these are maps $\theta(\sigma^i) : \Sigma_3 \to M_{11}$), with their Dirac conjugate $\bar{\theta} \equiv \theta^\dagger C = \theta^T C$ (since $\theta$ is real in 11-dimensions [17]), these transform as spinors in space-time and anti-commute $\theta_1\theta_2 = -\theta_2\theta_1$ (Grassmann variables) with each other but commute with the bosonic maps $\theta X = X\theta$. The bosonic coordinates (maps) are denoted by $X^m$ as in the previous chapters. We also introduce the super-space coordinates $Z^M = (X^m, \theta^\mu)$. The meaning of this is that we embed the membrane in a supersymmetric space-time. For an overview of gamma matrices and spinors see Appendix B

## 5.1 LAGRANGIAN

### 5.1.1 *Lagrangian formulation in super-gravity*

The Lagrangian density is given by [18, 19]

$$\mathscr{L} = \mathscr{L}_{\text{vol}} + \mathscr{L}_{\text{wz}}, \tag{126}$$

where $\mathscr{L}_{vol}$ is a kinetic term proportional to the world volume and $\mathscr{L}_{wz}$ is the Wess-Zumino term which specifies the minimal coupling to a 3-form potential under which the membrane carries charge

$$\mathscr{L}_{\text{vol}} = -\int \mathrm{d}^3\sigma \sqrt{-\det(h_{ij})}, \tag{127a}$$

$$\mathscr{L}_{\text{wz}} = \int C \tag{127b}$$

with the induced metric $h_{ij} = E_i{}^m E_j{}^n \eta_{mn}$, the pullback of the anti-symmetric tensor gauge field $C_{ijk} = E_k{}^C E_j{}^B E_i{}^A C_{ABC}$ and $C = \frac{1}{6}\epsilon^{ijk} C_{ijk}$ with the pullback ($\partial_i Z^M$) of the super-vielbein to the membrane world volume $E_i{}^A = \partial_i Z^M E_M^A$ where the target-space super-vielbein is $E_M^A$. The $\epsilon^{ijk}$ is the anti-symmetric Levi-Civita symbol, with $\epsilon^{123} = +1$.

### 5.1.2 *Lagrangian formulation in flat super-space*

In flat target-space the super-vielbien becomes [19]

$$E_i^m \equiv \partial_i X^m + \bar{\theta}\Gamma^m \partial_i \theta. \tag{128}$$

and the Lagrangian becomes

$$\mathscr{L}_{\text{vol}} = \frac{1}{2}\sqrt{-h}\left(1 - h^{ij}E_i^m E_j^n \eta_{mn}\right), \tag{129a}$$

$$\mathscr{L}_{\text{WZ}} = -\epsilon^{ijk}\left(\frac{1}{2}\partial_i X^m\left(\partial_j X^n + \bar{\theta}\Gamma^n\partial_j\theta\right) + \frac{1}{6}\bar{\theta}\Gamma^m\partial_i\theta\bar{\theta}\Gamma^n\partial_j\theta\right)\bar{\theta}\Gamma_{mn}\partial_k\theta. \tag{129b}$$

The gamma matrices generate the target-space Clifford algebra

$$\Gamma^m\Gamma^n + \Gamma^n\Gamma^m = 2\eta^{mn} \tag{130}$$

and $\Gamma_{mn} = \frac{1}{2}\left(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m\right)$.

## 5.2 EQUATIONS OF MOTIONS

The equations of motion for the super-membrane in flat super-space are [19, 7]

$$\partial_i\left(\sqrt{-h}h^{ij}E_j{}^m\right) - \epsilon^{ijk}E_i{}^n\partial_j\bar{\theta}\Gamma^m_n\partial_k\theta = 0, \tag{131}$$

$$(1+\Gamma)h^{ij}\Gamma_m E_i{}^m\partial_j\theta = 0, \tag{132}$$

$$h_{ij} - E_i{}^m E_j{}^n\eta_{mn} = 0, \tag{133}$$

with

$$\Gamma = \frac{1}{3!\sqrt{-h}}\epsilon^{ijk}\Gamma_{ijk}. \tag{134}$$

## 5.3 $\kappa$-SYMMETRY

Since we have supersymmetry, which means that we relate fermions to bosons, we need the fermions and the bosons to have the same physical degrees of freedom. A Majorana spinor in 11-dimensions has 32 components. This is reduced to half by the equations of motion. We need one more symmetry which reduces the components by half, this is what the so called local $\kappa$-symmetry does, which we will now show.

The $\kappa$-symmetry is generated by a super-space vector that only points in the fermionic directions $\kappa = \kappa^M\partial_M = \kappa^\alpha E_\alpha{}^M\partial_M$ (i.e. $\kappa^A = (\kappa^a, \kappa^\alpha) = (0, \kappa^\alpha)$). The coordinates transforms as $\delta_\kappa Z^m = \kappa^M$. To see how the Lagrangian transform we take the Lie-derivative. We use the general super-gravity formulation (127). The transformations are (with pullbacks suppressed)

$$\delta_\kappa C = i_\kappa \underbrace{dC}_{H} + d i_\kappa C \tag{135}$$

and for the vielbein

$$
\begin{aligned}
\delta_\kappa E^A =& i_\kappa dE^A + d \underbrace{i_\kappa E^A}_{=\kappa^M E_M^A = \kappa^A} \\
=& i_\kappa \Big( \underbrace{DE^A}_{\equiv T^A} - \omega^A{}_B E^B \Big) + D\kappa^A - \omega^A{}_B \kappa^B \\
=& \Big[ D\kappa^A + i_\kappa T^A \Big] - i_\kappa (\omega^A{}_B E^B) - \omega^A{}_B \kappa^B \\
=& \Big[ D\kappa^A + i_\kappa T^A \Big] - i_\kappa \omega^A{}_B E^B + \underbrace{\omega^A{}_B \underbrace{i_\kappa E^B}_{=\kappa^B} - \omega^A{}_B \kappa^B}_{=0} \\
=& D\kappa^A + i_\kappa T^A
\end{aligned} \tag{136}
$$

where $i_\kappa \omega^A{}_B E^B$ is Lorentz transformed away and $T^A$ is the super-space torsion. For the induced metric we get

$$
\begin{aligned}
\delta_\kappa h_{ij} =& \delta_\kappa \big( E_i{}^a E_j{}^b \eta_{ab} \big) \\
=& 2 \big( D \underbrace{\kappa^a}_{=0} + i_\kappa T^a \big)_{(i} E_{j)}^b \eta_{ab} \\
=& 2 E_{(i}{}^B E_{j)}^a \kappa^\alpha T_{\alpha B}{}^b \eta_{ab}
\end{aligned} \tag{137}
$$

where we used $(i_\kappa T^a)_i = E_i{}^B \kappa^\alpha T_{\alpha B}{}^a$. To show the $\kappa$-symmetry we need $i_\kappa H$ and $i_\kappa T^a$. In 11-dimensions they are [18]; $H_{ab\alpha\beta} = 2(\Gamma_{ab})_{\alpha\beta}$ and $T_{\alpha\beta}{}^a = 2\Gamma^a_{\alpha\beta}$. Now we have everything we need to show the $\kappa$-symmetry:

$$
\begin{aligned}
\delta_\kappa S =& - \int d^3\sigma \left( \frac{1}{2} \sqrt{-h} h^{ij} \delta_\kappa h_{ij} \right) + \int \Big( i_\kappa H + \underbrace{d i_\kappa C}_{\text{boundary}} \Big) \\
=& - \int d^3\sigma \sqrt{-h} h^{ij} E_{(i}{}^B E_{j)}^a \kappa^\alpha T_{\alpha B}{}^b \eta_{ab} + \int i_\kappa H \\
=& -2 \int d^3\sigma \sqrt{-h} E_i{}^B \underbrace{h^{ij} E_j{}^a \kappa^\alpha \Gamma_{\alpha B}{}^b \eta_{ab}}_{=(\Gamma^i)_{\alpha\beta}\kappa^\alpha} + \int \frac{1}{3!} \kappa^A \epsilon^{ijk} \underbrace{H_{Aijk}}_{2E_i{}^\beta (\Gamma_{jk})_{A\beta}} d^3\sigma \\
=& -2 \int d^3\sigma \sqrt{-h} E_i{}^B \left( \Gamma^i - \frac{1}{3!\sqrt{-h}} \epsilon^{ijk} \Gamma_{jk} \right)_{\alpha\beta} \kappa^\alpha
\end{aligned} \tag{138}
$$

$$
= -2 \int d^3\sigma \sqrt{-h} E_i{}^B \left( \Gamma^i \left[ 1 - \frac{1}{3!\sqrt{-h}} \epsilon^{ijk} \Gamma_{ijk} \right] \right)_{\alpha\beta} \kappa^\alpha \tag{139}
$$

$$
= -2 \int d^3\sigma \sqrt{-h} E_i{}^B \left( \Gamma^i [1 - \Gamma] \right)_{\alpha\beta} \kappa^\alpha = 0. \tag{140}
$$

Requiring this to be zero halves the components of the spinors since the square bracket is a projection matrix[1], projecting out half of the spinor components [18]. The $\kappa$-symmetry hold only on-shell for supergravity [18]. We are left with 8 fermionic degrees of freedom and 8 bosonic degrees of freedom.

1 Since $\Gamma^2 = 1$ and $\mathrm{Tr}(\Gamma) = 0$. This is explicitly proven for the case of flat super-space in [13].

# 6 HAMILTONIAN FORMALISM

Since we are interested in going from the light-cone gauge to matrix mechanics and already looked at the bosonic membrane from different viewpoints we proceed directly to the light-cone gauge in the Hamiltonian formalism.

## 6.1 LIGHT-CONE HAMILTONIAN

We define the light-cone coordinates as before

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^{10} \pm X^{0}\right), \qquad \Gamma^{\pm} = \frac{1}{\sqrt{2}}\left(\Gamma^{10} \pm \Gamma^{0}\right) \tag{141}$$

and impose the light-cone gauge

$$\partial_i X^{+} = \delta_{i,0}, \qquad \Gamma^{+}\theta = 0. \tag{142}$$

The equations of motion (133) for the induced metric becomes

$$h_{ij} = \delta_{i,0}\left(\partial_j X^{-} + \bar{\theta}\Gamma^{-}\partial_j\theta\right) + \delta_{j,0}\left(\partial_i X^{-} + \bar{\theta}\Gamma^{-}\partial_i\theta\right) + \partial_i X^{I}\partial_j X^{I'}\eta_{II'}, \tag{143}$$

that is

$$h_{00} = 2\partial_0 X^{-} + 2\bar{\theta}\Gamma^{-}\partial_0\theta + \partial_0 X^{I}\partial_0 X^{I'}\eta_{II'}, \tag{144}$$

$$h_{0r} \equiv u_r = \partial_r X^{-} + \bar{\theta}\Gamma^{-}\partial_r\theta + \partial_0 X^{I}\partial_r X^{I'}\eta_{II'}, \tag{145}$$

$$h_{rs} = \partial_r X^{I}\partial_s X^{I'}\eta_{II'}. \tag{146}$$

The Lagrangian becomes, using Leibniz formula for determinants and the equation of motion for $h_{ij}$

$$\mathscr{L} = -\sqrt{h'U} + \epsilon^{rs}\partial_r X^{I}\bar{\theta}\Gamma^{-}\Gamma_I\partial_s\theta \tag{147}$$

with $h' = \det(h_{rs})$ and $U = -h_{00} + u_r h^{rs} u_s$.

We define the conjugate momenta densities

$$\mathscr{P}^{I} \equiv \frac{\partial\mathscr{L}}{\partial(\partial_0 X_I)} = \sqrt{\frac{h'}{U}}\left(\partial_0 X^{I} - \partial_r X^{I} h^{rs} u_s\right), \tag{148}$$

$$\mathscr{P}^{+} \equiv \frac{\partial\mathscr{L}}{\partial(\partial_0 X^{-})} = \sqrt{\frac{h'}{U}}, \tag{149}$$

$$\mathscr{S} \equiv \frac{\partial\mathscr{L}}{\partial_R(\partial_0\bar{\theta})} = -\sqrt{\frac{h'}{U}}\Gamma^{-}\theta \iff \bar{\mathscr{S}} \equiv \frac{\partial\mathscr{L}}{\partial_L(\partial_0\theta)} = \sqrt{\frac{h'}{U}}\bar{\theta}\Gamma^{-} \tag{150}$$

where $\partial_L$ and $\partial_R$ are left and right derivative respectively, this is needed since these fermionic derivatives anti-commute with the spinors. From these we get the primary constraints, multiplying (148) by $\partial_r X^I$ yields

$$\phi_r = \mathscr{P}^I \partial_r X^{I'} \eta_{II'} + \mathscr{P}^+ \partial_r X^- + \bar{\mathscr{S}} \partial_r \theta \approx 0 \tag{151}$$

We construct the Hamiltonian density[1]

$$\begin{aligned}\mathscr{H} =& \mathscr{P}^I \partial_0 X_I + \mathscr{P}^+ \partial_0 X^- + \bar{\mathscr{S}} \partial_0 \theta - \mathscr{L} \\ =& \frac{\mathscr{P}^I \mathscr{P}_I}{2\mathscr{P}^+} + \frac{h'}{2\mathscr{P}^+} + \theta^T \gamma_I \left\{ X^I, \theta \right\}.\end{aligned} \tag{152}$$

Then we add the constraints according to Dirac's theory

$$\mathscr{H}' = \frac{\mathscr{P}^I \mathscr{P}_I}{2\mathscr{P}^+} + \frac{h'}{2\mathscr{P}^+} + \theta^T \gamma_I \left\{ X^I, \theta \right\} + C^r \phi_r. \tag{153}$$

Now we have to check for inconsistencies, second constraints or equations for $C^r$. This is done by checking that the constraints time evolution are weakly zero. That is, checking that

$$\dot{\phi}_r = \left\{ \mathscr{H}', \phi_r \right\} \approx 0. \tag{154}$$

It turns out that the time derivative for all these are weakly zero [7]. In analogy with the bosonic case we can set[2] $u_r = 0$ which when using Hamilton's equations reveals that $C^r = 0$. Just as in the bosonic case this gives rise to a Gauss constraint, which is given by

$$\left\{ \partial_0 X^I, X_I \right\} + \left\{ \theta^T, \theta \right\} = 0 \tag{155}$$

So we get the total Hamiltonian

$$H = \frac{1}{2\lambda} \int d^2\sigma \left( \frac{\mathscr{P}^I \mathscr{P}_I}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2} \left\{ X^I, X^{I'} \right\}^2 + 2\lambda \sqrt{w(\omega)} \theta^T \gamma_I \left\{ X^I, \theta \right\} \right) \tag{156}$$

and in analogy with the bosonic case the mass becomes

$$M^2 = \int d^2\sigma \left( \frac{[\mathscr{P}^I \mathscr{P}_I]'}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2} \left\{ X^I, X^{I'} \right\}^2 + 2\lambda \sqrt{w(\omega)} \theta^T \gamma_I \left\{ X^I, \theta \right\} \right). \tag{157}$$

---

1 Taking a look in appendix B we see that $\Gamma^- \Gamma_I = \sqrt{2} \begin{pmatrix} 0 & 0 \\ \gamma_I & 0 \end{pmatrix}$ and the gauge condition $\Gamma^+ \theta = 0$ gives $\theta_{32\times 1} = \begin{pmatrix} \theta_{16\times 1} \\ 0_{16\times 1} \end{pmatrix}$. We therefore rewrite $\bar{\theta} \Gamma^- \Gamma_I \partial_s \theta$ (in $SO(10,1)$) as $-\theta^T \gamma_I \partial_s \theta$ (in $SO(9)$) where we absorbed the factor $\sqrt{2}$ into the fields.

2 This choice removes the constraints given by (151) just as in the bosonic case.

## 6.2 FROM SUPER-MEMBRANE TO M(ATRIX) THEORY

As stated above the mass of the super-membrane is given by (157) which is just $M^2 = M^2_{\text{bosonic}} +$ fermionic terms. In analogy with the bosonic case we have residual gauge invariance, namely APD, which for $\delta\sigma^r = \xi^r$ have to satisfy $\partial_r \left(\sqrt{w}\xi^r\right) = 0$, that is, be generated by divergence free vector fields.

In analogy with the bosonic case we expand the coordinates in a complete set of orthonormal functions, which are different for different topologies. We then translate the Hamiltonian to super M(atrix) theory[3] [4]

$$H = \frac{1}{2\lambda}\text{Tr}\left(P^I P_I + \frac{1}{2}\left[X^I, X^{I'}\right]^2 + \theta^T\gamma_I\left[X^I, \theta\right]\right), \tag{158}$$

The matrices here are in $U(N)$ since we haven't omitted the zero modes (center of mass). Note that the contraction for the indices are done with the Euclidean metric ($P^I P_I = P^I P^{I'}\delta_{II'}$) now. This Hamiltonian need to be supplemented with the Gauss constraint found in (155).

We will look more into this Hamiltonian in the M(atrix) theory before we, in the final chapter, calculate the first order scattering amplitude for two D0-branes and compare the results to super-gravity.

## 6.3 SUMMARY

The super-membrane theory is now completely described by the Hamiltonian

$$H = \frac{1}{2\lambda}\int d^2\sigma\left(\frac{\mathscr{P}^I\mathscr{P}_I}{\sqrt{w(\sigma)}} + \frac{\sqrt{w(\sigma)}}{2}\left\{X^I, X^{I'}\right\}^2 + 2\lambda\sqrt{w(\omega)}\theta^T\gamma_I\left\{X^I, \theta\right\}\right), \tag{159}$$

the Gauss constraint $\left\{\mathscr{P}^I, X_I\right\} + \left\{\theta^T, \theta\right\} = 0$ and the residual gauge invariance for the bosonic coordinates described by $\delta\sigma^r = \xi^r$ with $\partial_r(\sqrt{w}\xi^r) = 0$.

3 We let $\theta \to \sqrt{2\lambda}\theta$.

4 $w(\sigma)$ depends on the membranes topology. Choosing for example a torus where $w(\sigma)$ is just a constant makes the transition easier.

# Part V

## M(ATRIX) THEORY

We give an introduction to M(atrix) theory and explain how M(atrix) theory connects to, not just membrane theory, but to other theories as well. We develop some needed tools for our scattering calculation, e.g. Schwinger action principle and the background field method. Then we take a closer look at the M(atrix) theory to get a feeling for the theory. In the last section we calculate the scattering amplitude to first order in quantum corrections in the M(atrix) theory and compare it to the results of supergravity (SUGRA) in 11D.

# 7

## INTRODUCTION TO M(ATRIX) THEORY

M(atrix) theory is a supersymmetric quantum mechanics with matrix degrees of freedom. The Lagrangian of the theory is built from 9 bosonic $N \times N$ matrices with their corresponding 16 components (with every component a matrix[1]) fermionic partners $\theta$. There is a gauge invariance for the bosonic coordinates given by $\delta X^a = [\Lambda, X^a]$ which are the APD in the membrane theory. This reduces the bosonic degrees of freedom down to 8.

Since $N$ is finite we have a finite degrees of freedom making the theory manifestly well defined, note that this is not a field theory[2] (which has infinite degrees of freedom) so we should not run into any problems of re-normalization.

It is believed that M(atrix) theory describes M-theory in an infinite momentum frame, in light-cone coordinate, in flat space-time[3]. Thus M(atrix) theory provides a calculational framework for M-theory, however it has proved itself to be a theory where detailed calculations gets hard quickly, even for small $N$. People have however, preformed detailed calculations to show that 11-dimensional classical super-gravity is produces by M(atrix) theory to some low orders. We shall, later on, show this for the first term which we get from a first order loop calculation in M(atrix) theory.

The action and Hamiltonian for M(atrix) theory is given by[4]

$$S = \frac{1}{2\lambda} \int \mathrm{d}t \, \mathrm{Tr} \left( \partial_t X^a \partial_t X_a - \frac{1}{2} \left[ X^a, X^b \right]^2 + i\theta^T \partial_t \theta - \theta^T \gamma_a \left[ X^a, \theta \right] \right), \tag{160}$$

$$H = \frac{1}{2\lambda} \mathrm{Tr} \left( P^a P_a + \frac{1}{2} \left[ X^a, X^b \right]^2 + \theta^T \gamma_a \left[ X^a, \theta \right] \right), \tag{161}$$

where $X^a$ are Hermitian $N \times N$ matrices, $a = 1, \ldots, 9$ and $\theta$ is a 16 component matrix valued real spinor, i.e 16 Grassmann matrices of $SO(9)$. Here we use the Euclidean metric $\delta_{ab}$ to raise or lower the bosonic indices. This Hamiltonian is obviously the same as in the truncated membrane theory (158).

1 Since the equations of motion halves the degrees of freedom for the fermions we have equal degrees of freedom for the bosonic and fermionic parts. As we should have in a supersymmetric theory.

2 Although we will often call the objects $X^a$ and $\theta$ for fields anyway.

3 This conjecture is know as the BFSS conjecture, given in [20].

4 One can of course derive this Hamiltonian from the action, so they describe the same theory.

This theory is in $(0+1)$ dimensions, meaning that the fields only depend on time[5].

We will show how this theory can be interpreted as $N$ numbers of D0-branes, elaborate on how this is a multi-particle theory and calculate the first loop order for the scattering of two D0-branes.

5 We can also arrive at this theory by doing dimensional reduction of the $(9+1)$ dimensional Yang-Mills theory, which will be shown later.

# 8 TOOLS AND FORMULAS

First we need to take a detour to develop some tools which tells us how to compute the first order loop amplitude. This is done by something called the background field theory, where we expand the fields in a classical background with some quantum fluctuations.

This chapter follows [21], which is a great introduction to effective potentials and Schwinger action principle if one feels the need for a more complete overview of the subject, and [22] which is a great introduction to the background field method.

## 8.1 SCHWINGER ACTION PRINCIPLE

We will write[1] $|a\rangle$ as the eigenket of an operator $A$ and $|b\rangle$ as the eigenket of an operator $B$, and so forth. We won't write out the time of these kets since we want to keep the readability, thus it is understood that $|a\rangle = |a, t_a\rangle$ and $|b\rangle = |b, t_b\rangle$ etc.

We can write

$$\sum_b \langle a|b\rangle\langle b|c\rangle = \langle a|c\rangle \tag{162}$$

since we have a completeness relation

$$\sum_b |b\rangle\langle b| = 1 \tag{163}$$

where the 1 on the right side is the identity operator.

If $|a\rangle$ and $|a'\rangle$ are both eigenkets of the same operator $A$ we have the orthogonal relation

$$\langle a|a'\rangle = \delta(a-a'). \tag{164}$$

The bras $\langle a|$ are dual vectors to the kets $|a\rangle$ which means that we have

$$\Big(\langle b|a\rangle\Big)^\dagger = \langle a|b\rangle \tag{165}$$

where † is the Hermitian conjugate (transpose and complex-conjugate).

Varying (162) yields

$$\delta\langle a|c\rangle = \sum_b \Big[\Big(\delta\langle a|b\rangle\Big)\langle b|c\rangle + \langle a|b\rangle\Big(\delta\langle b|c\rangle\Big)\Big] \tag{166}$$

1 This is basic quantum mechanical notation called "brackets" and can all be found in e.g. [23].

and varying (165) yields

$$\delta\Big(\langle b|a\rangle\Big)^{\dagger} = \delta\langle a|b\rangle. \tag{167}$$

We define an operator $\delta W_{ab}$ through the equation

$$\delta\langle a|b\rangle = i\langle a|\delta W_{ab}|b\rangle \tag{168}$$

where we choose the normalization $i$ to make $\delta W_{ab}$ Hermitian (we will show the Hermitian property of $\delta W_{ab}$ soon).

We rewrite (166) with the aid of (168)

$$\begin{aligned}\langle a|\delta W_{ac}|c\rangle &= \sum_b \Big[\Big(\langle a|\delta W_{ab}|b\rangle\langle b|c\rangle + \langle a|b\rangle\langle b|\delta W_{bc}|c\rangle\Big] \\ &= \langle a|\delta W_{ab} + \delta W_{bc}|c\rangle\end{aligned} \tag{169}$$

where we used the completeness relation in the last step. We then get the relation

$$\delta W_{ac} = \delta W_{ab} + \delta W_{bc}. \tag{170}$$

Varying (164) yields

$$\delta W_{aa} = 0. \tag{171}$$

We set $a = c$ in (170) and use the above equation to get

$$\delta W_{ab} = -\delta W_{ba}. \tag{172}$$

Using (167) and (168) gives the equation

$$\Big(i\langle b|\delta W_{ba}|a\rangle\Big)^{\dagger} = i\langle a|\delta W_{ab}|b\rangle \tag{173}$$

reveling the property

$$\delta W_{ab}^{\dagger} = \delta W_{ab} \tag{174}$$

which means that $\delta W_{ab}$ is Hermitian.

Now we assume that the infinitesimal operator $\delta W_{ab}$ can be obtained as an variation of an operator $W_{ab}$ (called the action operator), which then have to obey the properties above, just derived

$$\begin{aligned}W_{ac} &= W_{ab} + W_{bc}, \\ W_{aa} &= 0, \\ W_{ab} &= -W_{ba} = W_{ab}^{\dagger}.\end{aligned} \tag{175}$$

If the time evolution from a state $|a\rangle$ to a state $|b\rangle$ is continuous in time we may write

$$W_{ba} = \int_{t_a}^{t_b} \mathrm{d}t\, L(t), \tag{176}$$

where $L(t)$ is the Lagrangian operator which must be Hermitian $L^\dagger(t) = L(t)$ due to (175). This gives us the Schwinger action principle

$$\delta\langle b|a\rangle = i\langle b|\delta S|a\rangle \tag{177}$$

where $S = \int_{t_a}^{t_b} \mathrm{d}t\, L(t)$ is the action.

### 8.1.1 *Double variation of the Lagrangian*

We will need the formula for two variations of an amplitude later on so we will derive it here. Putting in two completeness relations in (177) and giving the variation a label since we are going to take two variations yields

$$\delta_1\langle b|a\rangle = i\int_{t_a}^{t_b} \mathrm{d}t \sum_{c,d} \langle b|c\rangle\langle c|\delta_1 L(t)|d\rangle\langle d|a\rangle. \tag{178}$$

Now we perform another variation $\delta_2$ which is independent of the first variation[2]

$$\begin{aligned}\delta_2\delta_1\langle b|a\rangle =& i\int_{t_a}^{t_b} \mathrm{d}t \sum_{c,d} \Big[\Big(\delta_2\langle b|c\rangle\Big)\langle c|\delta_1 L(t)|d\rangle\langle d|a\rangle \\ &+ \langle b|c\rangle\langle c|\delta_1 L(t)|d\rangle\Big(\delta_2\langle d|a\rangle\Big)\Big].\end{aligned} \tag{179}$$

We can apply the Schwinger action principle (177) and expand out the variation on the right hand side of the above equation as

$$\delta_2\langle b|c\rangle = i\int_{t}^{t_b} \mathrm{d}t'\langle b|\delta_2 L(t')|c\rangle, \tag{180}$$

$$\delta_2\langle d|a\rangle = i\int_{t_a}^{t} \mathrm{d}t'\langle d|\delta_2 L(t')|a\rangle, \tag{181}$$

with $t_a \leq t \leq t_b$. We plug this back into (179) and use the completeness relations to get

$$\begin{aligned}\delta_2\delta_1\langle b|a\rangle =& i^2\int_{t_a}^{t_b} \mathrm{d}t \left(\int_{t}^{t_b} \mathrm{d}t'\langle b|\delta_2 L(t')\delta_1 L(t)|a\rangle\right. \\ &\left.+ \int_{t_a}^{t} \mathrm{d}t'\langle b|\delta_1 L(t)\delta_2 L(t')|a\rangle\right).\end{aligned} \tag{182}$$

We can write this more compact by defining the time ordering operator T

$$\mathrm{T}\left\{\delta_1 L(t)\delta_2 L(t')\right\} = \theta(t-t')\delta_1 L(t)\delta_2 L(t') + \theta(t'-t)\delta_2 L(t')\delta_1 L(t) \tag{183}$$

2 Meaning that terms such as $\delta_2\langle c|\delta_1 L(t)|d\rangle$ are zero.

where $\theta(t-t')$ is the Heaviside step function

$$\theta(t-t') = \begin{cases} 1 & \text{for } t-t' \geq 0 \\ 0 & \text{for } t-t' < 0 \end{cases}. \tag{184}$$

The double variation can then be written as

$$\delta_2\delta_1\langle b|a\rangle = i^2 \int_{t_a}^{t_b} \mathrm{d}t \int_{t_a}^{t_b} \mathrm{d}t' \langle b|\mathrm{T}\left\{\delta_1 L(t)\delta_2 L(t')\right\}|a\rangle. \tag{185}$$

This can be generalized to arbitrary many variations but two is sufficient for us.

## 8.2 EFFECTIVE ACTION AT FIRST LOOP ORDER

We consider a free field theory with a field operator $\phi^I(\mathbf{x},t)$ where $I$ runs over whatever indices we might have. We will use a very condense notation due to Dewitt, where we will label the field operators by $\phi^i$ and let $i$ stand for all the labels of the field operators, i.e. $i = (I,\mathbf{x},t)$. In this notation it is understood that for any repeated indices we sum over all the $I$ and integrate over all $\mathbf{x}$ and $t$. Two examples are

$$J_i\phi^i = \sum_I \int \mathrm{d}t \int \mathrm{d}^D x\, J_I(\mathbf{x},t)\phi^I(\mathbf{x},t), \tag{186}$$

$$\mathcal{O}_{ij}\phi^i\phi^j = \sum_{I,J} \int \mathrm{d}t \int \mathrm{d}^D x \int \mathrm{d}t' \int \mathrm{d}^D x'\, \mathcal{O}_{IJ}(\mathbf{x},t;\mathbf{x}',t')\phi^I(\mathbf{x},t)\phi^J(\mathbf{x},t). \tag{187}$$

We will consider a free theory (that is, in the limit where the interactions gets small) where the action $S$ is given by

$$S[\phi] = -\frac{1}{2}\mathcal{O}_{ij}\phi^i\phi^j + J_i\phi^i \tag{188}$$

where $\mathcal{O}_{ij}$ is the wave operator $(\Box_x + m^2 + i\epsilon)\delta(x,x')$ and $J_i$ is called the source function, we add $J_i$ since it is convenient as we will see. It can be set to zero at any time and no physical results are to depend on $J_i$, it's just a mathematical trick.

The equations of motion for $\phi^i$ following from the action (188) is

$$\mathcal{O}_{ij}\phi^j = J_i. \tag{189}$$

We define the Feynman Green function (also called the propagator) $G^{jk}$ in this notation as the inverse of $\mathcal{O}_{ij}$

$$\mathcal{O}_{ij}G^{jk} = \delta_i^k. \tag{190}$$

We can solve (189) in terms of the Green function

$$\phi^i = G^{ij}J_j. \tag{191}$$

Choosing the variation to be with respect to $J_i$ for the Schwinger action principle (177) we get[3]

$$\frac{\delta}{\delta J_i}\langle b|a\rangle = i\langle b|\phi^i|a\rangle. \tag{192}$$

We introduce the standard notation

$$\langle O[\phi^i]\rangle = \frac{\langle b|\mathrm{T}\{O[\phi^i]\}|a\rangle}{\langle b|a\rangle} \tag{193}$$

where $O[\phi^i]$ is any operator and are also allowed to be a functional of $\phi^i$ and $T$ is the time ordering operator defined in (183).

Since $\langle b|a\rangle$ is just a complex number we can write it as

$$\langle b|a\rangle = e^{iW[J]}. \tag{194}$$

Varying (194) with respect to $J_i$ gives

$$\frac{\delta}{\delta J_i}\langle b|a\rangle = i\frac{\delta W[J]}{\delta J_i}e^{iW[J]} = i\frac{\delta W[J]}{\delta J_i}\langle b|a\rangle. \tag{195}$$

Comparing (192) and (195) we get

$$\frac{\delta W[J]}{\delta J_i} = \langle\phi^i\rangle = \langle G^{ij}J_j\rangle \tag{196}$$

where the last equality follows from (191). (196) is a functional differential equation for $W[J]$ with the solution

$$W[J] = \langle\frac{1}{2}J_i G^{ij} J_j\rangle + W[0] \tag{197}$$

where $W[0]$ is independent of the source $J$ and can be thought of as an integration constant.

Next we need to determine the explicit form of $W[0]$.

Taking the variation of the action (188) with respect to $\mathcal{O}_{ij}$ yields

$$\delta S = -\frac{1}{2}\delta\mathcal{O}_{ij}\phi^i\phi^j, \tag{198}$$

taking the variation of the Schwinger action principle (177) and using (195) and (198) yields

$$\delta\langle b|a\rangle = \langle b|a\rangle i\delta W = -i\frac{1}{2}(\delta\mathcal{O}_{ij})\langle b|\mathrm{T}\{\phi^i\phi^j\}|a\rangle \tag{199}$$

which gives us the relation, for $J = 0$,

$$\delta W[0] = -\frac{1}{2}(\delta\mathcal{O}_{ij})\langle\mathrm{T}\{\phi^i\phi^j\}\rangle. \tag{200}$$

3 Here and for the rest of this section we take the states to be in the infinitely far past $t_a \to -\infty$ and infinitely far future $t_b \to +\infty$ , so that we start with a state that is the direct product of non-interaction states and end with such a state. These states are often denoted $\langle\mathrm{Out}|$ and $|\mathrm{in}\rangle$, we will however not use these conventions.

From the double variation (185) we know that

$$\frac{\delta^2\langle b|a\rangle}{\delta J_i \delta J_j} = i^2\langle b|\mathrm{T}\{\phi^i\phi^j\}|a\rangle \tag{201}$$

we note that (196) gives $\frac{\delta W}{\delta J_i}\big|_{J_i=0} = 0$ and rewrite (201) with the help of (194)

$$\frac{\delta^2 W}{\delta J_i \delta J_j}\bigg|_{J_i=0} = ie^{-iW[0]}\langle b|\mathrm{T}\{\phi^i\phi^j\}|a\rangle. \tag{202}$$

We get the left hand side from (197) and note that $e^{-iW[J]}\big|_{J=0} = \frac{1}{\langle b|a\rangle}\big|_{J=0}$ by (194), giving us

$$-iG^{ij} = \langle \mathrm{T}\{\phi^i\phi^j\}\rangle. \tag{203}$$

Using the result in (200) yields

$$\delta W[0] = \frac{i}{2}(\delta\mathcal{O}_{ij})G^{ij}. \tag{204}$$

But $G^{ij}$ and $\mathcal{O}_{ij}$ are inverses and we are summing over both the indices, that is, taking the trace, in crude notation[4]

$$\begin{aligned}(\delta\mathcal{O}_{ij})G^{ij} &= \mathrm{Tr}(\delta\mathcal{O}\,G)\\ &= \delta\mathrm{Tr}(\ln(\mathcal{O})), \qquad (205)\\ \therefore \quad W[0] &= \mathrm{Tr}(\ln(\mathcal{O})). \qquad (206)\end{aligned}$$

Combining this with (197) we get

$$W[J] = \langle\frac{1}{2}J_i G^{ij} J_j\rangle + \frac{i}{2}\mathrm{Tr}(\ln(\mathcal{O})). \tag{207}$$

But we are not quite there yet. The source function $J$ was just a mathematical trick, we need to change these variables into the field variables, remember that nothing were to depend on the source function. To eliminate $J$ we do a Legendre transformation, note that $\frac{\delta W[J]}{\delta J_i} = \langle\phi^i\rangle$ (which is how we usually define conjugate momenta when we go from the Lagrangian to the Hamiltonian.) The Legendre transform is given by

$$\begin{aligned}\Gamma\left[\langle\phi^i\rangle\right] &= W[J] - J_i\langle\phi^i\rangle \qquad (208)\\ &= -\frac{1}{2}\mathcal{O}_{ij}\langle\phi^i\rangle\langle\phi^j\rangle + \frac{i}{2}\mathrm{Tr}(\ln(\mathcal{O})), \qquad (209)\end{aligned}$$

4 One could think of this as matrix notation, but this is not quite true for this case since we here have to sum over all indices and set the arguments of the functions the same and integrate over all space-time.

we call $\Gamma\left[\langle\phi^i\rangle\right]$ the effective action since the first term is just the classical Feynman diagram for a free theory and the second term is the first quantum correction, which is a loop for free theories. So the one loop effective action is given by

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}(\ln(\mathcal{O})) \tag{210}$$

where $\mathcal{O}$ is the wave operator and the inverse of the propagator. This formula is however derived with the Minkowski metric. We want the Euclidean version.

If we look at the full quantum action $S[\phi]$ and expand $z = b + \varphi$ where $b$ is the classical background and $\varphi$ are the quantum fluctuations, then we can expand the action as

$$S[\phi] = S[b] + \frac{\delta S}{\delta\phi(x)}[b]\varphi(x) + \frac{1}{2}\varphi(y)\frac{\delta^2 S}{\delta\phi(y)\delta\phi(x)}[b]\varphi(x) + \dots \tag{211}$$

The first term is just a constant, the second term is zero if $b$ is on-shell, which we choose it to be. The third term is the background dependent kinetic term $\varphi\mathcal{O}\varphi$. Going from the Minkowski theory to the Euclidean theory we multiply the Minkowski action with $-i$ and take $t \to it$, this gives the Euclidean action and hence the one loop effective action for the Euclidean theory is

$$\Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}(\ln(\mathcal{O})) \tag{212}$$

### 8.2.1 *Heat kernel and effective action*

We will introduce the heat kernel which we relate to a number of quantities since this simplifies computations.

The heat kernel[5] $h(x,y,\chi)$ is defined through the PDE it satisfy

$$\mathcal{O}h(x,y,\chi) = -\frac{\partial}{\partial\chi}h(x,y,\chi) \tag{213}$$

where $\mathcal{O}$ is the wave operator and therefore the inverse to the propagator. The heat kernel also have a boundary condition to satisfy

$$\lim_{\chi\to o} h(x,y,\chi) = \delta(x,y). \tag{214}$$

One does not have to stare long at these equations to see that the explicit form for the heat kernel is

$$h(x,y,\chi) = e^{-\chi\mathcal{O}}\delta(x,y). \tag{215}$$

We can express the propagator $G(x,y)$ in terms of the heat kernel

$$G(x,y) = \int_0^\infty \mathrm{d}\chi\, h(x,y,\chi) \tag{216}$$

5 Often denoted by $K$ in the literature, but we will call it $h$.

which indeed is the inverse to $\mathcal{O}$

$$\begin{aligned} G(x,y)\mathcal{O} &= \int_0^\infty \mathrm{d}\chi\, \mathcal{O}h(x,y,\chi) \\ &= \int_0^\infty \mathrm{d}\chi \left( -\frac{\partial}{\partial\chi} h(x,y,\chi) \right) \\ &= -\, h(x,y,\chi)\Big|_{\chi=0}^{\chi=\infty} \\ &= \delta(x,y) \end{aligned} \tag{217}$$

Armed with this we can express (212) in terms of the heat kernel, first we ask how the effective action varies as the wave operator $\mathcal{O}$ varies

$$\begin{aligned} \delta\Gamma^{(1)} &= \frac{1}{2}\delta \mathrm{Tr}(\ln(\mathcal{O})) \\ &= \frac{1}{2}\mathrm{Tr}(G(x,y)\delta\mathcal{O}) \\ &= -\frac{1}{2}\mathrm{Tr}\left( \int_0^\infty \frac{\mathrm{d}\chi}{\chi} \delta h(x,y,\chi) \right). \end{aligned} \tag{218}$$

Now we integrate over the variation and arrive at an expression for the effective action in terms of the heat kernel

$$\Gamma^{(1)} = -\frac{1}{2}\mathrm{Tr}\left( \int_0^\infty \frac{\mathrm{d}\chi}{\chi} h(x,y,\chi) \right), \tag{219}$$

remember that the trace here is over all indices and then we set $x$ to $y$ and integrate over space-time, just as we defined the notation in the section above.

### 8.2.2 *Heat kernel a free field of mass n*

For a free scalar field $x(t)$ of mass $n$ in 1 dimension we have the heat kernel $h$ (imaginary time propagator)

$$h\big(T, x(0), x(T)\big) = \int \mathrm{D}x \, \exp\left( -\int_0^T \mathrm{d}t \left[ \frac{1}{2}\dot{x}^2 + \frac{n^2}{2}x^2 \right] \right) \tag{220}$$

which is of the same form as the heat kernel for the harmonic oscillator $Z_{HO}$ (derived in appendix C). We use this result with the identification $m = \frac{1}{2}$ and $n = \omega$ ( and rename $X_T = t$, $X_0 = t'$, $T = 2\chi$) to get

$$\begin{aligned} h(t,t',\chi) =& \sqrt{\frac{n}{2\pi \sinh(2n\chi)}} \\ & \times \exp\left( -\frac{n}{2\sinh(2n\chi)} \left[ (t^2 + t'^2)\cosh(2n\chi) - 2tt' \right] \right) \end{aligned} \tag{221}$$

which can be rewritten as

$$h(t,t',\chi) = \sqrt{\frac{n}{2\pi\sinh(2n\chi)}}\exp\left(-nt_-^2\coth(n\chi) - nt_+^2\tanh(n\chi)\right) \tag{222}$$

for $t_\pm = \frac{t+t'}{2}$.

# 9 MATRIX MODELS

## 9.1 D0-BRANES AS PARTICLES

The bosonic part of the Lagrangian building the action (160) is given by

$$L = \frac{1}{2\lambda} \mathrm{Tr} \left( \partial_t X^a \partial_t X_a - \frac{1}{2} \left[ X^a, X^b \right]^2 \right). \tag{223}$$

Small energies $E \ll 1$ implies large distances since $[Energy] = [Length]^{-1}$ by the equation $E = \hbar\omega$. Then by locality the potential energy $V = \frac{1}{4\lambda} \left[ X^a, X^b \right]^2$ must be small since it is built out of commutators of the dynamical variables $X^a$ which are largely separated, resulting in $\left[ X^a, X^b \right] = 0 \ \forall a, b$, that is, all the dynamical variables commute. By standard result in linear algebra or elementary quantum mechanics [23] this means that we can diagonalize all the matrices $X^a$ simultaneously

$$X^a = \begin{pmatrix} \lambda_1^a(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N^a(t) \end{pmatrix} \tag{224}$$

Plugging this into the Lagrangian (223) (with $V = 0$) and deriving the Euler-Lagrange equations of motion we find

$$\partial_t^2 \lambda_i^a(t) = 0, \quad i = 1, \dots, N \tag{225}$$

which means that we can write the eigenvalues $\lambda_i^a$ of $X^a$ as $\lambda_i^a(t) = v_i^a t + b_i^a$, where $v$ is the velocity and $b$ is called the impact parameter. If we place all these in vectors according to $\vec{\lambda}_i(t) = (\lambda_i^1, \dots, \lambda_i^9)$ where $i = 1, \dots N$ and $N$ is the dimension of $X^a$, we can interpret the vectors $\vec{\lambda}_i(t)$ as position vectors for $N$ D0-branes. Thus this theory can also be seen as an interaction theory for $N$ D0-branes.

## 9.2 MULTI-PARTICLE THEORY

The M(atrix) theory is a multi-particle theory[1], just as the membrane theory. However we got a restriction in the number of particles we

1 An old word for this, which is often used, is "a second quantized theory". It is due to the fact that if one quantize one, say scalar particle, one gets an equation for a probability wave function. On the other side, if one quantize, say a scalar field, one gets an quantized field built from an infinite number of harmonic oscillators. It just so happens that the structure of the one particle probability wave and the classical

can have in the M(atrix) theory, so this is not a quantum field theory but a multi-particle quantum mechanics, a regulated theory. Our dynamical variables in the (bosonic) M(atrix) theory are the $N \times N$ matrices $X^m$. This is also the number of maximal particles we can have in our theory as we will now show.

The bosonic part of the classical Lagrangian is given by, when rearranged (use the cyclicity of the trace $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$)

$$L = \frac{1}{2\lambda}\mathrm{Tr}\left(\partial_t X^a \partial_t X_a - \frac{1}{2}X_a\left[X_b,\left[X^a,X^b\right]\right]\right) \tag{226}$$

Using Euler-Lagrange equations we arrive at the M(atrix) theory's equations of motion

$$\partial_t^2 X^a = \left[X_b\left[X^a,X^b\right]\right]. \tag{227}$$

If the matrices are block diagonal

$$X^a = \left(\begin{array}{c|c} A^a & 0 \\ \hline 0 & B^a \end{array}\right), \tag{228}$$

the two blocks decouple and satisfy their own equations of motion

$$\begin{aligned} \partial_t^2 A^a &= \left[A_b\left[A^a,A^b\right]\right], \\ \partial_t^2 B^a &= \left[B_b\left[B^a,B^b\right]\right], \end{aligned} \tag{229}$$

showing that we can think of them as two separate particles. We can do this maximally $N$ times if the matrices $X^a$ are $N \times N$. Particles in M(atrix) theory can thus merge and unmerge, be created or annihilated.

field are the same. This made people to think that they somehow quantized the one particle two times. Both this is not at all what actually happened. The scalar field and the probability wave can not be interpret the same way at all. Therefore we will drop the old term and call it for what it actually means these days, a multi-particle theory.

# 10 FIRST ORDER EFFECTIVE POTENTIAL

## 10.1 Gauge inv. [super Yang-Mills to M(atrix) theory]

Another way to arrive at the M(atrix) theory is the dimensional reduction of the $9+1$ dimensional[1] super Yang-Mills theory[2] to a $0+1$ dimensional theory. The advantage of this approach is that one arrives at an gauge invariant formulation of the super M(atrix) theory, which for the gauge choice $A = 0$ gives the case we arrive at through the regularization of the super-membrane.

The super Yang-Mills action in flat Minkowski space is usually given by [24]

$$S = \int \mathrm{d}^{10}\sigma \,\mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \theta^T\gamma^\mu D_\mu\theta\right) \qquad (230)$$

with the field strength defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$, $A_\mu$ being a $U(N)$ Hermitian gauge field in the adjoint representation, $\theta$ being a $16\times 1$ Majorana spinor of[3] $SO(9,1)$ in the adjoint representation and $\mu = 0,\ldots,9$. The covariant derivative is given by $D_\mu\theta = \partial_t\theta - ig\left[A_\mu, \theta\right]$.

We re-scale the fields by $A_\mu \to \frac{i}{g}A_\mu$ and let $g^2 \to \lambda$ which gives us

$$S = \int \mathrm{d}^{10}\sigma \,\mathrm{Tr}\left(\frac{1}{4\lambda}F_{\mu\nu}F^{\mu\nu} - \theta^T\gamma^\mu D_\mu\theta\right) \qquad (231)$$

with the field strength defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right]$ and the covariant derivative $D_\mu\theta = \partial_t\theta + \left[A_\mu, \theta\right]$. $A_\mu$ being anti-Hermitian now.

### 10.1.1 *Dimensional reduction*

Now we perform a dimensional reduction from $9+1$ dimensions to $0+1$ dimensions, so that all the fields only depend on time, thus all spatial derivatives vanish i.e. $\partial_a(\text{Anything}) = 0$. The 10-dimensional vector field decomposes into 9 scalar fields $A_a$ which we rename $X^a$ and one gauge field $A_0$ which we rename $A$. This gives[4]

$$F_{0a} = \partial_t X^a + \left[A, X^a\right], \quad F_{ab} = +\left[X^a, X^b\right] \qquad (232)$$

$$\gamma^t D_t\theta = \partial_t\theta + \left[A, \theta\right], \quad \gamma^a D_a\theta = \gamma_a\left[X^a, \theta\right] \qquad (233)$$

1 9 spatial dimensions and 1 time dimension.
2 Non-Abelian gauge theories are called Yang-Mills theories by physicists.
3 Requiring that half of these are zero as in the super-membrane case we can decompose these into $SO(9)$.
4 Note that $\gamma^t = \mathbb{I}$ and that $\gamma^a = \gamma_a$.

The action for this theory is then

$$S = \int \mathrm{d}t \, \mathrm{Tr} \left( \frac{1}{2\lambda} \left\{ -(D_t X^a)^2 + \frac{1}{2} \left[ X^a, X^b \right]^2 \right\} - \theta^T D_t \theta - \theta^T \gamma_a \left[ X^a, \theta \right] \right) \tag{234}$$

with the covariant derivative defined as $D_t X^a = \partial_t X^a + [A, X^a]$ and $D_t \theta = \partial_t \theta + [A, \theta]$.

### 10.1.2 *Wick rotation*

The degrees of freedom we have left in the truncated super-membrane lives in Euclidean space. To match the dimensional reduced Yang-Mills theory we have to perform a Wick rotation, that is, we make the time coordinate imaginary[5] $t \to i\tau$ which gives the transformation of the derivative $\partial_t \to -i\partial_\tau$. To keep the covariant derivative covariant, we have to transform the gauge field $A \to -iA$ so that the covariant derivative transforms as $D_t X^a \to -iD_\tau X^a$. We define the Euclidean action without the $i$ as

$$iS_{\mathrm{Euc}} = i \int \mathrm{d}\tau \, \mathrm{Tr} \left( \frac{1}{2\lambda} \left\{ (D_\tau X^a)^2 + \frac{1}{2} \left[ X^a, X^b \right]^2 \right\} + i\theta^T D_\tau \theta - \theta^T \gamma_a \left[ X^a, \theta \right] \right). \tag{235}$$

Now we have a Euclidean metric as in the truncated super-membrane case where we only have the transverse degrees of freedom left from the super-membrane theory.

### 10.1.3 *Yang-Mills - Super-membrane correspondence*

Setting the gauge field to zero, $A = 0$ we get the action (we drop the *Euc* subscript on the action $S$ and rename $\tau$ to $t$)[6]

$$S = \int \mathrm{d}t \, \mathrm{Tr} \left( \frac{1}{2\lambda} \left\{ (\partial_t X^a)^2 - \frac{1}{2} \left[ X^a, X^b \right]^2 \right\} + i\theta^T \partial_t \theta - \theta^T \gamma_a \left[ X^a, \theta \right] \right), \tag{236}$$

this gives us the Hamiltonian density (where we re-scaled the fermionic fields $\theta \to \frac{1}{\sqrt{2\lambda}}\theta$ and changed the matrices to Hermitian matrices)

$$\mathscr{H} = \frac{1}{2\lambda} \mathrm{Tr} \left( \mathscr{P}^a \mathscr{P} + \frac{1}{2} \left[ X^a, X^b \right]^2 - \bar{\theta} \gamma^a \left[ X_a, \theta \right] \right) \tag{237}$$

5 Which transforms $-dt^2 \to +d\tau^2$, making the metric positive definite, i.e. $X^a \delta_{ab} X^b \geq 0$, where the equality only holds when $X^a = 0$.

6 Note that we changed the sign for the potential energy. In the action (235) the $X^a$ are anti-Hermitian since we let $A_\mu \to \frac{i}{g} A_\mu$. The commutator of anti-Hermitian matrices are also anti-Hermitian. So if we have (like for the potential energy) $\mathrm{Tr}(M^2)$, with $M$ being anti-Hermitian, then we can write it as $\mathrm{Tr}(M^2) = -\mathrm{Tr}((iM)^2)$, with $iM$ being Hermitian. Since the eigenvalues of an Hermitian matrix is real and we take the trace of the square it, it follows that $\mathrm{Tr}(M^2) \leq 0$. Changing the anti-Hermitian matrices to Hermitian matrices yields (236).

which is the same as for the truncated super-membrane (158) and our starting point for M(atrix) theory in (161).

## 10.2 EXPANSION OF THE FIELDS

The background field method allows us to keep all the terms gauge invariant, so we will start from the gauge invariant Euclidean Lagrangian (scaling $\lambda \to \frac{1}{2}\lambda$, dropping the *Euc* subscript on the action $S$ and rename $\tau$ to $t$ from (235))

$$S = \int \mathrm{d}t\, \mathrm{Tr}\Bigg( \frac{1}{\lambda}\Big\{ (D_t X^a)^2 + \frac{1}{2}\left[X^a, X^b\right]^2 \Big\} + i\theta^T D_t \theta - \theta^T \gamma_a \left[X^a, \theta\right] + \frac{1}{\lambda}(\partial_t A + \sqrt{\lambda}\left[B^a, Y_a\right])^2 \Bigg) + S_{\text{ghosts}} \tag{238}$$

where we added the background gauge fixing term[7] $(D^\mu A_\mu)^2$ and ghosts, which will be given in explicit form later.

We expand the fields in a classical background $B^a$ and quantum fluctuations $Y^a$ according to $X^a = B^a + \sqrt{\lambda} Y^a$, where we introduced a coupling constant to count the loop order, we also re-scale $A \to \sqrt{\lambda} A$.

Expanding the bosonic+gauge-fixing part of the Lagrangian

$$L_{\text{Bosonic+Gauge}} = \frac{1}{\lambda}\mathrm{Tr}\Bigg( (D_t X^a)^2 + \frac{1}{2}\left[X^a, X^b\right]^2 + (\sqrt{\lambda}\partial_t A + \sqrt{\lambda}\left[B^a, Y_a\right])^2 \Bigg) \tag{239}$$

and only keeping terms of the order $\lambda^n$ with $n \geq 0$, since the rest must be fulfilled by the background, we find for the quantum fluctuations $Y$

$$L_Y = \mathrm{Tr}\Bigg( (\partial_t Y^a)^2 + \left[B^a, Y^a\right]^2 + \left[B^a, Y^b\right]^2 + \left[B^a, B^b\right]\left[Y^a, Y^b\right] + \left[B^a, Y^b\right]\left[Y^a, B^b\right] + 2\sqrt{\lambda}\left[B^a, Y^b\right]\left[Y^a, Y^b\right] + \frac{\lambda}{2}\left[Y^a, Y^b\right]^2 \Bigg) \tag{240}$$

and for the gauge field $A$

$$L_A = \mathrm{Tr}\Bigg( (\partial_t A)^2 + \left[A, B^a\right]^2 + 4\partial_t B^a \left[A, Y^a\right] + 2\sqrt{\lambda}\partial_t Y^a \left[A, Y^a\right] + 2\sqrt{\lambda}\left[A, B^a\right]\left[A, Y^a\right] + \lambda\left[A, Y^a\right]^2 \Bigg). \tag{241}$$

7 To get from $(D^\mu A_\mu)^2$ to $(\partial_t A + \sqrt{\lambda}\left[B^a, Y_a\right])^2$ we have used the dimensional reduction, Wick rotated and let $A \to -iA$ to keep the covariant derivative covariant, as before. $B$ and $Y$ are from $X^a = B^a + \sqrt{\lambda} Y^a$ as will be explained below.

We write the action in terms of the $U(2)$ generators (which are just the Pauli matrices divided by 2)

$$\begin{aligned} A =& \frac{1}{2}\left(A_0 \mathbb{I}_{2\times 2} + A_i \sigma^i\right), \quad X^a = \frac{1}{2}\left(X_0^a \mathbb{I}_{2\times 2} + X_i^a \sigma^i\right), \\ \theta =& \frac{1}{2}\left(\theta_0 \mathbb{I}_{2\times 2} + \theta_i \sigma^i\right), \end{aligned} \tag{242}$$

with $\sigma^i$ being the anti-hermitian ($\left(\sigma^i\right)^\dagger = -\sigma^i$) Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{243}$$

with the identities

$$\left[\sigma^i, \sigma^j\right] = 2\epsilon_{ijk}\sigma^k, \quad \epsilon_{123} = +1, \quad \sigma^i\sigma^j = \epsilon_{ijk}\sigma^k - \delta_{ij}\mathbb{I}_{2\times 2}. \tag{244}$$

We will not consider the zero-modes any further since they only describe the center of mass motion which we always can take to zero if we choose such a frame of reference, so it is not important.

We choose the background field $B^a$

$$B^1 = \frac{vt}{2}\sigma^3, \quad B^2 = \frac{b}{2}\sigma^3, \quad B^{a\neq 1,2} = 0, \quad \vec{r} = \vec{v}t + \vec{b} \tag{245}$$

which describe two D0-branes traveling on straight lines with relative velocity $\vec{v}$ and impact parameter $\vec{b}$. We choose $\vec{v}$ to be orthogonal to $\vec{b}$ which always can be accomplished by a shift in $t$.

Plugging in this in the Lagrangians (240) and (241) we arrive at

$$\begin{aligned} L_Y =& \frac{1}{2}Y_1^a(\partial_t^2 - r^2)Y_1^a + \frac{1}{2}Y_2^a(\partial_t^2 - r^2)Y_2^a + \frac{1}{2}Y_3^a\partial_t^2 Y_3^a \\ & - \sqrt{\lambda} r^a Y_i^b Y_j^a Y_k^b \epsilon^{3ix}\epsilon^{ikx} - \frac{\lambda}{4}Y_i^a Y_j^b Y_k^a Y_l^b \epsilon^{ijx}\epsilon^{klx}, \end{aligned} \tag{246}$$

$$\begin{aligned} L_A =& \frac{1}{2}A_1(\partial_t^2 - r^2)A_1 + \frac{1}{2}A_2(\partial_t^2 - r^2)A_2 + \frac{1}{2}A_3\partial_t^2 A_3 - 2v^a A_i Y_j^a \epsilon^{ij3} \\ & + \sqrt{\lambda} r^a A_i A_j Y_k^a \epsilon^{ix3}\epsilon^{jkx} - \sqrt{\lambda}\partial_t Y_i^a A_j Y_k^a \epsilon^{ijk} - \frac{\lambda}{2}A_i A_j Y_k^a Y_l^a \epsilon^{ijx}\epsilon^{klx}. \end{aligned} \tag{247}$$

As for the fermionic fields the expansion gives us

$$\begin{aligned} L_\theta =& i\Big[i\theta_+^T(\partial_t - \gamma^1 vt - \gamma^2 b)\theta_- + \frac{i}{2}\theta_3\partial_t\theta_3 + \sqrt{\lambda}\theta_+^T A_3\theta_- - \sqrt{\lambda}\theta_+ Y_3^a\gamma^a\theta_- \\ & + \sqrt{\frac{\lambda}{2}}\left(\theta_+^T\theta_3(iA_2 - A_1) + \theta_-^T\theta_3(iA_2 + A_1)\right) \\ & + \sqrt{\frac{\lambda}{2}}\left(\theta_+^T\gamma^a\theta_3(Y_1^a - iY_2^a) + \theta_3^T\gamma^a\theta_-(Y_1^a + iY_2^a)\right)\Big]. \end{aligned} \tag{248}$$

with $\theta_\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm i\theta_2)$. Varying the gauge fixing term with respect to the gauge transformation gives us the ghosts, we won't derive them here, we'll just quote them from [25]

$$S_{\text{ghost}} = \int \mathrm{d}t \Big( C_1^*(-\partial_t^2 + r^2)C_1 + C_2^*(-\partial_t^2 + r^2)C_2 - C_3^*\partial_t^* C_3 + \sqrt{\lambda}\epsilon^{ijk}\partial_t C_i^* C_j A_k - \sqrt{\lambda}\epsilon^{i3x}\epsilon^{kjx} B_3^a C_i^* C_j Y_k^a \Big), \tag{249}$$

where $C$ is a complex Grassmann variable, we have two complex bosons with mass $r$ and one massless.

### 10.2.1 *Masses of fields*

First of all we have to diagonalize the mass matrix in (247) given by the term $-2v^a A_i Y_j^a \epsilon^{ij3}$ (and the $r^2$ terms for the relevant fields) , this gives us two mass matrices

$$-\left(Y_1^1, A_2\right)\begin{pmatrix} r^2 & -2v \\ -2v & r^2 \end{pmatrix}\begin{pmatrix} Y_1^1 \\ A_2 \end{pmatrix}, \quad -\left(Y_2^1, A_1\right)\begin{pmatrix} r^2 & 2v \\ 2v & r^2 \end{pmatrix}\begin{pmatrix} Y_2^1 \\ A_1 \end{pmatrix} \tag{250}$$

which both have eigenvalues $r^2 \pm 2v$. So there are two bosons with mass $r^2 + 2v$ and two bosons with mass $r^2 - 2v$. We have 16 bosons left in (246) from $Y_1^{a\neq 1}$ and $Y_2^{a\neq 1}$ which all have mass $r^2$ each. We have 9 massless bosons from $Y_3^a$ and 1 massless boson from $A_3$.
The ghost are complex and thus we have 2 fermions of mass $r^2$ from $C_1$ and 2 from $C_2$, we also have 2 massless from $C_3$, the ghosts are Grassmann variable and anti-commute so we get a weight factor of $-1$.
For $\theta_+$ and $\theta_-$ we have to diagonalize the mass matrix $\gamma^1 vt + \gamma^2 b$. There is however a trick we could use, where we rather then diagonalize the mass matrix calculate it's square. The wave operator for the fermions $\theta_+$ and $\theta_-$ is given by (ignoring the interactions) $\mathscr{O}_\theta = \partial_t - \gamma^1 vt - \gamma^2 b$ and it's Hermitian conjugate by $\mathscr{O}_\theta^\dagger = -\partial_t - \gamma^1 vt - \gamma^2 b$ since the gamma matrices in $SO(9)$ is real and the Hamiltonian is Hermitian and proportional to $i\partial_t$, which makes $\partial_t$ anti-Hermitian. The effective action for these fermions is given by

$$\Gamma_\theta^{(1)} = \frac{1}{2}\mathrm{Tr}(\ln(\mathcal{O}_\theta)), \tag{251}$$

but noting that $\ln(\mathscr{O}_\theta \mathscr{O}_\theta^\dagger) = \ln(\mathscr{O}_\theta) + \ln(\mathscr{O}_\theta^\dagger) = 2\ln(\mathscr{O}_\theta)$, we could rather calculate

$$\begin{aligned}\Gamma_\theta^{(1)} =& \frac{1}{4}\ln(\mathscr{O}_\theta \mathscr{O}_\theta^\dagger) \\ =& \frac{1}{4}\ln(-\partial_t^2 - \left[\partial_t, \gamma^1 vt + \gamma^2 b\right] + r^2) \\ =& \frac{1}{4}\ln(-\partial_t^2 - \gamma^1 v + r^2).\end{aligned} \tag{252}$$

This is still not diagonal, but since $\mathrm{Tr}(\gamma^1) = 0$ and $(\gamma^1)^2 = \mathbb{I}$, we know that half of the 16 complex component of both $\theta_\pm$ are $m^2 = r^2 - v$ and the other 16 are of mass $m^2 = r^2 + v$, we will have to weight this with a factor $-\frac{1}{2}$ to compensate for calculating $2\Gamma^{(1)}_\theta$ rather then $\Gamma^{(1)}_\theta$ when we add all the field together in the next section. We do the same trick for $\theta_3$ to easily match it's propagator the that of an harmonic oscillator.

We summaries the masses in table (1) below.

## 10.3 CALCULATION OF THE FIRST LOOP

The masses found in the previous section is collected in the table below, where we also present the corresponding factor to the heat kernel as calculated in subsection 8.2.2.

| Real components | $(\mathrm{Mass})^2$ | Weight factor | Factor to heat kernel |
|---|---|---|---|
| 16 | $r^2$ | $1$ | $16e^{-\chi r^2}$ |
| 2 | $r^2 - 2v$ | $1$ | $2e^{-\chi r^2}e^{2v}$ |
| 2 | $r^2 + 2v$ | $1$ | $2e^{-\chi r^2}e^{-2v}$ |
| 10 | $0$ | $1$ | $10$ |
| 4 | $r^2$ | $-1$ | $-4e^{-\chi r^2}$ |
| 2 | $0$ | $-1$ | $-2$ |
| 16 | $r^2 - v$ | $-\frac{1}{2}$ | $-8e^{-\chi r^2}e^{v\chi}$ |
| 16 | $r^2 + v$ | $-\frac{1}{2}$ | $-8e^{-\chi r^2}e^{-v\chi}$ |
| 16 | $0$ | $-\frac{1}{2}$ | $-8$ |

Table 1: Mass and factor to the heat kernel table.

The sum of all of these terms gives us the effective action

$$\begin{aligned}\Gamma^{(1)} &= -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{\mathrm{d}\chi}{\chi} e^{-\chi r^2}\left[12 + 4\cosh(2\chi v) - 16\cosh(\chi v)\right] h(t,t',\chi)\\ &= -\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\int_0^\infty \frac{\mathrm{d}\chi}{\chi} e^{-\chi r^2}\left[12 + 4\cosh(2\chi v) - 16\cosh(\chi v)\right] h(t,t,\chi)\\ &= -\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\frac{v^4}{\sqrt{\pi}}\underbrace{\int_0^\infty \mathrm{d}\chi e^{-\chi r^2}\chi^{\frac{5}{2}}}_{\frac{15\sqrt{\pi}}{8r^7}} + \mathcal{O}(v^6)\\ &= \int_{-\infty}^{\infty}\mathrm{d}t\left(-\frac{15v^4}{16r^7}\right) + \mathcal{O}(v^6)\end{aligned} \tag{253}$$

which agrees with super-gravity in 11D [26, 20, 25, 27].

# 11 SUMMARY AND CONCLUSIONS

We wrote down the action for a membrane in space-time and found a field theory on its world-volume. It turned out that there was ways to truncate this field theory to arrive at a regularized theory, namely M(atrix) theory. This was however only one of the ways to arrive at M(atrix) theory, another one was the dimensional reduction of a $9+1$ dimensional super Yang-Mills theory to a $0+1$ dimensional theory. This path was in some way more general since this reduced in a particular gauge choice, $A = 0$, to the membrane point of arrival. It has been pointed out[20] that M-theory should in its low energy limit reduce to super-gravity. We found that the M(atrix) theory does this! Therefore both the membrane theory and the $9+1$ Yang-Mills theory become especially interesting since both of these reduces to M(atrix) theory. M-theory is an ultimate theory of everything and since membrane theory is a candidate for this theory, as we have proved, it is also a candidate for the ultimate theory of everything. This thesis have shown that in indeed reduces to the right limit, at least in first order, and that it is a multi-particle theory, as an ultimate theory of everything has to be. This theory is however computationally hard and therefore only little bits of the theory have been uncovered.

# Part VI

# APPENDIX

In this appendix we take a look at the Dirac theory which is a generalization of the Hamiltonian formalism to incorporate constrained systems and systems with gauge freedom. We will also take a look at the Dirac matrices, how to raise and lower the indices and give explicit representations to the ones that we use. In the last section we derive the propagator for a free particle and the propagator for the harmonic oscillator.

# A DIRAC THEORY

Here we will give a short overview of the generalization of the Hamiltonian formalism for systems with gauge freedoms and constraint systems, due to Dirac [28]. $q_n$ are the generalized coordinates and $\dot{q}_n = \frac{dq_n}{dt}$ are the generalized velocities. In this chapter we consider theories in Euclidean space so that $A^b = A_b$, furthermore we sum over repeated indices.

## A.1 LAGRANGIAN

The Lagrangian $L$ is defined as

$$S = \int L(q_n, \dot{q}_n, t)\, dt. \tag{254}$$

The equations of motion are given by the Euler-Lagrange equations which one gets by varying $S$ with respect to $q_n$ and $\dot{q}_n$ and demanding the action to be stationary, $\delta S = 0$. This yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n}. \tag{255}$$

## A.2 HAMILTONIAN

The canonical momenta $p_n$ are defined as

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \tag{256}$$

From the above equation (256) we get "primary constraints". That is, functions of only the coordinates and momenta

$$\phi_m(q, p) \approx 0, \tag{257}$$

with $m = 1, \ldots, M$. We write these as weakly zero($\approx$) since these only hold if the equations of motion are satisfied. No weak equations are to be used before evaluation of any derivatives in the theory. These equations are not invertible for $\dot{q}_n$ since they are independent of them. If they are invertible they are not primary constraints and they will be used to replace $\dot{q}_n$ in favor of $p_n$ when we build the Hamiltonian.

Now we build the Hamiltonian the usual way, through the Legendre transformation

$$H(p, q, t) = p^n \dot{q}_n - L(\dot{q}, q, t). \tag{258}$$

But a more general Hamiltonian would be

$$H^* = H + c^m \phi_m \tag{259}$$

with $c^m = c^m(q,p)$ being arbitrary functions of coordinates and momenta. Since the constraints are zero on shell, $\phi_m \approx 0$, we get the same dynamics for $H^*$ as for $H$.

Using the more general $H^*$ instead of $H$ we get the Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \tag{260}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - u^m \frac{\partial \phi_m}{q_n}. \tag{261}$$

We can rewrite these equations with "Poisson brackets" $\{\cdot,\cdot\}_p$, defined as

**Definition 28. Poisson bracket**

$$\{f,g\}_p = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p^n} - \frac{\partial f}{\partial p^n}\frac{\partial g}{\partial q_n} \tag{262}$$

*Notice the sum over n.*

or for the continuous case the Poisson bracket becomes a functional

$$\{f,g\}_p = \int \left( \frac{\delta f(x)}{\delta q(x)} \frac{\delta g(x)}{\delta p(x)} - \frac{\delta f(x)}{\delta p(x)} \frac{\delta g(x)}{\delta q(x)} \right) \mathrm{d}x. \tag{263}$$

◇

Since the Poisson brackets are formed from derivatives these must be calculated before one uses any of the primary constraints (257).

*Claim* 9. Poisson bracket properties.

From the definition we get the relations

$$\{f,g\}_p = -\{g,f\}_p, \tag{264}$$

$$\{f_1+f_2,g\}_p = \{f_1,g\}_p + \{f_2,g\}_p, \tag{265}$$

$$\{f_1 f_2,g\}_p = f_1\{f_2,g\}_p + \{f_1,g\}_p f_2 \tag{266}$$

and the Jacobi identity

$$\{f,\{g,h\}\}_p + \{g,\{h,f\}\}_p + \{h,\{f,g\}\}_p = 0. \tag{267}$$

◇

For any function $f = f(q,p)$ we have

$$\dot{f} = \{f,H'\} \approx \{f,H\} + u^m \{f,\phi_m\}_p \tag{268}$$

with $H' = H + u^m \phi_m$. Notice that $\{f,u^m\}_p \phi_m \approx 0$ since $\phi_m \approx 0$ (we have dropped this term in the above equation).

The constraints are to hold for all times. So we have to check that the time-evolution of the constraints are weakly zero

$$\dot{\phi}_m \approx 0 \approx \{\phi_m, H'\}_p. \tag{269}$$

We can split this into 4 different cases:

1. We get an inconsistency, e.g. $1 = 0$, which would follow from the Lagrangian $L = q$. This would mean that the Lagrangian equations of motion would be inconsistent. I.e. we have described the system wrong.

2. We get that this is identically satisfied $0 \approx 0$. Where we may have used some of the primary constraints (257) after we worked out the derivatives.

3. We could get another function of only coordinates and momenta $\chi_k(q, p) \approx 0$, i.e no dependence of $u^m$. This would be called a "secondary constraint". We have to add this constraint to $H'$ and do the checks again, until this case no longer appear.

4. We could get a function that involves $u^m$, i.e $\varphi_\ell(q, p, u) \approx 0$. Each $\varphi$ specifies one of the $u^m$.

For our purposes we will treat secondary constraints $\chi$ as primary constraints $\phi$. So we add all of these to a new $\phi$

$$\phi_j = \{\{\phi_m\} \cup \{\chi_k\}\} \approx 0. \tag{270}$$

with $j = 1, \ldots, J$ and $J = \#\{\phi_m\} + \#\{\chi_k\}$.

Solving for $u^m$ from the 4:th case above $\varphi_\ell(q, p, u) \approx 0$, i.e.

$$\{\phi_\ell, H\}_p + u^m \{\phi_\ell, \phi_m\}_p \approx 0. \tag{271}$$

We get (assuming that the Lagrangian equations are not inconsistent)

$$u^m = U^m(q, p) + V^m(q, p) \tag{272}$$

with $U^m$ the particular solution and $V^m$ the general solution to the homogeneous equation

$$V^m \{\phi_\ell, \phi_m\}_p \approx 0. \tag{273}$$

So the most general solution is

$$u^m = U^m(q, p) + v^a(t) V^m_a(q, p) \tag{274}$$

where $a$ runs from 1 to the number of nonphysical degrees of freedom, which are the number of $u^m$ minus the number of equations for $u^m$. $v^a(t)$ are arbitrary functions of time.

The total Hamiltonian becomes

$$H_T = H + U^m \phi_m + v^a V^m_a \phi_m. \tag{275}$$

The equations of motions for all gauge invariant quantities, i.e. physical quantities are

$$\dot{f} \approx \{f, H_T\}_p + \frac{\partial f}{\partial t}. \tag{276}$$

## A.3 DIRAC BRACKETS

To canonically quantize the system we need to generalize the Poisson brackets to "Dirac brackets". We start of with some terminology.

If $\left\{f,\phi_j\right\}_p \approx 0$, $\forall j$, then $f$ is called "first-class" else $f$ is called "second-class".

The phase space variables in the theory that are weakly equal to zero are combinations of $\phi_j$ since these are the variables we defined as weakly zero and everything else has been built up from them. So if $f$ is first class then the bracket can be written as

$$\left\{f,\phi_j\right\}_p = r_{jj'}\phi_{j'} \tag{277}$$

for some $r_{jj'}$. One can then prove that if $f$ and $g$ are both first-class so is $\{f,g\}_p$, i.e. $\{f,\phi_i\}_p \approx 0$ and $\left\{g,\phi_j\right\}_p \approx 0 \Rightarrow \left\{\{f,g\}_p,\phi_k\right\}_p \approx 0$, $\forall i,j,k$ . Although the proof is simple we do not prove it, see [28] page 18.

If we quantize the theory using the correspondence principle $\{\cdot,\cdot\}_p \rightarrow \frac{1}{i\hbar}[\hat{\cdot},\hat{\cdot}]$ without doing anything about the second-class constraints, denoted by $\tilde{\phi}_i$ , we see that

$$\left\{\tilde{\phi}_i,\tilde{\phi}_j\right\}_p = M_{ij} \rightarrow \frac{1}{i\hbar}\left[\hat{\tilde{\phi}}_i,\ \hat{\tilde{\phi}}_j\right] = \hat{M}_{ij} \tag{278}$$

where the lhs of the quantum theory vanishes on any state by definition of the constraints $\phi_k$, while the rhs do not vanish. This leads to inconsistencies. We need new brackets that respects the constraints in the quantized theory. They should

1. be bi-linear
2. anti-symmetric
3. satisfy Jacobi identity (267)
4. reduce to Poisson brackets when there are no constraints
5. for any constraint $\phi_i$ and any quantity $f$ ($f$ is allowed to be $\phi_k$) we must have $\{\phi_i,f\} \approx 0$.

We define the "Dirac bracket"

**Definition 29. Dirac bracket**

$$\{f,g\}_D = \{f,g\}_p - \{f,\tilde{\phi}_i\}\, M^{-1}_{ij}\, \{\tilde{\phi}_j,g\}\,. \tag{279}$$

◇

Dirac proved that there always exists an inverse $M^{-1}$ if $M$ exists.

## A.4 PROCEDURE

1. Get canonical momenta $p_n = \frac{\partial L}{\partial \dot{q}_n}$.
2. Get the primary constraints from $p_n - \frac{\partial L}{\partial \dot{q}_n} = 0$.
3. Build $H' = H + u^m \phi_m$, with $H = p_n \dot{q}^n - L$.
4. Check for inconsistencies, second constraints, equations for $u^m$ or $0 \approx 0$. $\{\phi_j, H'\} = \dot{\phi}_j$
5. Repeat step 3 and 4 until the only thing we get are $0 \approx 0$.
6. Solve equations for $u^m$ to get $u^m = U^m + v^a V_a^m$.
7. Build the total Hamiltonian $H_T = H + U^m \phi_m + v^a(t) V_a^m \phi_m$. Now we have the equations of motion $\dot{f} \approx \{f, H_T\} + \frac{\partial f}{\partial t}$ for all physical quantities.
8. Check for second-class constraints $\{\tilde{\phi}_i, \tilde{\phi}_j\}_p = M_{ij}$.
9. Build Dirac bracket $\{f, g\}_D = \{f, g\}_p - \{f, \tilde{\phi}_i\} M_{ij}^{-1} \{\tilde{\phi}_j, g\}$.
10. Quantize $\{\cdot, \cdot\}_p \to \frac{1}{i\hbar} [\hat{\cdot}, \hat{\cdot}]$.

*Or $p_n - \frac{\partial L}{\partial \dot{q}_n} \approx 0$ to remind our self that they only hold on-shell.*

# B

# GAMMA MATRICES AND SPINORS IN 11-D

The gamma matrices $(\Gamma^\mu)^\beta{}_\alpha$ are $32 \times 32$ matrices and are defined through

$$\{\Gamma^\mu, \Gamma^\nu\} = \Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}\mathbb{I}_{32\times 32}, \tag{280}$$

where $\mathbb{I}_{32\times 32}$ is the identity matrix which is $32 \times 32$, this is often left in-explicit, as well as the spinor indices $\alpha, \beta$. The Lorentz-indices $\mu, \nu$ ranges from 0 to 10. The algebra in (280) is called a "Clifford-algebra".

## B.1 REPRESENTATIONS OF GAMMA MATRICES

There are procedures to construct the representations (denoted by $\doteq$), see for e.g. [17]. We will however just list a set of matrices that fulfills the Clifford algebra (280).

$$(\Gamma^0)^\beta{}_\alpha \doteq \begin{pmatrix} 0 & +\mathbb{I}_{16\times 16} \\ -\mathbb{I}_{16\times 16} & 0 \end{pmatrix}, \qquad (\Gamma^{10})^\beta{}_\alpha \doteq \begin{pmatrix} 0 & +\mathbb{I}_{16\times 16} \\ +\mathbb{I}_{16\times 16} & 0 \end{pmatrix},$$
$$(\Gamma^I)^\beta{}_\alpha \doteq \begin{pmatrix} -\gamma^I & 0 \\ 0 & +\gamma^I \end{pmatrix}. \tag{281}$$

Here the gamma matrices $\gamma^I$ are of dimension $16 \times 16$ for 9 dimensional space-time, with $I = 1, 2, \ldots 9$.

### B.1.1 *Representations in the light-cone coordinates*

In light-cone coordinates we build two different matrices from $\Gamma^0$ and $\Gamma^{10}$, namely

$$\Gamma^\pm = \frac{1}{\sqrt{2}}\left(\Gamma^0 \pm \Gamma^{10}\right). \tag{282}$$

The new matrices are then

$$(\Gamma^+)^\beta{}_\alpha \doteq \sqrt{2}\begin{pmatrix} 0 & +\mathbb{I}_{16\times 16} \\ 0 & 0 \end{pmatrix}, \qquad (\Gamma^-)^\beta{}_\alpha \doteq \sqrt{2}\begin{pmatrix} 0 & 0 \\ -\mathbb{I}_{16\times 16} & 0 \end{pmatrix},$$
$$(\Gamma^I)^\beta{}_\alpha \doteq \begin{pmatrix} -\gamma^I & 0 \\ 0 & +\gamma^I \end{pmatrix}. \tag{283}$$

The light-cone matrices obeys

$$\left\{\Gamma^\pm, \Gamma^I\right\} = 0, \qquad \{\Gamma^+, \Gamma^-\} = -2, \qquad \left\{\Gamma^I, \Gamma^{I'}\right\} = 2\eta^{II'}.$$

## B.2 THE DIFFERENT GAMMA MATRICES

We can construct new matrices that also fulfill the Clifford-algebra. Here is a table of these matrices

| Object | Spinor-index | | Lorentz-index | | # of obj. |
|---|---|---|---|---|---|
| | Sym | Type | Sym | Type | |
| $C_{\alpha\beta}$ | A | Spinor | / | Scalar | $\binom{11}{0} = 1$ |
| $(\Gamma^{\mu})_{\alpha\beta}$ | S | Spinor | / | Vector | $\binom{11}{1} = 11$ |
| $(\Gamma^{\mu\nu})_{\alpha\beta} = (\Gamma^{[\mu}\Gamma^{\nu]})_{\alpha\beta}$ | S | Spinor | A | Bi-vector | $\binom{11}{2} = 55$ |
| $(\Gamma^{\mu\nu\rho})_{\alpha\beta} = (\Gamma^{[\mu}\Gamma^{\nu}\Gamma^{\rho]})_{\alpha\beta}$ | A | Spinor | A | Tri-vector | $\binom{11}{3} = 165$ |
| $(\Gamma^{\mu\nu\rho\sigma})_{\alpha\beta}$ | A | Spinor | A | Quad-vector | $\binom{11}{4} = 330$ |
| $(\Gamma^{\mu\nu\rho\sigma\xi})_{\alpha\beta}$ | S | Spinor | A | Penta-vector | $\binom{11}{5} = 462$ |
| Total:$1024 = 32^2 = 2^{10}$ | | | | | |

Table 2: Gamma matrices

where $(\Gamma^{\mu\nu\rho\sigma})_{\alpha\beta}$ and $(\Gamma^{\mu\nu\rho\sigma\xi})_{\alpha\beta}$ are also anti-symmetrized in the Lorentz-indices the same way as the others. Gamma matrices with more than 5 Lorentz-indices can be built as a linear combination of the ones listed in the table.

### B.2.1 *Raise and lower spinor indices*

We define the charge conjugation matrix[1]

$$C^{\alpha\beta} \doteq \begin{pmatrix} 0 & +\mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \qquad \text{and its inverse } C_{\alpha\beta} \doteq \begin{pmatrix} 0 & -\mathbb{I} \\ +\mathbb{I} & 0 \end{pmatrix}. \tag{284}$$

This enables us to raise and lower spinor indices

$$\begin{aligned} \theta^{\alpha} &= C^{\alpha\beta}\theta_{\beta} = \theta_{\beta}C^{\beta\alpha}, \\ \theta_{\alpha} &= \theta^{\beta}C_{\beta\alpha} = -\, C_{\alpha\beta}\theta^{\beta}, \end{aligned} \tag{285}$$

with $\theta$ a spinor.

## B.3 SPINORS

We denote the $32 \times 1$ spinors by $\theta \doteq \begin{pmatrix} \psi \\ \chi \end{pmatrix}$, where $\psi$ and $\chi$ are $16 \times 1$ blocks of the spinors. We denote it's dual $1 \times 32$ by $\bar{\theta}$. Since we are

1 If one chooses a particular basis then the charge conjugation matrix $C$ and the first gamma matrix $\Gamma^0$ will numerically be the same (they are numerically the same in our basis). So $\Gamma^0$ is often chosen as $C$, but in fact there is really no good reason to do that choice, they do not even have the same index structure.

in 11 dimensional space-time we can choose these to be real representations [17]. Spinors need to fulfill $\bar{\theta}\theta = \chi\psi - \psi\chi$, so we choose to represent the dual spinor by $\bar{\theta} \doteq (\chi, -\psi)$.

$$\theta^\alpha = \theta \doteq \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \qquad \theta_\alpha = \theta^\beta C_{\beta\alpha} = \bar{\theta} \doteq (\chi, -\psi),$$

## B.4 GAUGE FIXING

The gauge fixing condition $\Gamma^+\theta = 0$ in this representation implies

$$\Gamma^+\theta = 0 \doteq \begin{pmatrix} 0 & \mathbb{I}\sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \sqrt{2} \begin{pmatrix} \chi \\ 0 \end{pmatrix} = 0 \tag{286}$$

that $\chi = 0$. This implies something for the dual vector, namely

$$\bar{\theta}\Gamma^+ = 0 \doteq (\chi, -\psi) \begin{pmatrix} 0 & \mathbb{I}\sqrt{2} \\ 0 & 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0, & \chi \end{pmatrix} = 0. \tag{287}$$

# C

# PROPAGATORS AND HEAT KERNELS

Since we use these propagators in Euclidean space we will do them for Euclidean space, so we make a Wick rotation from Minkowski space by[1] $t \to it$. Through out we will slice the time interval $T$ into $N$ equally big slices $\epsilon_i = t_i - t_{i-1}$ and use the path integral measure

$$\mathrm{D}x = \lim_{\epsilon_i \to 0} \sqrt{\frac{m}{2\pi\epsilon_N}} \prod_{i=1}^{N-1} \sqrt{\frac{m}{2\pi\epsilon_i}} \mathrm{d}x_i. \tag{288}$$

## C.1 FREE PARTICLE

The path integral for a free particles is

$$Z_{\mathrm{FP}} = \int \mathrm{D}x \, \exp\left( -\int_0^T \mathrm{d}t \, \frac{m}{2}\dot{x}^2 \right) \tag{289}$$

with the Wick rotated Hamiltonian $H = -\frac{p^2}{2m}$. We calculate this using brackets, with $x_T = x(T)$ and $x_0 = x(0)$,

$$\begin{aligned}
Z_{\mathrm{FP}} =& \langle x_T, T | x_o, 0 \rangle \\
=& \int \mathrm{d}p \, \langle x_T | p \rangle \langle p | e^{HT} | x_o \rangle \\
=& \int \mathrm{d}p \, \exp(-\frac{p^2 T}{2m}) \langle x_T | p \rangle \langle p | x_o \rangle \\
=& \frac{1}{2\pi} \int \mathrm{d}p \, \exp(-\frac{p^2 T}{2m} + ip(x_t - x_0)) \\
=& \int \mathrm{d}p \, \frac{1}{2\pi} \exp(-\frac{T}{2m}[p - \frac{im}{T}(x_t - x_0)]^2 + \frac{m^2}{T^2}(x_t - x_0)^2) \\
=& \sqrt{\frac{m}{2\pi T}} \exp(-\frac{m}{2T}(x_t - x_0)^2).
\end{aligned} \tag{290}$$

1 Note that if we calculate the propagator in Minkowski space and make a Wick rotation we get the heat kernel instead.

## C.2 HARMONIC OSCILLATOR

We will follow the elegant derivation presented in [29], we will provide it here for completeness. The path integral for the imaginary time harmonic oscillator is given by

$$\begin{aligned}
Z_{\rm HO} &= \int \mathrm{D}x \exp\left(-\int_0^T \mathrm{d}t \left[\frac{m}{2}\dot{x}^2 + \frac{m\omega^2}{2}x^2\right]\right) \\
&= \int \mathrm{D}x \exp\left(-\int_0^T \mathrm{d}t \left\{\frac{m}{2}\left[\dot{x}+\omega x\right]^2 - \frac{m\omega}{2}\frac{dx^2}{dt}\right\}\right) \\
&= \exp\left(\frac{m\omega}{2}(x_T^2 - x_0^2)\right)\int \mathrm{D}x \exp\left(-\int_0^T \mathrm{d}t\, \frac{m}{2}\left[\dot{x}+\omega x\right]^2\right).
\end{aligned} \tag{291}$$

Now we make the transformation

$$\begin{cases} x(t) = & z(t)e^{-\omega t} \\ dx_i = & e^{-\omega t_i}dz_i \end{cases} \tag{292}$$

and note that $\prod_{i=1}^{N-1} e^{-\omega t_i} = e^{-\frac{\omega}{2}(N-1)(t_1+t_{N-1})} = e^{-\frac{\omega}{2}NT+\frac{\omega T}{2}}$

$$Z_{HO} = \exp\left(\frac{m\omega}{2}(x_T^2 - x_0^2)\right)\int \mathrm{D}z \exp(-\frac{\omega}{2}NT + \frac{\omega T}{2}) \exp\left(-\int_0^T \mathrm{d}t\, \frac{m}{2}e^{-2\omega t}\dot{z}^2\right). \tag{293}$$

Now we use time reparametrization to absorb the exponential in the integrand for $t$. We take $\bar{t} = \frac{e^{2\omega t}}{2\omega}$, which will change the integration limits $0 \to t_a = \frac{1}{2\omega}$, $T \to t_b = \frac{e^{2\omega T}}{2\omega}$ and thus

$$\begin{aligned}
\lim_{\epsilon_i\to 0}\epsilon_i &\to \lim_{\bar{\epsilon}_i\to 0}\bar{\epsilon}_i \\
&= \lim_{\epsilon_i\to 0}\frac{1}{2\omega}(e^{2\omega t_i} - e^{2\omega t_{i-1}}) \\
&= \lim_{\epsilon_i\to 0}\frac{e^{2\omega t_i^*}}{\omega}\sinh(\omega\epsilon_i) \\
&\simeq \lim_{\epsilon_i\to 0}\epsilon_i e^{2\omega t_i^*},
\end{aligned}$$

with $t_i^* = (t_i - t_{i-1})/2$ we also define $\bar{z}(\bar{t}) = z(t)$. This gives the change

$$\begin{aligned}
\mathrm{D}z \exp(-\frac{\omega}{2}NT + \frac{\omega T}{2}) &= \lim_{\epsilon_i\to 0}\exp(-\frac{\omega}{2}NT + \frac{\omega T}{2})\sqrt{\frac{m}{2\pi\epsilon_N}}\prod_{i=1}^{N-1}\sqrt{\frac{m}{2\pi\epsilon_i}}\mathrm{d}z_i \\
&\to \lim_{\epsilon_i\to 0}\exp(\frac{\omega T}{2})\sqrt{\frac{m}{2\pi\epsilon_N e^{-\omega NT^*}}}\prod_{i=1}^{N-1}\sqrt{\frac{m}{2\pi\epsilon_i e^{2\omega t_i^*}}}\mathrm{d}z_i \\
&= \lim_{\epsilon_i\to 0}\exp(\frac{\omega T}{2})\sqrt{\frac{m}{2\pi\bar{\epsilon}_N}}\prod_{i=1}^{N-1}\sqrt{\frac{m}{2\pi\bar{\epsilon}_i}}\mathrm{d}\bar{z}_i \\
&= e^{\frac{\omega T}{2}}\mathrm{D}\bar{z}.
\end{aligned} \tag{294}$$

The propagator for the harmonic oscillator is then given by

$$Z_{\text{HO}} = \exp\left(\frac{m\omega}{2}(x_T^2 - x_0^2) + \frac{\omega T}{2}\right) \int \mathrm{D}\bar{z}\, \exp\left(-\int_{t_a}^{t_b} \mathrm{d}t\, \frac{m}{2}\dot{\bar{z}}^2\right). \tag{295}$$

Now we use the propagator for the free particle (290) which we derived in the section above, and some straight forward algebra to arrive at

$$Z_{\text{HO}}(x_T, x_0, T) = \sqrt{\frac{m\omega}{2\pi \sinh(\omega T)}} \exp\left(-\frac{m\omega}{2\sinh(\omega T)}\left[(x_T^2 + x_0^2)\cosh(\omega T) - 2x_T x_0\right]\right) \tag{296}$$

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