QUANTUM MODELS WITHOUT CANONICAL QUANTIZATION*

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Conventional approach to constructing a quantum model,consisting in canonical quantization of a related classical model,can be applied only if we know the classical model and its Hamiltonian formulation in advance. Approach based on a set of postulates reflecting properties of physical configuration space is more general. As an example,quantum mechanics of a particle on a space consisting of just two points is constructed.

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1. Introduction

The aim of this paper is to advocate certain non-canonical approach to constructing quantum models. We point out shortcomings of the conventional,based on canonical quantization,procedure for constructing quantum models, and we show by presenting an example that a slightly less traditional approach,based on considerations of physical configuration space only, can be more advantageous.

The approach based on canonical quantization of a classical model has several unpleasant features. In our opinion the main one is that the construction of quantum model is based on preceding considerations of a classical model. This order of appearence of the models does not seem to be consistent with the fact that quantum description is apparently much closer to the real workings of Nature than the classical description (unless there are some yet undiscovered hidden variables). Therefore, it is the quantum model which should be introduced first, on basis of some fundamental

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physical assumptions, and the classical model should appear only as a secondary one, derived from the quantum model merely as an approximation. In particular, the classical canonical formalism should be obtained from the quantum description.

The canonical approach has also the disadvantage that it can not be applied in cases when it is not known in advance what is the relevant classical model to be quantized. One encounters this problem for example when trying to construct quantum mechanics of a particle on a non-smooth configuration space,e.g. on a lattice.

There are also other difficulties, like problems generated by presence of constraints in the classical system [1], or the problem of dependence of physical contents of quantum models on choice of canonical variables used for the quantization, while such choice is irrelevant on the classical level as long as the various choices of variables are related by canonical transformations. Yet another shortcoming of the canonical quantization is its nonuniqueness — in many cases canonical commutation relations have several inequivalent representations.

For these reasons, we would like to have a more self-contained approach to constructing quantum models which would not require foreknowledge of related classical models.

Very popular path integral formulation of quantum models does not provide the replacement for the canonical approach. This formulation is in fact nothing more than a special formula for matrix elements of the time evolution operator $U(t_1, t_0)$ giving them as a sum over paths. Nothing new is said about Hilbert space of states, observables, their commutation relations, *etc.* These important elements of definition of the quantum model are taken over from the operatorial approach. Moreover, in most cases in order to write the path integral formula a classical action is used, so again in some sense one starts from the classical theory, even if this is not so plain as in the case of canonical quantization.

It seems that more promising in this respect is the non-canonical approach based on a set of postulates which reflect properties of physical configuration space. The idea of non-canonical approach to constructing quantum models is not new. Very early example of non-canonical approach is provided by Dirac equation for a relativistic spin-$\frac{1}{2}$ particle. Original arguments leading to this equation were not based on canonical quantization of a classical model. Only afterwards the relevant classical model has been developed and canonically quantized to yield the Dirac equation. For a review and literature on this very interesting topic see, *e.g.* [2, 3, 4]. Later, the classic analysis of position operators [5, 6] was carried out without much emphasis on the canonical quantization. Ideas presented in the paper [7] in connection with the problem of nonrenormalizability in quantum field theory also point to a non-canonical approach. Quite recently, the role of the configuration space in construction of quantum models has been emphasized in [8].

In our present paper we give two elementary examples of consequent construction of full quantum models in the non-canonical approach. We show that this approach is very simple and natural. In both examples pertinent classical models and their canonical formulations are derived from the quantum theory as classical approximations.

We specify the main steps of the non-canonical approach in Section 2, using for this purpose the standard quantum mechanics of a spinless particle on the R^3 configuration space. We show how to obtain this quantum model without referring to classical Poisson brackets. Because our goal is to recover the standard quantum mechanics, ordinary canonical commutation relations for momentum and position operators can not be avoided. The point is to obtain them from postulates which do not allude to classical mechanics. In our approach the classical canonical formalism follows from the quantum mechanics in a classical limit performed with the help of Weyl transforms and Wigner densities.

In Section 3 we test usefulness of the non-canonical approach by applying it in a less obvious case. Namely, we construct quantum mechanics of a particle on a space consisting of two points. In this case we do not know relevant classical mechanics beforehand — in fact we shall see that it would not be quite easy to guess it because the classical phase space is completely different from the physical one. It turns out that the non-canonical approach works very effectively also in this case. It is interesting that in the classical limit of this quantum model we obtain straight away from the quantum theory a classical Dirac bracket, consistent with a constraint present in the classical model.

Section 4 contains ending remarks and suggestions for other applications of the non-canonical approach.

2. Particle on the R^3 configuration space

We would like to construct the quantum model with R^3 as the configuration space. By this we mean that the model should fulfill the following rather natural requirements.

(i) There exists observable (position operator) $\vec{\hat{x}} = (\hat{x}^i)$, i=1,2,3, whose spectrum covers the whole configuration space R^3. The operators $\hat{x}^i$ commute,

$$[\hat{x}^i, \hat{x}^k] = 0\,. \tag{1}$$

The role of the requirement (1) is to ensure that there exist common eigenvectors $\mid \vec{x}\rangle$ of the operators $\hat{x}^i$.

(ii) The (improper in this case) eigenvectors $\mid \vec{x}\rangle$ of the operator $\hat{\vec{x}}$ span the whole Hilbert space $\mathcal{H}$ of physical states of the particle.
Thus, any normalizable state $\mid \psi\rangle \in \mathcal{H}$ can be written in the form

$$\mid \psi\rangle = \int d^3x \psi(\vec{x}) \mid \vec{x}\rangle .$$

The eigenvectors $\mid \vec{x}\rangle$ are normalised in the usual way,

$$\langle \vec{x} \mid \vec{x'}\rangle = \delta(\vec{x} - \vec{x'}) . \tag{2}$$

If the requirement *(ii)* was not imposed, in $\mathcal{H}$ would exist a normalizable state $\mid \psi_0\rangle$ such that $\langle \vec{x} \mid \psi_0\rangle = 0$ for all $\vec{x} \in R^3$. According to the standard interpretation of quantum mechanical formalism this would mean that the particle in the state $\mid \psi_0\rangle$ was wandering beyond the R^3 space, *i.e.* R^3 was not the full configuration space.

(iii) The eigenvectors $\mid \vec{x}\rangle$ of the operators $\hat{\vec{x}}$ are nondegenerate.
The content of this requirement is that the particle does not have degrees of freedom other than the position, e.g. spin degrees of freedom.

The completeness of the set of states $\{\mid \vec{x}\rangle\}$ allows us to define linear operators in the Hilbert space $\mathcal{H}$ which correspond to various transformations of the R^3 space. For instance, the prescription

$$T(\vec{a}) \mid \vec{x}\rangle \stackrel{\text{df}}{=} \mid \vec{x} + \vec{a}\rangle \tag{3}$$

for $\vec{a} \in R^3$, defines unitary operator $T(\vec{a})$ corresponding to translations; the prescription

$$V(R) \mid \vec{x}\rangle \stackrel{\text{df}}{=} \mid R\vec{x}\rangle, \quad R \in \mathrm{SO}(3) , \tag{4}$$

gives unitary operator $V(R)$ corresponding to rotations; operator $D(\lambda)$ defined by the formula

$$D(\lambda) \mid \vec{x}\rangle \stackrel{\text{df}}{=} \lambda^{\frac{3}{2}} \mid \lambda\vec{x}\rangle, \ \lambda \in R, \ \lambda > 0 , \tag{5}$$

corresponds to dilatations. The factor $\lambda^{\frac{3}{2}}$ in formula (5) is necessary for unitarity of the operator $D(\lambda)$, because

$$\langle \lambda\vec{x} \mid \lambda\vec{y}\rangle = \lambda^{-3}\langle \vec{x} \mid \vec{y}\rangle ,$$

as follows from formula (2).

It follows from formulae (4), (5) that the operators $V(R)$, $D(\lambda)$ are related to the translation operator $T(\vec{a})$:

$$V(R) \mid \vec{x}\rangle = T(R\vec{x} - \vec{x}) \mid \vec{x}\rangle , \tag{6}$$

$$D(\lambda) \mid \vec{x}\rangle = \lambda^{\frac{3}{2}} T(\lambda\vec{x} - \vec{x}) \mid \vec{x}\rangle . \tag{7}$$

These formulae reflect the fact that rotations and dilatations (as well as any other transformation of R^3) can be regarded as local, *i.e.* $\vec{x}$-dependent translations. For this reason the translation operator $T(\vec{a})$ can be regarded as more basic than operators corresponding to the other transformations. From (6), (7) it easy to derive well-known formulae expressing generators of infinitesimal rotations and dilatations by the position operators $\hat{x}^i$ and generators $\hat{p}^i$ of infinitesimal translations.

The unitary translation operator $T(\vec{a})$ can be written in the exponential form,

$$T(\vec{a}) = \exp\{-ia^k\hat{p}^k\}\,, \tag{8}$$

where $\hat{p}^i$, $i = 1,2,3$ are hermitean operators. We do not know yet that they are related to the momentum. From definition (3) it follows that

$$T(\vec{a})T(\vec{b}) = T(\vec{b})T(\vec{a}) \tag{9}$$

for any $\vec{a}, \vec{b} \in R^3$. Relation (9) implies that the operators $\hat{p}^i$ commute,

$$[\hat{p}^i, \hat{p}^k] = 0\,. \tag{10}$$

Furthermore, from formula (3) it also follows that

$$\hat{x}^i T(\vec{a}) = T(\vec{a})(\hat{x}^i + a^i I)\,, \tag{11}$$

and consequently that

$$[\hat{x}^k, \hat{p}^l] = i\delta_{kl} I\,, \tag{12}$$

where I denotes the identity operator. Notice that we have obtained the commutation relation (12) without any reference to the canonical formalism. Notice also that there is not any $\hbar$ constant in formula (12) and that $\hat{p}^k$ has the dimension cm^{-1}. To avoid possible misunderstanding, let us stress that we do not use the units $c = \hbar = 1$ in this paper. Introducing the $\hbar$ constant in formulae (8), (12) and the corresponding change of dimension of the $\hat{p}^i$ operators would be artificial at this point, because we have not shown yet that the $\hat{p}^i$ operators are related to the classical momentum.

Taking matrix elements of the both sides of relation (12) we obtain the following equation

$$(x^k - x'^k)\langle \vec{x} \mid \hat{p}^l \mid \vec{x}'\rangle = i\delta_{kl}\delta(\vec{x} - \vec{x}')\,. \tag{13}$$

Its general solution has the form

$$\langle \vec{x} \mid \hat{p}^l \mid \vec{x}'\rangle = -i\frac{\partial}{\partial x^l}\delta(\vec{x} - \vec{x}') + c_0^l(\vec{x})\delta(\vec{x} - \vec{x}')\,, \tag{14}$$

where $c_0^l(\vec{x})$ are arbitrary real functions of $\vec{x}$. However, it is easy to show that we have to choose $c_0^l(\vec{x}) = 0$, otherwise the operator $T(\vec{a})$ computed from formulae (8) and (14) does not satisfy the definition (3). The representation (14) can be used to obtain generalised eigenvectors and spectrum of the operators $\hat{p}^k$. Not surprisingly, they turn out to be plane waves and R^1, respectively.

Remaining element of quantum mechanics of the spinless particle on R^3 we have yet to recover in our approach is Schrödinger equation. It is needed in order to determine time evolution of states. General form of this equation is a consequence of the following requirement.

(iv) There exists unitary time-evolution operator $U(t, t_0)$ which obeys the usual composition law

$$U(t_1, t)\, U(t, t_0) = U(t_1, t_0)\,. \tag{15}$$

Differentiating both sides of formula (15) with respect to t_1 and putting $t_1 = t$ we obtain

$$i\frac{\partial}{\partial t}U(t, t_0) = \hat{H}\, U(t, t_0)\,, \tag{16}$$

where the hermitean operator $\hat{H}$ (the quantum Hamiltonian) is defined as

$$\hat{H} \stackrel{\mathrm{df}}{=} i\frac{\partial U(t_1, t)}{\partial t_1}\,|_{t_1 = t}\;. \tag{17}$$

Equation (16) is equivalent to the Schrödinger equation. Notice however that there is no $\hbar$ constant in it, and that the operator $\hat{H}$ has the dimension $\sec^{-1}$.

The last step is to choose a concrete form of the quantum Hamiltonian $\hat{H}$. For a free particle on R^3 one could use a requirement of Galilean or Poincaré invariance [9, 10]. Then there is no need at all to consider a classical limit of our quantum model. In general case however, it is useful to investigate the classical limit. Let us stress that we consider this classical limit merely as a help in choosing the concrete form of the quantum Hamiltonian $\hat{H}$. The basic commutation relations (1), (10) and (12) have already been found without any reference to classical mechanics.

In the present context it is particularly convenient to work with an equivalent reformulation of our quantum model in terms of Wigner distributions on $\{(\vec{p}, \vec{x})\}$ space, see *e.g.* [11]. This reformulation can be obtained solely on the basis of commutation relations (1), (10), (12) and the assumption that the spectrum of the $\hat{\vec{x}}$ operator is R^3. We do not have to know the precise form of the quantum Hamiltonian $\hat{H}$ and there are no particular restrictions on the wave functions. In our case there is no $\hbar$ constant as yet, therefore the relevant formulae should be taken with appropriate changes.

Let us recall that the Wigner distribution $\rho(\vec{p},\vec{x},t)$ corresponding to the state $|\,\psi\rangle \in \mathcal{H}$ is given by the following formula

$$\rho(\vec{p},\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3v\; e^{i\vec{p}\vec{v}}\psi^*(\vec{x}+\tfrac{1}{2}\vec{v},t)\;\psi(\vec{x}-\tfrac{1}{2}\vec{v},t)\,, \tag{18}$$

where $\vec{x}\,(\vec{p})$ belongs to the spectrum of the operator $\hat{\vec{x}}\,(\hat{\vec{p}})$. Operators are described by Weyl transforms — for an operator $\hat{A}_S$ considered in Schrödinger picture

$$\tilde{A}_S(\vec{p},\vec{x}) \stackrel{\mathrm{df}}{=} \int d^3v\; e^{i\vec{p}\vec{v}}\langle \vec{x}-\tfrac{1}{2}\vec{v}\mid \hat{A}_S \mid \vec{x}+\tfrac{1}{2}\vec{v}\rangle \tag{19}$$

(we assume for simplicity that $\hat{A}_S$ does not depend on time). Then

$$\langle \psi \mid \hat{A}_S \mid \psi\rangle \;=\; \int d^3p d^3x\; \rho(\vec{p},\vec{x})\tilde{A}_S(\vec{p},\vec{x})\,. \tag{20}$$

Schrödinger equation (16) can be replaced by Heisenberg equation for operators $\hat{A}_H(t)$ in Heisenberg picture

$$\frac{d}{dt}\hat{A}_H(t) = \; i\,[\hat{H},\; \hat{A}_H(t)\,]\,. \tag{21}$$

Using a formula for the Weyl transform of a commutator [11], one can show that the Weyl transform $\tilde{A}_H(\vec{p},\vec{x},t)$ of the operator $\hat{A}_H(t)$ obeys the following equation

$$\frac{\partial \tilde{A}_H(\vec{p},\vec{x},t)}{\partial t} = 2\sin\left[\frac{1}{2}\left(\frac{\partial^{(\tilde{A})}}{\partial\vec{x}}\frac{\partial^{(\tilde{H})}}{\partial\vec{p}} - \frac{\partial^{(\tilde{A})}}{\partial\vec{p}}\frac{\partial^{(\tilde{H})}}{\partial\vec{x}}\right)\right]\tilde{A}_H(\vec{p},\vec{x},t)\tilde{H}(\vec{p},\vec{x})\,, \tag{22}$$

where $\partial^{(\tilde{H})}$ $(\partial^{(\tilde{A})})$ means that the differential operator acts only on $\tilde{H}$ $(\tilde{A})$. For simplicity we have assumed that $\hat{H}$ does not depend on time. Expanding in power series the sinus function on the r.h.s. of formula (22) gives

$$\frac{\partial \tilde{A}_H(\vec{p},\vec{x},t)}{\partial t} = \{\tilde{A}_H,\tilde{H}\} + (\text{terms with higher derivatives})\,, \tag{23}$$

where $\{\;,\;\}$ denotes Poisson bracket,

$$\{\tilde{A}_H,\tilde{H}\} \stackrel{\mathrm{df}}{=} \frac{\partial \tilde{A}_H}{\partial\vec{x}}\frac{\partial\tilde{H}}{\partial\vec{p}} - \frac{\partial \tilde{A}_H}{\partial\vec{p}}\frac{\partial\tilde{H}}{\partial\vec{x}}\,. \tag{24}$$

Equations (22) or (23) are equivalent to Schrödinger equation (16). As yet there has not been any classical limit involved.

Formula (23) is a convenient starting point for discussion of the classical limit. Classical limit of quantum models is a vast subject, and it is not the goal of this paper to investigate it in detail. We shall be satisfied with several simple observations.

First, because we do not have the $\hbar$ constant in our quantum model, it is not possible to consider this limit as the formal $\hbar \to 0$ limit. We shall consider Ehrenfest type (*i.e.* wave-packet type) of the classical limit, in which one analyses equations of motion for expectation values of quantum observables. In the Wigner formalism the expectation values are given by formula (20), and equations of motion for them are obtained from equation (23) by integrating the both sides of it with the $\rho(\vec{p},\vec{x})$ function which is time-independent because we work in the Heisenberg picture. It is clear that in general nothing simple is obtained.

However, in the particular case when the functions $\frac{\partial \tilde{H}}{\partial \vec{p}}, \frac{\partial \tilde{H}}{\partial \vec{x}}$ are approximately constant for $\vec{p},\vec{x}$ belonging to the support of the $\rho(\vec{p},\vec{x})$ function, the contribution from the higher derivative terms on the r.h.s. of formula (23) is small, and in the Poisson bracket term the functions $\frac{\partial \tilde{H}}{\partial \vec{p}}$ and $\frac{\partial \tilde{H}}{\partial \vec{x}}$ can be taken out of the integral over $\vec{p}$ and $\vec{x}$. Now let us take the distribution function ρ concentrated around $\vec{P},\vec{X}$ and having the form $\rho(\vec{p}-\vec{P},\vec{x}-\vec{X})$. For instance, it could correspond to an appropriate coherent state. Then the expectation value A of the operator $\hat{A}_H(t)$ is a function of $\vec{P},\vec{X},t$,

$$\int d^3p d^3x \; \rho(\vec{p}-\vec{P},\vec{x}-\vec{X}) \; \tilde{A}_H(\vec{p},\vec{x},t) = A(\vec{P},\vec{X},t)\,,$$

and

$$\int d^3p d^3x \; \frac{\partial \tilde{A}_H}{\partial \vec{x}} \rho = \frac{\partial A(\vec{P},\vec{X},t)}{\partial \vec{X}}\,,$$

$$\int d^3p d^3x \; \frac{\partial \tilde{A}_H}{\partial \vec{p}} \rho = \frac{\partial A(\vec{P},\vec{X},t)}{\partial \vec{P}}\,.$$

Taking all this into account, it is easy to obtain from Eq. (23) the following approximate equation

$$\frac{\partial A(\vec{P},\vec{X},t)}{\partial t} = \{A, \tilde{H}\}\,, \tag{25}$$

where now the Poisson bracket is taken with respect to the $\vec{P},\vec{X}$ variables.

In this way we have derived from the quantum mechanics the classical Hamiltonian dynamics. The only difference is that in formula (25) $\tilde{H}$ and $\vec{P}$ have dimensions $\sec^{-1}$ and cm^{-1}, respectively. In order to identify $\tilde{H}$ and $\vec{P}$ with classical Hamiltonian and momentum we have to change their dimensions by multiplying them by the dimensionful constant $\hbar$, *i.e.* $\hbar\tilde{H} \to \tilde{H}$, $\hbar\vec{P} \to \vec{P}$. This is equivalent to the following rescaling of operators

$$\hbar\hat{H} \to \hat{H}, \ \ \hbar\hat{p}^k \to \hat{p}^k \,. \tag{26}$$

After the rescaling, the operators $\hat{p}^i$, Schrödinger equation (16) and Heisenberg equation (21) acquire the standard form with the $\hbar$ constant in the right places. The Poisson bracket term on the r.h.s. of formula (23) is not changed by the rescaling, while the terms with higher derivatives acquire positive (even) powers of the Planck constant $\hbar$.

Notice that $\tilde{H}$ and $\vec{P}$ have to be multiplied by the same constant $\hbar$ if we want to preserve the standard form of classical Hamilton equations of motion. Also interesting is the fact that the Planck's constant appears only when relating the quantum model with the classical physics — our quantum model is constructed essentially without that constant.

Now we can conclude our search for the concrete form of the $\hat{H}$ operator. We know that physically interesting models in classical mechanics of a particle on R^3 configuration space have classical Hamiltonians of the form

$$H_{\mathrm{cl}} \ = \ \frac{\vec{P}^{\,2}}{2m} \ + \ V(\vec{X}) \,.$$

On the basis of Eq. (25), which summarizes the classical limit of our quantum model, it is therefore natural to choose

$$\tilde{H} \ = \ H_{cl} \ + \ \mathcal{O}(\hbar^2) \,.$$

The inverse Weyl transform then gives

$$\hat{H} \ = \ \frac{\vec{\hat{p}}^{\,2}}{2m} + V(\vec{\hat{x}}) + \mathcal{O}(\hbar^2) \,. \tag{27}$$

The $\mathcal{O}(\hbar^2)$ terms are allowed because of the presence of higher derivative terms on the r.h.s. of Eq. (23). There are many quantum Hamiltonians which have identical classical limits. When identifying the $\mathcal{O}(\hbar^2)$ terms, the $\hbar$ present in the $\hat{p}^i$ operators should not be counted. For consistency, one should check that in the case of quantum Hamiltonian given by formula (27) it is possible to satisfy the conditions for the classical limit. Discussion of this point is completely standard, therefore we shall not present it here.

3. Particle on the two-point space

In the previous Section we have presented the main steps of the non-canonical approach to constructing quantum models, and we have shown that in this approach one can easily recover the standard quantum mechanics of spinless particle on R^3. However a true advantage of the non-canonical approach is that it can be applied without any essential changes also in less obvious cases, when it is not clear at all what is the underlying classical model which could be canonically quantized to yield the quantum model. To illustrate this point we construct the quantum mechanics of a particle on the configuration space consisting of just two points. We shall proceed by following exactly the steps presented and motivated in the previous Section.

In this case, measurements of position of the particle can give only two values, a or $-a$ on a coordinate axis. The corresponding position operator $\hat{q}$ can be written in the form

$$\hat{q} = a \mid a\rangle\langle a \mid +(-a) \mid -a\rangle\langle -a \mid , \tag{28}$$

where $\mid \pm a\rangle$ are orthonormal eigenvectors of $\hat{q}$. The Hilbert space $\mathcal{H}_2$ of states is 2-dimensional

$$\mathcal{H}_2 \ni\mid \psi\rangle = \psi^1 \mid a\rangle + \psi^2 \mid -a\rangle . \tag{29}$$

In this manner we have satisfied the requirements *(i)–(iii)* of Section 2. Again we have assumed that the particle has no other degrees of freedom than the position — in particular it is spinless. Expectation values of the position operator $\hat{q}$ in a normalized state $\mid \psi\rangle$ are

$$q \stackrel{df}{=} \langle\psi \mid \hat{q} \mid \psi\rangle = a(\mid \psi^1 \mid^2 - \mid \psi^2 \mid^2) , \tag{30}$$

where $\mid \psi^1 \mid^2 + \mid \psi^2 \mid^2 = 1$ because of the normalization. Thus, q can take any value in the interval $[-a, a]$. For this reason we expect after all that there exists a related classical model — classical variables correspond to expectation values and these have continuous values. We shall derive this classical model below.

It is clear that our quantum model has the following two-dimensional matrix realisation:

$$\hat{q} \leftrightarrow a\sigma_3, \mid a\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mid -a\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mid \psi\rangle \leftrightarrow \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} , \tag{31}$$

where σ_3 is the Pauli matrix.

In the Hilbert space $\mathcal{H}_2$ one can introduce unitary operators corresponding to translations. It is natural to require that these operators act on the basis states in the following manner

$$T(\pm 2a) \mid \mp a\rangle =\mid \pm a\rangle . \tag{32}$$

In order to have a complete definition we also have to specify how $T(\pm 2a)$ act on the states $\mid \pm a\rangle$. We shall be guided by the following intuition. For the considered particle, in its world consisting of just two points there is no intrinsic notion of direction. The fact that we imagine the two points $\pm a$ as located on the oriented axis is irrelevant. Whichever point the particle is located at, the other point is "ahead", that is the both movements, from $-a$ to $+a$ and from $+a$ to $-a$, are movements forward. Any asymmetry between the two points should be interpreted as the effect of an external potential rather than intrinsic property of the space itself. Therefore it is natural to assume that operators of translations by $2a$ and $-2a$ coincide,

$$T(2a) = T(-2a)\,. \tag{33}$$

Then, the complete definition of the operator $T(2a)$ is

$$T(2a) \mid \mp a\rangle = \mid \pm a\rangle\,. \tag{34}$$

This operator is linear by definition, and it is unitary. In the matrix representation

$$T(2a) \leftrightarrow \sigma_1\,. \tag{35}$$

For any state $\mid \psi\rangle \in \mathcal{H}_2$ we can consider the transformation

$$\mid \psi\rangle \to \mid \psi'\rangle = T(2a) \mid \psi\rangle\,. \tag{36}$$

Under this transformation the components ψ^1, ψ^2 of $\mid \psi\rangle$ are interchanged, and therefore the expectation value of the position operator $\hat{q}$ changes its sign

$$q' \stackrel{df}{=} \langle \psi' \mid \hat{q} \mid \psi'\rangle = -\langle \psi \mid \hat{q} \mid \psi\rangle = -q\,. \tag{37}$$

The operator $T(2a)$ can be written in exponential form. This form of $T(2a)$ defines a quasi-momentum $\hat{p}$ of the particle:

$$T(2a) = \exp(-i2a\hat{p})\,. \tag{38}$$

From definitions (34), (38) it follows that we can take in the matrix representation

$$\hat{p} = \frac{\pi}{4a}(\sigma_0 - \sigma_1)\,, \tag{39}$$

where σ_0 is the 2 by 2 unit matrix. The operator $\hat{p}$ has the following eigenvalues and eigenvectors

$$p_+ = \frac{\pi}{2a},\ \mid p_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix};\ \ p_0 = 0,\ \mid p_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{40}$$

Expectation values p of the $\hat{p}$ operator take continuous values in the interval $[\pi/2a\,,0]$.

Amusingly, the $\hat{p}$ operator can be related to a discrete version of derivative. For a function $f(x), x = \pm a$,on the two-point space one can define two "derivatives"

$$(\partial_+ f)(x) \stackrel{df}{=} \frac{f(x+2a) - f(x)}{2a}, \quad (\partial_- f)(x) \stackrel{df}{=} \frac{f(x) - f(x-2a)}{2a}. \qquad (41)$$

It is understood that in these formulae

$$f(a+2a) \equiv f(-a), \quad f(-a-2a) \equiv f(a).$$

It is easy to check that $\partial_+ f = -\partial_- f$, and that in the matrix representation

$$\partial_+ \leftrightarrow \frac{1}{2a}(\sigma_1 - \sigma_0).$$

Comparing this formula with formula (39) we find that

$$\hat{p} = \frac{\pi}{2}\partial_- .$$

Definition (34) could be modified by multiplying the vectors $\mid \pm a\rangle$ on the r.h.s. of that formula by phase factors. However by an appropriate change of phase of one of the states $\mid \pm a\rangle$ one can achieve that

$$T(2a) \mid \pm a\rangle = e^{i\phi} \mid \mp a\rangle,$$

and this is merely a redefinition of the $T(2a)$ operator by the phase factor $e^{i\phi}$. If we still define the quasi-momentum operator $\hat{p}$ by formula (38), then $\hat{p}$ and its spectrum are shifted by the constant $\phi/2a$.

Because the Hilbert space $\mathcal{H}_2$ is two-dimensional, the most general Hamiltonian $\hat{H}$ for our particle has only four real parameters and it can be written in the following form

$$\hat{H} = h_0\sigma_0 + \vec{h}\vec{\sigma}. \qquad (42)$$

The parameters $h_0, \vec{h}$ are fixed externally. They can depend on time. The Hamiltonian (42) can be expressed by the $\hat{q}$, $\hat{p}$ operators,

$$\hat{H} = (h_0 + h^1)\sigma_0 - \frac{4a}{\pi}h^1\hat{p} + \frac{2i}{\pi}h^2[\hat{q},\hat{p}] + \frac{h^3}{a}\hat{q}. \qquad (43)$$

There are three independent, nontrivial observables altogether: σ_i, $i = 1,2,3$, or equivalently $\hat{q}, \hat{p}$ and $\hat{c} \stackrel{df}{=} i[\hat{q},\hat{p}]$. We shall denote expectation values of them by q, p, and c, correspondingly. These three expectation values determine completely the density matrix $|\psi\rangle\langle\psi|$,

$$|\psi\rangle\langle\psi| = \frac{1}{2}\sigma_0 + \left(\frac{1}{2} - \frac{2ap}{\pi}\right)\sigma_1 + \frac{c}{\pi}\sigma_2 + \frac{q}{2a}\sigma_3\,, \tag{44}$$

and therefore also expectation value of any observable. Using formula (44) on the both sides of the identity

$$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\,,$$

we obtain the following constraint

$$\left(1 - \frac{4ap}{\pi}\right)^2 + \frac{4c^2}{\pi^2} + \frac{q^2}{a^2} = 1\,, \tag{45}$$

i.e. q, p, and c lie on an ellipsoide.

Heisenberg equations of motion for an observable $\hat{A}_H(t)$,

$$\frac{d\hat{A}_H(t)}{dt} = i[\hat{H}, \hat{A}_H(t)] + \left(\frac{\partial \hat{A}_S}{\partial t}\right)_H\,, \tag{46}$$

in the case of observables $\hat{q}$, $\hat{p}$ and $\hat{c}$ can be written in the form

$$\frac{d\hat{Q}^i_H(t)}{dt} = 2\epsilon_{iks} h^k \hat{Q}^s_H(t)\,, \tag{47}$$

where

$$\vec{\hat{Q}}_H(t) = \begin{pmatrix} \sigma_{1H}(t) \\ \sigma_{2H}(t) \\ \sigma_{3H}(t) \end{pmatrix} = \begin{pmatrix} \sigma_0 - \frac{4a}{\pi}\hat{p}_H \\ \frac{2i}{\pi}[\hat{q}, \hat{p}]_H \\ a^{-1}\hat{q}_H \end{pmatrix}. \tag{48}$$

Equations (47) are formally identical with equations for precession of a spin $\frac{1}{2}$ in an external magnetic field $\vec{B} = 2\vec{h}$. For constant in time $\vec{h}$ their solutions are easy to obtain. For time-dependent $\vec{h}$ situation is more complicated - in particular, Berry's phase factor may appear. Of course, this relationship with spin $\frac{1}{2}$ is purely mathematical one. Any model dealing with 2 by 2 matrices can be expressed in terms of the spin $\frac{1}{2}$. Physical meaning of our model has nothing to do with spin, because we do not consider any rotations.

Finally, let us turn to the problem of classical limit of our quantum model. A classical model is a good approximation to the quantum model if expectation values of all quantum observables can be expressed by the classical dynamical variables with good accuracy. In the quantum mechanics of a spinless particle on R^3 this is the case when we consider wave packets with small dispersions of the $\hat{x}^i$ and $\hat{p}^k$ observables. Then, with good accuracy,

$$\langle f(\hat{x}^i, \hat{p}^k)\rangle \cong f(\langle\hat{x}^i\rangle, \langle\hat{p}^k\rangle)$$

for a wide class of observables $f(\hat{x}^i, \hat{p}^k)$. In the case of our quantum model on the two-point space a generic time-independent observable $\hat{A}$ in the Schrödinger picture has the form

$$\hat{A} = a_0\sigma_0 + \vec{a}\vec{\sigma}\,,$$

where a_0, $\vec{a} = (a^i)$ are real constants. The expectation value $A(t) \equiv \langle\psi \mid \hat{A}_H(t) \mid \psi\rangle$ of the Heisenberg picture counterpart $\hat{A}_H(t)$ of the observable $\hat{A}$ can be written in the form

$$A(t) = a_0 + \vec{Q}(t)\vec{a}\,, \tag{49}$$

where $a_0, \vec{a}$ are constant in time, and $\vec{Q}(t)$ is expectation value of $\hat{\vec{Q}}_H(t)$. The constraint (45) means that $\vec{Q}^2 = 1$. Taking expectation value of Heisenberg equation of motion (46) we find that A obeys the following equation

$$\frac{dA}{dt} = 2(\vec{a} \times \vec{h})\vec{Q}\,. \tag{50}$$

This equation can be written in the form of Hamilton equation of motion

$$\frac{dA}{dt} = \{A, H\}\,, \tag{51}$$

where we have introduced a classical Dirac bracket

$$\{A, H\} = 2\epsilon_{iks}Q^s \frac{\partial A}{\partial Q^i}\frac{\partial H}{\partial Q^k}\,, \tag{52}$$

and

$$H = h_0 + \vec{h}\vec{Q} \tag{53}$$

is the expectation value of the Hamiltonian $\hat{H}$. It is easy to check that the Dirac bracket defined by formula (52) has all necessary properties. It is bilinear, antisymmetric, it satisfies Lie identity and Leibniz rule

$$\{A, BC\} = B\{A, C\} + \{A, B\}C\,.$$

We call it the Dirac bracket because it is compatible with the constraint $\Phi \equiv \vec{Q}^2 - 1 = 0$, in the sense that for any function $A(\vec{Q})$

$$\{\Phi, A\} = 0\,.$$

The Q^i variables should be regarded as canonical variables for our classical system. The $\vec{Q}$ vector has values in the classical phase space which is the sphere $\vec{Q}^2 = 1$.

Because any nontrivial observable is a linear combination of σ_0 and $\hat{q}$, $\hat{p}$, $\hat{c}$ operators, and because their expectation values q, p, c determine the density matrix (44), the classical equation of motion (50) is exactly equivalent to the quantum one given by formula (47). This is in contradistinction with the case of particle on R^3, where the relation between classical and quantum equations of motion was approximate, and restricted to particular quantum states namely, to wave packets. Of course, that mathematical equivalence of equations of motion does not imply that the classical and quantum models are physically equivalent, *e.g.* spectra of observables are different.

The classical model in its canonical formulation defined by equations (51), (52) has been derived straight away from the quantum theory. The classical phase space does not show in any apparent way that the physical configuration space consists of just two points. One could hardly guess in advance that this is the relevant classical model.

This classical model can be canonically quantized by passing to operators and replacing the Dirac bracket by $-i$ times commutator. Then one can have infinitely many inequivalent matrix models, characterised by the dimension of the matrices. This follows from the fact that the classical Dirac bracket algebra of the Q^i observables mathematically coincides with algebra of angular momentum, so all spin representations are possible. We see that also in this example canonical quantization does not give unique result.

4. Remarks

The two examples considered above show effectiveness of the non--canonical quantization. We hope that this approach will turn out very useful also in other cases. For example, one could consider a quantum particle on a cone (which is not a smooth manifold), or on a lattice bigger than the two-point one. We also hope that this approach can be generalised to field theory. There, interesting problems are provided by models with fields taking values in target spaces which are not smooth manifolds.

It would also be interesting to apply the non-canonical approach to models which in the canonical approach are plagued by constraints. Important type of such models is the class of models with local gauge invariance.Perhaps one could avoid tedious identification and classification of constraints, as well as computation of Dirac brackets. Taking classical limit of the quantum model constructed with the help of the non-canonical approach we could derive classical Dirac brackets from quantum theory. Example of such derivation we have seen in Section 3.

In the examples considered in Sections 2 and 3 we have investigated the classical limit of the quantum models. In the case of particle on R^3 we have recovered the standard classical canonical formalism from the quantum mechanics. We expect that this example can be generalised to cases in which configuration space is a more general smooth manifold. It would be interesting to recover classical canonical formalism from the quantum mechanics also in such more general case.

In the case of particle on the two-point space, the classical model has different phase space than the quantum one. This is due to the fact that classical observables are identified with expectation values, while measured values of quantum observables (*e.g.* the position or the quasi-momentum) are restricted to the discrete set of their eigenvalues. This renders the canonical quantization practically useless, because in order to apply it we have to know the classical model in advance, and in the case at hand it is not easy to guess the correct one. This difficulty will appear also in other models with discrete physical configuration space. We think that in such cases the non-canonical approach will turn out to be much more effective.

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