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# Construction of new solitary wave solutions of generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony and simplified modified form of Camassa-Holm equations

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**Abstract:** In this research work, for the first time we introduced and described the new method, which is modified extended auxiliary equation mapping method. We investigated the new exact traveling and families of solitary wave solutions of two well-known nonlinear evaluation equations, which are generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony and simplified modified forms of Camassa-Holm equations. We used a new technique and we successfully obtained the new families of solitary wave solutions. As a result, these new solutions are obtained in the form of elliptic functions, trigonometric functions, kink and antikink solitons, bright and dark solitons, periodic solitary wave and traveling wave solutions. These new solutions show the power and fruitfulness of this new method. We can solve other nonlinear partial differential equations with the use of this method.

**Keywords:** Modified extended auxiliary equation mapping method; GZK-BBM equation; modified form of CH equation; exact and Traveling wave solutions; solitary wave solutions

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## 1 Introduction

A few years ago many authors found the different types of exact traveling and solitary wave solutions of both nonlinear models GZK-BBM and GCH-equations. These both equations are well-known nonlinear evaluation equations and play important role in many scientific fields. The GZK-BBM equation used in the studies of acoustic waves, acoustic-gravity waves, surface waves with long wavelength, hydromagnetic waves, these all mentioned waves have source harmonic crystals, compressible fluids, cold plasma and liquids, respectively. The GCH equation play important role in shallow water waves. In 2005 Wazwaz [1] studied the GZK-BBM equation for the first time and found some complex solutions, kink type solutions, periodic wave solutions and solitons solutions with the help of the sine-cosine method. Wazwaz [2] found the two types of compactons and solitary patterns wave solutions of ZK-BBM equation by applying the extended tanh method. Abdou [3] found the set of exact solutions of ZK-BBM equation with the help of extended F-Expansion method. Mahmoudi et al. [4] investigated the periodic solitary wave solutions of ZK-BBM equation by applying the exp-function method. Wang and Tang [5] studied the existence property of smoothness of traveling wave solutions of ZK-BBM equation by apply the bifurcation theory of planner. Song and Yang [6] with the help of bifurcation technique found the traveling wave, solitary wave and kink type solutions of ZK-BBM equation.

Camassa and Holm [7] derived a Camassa-Holm equation (CH-equation) by using the Hamiltonian methods, which is a completely integrable dispersive water waves equation by holding two terms, which are neglected in the limit of shallow water waves, having small amplitude. After that many authors started to investigate the different types of travelling solitary wave solutions of CH equation by using various methods. Cooper and Shepard [8] found the solitary wave solutions of GCH-equation by using the variational function. Liu et al. [9] improved? CH equation and found traveling wave solutions. Zhang and Bi [10] studied the bifurcation technique of CH-equation. Liu and Tang [11] investigated the bifurcation phenomena

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and found the periodic solutions of GCH-equation with the help of integrated scheme. Deng et al. [12] found the compacton, kink and anti-kink, periodic solitary wave and solitons solutions of GCH-Degaspersi-Procesi-equation. Kalla and Klein [13] found the multidimensional theta functions independent derivation solutions of GCH equation with the help of technique that is related to Fay's identity.

Recently, Liu and Song [14] found the smooth periodic and blow-up periodic solutions of GZK-BBM equation by applying the bifurcation method. Khadijo Adem and Masood Khalique [15] investigated the traveling waves solutions and conservation laws of GZK-BBM equation with the help of $(G^{'}/G)$– expansion method. Harun-Or-Roshid et al. [16] found the families of solitary waves solutions of GZK-BBM and RLW equations by using the modified simple equation method. Seadawy et al. [17] found the families of exact travelling and solitary wave solutions of GZK-BBM equation with the help of $\exp(-\varphi(\xi))$-expansion method. Many other authors have investigated the travelling solitary wave solutions of GZK-BBM equation and GCH-equation see Ref. [18–20, 26–30].

The nonlinear system of partial differential equations is very useful to study the physical nature in many different scientific fields, such as engineering, physics, geophysics, optics, chemistry, biology, material science, computer science, mechanics, electricity, ultrasound, thermodynamics and so on. The solitary and travelling wave solutions of NPDEs have many applications to understanding the process and physical phenomena in many areas of applied science. In the last five decades a lot of new methods have been developed by many groups of mathematicianas and engineers to investigate the (NPDEs). For example some important methods such as, exp-function method; modified Extended tanh-expansion method; modified simple equation method; homotopy perturbation method; novel $(G^{'}/G)$– expansion method; extended modified direct algebraic method; generalized kudryashov method; modified extended Kudryashov method; $\exp(-\varphi(\xi))$-expansion method; extended Jacobian method; extended trial equation method and so on [31, 32, 35–50].

The main aim of this research is to investigate the exact traveling and solitary wave solutions of GZK-BBM and simplified modified form of CH-equations. These new solutions are obtained with the help of new method, which is modified extended auxiliary equation mapping method. The arrangement of this article is organized as fellows. Description of the modified extended auxiliary equation mapping method is given in Section 2. Section 3 deals with the investigation of the solitary wave solutions of GZK-BBM-equation and simplified modified CH-equation by using the described method. Finally, the conclusion are presented in Section 4.

## 2 Modified extended auxiliary equation mapping method

Consider the general form of (2+1)-dimensional NPDEs as

$$F(U, U_t, U_x, U_y, U_{xt}, \ldots) = 0, \tag{1}$$

here $F$ denotes the polynomial function of $U(x, y, t)$ and its all derivatives which contained highest order nonlinear terms and highest order partial derivatives. Here we explain the important steps of the new method as:

**Step1**. We apply the traveling wave transformations as

$$U(x, y, t) = U(\xi), \xi = lx + my + \omega t, \tag{2}$$

where l and m are the wave numbers and $\omega$ is the frequency of the wave. We obtained the ODE of Eq.(1) as

$$P(U, U^{'}, U^{''}, U^{'''}, \ldots) = 0, \tag{3}$$

here $P$ is the polynomial function in $U(\xi)$ and its derivatives.

**Step2**. We consider the general solution of $Eq.(2)$, in the following form

$$U(\xi) = \sum_{i=0}^{n} a_i \Psi(x)^i + \sum_{i=-1}^{-n} b_{-i} \Psi(x)^i + \sum_{i=2}^{n} c_2 \Psi(x)^{i-2} \Psi^{'}(x) + \sum_{i=1}^{n} d_i \left(\frac{\Psi'(x)}{\Psi(x)}\right)^i \tag{4}$$

where $a_0, a_1, \ldots a_n, b_1, b_2, \ldots b_n, c_2, c_3, \ldots c_n, d_1, d_2, \ldots d_n$ are constants parameter to be find later, the values of $\Psi(\xi)$ and its derivative $\Psi^{'}(\xi)$ satisfy to the given auxiliary equation

$$\begin{aligned}
\Psi^{'}(\xi) &= \sqrt{\beta_1 \Psi^2(\xi) + \beta_2 \Psi^3(\xi) + \beta_3 \Psi^4(\xi)};\\
\Psi^{''}(\xi) &= \beta_1 \Psi(\xi) + \frac{3}{2}\beta_2 \Psi^2(\xi) + 2\beta_3 \Psi^3(\xi);\\
\Psi^{'''}(\xi) &= \left(\beta_1 + 3\beta_2 \Psi(\xi) + 6\beta_3 \Psi^2(\xi)\right) \Psi^{'}(\xi);\\
\Psi^{''''}(\xi) &= \frac{1}{2}\Psi(\xi)(2\beta_1^2 + 15\beta_1\beta_2\Psi(\xi) + 5(3\beta_2^2 + 8\beta_1\beta_3)\Psi^2(\xi)\\
&\quad + 60\beta_2\beta_3\Psi^3(\xi) + 48\beta_3^2\Psi^4(\xi)).
\end{aligned} \tag{5}$$

Where $\beta_i^{'}s$ are real constants, which determine later such that $\beta_n \neq 0$.

**Step3**. Balance of the highest order nonlinear terms and highest order partial derivatives in Eq. (3) is found to be the positive integer $n$ of Eq. (3).

**Step4**. Substituting Eq. (5) into Eq. (4) and combining each coefficients of $\Psi^{i}(\xi)(i = 1,2,3,...n)$, then making a every coefficient equal to zero and obtaining a families of algebraic equations, solving this system of equations with the help of Mathematica, the constants $a_0, a_1, ...a_n, b_1, b_2, ...b_n, c_2, c_3, ...c_n, d_1, d_2, ...d_n$ can be determined.

**Step5**. Substituting the values of obtained constants and $\Psi(\xi)$ into Eq. (3), then required solutions of Eq. (1) are obtained.

# 3 Application of the proposed method

Now we applying the modified extended auxiliary equation mapping method to investigate the families of new solitary wave solutions for the (2+1)-dimensional GZK-BBM-equation and simplified modified form of CH-equation.

## 3.1 Generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation

We consider a (2+1)-dim GZK-BBM-equation as

$$U_t + U_x + \alpha(U^3)_x + \gamma\left(U_{\text{xt}} + U_{\text{yy}}\right)_x = 0, \tag{6}$$

where $\gamma$ and $\alpha$ are non zero constants. Consider the traveling wave transformation $U(x,y,t) = U(\xi)$, $\xi = lx+my+\omega t$, by this transformation we obtained ordinary differential equation of Eq.(6) as

$$\omega U' + \mathrm{l}U' + 3\alpha \mathrm{l}U^2 U' + \gamma m^3 U''' + \gamma \mathrm{l}^2 \omega U''' = 0, \tag{7}$$

we integrate the Eq.(7) once time according to $\xi$ and integration constant equal to zero, then we obtained as

$$\omega U + \mathrm{l}U + \alpha l U^3 + \gamma l^2 \omega U'' + \gamma m^3 U'' = 0, \tag{8}$$

Balance the highest order nonlinear term and highest order partial derivative in Eq.(8) obtained the value of $n = 1$. The general solution of Eq.(8) takes form of

$$U(\xi) = a_0 + a_1\Psi(\xi) + \frac{b_1}{\Psi(\xi)} + d_1\frac{\Psi'(\xi)}{\Psi(\xi)} \tag{9}$$

Substituting Eq. (9) into Eq. (8) and combining each coefficients of $\Psi'^{j}(\xi)\Psi^{i}(\xi)(j = 0,1; i = 1,2,3,...n)$, then making a every coefficient equal to zero and obtaining a set of algebraic equations. We solve this system of equations with the aid of Mathematica. The parameters $a_0, a_1, b_1, d_1$ can be determined as

**Case-I**

$$a_0 = 0,\ a_1 = a_1,\ b_1 = 0,\ d_1 = \pm\frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{\beta_1}\sqrt{l}},\ \beta_3 = \pm\frac{\alpha a_1^2\beta_1 l}{l+\omega},$$
$$m = \pm\frac{(-1)^{2/3}\sqrt[3]{\beta_1(-\gamma)l^2\omega + 2l + 2\omega}}{\sqrt[3]{\beta_1}\sqrt[3]{\gamma}}. \tag{10}$$

Substituting the Eq.(10), only for the positive value of $d_1$ in Eq. (9), then solitary wave solutions of Eq. (6) are obtained in simplified forms as:

$$U_1(x,y,t) = -\frac{\epsilon\sqrt{-l-\omega}\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]^2}{\sqrt{\alpha}\sqrt{l}\left(2\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+2\right)} - \frac{a_1\beta_1\left(1+\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)}{\beta_2} \tag{11}$$

$$\begin{aligned}U_2(x,y,t) = \Big(&2\epsilon\sqrt{-l-\omega}\left(\eta\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1\right)\\ &-\sqrt{\alpha}\sqrt{l}\Big(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\\ &+\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\Big)^2 a_1\sqrt{\frac{\beta_1}{\beta_3}}\Big)\Big/\\ &\left(2\sqrt{\alpha}\sqrt{l}\left(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\right.\\ &\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\\ &\left.+\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\end{aligned} \tag{12}$$

$$\begin{aligned}U_3(x,y,t) = &\epsilon\sqrt{-l-\omega}\Big(\eta\sqrt{p^2+1}\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\\ &-p\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1\Big)\Big/\\ &\Big(\sqrt{\alpha}\sqrt{l}\Big(\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\\ &+\eta\sqrt{p^2+1}\Big)\Big(\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\\ &+\eta\sqrt{p^2+1}+\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\right)\Big)\Big)\\ &+a_1\left(-1-\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}\right)\end{aligned} \tag{13}$$

**Case-II**

$$a_0 = \pm\frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{l}}, a_1 = a_1, b_1 = d_1 = 0, m = \pm\frac{\sqrt[3]{2(l+\omega)-\beta_1\gamma l^2\omega}}{\sqrt[3]{\beta_1}\sqrt[3]{\gamma}}, \beta_2 = \pm\frac{\sqrt{\alpha}a_1\beta_1\sqrt{l}}{\sqrt{-l-\omega}}, \beta_3 = -\frac{\alpha a_1^2\beta_1 l}{4(l+\omega)}. \tag{14}$$

Substituting Eq.(14), only the positive value of $a_0$ into Eq.(9), the solutions of Eq.(6) are given as:

$$U_4(x,y,t) = \frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{l}} - \frac{a_1\beta_1\left(1+\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)}{\beta_2} \tag{15}$$

$$U_5(x,y,t) = \frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{l}} - \frac{1}{2}a_1\sqrt{\frac{\beta_1}{\beta_3}}\left(1+\frac{\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]}{\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]}\right) \tag{16}$$

$$U_6(x,y,t) = \frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{l}} + a_1\left(-1-\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}\right) \tag{17}$$

**Case-III**

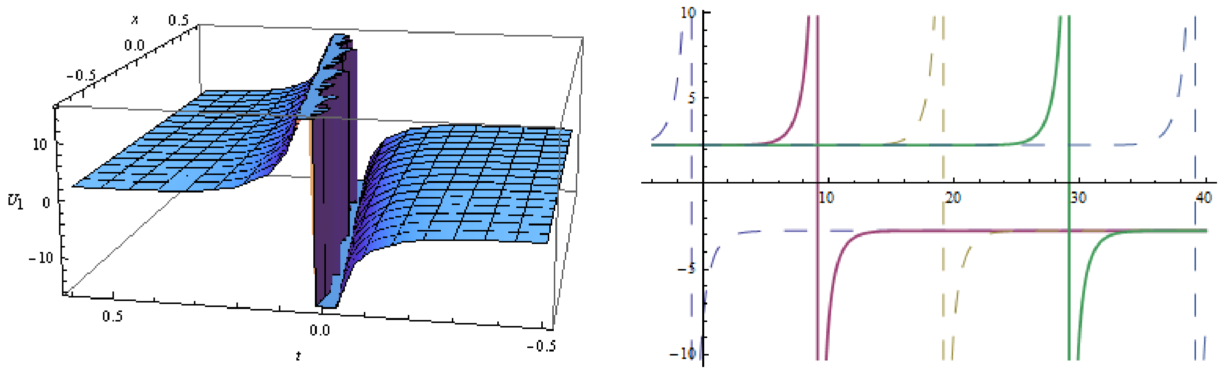
$$a_0 = a_1 = b_1 = 0, d_1 = \pm\frac{\sqrt{-l-\omega}}{\sqrt{\alpha}\sqrt{\beta_1}\sqrt{l}}, m = \pm\frac{(-1)^{2/3}\sqrt[3]{2(l+\omega)-\beta_1\gamma l^2\omega}}{\sqrt[3]{\beta_1}\sqrt[3]{\gamma}}. \tag{18}$$

Substituting Eq.(18) into Eq.(9), then the solutions of Eq.(6) can be obtained as:

$$U_7(x,y,t) = \frac{\epsilon\sqrt{-l-\omega}\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]^2}{\sqrt{\alpha}\sqrt{l}\left(2\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+2\right)} \tag{19}$$

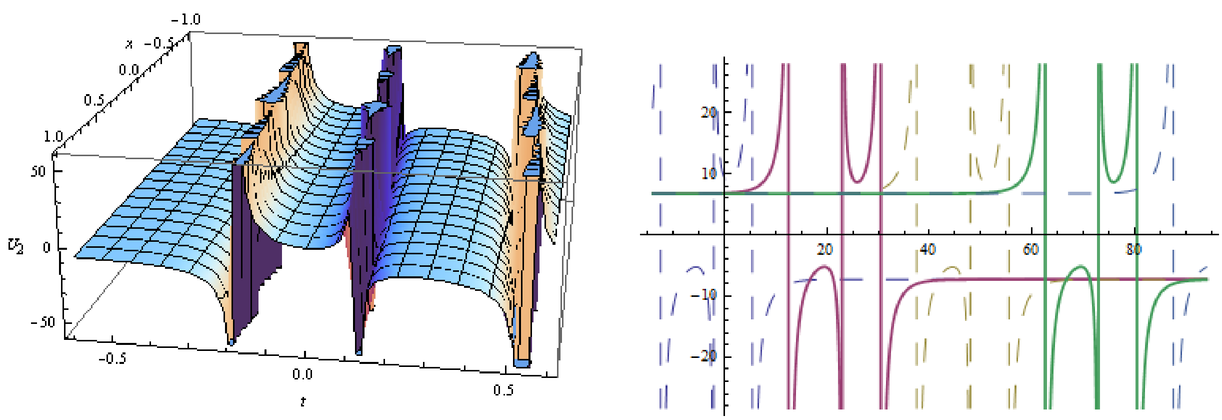
$$\begin{aligned} U_8(x,y,t) = &-\epsilon\sqrt{-l-\omega}\left(\eta\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1\right) / \left(\sqrt{\alpha}\sqrt{l}\left(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\right. \\ &\left.\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right) \end{aligned} \tag{20}$$

$$\begin{aligned} U_9(x,y,t) = &\epsilon\sqrt{-l-\omega}\left(\eta\left(-\sqrt{p^2+1}\right)\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]-1\right) / \\ &\left(\sqrt{\alpha}\sqrt{l}\left(\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}\right)\left(\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}\right.\right. \\ &\left.\left.+\ \epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\right)\right)\right) \end{aligned} \tag{21}$$



(a) (b)

**Figure 1:** Solitary wave solution given in Eq.(11) when $a_1 = 0.5, \beta_1 = 2, \beta_2 = 4, \epsilon = 10, \eta = -8, \xi_0 = 0.5, l = 0.8, m = 0.2, \omega = -8, y = 1, \alpha = 1$



(a) (b)

**Figure 2:** Solitary wave solution given in Eq.(12) when $a_1 = 0.5, \beta_1 = 4, \beta_2 = 8, \beta_3 = 4, \epsilon = 28, \eta = -18, \xi_0 = 0.3, l = 0.2, m = 0.4, \omega = -5, y = 1, \alpha = 1$

**Case-IV**

$$a_0 = \pm\frac{\sqrt{-l-\omega}}{2\sqrt{\alpha}\sqrt{l}}, a_1 = b_1 = 0, d_1 = \pm\frac{\sqrt{-l-\omega}}{2\sqrt{\alpha}\sqrt{\beta_1}\sqrt{l}} \tag{22}$$

Substituting Eq.(22), only the positive value of $a_0$ into Eq.(9), then the solutions of Eq.(6), can be given as:

$$U_{10}(x,y,t) = \frac{\sqrt{-l-\omega}\left(\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]-1\right)\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]^2+2\right)}{4\sqrt{\alpha}\sqrt{l}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1\right)} \tag{23}$$

$$U_{11}(x,y,t)=\left(\sqrt{-l-\omega}\left(1+2\epsilon+2\eta^2+2\eta(\epsilon+2)\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\cosh\left[2\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+2\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\left(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\right)\right)\Big/\left(4\sqrt{\alpha}\sqrt{l}\left(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\left(\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]\right)\right) \tag{24}$$

$$U_{12}(x,y,t)=\frac{\sqrt{-l-\omega}\left(\frac{\epsilon\left(\eta\sqrt{p^2+1}\cosh\left[\sqrt{\beta_1}(lx+my+\xi_0+t\omega)\right]-p\sinh\left[\sqrt{\beta_1}(lx+my+\xi_0+t\omega)\right]+1\right)}{\left(\cosh\left[\sqrt{\beta_1}(lx+my+\xi_0+t\omega)\right]+\eta\sqrt{p^2+1}\right)\left(\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\xi_0+t\omega)\right]+p\right)+\cosh\left[\sqrt{\beta_1}(lx+my+\xi_0+t\omega)\right]+\eta\sqrt{p^2+1}\right)}+1\right)}{2\sqrt{\alpha}\sqrt{l}} \tag{25}$$

**Case-V**

$$a_0=a_0,\ a_1=0,\ b_1=\frac{2a_0\beta_1}{\beta_2},\ d_1=0,\ m=\frac{\sqrt[3]{-1}\sqrt[3]{\omega}}{\sqrt[3]{\beta_1}\sqrt[3]{\gamma}} \tag{26}$$

Substituting Eq.(26) into Eq.(9), then the solutions of Eq.(6) can be obtained as:

$$U_{13}(x,y,t)=a_0\left(1-\frac{2}{\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1}\right) \tag{27}$$

$$U_{14}(x,y,t)=a_0\left(1-\frac{4\sqrt{\frac{\beta_1}{\beta_3}}\beta_3}{\beta_2+\frac{\beta_2\epsilon\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]}{\eta+\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]}}\right) \tag{28}$$

$$U_{15}(x,y,t)=a_0\left(1-\frac{2\beta_1}{\beta_2\left(\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}+1\right)}\right) \tag{29}$$

**Case-VI**

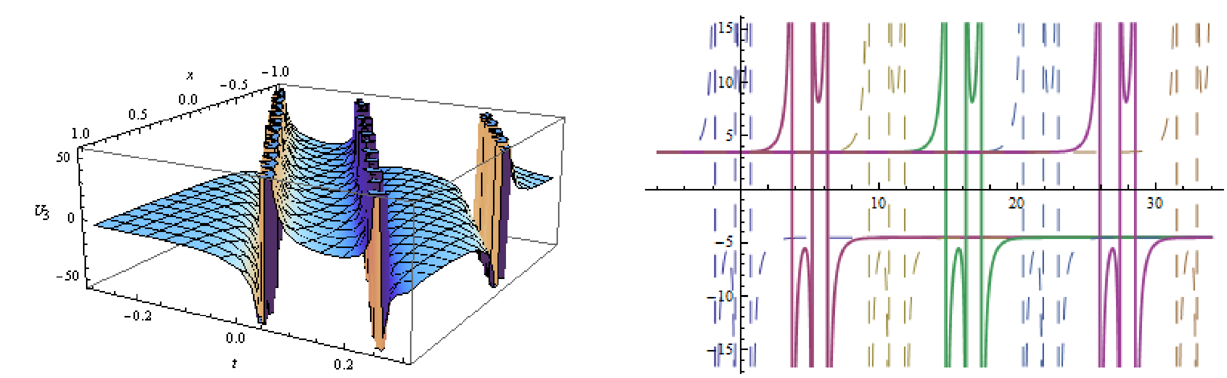
$$a_0=\pm\frac{\sqrt{6\beta_1\gamma\omega^2+9}}{2\sqrt{\alpha\left(4\beta_1^2\gamma^2\omega^4-9\right)}},\ a_1=b_1=0,\ d_1=\pm\frac{\sqrt{6\beta_1\gamma\omega^2+9}}{2\sqrt{\alpha\beta_1\left(4\beta_1^2\gamma^2\omega^4-9\right)}},\quad l=\frac{3}{2\beta_1\gamma\omega}-\omega,\ m=\pm\frac{\sqrt[3]{-1}\sqrt[3]{3-4\beta_1\gamma\omega^2\left(\beta_1\gamma\omega^2-3\right)}}{2^{2/3}\beta_1^{2/3}\gamma^{2/3}\sqrt[3]{\omega}}. \tag{30}$$

Substituting Eq.(30), only positive value of $a_0$ and $d_1$ into Eq.(9), the solitary wave solutions of Eq.(6) can be obtained in the simplified form as:

$$U_{16}(x,y,t)=\frac{1}{4}\sqrt{6\beta_1\gamma\omega^2+9}\left(\frac{2}{\sqrt{\alpha\left(4\beta_1^2\gamma^2\omega^4-9\right)}}-\frac{\sqrt{\beta_1}\epsilon\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]^2}{\sqrt{\alpha\beta_1\left(4\beta_1^2\gamma^2\omega^4-9\right)}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+1\right)}\right) \tag{31}$$

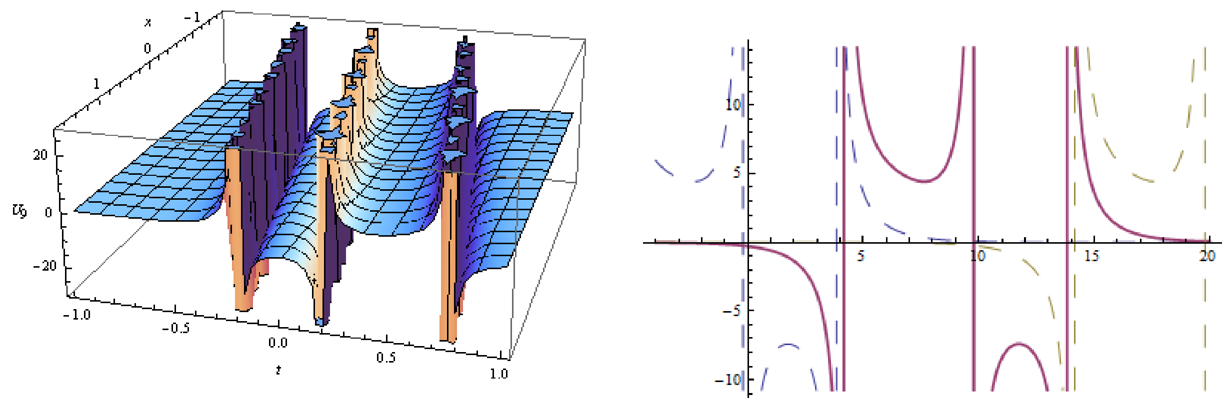
$$\begin{aligned}U_{17}(x,y,t)=&\frac{1}{2}\sqrt{6\beta_1\gamma\omega^2+9}\left(\frac{1}{\sqrt{\alpha\left(4\beta_1^2\gamma^2\omega^4-9\right)}}+\sqrt{\beta_1}\epsilon\left(\eta\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+1\right)\Big/\right.\\&\left(\eta+\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right)\left(\eta+\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right.\\&\left.\left.+\epsilon\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right)\sqrt{\alpha\beta_1\left(4\beta_1^2\gamma^2\omega^4-9\right)}\right)\end{aligned} \tag{32}$$

$$\begin{aligned}U_{18}(x,y,t)=&\frac{1}{2}\sqrt{6\beta_1\gamma\omega^2+9}\left(\frac{1}{\sqrt{\alpha\left(4\beta_1^2\gamma^2\omega^4-9\right)}}+\left(\epsilon\left(\eta\sqrt{p^2+1}\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+1\right.\right.\right.\\&\left.\left.-\,p\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right)\sqrt{\beta_1}\right)\Big/\left(\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+\eta\sqrt{p^2+1}\right.\\&\left(\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+\eta\sqrt{p^2+1}+\epsilon\left(\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+p\right)\right)\\&\left.\left.\sqrt{\alpha\beta_1\left(4\beta_1^2\gamma^2\omega^4-9\right)}\right)\right)\end{aligned} \tag{33}$$




(a) (b)

**Figure 3:** Solitary wave solution given in Eq.(13) when $a_1 = 0.5, \beta_1 = 4, \epsilon = 8, \eta = -5, \xi_0 = 0.3, l = 0.9, m = 0.2, \omega = -5, y = 1, a = 0.4, p = 0.2$




(a) (b)

**Figure 4:** Solitary wave solution given in Eq.(21) when $a_1 = 0.4, \beta_1 = 2, \epsilon = 18, \eta = -15, \xi_0 = 0.3, l = 0.5, m = 0.2, \omega = -5, y = 1, \alpha = 0.4, p = 0.2$

**Case-VII**

$$a_0 = \pm \frac{\sqrt{1-\beta_1^2\gamma^2\omega^4}}{\sqrt{2}\sqrt{\alpha\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}}, a_1 = b_1 = 0,$$
$$d_1 = \pm \frac{\sqrt{1-\beta_1^2\gamma^2\omega^4}}{\sqrt{2}\sqrt{\alpha\beta_1\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}}, l = \frac{2}{\beta_1\gamma\omega}+\omega, m = -(-1)^{2/3}\omega. \tag{34}$$

Substituting Eq.(34), only the positive value of $a_0$ and $d_1$ into Eq.(9), then the solutions of Eq.(6) can be get in the simplified form as:

$$U_{19}(x,y,t) = \frac{\sqrt{1-\beta_1^2\gamma^2\omega^4}\left(\frac{2}{\sqrt{\alpha(\beta_1\gamma\omega^2-1)(\beta_1\gamma\omega^2+2)}} - \frac{\sqrt{\beta_1}\epsilon\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]^2}{\sqrt{\alpha\beta_1(\beta_1\gamma\omega^2-1)(\beta_1\gamma\omega^2+2)}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(lx+my+\omega t+\xi_0)\right]+1\right)}\right)}{2\sqrt{2}} \tag{35}$$

$$U_{20}(x,y,t) = \frac{1}{\sqrt{2}}\sqrt{1-\beta_1^2\gamma^2\omega^4}\left(\frac{1}{\sqrt{\alpha\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}} + \sqrt{\beta_1}\epsilon\left(\eta\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+1\right) / \right.$$
$$\left(\eta+\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\left(\eta+\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right.\right.$$
$$\left.\left.\left. + \epsilon\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right)\sqrt{\alpha\beta_1\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}\right)\right) \tag{36}$$

$$U_{21}(x,y,t) = \frac{1}{\sqrt{2}}\sqrt{1-\beta_1^2\gamma^2\omega^4}\left(\frac{1}{\sqrt{\alpha\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}} + \left(\epsilon\left(\eta\sqrt{p^2+1}\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+1\right.\right.\right.$$
$$\left.\left. - p\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]\right)\sqrt{\beta_1}\right) / \left(\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+\eta\sqrt{p^2+1}\right.$$
$$\left(\eta\sqrt{p^2+1}+\cosh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+\epsilon\left(\sinh\left[\sqrt{\beta_1}\left(lx+my+\omega t+\xi_0\right)\right]+p\right)\right)$$
$$\left.\left.\sqrt{\alpha\beta_1\left(\beta_1\gamma\omega^2-1\right)\left(\beta_1\gamma\omega^2+2\right)}\right)\right) \tag{37}$$

## 3.2 Simplified modified form of Camassa-Holm equation

We consider a simplified modified form of CH-equation as

$$U_t + 2\beta U_x - U_{xxt} + \delta U^2 U_x = 0, \tag{38}$$

where $\beta$ and $\delta$ are non zero constants. Consider the wave transformation as; $U(x, y, t) = U(\xi), \xi = kx + \omega t$. By this transformation we obtained ordinary differential equation of Eq.(38) as

$$\omega U' + 2\beta k U' - k^2 \omega U''' + \delta k U^2 U' = 0, \tag{39}$$

we integrate Eq.(39) once time according to the $\xi$ and constant of integration equal to zero, then we obtained as

$$\omega U + 2\beta k U - k^2 \omega U'' + \frac{1}{3}\delta k U^3 = 0. \tag{40}$$

We balance the nonlinear term and highest order derivative in Eq.(40) allow obtaining the value of $n = 1$. The general solution of Eq.(40) takes the form of:

$$U(\xi) = a_0 + a_1 \Psi(\xi) + \frac{b_1}{\Psi(\xi)} + d_1 \frac{\Psi'(\xi)}{\Psi(\xi)} \tag{41}$$

Substituting Eq.(41) into Eq.(40) and combining each coefficients of $\Psi'^{j}(\xi)\Psi^{i}(\xi)(j = 0, 1; i = 1, 2, 3, \ldots n)$, then making a every coefficient equal to zero and obtaining a set of algebraic equations. We solve this system of equations with the aid of Mathematica. The parameters $a_0, a_1, b_1, d_1$ can be determined as

**Case-1**

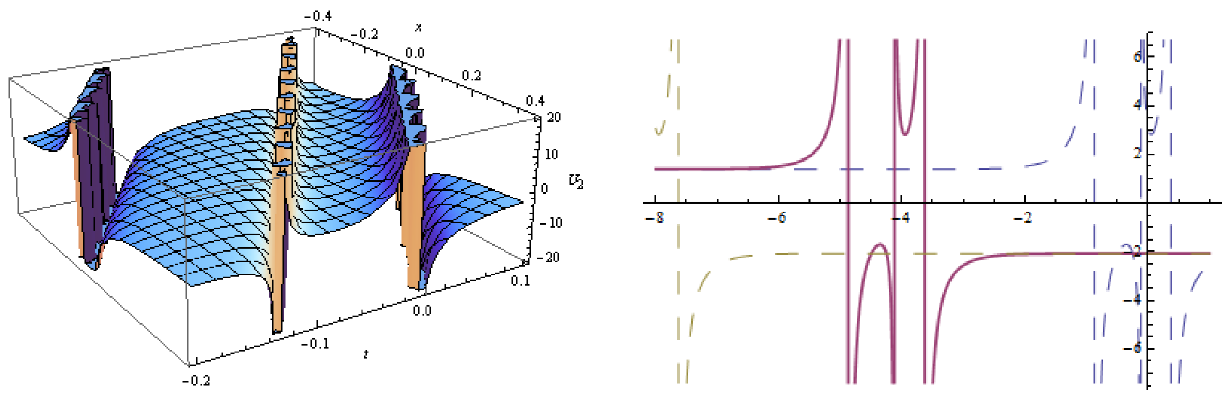
$$a_0 = 0, a_1 = \pm\frac{\sqrt{6}\sqrt{\beta}\sqrt{\beta_3}k}{\sqrt{\beta_1(-\delta)k^2 - 2\delta}}, b_1 = 0, d_1 = \pm\frac{\sqrt{6}\sqrt{\beta}k}{\sqrt{\beta_1(-\delta)k^2 - 2\delta}}, \omega = -\frac{4\beta k}{\beta_1 k^2 + 2}. \tag{42}$$

Substituting the Eq.(42), only for the positive value of $a_1$ and $d_1$ in Eq.(41), the solutions of Eq.(38) are given as:

$$U_1(x, y, t) = -\frac{\sqrt{\frac{3}{2}}\sqrt{\beta}k\left(2\beta_1\sqrt{\beta_3}\left(\epsilon \coth\left[\frac{1}{2}\sqrt{\beta_1}(kx + \omega t + \xi_0)\right] + 1\right)^2 + \sqrt{\beta_1}\beta_2 \epsilon \operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(kx + \omega t + \xi_0)\right]^2\right)}{\beta_2\sqrt{-\delta\left(\beta_1 k^2 + 2\right)}\left(\epsilon \coth\left[\frac{1}{2}\sqrt{\beta_1}(kx + \omega t + \xi_0)\right] + 1\right)} \tag{43}$$

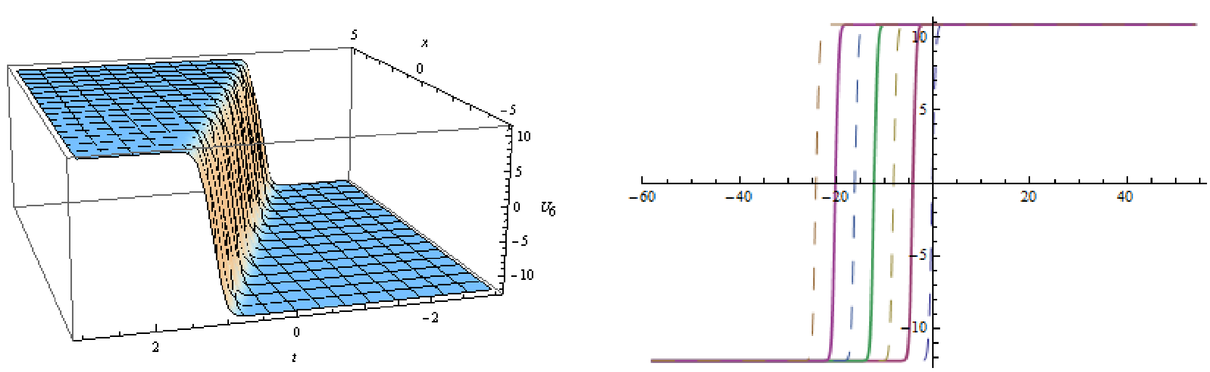
$$\begin{aligned} U_2(x, y, t) = -\Bigg(&\sqrt{\frac{3}{2}}k\sqrt{\beta}\Big(-2\sqrt{\beta_1}\epsilon\left(\eta \cosh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right] + 1\right) + \Big(\eta + \cosh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right] \\ &+ \epsilon \sinh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right]\Big)^2 \sqrt{\frac{\beta_1}{\beta_3}}\sqrt{\beta_3}\Big)\Bigg) \Bigg/ \Big(\left(\eta + \cosh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right]\right. \\ &\left(\eta + \cosh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right] + \epsilon \sinh\left[\sqrt{\beta_1}(kx + \omega t + \xi_0)\right]\right)\sqrt{-\delta\left(\beta_1 k^2 + 2\right)}\Big) \end{aligned} \tag{44}$$

$$
\begin{aligned}
U_3(x,y,t) = &-\left(\sqrt{6}\sqrt{\beta}k\left(\left(\epsilon\left(\eta\left(-\sqrt{p^2+1}\right)\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]-1 \quad \sqrt{\beta_1}\right.\right.\right.\right.\\
&+\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\left(2\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+2\eta\sqrt{p^2+1}\right.\\
&+\epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)\Big)\ \epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)\Big)\\
&\sqrt{-\delta\left(\beta_1k^2+2\right)}\left(\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(kx+\xi_0+t\omega)\right]+\eta\sqrt{p^2+1}}+1\right)
\end{aligned}
\tag{45}
$$




(a) (b)

**Figure 5:** Solitary wave solution given in Eq.(44) when $\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 5, \eta = -3, \xi_0 = 0.5, k = 2, \omega = 8, \alpha = 0.2, \delta = -2$




(a) (b)

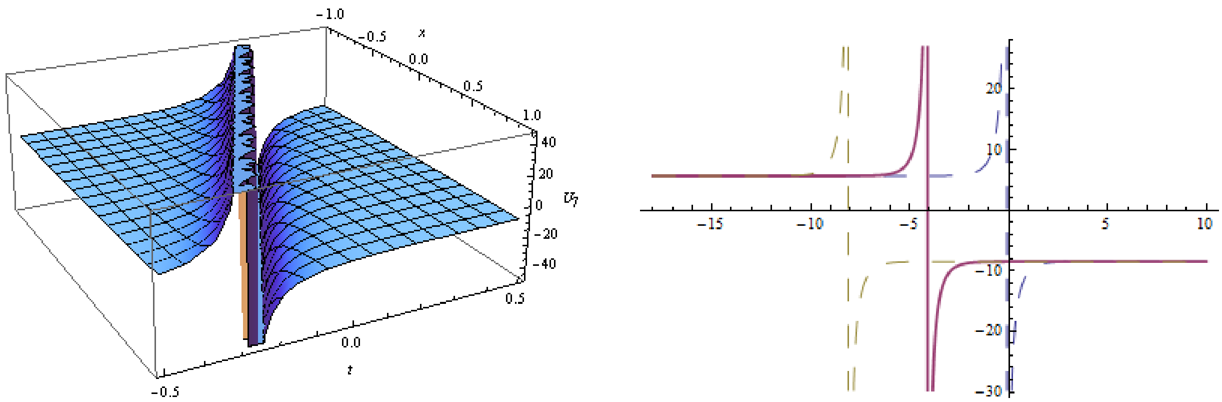
**Figure 6:** Solitary wave solution given in Eq.(49) when $\beta_1 = 4, \beta_3 = 2, \epsilon = -5, \eta = 2, \xi_0 = 0.3, k = 2, \omega = 8, p = 0.6, \alpha = 0.2, \delta = -4$

**Case-II**

$$
\begin{aligned}
&a_0 = \pm\frac{\sqrt{6}\sqrt{\beta}\sqrt{\beta_1}k}{\sqrt{\beta_1(-\delta)k^2-2\delta}}, a_1 = \pm\frac{2\sqrt{6}\sqrt{\beta}\sqrt{\beta_3}k}{\sqrt{\beta_1(-\delta)k^2-2\delta}}, b_1 = d_1 = 0,\\
&\omega = -\frac{4\beta k}{\beta_1k^2+2}, \beta_2 = \pm 2\sqrt{\beta_1}\sqrt{\beta_3}.
\end{aligned}
\tag{46}
$$

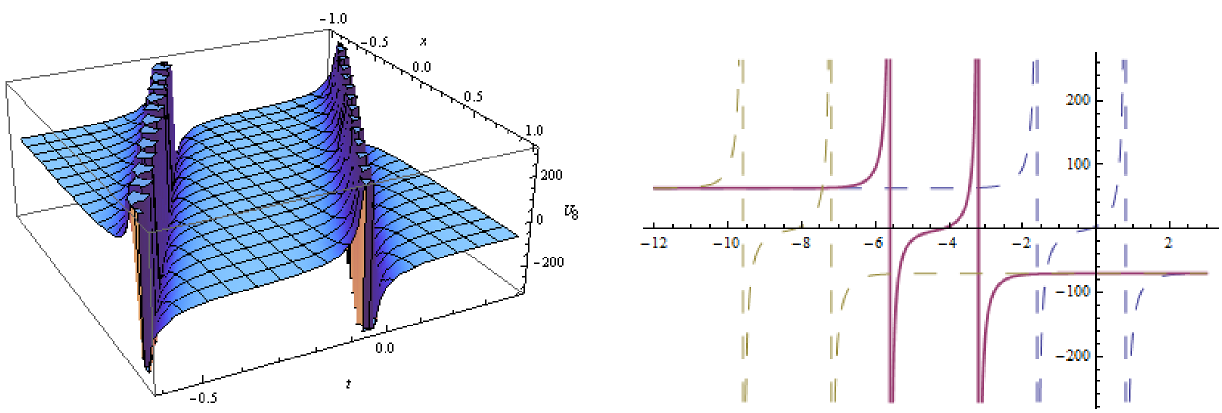
Substituting Eq. (46), only the positive value of $a_0$ and $a_1$ into Eq. (41), the solutions of Eq. (38) can be given as:

$$U_4(x,y,t) = -\frac{\sqrt{6}\sqrt{\beta}k\left(2\beta_1\sqrt{\beta_3}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+1\right)-\sqrt{\beta_1}\beta_2\right)}{\beta_2\sqrt{-\delta\left(\beta_1k^2+2\right)}} \tag{47}$$



(a) (b)

**Figure 7:** Solitary wave solution given in Eq. (51) when $\beta_1 = 1, \beta_2 = 2, \beta_3 = 1, \epsilon = 5, \eta = 3, \xi_0 = 0.2, k = 2, \omega = 8, \alpha = 1, \delta = 2$



(a) (b)

**Figure 8:** Solitary wave solution given in Eq. (52) when $\beta_1 = 2, \beta_2 = 4, \beta_3 = 2, \epsilon = 18, \eta = -15, \xi_0 = 0.8, k = 2, \omega = 8, \alpha = 8, \delta = 2$

$$U_5(x,y,t) = \left(\sqrt{6}k\sqrt{\beta}\left(\sqrt{\beta_1}\left(\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right)-\left(\eta+\cosh \mathit{left}[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right.\right.$$
$$\left.\left.+\epsilon\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right)\sqrt{\frac{\beta_1}{\beta_3}}\sqrt{\beta_3}\right)\right)\Big/\sqrt{-\delta\left(\beta_1k^2+2\right)}\left(\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right) \tag{48}$$

$$U_6(x,y,t) = \frac{\sqrt{6}\sqrt{\beta}k\left(\sqrt{\beta_1}+2\sqrt{\beta_3}\left(-\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}-1\right)\right)}{\sqrt{-\delta\left(\beta_1k^2+2\right)}} \tag{49}$$

**Case-III**

$$a_0 = 0, a_1 = \pm\frac{2\sqrt{3}\sqrt{\beta}\sqrt{\beta_3}k}{\sqrt{\beta_1\delta k^2-\delta}}, b_1 = d_1 = 0, \omega = \frac{2\beta k}{\beta_1 k^2 - 1}. \quad (50)$$

Substituting Eq.(50), only the positive value of $a_1$ into Eq.(41) the solutions of Eq.(38) can be given as:

$$U_7(x,y,t) = -\frac{2\sqrt{3}\sqrt{\beta}\beta_1\sqrt{\beta_3}k\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+1\right)}{\beta_2\sqrt{\delta\left(\beta_1k^2-1\right)}} \quad (51)$$

$$U_8(x,y,t) = -\frac{\sqrt{3}\sqrt{\beta}\sqrt{\frac{\beta_1}{\beta_3}}\sqrt{\beta_3}k\left(\frac{\epsilon\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]}{\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]}+1\right)}{\sqrt{\delta\left(\beta_1k^2-1\right)}} \quad (52)$$

$$U_9(x,y,t) = \frac{2\sqrt{3}\sqrt{\beta}\sqrt{\beta_3}k\left(-\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}-1\right)}{\sqrt{\delta\left(\beta_1k^2-1\right)}} \quad (53)$$

**Case-IV**

$$a_0 = a_1 = 0, b_1 = 0, d_1 = \pm\frac{2\sqrt{3}\sqrt{\beta}k}{\sqrt{-\delta-2\beta_1\delta k^2}}, \omega = -\frac{2\beta k}{2\beta_1k^2+1} \quad (54)$$

Substituting Eq. (54), only the positive value of $d_1$ into Eq. (41), the solutions of Eq. (38) can be given as:

$$U_{10}(x,y,t) = -\frac{\sqrt{3}\sqrt{\beta}\sqrt{\beta_1}k\epsilon\operatorname{csch}\left[\frac{1}{2}\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]^2}{\sqrt{-\delta\left(2\beta_1k^2+1\right)}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+1\right)} \quad (55)$$

$$U_{11}(x,y,t) = 2\sqrt{3}\sqrt{\beta}\sqrt{\beta_1}k\epsilon\left(\eta\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right.$$
$$\left.+1\right)/\left(\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right.$$
$$\sqrt{-\delta\left(2\beta_1k^2+1\right)}\left(\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right.$$
$$\left.\left.+\epsilon\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right)\right) \quad (56)$$

$$U_{12}(x,y,t) = 2\sqrt{3}\sqrt{\beta}\sqrt{\beta_1}k\epsilon\left(\eta\sqrt{p^2+1}\cosh\right.$$
$$\left.\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]-p\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+1\right)/$$
$$\left(\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]\right.$$
$$+\eta\sqrt{p^2+1}\left(\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}\right.$$
$$\left.\left.+\ \epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)\right)\sqrt{-\delta\left(2\beta_1k^2+1\right)}\right) \quad (57)$$

**Case-V**

$$a_0 = \pm\frac{\sqrt{2}\sqrt{\beta}}{\sqrt{\delta}}, a_1 = \pm\frac{2\sqrt{2}\sqrt{\beta}\sqrt{\beta_3}}{\sqrt{\beta_1}\sqrt{\delta}}, b_1 = d_1 = 0, k = \frac{1}{\sqrt{\beta_1}},$$
$$\omega = -\frac{4\beta}{3\sqrt{\beta_1}}, \beta_2 = \pm2\sqrt{\beta_1}\sqrt{\beta_3} \quad (58)$$

Substituting Eq. (58), only the positive value of $a_0$ and $a_1$ into Eq. (41), the solutions of Eq. (38) can be given as:

$$U_{13}(x,y,t) = \frac{\sqrt{2}\sqrt{\beta}\left(\beta_2-2\sqrt{\beta_1}\sqrt{\beta_3}\left(\epsilon\coth\left[\frac{1}{2}\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+1\right)\right)}{\beta_2\sqrt{\delta}}$$

$$U_{14}(x,y,t) = \frac{\sqrt{2}\sqrt{\beta}\left(1-\frac{\sqrt{\frac{\beta_1}{\beta_3}}\sqrt{\beta_3}\left(\frac{\epsilon\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]}{\eta+\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]}+1\right)}{\sqrt{\beta_1}}\right)}{\sqrt{\delta}} \quad (59)$$

$$U_{15}(x,y,t) = \frac{\sqrt{2}\sqrt{\beta}\left(\frac{2\sqrt{\beta_3}\left(-\frac{\epsilon\left(\sinh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+p\right)}{\cosh\left[\sqrt{\beta_1}(kx+\omega t+\xi_0)\right]+\eta\sqrt{p^2+1}}-1\right)}{\sqrt{\beta_1}}+1\right)}{\sqrt{\delta}}$$

## 4 Conclusion

We have successfully applied a new method on two nonlinear evaluation equations. We have obtained a new exact traveling and solitary wave solutions of GZK-BBM-equation and simplified modified form of CH-equation by applying the Modified extended auxiliary equation mapping method. As a results, these new solutions are obtained in the form of elliptic functions, trigonometric functions, kink and antikink solitions, bright and dark solitons, periodic solitary wave and travelling wave solutions and also show two and three dimensional graphs with the help of Mathematica. These new families of solutions show the power, effectiveness, capability, realizabilities and fruitfulness of this new method. We can solve other nonlinear physical phenomena, which are related to nonlinear evaluation equations with the help of this new method.

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