DEGENERATE REPRESENTATIONS FROM QUANTUM KINEMATICAL CONSTRAINTS

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1. Introduction

We present recent results /1/ concerning the classification of the finite-dimensional representations and of the Poisson bracket (PB) realizations of the real compact forms for all classical semisimple Lie algebras which satisfy *second-degree polynomial identities*. The expressions of these identities are presented in Section 2.

For the same algebras, a closely related problem, namely the classification of all pairs $\{\rho_\Lambda, \rho_M\}$ of finite-dimensional representations the Kronecker product $\rho_\Lambda \otimes \rho_M$ of which decomposes into two irreducible components has also been solved: in each such pair, ρ_Λ is a representation the highest weight of which is a minuscule weight Λ and ρ_M is a representation for which the adjoint orbit of the maximal weight vector is a Hermitian symmetric space.

A classical analogue of the Hannabuss operator /2/ associated with these Kronecker products can also be defined for any pair (f, ρ) in which f is a PB realization and ρ a finite dimensional representation. This analogue - which is a mathematical object defined on a symplectic manifold with values operators on the representation space /1,3/ - satisfies in our case second-degree polynomial equations which can be obtained as a classical limit of the equations satisfied by the Hannabuss operator /1/.

All these results are intimately related to the structure of completely integrable classical or quantum systems. For instance, the finite-dimensional representations on which the second-degree irreducible tensors in the envoloping algebra vanish are exactly the representations which can be extended to the representations of the Yangians obtained by Drinfeld /4/ in connection with the problem of solving the quantum Yang-Baxter equations; the representations ρ_M associated with Hermitian symmetric spaces are those used by Reshetikhin /5/ in his construction of the elementary realizations of Yang-Baxter-Zamolodchikov-Faddeev (YBZF) algebras /6/. For the elementary classical realizations defined by Reshetikhin /5/ a complete classification has been obtained /6/.

2. Tensorial identities associated with realizations of Lie algebras

The homogeneous identities for linear representations (for PB realizations) of a Lie algebra L result /1,7/ by equating to zero the irreducible tensors in the enveloping algebra U(L) (in the symmetric algebra S(L)). We list in the following the second-degree "tensorial identities" for the representations of the semisimple Lie algebras:

$$A_n(n\geq 3),\ B_n(n\geq 2),\ C_n(n\geq 2),\ D_n(n>5) \qquad (2.1)$$

The second-degree tensors in U(L) have been derived /1/ by reducing the symmetric part of the Kronecker square $(ad\otimes ad)_s$ of the adjoint representation. For the Lie algebras (1.1) the

Clebsch-Gordan series of $(\mathrm{ad}\otimes\mathrm{ad})_s$ is multiplicity-free and contains four terms /8,9/. To each irreducible tensor $T_{L,\Omega}$ in U(L), transforming under the representation $(\Omega)\subset(\mathrm{ad}\otimes\mathrm{ad})_s$ of highest weight Ω a tensorial identity is associated, by equating $T_{L,\Omega}$ to zero in a representation to be determined.

Let us denote by $\Lambda_1,\Lambda_2,\ldots,\Lambda_n$ the highest weights of the fundamental representations. The conventions adopted in /10/ have been used.

I) Algebras of type A_n. The generators A_{ij} $(i,j = 1,2,\ldots,n+1)$ $(\sum_{i=1}^{n+1} A_{ii} = 0)$ of the algebra $sl(n+1,C)$ satisfy the structure relations $[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{il}A_{kj}$. Denoting representations by their highest weights we have $\mathrm{ad} = (\Lambda_1+\Lambda_n)$ and $(\mathrm{ad}\otimes\mathrm{ad})_s = (0)\oplus(\Lambda_1+\Lambda_n)\oplus(\Lambda_2+\Lambda_{n-1})\oplus(2\Lambda_1+2\Lambda_n)$. Defining

$$\mathcal{A}_{pq}(\lambda) \equiv \sum_{i=1}^{n+1}[A_{pi}, A_{iq}]_+ - \frac{\delta_{pq}}{2\lambda}\sum_{i,j=1}^{n+1} A_{ij}A_{ji} \tag{2.2}$$

(where $[a,b]_+ = ab + ba$) we obtain the tensorial identities

$$T_{\Lambda_1+\Lambda_n}(p,q) = \mathcal{A}_{pq}(n+1) = 0 \quad (p,q = 1,2,\ldots,n+1) \tag{2.3}$$

$$\begin{aligned} T_{\Lambda_2+\Lambda_{n-1}}(p,q,r,s) &= [A_{pq}, A_{rs}]_+ - [A_{ps}, A_{rq}]_+ \\ &+ \frac{1}{n-1}\{-\delta_{qr}\mathcal{A}_{ps}(2n) - \delta_{ps}\mathcal{A}_{rq}(2n) + \delta_{pq}\mathcal{A}_{rs}(2n) + \delta_{rs}\mathcal{A}_{pq}(2n)\} = 0 \\ &\qquad (p,q,r,s = 1,2,\ldots,n+1) \end{aligned} \tag{2.4}$$

$$\begin{aligned} T_{2\Lambda_1+2\Lambda_n}(p,q,r,s) &= [A_{pq}, A_{rs}]_+ + [A_{ps}, A_{rq}]_+ \\ &- \frac{1}{n+3}\{\delta_{qr}\mathcal{A}_{ps}(2(n+2)) + \delta_{ps}\mathcal{A}_{rq}(2(n+2)) + \delta_{pq}\mathcal{A}_{rs}(2(n+2)) + \delta_{rs}\mathcal{A}_{pq}(2(n+2))\} = 0 \end{aligned} \tag{2.5}$$

II) Algebras of types B_n, C_n and D_n. Let us adopt a unifying notation for these series and denote their generators by X_{ij} $(i,j = 1,2,\ldots,N;\ N = 2n+1$ for B_n and $N = 2n$ for C_n and $D_n)$ with the Lie relations $[X_{ij}, X_{kl}] = g_{kj}X_{il} - g_{il}X_{kj} - g_{ik}X_{jl} + g_{lj}X_{ki}$. For C_n, $X_{ij} = X_{ji}$, $g_{ij} = \delta_{i,j+n} - \delta_{i+n,j}$ and $\mathrm{ad} = (2\Lambda_1)$; for B_n and D_n, $X_{ij} = -X_{ji}$, $g_{ij} = \delta_{ij}$ and $\mathrm{ad} = (\Lambda_2)$ (for B_n, $n > 3$ and D_n $n > 5$) for B_2, $\mathrm{ad} = (2\Lambda_2)$. We have

$$(\mathrm{ad}\otimes\mathrm{ad})_s = (0)\oplus\begin{matrix}(2\Lambda_1)\\(\Lambda_2)\end{matrix}\oplus\begin{matrix}(\Lambda_4)\\(4\Lambda_1)\end{matrix}\oplus(2\Lambda_2) \quad \text{for} \quad \begin{matrix}B_n\,(n>3),\ D_n\,(n>5)\\ C_n\,(n\geq 2)\end{matrix} \tag{2.6}$$

Here and in the following, for B_2 replace (Λ_4) by (Λ_1) and $(2\Lambda_2)$ by $(4\Lambda_2)$; for B_3 replace (Λ_4) by $(2\Lambda_3)$. Denoting

$$\mathcal{X}_{pq}(\lambda) \equiv \sum_{i,j=1}^{N} g_{ji}[X_{pi}, X_{jq}]_+ - \frac{g_{pq}}{2\lambda}\sum_{i,j,k,l=1}^{N} g_{ij}g_{kl}X_{il}X_{jk} \tag{2.7}$$

the tensorial identities associated with the nontrivial terms in (2.6) are

$$T_{\substack{(2\Lambda_1)\\(\Lambda_2)}}(p,q) = \mathcal{X}_{pq}(N) = 0 \quad (p,q = 1,2,\ldots,N) \tag{2.8}$$

$$T_{\substack{(\Lambda_4)\\(4\Lambda_1)}}(p,q,r,s) = [X_{pq}, X_{rs}]_+ + [X_{ps}, X_{qr}]_+ + [X_{pr}, X_{sq}]_+ = 0 \quad (p,q,r,s = 1,2,\ldots,N) \tag{2.9}$$

$$\begin{aligned} T_{(2\Lambda_2)}(p,q,r,s) &= \frac{1}{3}\{2[X_{pq}, X_{rs}]_+ - [X_{ps}, X_{qr}]_+ - [X_{pr}, X_{sq}]_+\} \\ &+ \frac{1}{N-2\varepsilon}\{-g_{sp}\mathcal{X}_{qr}(2(N-2\varepsilon)) + g_{pr}\mathcal{X}_{qs}(2(N-2\varepsilon)) - g_{rq}\mathcal{X}_{ps}(2(N-2\varepsilon)) + g_{qs}\mathcal{X}_{pr}(2(N-2\varepsilon))\} = 0 \end{aligned} \tag{2.10}$$

where $\varepsilon = +1$ for so(N) and $\varepsilon = -1$ for sp(N).

3. The finite-dimensional representationa on which the tensorial identities vanish

Theorem 1. Let L be one of the semisimple Lie algebras (2.1); let $T_{L\Omega}(x_1,\ldots,x_{\dim L}) \in U(L)$ (x_i = generators of L) be the second-degree tensor operator which transforms under the subrepresentation of $(\mathrm{ad}\otimes\mathrm{ad})_s$ of L with highest weight Ω and let $\rho_\Lambda(x_i)$ ($i = 1,\ldots,\dim L$) be the generators of the representation ρ_Λ of L of highest weight Λ. The finite-dimensional irreducible representations ρ_Λ and the tensors $T_{L\Omega}$ of L for which

$$T_{L\Omega}(\rho_\Lambda) = T_{L\Omega}(\rho_\Lambda(x_1),\ldots,\rho_\Lambda(x_{\dim L})) = 0 \tag{3.1}$$

are those listed in columns 4, 6 and in columns 5, 7 of Table 1, respectively.

The proof /1/ results by writing the identities in a Cartan-Weyl basis and by observing that a sufficient condition for a tensor operator to vanish on a finite-dimensional representation is its vanishing if applied on the highest-weight vector of the representation. The tensorial identities transform into equations for the highest weights, which can be solved.

The solutions of the tensorial identities have also been obtained by another proof which makes use of the Wigner-Eckart theorem.

The highest weights Λ of the solutions ρ_Λ of the equations $T_{L\Omega}(\rho_\Lambda) = 0$ can be characterized in a synthetic way using the coefficients c_i, d_i, $c_i^\vee$, $d_i^\vee$ of the highest long root ($\alpha_{hl} = \sum_{i=1}^{n} c_i \alpha_i$), of the highest short root ($\alpha_{hs} = \sum_{i=1}^{n} d_i \alpha_i$) and of their duals ($\alpha_{hl}^\vee = \sum_{i=1}^{n} c_i^\vee \alpha_i^\vee$) and ($\alpha_{hs}^\vee = \sum_{i=1}^{n} d_i^\vee \alpha_i^\vee$), respectively. (By $\alpha_1, \alpha_2, \ldots, \alpha_n$ we denoted the simple roots of L.) In column 4 of Table 1 we have separated the solutions with the highest weights $m\Lambda_i$ where i are the labels for which $c_i = 1$ and $m = 1,2,\ldots$. These representations have also been obtained in /11/from the condition that the orbits of their highest-weight vectors are Hermitian symmetric spaces and hence are in one-to-one correspondents with the non-semisimple Lie subalgebras of maximal rank of L /12/. The Lie algebras of type A_n present an exception; indeed, for these algebras $T_{A_n(\Lambda_1+\Lambda_2)}(\rho_{m\Lambda_k}) = 0$ only for $k = (n+1)/2$ because

$$T_{A_n,(\Lambda_1+\Lambda_n)}(\rho_{m\Lambda_k}) = 2\left(m - k + \frac{n+1}{2} - \frac{2mk}{n+1}\right)\rho_{m\Lambda_k} \tag{3.2}$$

i.e. $\rho_{m\Lambda_k}$ is the solution of a non-homogeneous tensorial equation of degree two. In column 6 we have separated the solutions with highest weights Λ_k where k are the labels for which $c_k = \ell_k = (\alpha_{hl},\alpha_{hl})/(\alpha_k,\alpha_k)$. We remark that all the minuscule weights /13/ given in column 3 (and characterized by the condition "Λ_i is a minuscule weight if $d_i^\vee = 1$") belong to this last class of solutions but do not exhaust it.

For the exceptional semisimple Lie algebras we do not have solutions of the first type (i.e. with highest weights $m\Lambda_j$) in spite of the fact that for the Lie algebras E_6 and E_7 there exist representations $(m\Lambda_1)$, $(m\Lambda_6)$ and $(m\Lambda_7)$, respectively with the property that the orbits of their highest-weight vectors are symmetric spaces. For the exceptional Lie algebras there exist only solutions of the second type, which are precisely representations the highest weights of which are minuscule weights, for E_6 and E_7, or fundamental representations with highest weights given by the highest short roots, for F_4 and G_2.

The representations given in columns 4 and 6 coincide with the representations obtained by Drinfeld /4/ from the condition that a set of third-degree polynomials vanish (this vanishing being the condition that these representations generate representations of the "Yangian").

4. Identities obtained using the Hannabuss operator method

Let ρ_Λ be a finite-dimensional representation of highest weight Λ of a semisimple Lie algebra L acting on the vector space V_Λ ; Let $c_2(\Lambda)$ be the second-degree Casimir operator associated with representation

$$c_2(\Lambda) \equiv \sum_{i=1}^{\dim L} \rho_\Lambda(e_i)\ \rho_\Lambda(e^i) \tag{4.1}$$

In Eq.(4.1) $\{e_i,\ i = 1,\dots,\dim L\}$ is a basis in L and $\{e^i,\ i = 1,\dots,\dim L\}$ is the basis of L dual to $\{e_i\}$ with respect to the Cartan-Killing bilinear form: $(e^i,e_j) = \delta_{ij}$.

Definition. We call Hannabuss operator associated with the pair of representations ρ_Λ and ρ_M of the semisimple Lie algebra L the operator $\mathcal{E}_{\Lambda,M}$ defined by

$$\mathcal{E}_{\Lambda,M} \equiv \sum_{i=1}^{\dim L} \rho_\Lambda(e_i) \otimes \rho_M(e^i) \tag{4.2}$$

The Hannabuss operator $\mathcal{E}_{\Lambda,M}$ commutes with $\rho_\Lambda \otimes 1 + 1 \otimes \rho_M$ and can be expressed as a function of the Casimir operators $c_2(\rho_\Lambda \otimes \rho_M)$, $c_2(\rho_\Lambda)$, $c_2(\rho_M)$. The expression of the minimal polynomial satisfied by $\mathcal{E}_{\Lambda,M}$ is

$$\prod_{\omega \in CG(\Lambda,M)} \left[\mathcal{E}_{\Lambda,M} - \frac{1}{2}\left((\omega+2\delta,\omega) - (\Lambda+2\delta,\Lambda) - (M+2\delta,M)\right)\right] \tag{4.3}$$

where $CG(\Lambda,M)$ is the set of distinct highest weights in the Clebsch-Gordan series of the product $\rho_\Lambda \otimes \rho_M$, 2δ is the sum of the positive roots of L and the expression $c_2(\Lambda)=(\Lambda+2\delta,\Lambda)$ for the Casimir operator of representation ρ_Λ has been used /2/.

The Hannabuss operator method for the determination of the polynomial relations satisfied by a representation ρ_Λ of L consists in taking the matrix elements of the polynomial relation obtained by equating (4.3) to zero, between basis vectors of the representation ρ_M /2,14,15/. Thus, in order to obtain the second-degree polynomial relations satisfied by ρ_Λ it is necessary to determine the representations ρ_M for which the Kronecker product $\rho_\Lambda \otimes \rho_M$ decomposes into precisely two terms, i.e. the set of weights $CG(\Lambda,M)$ contains only two elements.

Theorem 2. For the semisimple Lie algebras (2.1) the pairs $\{\rho_\Lambda, \rho_M\}$ of representations whose Kronecker product decomposes into a direct sum of two inequivalent irreducible representations are those listed in column 8 of Table 1.

It may be observed that a pair $\{\Lambda, M\}$ is always composed of a minuscule weight Λ and a weight M for which the adjoint orbit of the highest-weight vector of ρ_M is a Hermitian symmetric space /11/. This result explains the one-to-ne correspondents between the elementary realizations given by Reshetikhin /5/ for the YBZF algebras and the Hermitian symmetric spaces for the Lie algebras of types A_n, B_n, C_n, D_n. Reshetikhin's L and R operators are constructed in fact as linear combinations of the identity operator I and the

Hannabuss operator $\mathcal{O}_{m\Lambda_j,\Lambda_i}$ acting on $V_{m\Lambda_j} \otimes V_{\Lambda_i}$ where the pairs $\{\Lambda_i, m\Lambda_j\}$ are those given in the last column of Table 1. Hence, the highest weights of the auxiliar representations Λ_i in /5/ are always defined by minuscule weights and the highest weights of the physical state space $V_{m\Lambda_j}$ is the corresponding highest weight from the pair to which Λ_i belongs.

Curiously enough, some of the representations which appear in column 6 of Table 1 and satisfy second-degree polynomial identities do not have companions with which to form pairs in the sense of theorem 2. These are the representations $\Lambda_2, \Lambda_3, \ldots, \Lambda_{n-1}$ of the Lie algebras of type C_n and the representations Λ_1, Λ_6 for the Lie algebra E_6, Λ_7 for E_7, Λ_4 for F_4 and Λ_1 for G_2.

We remark also that the representations $(m\Lambda_1)$, $(m\Lambda_6)$ and $(m\Lambda_7)$ of the Lie algebras E_6 and E_7, respectively, - which possess Hermitian symmetric highest weight orbits - do not satisfy any second-degree tensorial identity.

Finally, as already remarked, (cf. eq.(3.2)), for the Lie algebras of type A_n, a number of pairs of representations in column 8 of Table 1 lead to inhomogeneous tensorial identities which are ab initio excluded by the procedure outlined in section 2 (cf. columns 4, 5 of Table 1).

5. Classical analogue of the Hannabuss operator

Let R_Λ (R_M) be irreducible representations of the Lie group G defined by the highest weights Λ (M) and acting on the linear spaces V (W). Let ϱ_Λ (ϱ_M) be the corresponding representations of the Lie algebra L of G, assumed to posess the property

$$\varrho_\Lambda \otimes \varrho_M = \theta_1 \oplus \theta_2 \tag{5.1}$$

with θ_1, θ_2 irreducible representation.

It is known /17/ that to any finite-dimensional representation R_Λ of a compact group G we can associate a PB realization of its Lie algebra L defined on the coadjoint orbit $\mathcal{M}_\Lambda$ through the highest weight $\Lambda \in L^*$ of R_Λ in the following way

$$x \in L \quad \rightarrow \quad f_x(\xi) \equiv \mathrm{Tr}\, P_\Lambda\, \varrho_\Lambda(Ad(g)x) \in C^\infty(\mathcal{M}_\Lambda) \tag{5.2}$$

where $\xi \equiv Ad(g)^*\Lambda \in \mathcal{M}_\Lambda$ and P_Λ is the projection operator on the one-dimensional subspace spanned by the highest-weight vector of R_Λ. From the definition (5.2) we have

$$f_x(Ad(g)^*\Lambda) = f_{Ad(g)x}(\Lambda) \tag{5.3}$$

From the properties of the Hannabuss operator we get

$$(R_\Lambda(g) \otimes I)\, \mathcal{O}_{\Lambda,M}\, (R_\Lambda(g^{-1}) \otimes I) = (I \otimes R_M(g^{-1}))\, \mathcal{O}_{\Lambda,M}\, (I \otimes R_M(g)) \tag{5.4}$$

whence

$$\begin{aligned} \mathcal{H}(\xi) &\equiv \mathrm{Tr}\, P_\Lambda (R_\Lambda(g) \otimes I)\, \mathcal{O}_{\Lambda,M} (R_\Lambda(g^{-1}) \otimes I) = \sum_{i=1}^{\dim L} f_{Ad(g)x_i}(\Lambda)\, \varrho_M(x^i) \\ &= \sum_{i=1}^{\dim L} f_{x_i}(\xi)\, \varrho_M(x^i) = R_M(g^{-1}) \Big(\sum_{i=1}^{\dim L} f_{x_i}(\Lambda)\, \varrho_M(x^i) \Big) R_M(g) = R_M(g^{-1})\, \mathcal{H}(\Lambda)\, R_M(g) \end{aligned} \tag{5.5}$$

The mapping $\mathcal{K}: \mathcal{M}_\Lambda \to \mathrm{End}\, W$ defined by Eq.(5.5) is the classical analogue of the Hannabuss operator $\mathcal{O}_{\Lambda,M}$.

In fact, such an object can be defined for any pair (f, ϱ_M) in which f is an equivariant PB realization of a Lie algebra L defined on a symplectic manifold $\mathcal{M}$ and ϱ is a representation of L, by the expression

$$\mathcal{K}(p) \equiv \sum_{i=1}^{\dim L} f_{x_i}(p)\, \varrho(x^i) \tag{5.6}$$

for any $p \in \mathcal{M}$. The equivariance property

$$\mathcal{K}(g \cdot p) = R(g^{-1})\, \mathcal{K}(p)\, R(g) \tag{5.7}$$

is an immediate consequence of the definiton (5.5). From the equivariance property it folows that if, for a fixed point $p \in \mathcal{M}$, a polynomial identity $\mathcal{P}(\mathcal{K}(p)) = 0$ is valid, then it is valid for any point of the G-orbit through p.

In order to obtain from the equation satisfied by the Hannabuss operator $\mathcal{O}_{\Lambda,M}$ the corresponding equation for the $\mathcal{K}$-mapping (5.6), we must take a classical limit /16/. To do that, let us consider the representations $\varrho_{m\Lambda}$ with highest weights $m\Lambda$ with $m = 1,2,\ldots$ and the corresponding $\mathcal{K}$-mappings. From the definition, it follows that

$$\mathcal{K}(\xi) = \frac{1}{m}\, \mathrm{Tr}\, P_{m\Lambda} \left(R_{m\Lambda}(g) \otimes I\right) \mathcal{O}_{m\Lambda,M} \left(R_{m\Lambda}(g^{-1}) \otimes I\right) \tag{5.8}$$

for any $m = 1,2,\ldots$. We have /16/

$$\mathcal{K}(\xi)^2 = \lim_{m\to\infty} \frac{1}{m^2}\, \mathrm{Tr}\, P_{m\Lambda} \left(R_{m\Lambda}(g) \otimes I\right) \mathcal{O}^2_{m\Lambda,M} \left(R_{m\Lambda}(g^{-1}) \otimes I\right) \tag{5.9}$$

whence from the equation satisfied by $\mathcal{O}_{\Lambda,M}$

$$\left(\mathcal{O}_{\Lambda,M} - \mu_1 I\right)\left(\mathcal{O}_{\Lambda,M} - \mu_2 I\right) = 0 \tag{5.10}$$

we obtain

$$\mathcal{K}(\xi)^2 + b\, \mathcal{K}(\xi) + c\, I = 0 \tag{5.11}$$

with $b = -\lim(1/m)(\mu_1 + \mu_2)$ and $c = \lim(1/m^2)(\mu_1 \mu_2)$.

The validity of property (5.1) for all the pairs of representations $(\varrho_{m\Lambda}, \varrho_M)$ $(m = 1,2,\ldots)$ with M fixed is essential for the derivation for the second-degree classical limit (5.11) for the equation satisfied by the Hannabuss operator.

By taking the classical limit considered above, we obtain the equations satisfied by PB realizations which can be obtained also in a direct way. Applying this procedure to all pairs from the last column of Table 1 we obtain the classification of all pairs formed with a PB realization of L and a representation of L, with the property that the corresponding $\mathcal{K}$-operator satisfies a second-degree polynomial equation. Using this property of the $\mathcal{K}$-operator a simple proof results /6/ for the existence of the elementary classical realizations of YBZF algebras given by Reshetikhin /5/ for any such PB realization of L. Hence, the last column of Table 1 classifies also these elementary realizations.

TABLE 1

Lie algebra L	Dynkin diagrams and coefficients of the highest long roots (c_i) highest short roots(d_i) and their duals($c_i^\vee, d_i^\vee$) / Non-semisimple subalgebras of maximal rank	Minuscule weights ($d_i^\vee=1$)	Representations (Λ) on which second-degree tensors $T_{L\Omega}$ vanish (m=1,2,...): ($c_i=1$) Λ	($c_i=1$) Ω	($c_i^\vee=1$) Λ	($c_i^\vee=1$) Ω	Pairs of representations the Kronecker product of which decomposes in two irreducible components (m=1,2,...)
A_n	i: 1 2 … n−1 n c_i: 1 1 … 1 1 $(c_i = d_i = c_i^\vee = d_i^\vee)$ $\mathbb{R} \oplus A_p \oplus A_{n+1-p}$ $(p=1,2,\dots,n)$	Λ_1 Λ_2 $\vdots$ Λ_{n-1} Λ_n	$m\Lambda_1$ $m\Lambda_2$ $\vdots$ $m\Lambda_{\frac{n-1}{2}}$ $m\Lambda_{\frac{n+1}{2}}$ $m\Lambda_{\frac{n+3}{2}}$ $\vdots$ $m\Lambda_{n-1}$ $m\Lambda_n$	$\Lambda_2+\Lambda_{n-1}$ — $\vdots$ — $\Lambda_1+\Lambda_n$ — $\vdots$ — $\Lambda_2+\Lambda_{n-1}$	Λ_1 Λ_2 $\vdots$ Λ_{n-1} Λ_n	$2\Lambda_1+2\Lambda_n$ $2\Lambda_1+2\Lambda_n$ $\vdots$ $2\Lambda_1+2\Lambda_n$ $2\Lambda_1+2\Lambda_n$	$\{\Lambda_1, m\Lambda_k\}$ $\{\Lambda_k, m\Lambda_1\}$ $\{\Lambda_{n-k+1}, m\Lambda_n\}$ $\{\Lambda_n, m\Lambda_{n-k+1}\}$ $k=1,2,\dots,n$
B_n	i: 1 2 … n−1 n c_i: 1 2 … 2 2 d_i: 1 1 … 1 1 $c_i^\vee$: 1 2 … 2 1 $d_i^\vee$: 2 2 … 2 1 $\mathbb{R} \oplus B_{n-1}$	Λ_n	$m\Lambda_1$	Λ_4	Λ_1 Λ_n	$2\Lambda_2$ $2\Lambda_1$ $2\Lambda_2$	$\{\Lambda_n, m\Lambda_1\}$
C_n	i: 1 2 … n−1 n c_i: 2 2 … 2 1 d_i: 1 2 … 2 1 $c_i^\vee$: 1 1 … 1 1 $d_i^\vee$: 1 2 … 2 2 $\mathbb{R} \oplus A_{n-1}$	Λ_1	$m\Lambda_n$	Λ_2	Λ_1 Λ_2 $\vdots$ Λ_{n-1} Λ_n	$4\Lambda_1$	$\{\Lambda_1, m\Lambda_n\}$
D_n	i: 1 2 … n−2 n−1 n c_i: 1 2 … 2 1 1 $(c_i = d_i = c_i^\vee = d_i^\vee)$ $\mathbb{R} \oplus D_{n-1}$; $\mathbb{R} \oplus A_{n-1}$	Λ_1 Λ_{n-1} Λ_n	$m\Lambda_1$ $m\Lambda_{n-1}$ $m\Lambda_n$	Λ_4 $2\Lambda_1$	Λ_1 Λ_{n-1} Λ_n	$2\Lambda_2$	$\{\Lambda_1, m\Lambda_{n-1}\}$ $\{\Lambda_1, m\Lambda_n\}$ $\{\Lambda_{n-1}, m\Lambda_1\}$ $\{\Lambda_n, m\Lambda_1\}$
E_6	i: 1 3 4 5 6 (2) c_i: 1 2 3 2 1 (2) $(c_i = d_i = c_i^\vee = d_i^\vee)$ $\mathbb{R} \oplus D_5$	Λ_1 Λ_6	$m\Lambda_1$ $m\Lambda_6$	— —	Λ_1 Λ_6	$2\Lambda_2$	—
E_7	i: 1 3 4 5 6 7 (2) c_i: 2 3 4 3 2 1 (2) $(c_i = d_i = c_i^\vee = d_i^\vee)$ $\mathbb{R} \oplus E_6$	Λ_7	$m\Lambda_7$	—	Λ_7	$2\Lambda_1$	—
E_8	i: 1 3 4 5 6 7 8 (2) c_i: 2 4 6 5 4 3 2 (3) $(c_i = d_i = c_i^\vee = d_i^\vee)$	—	—	—	—	—	—
F_4	i: 1 2 3 4 $c_i^\vee$: 2 3 2 1 $(c_i > 1)$ $(d_i^\vee > 1)$	—	—	—	Λ_4	$2\Lambda_1$	—
G_2	i: 1 2 $c_i^\vee$: 1 2 $(c_i>1 ; d_i^\vee>1)$	—	—	—	Λ_1	$2\Lambda_2$	—

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