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Can one hear the Riemann zeros in black hole ringing?

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Abstract. We elaborate on an entry of the AdS/CFT dictionary relating functional determinants: the determinant of the one-loop contribution to the effective gravitational action by bulk scalars in an asymptotically locally AdS background X, and the determinant of the two-point function of the dual operator (a.k.a. scattering matrix) at the conformal boundary M. The formula originates from AdS/CFT heuristics that map a quantum contribution in the bulk gravitational partition function to a subleading large-N contribution in the boundary CFT partition function:

$$\frac{\det_-(\Delta_X+\lambda(\lambda-n))}{\det_+(\Delta_X+\lambda(\lambda-n))}=\det\, S_M(\lambda)\ .$$

The formula applies to quotients of AdS as well [1]. In the particular case of the BTZ black hole, a closed expression can be worked out in terms of an associated Patterson-Selberg zeta function $Z_{BTZ}(\lambda)$ [2]. The determinants can then be thought of as regularized products of either zeta zeros, scattering resonances or quasinormal frequencies [3]. In this sense, one could say that the zeros of $Z_{BTZ}(\lambda)$ *can be heard* in the spectrum of quasinormal modes of the BTZ black hole. The question we want to pose is whether a similar situation might exist for the celebrated zeros of the Riemann zeta function.

1. Introduction

The aim of this note is to describe a physical scenario that might shed light on the celebrated *Riemann hypothesis*. The question we pose is evidently inspired by the classical paper of Mark Kac (1966) *Can one hear the shape of a drum?*; however, in contrast to most of the physical approaches that look after the (as yet elusive) Hilbert-Pólya hamiltonian, our focus is the spectrum of black hole quasinormal modes or, for short, *black hole ringing*. We have an explicit example at hand, the BTZ black hole, where a connection can be established between the quasinormal spectrum and the zeros of an associated Patterson-Selberg zeta function

$$\boxed{\text{BTZ bh} \Rightarrow \text{quasinormal freq's} \Rightarrow \text{Patterson-Selberg zeros.}}$$

The natural, though perhaps naive, question that comes to mind is whether a similar construction exists, involving again a quotient of AdS space, where the Patterson-Selberg zeros are traded by the Riemann zeros

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$$\boxed{AdS/\Gamma\ ?\ \ \Leftarrow \text{quasinormal freq's ?}\ \ \Leftarrow \text{Riemann zeros.}}$$

We first present the explicit AdS/CFT result for the BTZ black hole and then elaborate on the possibility to have the Riemann zeta, instead of the Patterson-Selberg zeta, and the restriction on the location of its nontrivial zeros.

2. The Rosetta stone of BTZ black hole

Ever since its appearance in the late 1990s in the form of Maldacena's conjecture, the AdS/CFT correspondence has evolved as a theoretical tool to address questions concerning strongly coupled physical systems within a dual holographic framework. It relates two seemingly different theories, gauge theory and gravity, via two deeply rooted ideas in physics, namely, the string emerging at the large-N limit of the gauge theory and the holographic principle. Many developments are, perhaps more appropriately, embraced under the name gauge/gravity dualities for they depart from the original canonical examples and the boundary theories capture the physics of condensed matter, liquids, superconductors, heavy-ion collisions, etc. These recent and exciting applications to physically relevant situations are more heuristic, and therefore mathematically exact results become rare; nonetheless, we will now exploit a *holographic formula*, one of the entries of the AdS/CFT dictionary that seems deeply rooted in spectral theory and conformal geometry.

2.1. Holographic formula

Our favorite entry of AdS/CFT dictionary is a relation (holographic formula) between the determinant of the one-loop contribution to the effective gravitational action by bulk scalars in an asymptotically locally AdS background X and the determinant of the two-point function of the dual operator (a.k.a. scattering matrix) at the conformal boundary M. The formula originates from AdS/CFT heuristics that map a quantum contribution in the bulk gravitational partition function to a subleading large-N contribution in the boundary CFT partition function:

$$\frac{\det_-(\Delta_X + \lambda(\lambda-n))}{\det_+(\Delta_X + \lambda(\lambda-n))} = \det\, S_M(\lambda)\ .$$

The bulk determinant $\det_+$ is computed in the usual way (via Green functions or heat kernel) and $\det_-$ is obtained by analytic continuation in the conformal dimension $\lambda_+ \to \lambda_- = n - \lambda$.

2.2. The case of BTZ black hole

Explicit results can be obtained for pure AdS and for quotients of the form AdS/Γ as well [1]. In particular, the determinants in the BTZ black hole background and the determinant on its boundary torus can be explicitly computed in terms of a Patterson-Selberg zeta function $Z_{BTZ}(\lambda)$ [2] defined by

$$Z_{BTZ}(\lambda) = \prod_{k_1,k_2\geq 0}\left[1-\alpha_1^{k_1}\alpha_2^{k_2}e^{-(k_1+k_2+\lambda)l}\right]$$

where the parameters $\alpha_1 = e^{i\theta}$, $\alpha_2 = e^{-i\theta}$ and $l = 2\pi r_+$, $\theta = 2\pi|r_-|$ relate the horizon radii r_+, r_- of the black hole with the modular parameter of the boundary torus,

$$\frac{\det_-(\Delta_{BTZ} + \lambda(\lambda-2))}{\det_+(\Delta_{BTZ} + \lambda(\lambda-2))} = \det\, S_{T^2}(\lambda) = \frac{(Z_{BTZ}(2-\lambda))^2}{(Z_{BTZ}(\lambda))^2}\ .$$

Here we find a first connection: scattering poles (poles of the determinant S_{T^2}) are mapped to zeros of Z_{BTZ}. That is, the scattering poles are also the Patterson-Selberg zeros.

The second connection is provided by an independent result valid for the non-spinning black hole ($r_- = 0$): the bulk determinant $\det_+(\Delta_{BTZ}+\lambda(\lambda-2))$ for the massive scalar can be written as (regularized) product of quasinormal ω_{QN} and Matsubara frequencies[3]

$$Z_{BTZ} = \prod_{Matsub.,QN} (N + i\omega_{QN})\,.$$

In all, the Patterson-Selberg zeta attached to the BTZ black hole can then be written in three equivalent ways[1]:

$$Z_{BTZ}(\lambda) \sim \prod_{P-Szeros} (\lambda-\rho) \sim \prod_{scatt.poles} (\lambda-\xi) \sim \prod_{Matsub.,QN} (N + i\omega_{QN})$$

and, therefore, the Patterson-Selberg zeros can be 'heard' in the static BTZ black hole ringing.

3. From Patterson-Selberg to Riemann zeros

The challenge now is to trade the Patterson-Selberg zeros by the Riemann ones. We briefly report some pros and cons of this program.

The first thing to notice is that it is certainly possible to obtain the connection 'scattering poles = Riemann zeros'. This occurs for scattering in a *non-compact* quotient of H^2 (Euclidean AdS_2) of finite area with one cusp, as found in the early 70's by Faddeev and Pavlov; alternatively, a decade after, Gutzwiller found the same interesting result for scattering in the 'leaky torus'. The difficulty in establishing the connection with black hole ringing is that the quotient is taken by a parabolic element of the isometry group and this points to *extremal* black holes, whereas the quasi normal mode factorization only works for static situations.

Secondly, even if one succeeded in finding the Riemann zeros in a black hole quasinormal spectrum, there is no really a compelling reason that restricts their location implying the Riemann hypothesis. This was certainly the case of scattering in the modular domain, where Lax and Phillips were "able to contemplate the Riemann hypothesis peeking around the corner". Having said so, we still believe that a holographic approach via AdS/CFT may open a window to, at least, the known *semiclassical* prospects for the Hilbert-Pólya operator and provide a unifying view. These include random matrices and quantization of classical chaotic systems among many others (cf. [4] and [5] for recent progress). Beyond the semiclassical regime, it remains the hope that gauge/gravity dualities in a broader sense will bring in a physical reason that the Riemann hypothesis should be true.

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References

[1] Aros R and Diaz D E 2010 *Functional determinants, generalized BTZ geometries and Selberg zeta function* J. Phys. A: Math. Theor. 43 205402.

[2] Perry P and Williams F 2003 *Selberg zeta function and trace formula for the BTZ black hole* Int. J. Pure Appl. Math. 9 1.

[3] Denef F, Hartnoll S A and Sachdev S 2010 *Black hole determinants and quasinormal modes* Class. Quantum Grav. 27 125001.

[4] Pando-Zayas L and Reichmann D 2013 *A String Theory Explanation for Quantum Chaos in the Hadronic Spectrum* JHEP 1304 083.

[5] Dueñas J G, Svaiter N F and Menezes G 2014 *One-loop effective action and the Riemann Zeros* Int. J. Mod. Phys. A 29 31

[1] Hence Z_{BTZ} is our Rosetta stone (paraphrasing G. Sierra).