AEI-2002-034

Introductory Lectures on String Theory and the AdS/CFT Correspondence

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Summary: The first lecture is of a qualitative nature. We explain the concept and the uses of duality in string theory and field theory. The prospects to understand QCD, the theory of the strong interactions, via string theory are discussed and we mention the AdS/CFT correspondence. In the remaining three lectures we introduce some of the tools which are necessary to understand many (but not all) of the issues which were raised in the first lecture. In the second lecture we give an elementary introduction to string theory, concentrating on those aspects which are necessary for understanding the AdS/CFT correspondence. We present both open and closed strings, introduce D-branes and determine the spectra of the type II string theories in ten dimensions. In lecture three we discuss brane solutions of the low energy effective actions, the type II supergravity theories. In the final lecture we compare the two brane pictures – D-branes and supergravity branes. This leads to the formulation of the Maldacena conjecture, or the AdS/CFT correspondence. We also give a brief introduction to the conformal group and AdS space.

[1]Based on lectures given in July 2001 at the Universidad Simon Bolivar and at the Summer School on "Geometric and Topological Methods for Quantum Field Theory" in Villa de Leyva, Colombia.

Lecture 1: Introduction

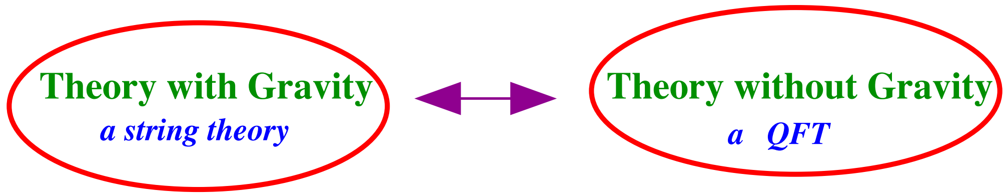
There are two central open problems in theoretical high energy physics:
• the search for a quantum theory of gravity and
• the solution of QCD at low energies.

The first problem is apparent if one considers the Einstein equations which couple the dynamics of the gravitational field to that of matter and radiation. Since matter and radiation follow the laws of quantum mechanics this must also be the case for the gravitational field. If one applies the methods of perturbative quantum field theory, which have been very successful for the electromagnetic, the weak and the strong interactions, to the theory of gravity as formulated by Einstein in his general theory of relativity, one gets stuck at a problem which is often stated as the non-renormalizability of a quantum field theory of gravity.

What is meant by the second problem is that we have no analytic tools to prove e.g. the existence of a mass gap in QCD and the phenomenon of quark confinement, i.e. the fact that at low energies there are neither massless gluons, the gauge particles of the strong interactions, nor free quarks, but rather there are massive 'colourless' mesons and baryons; for a precise description of the problem, see [1].

Solutions to both problems have been proposed which stay within the realm of theoretical concepts which have proven successful in the past. There is e.g. the approach of Ashtekar to the canonical quantization of gravity (see e.g. [2] for reviews) and there are lattice simulations of QCD [3]. We will not say anything about these approaches. Instead we take the (unproven) point of view that in order to solve these two problems, we have to go beyond known and established theoretical frameworks and introduce new ones. This would be in line with the history of physics where apparent conflicts forced completely new lines of thought upon us. As an example consider the special theory of relativity which arose from reconciling discrepancies between the predictions from Newtonian mechanics and electromagnetism, or quantum field theory which combines special relativity and quantum mechanics. We consider this an optimistic point of view since, if true, it would eventually provide new and exciting theories along with their ramifications and implications.

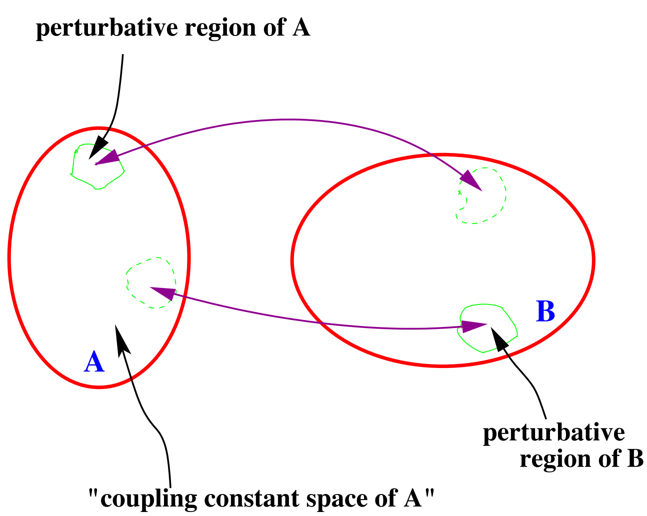
Our basic assumption for these lectures is that string theory is the solution to both problems. We will, however, not discuss at any length why string theory is believed to be a consistent (perturbative) theory of quantum gravity. Our goal is rather to provide the background for understanding the evidence for a *duality* between



Duality means an exact quantum equivalence of the two theories, which thus really represent only one theory, albeit in very different guises.

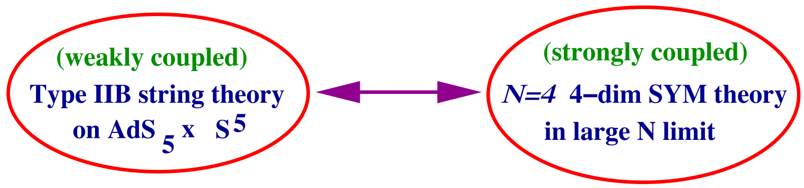
To establish such a duality, we must (1) identify the pair of theories which are proposed to be dual to each other and (2) find the duality map $\Leftrightarrow$.

A duality between two theories **A** and **B** is most useful if we can learn about the non-perturbative behavior (*strong coupling*) of one theory from the computable perturbative behavior (*weak coupling*) of the other. Schematically



Of course, to establish such a duality is very difficult and one will be mostly, at least for the time being, working at the level of conjectures. However, in concrete examples one has gathered compelling evidence for the duality conjecture and one has performed non-trivial tests.

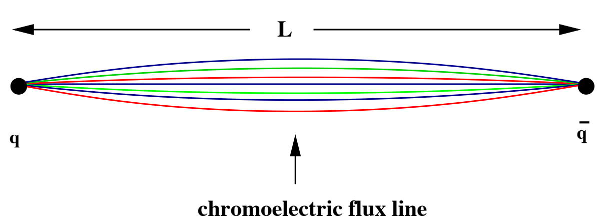
A well known example is the particle-wave duality of quantum mechanics: depending on the experiment, either the particle or the wave aspect of light or matter gives the simpler description. An example from (two-dimensional) quantum field theory is the duality between the sine-Gordon model and the massive Thirring model [4]. More recent examples are the Olive-Montonen duality of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [5] and various (perturbative and non-perturbative) string dualities [6]. Except for the first example, the duality is between two theories of the same kind, e.g. a duality between two string theories. The duality to be discussed in these lectures is not of this type but a duality between a gauge field theory and a string theory:



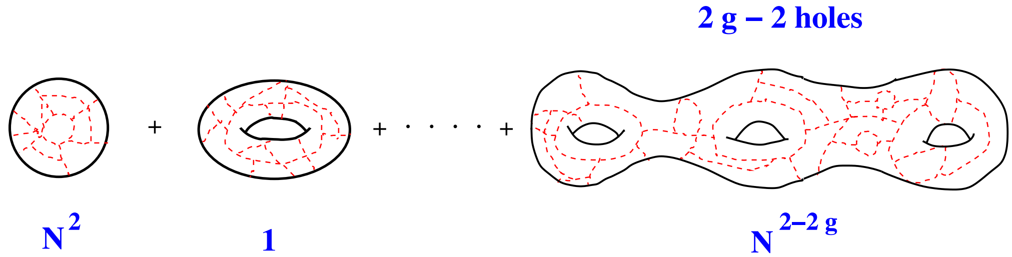
This dual pair was first conjectured by Maldacena in 1997 [7]. For reasons that will be become clear later, it is known as the AdS/CFT correspondence. The duality map was constructed by Gubser, Klebanov and Polyakov [8] and by Witten [9]. Note that the gauge theory does not contain gravity whereas the type IIB string does not contain gauge degrees of freedom in its (perturbative) spectrum. Another feature of the AdS/CFT correspondence is the fact that the two theories which are dual to each other are formulated in different numbers of dimensions: a four-dimensional gauge theory in Minkowski space and a string theory compactified on $AdS_5 \times S^5$. One implication of this duality is that all information of the string theory is encoded in the lower-dimensional field theory. With reference to a similar phenomenon in optics, this is called holography. We will return to it below.

A word of caution to prevent confusion: in these lectures we do not discuss gauge theory as a low-energy effective field theory of, say, the heterotic string. The relation between string theory and gauge theory which is implied by the AdS/CFT correspondence is more subtle.

The idea that a gauge theory has a description as a string theory is in fact an old one. At low energies, QCD is a confining theory. This means that one sees neither free gluons nor quarks but mesons ($q\bar{q}$) and baryons (qqq). There is a linear – rather than Coulomb like – potential between a quark and an anti-quark, $V(L) = \sigma L$. The chromoelectric flux lines are confined to a flux tube or string. σ is the tension of the QCD string.

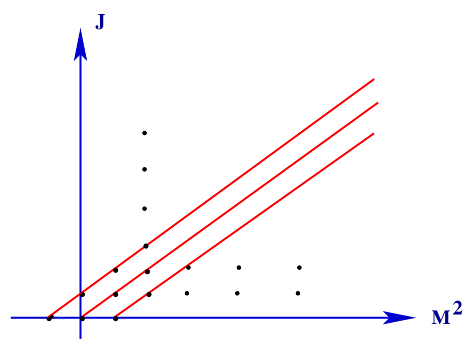


The hope is now that this can be described by a string theory. That this hope is not completely futile can be seen if one considers a gauge theory with gauge group $SU(N)$ for large N. Rather than making a perturbation series in a small coupling constant, which does not exist for QCD at low energies where it is strongly coupled, one makes an expansion in powers of $1/N$. This was first done by 't Hooft [10] (for reviews, see [11, 12, 13]) who showed that all Feynman diagrams which contribute to a given order in $1/N$ can be drawn (without any lines crossing) on a Riemann surface whose Euler number $\chi = 2 - 2g$ (g being the genus) is precisely the power of N to which the diagram contributes.



We will see in lecture 2 that this is very much like the perturbation series of string theory where the expansion parameter is g_s, the string coupling constant, rather than $1/N$. The Riemann surface is the world-sheet of the string. In the figure above we have shown only surfaces without boundary which are the only ones occurring in a pure gauge theory without quarks in the fundamental representation. In the string theory this correponds to a loop expansion of the vacuum amplitude.

One can indeed show that fundamental strings reproduce some of the features of the physics of strong interactions. The excitations of the string satisfy a linear relation between their $(mass)^2 = M^2$ and their spin J: $M^2 = J/\alpha' + \text{const}$, where α' is related to the string tension which becomes a fundamental dimensionful parameter in the theory. This was in qualitative agreement with the Regge trajectories for hadronic resonances which were found in experiments, provided α' was chosen $1/\sqrt{\alpha'} \sim 100$ MeV, which is the typical energy scale of strong interaction physics:



Soon after these results were obtained it was realized that string theory could not correctly reproduce the high energy behaviour of hadronic scattering amplitudes. In addition it was observed that the spectrum of the string contains a massless spin two particle with many of the properties of the graviton, the exchange particle which mediates the gravitational force. At around the same time, quantum-chromodynamics (QCD) was developed as a gauge theory of the strong interaction, with gauge group $SU(3)$. For these, and other reasons, string theory was abandoned as a theory of the strong interactions and was elevated to a candidate for a theory of quantum gravity. The natural energy scale is now $1/\sqrt{\alpha'} \sim 10^{19}$ GeV, the Planck energy. This was around 1975.

If one attempts to quantize string theory in four-dimensional Minkowski space-time one finds a dependence on the size of the world-sheet, and this dependence enters as a new field which can be interpreted as an additional space-time coordinate. This means that we need at least five dimensions for a consistent quantization of string theory [14]. If we want all coordinates to span a d-dimensional Minkowski space-time, we find $d = 26$ for the bosonic string and $d = 10$ for the fermionic string. These are the *critical dimensions*.

We now return to the concept of holography. One interesting physical system where it is realized is a black hole. The Schwarzschild solution of the vacuum Einstein equations, $R_{\mu\nu} = 0$, is the simplest example. This solution depends on one parameter, the mass M of the black hole. Classically, black holes are black, but due to quantum processes they emit so-called Hawking radiation. They radiate like a black body with temperature $T_{\rm BH} = \frac{1}{8\pi M}\left[\frac{\hbar c^3}{G_N k_B}\right]$. As radiating systems black holes are expected to obey the laws of thermodynamics. If one defines the black hole entropy, as first proposed by Bekenstein and Hawking, as $S = 4\pi M^2\left[\frac{k_B G_N}{\hbar c}\right]$, one indeed verifies e.g. the second law $d(Mc^2) = T_{\rm BH} dS$. A quantum theory of gravity should provide the framework for a microscopic computation of the black hole entropy. In the search for such a theory one might turn the logic around and start from the expression for the black hole entropy and try to deduce certain properties that the quantum theory of gravity must possess in order to lead to such an entropy formula. The simple fact that $S = \frac{1}{4}A\left[\frac{k_B c^3}{G_N \hbar}\right]$, where $A = 4\pi\left(\frac{2MG_N}{c^2}\right)^2$ is the area of the black hole horizon, leads to the concept of holography. The information contained inside the region enclosed by the horizon is represented as a hologram on the horizon: all the information about the inside is stored on the holographic screen.[2] This is in sharp contrast with what we expect from statistical mechanics and local quantum field theory where the entropy is an extensive quantity and thus should be proportional to the volume of the system. The lesson we learn from this is that the nature of the degrees of freedom of quantum gravity is quite different from that of a local quantum field theory. In fact, string theory provides a microscopic account of the states and thus entropy of certain (extremal and near-extremal) black holes [17, 18].

Let us now suppose that a d-dimensional quantum field theory 'lives' on the horizon of a $(d+1)$-dimensional black hole with entropy $S_{\rm QFT} \sim A_d$. One wonders how quantum gravity in $(d+1)$ dimensions can be related to a local QFT in d dimensions. The $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence gives an answer to this question. This correspondence goes far beyond the matching of entropies. It is conjectured to be an exact duality in the sense described before.

In the remaining three lectures we provide the necessary background to understand the conjecture. This requires a crash course in string theory and supergravity branes. Much more can be said about any of the issues that we touch upon. Good referencs for string theory are [19, 20, 21, 22, 23]. D-branes are reviewed in [24, 25, 26, 27], brane solutions of supergravity theories in [28, 29] and the AdS/CFT correspondence in [30, 31, 32, 33, 34, 35]. It migth also be fun to brouse through the 'official' string

[2]More generally, the holographic principle asserts that the information contained in some region of space can be represented as a 'hologram' - a theory which 'lives' on the boundary of that region. It furthermore asserts that the theory on the boundary of the region of space in question should contain at most one degree of freedom per Planck area $\hbar$. The black hole precisely satisfies that bound. See [15, 16] for reviews of the holographic principle.

theory web-site [37].

Lecture 2: Elementary introduction to string theory

In this lecture we provide some aspects of string theory needed to introduce the AdS/CFT correspondence. We first discuss the bosonic string and then the fermionic string. There are many aspects of string theory which we will not mention at all. Some of the major omissions are conformal field theory, heterotic and type I strings, compactifications, string dualities, orientifolds, *etc.* For this we have to refer to the literature [20, 23, 19, 21, 22, 27].

We start by comparing the classical mechanics of a zero-dimensional object – a relativistic point particle – and a one-dimensional object – a string – moving in D-dimensional Minkowski space-time with metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,\cdots,+)$, $\mu,\nu = 0,\ldots,D-1$.

As a particle moves through space-time it sweeps out a one-dimensional world-line $\mathcal{C}$ whose embedding in space-time is specified by D functions $X^\mu(\tau)$, τ being an arbitrary parameterization of the world-line. The simplest Poincaré-invariant action that does not depend on the parameterization is

$$S_{\mathrm{pp}} = -mc\int_{\mathcal{C}} d\tau\,\sqrt{-\dot{X}\cdot\dot{X}}\,, \tag{1}$$

the integrand being the infinitesimal path length, $\dot{X}^\mu \equiv \partial_\tau X^\mu$, $X\cdot X = X^\mu X^\nu \eta_{\mu\nu}$, c is the speed of light and m is the particle's mass as can be seen from the nonrelativistic limit.

Analogously, a one-dimensional object sweeps out a two-dimensional world-sheet Σ in space-time and its embedding is described by D functions $X^\mu(\sigma,\tau)$. Again, physics must depend only on the embedding and not on the parameterization of the world-sheet. The simplest invariant action – the Nambu-Goto action – is ($X' = \partial_\sigma X$)

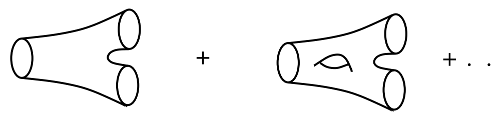
$$S_{\mathrm{NG}} = -Tc\int_\Sigma d\tau d\sigma\,\sqrt{-\det\, h_{\alpha\beta}} = -Tc\int_\Sigma d\tau d\sigma\,\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2\cdot X'^2}\,. \tag{2}$$

T is the string tension, a new fundamental constant of nature [3] of dimension $mass/length$ and $h_{\alpha\beta}$ is the induced metric on the world-sheet

$$h_{\alpha\beta} = \frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}\eta_{\mu\nu}\,, \qquad \xi^\alpha = (\tau,\sigma)\,. \tag{3}$$

[3] There are three fundamental constants of nature which, in pre-string physics, are the speed of light c, the gravitational constant G_N (N for Newton) and Planck's constant $\hbar$. For a discussion about the number of fundamental constants, see [38]. The limit $\hbar\to 0$ is called the classical limit, $c\to\infty$ the non-relativistic limit and $G_N\to 0$ the decoupling of gravity. From these three constants one can construct the so-called Planck-units, namely three fundamental scales, for length ($l_P = \sqrt{G_N\hbar/c^3}$), time ($t_P = \sqrt{G_N\hbar/c^5}$) and mass ($m_P = \sqrt{c\hbar/G_N}$). In string theory the tension replaces G_N as a fundamental constant with the relation (valid in four dimensions) $G_N \sim c^2/T$. One also introduces the string scale $\alpha' = l_s^2$ via the relation $T = \hbar/(2\pi c\alpha')$. l_s is the typical length scale in string theory instead of l_P. The characteristic energy scale is $E_s = \sqrt{\hbar c^3 T}$. At infinite tension, the extension of the string becomes zero. The limit $T\to\infty$ or, equivalently, $\alpha'\to 0$ is called the point particle or field theory limit.

(2) is the straightforward generalization of (1) to an extended object. It is the area of its world-sheet. One distinguishes between open and closed strings. The world-sheet of a free closed string has the topology of a cylinder, that of a free open string the topology of a strip. Interactions are taken into account by considering topologically non-trivial world-sheets. For instance, the decay of one closed string into two, will correspond to the the following world-sheets, where we have indicated only the first two terms in an infinite perturbation series (time runs from left to right):



If we denote the strength of the basic closed string interaction, which is given by the left diagram, by g_s, we see that the second diagram has strength g_s^3. In general, a given world-sheet is weighted by ${g_s}^{-\chi+n_o/2}$ where χ is its Euler number and n_o is the number of external open strings. g_s is called the string coupling constant. We will say a little more about it later.

Due to the $\sqrt{\ }$, the Nambu-Goto action is difficult to deal with. One can remove the square root at the expense of introducing an additional (auxiliary) field on the world-sheet which, however, should not introduce new dynamical degrees of freedom. This field is the world-sheet metric $\gamma_{\alpha\beta}$ (with signature $(-,+)$). The resulting action – the Polyakov action – couples the D massless world-sheet scalar fields X^μ to two-dimensional gravity $\gamma_{\alpha\beta}$[4]

$$S_{\rm P} = -\frac{T}{2}\int_\Sigma d\tau d\sigma\,\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,. \tag{4}$$

The Polyakov and Nambu-Goto actions are in fact classically equivalent. To see this one uses the equation of motion for $\gamma_{\alpha\beta}$, i.e.[5]

$$\delta S = \int d\tau d\sigma \sqrt{-\gamma} T_{\alpha\beta}\,\delta\gamma^{\alpha\beta} = 0 \quad \rightarrow \quad T_{\alpha\beta} = \frac{T}{2}(\frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\sigma}\partial_\rho X\cdot\partial_\sigma X - \partial_\alpha X\cdot\partial_\beta X) = 0 \tag{5}$$

to eliminate $\gamma_{\alpha\beta}$ from $S_{\rm P}$. One then obtains $S_{\rm NG}$ back. Note that the equations $T_{\alpha\beta} = 0$ impose constraints on the dynamical variables X^μ. In the following we will use the Polyakov action.

We now discuss the symmetries of the Polyakov action. In addition to global D-dimensional Poincaré invariance ($X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$, $\Lambda\eta\Lambda^{\rm T} = \eta$), Λ, a constant, $S_{\rm P}$ has the following *local* symmetries

(I) (world-sheet) diffeomorphism invariance:

$$\begin{aligned} X'^\mu(\tau',\sigma') &= X^\mu(\tau,\sigma)\,,\\ \frac{\partial\xi'^\gamma}{\partial\xi^\alpha}\frac{\partial\xi'^\delta}{\partial\xi^\beta}\gamma'_{\gamma\delta}(\tau',\sigma') &= \gamma_{\alpha\beta}(\tau,\sigma)\,, \end{aligned} \tag{6}$$

for new coordinates $\xi'^\alpha = \xi'^\alpha(\xi)$.

[4]From here on we set $c \equiv 1$ and $\gamma = \det(\gamma_{\alpha\beta})$

[5]$T_{\alpha\beta}$ is the energy-momentum tensor which measures the response of the action to a change of the metric.

(II) 2-dimensional Weyl invariance:

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= X^{\mu}(\tau,\sigma)\,, \\ \gamma'_{\alpha\beta}(\tau,\sigma) &= \exp\big(2\omega(\tau,\sigma)\big)\gamma_{\alpha\beta}(\tau,\sigma)\,, \end{aligned} \tag{7}$$

for arbitrary $\omega(\tau,\sigma)$.

The Weyl invariance, a local rescaling of the world-sheet metric is an extra redundancy of the Polyakov formulation and has no analog in the Nambu-Goto form. Weyl invariance guarantees that $\gamma^{\alpha\beta}T_{\alpha\beta} = 0$, which is easily checked.

The local symmetries can be used to simplify the action and the equations of motion. The diffeomorphism symmetry can be used to go to conformal gauge in which the metric has the form

$$\gamma_{\alpha\beta} = \exp\big(2\omega(\tau,\sigma)\big)\eta_{\alpha\beta}\,, \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{8}$$

Due to (classical) Weyl invariance $\omega(\tau,\sigma)$ decouples from the action which is now

$$S_{\rm P} = \frac{T}{2}\int d\sigma d\tau (\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X) \tag{9}$$

The equations of motion for the scalars X^μ are then obtained as usual

$$\begin{aligned} \delta S_{\rm P} &= -T\int d\tau \int_0^L d\sigma\, \partial_\alpha X^\mu \partial^\alpha \delta X_\mu \\ &= T\int d\tau \int_0^L d\sigma\, \Box X^\mu \delta X_\mu - T\int d\tau\, \underbrace{\big[\partial_\sigma X^\mu \delta X_\mu\big]\big|_0^L}_{\text{boundary term}}\,, \end{aligned} \tag{10}$$

where L is the length of the string. Thus, the equation of motion for X^μ is the two-dimensional wave equation

$$\Box X^\mu = \big(-\partial_\tau^2 + \partial_\sigma^2\big)X^\mu = 0\,, \tag{11}$$

subject to the vanishing of the boundary term. Here we can distinguish two cases:

- closed string $\Rightarrow$ no boundary term

but X^μ must be L-periodic in σ, i.e.

$$X^\mu(\tau,\sigma+L) = X^\mu(\tau,\sigma) \tag{12}$$

and there is no boundary term. The general solution to the wave equation (11) with the periodicity condition (12) is

$$X^\mu(\tau,\sigma) = q^\mu + \frac{1}{TL}p^\mu\tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\left\{\alpha_n^\mu e^{-2\pi i n(\tau+\sigma)/L} + \tilde{\alpha}_n^\mu e^{-2\pi i n(\tau-\sigma)/L}\right\}\,. \tag{13}$$

The powers of T are required for dimensional reasons (X has dimension *length*). The numerical coefficients have been chosen to make the canonical commutation relations which we will discuss below free of numerical constants. The solution (13) is the sum of the center-of-mass motion (given by the first two terms) and left– and right–moving waves with amplitudes α_n^μ and $\tilde{\alpha}_n^\mu$ respectively. The c.o.m. position is

$$q^\mu(\tau) = \frac{1}{L}\int_0^L d\sigma X^\mu(\tau,\sigma) = q^\mu + \frac{1}{LT}p^\mu\tau \tag{14}$$

and the total momentum of the string is

$$p^\mu(\tau) = \int_0^L P^\mu(\tau,\sigma) = T\int_0^L d\sigma \dot{X}^\mu(\tau,\sigma)\,; \tag{15}$$

the momentum density $P^\mu(\sigma,\tau) = T\partial_\tau X^\mu(\sigma,\tau)$ is the variable conjugate to $X^\mu(\sigma,\tau)$. Reality of X^μ requires that $\alpha_{-n}^\mu = \left(\alpha_n^\mu\right)^*$ and analogously for $\tilde{\alpha}_n^\mu$.

In the second case

- open string $\Rightarrow$ boundary term

we have to impose boundary conditions

$$\partial_\sigma X^\mu \delta X_\mu = 0\,, \quad \text{at } \sigma = 0 \text{ and } \sigma = L\,. \tag{16}$$

There are two possibilities to satisfy the open string boundary conditions (b.c.)

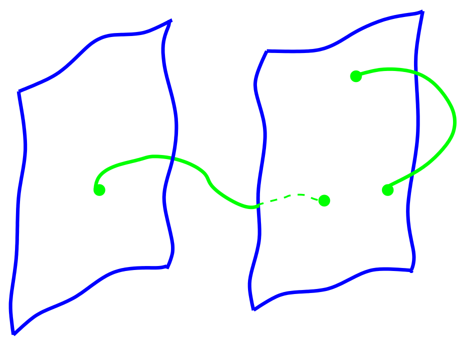
(N) Neumann: $\partial_\sigma X^\mu|_{\text{bndy}} = 0\,,$

(D) Dirichlet: $\delta X^\mu|_{\text{bndy}} = 0\,.$

These can be imposed independently for each space-time direction μ and each of the two ends of the open string. It is important to understand the physical meaning of these different boundary conditions. The total momentum $p^\mu(\tau) = \int_0^L d\sigma\, P^\mu(\tau,\sigma)$ is conserved for Neumann boundary conditions which place no restriction on the position of the endpoint of the open string. On the other hand, it is easy to see that the space-time momentum is not conserved for Dirichlet boundary conditions. This should not come as a surprise since Dirichlet b.c. fix the endpoints of the open string to lie on hyper-surfaces in space-time; these necessarily break translational invariance. The non-conservation of momentum in directions with Dirichlet b.c. is thus to be expected. Where does the momentum flow to? The only candidates are the hyper-surfaces on which the open strings end, so these themselves have to be dynamical objects which absorb the open string momentum. These dynamical objects are called Dirichlet-branes or D-branes.[6] More specifically, if we have NN b.c. along (say) $\mu = 0,\ldots,p$ and DD

[6]We will see later that they are not only the loci in space-time where open strings can end but they are also the source for modes of the closed string.

b.c. along $\mu = p+1, \ldots, D-1$ the end-point of the string lies on a Dp-brane. The two ends of an open string can, of course, lie on two different D-branes, say one end on a Dp and the other end on a Dp$'$ brane. This is illustrated in the figure below for $p = p' = 2$. No end of an open string can end in 'free space'. It must lie on a D-brane, which can move through space-time.



Notice that D-branes have appeared in two ways: (1) as hyper-surfaces in space-time on which open strings end and (2) as dynamical objects with which open strings exchange momentum. This latter fact is illustrated in the figure by the 'wiggly' shape of the branes.

Finally, we list the solutions to $\Box X = 0$ with the various possible boundary conditions. For simplicity of notation and without loss of generality, we have set $L = \pi$ (recall that we have defined $\alpha' = \frac{1}{2\pi T}$):

(NN) $\partial_\sigma X|_{\sigma=0,\pi} = 0$

$$X(\tau,\sigma) = q + 2\alpha' p\tau + i\sqrt{2\alpha'} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n \cos(n\sigma) e^{-in\tau}\,;$$

(DD) $X|_{\sigma=0} = q_i\,,\ X|_{\sigma=\pi} = q_f$ (the positions of two D-branes)

$$X(\tau,\sigma) = q_i + \frac{1}{\pi}(q_f - q_i)\sigma + \sqrt{2\alpha'} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n \sin(n\sigma) e^{-in\tau}\,;$$

(ND) $\partial_\sigma X|_{\sigma=0} = 0\,,\ X|_{\sigma=\pi} = q_f$

$$X(\tau,\sigma) = q_f + i\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\alpha_r \cos(r\sigma) e^{-ir\tau}\,.$$

Notice that only NN boundary conditions allow for a center-of-mass motion. In contrast to the closed string the open string has only one set of oscillators; left- and right movers get tied up through the boundary conditions: a left-moving wave is reflected and returns as a right-moving wave, and vice versa. Depending on the type of boundary condition the wave is reflected with phase shift 0 (N) or π

(D). We observe that for mixed b.c. the oscillators have half-integer moding. Factors of i are chosen such that the reality condition on X translates to $\alpha_{-n} = (\alpha_n)^*$ and $\alpha_{-r} = (\alpha_r)^*$.

So far the discussion was entirely classical. This means that Planck's constant $\hbar$ has not appeared yet. It will enter through quantization. There are various ways to subject a given classical system to the laws of quantum mechanics, the most familiar one being canonical quantization. Here the Poisson bracket $\{q, p\} = 1$ between a coordinate q and its canonically conjugate momentum p is replaced by the commutator of the position and momentum operators (which we also denote by q and p) $[q, p] = i\hbar$. Part of the quantization procedure consists of specifying the Hilbert space on which the operators act. The probabilistic interpretation of the Schrödinger wave-function, which is a vector in the Hilbert space, requires that the Hilbert space be positive definite (this is the requirement of unitarity of the quantum theory). Naive canonical quantization of the string leads, as a consequence of the indefiniteness of the Minkowski metric, to the existence of negative norm states. One then has to ensure that these unphysical states decouple from the theory. This can be shown to be a consequence of the constraint equations $T_{\alpha\beta} = 0$. For our purposes the so-called light-cone quantization is most appropriate. In this scheme only physical degrees of freedom appear and it offers the quickest route to the excitation spectrum of the closed and open string. The disadvantage is that explicit space-time Lorentz-covariance is lost. A treatment of covariant quantization procedures can be found in the references on string theory.

In light-cone quantization the constraints $T_{\alpha\beta} = 0$ are taken care of by solving them explicitly. This becomes possible by choosing the so-called light-cone gauge for the local symmetries of the Polyakov action. In conformal gauge the constraints can be expressed as $\partial_+ X \cdot \partial_+ X = 0$ and $\partial_- X \cdot \partial_- X = 0$ where $\partial_\pm$ are derivatives with respect to $\sigma^\pm \equiv \sigma \pm \tau$. We observe that even after going to conformal gauge, the diffeomorphism invariance is still not completely fixed. The easiest way to see this is to express the metric in the coordinates $\sigma^\pm$, in which $ds^2 = e^{2\omega} d\sigma^+ d\sigma^-$. Under coordinate transformations $\sigma^+ \to f^+(\sigma^+)$ and $\sigma^- \to f^-(\sigma^-)$ the only change in the metric is a change of the conformal factor $e^{2\omega}$, i.e. we are still in conformal gauge, provided $f^\pm$ are arbitrary (non-constant) solutions of the wave equation which respect the b.c. or periodicity condition. We can now use this freedom to identify τ with any one of the fields X^μ which solve the wave equation. We choose $X^+ = 2\alpha' p^+ \tau + q^+$ where $X^\pm \equiv X^0 \pm X^{D-1}$ are called light-cone coordinates. The remaining $D - 2$ coordinates are the transverse coordinates. If we insert this into the action and perform a Legendre transformation we find the light-cone Hamiltonian

$$H_{\text{l.c.}} = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \sum_{i=1}^{D-2} \left\{ (\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2 \right\} . \tag{17}$$

$H_{\text{l.c.}}$ contains only the transverse coordinates X^i. The virtue of the choice $X^+ \propto \tau$ rather than, say, $X^0 \propto \tau$ is that X^- does not appear in $H_{\text{l.c.}}$ and that it can be expressed, via the constraint equations $T_{\pm\pm} = 0$, up to an integration constant q^-, in terms of the X^i. In particular one finds that

$p^- = T\int d\sigma\partial_\tau X^- = \frac{1}{\alpha' p^+}H_{\rm l.c.}$. The dynamical degrees of freedom are thus p^+, q^- and the complete X^i (zero-mode and oscillator parts).

Once we are in light-cone gauge, we can go ahead with canonical quantization. Imposing standard equal time commutators on the X^i and their momenta these become operators in the single-string Hilbert space and so do the q^i, p^i, α^i_n and $\tilde\alpha^i_n$. For these the commutation relations are (from now on we set $\hbar = 1$)

$$[q^i, p^j] = i\delta^{ij}\,, \qquad [\alpha^i_n, \alpha^j_m] = n\,\delta_{n+m,0}\,\delta^{ij}\,, \qquad [\tilde\alpha^i_n, \tilde\alpha^j_m] = n\,\delta_{n+m,0}\,\delta^{ij}\,. \tag{18}$$

All other commutators vanish. Of course, for the open string we have only one kind of oscillators. They satisfy the same commutation relation as, say, the α^i_n, where n is now either integer or half-integer, depending on the boundary conditions.

The mass spectrum of the vibrating string is obtained from the eigenvalues of the mass operator which is defined in the usual way as $m^2 = p^+p^- - \sum_{i=1}^{D-2}(p^i)^2$. With $p^+p^- = H_{\rm l.c.}/\alpha'$ this is $m^2 = \frac{1}{\alpha'}H_{\rm l.c.} - \sum(p^i)^2$. This expression is correct for both, open and closed strings. In the case of open strings the sum extends only over the directions with NN b.c.'s; in the other directions there is no c.o.m. momentum.[7] Straightforward calculation yields

$$\alpha' m^2 = 2(N + \tilde N) + a + \tilde a \qquad \text{(closed strings)} \tag{19}$$

and

$$\alpha' m^2 = N_{\rm NN+DD} + N_{\rm ND+DN} + a + \frac{\alpha'}{(2\pi\alpha')^2}(\Delta q)^2 \qquad \text{(open string)} \tag{20}$$

where

$$N = \sum_{n>0}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n \tag{21}$$

measures the total occupation number of α excitations and likewise for $\tilde N$, *etc.* Several explanations are in order: (i) It is easy to verify, using the canonical commutation relations, that e.g. for the left-movers of the closed string, $m^2\alpha^i_n = \alpha^i_n(m^2 - \frac{2}{\alpha'}n)$, which tells us that the α^i_n for $n > 0$ lower the mass of a state whereas the modes with $n < 0$ raise its mass. The Fock vacuum $|0\rangle$ is defined to be the state which is annihilated by all lowering operators. Although in the classical expressions the ordering of the oscillators is irrelevant, this is not true in the quantized theory. In the expressions (19) and (20) we have brought the operators to normal ordered form, i.e. we have moved all lowering operators to the right of all raising operators (note that the raising and the lowering operators commute among themselves). This means that the vacuum state $|0\rangle$ is the state with the lowest mass which is in fact given by the normal ordering constants a and $\tilde a$, which we will determine below. (ii) The last term in (20), $(T\Delta q)^2$, is the contribution to the mass from the stretching of the open string whose endpoints lie on two D-branes which are separated by the distance Δq.

[7] Note that when going to the light-cone gauge we need to assume that we have NN b.c. in the $X^\pm$ directions.

We now discuss the normal ordering constants. For each transverse directions they are proportional to $\sum_{n\geq 0}(n+\nu)$ where $\nu = 0$ for integer moded oscillators and $\nu = \frac{1}{2}$ for half-integer moded oscillators. Clearly these are divergent sums which can, however, be given a precise mathematical meaning (see e.g. [39]) as follows. For $\mathrm{Re}(s) > 1$ the generalized Riemann zeta function $\zeta(s, \nu)$ can be written as the infinite sum $\zeta(s, \nu) = \sum_{n=0}^{\infty}(n + \nu)^{-s}$. For other values of s it is defined by analytic continuation. We then define the normal ordering constant as the value of the ζ-function at $s = -1$ and thus obtain

$$\sum_{n=0}^{\infty}(n+\nu) \equiv \zeta(-1,\nu) = -\frac{1}{12}\big(1 - 6\nu(1-\nu)\big)\,. \tag{22}$$

This leads to the following normal ordering constants:

$$\text{(closed string)}: \qquad a = \tilde{a} = -\frac{D-2}{12} \tag{23}$$

$$\text{(open string)}: \qquad a = -\frac{D-2}{24} + \frac{d}{16} \tag{24}$$

where $d = \#$(ND+DN directions). The way we have fixed the normal ordering constant looks like a mathematical trick. The procedure of making sense of infinities in a quantum field theory, such as infinite zero-point energies, is called regularization. We can subtract the infinity by modifying the Polyakov action through the addition of a 'cosmological constant' proportional to $\int d\sigma d\tau\sqrt{-\gamma}$. The only choice which is consistent with Weyl invariance is the one which leads to the result above.

We are now ready to determine the mass spectrum of the string. First, consider the open string with only NN b.c.'s. Then the ground state has mass $m^2 = -\frac{(D-2)}{24\alpha'}$, i.e. it is tachyonic. The first excited states are

$$\alpha^i_{-1}|0\rangle\,, \quad \text{with mass} \quad m^2 = \frac{1}{\alpha'}\frac{26-D}{24}\,. \tag{25}$$

There are precisely $(D-2)$ of these states and they transform as a vector under $SO(D-2)$, the rotation group in the transverse space, which is the little group for massless particles. What this means is the following. Consider first a massive particle moving through D-dimensional Minkowski space-time. Since any massive particle necessarily moves with a speed less than c, we can make a Lorentz boost and go to its rest frame. In this frame the particle's momentum is $p^\mu = (m, 0, \ldots, 0)$ with $-p^2 = m^2$ whose invariance subgroup (isotropy group, little group) is $SO(D-1)$. This means that massive particles can be classified by representations of $SO(D-1)$. For a massless particle the situation is different. Since they necessarily move at the speed of light and satisfy $p^2 = 0$, we can choose a frame in which its momentum is $p^0 = (E, 0, \ldots, 0, E)$. The invariance subgroup of this vector is $E(D-2)$, the group of motions in $(D-2)$-dimensional Euclidean space. Massless string states form, however, finite dimensional representations of a $SO(D-2)$ subgroup.

Back to the string spectrum: since the first excited states form a vector of $SO(D-2)$ and since there are no other states of the same mass, these states must form a massless vector. Otherwise Lorentz invariance will be broken. Requiring the mass of these states to vanish fixes $D = 26$ which is the so-called critical dimension of the bosonic string. We have thus shown that Lorentz invariance of the quantized bosonic string theory requires that space-time has dimension 26. The classical theory was Lorentz invariant for any D but this symmetry is not preserved by the quantization, i.e. there is a Lorentz anomaly except in the critical dimension. This fact holds for general quantization schemes: the absence of anomalies requires the critical dimension $D = 26$.

We have only considered the ground state and the first excited state of the open string with all directions being NN. It is straightforward to consider higher excited states and with different boundary conditions. Any state which can be reached by acting with an arbitrary number of creation operators on the vacuum is allowed. Since these states will not be of interest to us, we will leave their exploration as an exercise. We only want to make two remarks: (1) The masses of the massive states are n/α', $n = 1, 2, \ldots$. In the field theory limit where $\alpha' \to 0$ or, equivalently, $T \to \infty$, they become infinitely massive and decouple. (2) For, say, all directions NN, the states at each mass level must arrange themselves into representations of $SO(25)$. In the presence of a Dp-brane, for instance, massless (massive) states must come in representations of $SO(p-1)$ ($SO(p)$) since the Dp-brane breaks $SO(1,25)$ to $SO(1,p)$.

We have remarked that the end-points of open strings must lie on D-branes. In the presence of several D-branes we must label the open string states by two additional labels to indicate on which of the D-branes the two end-points of the open string lie. If the open string is oriented we can distinguish its two end-points from each other and we denote the states by $|N; p; i, \bar{\jmath}\rangle$, where N denotes the oscillator numbers, p the c.o.m. momentum of the string and i and $\bar{\jmath}$ are the so-called Chan-Paton (CP) indices of the $\sigma = 0$ and $\sigma = \pi$ endpoints, respectively.[8] We write these states in the form

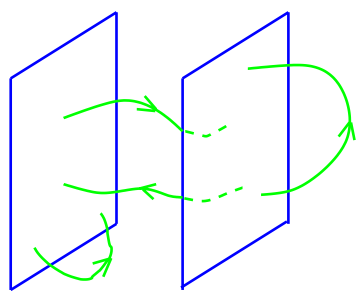
$$|N; p; i, \bar{\jmath}\rangle = \sum_a |N; p; a\rangle \lambda^a_{ij} \, . \tag{26}$$

Later on we will be interested in the case of N parallel and coincident Dp-branes. In this case one can show, by looking at the interactions[9] of excited strings, that the allowed matrices λ^a_{ij} generate the

[8] We will not consider unoriented strings in these lectures, even though they are also of interest. The possible gauge groups are then SO and USp. They require, in addition to D-branes, also so-called orientifold planes. For a recent review, see [27].

[9] Roughly speaking, interactions are taken into account by looking at topologically non-trivial world-sheets, as we have alluded to before. More concretely, there is a correspondence between states and operators, the so-called vertex operators and one computes the interaction of strings with given excitation as correlation functions of the corresponding vertex operators. The vertex operators for closed string states are inserted in the interior of the world-sheet, those for open string states at the boundary. As long as the external momenta are small compared to the masses of the massive string excitations, one can reproduce the scattering amplitudes of the massless states by a low-energy effective field theory action. In the case of the gauge bosons, this action is, to lowest order in α', the $U(N)$ Yang-Mills action in $p+1$ dimensions with gauge coupling $g^2_{\rm YM} \sim g_s(\alpha')^{(p-3)/2}$. In the same way the other massless modes that we will encounter below are identified. The fields corresponding to excitation modes of the open string are confined to the world-volume of the brane on which the string ends whereas those corresponding to the modes of the closed string can propagate anywhere in the bulk, but also interact with the open string modes. Correspondingly the low-energy effective field theories are

group $U(N)$. The massless states $|1;p;a\rangle = \sum_{i,j=1}^N \alpha_{-1}^m |0;p;i\bar{j}\rangle \lambda_{ij}^a$ are identified as the gauge bosons of an $U(N)$ gauge symmetry if m labels a direction along the brane and as massless scalars transforming in the adjoint representation of the gauge group if m labels a direction transverse to the brane (we use the normalization $\mathrm{Tr}(\lambda^a\lambda^b) = \delta^{ab}$). If we separate the N D-branes (but keep them parallel) into two stacks of N_1 and N_2 branes, the $(T\Delta q)^2$ term in the mass formula (20) contributes to the mass of the excitations of those strings whose endpoints lie on separated branes. The corresponding states become massive and the gauge symmetry is broken to $U(N_1)\times U(N_2)$. This is a brane realization of the familiar Higgs effect. The extra degrees of freedom needed to give mass to the gauge bosons which lie in the coset $U(N)/(U(N_1)\times U(N_2))$ are provided by the scalar fields. The figure illustrates the situation for $U(2)$. The massless excitations of the four possible oriented open strings represent the four gauge bosons. If the two branes are separated, as in the figure, two gauge bosons are massive, their mass being the string tension times the separation of the two branes. The unbroken gauge group is $U(1)\times U(1)$.



We now discuss the closed string spectrum. It is very similar to the discussion of the open string spectrum, without the complication of different boundary conditions and CP factors. However, there is one further constraint we have to impose on the closed string states. This comes about as follows. After going to conformal gauge we have used the remaining diffeomorphisms to go to light-cone gauge. In the case of the open string this fixes the diffeomorphisms completely. However, in the closed string we are still allowed to make constant shifts in $\sigma \to \sigma + \sigma_0$. Using the mode expansion and the commutation relations it is not difficult to show that the operator $U_{\sigma_0} = \exp(i\sigma_0(N-\tilde{N}))$ satisfies $U_{\sigma_0} X(\tau,\sigma) U_{\sigma_0}^{-1} = X(\tau,\sigma+\sigma_0)$. We thus have to impose the following physical state condition (level matching condition) on closed string states:

$$(N-\tilde{N})|\mathrm{state}\rangle = 0\,. \tag{27}$$

We can now determine the spectrum of the closed bosonic string. Again, the ground state is tachyonic. The first excited states are $\alpha_{-1}^i\tilde{\alpha}_{-1}^j|0\rangle$ of which there are $(D-2)^2$. They transform as reducible rank-two tensor representation of $SO(D-2)$. Its irreducible components are the symmetric traceless, the antisymmetric and the trace parts. The same group theoretical argument as for the open string

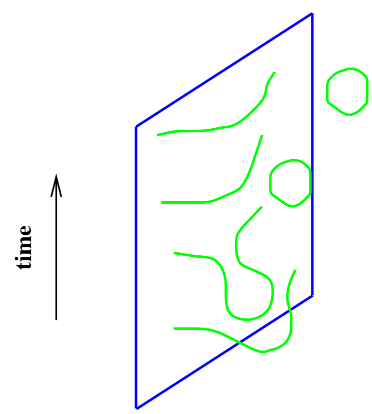
formulated on the branes and in the bulk, respectively.

requires that these states are massless and hence we conclude that also for the closed bosonic string $D = 26$. The irreducible components of the massless states can be identified, via their interactions as the graviton G_{ij}, an antisymmetric tensor particle B_{ij} and the so-called dilaton Φ:

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \,\hat{=}\, \underbrace{G_{ij}}_{\substack{\text{symmetric}\\\text{traceless}}} + \underbrace{B_{ij}}_{\text{antisymmetric}} + \underbrace{\Phi}_{\text{trace}} . \tag{28}$$

As we argue below, Φ is related to the string coupling constant g_s via $g_s = e^{\Phi_0}$ where Φ_0 is the vacuum expectation value of the dilaton. Again, we will not discuss the massive spectrum.

Notice that the space-time fields corresponding to the excitations of the open string are confined to live on the world-volume of the D-branes on which the open strings end. There is no such restriction for the excitations of the closed string. D-branes interact with each other via the exchange of closed strings. The figure shows the emission of a closed string from a D-brane. In a time-reversed process the closed string can be absorbed by another D-brane.



The figure also illustrates that while open strings are attached to D-branes, closed strings can move in 10-dimensional space-time.

One can easily generalize the Polyakov action in such a way that all the massless string modes appear:[10]

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\sqrt{-\gamma}\left\{\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X) + \alpha' R(\gamma)\Phi(X)\right\} . \tag{29}$$

This describes the motion of the string through a space-time with metric $G_{\mu\nu}$ and a background anti-symmetric tensor and dilaton field. In the last term $R(\gamma)$ is the Ricci scalar for the metric γ. If we now separate Φ into a constant background value Φ_0 and a fluctuating piece, $\Phi = \Phi_0 + \phi$, then the contribution of Φ_0 is proportional to Einstein action for the metric γ which, in two dimensions, is

[10]There is also a term over the boundary of the world-sheet which contains the massless gauge bosons of the open string spectrum, but we will not write it.

proportional to the Euler number of the world-sheet (for open strings there are additional boundary terms).[11] This means that in a path integral evaluation of string scattering amplitudes a world-sheet with Euler number χ carries a weight $e^{-\chi\Phi_0} \equiv g_s^{-\chi}$.

Note that unless $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are constant, i.e. independent on X, the world-sheet action is that of an interacting field theory which can only be quantized perturbatively. In the quantum theory we have to make sure that the local symmetries of (4), which allowed the elimination of the degrees of freedom contained in $\gamma_{\alpha\beta}$, are still present. This requirement imposes severe restrictions on the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. The conditions they have to satisfy are in fact equivalent to the equations of motions of these fields which follow from the low energy effective action.

So far we have dealt with the bosonic string whose world-sheet description involves only bosonic fields and whose excitations all transform in tensor representations of the little group (we have shown this explicitly for the massless states, but this is also true for the massive states) and they are thus space-time bosons. Clearly, for a realistic theory of nature we also need space-time fermions. They will appear in the spectrum of the fermionic string which we will now discuss.

The world-sheet action of the fermionic string contains bosons and fermions.[12] In light-cone gauge the action is[13]

$$S = -\frac{1}{2}\int d\tau d\sigma \left\{ T\partial_\alpha X^i \partial^\alpha X^i - \frac{i}{\pi}\bar{\Psi}^i \rho^\alpha \partial_\alpha \Psi^i \right\} . \tag{30}$$

The Ψ^i form, as the X^i, a vector of $SO(D-2)$. ρ^α are 2-dimensional Dirac matrices obeying $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$. In the basis $\rho^0 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\rho^1 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ the components of $\Psi^T = (\tilde{\psi}, \psi)$ can be chosen to be real (Ψ is thus a Majorana spinor). Then $\bar{\Psi} = \Psi^\dagger \rho^0 = \Psi^T \rho^0$. The equations of motion are the wave equation for the X^i and the massless Dirac equation for the Ψ^i:

$$\Box X^i = 0\,, \qquad \rho^\alpha \partial_\alpha \Psi^i = 0 \leftrightarrow \begin{cases} \partial_- \psi^i = 0\,, \\ \partial_+ \tilde{\psi}^i = 0\,. \end{cases} \tag{31}$$

Again we have to distinguish between the open and the closed string. We start with the open string.

For the open string the fermions are subject to the boundary conditions

$$\Psi^T \rho^0 \rho^1 \delta\Psi\big|_{\sigma=0,\pi} = (\tilde{\psi}\delta\tilde{\psi} - \psi\delta\psi)\Big|_{\sigma=0,\pi} = 0\,, \tag{32}$$

[11] This makes sense if we perform a Wick rotation and change the signature of the world-sheet from $(-,+)$ to $(+,+)$.

[12] We are using the NSR (Neveu-Schwarz-Ramond) formulation. In the Green-Schwarz formulation, one introduces additional world-sheet scalars which are, however, space-time spinors.

[13] One can derive this action by starting with a generalization of the Polyakov action. While in the bosonic case this is a two-dimensional field theory coupled to gravity, we would now consider a two-dimensional field theory with bosons and fermions which couples to supergravity. Ψ^μ is the supersymmetry partner of X^μ. The partner of the world-sheet metric γ is a world-sheet vector-spinor, the gravitino. The gravitino can be gauged away via the local fermionic symmetries (supersymmetry and super-Weyl). This defines the super-conformal gauge in which the equations of motion for the gravitino are $\psi \cdot \partial_+ X = \tilde{\psi} \cdot \partial_- X = 0$ which have to be imposed as constraints. The remaining gauge freedom again allows to go to light cone gauge which sets the components ψ^+ and $\tilde{\psi}^+$ to zero and the components ψ^- and $\tilde{\psi}^-$ can be expressed in terms of the ψ^i and $\tilde{\psi}^i$, respectively. The resulting action is (30).

which couple the left- and right moving fermions. The relative sign between ψ and $\tilde{\psi}$ is a matter of convention, which we choose such that

$$\begin{aligned} \psi^i(\tau,0) &= \tilde{\psi}^i(\tau,0)\,, \\ \psi^i(\tau,\pi) &= \eta\,\tilde{\psi}^i(\tau,\pi)\,,\ \eta=\pm 1\,, \end{aligned} \tag{33}$$

which solves (32). The two choices $\eta = \pm 1$ define two sectors of the theory, the Neveu-Schwarz (NS) sector for $\eta = -1$ and the Ramond (R) sector for $\eta = +1$. As we will see below, fields corresponding to states having excitations in the NS–sector are space-time bosons, whereas excitations in the R–sector lead to space-time fermions. The solution of the equations of motion which respect the b.c. is[14]

$$\psi^i = \sum_r \psi^i_r e^{-ir(\tau+\sigma)}\,, \quad \tilde{\psi}^i = \sum_r \psi^i_r e^{-ir(\tau-\sigma)} \quad \text{with} \quad \begin{cases} r\in\mathbb{Z}+\frac{1}{2} & \text{NS sector,} \\ r\in\mathbb{Z} & \text{R sector.}\end{cases} \tag{34}$$

The fact that there is only one set of fermionic oscillators is due to the boundary conditions. For the contribution of the fermions to the light-cone Hamiltonian one finds

$$H_{\text{l.c.}} = \sum_r r\psi^i_{-r}\psi^i_r \tag{35}$$

with r integer (half-integer) in the R (NS) sector. The fermionic oscillators are quantized by imposing anti-commutation relations:

$$\{\psi^i_r, \psi^j_s\} = \delta_{r+s,0}\,\delta^{ij}\,. \tag{36}$$

They also contribute to the mass of a state. The mass operator for the open string in either sector is now

$$\alpha' m^2 = \sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r + \begin{cases} -\frac{D-2}{16} & \text{NS,} \\ \quad 0 & \text{R.}\end{cases} \tag{37}$$

The normal ordering constants arise from putting the fermionic oscillators in normal ordered form (again, the positive modes are lowering and the negative modes are raising operators). They are (c.f. (22)) $a_{\text{R}} = -(D-2)\zeta(-1,0) = \frac{D-2}{24}$ in the R–sector and $a_{\text{NS}} = -(D-2)\zeta(-1,1/2) = -\frac{D-2}{48}$ in the NS–sector. Note that the total zero-point energy vanishes in the R–sector. Here we have assumed that we have only NN or DD b.c.'s. If we have d ND plus DN b.c.'s, the normal ordering constant is again zero in the R sector and $-\frac{D-2}{16} + \frac{d}{8}$ in the NS sector.

The NS–sector has a unique ground state which is tachyonic. It is a space-time boson and so are all excited states in the NS sector which can be reached by acting with creation operators $\psi^i_{-n+1/2}$, $n \in \mathbb{Z}_+$ on the NS ground state. The first excited states in the NS–sector is the $SO(D-2)$ vector

$$\psi^i_{-\frac{1}{2}}|0\rangle \quad \text{with mass} \quad \alpha' m^2 = \frac{3(10-D)}{48} \tag{38}$$

[14]When associating (half)integer moded fermionic oscillators for ψ^i with the (NS) R sectors, we have assumed that the the bosonic field X^i has integer moded oscillators, i.e. satisfies NN or DD boundary conditions. The precise definition is that in the R sector the moding of bosons and fermions is the same whereas in the NS sector they are different. This means in particular that for ND and DN b.c.'s the (half) integer moded fermions belong to the NS (R) sector.

and, therefore, the critical dimension of the fermionic string is $D = 10$. One can show that at each positive mass level the states combine into (reducible) tensor representations of $SO(9)$. Note that when enumerating all states in the NS sector, one has to take into account that $(\psi^i_r)^2 = 0$ for each i and r. This is the Pauli exclusion principle for the world-sheet fermions. Note that even though we are discussing world-sheet fermions here, the excited string states they create are space-time bosons. This will be different in the R–sector which we discuss next.

In the R–sector we have the zero modes ψ^i_0 which require special treatment. They satisfy the Clifford algebra

$$\{\psi^i_0, \psi^j_0\} = \delta^{ij}\,, \qquad i, j = 1, \ldots, 8\,. \tag{39}$$

The representation of (39) is essentially unique[15] and given in terms of Dirac matrices, i.e. $\psi^i_0 = \frac{1}{\sqrt{2}}\Gamma^i$, Γ^i being the 16×16 Dirac-matrices of $SO(8)$. The ψ^i_0 commute with the mass operator which means that the ground state in the R–sector is degenerate. In fact it has zero mass since the zero-point energy vanishes in the R–sector. The different ground states are transformed into each other via the action of ψ^i_0. But this means that the ground-state in the R–sector, which we will denote by $|A\rangle$, $A = 1, \ldots, 16$, transforms as a spinor of $SO(8)$ and that

$$\psi^i_0|A\rangle = \frac{1}{\sqrt{2}}\big(\Gamma^i\big)^A{}_B|B\rangle\,. \tag{40}$$

We can reach excited states by acting on the ground states with oscillators ψ^i_{-n} with $n \in \mathbb{Z}_+$. Of course, the Pauli exclusion principle $(\psi^i_n)^2 = 0$, $\forall n$ and i, has to be taken into account when enumerating the states. Since the oscillators carry a $SO(8)$ vector index, all states in the R–sector transform in a spinor representation of $SO(8)$ and are thus space-time fermions.

The 16-dimensional spinor representation of $SO(8)$ is reducible, its irreducible components being the eight-dimensional chiral spinors which span the subspaces with eigenvalue ± 1 of the chirality operator $\Gamma^9 \equiv \Gamma^1 \cdots \Gamma^8$ which satisfies $(\Gamma^9)^2 = 1$. (Spinors with definite Γ^9 eigenvalue are called Weyl spinors.[16]) To distinguish the two irreducible components we split the spinor index as follows: $A = (a, \dot{a})$ and thus $|A\rangle = |a\rangle \oplus |\dot{a}\rangle$ with $\Gamma^9|a\rangle = +|a\rangle$ and $\Gamma^9|\dot{a}\rangle = -|\dot{a}\rangle$. The two eight-dimensional spinor representations are often denoted as $\underline{8}_s$ and $\underline{8}_c$. $SO(8)$ has a third eight-dimensional representation – the vector on which $SO(8)$ acts as a rotation – often denoted as $\underline{8}_v$.

For the closed string we have to impose periodicity conditions. In the case of real fermions there

[15] In even dimensions the Clifford algebra has only one inequivalent irreducible representation whereas in odd dimensions it has two. This representation is in terms of Dirac matrices of dimension $2^{[D/2]}$ where $[D/2]$ is the integer part of $D/2$. The two representations for D odd differ by the sign of $\Gamma^D \propto \Gamma^1 \cdots \Gamma^{D-1}$. A proof of this statement and many other useful properties of Dirac matrices in arbitrary dimensions can be found in [40].

[16] Weyl spinors exist in all even dimensions. For $D = 2n$ we define $\Gamma^{2n+1} = \alpha \prod_{i=1}^{2n} \Gamma^i$ with the phase α chosen such that $(\Gamma^{2n+1})^2 = 1$. Then Weyl spinors are eigenspinors of Γ^{2n+1}. In odd dimensions they do not exist since there $\prod_{i=1}^{D} \Gamma^i \propto \mathbf{1}$.

are two options: periodic or anti-periodic. Both options leave the action invariant.[17] This leads to the following options:[18]

$$\psi^i(\tau,\sigma+\pi)=\eta\psi^i(\tau,\sigma) \quad \text{and} \quad \tilde\psi^i(\tau,\sigma+\pi)=\tilde\eta\psi^i(\tau,\sigma) \qquad \text{with} \qquad \eta,\tilde\eta=\pm 1\,. \tag{41}$$

If we now make a mode expansion we need to introduce two sets of oscillators, ψ^i_r and $\tilde\psi^i_r$, where, depending on the choices for η and $\tilde\eta$, the mode number r is either integer (periodic) or half-integer (anti-periodic). This gives four possible sectors

- (NS,NS) : $\eta=\tilde\eta=-1\,,$ (R,R) : $\eta=\tilde\eta=1$ (space-time bosons),
- (NS,R) : $\eta=-\tilde\eta=-1\,,$ (R,NS) : $\eta=-\tilde\eta=1$ (space-time fermions).

Quantization proceeds as for the open string only that we now have two sets of fermionic oscillators, each contributing to the Hamiltonian as in (35) and each satisfying the anti-commutation relations (36). Also each set of fermionic oscillators contributes to the zero point point energy and the level matching condition (27) now involves the number operators for bosonic and fermionic oscillators. Note that the level-matching condition forbids e.g. a tachyon in the (NS,R) sector.

(R,R)–sector ground states are bispinors

$$|A\rangle_L\otimes|B\rangle_R=|a\rangle_L\otimes|b\rangle_R\oplus|a\rangle_L\otimes|\dot b\rangle_R\oplus|\dot a\rangle_L\otimes|b\rangle_R\oplus|\dot a\rangle_L\otimes|\dot b\rangle_R. \tag{42}$$

As for the open string the zero-point energy vanishes in the R–sectors and these states give rise to massless bosonic fields in space-time (cf. below).

It turns out that the theory we have constructed is not consistent. One sign of the inconsistency is the appearance of the tachyonic ground state in the (NS,NS)-sector. A more severe inconsistency is the lack of modular invariance of the one-loop partition function. We do not intend to elaborate on this very much but try to convey the main point of the argument and then simply state the consequences.

We had discussed diffeomorphism invariance of the Polyakov action. Together with Weyl invariance it is necessary to ensure that with the introduction of the world-sheet metric no new degrees of freedom are added, or in other words, that the three degrees of freedom of $\gamma_{\alpha\beta}$ can be gauged away. This is a non-trivial requirement for the quantized theory and, in fact, quantization often breaks symmetries which the classical action possesses. One then speaks of anomalies. In string theory one must ensure that diffeomorphism and Weyl invariance are still present after quantization. One can show that this requirement also fixes the critical dimension to the values we found. However there are additional restrictions which one encounters when one studies the diffeomorphism invariance of correlation functions on world-sheets of higher genus, e.g. the closed string zero-point function at one

[17] The interpretation of X^i as space-time coordinates does not allow for more complicated periodicity conditions of the bosons if we want to describe a string moving in Minkowski space-time, even though other possibilities are compatible with the reality of X and the invariance of the action. When one considers compactifications of string theory such possibilities are, however, considered.

[18] We have to impose the same conditions for all i if we want to preserve $SO(D-2)$ invariance.

loop, i.e. with the world-sheet being a torus. This is also called the closed string partition function and it can be shown to be $\mathrm{Tr}\big(e^{2\pi i\tau H_{\mathrm{l.c.}}}e^{-2\pi i\bar\tau\tilde H_{\mathrm{l.c.}}}\big)$. The trace is over all states of the closed string, not necessarily satisfying the physical state condition. We have split the contributions from left and right movers to the Hamiltonian and have denoted the two contributions by $H_{\mathrm{l.c.}}$ and $\tilde H_{\mathrm{l.c.}}$. τ is the modular parameter of the torus. We can define the torus as $\mathbb{R}/\Lambda$, where Λ is a two-dimensional lattice. This lattice can be specified by fixing a point $\tau=\tau_1+i\tau_2$, $\tau_2>0$ in the upper half of the complex plane. The generators of the lattice are then the two vectors $(1,0)$ and (τ_1,τ_2), where we have used Weyl invariance to scale the length of the first generator to one. One now uses the fact that not all choices for τ in the upper half-plane lead to diffeomorphically different tori: if τ and τ' are related by a $PSL(2,\mathbb{Z})$ transformation, i.e. if $\tau'=\frac{a\tau+b}{c\tau+d}$ with $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$, the two tori defined by τ and τ' are diffeomorphic. In fact, the two lattices are the same, only the choice of generators is different. The diffeomorphism cannot be smoothly deformed to the identity; it is a so-called large diffeomorphism. Nevertheless, the partition function should be invariant under these diffeomorphisms or, in other words, it must be modular invariant ($PSL(2,\mathbb{Z})$ is the modular group).

This condition, namely that the one-loop partition function be invariant under modular transformations, will not be satisfied if one sums over all states of the closed fermionic string. To get a modular invariant expression, one has to truncate the spectrum, or, in other words, one has to introduce a suitable projection operator $\mathcal{P}$ and compute $\mathrm{Tr}(\mathcal{P}e^{2\pi i\tau H_{\mathrm{l.c.}}}e^{-2\pi i\bar\tau\tilde H_{\mathrm{l.c.}}})$. The necessity for such a projection to arrive at a consistent string theory was first realized by Gliozzi-Scherk-Olive and is called GSO-projection. For the closed fermionic string there are two possible GSO projections which lead to a tachyon free spectrum. We will not describe them in any detail but simply state the resulting massless spectra.

In both cases the (NS,NS)–sector contributes a graviton G_{ij}, an anti-symmetric tensor B_{ij} and a dilaton Φ. In contrast to the bosonic theory these states are created from the (NS,NS) vacuum with fermionic oscillators, i.e. $\psi^i_{-1/2}\tilde\psi^j_{-1/2}|0\rangle_{\mathrm{NS,NS}}$.

The (R,R)–sectors of the two theories are different. The two consistent choices are:[19]

- type IIA with (R,R) ground state $|a\rangle_L\otimes|\dot b\rangle_R$ (not chiral),
- type IIB with (R,R) ground state $|a\rangle_L\otimes|b\rangle_R$ (chiral).

The statement about the chirality means the following: in the type IIA theory the part of the spectrum with Γ^9 eigenvalue $+1$ is identical to the part with eigenvalue -1. For the type IIB theory this is not true. This is already obvious from looking at the (R,R)–ground states.

The (R,R) ground states transform under reducible components of $SO(8)$, namely as $\underline{8}_s\times\underline{8}_c$ and $\underline{8}_s\times\underline{8}_s$ for type IIA and IIB, respectively. To extract the irreducible pieces we make a short aside and

[19] $|\dot a\rangle_L\otimes|b\rangle_R$ and $|\dot a\rangle_L\otimes|\dot b\rangle_R$ are equivalent to these two choices.

discuss the index structure of Dirac matrices. An arbitrary Dirac matrix Γ can be decomposed into block form:[20] $\Gamma_{AB} = \begin{pmatrix} \Gamma_{ab} & \Gamma_{a\dot{b}} \\ \Gamma_{\dot{a}b} & \Gamma_{\dot{a}\dot{b}} \end{pmatrix}$. We now define the anti-symmetrized products of Dirac matrices: $\Gamma^{i_1 \ldots i_p} = \Gamma^{[i_1 \ldots i_p]} \equiv \frac{1}{p!}(\Gamma^{i_1} \cdots \Gamma^{i_p} \pm \text{permutations})$. For $p = 0$ this is the charge-conjugation matrix which is also used to raise and lower spinor indices. One can choose a basis in which either the two diagonal or the two off-diagonal blocks of each of these matrices is zero. More concretely, one finds: $\underline{D = 4n}$: For p even the blocks with mixed indices vanish and for p odd the blocks with the same indices vanish. $\underline{D = 4n + 2}$: here the situation is opposite w.r.t. p. In this basis $\Gamma^{D+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$. For $\underline{D = 2n + 1}$ there is no chirality and hence no distinction between dotted and un-dotted indices. Using these results, which can be proven by simple $SO(n)$ group theory or by explicitly constructing the Dirac matrices, we get the following decompositions into irreducible components:

$$\begin{aligned} \text{Type IIA:} \quad & |a\rangle \otimes |\dot{b}\rangle = \Gamma^{i}_{a\dot{b}}|i\rangle \oplus \Gamma^{ijk}_{a\dot{b}}|ijk\rangle \\ \text{Type IIB:} \quad & |a\rangle \otimes |b\rangle = C_{ab}|\cdot\rangle \oplus \Gamma^{ij}_{ab}|ij\rangle \oplus \Gamma^{ijkl}_{ab}|ijkl\rangle \end{aligned}$$

Another way of writing this is $\underline{8}_s \otimes \underline{8}_c = \underline{8}_v \oplus \underline{56}$ and $\underline{8}_s \otimes \underline{8}_s = \underline{1} \oplus \underline{28} \oplus \underline{35}^+$, where $\underline{35}^+$ denotes the self-dual fourth rank tensor representation of $SO(8)$. In fact, one can show that Γ^{ijkl}_{ab} is self-dual whereas $\Gamma^{ijkl}_{\dot{a}b}$ is anti-self-dual.

To summarize: the massless bosonic excitation spectra of type II theories are

(NS,NS)	IIA/IIB:	$G_{\mu\nu}$, $B_{\mu\nu}$ and Φ
(R,R)	IIA:	1-form A_μ, 3-form $A_{\mu\nu\rho}$
	IIB:	scalar χ, 2-form $B'_{\mu\nu}$, 4-form $A_{\mu\nu\rho\sigma}$ with self-dual field strength $F = dA$

We have written the space-time fields in covariant form. The light-cone components are not directly visible in the light-cone gauge, but in the critical dimension Lorentz-invariance is preserved and all bosonic fields must transform as tensors of the full Lorentz group $SO(1, 9)$. Nevertheless, the number of physical degrees of freedom they present is given by the counting in light-cone gauge. The reduction is due to gauge invariances and the equations of motion these space-time fields satisfy.

So far we have only discussed the bosonic degrees of freedom. The remaining two sectors, (R,NS) and (NS,R) contain space-time fermions. In fact, one can show that the resulting spectrum has $\mathcal{N} = 2$ space-time supersymmetry.

Supersymmetry (SUSY) is a generalization of Poincaré symmetry. In addition to having only bosonic generators which transform in tensor representations of the Lorentz-group (Lorentz-transformations, rotations and translations) the supersymmetry algebra also has fermionic generators, called supercharges. The algebra is a $\mathbb{Z}_2$ graded Lie-algebra, since the supercharges satisfy anti-commutation relations.[21] The corresponding transformation parameters ε are fermionic. They transform in spinor

[20] Properties of Dirac matrices are discussed in some detail in the lectures by J. Zanelli in these volume.

[21] More details on Supersymmetry and Supergravity and on the supersymmetry algebra can be found in the lectures by J. Zanelli.

representations and transform bosons into fermions and vice versa. Schematically $\delta_\varepsilon B = \varepsilon F$ and $\delta_\varepsilon F = \varepsilon \nabla B$. Saying that we have $\mathcal{N} = 2$ supersymmetry in $D = 10$ means that we have altogether 32 supercharges which form two Majorana-Weyl spinors of $SO(1,9)$ which we will call Q_L and Q_R, where the subscript means that one originates from the left and the other from the right-moving sector of the closed string theory. The distinction between type IIA and IIB is that in the former case the two Majorana-Weyl spinors have opposite chirality whereas they have equal chirality in the latter case. The same holds for the supersymmetry parameters which we will call ε_L and ε_R.

One of the hallmarks of linear representations of supersymmetry is that each of its irreducible representations contains the same number of bosonic and fermionic degrees of freedom; see, however, [41] and J. Zanelli's contribution in this volume. Since we will not need the fermionic part of the spectrum in the subsequent discussion, we will not discuss it. It can be reconstructed from the bosonic part of the spectrum via the supersymmetry algebra.

Supersymmetric field theories possess a supersymmetric spectrum and the action is invariant under the supersymmetry algebra. If the SUSY parameters are constants, one deals with global supersymmetry. If they depend on space-time, the theory necessarily contains gravity and one has a supergravity (SUGRA) theory. In the case of the type II string theories, one finds that their low-energy effective actions are in fact the type IIA and type IIB $\mathcal{N} = 2$ supergravity theories in ten dimensions.

One can show that the space-time supersymmetry of the type II string theories is a consequence of a world-sheet supersymmetry of the Polyakov action for the fermionic string. This is the generalization of the statement that the absence of anomalies of the local world-sheet symmetries in the bosonic string leads to the critical dimension which also guarantees space-time Lorentz symmetry. Here anomaly freedom of the world-sheet supersymmetry leads to space-time supersymmetry.

So far we have discussed the supersymmetry of the closed string sector. If we add D-branes we get theories with open and closed strings. We have seen that in the closed string Q_L and Q_R are associated with the left– and right–moving sectors of the world-sheet theory. Since they are coupled by the open string boundary conditions, one gets a reduction of the number of independent supercharges from 32 to 16.

One can show that in the presence of a Dp-brane whose world-volume fills the $x^0, \ldots, x^p$ directions, the surviving SUSY generators are $\bar{\varepsilon}_L Q_L + \bar{\varepsilon}_R Q_R$ where $\varepsilon_{L,R}$ are related as

$$\varepsilon_L = \pm \Gamma^0 \Gamma^1 \cdots \Gamma^p \varepsilon_R \,. \tag{43}$$

The sign choice distinguishes between a brane and an anti-brane. We multiply both sides of this equation by the $SO(1,9)$ chirality matrix $\Gamma = \Gamma^0 \cdots \Gamma^9$ and commute Γ on the r.h.s. through the $(p+1)$ Γ^i's. This produces a factor $(-1)^{p+1}$. If we now use that $\Gamma\varepsilon_L = \Gamma\varepsilon_R$ for IIB and $\Gamma\varepsilon_L = -\Gamma\varepsilon_R$ for IIA, we find that SUSY preserving Dp-branes exist in type IIB for p odd and in type IIA for p even. In the other cases one finds that the spectrum contains tachyons so that e.g. a D3 brane in type

IIA is unstable.[22] One can also work out the condition under which different species of branes, either for different p and p' and/or for different orientations in space-time, preserve some supersymmetry. One finds e.g. that for non-parallel branes one can preserve at most eight SUSY charges.

SUSY preserving branes are so-called BPS [23] configurations. They can be characterized by the representation theory of the SYSU algebra. We will illustrate this on a simple example, which is relavant to supersymmetric quantum mechanics, but the idea generalizes to field theory. Consider the algebra generated by two bosonic generators H and Z and two fermionic generators Q_1 and Q_2 (supercharges). The only non-zero (anti)commutators are $\{Q_1, Q_1^\dagger\} = H + Z$, $\{Q_2, Q_2^\dagger\} = H - Z$. In particular, since $[H, Z] = 0$, they can be diagonalized simultaneously. Consider an eigenstate $|\psi\rangle$. Its eigenvalues h and z satisfy the inequality $h \mp z = \langle\psi|[Q_{1,2}, Q_{1,2}^\dagger]|\psi\rangle = \|Q_{1,2}^\dagger|\psi\rangle\|^2 + \|Q_{1,2}|\psi\rangle\|^2 \geq 0$, *i.e.* $h \geq |z|$. This is the BPS-bound. For $h > |z|$, an irreducible SUSY multiplet consists of four states: the rescaled generators $q_{1,2} = Q_{1,2}/\sqrt{h \pm z}$ satisfy the algebra of two fermionic oscillators, and the multiplet consists of the following four states: $|0\rangle$, $q_1^\dagger|0\rangle$, $q_2^\dagger|0\rangle$, $q_1^\dagger q_2^\dagger|0\rangle$. The vacuum $|0\rangle$ satisfies $q_{1,2}|0\rangle = 0$. If, however, $h = |z|$, one of the two supercharges decouples and we are left with just one fermionic oscillator. The multiplet then consists of two rather than four states. The states of these 'short multiplets' are called BPS states. *E.g.* for $h = z$, Q_2 decouples and both Q_2 and $Q_2^\dagger$ annihilate the eigenstate $|\psi\rangle$. In the context of our discussion of branes in lecture 3, h is their mass and z their (R,R)-charge.

Given a SUSY preserving brane, we can always add more branes of the same type without breaking SUSY further. These branes do not have to be coincident. As long as they are parallel one also obtains a BPS configuration. These configurations are, as a consequence of SUSY, stable, i.e. no *net* force acts between the branes. However, a brane-anti-brane system breaks all supersymmetries and is unstable. This is also true for generic configurations of Dp- and Dp$'$-branes. The equations (43) have no solution.

The low-energy effective actions of the massless excitations of the open string are, to lowest order in α', Super-Yang-Mills (SYM) theories on the world-volumes of the branes where the gauge group is determined by the brane configuration. For instance, in the case of N parallel D3 branes one obtains a four-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$.

At the end of our discussion of the bosonic string we briefly discussed the generalization of the Polyakov action which incorporates background values for the massless space-time fields. The same generalization also holds for the massless fields of the (NS,NS) sector of the type II string theories. However, within the NSR formulation that we have been using, no such coupling to fields in the (R,R) sector is known.

Lecture 3: Branes from supergravity

[22] The fate of tachyonic theories has been much discussed recently, see e.g. [42].

[23] Bogomolnyi-Prasad-Sommerfield

In the previous lecture we have seen that Dirichlet boundary conditions of the open string ends implies the existence of D-branes and we have argued that they are dynamical objects of the theory. One might wonder whether they are a necessity. After all, one might decide to impose only Neumann boundary conditions. This would correspond to the presence of space-time filling D9 branes which can have no dynamics. There are, however, various ways to show that lower dimensional branes must also be considered. One is based on T-duality, which we will not discuss here, except for saying that this is a symmetry of string theory which changes the boundary conditions from Neumann to Dirichlet and vice versa. Another argument is based on the low-energy effective action for the massless string excitations where one finds brane solutions as solitonic solutions of the classical equations of motion. This is the route we will follow.

The discussion in this section mainly involves the bosonic fields. For the type II theories they were summarized above. Their low energy dynamics is governed by a low energy effective action (*leea*). In the limit $g_s \to 0$, which suppresses string loop effects and $\alpha' \to 0$, which renders all massive string modes infinitely heavy and they thus decouple, the *leea*'s are the type IIA and IIB supergravity theories. They involve only terms with at most two derivatives. Higher derivative terms, such as $\mathcal{R}^2$ would be multiplied by additional powers of α'. They are suppressed in the low energy limit where all external momenta satisfy $k \ll 1/\sqrt{\alpha'}$. We will not write down the complete *leea*; the interested reader may find it e.g. in [23]. We will write down the relevant terms which are needed for finding brane solutions. But first we want to understand which fields such a solution will involve. For this purpose we clarify the relevance of the anti-symmetric tensor fields which appear in the massless spectra.

Recall that a charged particle in four dimensions couples to a background vector (1-form) potential A_μ via the term

$$q_e \int_{\mathcal{C}_1} A \tag{44}$$

in the action. Here q_e is the (electric) charge of the particle and $\mathcal{C}_1$ its world-line through space-time. The charged particle is also a source for the field. If $F = dA$ is the total field strength (including the one generated by the particle), q_e can be determined by integrating the dual field strength $*F$ over a 2-sphere surrounding the particle (but no other source for F)

$$q_e = \int_{S^2} *F \,. \tag{45}$$

One can modify Maxwell theory to allow magnetically charged objects. In $3+1$ dimensions these are also point particles (magnetic monopoles) and their magnetic charge is

$$q_m = \int_{S^2} F \,. \tag{46}$$

The electric and magnetic charges are not independent from each other. They satisfy the Dirac quantization condition

$$q_e q_m = 2\pi n \,, \quad n \in \mathbb{Z} \,. \tag{47}$$

This can be derived by requiring single-valuedness of the wave-function of an electrically charged particle in the presence of a magnetic monopole.

There is a two-fold generalization to the above, namely going from four to D dimensions and introducing higher-dimensional objects [43, 44]. A p-dimensional (electrically) charged object couples to a $(p+1)$-form potential via

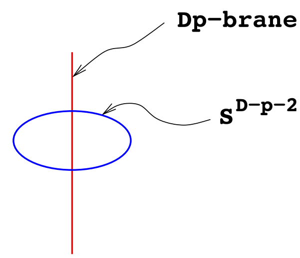
$$q_e \int_{\mathcal{C}_{p+1}} A^{(p+1)} \,, \tag{48}$$

where $\mathcal{C}_{p+1}$ is the object's world-volume. As an example, consider the coupling of the fundamental string to the (NS,NS) 2-form B-field

$$\int_\Sigma B = \int d\tau d\sigma \, \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \,. \tag{49}$$

Here $B_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}$ is the pull-back of the B-field from space-time to the string world-sheet. We have encountered this coupling in eq.(29). The objects which couple to the $(p+1)$-form potentials originating from the (R,R)-sector are called p-branes. The electric charge of a p-brane is

$$q_e = \int_{S^{D-p-2}} (*F)^{(D-p-2)} \tag{50}$$



The position of a p-brane is given by a point in the $(D-p-1)$-dimensional space transverse to it. In this transverse space it can be surrounded by a (large) $(D-p-2)$-sphere. This expression for the electric charge corresponds to an action of the form $S = -\frac{1}{2(p+2)!}\int F^2_{(p+2)} + q_e \int_{\mathcal{C}_{p+1}} A^{(p+1)}$. It leads to the equation of motion $d * F = q_e \delta^{\|}$, which, upon integration over the transverse space and use of Stokes's theorem gives (50); $\delta^{\|}$ is the delta function with support along the world-volume of the brane.

Given the electrically charged branes, what are the dual magnetically charged objects? Their charges should be given by

$$q_m = \int_{S^{p+2}} F^{(p+2)} \,. \tag{51}$$

Since a $(p+2)$-sphere surrounds a $(D-p-4)$-dimensional object, we have an electric-magnetic duality between p-branes and $(D-p-4)$-branes. Again, their charges must obey the Dirac quantization condition. A brief comment: in four dimensions, in addition to electrically charged particles and magnetically charged monopoles one can also have dyons, which carry electric and magnetic charge. For higher dimensional objects this is in general not possible as the dual objects carrying electric and magnetic charge generally have different dimensions. The same holds for self-dual objects which couple to a potential with self-dual field strength. In $D = 10$ the three brane of type IIB string theory is self-dual.

From the above discussion we conclude that a p-brane solution of the equations of motion should contain a non-trivial (R,R) $A^{(p+1)}$ background field configuration.[24] In addition it must contain the space-time metric which couples to the energy-momentum of the $A^{(p+1)}$. We now write those terms of the *leea* which contain the metric and $A^{(p+1)}$:

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(p+2)!}e^{\frac{3-p}{2}\phi}F^2_{(p+2)}\right) \tag{52}$$

where $F^2_{(p+2)} = F_{\mu_0\ldots\mu_{p+1}}F^{\mu_0\cdots\mu_{p+1}}$ and $\kappa^2 = \frac{1}{2}(2\pi)^7\alpha'^4 g_s^2$. We have chosen the action in the *Einstein frame* in which the metric $g_{\mu\nu}$ is related to the *string-frame* metric $G_{\mu\nu}$ via $G_{\mu\nu} = e^{\frac{1}{2}\phi}g_{\mu\nu}$. The equation of motion for the dilaton, $\nabla^2\phi = \frac{3-p}{4(p+2)!}e^{\frac{3-p}{2}\phi}F^2_{(p+2)}$, implies that for non-vanishing $F^2_{(p+2)}$ the dilaton cannot be constant unless $p = 3$. This is in fact the case we will eventually be interested in. However, for $p = 3$ the above action is not valid since $F^2_{(5)} = 0$ for self-dual $F_{(5)}$. Nevertheless, the equations of motion are the same as those following from (52) if one multiplies the F^2 term by 1/2 and imposes the self-duality condition separately [45]. For $p = 3$ we give the solution in (62).

It is now easy to set up the complete system of equations of motion. To solve them is an entirely different matter altogether. For this we have to make an ansatz which reflects the symmetries of the solution we are looking for. We are interested in brane solutions which extend in say, the $(x^0, x^1, \ldots, x^p)$ directions, which we call x^μ. The ansatz for the metric then has to respect Poincaré symmetry along the brane or, in other words, the complete solution should not depend on x^μ. In addition we assume spherical symmetry in the transverse space with coordinates y^m, $m = 1, \ldots, 9-p$. This leads to the following ansatz for an electrically charged brane at $y = 0$:[25]

$$ds^2 = A(r)\eta_{\mu\nu}dx^\mu dx^\nu + B(r)\delta_{mn}dy^m dy^n\,, \tag{53}$$

$$e^\phi = C(r)\,, \tag{54}$$

$$A_{\mu_0\cdots\mu_p} = \varepsilon_{\mu_0\cdots\mu_p}D(r)\,, \tag{55}$$

with $r^2 = \delta_{mn}y^m y^n$. Inserting this ansatz into the equations of motion leads to a system of second order ordinary non-linear differential equations. The simplest non-trivial solution which approaches ten-dimensional empty Minkowski-space for $r \to \infty$ and which is valid for all $r > 0$ is (details can be found e.g. in [28, 29]):[26,27]

$$A(r) = f(r)^{\frac{p-7}{8}}\,, \qquad B(r) = f(r)^{\frac{p+1}{8}}\,, \qquad C(r) = f(r)^{\frac{3-p}{4}}\,, \qquad D(r) = f(r)^{-1}, \tag{56}$$

[24] For the fundamental string and its dual object, the NS five-brane, we would need a non-trivial (NS,NS) B-field background. We will not discuss them here.

[25] For the magnetic dual we would make an ansatz for $*F$ and for $p = 3$ an ansatz which leads to a self-dual field strength.

[26] Since we are looking for supersymmetry preserving solutions, it is in fact simpler to analyze the SUSY condition. This amounts to requiring that the supersymmetry transformation of any fermionic field F vanishes, *i.e.* $\delta_\epsilon F = 0$. Here ϵ parametrizes the unbroken supersymmetries. This leads to first oder differential equations from which the second order equations derived from the effective SUGRA action can be recovered by iteration. The SUSY preserving solutions are in fact the simplest ones; they have the highest symmetry.

[27] This solution is valid for $p < 7$. For $p = 7$, $(f-1) \propto \ln(r)$.

where

$$f(r) = 1 + \frac{k_p N}{r^{7-p}}\,, \qquad N \in \mathbb{Z} \tag{57}$$

is a solution to the Laplace equation in the transverse $(9-p)$-dimensional space and

$$k_p = \frac{2\kappa^2 \tau_p}{(7-p)\Omega_{8-p}}\,. \tag{58}$$

Ω_{8-p} is the volume of S^{8-p}.[28] τ_p is the tension (mass per unit volume) of the brane which is defined via

$$N\tau_p = \int d^{9-p}x\Theta_{00}\,. \tag{59}$$

The integral is over the transverse space. Here $\Theta_{\mu\nu}$ is energy-momentum pseudo-tensor of the system which is defined as follows. Expand the metric which we found above around flat space as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and define $\mathcal{R}_{\mu\nu} = \mathcal{R}^{(1)}_{\mu\nu} + \mathcal{O}(h^2)$.[29] Then $\Theta_{\mu\nu}$ is defined as $\mathcal{R}^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{R}^{(1)} \equiv \kappa^2\Theta_{\mu\nu}$ ($\mathcal{R}^{(1)} = \eta^{\mu\nu}\mathcal{R}^{(1)}_{\mu\nu}$). One can read this as the wave equation for a spin-two particle (the graviton) with source given by the energy of the gravitational field and matter. For the metric (53) $2\kappa^2\Theta_{00} = p\Box A(r) + (8-p)\Box B(r)$ where $\Box = \partial_r^2 + \frac{8-p}{r}\partial_r$. What we have computed here is the ADM (for Arnowitt-Deser-Misner) tension of the brane. For further reading, see e.g. [46, 47]. If one computes the electric charge density of the brane, c.f. eq.(50) one finds $q_e = N\sqrt{2}\kappa\tau_p$. This follows most easily from the following observation. $A^{(p+1)}$ satisfies $\partial_\mu(\sqrt{g}e^{\frac{3-p}{2}\phi}F^{\mu 01\ldots p}) = 2\kappa^2\tau_p N\delta^{(9-p)}(y)$, i.e. there is a source for $A^{(p+1)}$, the brane.[30] In fact, one can incorporate the source term into the action by adding

$$S_{\text{brane}} = -N\tau_p \int_{\mathcal{C}_{(p+1)}} d^{p+1}\xi e^{\frac{p-3}{4}\phi}\sqrt{-\det \hat{g}_{\mu\nu}} + N\tau_p \int_{\mathcal{C}_{(p+1)}} A^{(p+1)}\,. \tag{60}$$

where $\hat{g}_{\mu\nu}$ is the induced metric on the brane which, in static gauge, where we identify $\xi^\mu = x^\mu$, is simply $\hat{g}_{\mu\nu} = A(r)\eta_{\mu\nu}$. The first term is the analogue of the Nambu-Goto action for the fundamental string: it is the 'area' of the world-volume in string frame (expressed in the Einstein frame metric).

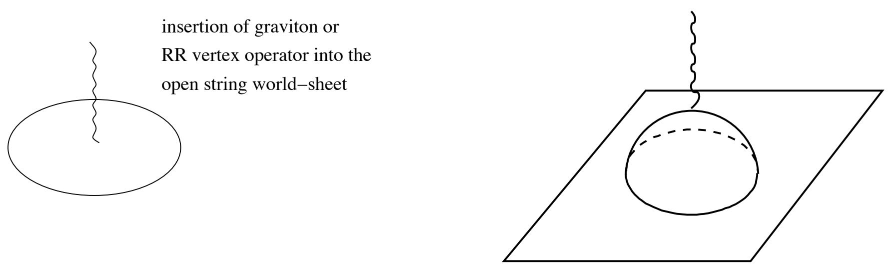
The crucial observation of Polchinski [54] was that the SUSY preserving p-brane solutions of the SUGRA equations of motion are the same objects as the Dp-branes. Before we discuss the implications (only closed strings in type II, no explicit gauge fields, etc) let us briefly review the arguments in favour of this identification. D-branes are BPS states. One consequence is that a stack of parallel D-branes is stable. The p-brane solution (56) can be easily generalized to a stack of parallel branes at $\vec{y}_i$ by the substitution $f \to 1 + \sum_{i=1}^N \frac{k_p}{|\vec{y}-\vec{y}_i|^{7-p}}$. This is a static solution which can be shown to be stable. From the results of the previous lecture it follows that the allowed dimensions of half SUSY preserving D-branes are correlated with the massless (R,R) states. This is also true for the SUGRA branes which carry (R,R) charge. It thus remains to be shown that D-branes also carry (R,R) charge and that the ratio between their charge and tension is the same as for the p-brane. We do not give details of this

[28] $\Omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$.

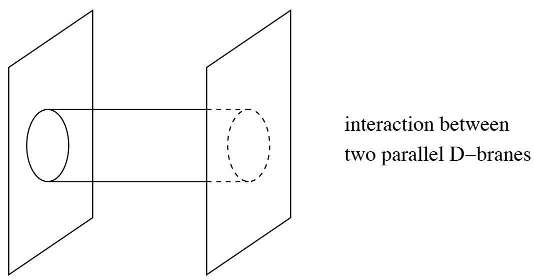
[29] Explicitly, this is $\mathcal{R}^{(1)}_{\mu\nu} = \frac{1}{2}(\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial^\rho\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\nu h^\rho{}_\rho)$ and $\mathcal{R}^{(1)} = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^\rho\partial_\rho h^\mu{}_\mu$.

[30] The definition of electric charge that follows from (52) is $q_e = \frac{1}{\sqrt{2}\kappa}\int_{S^{8-p}} e^{\frac{3-p}{2}\phi} * F$.

calculation which was first performed by Polchinski but we will try to convey the idea. In order to compute the tension of the brane we need to find the strength with which it couples to a graviton and in order to find its charge we need the coupling to the (R,R) fields. The string diagrams which are responsible for these couplings are a disk with a graviton or (R,R) p-form vertex operator inserted.



The boundary of the disc is stuck on the D-brane, as shown in the picture on the right. In the type II string theories D-branes are incorporated as so-called boundary states. The subtlety with the calculation is to fix the absolute normalization of the boundary states which is needed in order to get the correct coupling. The way one does the calculation is to compute the interaction between two parallel D-branes. They interact via exchange of closed strings. The relevant diagram is shown below.



In this closed string *tree-level* exchange diagram world-sheet time runs *along* the axis of the cylinder. However, we may equally well consider this diagram as an open string *one-loop* diagram, with world-sheet time running *around* the cylinder. This is just the open string partition function which can be computed straightforwardly. One finds a zero result which can be understood from the fact that this brane configuration preserves half of the supersymmetry. The attractive force mediated by the exchange of dilatons and gravitons and the repulsive force due to anti-symmetric tensor exchange cancel. This is the BPS or no-force property. One can separate these two contributions as they belong to different sectors ((NS,NS), *vs.* (R,R)). Comparing these two contributions to the amplitude to a field theoretic calculation one finds for the tension of a single Dp-brane

$$\tau_p = \frac{\sqrt{\pi}}{\kappa}(2\pi l_s)^{(3-p)} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \,. \tag{61}$$

Furthermore, one finds the same ratio between charge and tension as for the SUGRA p-brane solutions. Up to numerical factors the expression for the tension can be easily understood: the powers of $l_s = \sqrt{\alpha'}$ are needed for dimensional reasons. In natural units, where $\hbar = c = 1$, the tension, which is $mass/volume$ has dimension $(length)^{-(p+1)}$. The dependence on the string coupling constant, $\tau_p \sim 1/g_s$ follows from the fact that the tension is computed from a disk diagram with Euler number $+1$. At weak coupling, $g_s \to 0$, the D-branes are very heavy and are not visible in the perturbative excitation spectrum of the type II string theories.

Following Polchinski, we have argued that the D-branes, which have a microscopic description in open string theory, are in fact the same objects as the classical p-brane solutions of the low energy effective SUGRA theories which know nothing about open string modes. The two descriptions are good in different regimes of the parameter space. The D-brane picture is good at weak string coupling where the string is perturbative. In the presence of N D-branes, the effective coupling is Ng_s, which must stay small. Also, we have developed the D-brane picture in Minkowski space-time. This assumes that we can neglect the back reaction of the brane on the background geometry. This is justified if the number of branes which carry energy-momentum, is small. The SUGRA picture also requires weak string coupling since that was assumed in the construction of the *leea*. In addition, the curvature of space-time must stay small everywhere (in string units). This requires that $g_s N$ is large since this is the condition that the characteristic length scale in the solution is bigger than the string scale.

An important difference of the two descriptions is that while the D-branes couple to open strings and carry a gauge theory on their world-volume, there are no signs of open strings and gauge fields in the p-brane picture. Nevertheless, if the two descriptions are 'dual' to each other, they should describe the same physics. The AdS/CFT correspondence, which we will discuss in the final lecture, establishes the relation between the two pictures.

The solutions we have discussed here are *extremal* p-brane solutions. In appropriately chosen units they satisfy the equality $mass = charge$ (i.e. the coefficients in front of the two terms in (60) are equal). This equality is known as the Bogomolnyi bound and general solutions satisfy $mass \geq charge$. The solutions which do not satisfy the bound are called *non-extremal*. They break all 32 supercharges. The nomenclature here is the same as for charged (Reissner-Nordstroem) black holes who are characterized by two parameters, their mass and charge. The extremal solution has a degenerate horizon at $r = 0$, g_{tt} has a double zero there. The non-extremal solutions have an inner and an outer horizon. The curvature blows up at the inner horizon but the singularity disappears in the extremal limit. The construction of non-extremal solutions can be found in [28, 29].

Lecture 4: The AdS/CFT correspondence

In lectures two and three we have provided background material of string theory and classical solutions of the supergravity equations of motion, which describe the dynamics of the massless string excitations at low energies. One of the main results was the identification of D-branes and brane solutions in supergravity as two descriptions of the same objects. In this lecture we will, after providing additional background material, formulate the AdS/CFT correspondence, also known as Maldacena conjecture. This provides a precise identification between supergravity on the one side and gauge theory (on the brane) on the other side. Of course, this assumes that we are in a particular corner of parameter space, which we will specify. From now on we only consider the case $p = 3$, the self-dual three-brane solution of type IIB supergravity with a four-dimensional world-volume. For N coincident three-branes the solution is

$$\begin{aligned} ds^2 &= f(r)^{-1/2}\left(-dt^2 + d\mathbf{x}^2\right) + f(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \\ F_5 &= (1+*)df^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ \Phi &= \phi_0 = \text{const.} \\ f(r) &= 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N . \end{aligned} \tag{62}$$

$d\Omega_5^2$ is the length element on S^5 and $*$ the Hodge star. The constant term in $f(r)$ is an integration constant and it was chosen such that

$$\left(ds^2\right)_{\text{brane}} \longrightarrow \left(ds^2\right)_{\text{Mink.}} \tag{63}$$

as $r \to \infty$. We can also look at the limit of the metric as $r \to 0$ (i.e. the near-horizon limit). Then

$$f(r) = 1 + \frac{R^4}{r^4} \longrightarrow \frac{R^4}{r^4}, \tag{64}$$

and

$$\left(ds^2\right)_{\text{brane}} \longrightarrow \frac{r^2}{R^2}\left(-dt^2 + d\mathbf{x}^2\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 . \tag{65}$$

After the change of variables $\rho \equiv \frac{R^2}{r}$ this metric is

$$ds^2 = \underbrace{\frac{R^2}{\rho^2}\left(-dt^2 + d\mathbf{x}^2 + d\rho^2\right)}_{\left(ds^2\right)_{AdS_5}} + \underbrace{R^2 d\Omega_5^2}_{\left(ds^2\right)_{S^5}} \tag{66}$$

which may be recognized as the metric on the product-space $AdS_5 \times S^5$.

Before saying more about anti-de Sitter space we make some remarks. As shown above, the 3-brane metric interpolates between 10-dimensional Minkowski space, being the asymptotic space-time for $r \to \infty$ and the near-horizon geometry $AdS_5 \times S^5$ at $r \to 0$. The geometries in the extreme regions have higher symmetries (bigger isometry groups and more supersymmetry) than the brane solution has. Specifically, while (62) preserves 16 supercharges, both Minkowski space and $AdS_5 \times S^5$ are invariant under 32 supercharges. The Minkowski space is of course just the type IIB vacuum which is an exact perturbative ground state of string theory, i.e. to all orders in α' and g_s. The same can be shown to be true for $AdS_5 \times S^5$, but it is not true for the interpolating solution (62) [48].

Both AdS_d and S^d are symmetric spaces with curvature tensors[31]

$$\mathcal{R}_{ijkl} = \pm\frac{1}{R^2}\big(g_{ik}g_{jl} - g_{il}g_{jk}\big)\,, \quad \mathcal{R}_{ij} = \pm\frac{d-1}{R^2}g_{ij}\,, \quad \mathcal{R} = \pm\frac{d(d-1)}{R^2}\,. \tag{67}$$

The upper (lower) sign is for anti-de Sitter space (the sphere). They are conformally flat, i.e. their metrics are proportional to the flat Minkowski (Euclidean) metric.[32] They thus have have vanishing Weyl-tensor. They are solutions of the Einstein equations with cosmological constant derived from the action

$$S = \int d^d x\sqrt{-g}(\mathcal{R}+\Lambda)\,, \qquad \Lambda = \mp\frac{(d-2)(d-1)}{R^2}\,. \tag{68}$$

For the AdS/CFT correspondence both anti-de-Sitter space and the conformal group play a central rôle. We will return to AdS space after the following brief introduction to the latter.

The Poincaré group is familiar from introductory physics courses as the invariance group of $(length)^2$ in Minkowski space, i.e. the invariance group of $(ds)^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, $\mu,\nu = 0,1,\dots,D-1$. The Poincaré transformations are $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$ where the constant matrix $\Lambda_\mu{}^\nu$ satisfies $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ and a^μ is a constant vector. Λ generates Lorentz transformations and a translations. The conformal group is the invariance group of the light-cone, i.e. of all transformations which leave $(ds)^2 = 0$ invariant. Clearly this group contains the Poincaré group as a subgroup, but it is strictly bigger. For instance, constant rescalings $x^\mu \to e^\lambda x^\mu$ and inversion $x^\mu \to x^\mu/x^2$ also leave the light-cone invariant. If we follow an inversion by a translation by b and a second inversion, we arrive at the special conformal transformations $x^\mu \to \frac{x^\mu + x^2 b^\mu}{1+2b\cdot x + b^2x^2}$ which, in contrast to the inversion, can be expanded around the identity transformation.

We will now proceed as follows. We will show that infinitesimal translations P_μ, Lorentz transformations $L_{\mu\nu}$, rescalings (D) and special conformal transformations K_μ generate the $\frac{1}{2}(D+1)(D+2)$ parameter conformal group in D-dimensional Minkowski space. We will then show that the conformal group is isomorphic to $SO(D,2)$. It is clear from the form of the special conformal transformations that the conformal group acts non-linearly on Minkowski-space. But being isomorphic to $SO(D,2)$ it acts linearly on $\mathbb{R}^{D+2}$ endowed with a metric with signature $((+)^D,(-)^2)$. We will then define anti-de Sitter space as a hypersurface in this space on which $SO(D,2)$ acts isometrically.

Under infinitesimal diffeomorphisms $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ the Minkowski metric changes as $\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. This is proportional to the original metric if $\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = f\eta_{\mu\nu}$ for some function $f(x)$. Taking the trace of this equation gives $f = \frac{2}{d}(\partial\cdot\xi)$ which leads to the conformal Killing equation

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{2}{d}(\partial\cdot\xi)\eta_{\mu\nu}\,. \tag{69}$$

[31] Our conventions for the curvature tensors are $[\nabla_m,\nabla_n]V_p = -\mathcal{R}_{mnp}{}^q V_q$, $\mathcal{R}_{mn} = \mathcal{R}^p{}_{mpn}$, $\mathcal{R} = g^{mn}\mathcal{R}_{mn}$.

[32] This is obvious for AdS from (66). For S^d, defined as $\sum_{i=1}^{d+1}(x^i)^2 = R^2$ one sees this after defining the stereographic coordinates (say, on the Northern hemisphere) $x^i = \frac{y^i}{v}$ for $i=1,\dots,d$ and $x^{d+1} = R(1-\frac{y^2}{2vR^2})$ which solves the defining equation for $v = 1+\frac{y^2}{4R^2}$. The metric is $ds^2 = \sum_{i=1}^{d+1}(dx^i)^2 = g_{ij}dy^i dy^j$ with $g_{ij} = \frac{1}{v^2}\delta_{ij}$.

One may show that the most general solution to this equation is $\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu - 2(b\cdot x)x^\mu + x^2 b^\mu$ where λ is a constant, a^μ and b^μ are constant vectors and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ a constant antisymmetric matrix ($\omega^\mu{}_\nu = \eta^{\mu\rho}\omega_{\rho\nu}$).

a^μ, $\omega_{\mu\nu}$, λ and b^μ parametrize infinitesimal translations, Lorentz-transformations, rescalings and special conformal transformations of x^μ, respectively. They are generated by $P_\mu = \partial_\mu$, $L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, $D = x\cdot\partial$ and $K_\mu = -2x_\mu x\cdot\partial + x^2\partial_\mu$ whose algebra is easily worked out. The non-vanishing commutators are

$$\begin{aligned}
[D, P_\mu] &= -P_\mu\,, \\
[D, K_\mu] &= \quad K_\mu\,, \\
[P_\mu, K_\nu] &= -2\eta_{\mu\nu}D + 2L_{\mu\nu}\,, \\
[L_{\mu\nu}, P_\rho] &= -\eta_{\mu\rho}P_\nu + \eta_{\nu\rho}P_\mu\,, \\
[L_{\mu\nu}, K_\rho] &= -\eta_{\mu\rho}K_\nu + \eta_{\nu\rho}K_\mu\,, \\
[L_{\mu\nu}, L_{\rho\sigma}] &= -\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} + \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma}\,.
\end{aligned} \tag{70}$$

If one defines $L_{D,D+1} = -D$, $L_{\mu D} = \frac{1}{2}(P_\mu - K_\mu)$ and $L_{\mu,D+1} = -\frac{1}{2}(P_\mu + K_\mu)$ the above commutation relations can be combined into the following single relation:

$$[L_{MN}, L_{PQ}] = -\eta_{MP}L_{NQ} - \eta_{NQ}L_{MP} + \eta_{MQ}L_{NP} + \eta_{NP}L_{MQ} \tag{71}$$

where $M, N, \dots = 0, 1, \dots, D+1$ and $\eta_{MN} = \mathrm{diag}(-1, +1, \dots, +1, -1)$ is the invariant metric of $SO(D,2)$. This establishes the isomorphism of the conformal algebra of D-dimensional Minkowski space with $so(D,2)$, the Lie algebra of $SO(D,2)$.[33] $SO(D,2)$ acts linearly on $\mathbb{R}^{D+2}$ with metric $(ds)^2 = \eta_{\mu\nu}dy^\mu dy^\nu + (dy^D)^2 - (dy^{D+1})^2$.

We can now identify D-dimensional Minkowski space as a subspace of $\mathbb{R}^{D+2}$ and describe the non-linear action of the conformal group on it. To this end, consider the subspace defined by the constraint $\eta_{\mu\nu}y^\mu y^\nu = uv$ where we have defined $u = y^{D+1} + y^D$ and $v = y^{D+1} - y^D$. Note that this is a D-dimensional cone inside $\mathbb{R}^{D+2}$. Firstly, this equation constitutes one constraint. Secondly, if y is a solution of this constraint, then so is λy for any non-zero real λ. We can use this rescaling freedom to set $\sum_{i=1}^D (y^i)^2 = 1 = (y^0)^2 + (y^{D+1})^2$ which shows that this cone has the topology $(S^{D-1}\times S^1)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ accounts for the fact that rescaling by $\pm\lambda$ are equivalent. This is in fact the conformal compactification of Minkowski space on which the conformal group acts properly (we need to compactify since the inversion transformation maps the origin to infinity, which is not part of Minkowski space).

[33] Here we assume $D > 2$. In $D = 2$ the conformal algebra is infinite dimensional. This is in fact very relevant for the world-sheet aspects of string theory and two-dimensional conformal field theories in general. We did not encounter this in the lecture on string theory since we fixed conformal transformations when going to light-cone gauge.

We now solve the constraint locally (in a patch with $u \neq 0$) by defining coordinates x^μ via

$$y^M = \begin{pmatrix} y^\mu \\ u \\ v \end{pmatrix} = \begin{pmatrix} ux^\mu \\ u \\ ux^2 \end{pmatrix} . \tag{72}$$

The linear $SO(D,2)$ transformations on y^M induce conformal transformations on x^μ. Specifically, we find the following relation between linearly acting $SO(D,2)$ transformations of y and conformal transformations of x:

$$\begin{pmatrix} \delta_\mu^\nu & 0 & 0 \\ 0 & \frac{1}{\Lambda} & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \longleftrightarrow x^\mu \to \Lambda x^\mu , \qquad \begin{pmatrix} \delta_\mu^\nu & 0 & b^\nu \\ 2b_\mu & 1 & b^2 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow x^\mu \to \frac{x^\mu + x^2 b^\mu}{1 + 2(b\cdot x) + b^2 x^2} ,$$

$$\begin{pmatrix} \omega^\mu{}_\nu & a^\mu & 0 \\ 0 & 1 & 0 \\ 2a_\rho\omega^\rho{}_\nu & a^2 & 1 \end{pmatrix} \longleftrightarrow x^\mu \to \omega^\mu{}_\nu x^\nu + a^\mu . \tag{73}$$

Each of these matrices $M_M{}^N$ satisfies $M_M{}^P M_N{}^Q \eta_{PQ} = \eta_{MN}$. Consider now the following hypersurface in $\mathbb{R}^{D+2}$:

$$-(y^0)^2 + \sum_{i=1}^{D} (y^i)^2 - (y^{D+1})^2 = \eta_{\mu\nu} y^\mu y^\nu - uv = -R^2 . \tag{74}$$

This $(D+1)$-dimensional hypersurface, together with the induced metric, defines AdS_{D+1}, $(D+1)$-dimensional anti-de Sitter space. This constraint can be solved for v. Doing this and introducing the coordinates $x^\mu = \frac{Ry^\mu}{u}$ we find for the induced metric

$$ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu - du dv = \frac{R^2}{u^2} du^2 + \frac{u^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu . \tag{75}$$

$u \to \infty$ corresponds to the boundary of AdS, which is just compactified Minkowski space as discussed before. This can be seen by looking at the constraint equation (74) (after dividing by u^2 the r.h.s. vanishes for $u \to \infty$ whereas the l.h.s. stays finite) or by looking at the AdS metric in the form (75). If we introduce the coordinate $\rho = R^2/u$ the metric becomes

$$ds^2 = \frac{R^2}{\rho^2} (d\rho^2 + \eta_{\mu\nu} dx^\mu dx^\nu) \tag{76}$$

which coincides with the first part in (66). The boundary is now at $\rho = 0$. Simultaneously rescaling $u \to u/\lambda$ and $x^\mu \to \lambda x^\mu$ leaves the AdS metric invariant but induces a Weyl transformation of the metric on the boudary: $\eta_{\mu\nu} \to \lambda^2 \eta_{\mu\nu}$.

After having provided some background on AdS space and the conformal group, we will now return to branes, gravity and gauge theory. Take a single D3-brane. The fields living on its world-volume arise from the excitations of open strings ending on the brane. At low energies, lower than the string scale

$1/\sqrt{\alpha'}$, only the massless string states can be excited and their dynamics is governed by a low energy effective action on the world-volume. The massless open string states are the NS gauge field A_M and its fermionic superpartner, the gaugino, a Majorana-Weyl spinor in ten dimensions. Together these fields form the $\mathcal{N}=1$, $d=10$ Yang-Mills supermultiplet. The brane breaks the ten-dimensional Lorentz invariance $SO(1,9)\to SO(1,3)\times SO(6)$ to Lorentz transformations along the brane and rotations in the transverse space. The gauge field $\mathbf{10}$ decomposes as $\mathbf{10}=(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{6})$: $A_M=(A_\mu,\phi^i)$. The six scalars describe the fluctuations of the brane in the directions transverse to it. (They are the Goldstone bosons associated to the spontaneously broken translation symmetry.) The gaugino decomposes into four Weyl spinors which transform as $\mathbf{4}$ of $SO(6)$ and their complex conjugates. (They are the Goldstinos of the sixteen, in the presence of the brane, spontaneously broken supercharges.) Altogether we get one $\mathcal{N}=4$ $U(1)$ vector multiplet on the four-dimensional world-volume of the D3-brane. The generalization from one to N coincident D3 branes is straightforward leading to the gauge group $U(N)$ (c.f. the discussion in lecture 2). All fields are in the same supermultiplet and hence they all transform in the adjoint representation. The meaning of the $U(1)\subset U(N)$ factor is as above. The scalar in the $U(1)$ multiplet corresponds to the center-of-mass motion of the branes and those in the adjoint of $SU(N)$ to their relative motion.

The action of $\mathcal{N}=4$ SYM theory is highly restricted by the large amount of supersymmetry. In particular there is only one coupling constant. Another important consequence of the large amount of supersymmetry is that bosonic and fermionic contributions to divergences in Feynman diagram calculations cancel and the quantization procedure does not require introducing a scale into the theory. This in particular means that the beta function vanishes. This means that as a quantum theory, $\mathcal{N}=4$ SYM is conformally invariant. In other words, the conformal symmetry exhibited by the classical theory is not broken in the process of quantization. In fact, the theory is invariant under local $U(N)$ gauge transformation and under global super-conformal transformations which generate the supergroup[34]

$$SU(2,2|4)\supset SU(2,2)\times SU(4)_{\mathcal{R}}\simeq SO(4,2)\times SO(6)_{\mathcal{R}}\,. \tag{77}$$

On the r.h.s. we have written the bosonic subgroup. The $SO(4,2)$ factor is the conformal group in $d=4$ while $SO(6)_{\mathcal{R}}$ is called $\mathcal{R}$-symmetry group. It can be understood from the brane picture as the rotation group of the transverse space and we have seen how the various fields in the SYM multiplet transform under it. The fermionic generators are the sixteen supercharges Q of the $\mathcal{N}=4$ supersymmetry algebra plus sixteen *special supersymmetries* S which arise in the commutator between Q and the special conformal transformations K. The generators of the $\mathcal{R}$ symmetry appear in the $\{Q,S\}$ anti-commutators.

The dynamics of the massless fields can again be described by a low-energy effective action. For

[34] The corresponding superalgebra is discussed in the lectures by J. Zanelli.

the gauge fields this is the *Born-Infeld action* [50]:

$$S_{\text{B.I.}} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \text{Tr} \int_{\mathcal{C}_4} d^4\xi \sqrt{-\det\left(\hat{g}_{\alpha\beta} + e^{-\phi/2} 2\pi\alpha' F_{\alpha\beta}\right)} \,. \tag{78}$$

The integral is over the brane world-volume, Tr the trace over group (Chan-Paton) indices[35] and $\hat{g}_{\alpha\beta}$ the induced metric on the world-volume. We have omitted fermions, transverse scalars and Wess-Zumino couplings, as they are not relevant for our discussion. If we extract the $\mathcal{O}(F^2)$ term and compare with the usual gauge kinetic term $-\frac{1}{4g_{\text{YM}}^2} F^a_{\mu\nu} F^{a\mu\nu}$ we are led to the identification

$$g_{\text{YM}}^2 = 2\pi g_s \,. \tag{79}$$

Of course we also have perturbative closed string excitations in the bulk and the closed string modes interact with the open string modes which are localized on the brane. The complete effective action for all massless modes has the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \,, \tag{80}$$

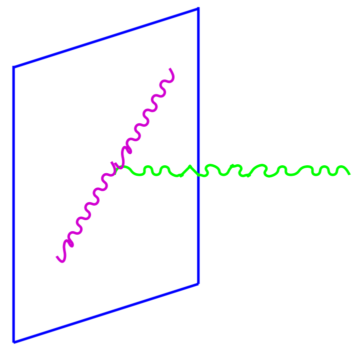
where S_{bulk} contains only closed string modes, S_{brane} only open string modes and S_{int} interaction terms between them. The coupling constant is proportional to κ, c.f. below. In the *decoupling limit* $\alpha' \to 0$ all higher derivative corrections as well as the interactions between closed and open strings can thus be neglected and we are left with pure four-dimensional $\mathcal{N} = 4$ SYM on the world-volume of the brane and free type IIB supergravity in the bulk (i.e. free gravitons and their SUSY partners) with no coupling between these two theories.

In the previous lecture we have collected evidence that the D-branes of string theory and the p-brane solutions of supergravity are complementary descriptions of one and the same object. We have just seen that the $\alpha' \to 0$ limit leads in the D-brane picture to two decoupled systems: SYM theory on the branes and free supergravity in the bulk.

The next step will be to find the correct decoupling limit in the SUGRA picture and to compare to the above. The following analysis, first performed in [51], will give an important clue.

If a dilaton hits the D-brane, it can be absorbed, thus exciting the D-brane. The quantum excitations of the D-brane are the open string modes. Indeed, as we see from the Born-Infeld action, the dilaton couples to the gauge bosons.

[35]This is unambiguous to $\mathcal{O}(F^2)$. At higher orders one has to be more specific about the prescription how to perform the trace; see [50].



At lowest order, this is the cubic coupling (c.f. (78)) $\frac{1}{g_s}\phi F^2$. To find the strength of this coupling one should normalize the fields such that they have canonical kinetic energies. This means that we have to rescale the gauge bosons by $\sqrt{g_s}$ and the dilaton by κ. This leads to a coupling constant $\propto \kappa$ (which does vanish in the decoupling limit). The (tree level) cross-section for a dilaton is then $\sigma \propto \kappa^2 E^3 N^2$ as we shall now explain. The κ^2 is clear as the cross-section involves the square of the amplitude which is $\propto \kappa$; N^2 because there are that many gluons into which the dilaton can decay. The cross-section for the scattering of a point-particle from a three-dimensional object in nine space dimensions has dimension $(length)^5$; the only dimensionful quantity to fix the dimension is E, and the factor E^3 indeed arises from the kinematics of the scattering process. A careful calculation gives for the absorption cross-section [51] of a dilaton incident at right angle (i.e. its momentum has no component parallel to the D3-brane)

$$\sigma_{\rm D3} = 2\pi^6 g_s^2 \alpha'^4 E^3 N^2 = \frac{\pi^4}{8} E^3 R^8 \,, \qquad R^4 = 4\pi g_s N \alpha'^2 \,. \tag{81}$$

On the supergravity side one solves the wave equation for the dilaton in the s-channel in the brane geometry. This exhibits at low energies $E \ll 1/R$ a potential barrier separating the two asymptotic regions $r \ll R$ and $r \gg R$, where r is the distance from the brane. One then obtains the absorption cross section from the tunneling probability through the barrier[52]. This calculation was also performed in [51] with the result

$$\sigma_{\rm D3} = \sigma_{\rm SUGRA} \,. \tag{82}$$

This also works with other SUGRA particles, e.g. the graviton and anti-symmetric three-form tensor; their absorption cross sections agree as well.

Eq.(82) is a very interesting result: in the D-brane picture a particle incident from infinity produces excitations of the gauge theory on the brane; in the SUGRA description of the brane a particle tunnels from the region $r \gg R$ to the region $r \ll R$ and produces an excitation there. The two à priori unrelated processes occur at exactly the same rate. One is tempted to identify the $\mathcal{N}=4$ SYM theory with gauge group $U(N)$ with the excitations in the near horizon region, $r \ll R$, of the brane geometry, which we already know is $AdS_5 \times S^5$. This gets further support from the following identification of two types of low-energy excitations, as measured by an observer at infinity (for this observer the coordinate t appearing in (62) is the time coordinate as $g_{tt}(r=\infty)=-1$). Due to the energy dependence of the cross section, $\sigma \propto E^3$, low-energy SUGRA modes in the region $r \gg R$ decouple from the near-horizon region. At $r \gg R$ we thus have a free[36] SUGRA theory. On the other hand, the energy of an excitation in the near horizon region appears redshifted for an observer at infinity, $E_\infty = [g_{tt}(r)/g_{tt}(\infty)]^{1/2}E_r = \frac{r}{R}E_r$. They cannot penetrate the energy barrier which separates the two asymptotic regions.

So, in the D3 and in the SUGRA picture we get two decoupled systems. In both cases the system in the bulk is free type IIB SUGRA. But then, if the D-brane and the SUGRA brane describe the same object, we should identify the two other systems: $\mathcal{N}=4$ SYM theory with gauge group $U(N)$ and type IIB string theory on $AdS_5 \times S^5$.

Before pursuing this further, we need to clarify two points. Firstly, we still need to be more specific about the precise form of the near horizon limit, which zooms into the region of the three-brane SUGRA solution which we want to identify with the gauge theory of the D3 picture. This limit should involve $\alpha' \to 0$, as this was the decoupling limit for the D3-brane, and is defined as follows:

$$\alpha' \to 0, \qquad r \to 0 \qquad \text{such that} \qquad U \equiv \frac{r}{\alpha'} \ \text{ fixed}\,. \tag{83}$$

In this limit α' scales out of the metric which becomes

$$ds^2/\alpha' = \frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\mathbf{x}^2) + \frac{\sqrt{4\pi g_s N}}{U^2}dU^2 + \sqrt{4\pi g_s N}\,d\Omega_5^2\,. \tag{84}$$

The limit is taken such that for the observer at infinity the string excitations in the horizon region, which have energies $E_0 \sim 1/\sqrt{\alpha'}$ and which are redshifted to $E_\infty \sim \frac{r}{\sqrt{\alpha'}}E_0 \sim U$ stay finite. This observer sees two decoupled systems: free SUGRA in the asymptotic region and type IIB string theory compactified on $AdS_5 \times S^5$.

The second point we need to clarify is the region of validity of the two calculations of the absorption cross section. In both pictures we have assumed that $E \ll /1\sqrt{\alpha'}$; otherwise massive string modes can be excited and their effect has to be taken into account. On the supergravity side we also have to require that (i) the typical length scale R of the geometry is large compared to the string scale, i.e. $R \gg \sqrt{\alpha'}$; otherwise we have to take higher derivative corrections $\sim (\alpha')^n \mathcal{R}^{n+1}$ to the supergravity action into account, which we did not. (ii) We also need $g_s \ll 1$ since we have neglected string loop effects. With the help of $R^4 \sim g_s N \alpha'^2$ and $g^2_{\rm YM} \sim g_s$ we can translate these restrictions to the following conditions on the gauge theory parameters:

$$\lambda \equiv g^2_{\rm YM}N = 2\pi N g_s = \frac{R^4}{2\alpha'^2} \gg 1 \qquad \text{and, since } g_s \ll 1, \qquad N \to \infty\,. \tag{85}$$

This specifies a large N YM theory at strong *'t Hooft coupling* λ. λ is the effective coupling constant and loop counting parameter in the large N limit of YM theories.

[36] The gravitational interaction is negligible at low energies. The dimensionless coupling constant is κE^4, where E is the typical energy of the interaction. For $E \ll 1/\sqrt{\alpha'}$ and $\kappa \propto \alpha'^2$ this is $\ll 1$.

We should thus identify the excitations in the near-horizon regions of the three-brane geometry, which, as we have seen above, is $AdS_5 \times S^5$, with the excitations of $\mathcal{N} = 4$ $U(N)$ SYM theory at large 't Hooft coupling and in the limit $N \to \infty$.[37] Based on the analysis presented above, this conclusion was first drawn by Maldacena in his famous paper [7] and is called Maldacena conjecture, or, since it involves AdS space on the one hand and a conformal field theory ($\mathcal{N} = 4$ SYM) on the other, the AdS/CFT correspondence.

In the weak form the conjecture states that *$\mathcal{N} = 4$ SYM with $U(N)$ gauge group at large 't Hooft coupling and in the limit $N \to \infty$ is equivalent (dual) to type IIB supergravity compactified on $AdS_5 \times S^5$.*

Note that even in this weak form the conjecture has far reaching implications. It relates a *classical* weakly coupled supergravity theory with a strongly coupled *quantum* field theory. This is a duality pair of the type we discussed in the beginning. The perturbative regimes of the two theories, $g^2_{\rm YM} N \sim g_s N \sim R^4/\alpha'^2 \ll 1$ for the gauge theory and $R^4/\alpha'^2 \gg 1$ for the supergravity theory, do not overlap. The good news is that such a duality is very useful for exploring the strongly coupled gauge theory; this is done by performing computations in a classical gravity theory. The bad news is that it is extremely difficult to prove such a duality conjecture.

Further support for this conjecture comes from comparing the symmetries. The isometries of $AdS_5 \times S^5$ are $SO(4,2) \times SO(6)$. But these are precisely the bosonic (non-gauge) symmetries of $\mathcal{N} = 4$ SYM theory (c.f. (77)). This discussion can be extended to the full supergroup $SU(2,2|4)$. This is the maximally extended supersymmetry algebra on AdS_5. It is realized as global symmetry of gauged $\mathcal{N} = 8$ supergravity on AdS_5 which can be obtained as Kaluza-Klein reduction of type IIB SUGRA on S^5. What is gauged is the isometry group of S^5. But $SU(2,2|4)$ is also the maximally extended superconformal symmetry in four-dimensional Minkowski space, i.e. the invariance group of $\mathcal{N} = 4$ SYM. Furthermore, the boundary of AdS_5 is four-dimensional Minkowski space on which the isometries of $AdS_5 \times S^5$ act as conformal and $\mathcal{R}$-symmetry transformations. This leads to the statement that the field theory which is dual to the string theory 'lives' on the boundary of AdS_5. Recall the observation made above that if we simultaneously rescale $U \to \lambda U$, $(t, \mathbf{x}) \to \lambda^{-1}(t, \mathbf{x})$ the metric does not change. This leads to an interpretation of U as the energy scale in the field theory: large U corresponds to the UV region and small U to the IR region (the boundary is at $U = \infty$).

The conjectured duality has several very remarkable features. First, it is a duality between a gravity theory and a field theory. In addition, these theories live in different numbers of dimensions and have completely different degrees of freedom: gravity *vs.* gauge degrees of freedom. There is the notion of the master field for large N QCD (see e.g. [11, 12]). One can show that fluctuations of gauge invariant observables vanish in the $N \to \infty$ limit. This is then analoguous to the classical limit $\hbar \to 0$ in which the functional integral is dominated by classical paths. In the same way, there should be a master field such that all Green functions are given by their value at the master field. What the

[37] Earlier we had mentioned that in the presence of N D-branes the effective coupling constant is $N g_s$.

Maldacena conjecture suggests is that this master field for $\mathcal{N} = 4$ SYM is in fact a gravity theory in ten dimensions. Also, the QCD string of these theories is simply the fundamental type IIB string which lives, however, not in four-dimensional space-time, but in ten-dimensional $AdS_5 \times S^5$. This is most convincingly demonstrated by the SUGRA computation of the gauge theory Wilson loop [53]: a static well separated $q\bar{q}$ pair must be viewed as the endpoints of an open type IIB string at the boundary of AdS_5. In order to minimize its length (and hence it energy) the string follows a geodesic. The geodesic does not lie in the boundary, which it would if the space were flat, but extends deep into the AdS space. The $q\bar{q}$ potential is then given by the (regularized) length of this geodesic.

One might wonder whether the duality only holds for the parameter region specified in (85), which corresponds to neglecting all $\mathcal{O}(\alpha'/R^2)$ and all string loop effects or whether it can be extended. Correlation functions of $AdS_5 \times S^5$ type IIB string theory will have a double expansion in powers of g_s and $\alpha'/R^2 \sim \lambda^{-1/2}$. In the large N limit we can write this as an expansion in powers of $1/N \sim g_s(\alpha'/R^2)^2$ where each coefficient has an expansion in powers of $\lambda^{-1/2}$. Clearly, a term at some power in the g_s expansion has the same power in the $1/N$ expansion. Since the type II string theory includes only closed oriented strings, we find the same general structure as in the field theory where each correlation function has an expansion in powers of $1/N^2$, each coefficient being some function of λ. The functions of λ have different expansion from the point of view of string theory and of field theory. A stronger version of the AdS/CFT correspondence states that both expansions give rise to the same function of λ at each power of $1/N^2$. The fact that one expansion is in powers of λ whereas the other is in powers of $\lambda^{-1/2}$ reflects the fact that the AdS/CFT duality is a strong/weak coupling duality.

In the strongest version of the conjecture the two theories are considered as exactly identical for all values of N and g_s. Here the corrections distort the space-time which is only required to be asymptotically $AdS_5 \times S^5$. The gauge theory then effectively sums over all such space-times.

As we have remarked in the first lecture, after identifying a pair of theories of which we have indications that they are dual to each other, we still need to find the map between them. This amounts to giving an explicit prescription of how to compute gauge theory correlation functions in the dual supergravity theory. This was done in [8, 9]. We will not review the results of these papers nor will we present any of the many applications. They can be found in the reviews on the AdS/CFT correspondence. There the interested reader may also find extensions of Maldacena's conjecture to other theories – in dimensions different from four and to theories with other gauge groups, less supersymmetry, not conformally invariant theories, theories at finite temperature (the latter involve the non-extremal brane solutions[38]) and theories on non-commutative Minkowski space (the latter require, in addition to the metric and (R,R) four-form potential, also a non-trivial (NS,NS)

[38]If one computes the Bekenstein-Hawking entropy and compares it to the entropy of $\mathcal{N} = 4$ SYM at temperature $T_{\rm BH}$, one finds that the field theory value is bigger by a factor 4/3. The fact that there is a discrepancy does not come as a surprise, since the SUGRA calculation is valid for $g_sN \to \infty$ whereas the field theory calculation assumed weak coupling, i.e. $g_sN \to 0$. One should thus view the SUGRA result as a prediction for the strongly coupled field theory.

two-form).

Maldacena's original paper has ignited a storm of activity.[39] Hopes were high that extensions of his conjecture would lead to new insight into realistic, i.e. non-supersymmetric QCD. However, all attempts to find its supergravity dual (at large N) have been futile. More optimistically, the AdS/CFT correspondence has pointed in a very interesting direction, namely the possible connection between gauge theories and gravity (string) theories. In fact, this might also provide insight into string theory in non-trivial backgrounds. The reason why the AdS/CFT correspondence has so far been mainly checked in its weak form is our inability to quantize string theory in an AdS background. However, besides flat Minkowski space-time and $AdS_5 \times S^5$ there is one other maximally supersymmetric background, which is the so-called pp-wave [56, 57]. In this background, which can be obtained from $AdS_5 \times S^5$ by a limiting process, called Penrose contraction, the string can be quantized exactly (in the Green-Schwarz light cone formulation) [58]. Recently a correspondence between the string theory in the pp-wave background and supersymmetric Yang-Mills has been conjectured [59]. This is presently being explored.

Acknowledgement: S.T. would like to take the opportunity to thank A. Font for the invitation to lecture in Caracas and the organizers and in particular H. Ocampo of the Summer school in Villa de Leyva for their invitation. In the preparation of these notes S.T. has has benefitted from many discussions with A. Font. The comments of G. Arutyunov and S. Kuzenko were also helpful and appreciated.

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