SLAC-PUB-338
August 1967

CURRENT ALGEBRA AT SMALL DISTANCES*

J. Bjorken

Stanford Linear Accelerator Center
Stanford University, Stanford, California

Lecture given at International School of Physics
"Enrico Fermi," XLI Course, Varenna, Italy

Selected Topics in Particle Physics
July 17 - 29, 1967

*Work supported by the U.S. Atomic Energy Commission

I. INTRODUCTION

The major successes of current algebra (Adler-Weisberger[1], Cabibbo-Radicati[2], Fubini sum rules[3], Callan-Treiman relations[4] for leptonic K-decays, etc.) are almost exclusively in the nature of low energy theorems. The basis for these low energy theorems lies in charge conservation or approximate "axial charge" conservation (PCAC). For example, the "soft pion" theorem

$$\langle A\, \pi^{\alpha} \left| J(o) \right| B \rangle \xrightarrow[q_{\pi} \to 0]{} a \langle A \left| \left[F_5^{\alpha},\, J(o) \right] \right| B \rangle \tag{1.1}$$

depends only upon the commutator of axial charge F_5^{α} with the local operator $J(o)$. The low-energy theorems for Compton scattering and the Kroll-Ruderman theorem for photoproduction are of a similar nature.

However, in quantum electrodynamics one has local current densities which are measurable and which commute at spacelike separations. It is a reasonable and generally accepted hypothesis that the hadronic currents which couple to leptons via weak and electro-magnetic interactions are likewise local; that is, there exists (in an experimentally meaningful sense) operators $j_{\mu}^{i}(x)$ which when sandwich between hadron states and coupled to a local lepton current gives the lowest order S-matrix element for a weak or electromagnetic process.

In these lectures we will discuss what happens when one tries to apply the same current algebra techniques for these local densities as one does for the charges. In doing so we run into two kinds of difficulties. The first is that there are far fewer direct applications. Essentially the only applications which are sensitive to distances $\ll 10^{-13}$cm are lepton scattering at high momentum transfers. These are elastic and inelastic electron- and muon-nucleon scattering and also the neutrino processes. The second difficulty is theoretical; we really do not have as much confidence in the commutation relations at the local level and, even worse, hardly any soundly based qualitative idea of the nature of the dynamics of lepton inelastic scattering at high momentum transfer. Nevertheless, that is what these lectures are about, and although there will be some speculations, they can be shot down by experiment. I also think that the problems raised here are quite fundamental, dealing, in what seems to be a direct way, with the question of whether there are any "elementary constituents" within the nucleon. Use of the leptons as a probe is a unique and possibly powerful way of attacking this problem.

We will first discuss the kinematics of lepton scattering processes. Once this chore is finished, we shall derive a sum rule for neutrino processes, and from it, a corollary for electron-scattering processes. We shall find these relations so perspicuous that, by an appeal to history, an interpretation in terms of "elementary constituents" of the nucleon is suggested. With the aid of this interpretation more predictions for inelastic lepton scattering can be made; we shall compare these with what little data exists.

Finally, we will discuss the radiative corrections to the Fermi part of β-decay and argue on the basis of the local algebra that these corrections diverge logarithmically, just like for point particles. The picture of the proton as composed of "elementary constituents" again is in accord with this result, which in fact is somewhat connected to the sum rules. There is one exception to the conclusion. If the constituents coupled by the weak process have mean charge of $-1/2$ and spin $1/2$ (such as μ and ν_μ in μ-decay), the divergent logarithm vanishes. In any case, we will compare the numbers with experiment, and find no disaster even if the "divergence" is present.

II. KINEMATICS

First consider the scattering of a lepton of momentum p from a nucleon of momentum P to final hadron state of momentum P_n. The scattering matrix T_{fi} looks like (see Fig. 1)

$$T_{fi} = \frac{-i}{(2\pi)^3} \frac{1}{\sqrt{4EE'}} \times \begin{cases} \frac{G}{\sqrt{2}} \bar{u}(p') \gamma_\mu (1-\gamma_5) u(p) \langle P_n | J^{\mu\dagger}(o) | P \rangle & \text{weak process} \\ \\ -\frac{4\pi\alpha}{q^2} \bar{u}(p') \gamma_\mu u(p) \langle P_n | j^\mu(o) | P \rangle & \text{e. m. process} \end{cases} \tag{2.1}$$

We neglect lepton mass and normalize lepton spinors such that $u^\dagger(p)\, u(p) = 2E$. p' is the final lepton momentum and $q = p - p'$ is the momentum transfer.

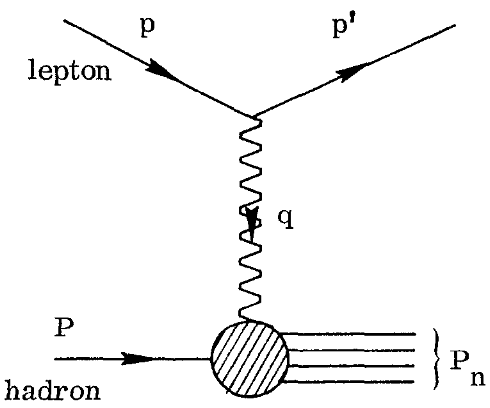

Fig. 1 -- Kinematics of lepton-nucleon scattering.

For our applications all hadron states of given momentum will be summed and initial spins averaged. A routine calculation then gives us the differential cross section, in angle and energy of the outgoing lepton but summed over hadron states of momentum $P_n = p + P - p'$. For electromagnetic scattering, we get

$$\frac{\pi}{EE'}\frac{d\sigma}{d\Omega\, dE'} = \frac{d\sigma}{dq^2 d\nu} = \left(\frac{E'}{E}\right)\frac{4\pi\alpha^2}{q^4}\left[\cos^2\frac{\theta}{2}\,\sigma_1\left(q^2,\nu\right) + \sin^2\frac{\theta}{2}\,\sigma_2\left(q^2,\nu\right)\right] \tag{2.2}$$

where we choose q^2 and $\nu = E - E' =$ virtual photon laboratory energy as the important variables. σ_1 and σ_2 describe the contents of the hadronic "black box," defined as follows $\left(M_p = 1\right)$

$$j_{\mu\nu}\left(q,P\right) = \frac{P_o}{M_p}\overline{\sum_n}\langle P|j_\mu(o)|n\rangle\langle n|j_\nu(o)|P\rangle\left(2\pi\right)^3\delta^4\left(P_n - P - q\right)$$

(spin averaged) (2.3)

$$= \left[P_\mu P_\nu\,\sigma_1\left(q^2,\nu\right) - \frac{g_{\mu\nu}}{2}\,\sigma_2\left(q^2,\nu\right) + \dots\right]$$

The terms left out are proportional to $P_\mu q_\nu$, $P_\nu q_\mu$, or $q_\mu q_\nu$ and do not contribute, because $q_\mu j^\mu$ (leptons) $= 0$. Also, elastic scattering is included in $j_{\mu\nu}$ by inclusion of pieces in σ_1 and σ_2 proportional to $\delta\left(q^2 + 2M\nu\right)$.

For weak processes we get

$$\frac{\pi}{EE'}\frac{d\sigma^{\bar\nu p}}{d\Omega\, dE'} = \frac{d\sigma^{\bar\nu p}}{dq^2 d\nu} = \left(\frac{E'}{E}\right)\frac{G^2}{2\pi}\left[\cos^2\frac{\theta}{2}\,\sigma_1^{\bar\nu p} + \sin^2\frac{\theta}{2}\,\sigma_2^{\bar\nu p} + \left(\frac{E+E'}{M}\right)\sin^2\frac{\theta}{2}\,\sigma_3^{\bar\nu p}\right] \tag{2.4}$$

and

$$J_{\mu\nu} = \frac{P_o}{M_p} \overline{\sum_n} \langle P | J_\mu(o) | n \rangle \langle n | J_\nu^\dagger(o) | P \rangle \left(2\pi\right)^3 \delta^4 \left(P_n - P - q\right)$$

$$= P_\mu P_\nu \sigma_1^{\bar{\nu}p}\left(q^2, \nu\right) - \frac{g_{\mu\nu}}{2} \sigma_2^{\bar{\nu}p}\left(q^2, \nu\right) - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} P^\alpha q^\beta \sigma_3^{\bar{\nu}p}\left(q^2, \nu\right) + \dots \tag{2.5}$$

The only difference, aside from the difference in G and e^2/q^2 is the extra form factor, arising from the presence of parity-violation in the weak interactions.

As $q^2 \to 0$, these cross sections become proportional to photoabsorption cross sections and pion scattering cross sections. The first is fairly clear, since the matrix elements $\langle P | j_\mu(o) | n \rangle$ as $q^2 \to 0$ are those needed for photoabsorption. In the neutrino case, as $q^2 \to 0$ the lepton current becomes a multiple of q_μ. That is, upon approximating $m_\ell \approx 0$, $\mathrm{Tr}\, \not{p}\, \gamma\, \not{p}'\, \gamma_\nu \approx 4\left(p_\mu p'_\nu + p'_\mu p_\nu\right) = \frac{8EE'}{\nu^2} q_\mu q_\nu$ because p_μ, p'_μ and q_μ become proportional to each other for forward scattering. Therefore, only $q_\mu q_\nu J^{\mu\nu}$ contributes to forward neutrino scattering. The divergence of the current is dominated, for $\Delta S = 0$ processes, by the divergence of the axial vector current which by PCAC is related to the pion field. Thus the neutrino scattering is related to pion-scattering, slightly off-the-mass-shell. In our normalization, PCAC may be written

$$i \langle p | q_\mu J^\mu(o) | n \rangle \approx \frac{\sqrt{2}\, g_A M \mu^2}{g_r} \langle p | \phi_\pi(o) | n \rangle = \frac{\sqrt{2}\, g_A M}{g_r} \langle p | j_\pi(o) | n \rangle$$

and putting this together with the lepton current gives us after some algebra

$$\frac{d\sigma^{(\bar{\nu} p)}}{dq^2\, d\nu} \begin{pmatrix} \Delta S = 0 \\ q^2 = 0 \end{pmatrix} \cong \left(\frac{E'}{E}\right) \frac{G^2}{2\pi} \left[\left(1 + g_A^2\right) \delta\left(\nu + \frac{q^2}{2M}\right) + \frac{2 g_A^2 M^2}{\pi g_r^2} \frac{\sqrt{\nu^2 - \mu^2}\, \sigma_{\pi p}\left(\mu^2, \nu\right)}{\nu^2}\right] \cos^2\theta_c$$

The $\left(1 + g_A^2\right)$ comes from the "elastic" scattering contribution, to which the vector current also contributes as $q^2 \to 0$.

III. SUM RULES

We turn now to the sum rules, which can be derived in various ways. We start with the "infinite-momentum" method[5] which is now quite easy, given the machinery at our disposal. Consider at fixed q^2 and $\nu = E - E'$, what happens as we increase the center-of-mass energy of lepton and hadron. We see that the scattering angle $\theta \longrightarrow 0$

$$\sin^2 \frac{\theta}{2} = 0\left(\frac{q^2}{E^2}\right) \longrightarrow 0$$

and

$$\frac{d\sigma^{ep}}{dq^2\, d\nu} \xrightarrow[E\to\infty]{} \frac{4\pi\alpha^2}{q^4}\sigma_1\left(q^2,\nu\right) \qquad \frac{d\sigma^{\bar{\nu}p}}{dq^2\, d\nu} \xrightarrow[E\to\infty]{} \frac{G^2}{2\pi}\sigma_1^{\bar{\nu}p}\left(q^2,\nu\right) \tag{3.1}$$

These equations are covariant, and we can consider them in the center-of-mass frame. In this frame, $P_\mu \longrightarrow \infty$ as well as p_μ and furthermore,

$$j_{\mu\nu} \longrightarrow P_\mu P_\nu\, \sigma_1\left(q^2,\nu\right) = \frac{P_\mu P_\nu}{P_o^2}\, j_{oo} \tag{3.2}$$

That is, the time-components of the currents alone determine the scattering. Also the momentum transfer becomes purely transverse!

$$\left(q + P\right)^2 = P_n^2 \quad \text{or} \quad q_o - q_z \approx \frac{M_n^2 - q^2 - M^2}{2P_o} \longrightarrow 0$$

and also from $\left(p - q\right)^2 \approx 0$ we get

$$q_o + q_z \approx \frac{q^2}{2E} \longrightarrow 0$$

which implies both $q_o \to 0$ and $q_z \to 0$ in this limit. This suggests that the electromagnetic scattering is "Coulombic" in character. A simple-minded picture of scattering would have a Lorentz-contracted disk of charge (the nucleon) scattering the lepton instantaneously in time via the Coulomb interaction with the charge density. We shall come back to this picture later in more detail.

We now go back to the neutrino process and try to utilize the commutator of charge densities proposed by Gell-Mann[6] and discussed by Adler in his lectures:

$$\left[J_o^+ \left(\underset{\sim}{x}, o\right), J_o^- (o)\right] = 2 J_o^3 (o) \, \delta^{(3)}\left(\underset{\sim}{x}\right) \tag{3.3}$$

sandwiched between proton states in the $P_z \to \infty$ limit).

$$\sum_n \left[\langle P | J_o^+ (o) | n \rangle \langle n | J_o^- (o) | P \rangle (2\pi)^3 \delta^{(3)} \left(\underset{\sim}{P}_n - \underset{\sim}{q} - \underset{\sim}{P}\right) - \begin{pmatrix} + \longleftrightarrow - \\ \underset{\sim}{q} \longleftrightarrow -\underset{\sim}{q} \end{pmatrix} \right] \tag{3.4}$$

$$= 2 \langle P | J_o^3 (o) | P \rangle = 2 \left(\cos^2 \theta_c + 2 \sin^2 \theta_c\right)$$

The right-hand side may be evaluated just by evaluating $\mathcal{F}_o^{\,3}(o)$ and $\mathcal{F}_o^{\,8}(o)$ in terms of Q and Y for the proton. A simple way to get the factors is to use the quark model directly, since the result is model-independent. In that model

$$J_o^+(x) = p^\dagger(x) \left(1 - \gamma_5\right) \left[n'(x) \cos \theta_c + \lambda'(x) \sin \theta_c\right] \equiv p'^\dagger \left(1 - \gamma_5\right) \tilde{n}$$

and

$$\begin{aligned} \langle P | \left[J_o^+ \left(\underset{\sim}{x}\right), J_o^- (o)\right] | P \rangle &= \delta^{(3)}\left(\underset{\sim}{x}\right) \langle P | 2 p'^\dagger \left(1 - \gamma_5\right) p' - 2 \tilde{n}^\dagger \left(1 - \gamma_5\right) \tilde{n} | P \rangle \\ &= 2 \, \delta^{(3)}\left(\underset{\sim}{x}\right) \langle P | p'^\dagger p' - n'^\dagger n' \cos^2 \theta_c - \lambda'^\dagger \lambda' \sin^2 \theta_c | P \rangle \\ &= 2 \, \delta^{(3)}\left(\underset{\sim}{x}\right) \left[2 - \cos^2 \theta_c\right] \end{aligned} \tag{3.5}$$

where the $\Delta S = \pm 1$ terms in J^3 have been dropped in the next to last line, and the last line follows from counting the quarks in the proton. Returning to the left-hand side, we see that there is a great similarity between the structure of the commutator and the structure of J_{oo}. In fact, for $\underset{\sim}{q}$ fixed we identify the commutator with

$$\int \frac{dq_o}{P_o} \left[J_{oo}^{\bar{\nu} p} - J_{oo}^{\nu p} \right] = P_o \int dq_o \left[\sigma_1^{\bar{\nu} p} \left(q^2, \nu\right) - \sigma_1^{\nu p} \left(q^2, \nu\right) \right]$$
$$= 2 \left[\cos^2 \theta_c + 2 \sin^2 \theta_c \right] \qquad (3.6)$$

Now in the limit $P_o \longrightarrow \infty$ from Eq. (3.2)

$$q_o \longrightarrow \frac{M_n^2 - M^2}{2P_o} = \frac{q^2 + 2M\nu}{2P_o} \longrightarrow 0$$

and

$$P_o \, dq_o \longrightarrow M d\nu = d\,\nu \quad \left(M = 1\right)$$

We thus arrive at the sum rule[7]

$$\int_0^\infty d\nu \left[\sigma_1^{\bar{\nu} p} \left(q^2, \nu\right) - \sigma_1^{\nu p} \left(q^2, \nu\right) \right] = 2 \left[\cos^2 \theta_c + 2 \sin^2 \theta_c \right] \qquad (3.7)$$

The question of interchange of limits is a little dirty and will be discussed by Adler. For a more critical discussion with regard to rigor, we shall choose in Section VI to discuss the process in terms of the Fubini, or dispersion method[3] of derivation.

Before doing that, however, there is some physics to do. Going to $q^2 = 0$ and extracting the $\Delta S = 0$ part by setting $\cos\theta_c = 1$ we see that

$$\lim_{E\to\infty}\int \frac{d\left(\sigma^{\bar{\nu}p} - \sigma^{\nu p}\right)}{dq^2\,d\nu}\Bigg|_{q^2=0} d\nu \approx \frac{G^2}{2\pi}\left[\left(1 + g_A^2\right) + \frac{2g_A^2 M^2}{\pi g_r^2}\int_0^\infty \frac{d\nu}{\nu^2}\sqrt{\nu^2-\mu^2}\left\{\sigma_{\pi^- p}\left(\mu^2,\nu\right) - \sigma_{\pi^+ p}\left(\mu^2,\nu\right)\right\}\right] \tag{3.8}$$

$$\simeq \frac{G^2}{2\pi}\int_0^\infty d\nu\left(\sigma_1^{\bar{\nu}p} - \sigma_1^{\nu p}\right) = \frac{G^2}{\pi}$$

which is the Adler-Weisberger sum rule.[1] Notice that for the forward neutrino reaction the sum rule is exact provided we do not replace σ_1 by $\sigma^{\pi p}$ using PCAC. Tests of the $q^2 \neq 0$ sum rules will be difficult. A related test, however, is provided by electromagnetic-scattering of leptons in the form of an inequality. We start with

$$\int d\nu\left[\sigma_{1p}\left(q^2,\nu\right) + \sigma_{1n}\left(q^2,\nu\right)\right]$$

$$= \int d\nu\left[\sigma_{1v}\left(q^2,\nu\right) + \sigma_{1s}\left(q^2,\nu\right)\right]$$

(σ_{1v} is isovector cross section. By averaging over p and n we cancel any interference terms)

$$\geq \int d\nu\ \sigma_{1v}\left(q^2,\nu\right)$$

$$\geq \int d\nu\ \sigma_{1v}\left(q^2,\nu\ ;\ I = \tfrac{1}{2}\ \text{final states only}\right)$$

$$\geq \int d\nu\left[\sigma_{1v}\left(q^2,\nu\right)_{I=\frac{1}{2}} - \sigma_{1v}\left(q^2,\nu\right)_{I=\frac{3}{2}}\right] \tag{3.9}$$

$$= \frac{1}{2}\int d\nu\left[\sigma_{1v}^{\bar{\nu}p}\left(q^2,\nu\right) - \sigma_{1v}^{\nu p}\left(q^2,\nu\right)\right]$$

$$= \frac{1}{2}$$

(This differs from the neutrino sum rule by a factor 2 because we leave out the contribution of axial current.) This leads to the inequality[8]

$$\lim_{E \to \infty} \frac{d\sigma_p}{dq^2} + \frac{d\sigma_n}{dq^2} \geq \frac{2\pi\alpha^2}{q^4} \tag{3.10}$$

The right-hand side is large, of order Rutherford scattering from a point particle, and it is this feature that makes the inequality interesting. We shall return to this point later on.

IV. AN APPEAL TO HISTORY

We have "derived" two interesting sum rules which bear a great similarity to sum rules obtained in non-relativistic quantum mechanics. For example, for electron scattering from an atom or nucleus we have the sum rule[9]

$$\begin{aligned} \int_0^\infty dE' \frac{d\sigma}{dq^2\, dE'} &= \frac{4\pi\alpha^2}{q^4} \left[Z + Z\,(Z-1)\, f_c\!\left(q^2\right) \right] \\ &\equiv \frac{4\pi\alpha^2}{q^4} \left[\sum_i Q_i^2 \right] \end{aligned} \tag{4.1}$$

where the Q_i are the charges of those constituents of the target which can be seen with the available spatial resolution $\left| \Delta \underset{\sim}{x} \right| = \frac{1}{|\underset{\sim}{q}|}$.

It is tempting to take over the same physical picture for the relativistic scattering at infinite momentum, because (1) the scattering goes via the charge density; (2) there is only spatial momentum transfer, essentially as in the non-relativistic case; (3) because of Lorentz contraction, the scattering occurs instantaneously in time provided we respect the uncertainty principle and do not look closely at the energy of the outgoing lepton.

The neutrino sum rule can now be interpreted in a simple way (set for simplicity $\theta_c = 0$)

$$\frac{d\sigma^{\bar{\nu} p}}{dq^2} - \frac{d\sigma^{\nu p}}{dq^2} = \frac{d\sigma}{dq^2}\,(\text{point}) = \frac{G^2}{\pi} \tag{4.2}$$

For point $I = I_3 = \frac{1}{2}$ spin $\frac{1}{2}$ particles only the $\bar{\nu}$ scatters. For $I_3 = -\frac{1}{2}$ particles only ν scatters. At large q^2, we expect the neutrino to scatter incoherently from the "elementary constituents" of the nucleon. If these constituents have spin and isospin $\frac{1}{2}$, then the sum rule simply says

$$\frac{d\sigma^{\bar{\nu} p}}{dq^2} = \langle N\uparrow \rangle \frac{G^2}{\pi} \qquad \frac{d\sigma^{\nu p}}{dq^2} = \langle N\downarrow \rangle \frac{G^2}{\pi}$$

and

$$\frac{d\sigma^{\bar{\nu} p}}{dq^2} - \frac{d\sigma^{\nu p}}{dq^2} = \langle N\uparrow - N\downarrow \rangle \frac{G^2}{\pi} = \frac{G^2}{\pi} \tag{4.3}$$

where $N\uparrow$ = no. of constituents with $I_3 = +\frac{1}{2}$. Of course the sum rule is more general than the model, but the same idea holds in the general case as well, independently of the magnitude of q^2.

For electron scattering $N\uparrow$ and $N\downarrow$ are replaced by $\sum_i Q_i^2$. Intuitively, we expect $\sum_i Q_i^2 \gtrsim 1$ because it would be hard to understand the quantized charge of observed states if the constitutents had charges small compared to 1. The inequality

$$\left[\sum_i Q_i^2\right]_{\text{proton}} + \left[\sum_i Q_i^2\right]_{\text{neutron}} \geq \frac{1}{2} \tag{4.4}$$

confirms that expectation. It is interesting to trace the origin of the result. It came from the assumed locality of isospin current. This means instantaneously in time the isospin is localized within the nucleon. Thus given a nucleon state $|P\rangle$, the state $J_\mu^\dagger(o)\,|P\rangle$ is meaningful and consists of one charge more at the origin of space (at t = 0). Therefore instantaneously in time, the constituents of the nucleon carrying isospin are localized in space. For simplicity, let the constituents have $I = 0$ and $I = \frac{1}{2}$. If for the proton,

$$\left[\sum_i Q_i^2\right]_{\text{proton}} = \sum_{i\left(I_3=+\frac{1}{2}\right)} \left(\bar{Q}_i + \frac{1}{2}\right)^2 + \sum_{j\left(I_3=-\frac{1}{2}\right)} \left(\bar{Q}_j - \frac{1}{2}\right)^2 + \sum_{k(I=0)} Q_k^2 \tag{4.5}$$

then charge symmetry gives for the neutron

$$\left[\sum_i Q_i^2\right]_{\text{neutron}} = \sum_i \left(\bar{Q}_i - \frac{1}{2}\right)^2 + \sum_j \left(\bar{Q}_j + \frac{1}{2}\right)^2 + \sum_k Q_k^2$$

and

$$\left[\sum Q_i^2\right]_p + \left[\sum_i Q_i^2\right]_n \geq \frac{1}{2} \times \left(\text{no. of } I = \frac{1}{2} \text{ constituents}\right) \geq \frac{1}{2}$$

Assuming this picture makes sense, the important question is what inelasticity ν is needed, for a given q^2, in order to see these supposed elementary constituents of the nucleon. Again we appeal to history: we suppose that at sufficiently high momentum transfer the scattering is quasi-elastic. This does not necessarily mean that the constituents be loosely bound or escape the nucleon. For example, for a non-relativistic particle bound in a harmonic-oscillator potential, the total scattering at fixed $\underset{\sim}{q}^2$ is always the Rutherford value, although the scattering to any given level is damped exponentially for large $\underset{\sim}{q}^2$. The important excited states are those which lie near the classical kinematics:

$$q_o = \Delta E = \frac{\underset{\sim}{q}^2}{2m}$$

The same relation might be expected relativistically as well, because of our "infinite momentum" kinematical argument

$$\nu \cong \frac{q^2}{2\bar{m}}$$

where $\bar{m}$ is the mean mass of the constituent in the nucleon. Because elastic scattering occurs for $\nu = \frac{q^2}{2M_p}$, we must have $\bar{m} < M_p$ in order for the kinematics to make sense.

A tempting choice is $\bar{m} \sim 300$ MeV corresponding to the "light quark" model[10], which explains well the magnetic moments of the nucleons in terms of Dirac moments of the constituent quarks. Unfortunately, this choice appears to have already been ruled out experimentally[11] by the data from CEA. We shall discuss this later in more detail. In any case, this order of magnitude $\bar{m}$ is easy to test with present machines; for $\nu \sim$ 10-15 BeV and $q^2 = 1\ (\mathrm{BeV})^2$ we can probe for a mean mass $\bar{m} \gtrsim 50$ MeV.

In the way we have phrased the question, the difficulty with finding "elementary constituents" of the nucleon is not that they are too heavy to be produced but that they are too light to be seen with the leptonic illumination at our disposal. For example, at $q^2 = 1\ (\mathrm{BeV})^2$, we need an incident lepton energy greater than 10^3 BeV just to "see" the electrons in the atom.

Thus far, we have discussed the possibility of a quasi-elastic peak in the scattered lepton energy spectrum. There will, in addition, probably be a broad continuum arising from diffraction production[12] of ρ^o and other 1^- mesons at high inelasticities $\nu = q_o = \Delta E$ at fixed q^2. The onset of these processes can be estimated on kinematical grounds. When the minimum momentum transfer to the nucleon is small compared to the nucleon size ($\sim$ 350 MeV) we can expect the coherent processes to go efficiently. This occurs when

$$\Delta_{min} = \frac{q^2 + m_\rho^2}{2q_o} \lesssim 350 \text{ MeV} \sim \bar{m}$$

or (4.6)

$$q_o = \nu \gtrsim \frac{q^2}{2\bar{m}} + m_\rho$$

which is, for large q^2, essentially the same estimate as from the quark model. Therefore, the "quasi-elastic peak," if any, may well merge into a continuum coming from the coherent production.

In terms of our model of constituents, we can interpret the coherent processes in a simple way. They are simply the scattering of leptons from the vacuum fluctuations of charge surrounding the nucleon; in the quark model it is the "meson cloud" of virtual quark-antiquark pairs surrounding the 3 constituent quarks. On these grounds, it would be reasonable to expect that the q^2 dependence of the diffractive contribution is again pointlike $\sim \left(q^2\right)^{-2}$ and may in all cases obscure any quasi-elastic bump. Because diffractive processes are expected to be the same strength for proton and neutron, in taking the difference of the yields they should disappear. We would predict (for large q^2) in the quark model

$$\lim_{E\to\infty}\left[\frac{d\sigma_{ep}}{dq^2} - \frac{d\sigma_{en}}{dq^2}\right] \xrightarrow[q^2 \to \infty]{}$$

$$\frac{4\pi\alpha^2}{q^4}\left[\left(\frac{4}{9} + \frac{4}{9} + \frac{1}{9}\right) - \left(\frac{1}{9} + \frac{1}{9} + \frac{4}{9}\right)\right] \qquad (4.7)$$

$$= \frac{1}{3}\left(\frac{4\pi\alpha^2}{q^4}\right)$$

V. FORMULAE

We now catalogue the sum rules which might conceivably be related to experiment. We first discuss the various kinematical regions in the light of the preceding arguments (see Fig. 2). Elastic scattering lies along the line $-q^2 = 2M\nu$.

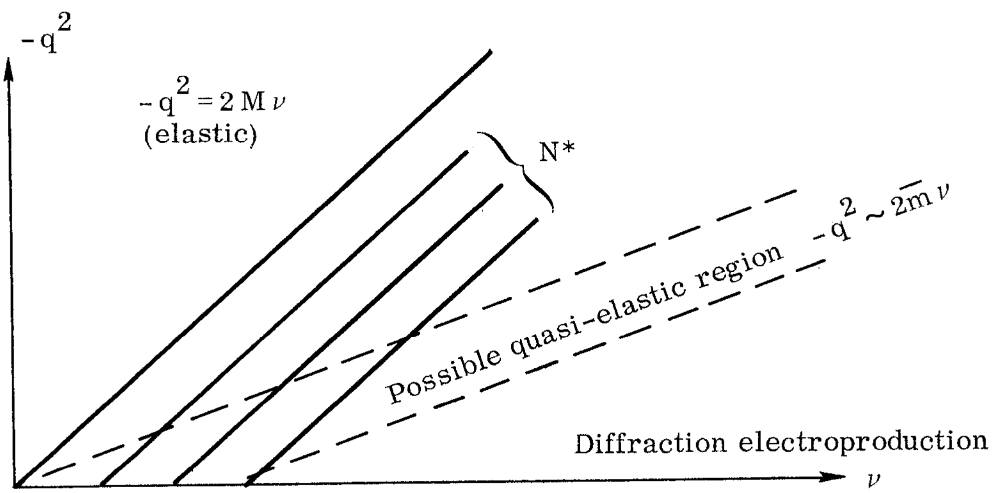

Fig. 2 -- Kinematical regions for inelastic lepton scattering.

Inelastic scattering to a given state lies along lines in the q^2, ν plane displaced parallel to the elastic line:

$$-q^2 = 2M\nu + M_n^2 - M^2 \tag{5.1}$$

The quasi-elastic region is

$$-q^2 = 2\bar{m}\nu \sim \frac{2}{3} M\nu \tag{5.2}$$

and cuts across all resonant states. This quasi-elastic region, where the largest scattering is expected, divides the region of asymptotic limits of form factors for electroproduction of resonant (or other) states from the region of diffraction production of 1^- mesons. It is interesting to plot the conjectured behavior of a form factor of a high-spin resonant state in this picture (Fig. 3). The rising behavior for small q^2 is characteristic of high-spin excitations in nuclear physics and has also been entertained in connection with electro-production of hadron resonances.[13]

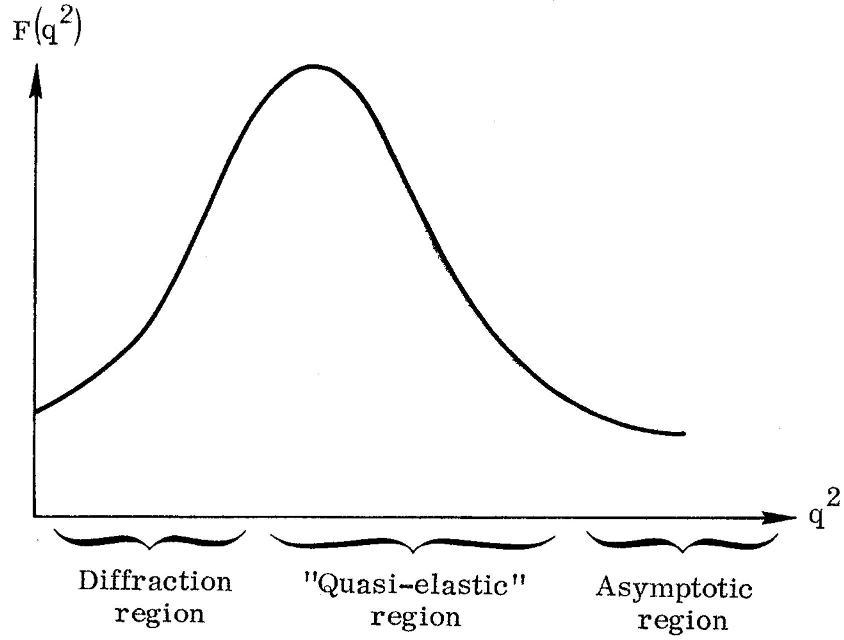

Fig. 3 -- Possible behavior for form factor of high-spin resonances

The formulae begin with the neutrino sum rule of Adler:

$$\int_{-q^2/2M}^{\infty} d\nu \left[\sigma_1\left(q^2,\nu\right)^{\bar{\nu} p} - \sigma_1^{\nu p}\left(q^2,\nu\right) \right] = 2 \tag{5.3}$$

On the basis of a diffraction-production model, we expect $\sigma_1 \longrightarrow \frac{\text{const}}{\nu}$ as $\nu \longrightarrow \infty$. Then, if $\sigma_1^{\bar{\nu} p} \longrightarrow \sigma_1^{\nu p}$ as $\nu \longrightarrow \infty$, it presumably does so as we enter the diffraction region and the sum rule should be saturated in the quasi-elastic region.

This is in accord with the quark model, and one has, in addition,

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \sigma_1^{\bar{\nu}p}\left(\nu, q^2\right)} \approx 4 \qquad \underset{\text{Quarks}}{\left(-q^2 \longrightarrow \infty\right)} \tag{5.4}$$

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \sigma_1^{\nu p}\left(\nu, q^2\right)} \approx 2 \qquad \underset{\text{Quarks}}{\left(-q^2 \longrightarrow \infty\right)} \tag{5.5}$$

and in any case

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \sigma_1\left(\nu, q^2\right)^{\bar{\nu}p}} > 1 \qquad \left(\text{all } q^2\right) \tag{5.6}$$

The electron-scattering inequality is

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \left[\sigma_{1p}\left(\nu, q^2\right) + \sigma_{1n}\left(\nu, q^2\right)\right]} > \frac{1}{2} \qquad \left(\text{all } q^2\right) \tag{5.7}$$

and for the quark model

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \sigma_{1p}\left(\nu, q^2\right)} \approx 1 \qquad \underset{\text{Quarks}}{\left(q^2 \longrightarrow \infty\right)} \tag{5.8}$$

$$\underset{\text{Quasi-elastic region}}{\int d\nu\ \sigma_{1n}\left(\nu, q^2\right)} \approx \frac{2}{3} \qquad \underset{\text{Quarks}}{\left(q^2 \longrightarrow \infty\right)} \tag{5.9}$$

$$\int_0^\infty d\nu \left[\sigma_{1p}\left(\nu, q^2\right) - \sigma_{1n}\left(\nu, q^2\right)\right] = \frac{1}{3} \qquad \underset{\text{Quarks}}{q^2 \longrightarrow \infty} \tag{5.10}$$

For backward scattering there exists another set of sum rules[14], valid only for large q^2, which we discuss in detail in Section VI. The right-hand side of these depends upon equal-time commutators of space-components of the current densities, highly model-dependent. In models in which the charge is carried by spinless objects this commutator vanishes, being of the form

$$\left[\phi_i(\underset{\sim}{x}) \underset{\sim}{\nabla} \phi_j(\underset{\sim}{x}),\ \phi_k(\underset{\sim}{y}) \underset{\sim}{\nabla} \phi_\ell(\underset{\sim}{y})\right] = 0 \tag{5.11}$$

In field algebra[15] the commutator also vanishes. In models in which the elementary constituents are spin 1/2 fermions, the commutator is[16]

$$\int d^3x \langle P| \left[J_z(\underset{\sim}{x}),\ J_z^\dagger(o)\right] |P\rangle = 2\int d^3x \langle P|\ J_o^3(\underset{\sim}{x}) |P\rangle = 2\left[\cos^2\theta_c + 2\sin^2\theta_c\right] \tag{5.12}$$

The sum rules, assuming spin 1/2 "quark" commutators, are

$$\frac{|q^2|}{2}\int \frac{d\nu}{\nu^2}\left[\sigma_2\left(q^2,\nu\right)^{\bar{\nu}p} - \sigma_2\left(q^2,\nu\right)^{\nu p}\right] = 2\left(\cos^2\theta_c + 2\sin^2\theta_c\right) \qquad \underset{\left[\substack{\text{Quark}\\ \text{model}}\right]}{q^2 \longrightarrow \infty} \tag{5.13}$$

Corresponding to (5.4) and (5.5) we have

$$\underset{\substack{\text{Quasi-elastic}\\ \text{region}}}{\frac{|q^2|}{2}\int \frac{d\nu}{\nu^2}\ \sigma_2\left(q^2,\nu\right)^{\bar{\nu}p}} \approx 4 \qquad \left(\substack{q^2 \longrightarrow \infty\\ \text{Quark model}}\right) \tag{5.14}$$

$$\frac{|q|^2}{2} \underset{\substack{\text{Quasi-elastic} \\ \text{region}}}{\int} \frac{d\nu}{\nu^2}\, \sigma_2\left(q^2,\nu\right)^{\nu p} \approx 2 \qquad \left(\begin{array}{c} q^2 \longrightarrow \infty \\ \text{Quark model} \end{array}\right) \tag{5.15}$$

For the electromagnetic processes we obtain an inequality by the same isospin manipulations leading from Eq. (5.3) to (5.7)

$$\frac{|q|^2}{2} \underset{\substack{\text{Quasi-elastic} \\ \text{region}}}{\int} \frac{d\nu}{\nu^2} \left[\sigma_{2p}\left(q^2,\nu\right) + \sigma_{2n}\left(q^2,\nu\right)\right] > \frac{1}{2} \qquad \left(\begin{array}{c} q^2 \longrightarrow \infty \\ \text{Spin } \frac{1}{2} \text{ space} \\ \text{commutators} \\ \text{assumed} \end{array}\right) \tag{5.16}$$

and in the quark model

$$\frac{|q|^2}{2} \underset{\substack{\text{Quasi-elastic} \\ \text{region}}}{\int} \frac{d\nu}{\nu^2}\, \sigma_{2p}\left(q^2,\nu\right) \approx 1 \qquad \left(\begin{array}{c} q^2 \longrightarrow \infty \\ \text{Quark model} \end{array}\right) \tag{5.17}$$

$$\frac{|q|^2}{2} \underset{\substack{\text{Quasi-elastic} \\ \text{region}}}{\int} \frac{d\nu}{\nu^2}\, \sigma_{2n}\left(q^2,\nu\right) \approx \frac{2}{3} \qquad \left(\begin{array}{c} q^2 \longrightarrow \infty \\ \text{Quark model} \end{array}\right) \tag{5.18}$$

Again, these sum rules have an immediate interpretation in terms of quasi-elastic scattering from the Dirac moments of elementary spin 1/2 constituents. In a model where the charge-bearing constituents are spinless, there is no quasi-elastic scattering in the backward direction. Therefore, comparison of forward and backward scattering might say something about the spins of the "elementary constituents."

Gottfried[17], using arguments similar to those presented here, has noticed that if one uses the quark model and writes a sum rule for σ_{1p} at small q^2 in analogy to the nuclear sum rule (4.1), the correlation term f_c vanishes because in the proton

$$\sum_{i > j} Q_i Q_j = -\frac{2}{9} - \frac{2}{9} + \frac{4}{9} = 0 \tag{5.19}$$

Therefore, for <u>all</u> q^2, one should have

$$\int_{\substack{\text{Quasi-elastic}\\ \text{region}}} d\nu \; \sigma_{1p}\left(q^2, \nu\right) = 1 \tag{5.20}$$

In the photoabsorption limit $q^2 \longrightarrow 0$, Gottfried has evaluated the sum, which agrees reasonably well with experiment.

VI. DERIVATIONS

We now derive the isovector part of the Adler sum rule[7], the Cabibbo-Radicati limit[2], and the sum rule for backward neutrino scattering.[14] For Adler's vector sum rule, we consider the process shown in Fig. 4.

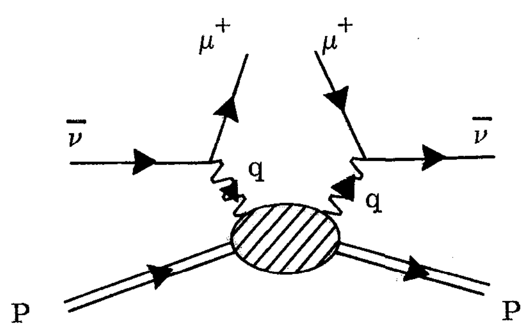

Fig. 4 -- Forward scattering of a current from a nucleon.

It is a "black box", now with two currents entering. After factoring out the lepton currents, the amplitude is described by a second-rank tensor

$$M_{\mu\nu} = \frac{iP_o}{M} \int d^4 x \, e^{-iqx} \left\langle P \left| T\left(J_\mu^{(v)}(x) \, J_\nu^{\dagger(v)}(o) \right) \right| P \right\rangle^c \tag{6.1}$$

$$+ \text{ Polynomial in } q \text{ and } P.$$

The superscript ^{c} means we systematically omit disconnected diagrams. The T-product is what is obtained by a naive application of perturbation theory, and the extra polynomial terms may or may not be present. They can be present because the ordered product of fields may be so singular at $x = 0$ that we cannot legally multiply by the step function $\theta(t)$ to form a time-ordered product. Thus we may err by terms proportional to $\delta^4(x)$ or derivatives thereof. By an expansion in intermediate states we obtain

$$\begin{aligned} M_{\mu\nu} &= \frac{P_o}{M} \sum_n \frac{\langle P | J_\mu^{(v)}(o) | n \rangle \langle n | J_\nu^{\dagger(v)}(o) | P \rangle^c}{E_p + q_o - E_n + i\epsilon} (2\pi)^3 \, \delta^3\left(\underset{\sim}{P_n} - \underset{\sim}{P} - \underset{\sim}{q}\right) \\ &+ \begin{pmatrix} J_\mu \longleftrightarrow J_\nu^\dagger \\ q \longleftrightarrow q \end{pmatrix} + \text{Polynomial} \end{aligned} \tag{6.2}$$

where the polynomial comes from the possible extra δ-function terms. With the identity

$$\begin{aligned} q^\mu \langle P | J_\mu(o) | n \rangle &= q_o \langle P | J_o | n \rangle - \langle P | \underset{\sim}{q} \cdot \underset{\sim}{J} | n \rangle \\ &= \left(q_o + E_p - E_n \right) \langle P | J_o(o) | n \rangle \end{aligned} \tag{6.3}$$

we obtain

$$q^{\mu} M_{\mu\nu} = \int \langle P \left| \left[J_o^{(v)}(\underset{\sim}{x}),\; J_\nu^{\dagger(v)}(o) \right] \right| P \rangle^{c} \; e^{i \underset{\sim}{q} \cdot \underset{\sim}{x}} \, d^3x \; + \; \text{Polynomial} \qquad (6.4)$$

The time component we evaluate, using the algebra of charge densities, and find

$$q^{\mu} M_{\mu o} = P_o \quad + \quad \text{Polynomial vanishing as } q \longrightarrow 0 \qquad (6.5)$$

We postulate:

$$q_\mu M^{\mu\nu} = P^{\nu} + \lambda q^{\nu}$$

with λ a constant undetermined and possibly zero. We rule out (by fiat only) higher polynomials, which however <u>must be covariant</u> because $M_{\mu\nu}$ is covariant. Indeed, from covariance and the divergence condition, it then follows that

$$M_{\mu\nu} = \left[q^2 P_\mu P_\nu - q \cdot P \left(q_\mu P_\nu + q_\nu P_\mu \right) + \left(q \cdot P \right)^2 g_{\mu\nu} \right] M_1 \left(q^2, \nu \right)$$
$$+ \left(q_\mu q_\nu - g_{\mu\nu} q^2 \right) M_2 \left(q^2, \nu \right) + \frac{\left(q_\mu P_\nu + q_\nu P_\mu - g_{\mu\nu} q \cdot P \right)}{q^2} + \lambda g_{\mu\nu} \qquad (6.6)$$

Notice that the absorptive part of $M_{\mu\nu}$ is related to $J_{\mu\nu}^{(v)}$ as follows from unitarity, or the definition of $M_{\mu\nu}$ we gave

$$\text{Im } M'_{\mu\nu} = - \frac{1}{2} \overline{\sum_n} \langle P | J_\mu^{(v)}(o) | n \rangle \langle n | J_\nu^{\dagger}(o)^{(v)} | P \rangle^{c} \left(2\pi \right)^4 \delta^4 \left(q + P - P_n \right)$$
$$= - \pi J_{\mu\nu}^{(v)} \left(q, P \right)^{\bar{\nu} p} \qquad \nu > 0 \qquad (6.7)$$

For $\nu < 0$ crossing gives

$$\text{Im } M_{\mu\nu} = + \pi J_{\mu\nu}^{(v)} \left(q, P \right)^{\nu p} \qquad \left(\nu < 0 \right) \qquad (6.8)$$

There are now two ways of obtaining the Cabibbo-Radicati sum rule:

a) Demand that the coefficient of $g_{\mu\nu}$ grow less rapidly than $q \cdot P = \nu$ as $\nu \longrightarrow \infty$

b) Demand the coefficient of $q_\mu P_\nu$ (as $\nu \to \infty$) $\longrightarrow 0$.

Regge-pole analysis (as well as perturbation theory) suggests that these conditions are true, although the application of such an analysis in these circumstances is dubious. If we demand condition (a), we learn that

$$\frac{1}{\nu}\left[\nu^2 M_1\left(q^2,\nu\right) - q^2 M_2\left(q^2,\nu\right) - \frac{\nu}{q^2}\right] \longrightarrow 0 \tag{6.9}$$

For $q^2 \longrightarrow 0$, we can ignore M_2 and find

$$M_1 \longrightarrow \frac{1}{\nu q^2} + 0\left(q^2\right) \qquad \begin{array}{l}\nu \longrightarrow \infty \\ q^2 \text{ small}\end{array} \tag{6.10}$$

and, because M_1 satisfies an unsubtracted dispersion relation, we find

$$M_1\left(q^2,\nu\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\nu' \, \mathrm{Im}\, M_1\left(q^2,\nu'\right)}{\nu' - \nu} \longrightarrow -\frac{1}{\pi\nu}\int_{-\infty}^{\infty} d\nu' \, \mathrm{Im}\, M_1\left(q^2,\nu'\right) \tag{6.11}$$

Identifying, by means of the definition of $J_{\mu\nu}$ and the formulae in problems 2 and 3,

$$\mathrm{Im}\, M_1 = \begin{cases} -\dfrac{\pi\sigma_1\left(q^2,\nu\right)^{\bar{\nu} p}}{q^2} - \dfrac{\pi}{q^2}\,\delta\left(\nu + \dfrac{q^2}{2m}\right)\left[F_{1v}^2\left(q^2\right) - \dfrac{\left(\kappa_p - \kappa_n\right)^2}{4M^2} F_{2v}^2\left(q^2\right)\right] & (\nu > 0) \\ + \dfrac{\pi\sigma_1\left(q^2|\nu|\right)^{\nu p}}{q^2} & (\nu < 0) \end{cases} \tag{6.12}$$

we can put everything together and obtain

$$\frac{1}{\nu q^2} + 0\left(q^2\right) \longrightarrow \frac{1}{\nu q^2}\left[F_{1v}^2\left(q^2\right) - \frac{\left(\kappa_p - \kappa_n\right)^2}{4M^2} q^2 \, F_{2v}^2\left(q^2\right)\right] - \frac{1}{\pi}\int_0^{\infty} \frac{d\nu}{\nu}\left[\sigma^{\gamma^-}(\nu) - \sigma^{\gamma^+}(\nu)\right] \tag{6.13}$$

As $q^2 \longrightarrow 0$, the singularity $\sim q^{-2}$ cancels $\left[F_{1v}(o) = F_{2v}(o) = 1\right]$ and we get

$$\frac{1}{3}\langle r^2\rangle_{F_1} - \frac{\left(\kappa_p - \kappa_n\right)^2}{4M^2} = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{\nu}\left[\sigma^{\gamma^-}(\nu) - \sigma^{\gamma^+}(\nu)\right] \tag{6.14}$$

Although "charged photons" do not exist, this sum rule can be evaluated by making an isospin rotation and relating the cross section to isovector photoabsorption cross sections, obtained by analyzing the cross section resonance by resonance. This was done by Gilman and Schnitzer[18], and by Adler and Gilman[19] who find agreement upon including contributions with $\nu \lesssim 1$ BeV. They divide the above formula by two and numerically get

$$\underset{\langle r^2\rangle_{F_1}}{\frac{.066}{M_\pi^2}} = \underset{\left(\kappa_p-\kappa_n\right)^2}{\frac{.059}{M_\pi^2}} + \underset{\substack{\text{S-wave} \\ E_1 \text{ thres-} \\ \text{hold prod.}}}{\frac{.016}{M_\pi^2}} - \underset{\substack{N^* \\ (1238)}}{\frac{.028}{M_\pi^2}} + \underset{\substack{N^* \\ (1520)}}{\frac{.016}{M_\pi^2}} + \underset{\substack{N^* \\ (1690)}}{0\left(\frac{.002}{M_\pi^2}\right)} = \frac{.063}{M_\pi^2} \tag{6.15}$$

The agreement with experiment is very good. However, it is not clear that the generalizations to fixed q^2 follow immediately. This requires that either $\nu^{-1}F_2 \longrightarrow 0$ as $\nu \longrightarrow \infty$ or that assumption (b) is valid. While the coefficient of $g_{\mu\nu}$ is the amplitude for scattering a real transverse "charged photon," these other amplitudes going as $P_\mu q_\nu$ have a less physical character. If we are given assumption (b), we clearly get

$$\nu M_1\left(q^2,\nu\right) \longrightarrow 1/q^2 \qquad \left(\nu \longrightarrow \infty\right) \tag{6.16}$$

and for all q^2 we get, as before,

$$\frac{1}{\pi}\int d\nu' \,\mathrm{Im}\, M_1\left(\nu',q^2\right) = -1/q^2 \tag{6.17}$$

In terms of $\sigma_1^{\bar{\nu}p}$ and $\sigma_1^{\nu p}$

$$\int_0^\infty d\nu' \left[\sigma_1^{(v)} \left(q^2, \nu'\right)^{\bar{\nu}p} - \sigma_1^{(v)} \left(q^2, \nu'\right)^{\nu p} \right] = 1 \qquad (6.18)$$

which is the same sum rule as we found with the infinite-momentum method.

Before going on, we comment on the implications of a violation in the sum rule. The weakest assumption made was that, essentially

$$\lim_{\nu \to \infty} \frac{M_{\mu\nu}\left(\nu, q^2\right) - M_{\mu\nu}\left(-\nu, q^2\right)}{\nu} = 0 \qquad (6.19)$$

that is, that part of the amplitude corresponding to $I = 1$ exchange in the t-channel grows less strongly with ν than that corresponding to exchange of an elementary $I = 1$, $J = 1$ particle. Thus Reggeized ρ-exchange models satisfy this assumption. This can be checked experimentally by looking for the energy dependence of coherent $\rho^\pm$ photo- and electroproduction.

We now derive the backward-scattering sum rules,[14] which involve a rather different technique. We choose a transverse polarization vector ϵ such that $\epsilon \cdot P = \epsilon \cdot q = 0$ and consider again

$$\epsilon_\mu \epsilon_\nu M^{\mu\nu} \equiv M_\perp\left(\nu, q^2\right) = q^2 M_2 - \nu^2 M_1 \qquad (6.20)$$

where we now include the axial terms as well. We have

$$\mathrm{Im}\, M_\perp = \begin{cases} -\frac{\pi}{2}\, \sigma_2^{\bar{\nu}p}\left(q^2, \nu\right) & \nu > 0 \\ +\frac{\pi}{2}\, \sigma_2^{\nu p}\left(q^2, |\nu|\right) & \nu < 0 \end{cases} \qquad (6.21)$$

which in turn is proportional to the cross section for 180° neutrino-muon scattering.

Now go back to the Low equation (6.2) for $M_\perp$ and let $q_o \longrightarrow i\infty$ (the reason for the i will come later). In this limit, we find that

$$M_\perp \longrightarrow [\text{Polynomial ?}] + \frac{1}{q_o}\frac{P_o}{M}\int d^3x\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} \langle P \left| \left[\underset{\sim}{\epsilon}\cdot \underset{\sim}{J}\left(\underset{\sim}{x},o\right),\ \underset{\sim}{\epsilon}\cdot \underset{\sim}{J}^\dagger(o) \right] \right| P \rangle \tag{6.22}$$

In the quark model, or any model based on spin 1/2 elementary constituents, the commutator is $\left(2P_o\right)\left(\cos^2\theta_c + 2\sin^2\theta_c\right)$. For spin 0 models or in the Lee-Zumino-Weinberg "field algebra"[15] the commutator vanishes. For definiteness, let us take the quark model, so that

$$M_\perp \longrightarrow (\text{Polynomial}) + \frac{2}{q_o}\left(\cos^2\theta_c + 2\sin^2\theta_c\right) \tag{6.23}$$

Now we evaluate $M_\perp$ another way, using a once-subtracted dispersion relation, valid if $\left(M_\perp/\nu\right) \longrightarrow 0$ (that assumption should be familiar!!) as $\nu \longrightarrow \infty$:

$$M_\perp\left(\nu, q^2\right) = M_\perp\left(o, q^2\right) + \frac{\nu}{\pi}\int_{-\infty}^{\infty} \frac{d\nu' \,\mathrm{Im}\, M_\perp\left(\nu', q^2\right)}{\nu'\left(\nu' - \nu - i\epsilon\right)} \tag{6.24}$$

$M_\perp\left(o, q^2\right)$ is even in q_o and therefore the q_o^{-1} term in (6.22) comes from the dispersion integral.

Now in the limit $q_o \longrightarrow i\infty$, $\underset{\sim}{q}$ fixed, $q^2 \longrightarrow -\infty$ and $\nu \longrightarrow q_o P_o = i\left|q_o\right| P_o$.

Since the values of ν' in the dispersion integral occur for $\nu' \gtrsim \frac{q^2}{2M} \approx \frac{q_o^2}{2M}$, then $\frac{\nu}{\nu'} \longrightarrow 0$ as $q_o \longrightarrow i\infty$. Thus, in the limit (in the absence of pathological cancellation), we get

$$M_\perp \xrightarrow[q_o \longrightarrow i\infty]{} \left[\text{Even in } q_o\right] + \frac{q_o P_o}{\pi}\int \frac{d\nu'}{\nu'^2}\, \mathrm{Im}\, M_\perp\left(q^2, \nu'\right) \tag{6.25}$$

The dispersion integral must be of order q^{-2} (at least) as $q^2 \longrightarrow -\infty$. That size is itself surprising, and we assume it is no worse (such as independent of q^2).

We conclude that

$$\lim_{q^2 \to -\infty} \frac{q^2}{\pi} \int \frac{d\nu}{\nu^2} \, \mathrm{Im}\, M_\perp\left(\nu, q^2\right) = \begin{cases} 2\left(\cos^2\theta_c + 2\sin^2\theta_c\right) & \text{quark model} \\ 0 & \text{spin 0 model} \end{cases} \quad (6.26)$$

Using the expression (6.21) for absorptive part

$$\lim_{q^2 \to -\infty} \frac{\left|q^2\right|}{2} \int \frac{d\nu}{\nu^2} \left[\sigma_2^{\bar{\nu} p}\left(q^2, \nu\right) - \sigma_2^{\nu p}\left(q^2, \nu\right)\right] = 2\left(\cos^2\theta_c + 2\sin^2\theta_c\right) \quad (6.27)$$

(quark model)

For lepton scattering there follows the inequality

$$\lim_{q^2 \to -\infty} \frac{|q|^2}{2} \int \frac{d\nu}{\nu^2} \left[\sigma_{2p}\left(q^2, \nu\right) + \sigma_{2n}\left(q^2, \nu\right)\right] > \frac{1}{2} \quad (6.28)$$

The physics of these sum rules is similar to those in nuclear physics. For large q^2, correlations vanish and the scattering is the sum of the scattering from the Dirac moments of the point spin 1/2 constituents (if they exist).

VII. COMPARISON WITH EXPERIMENTS

The experimental situation is not at all complete. The only relevant data on inelastic electron scattering comes from CEA[11], where there is an electron spectrum taken at $E = 4.9$ BeV, $\theta = 31^\circ$ and E' down to ~ 1.6 BeV just about where a quasi-elastic quark peak (for a 300 MeV quark) might be expected. We take Eq. (2.2) and write $\sigma_{1p} = \frac{F(x, q^2)}{\nu}$, $\sigma_{2p} \simeq 0$ and plot F versus x with $x = \frac{2M\nu}{|q^2|}$. For $x \gtrsim 3$ we enter (in the quark picture) into the quasi-elastic regions and since

$$\int_1^{\infty} \frac{dx}{x} F\left(x\,,\, q^2\right) \sim 1 \tag{7.1}$$

F should somewhere be $\gtrsim 1$ for $x \gtrsim 3$. The data are shown in Fig. 5.

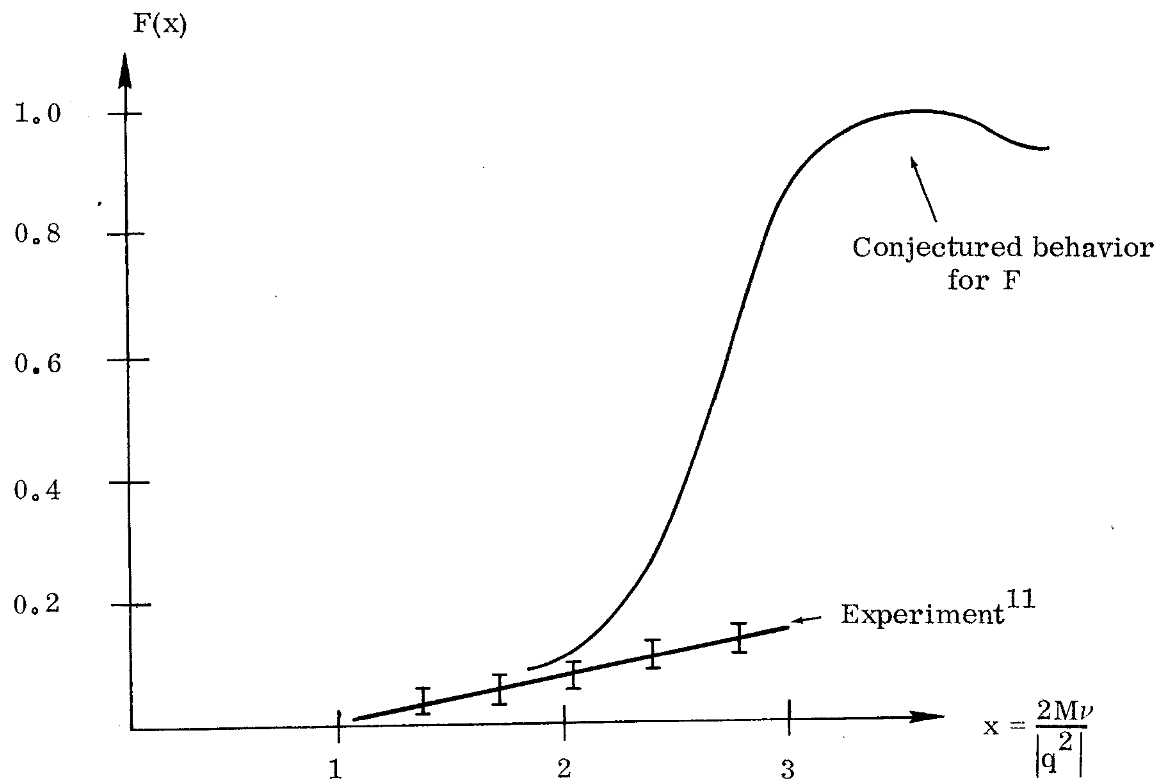

Fig. 5 -- Comparison of theoretical expectations with experiment.

The values of q^2 lie between 2 and 4 $(\mathrm{BeV/c})^2$; the maximum value of ν is 3.2 BeV. Less than 10% of the quark sum rule (5.8) is accounted for here; for the strict inequality (5.7) (assuming $\sigma_p \approx \sigma_n$) less than half. Evidently large inelasticities appear to be involved in this process in order to satisfy the sum rule.

The quasi-elastic quark model we described does not look too good. But we have not reached the diffraction region, and it will be of interest to look at very large inelasticity and dispose, without ambiguity, of the model completely.

There is one experiment on inelastic muon scattering.[20] It is in fact well fit by a form which has the $1/q^4$ asymptotic behavior, but the statistics are too sparse to draw any conclusion.

A test of the neutrino sum rule itself at $q^2 \gtrsim 1(\mathrm{BeV})^2$ must probably await higher energy machines. One other possibility lies in the underground experiments, although this is a fairly desperate hope. In these experiments, what is observed is a secondary muon from a neutrino produced in an air shower which then travels through the earth and interacts near the detector.[21] The neutrino spectrum from this source has been computed with rather high certainty ($\lesssim$ 30%) out to $E \sim 10^3$ BeV; it falls off as E_ν^{-3}. With this spectrum, the neutrino sum rule, and an assumption of how it saturates (such as the quasi-elastic mechanism), a lower bound can be placed on the rate which is comparable to that observed.[22] If one in addition accepts the quasi-elastic picture, the spectrum of secondary muons can be estimated to go like E_μ^{-1}, the two factors of E relative to the E_ν^{-3} neutrino spectrum coming for the following reasons:

1. The linear rise of the total cross section with E, coming from the point-like assumption:

$$\frac{d\sigma^{\bar{\nu}p}}{dq^2} \approx \frac{G^2}{\pi} \langle N\uparrow \rangle$$

$$\sigma_{tot} \cong \int_0^{2mE} \frac{d\sigma}{dq^2}\, dq^2 \sim \frac{G^2}{\pi} (2\overline{m}E) \langle N\uparrow \rangle \tag{7.2}$$

2. The range of the secondary muon, which determines the effective thickness of the target is roughly proportional to energy.

This picture predicts a relatively large number of muons ($\sim$ 20%) with energy greater than, say, 100 BeV, something which might be tested in the Utah experiment in particular.

Helen Quinn[22] has computed the spectrum using the quark model, and vector boson exchange

$$\frac{d\sigma^{\overline{\nu}p}}{dq^2} \sim \frac{G^2}{\pi} \langle N\uparrow\rangle \left(\frac{M_W^2}{|q^2| + M_W^2}\right)^2 \tag{7.3}$$

to cut off the linear rise of cross section with incident energy. Unfortunately, the distinction between various choices of M_W between 5 BeV and ∞ is probably not large enough to draw any firm conclusion. In any case, more will be known about the sum rules from electromagnetic processes before these experiments have accumulated enough statistics, which is characteristically $\sim$20 events/year.

VIII. RADIATIVE CORRECTIONS TO THE FERMI COUPLING IN β-DECAY

In this lecture we study radiative corrections to the vector (Fermi) -decay matrix element. The same locality assumption used to derive the sum rules here leads to the conclusion that the correction suffers the same logarithmic divergence as would be present in the absence of strong interactions. The coefficient of the divergent logarithm, however, is model-dependent, and it is possible to cancel the divergence in any model in which the "elementary constituents" (or fields) carrying isospin have $I = \frac{1}{2}$ and mean charge $-\frac{1}{2}$. We will elaborate this later in doing the details.

We assume that the corrections to β-decay come from three Feynman amplitudes (we assume there is no vector boson) shown in Fig. 6.

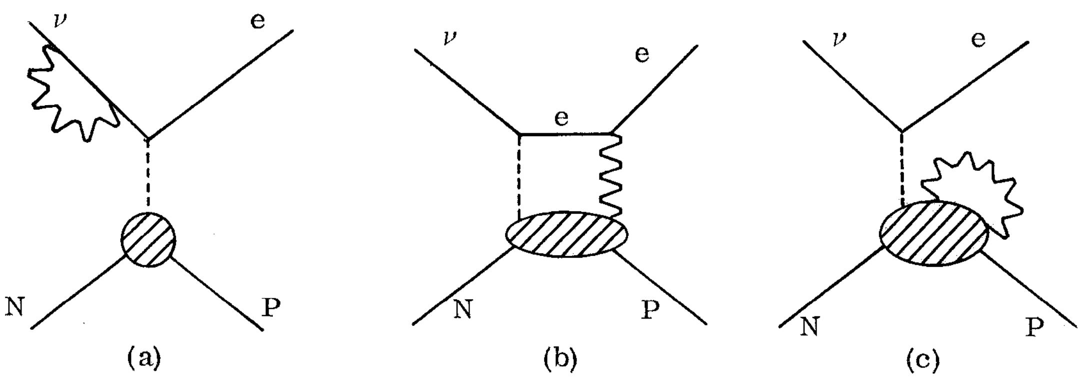

Fig. 6 -- Diagrams for β-decay radiative corrections.

There is no trouble in computing (a). It diverges logarithmically, except in Landau gauge; $D_{\mu\nu} = k^{-2}\left(g_{\mu\nu} - k_\mu k_\nu k^{-2}\right)$. For diagram (b) there are two contributions; the first comes from a combination of isovector photon and vector weak current, the other from axial current and isoscalar photon. This follows from looking at G-parity. Typical contributions are shown in Fig. 7:

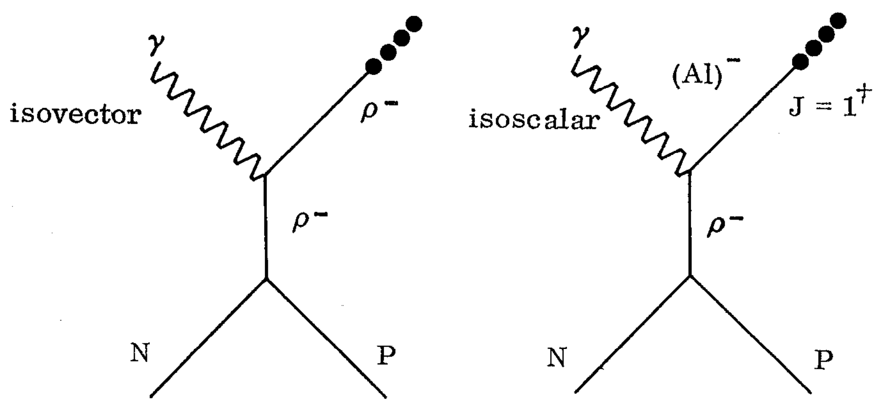

Fig. 7 -- Distinction between vector and axial contributions.

Diagram (c) can be evaluated using the Adler-Weisberger technique.[1] What is found is the amazing result that all but the isoscalar-axial vector piece of (b) and some portions involving soft photons are independent of the details of strong interactions, and are furthermore logarithmically divergent.[23,24,25] In addition, the isoscalar-axial piece is also divergent[25,26,27] as we shall show by the $q_o \to i\infty$ method used for the backward sum rules for $\sigma_2^{\nu p}$ and $\sigma_2^{\bar{\nu} p}$. We outline the calculation for the divergent part. First of all, we put in a low energy cutoff on the photon propagator to suppress soft photons. That is, we write

$$\frac{1}{k^2} = \frac{1}{k^2}\left(\frac{\lambda^2}{\lambda^2 - k^2}\right) + \frac{1}{k^2 - \lambda^2} \tag{8.1}$$

We choose $M^2 \gg \lambda^2 \gg M_e^2$, say $\lambda \sim 30$ MeV. For $k^2 \gg \lambda^2$ the first term is small, and we can use for the nucleon part of the amplitude the Born terms only, since these dominate for low k^2. That calculation is in the literature, done by Berman and Sirlin.[28] The second part reduces to $1/k^2$ for $k^2 \gg \lambda^2$ and is less singular as $k^2 \to 0$. In this part of the amplitude, we neglect the lepton momenta.

We now show that the vector part of the amplitude is logarithmically divergent and independent of details of hadron structure. We choose Feynman gauge $\left(D_{\mu\nu} = g_{\mu\nu} k^{-2}\right)$ for the calculation and start with diagram (b). The correction, from the vector current only is (ignoring now the soft-photon piece)

$$\delta M_b = -\frac{i\tilde{G}e^2}{\sqrt{2}} \int \frac{d^4k}{(2\pi)^4 \left(k^2 - \lambda^2\right)} \; \bar{u}_e \frac{\gamma^\mu \not{k} \gamma^\nu}{k^2} \left(1 - \gamma_5\right) u_\nu \; \Gamma_{\mu\nu}^{3+} \left(P, k\right) \tag{8.2}$$

where $\Gamma_{\mu\nu}^{3+}$ is the charge exchange amplitude for scattering a vector current from a nucleon of momentum P. We average the nucleon spin. $\Gamma_{\mu\nu}^{3+}$ is symmetric by PT invariance, and normalized such that its divergence condition is

$$k^\mu \Gamma^{3+}_{\mu\nu} = \frac{P_\nu}{M} = k^\nu \Gamma^{3+}_{\mu\nu} \tag{8.3}$$

We assume the absence of polynomials in k and P, as discussed in section VI (6.2); $\Gamma^{3+}_{\mu\nu}$ is then the matrix element (Fourier transformed) of the time ordered product of $J^{(v)}_\mu(x)$ and J^3_ν (o) between spin averaged nucleons. The normalization is such that

$$M = M_o + \delta M \qquad M_o = \frac{\widetilde{G}}{\sqrt{2}}\, \overline{u}_e \gamma_\mu \left(1 - \gamma_5\right) u_\nu \frac{P\mu}{M} \qquad \widetilde{G} = G \cos\theta_c \tag{8.4}$$

Because $\Gamma^{3+}_{\mu\nu}$ is symmetric, we may write

$$\begin{aligned} \gamma^\mu \not{k} \gamma^\mu &\longrightarrow \frac{1}{2}\left(\gamma^\mu \not{k} \gamma^\nu + \gamma^\nu \not{k} \gamma^\mu\right) \\ &= k^\mu \gamma^\nu + k^\nu \gamma^\mu - g^{\mu\nu} \not{k} \end{aligned} \tag{8.5}$$

and therefore, after some algebra

$$\delta M_b = -\frac{i\widetilde{G}e^2}{\sqrt{2}} \int \frac{d^4k}{\left(2\pi\right)^4} \frac{1}{k^2\left(k^2 - \lambda^2\right)} \left[2\overline{u}_e \frac{\not{P}}{M}\left(1 - \gamma_5\right) u_\nu - \overline{u}_e \not{k} \left(1 - \gamma_5\right) u_\nu \Gamma^{3+}_{\mu\mu}\left(P,k\right)\right. \tag{8.6}$$

Turning to diagram (c), we consider first the amplitude shown in Fig. 8, in the limit of small q

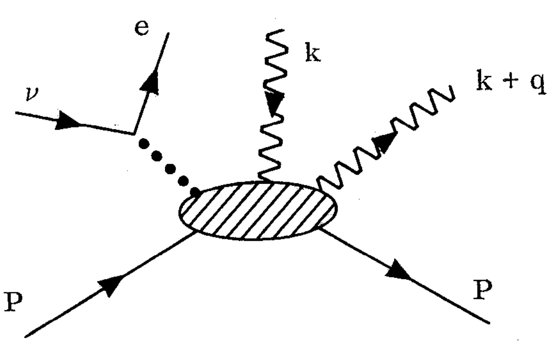

Fig. 8 -- Computation of diagram (c).

The wavy lines are photons, and the blob is the time-ordered product of two electromagnetic and one vector current $\Gamma_{\mu\nu\alpha}(P,k,q)$.

The divergence condition, obtained from the expression for the T-product, is (after some work!!)

$$q^{\alpha}\,\Gamma_{\mu\nu\alpha}(P,k,q) = \Gamma_{\mu\nu}^{+3}(P,k+q) - \Gamma_{\mu\nu}^{+3}(P,k) \tag{8.7}$$

Because the right-hand side is regular in the neighborhood of $q = 0$, we can write

$$\Gamma_{\mu\nu\alpha}(P,k,o) = \frac{\partial}{\partial k^{\alpha}}\,\Gamma_{\mu\nu}^{3+}(P,k) \tag{8.8}$$

This is not an obvious result, because individual diagrams in $\Gamma_{\mu\nu\alpha}$ are of order q^{-1} as $q \longrightarrow 0$; this result shows that the singular terms cancel out (Fig. 9):

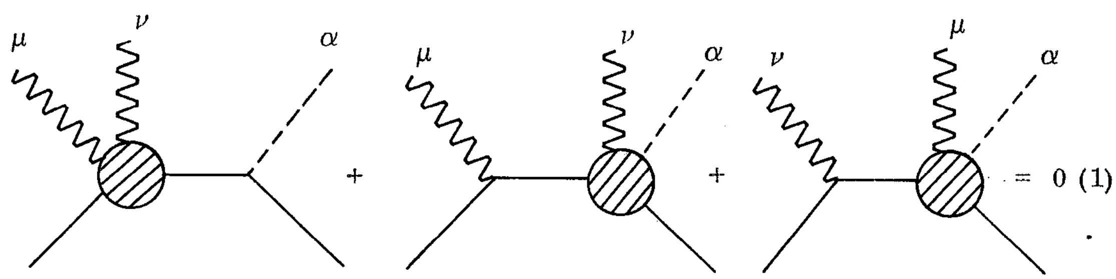

Fig. 9 -- Singular contributions to δM_c.

With this result we can write

$$\delta M_c = -\frac{ie^2\widetilde{G}}{\sqrt{2}}\,\overline{u}_e\gamma_\alpha(1-\gamma_5)u_\nu\,\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-\lambda^2}\,\frac{\partial}{\partial k_\alpha}\,\Gamma_{\mu\mu}^{3+}(P,k) \tag{8.9}$$

The factor 1/2 comes from the identity of the two photons; we would double-count otherwise.

An integration by parts now gives us the final result. To do this we first rotate the contour of the k_o integration (Fig. 10) to the imaginary axis, possible from the structure of the Low equation (6.2).

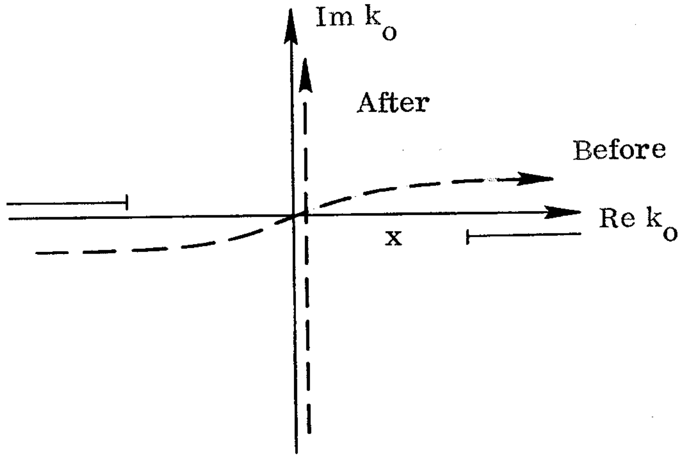

Fig. 10 — Contour rotation in the k_o plane.

We get $\left(k_o = ik'_o \; ; \; \underset{\sim}{k}' = \underset{\sim}{k}\right)$

$$\delta M_c = \frac{e^2 \tilde{G}}{\sqrt{2}} \int \frac{d^4 k'}{(2\pi)^4} \frac{\bar{u}_e \not{k}' (1-\gamma_5) u_\nu}{\left(k'^2 + \lambda^2\right)^2} \Gamma^{+3}_{\mu\mu}(P,k) \tag{8.10}$$

This must be joined onto δM_b, evaluated with the same contour integration:

$$\delta M_b + \delta M_c = \frac{e^2 \tilde{G}}{\sqrt{2}} \int \frac{d^4 k'}{(2\pi)^4} \left\{ \frac{2\bar{u}_e \not{P} (1-\gamma_5) u_\nu}{M k'^2 \left(k'^2 + \lambda^2\right)} + \frac{\bar{u}_e \not{k}' (1-\gamma_5) u_\nu \lambda^2}{k'^2 \left(k'^2 + \lambda^2\right)^2} \Gamma^{+3}_{\mu\mu}(P,k) \right\} \tag{8.11}$$

$$\approx \frac{\tilde{G}}{\sqrt{2}} \bar{u}_e \frac{\not{P}}{M} (1-\gamma_5) u_\nu \left(\frac{\alpha}{2\pi}\right) \log \frac{\Lambda^2}{\lambda^2} + \text{soft photons}$$

where we have put in an ultraviolet cutoff Λ. The structure-dependent term $\alpha \Gamma^{+3}_{\mu\mu}$ is rapidly convergent, and can be evaluated using the Born terms for $\Gamma^{+3}_{\mu\mu}$.

To this must be added δM_a

$$\delta M_a \cong - \frac{\alpha}{8\pi} \log \frac{\Lambda^2}{\lambda^2} M_o \tag{8.12}$$

leaving, for the vector-isovector contribution

$$\delta M \cong M_o \left(\frac{3\alpha}{8\pi} \log \frac{\Lambda^2}{\lambda^2} \right) + \text{soft photons} \tag{8.13}$$

The correction is universal, that is, structure independent. Therefore, for a given cutoff Λ the correction must be the <u>same</u> as the Berman-Sirlin calculation, based on structureless nucleons (the current algebra and assumptions of asymptotic growth are satisfied in that model). Thus to obtain the correct numerical contribution, all we must do is to set $g_A = 0$ in their calculation and extract their correction, now for a cutoff Λ independent of nucleon structure.

The isoscalar-axial contribution must now be considered. We shall argue that this term also diverges logarithmically by using the $q_o \longrightarrow i\infty$ method. Only in δM_b does it contribute and there we have

$$\delta M_b^{\text{Axial}} = -\frac{ie^2}{(2\pi)^4} \frac{\widetilde{G}}{\sqrt{2}} \int \frac{d^4k}{k^2} \, \bar{u}_e \frac{\gamma^\mu \not{k} \gamma^\nu}{k^2} \left(1 - \gamma_5\right) u_\nu \, \Gamma_{\mu\nu}^{\text{Axial}} (P,k) \tag{8.14}$$

$\Gamma_{\mu\nu}^{\text{Axial}} (P,k)$ is given by a time-ordered product of axial current and electromagnetic current and has the general form

$$\Gamma_{\mu\nu}^{\text{Axial}} (P,k) = \epsilon_{\mu\nu\alpha\beta} \, P^\alpha k^\beta \, F\left(k^2, \, k\cdot P\right)$$

$$= \sum_n \frac{\langle P | j_\mu^{em}(o) | n \rangle \langle n | J_\nu^{\text{Axial}}(o) | P \rangle \, (2\pi)^3 \, \delta^3 \left(\underset{\sim}{P}_n - \underset{\sim}{k} - \underset{\sim}{P}\right)}{k_o + E_p - E_n} \tag{8.15}$$

$$+ \left(k \longleftrightarrow -k, \; j_\mu \longleftrightarrow J_\nu^{\text{Axial}}\right)$$

From its structure as time-ordered product, as $k_o \longrightarrow i\infty$ $(\underset{\sim}{k} = 0)$ we have[23]

$$\Gamma_{\mu\nu} \xrightarrow[k_o \longrightarrow i\infty]{} \int \frac{\langle P | \left[j_\mu^{em}(o),\ J_\nu^{Axial}(\underset{\sim}{x}, o) \right] | P \rangle \, d^3x}{k_o} \tag{8.16}$$

We assume, as Regge exchange of ρ would suggest, that $F(k^2, k \cdot P)$ satisfies an unsubtracted dispersion relation

$$F(k^2, k \cdot P) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\nu' \, \mathrm{Im}\, F(k^2, \nu')}{\nu' - k \cdot P} \tag{8.17}$$

and as in the derivation of the backward sum rules

$$F \xrightarrow[k_o \longrightarrow i\infty]{} \frac{1}{\pi} \int \frac{d\nu'}{\nu'} \, \mathrm{Im}\, F(k^2, \nu') = F(k^2, o) \tag{8.18}$$

We find, then, that

$$F(k^2, o) \xrightarrow[k^2 \longrightarrow -\infty]{} \frac{1}{k^2} \int \frac{d^3x \langle P | \left[j_x^{em}(o),\ J_y^{Axial}(\underset{\sim}{x}, o) \right] | P \rangle}{\epsilon_{xyzo} \, P^z} \tag{8.19}$$

To study the commutator, we choose a model in which there is a single isospin doublet of spin 1/2 fields, all other fields having isospin zero. Then if the doublet has charge $Q = \overline{Q} + \frac{\tau_3}{2}$

$$\begin{aligned} \left[j_x^{em}(o),\ J_y^{axial}(\underset{\sim}{x}, o) \right] &= \psi^\dagger(o)\, \overline{Q} \left[\alpha_x,\ \tau^+ \alpha_y \gamma_5 \right] \psi(o)\, \delta^3(\underset{\sim}{x}) \\ &= 2\overline{Q}\, \psi^\dagger \tau^+ \alpha_z \psi\, \delta^3(\underset{\sim}{x}) = 2\overline{Q}\, J^z_{\text{vector}}(o)\, \delta^3(\underset{\sim}{x}) \end{aligned} \tag{8.20}$$

and inserting into (8.19), we find (spin 1/2 models only!!)

$$F\left(k^2, o\right) \xrightarrow[k^2 \to -\infty]{} - \frac{\left(2\overline{Q}\right)i}{k^2} \frac{\langle P | J^z_{(V)}(o) | P \rangle}{P_z} = - \frac{2i\overline{Q}}{k^2} \tag{8.21}$$

We now go back to $\delta M_b^{\text{Axial}}$, insert the general form for $\Gamma_{\mu\nu}^{\text{Axial}}$, do the spin algebra and rotate the k_o contour to the imaginary axis as before. To do the spin algebra, observe on general grounds

$$\gamma_\mu \not{k} \gamma_\nu \epsilon^{\mu\nu\alpha\beta} P_\alpha k_\beta = C \gamma^5 \left(P{\cdot}k \not{k} - k^2 \not{P}\right) \tag{8.22}$$

and evaluate C by letting $P^\mu = (1, 000)$, $k^\mu = (0, 1, 0, 0)$ obtaining

$$2 \gamma_2 \gamma_1 \gamma_3 \epsilon^{2301} (-1) = C \gamma^5 \gamma_0$$

or (8.23)

$$C = 2i$$

We get

$$\begin{aligned} \delta M_b^{\text{Axial}} &= - \frac{ie^2}{(2\pi)^4} \frac{\widetilde{G}}{\sqrt{2}} \int d^4k \, (-2i) \, \overline{u}_e \frac{\left(\not{k}(k \cdot P) - k^2 \not{P}\right)}{k^4} \left(1 - \gamma_5\right) u_\nu \, F\left(k^2, k \cdot P\right) \\ &= \frac{\widetilde{G}}{\sqrt{2}} \overline{u}_e \not{P} \left(1 - \gamma_5\right) u_\nu \left\{ \frac{3}{2} \frac{e^2}{(2\pi)^4} \int \frac{d^4k}{k^2} F\left(k^2, k \cdot P\right) \right\} \end{aligned} \tag{8.24}$$

We can only evaluate the integral for large k^2. However, we again can approximate in this region $F\left(k^2, k \cdot P\right)$ by $F\left(k^2, o\right)$. This is because after the contour rotation, we can write

$$k \cdot P = i \sqrt{-k^2} \, M \cos\theta \text{ with } |\cos\theta| < 1 \tag{8.25}$$

Then $|k\cdot P| < |k|M \ll \frac{|k^2|}{2M}$ which is the threshold of the dispersion integral in (8.17). So we finally find for the contribution from large $k'^2 = |k_o|^2 + |\underset{\sim}{k}|^2$

$$\delta M_b^{\text{Axial}} \approx M_o \left\{ \frac{3}{2} \frac{4\pi\alpha}{(2\pi)^4} (-i) \int dk'^2 \, F(-k'^2, o) \right\}$$
$$= \frac{3\alpha}{8\pi} \left(2\overline{Q}\right) M_o \int^{\infty} \frac{dk'^2}{k'^2} \tag{8.26}$$

The lower limit cannot be determined on general grounds, but we expect it to be of order of the nucleon mass. Therefore the final expression for the correction is

$$\frac{\delta M}{M_o} = \frac{3\alpha}{8\pi} \left[\log \frac{\Lambda^2}{\lambda^2} + 2\overline{Q} \log \frac{\Lambda^2}{M^2} \begin{array}{l} + \text{ soft photon portion} \\ + \text{ small corrections} \\ \quad \text{from axial Born term} \end{array} \right] \tag{8.27}$$

For $\overline{Q} = -\frac{1}{2}$ the correction is convergent, a fact which has stimulated models of the hadrons designed to avoid this divergence.[26,27] However, I do not believe that the disaster exhibited here is all so serious. I believe an upper limit on the cutoff is somewhere such that $G\Lambda^2 \sim 1$, say $\Lambda \sim 300$ BeV. For if nothing new has happened by that time, higher orders of weak interaction, neglected in this calculation, will become important. If we cut off at 300 BeV, the determination of the Fermi constant is modified by 1%. That is, in the past, using a 1 BeV cut-off, one compared

$$\widetilde{G} = G \cos \theta_c^{(\beta)} \left(1 + \delta\right)$$

with experiment (decay of 0^{14}) and obtained[29]

$$\cos \theta_c^{(\beta)} = 0.978$$

If we replace δ by $\delta + \frac{3\alpha}{8\pi}\left(1 + 2\overline{Q}\right) \log \frac{\Lambda^2}{M^2}$ with $\overline{Q} = \frac{1}{6}$ (quark model)

and $\Lambda \sim 300$ BeV, we <u>decrease</u> $\cos\theta_c$ by $\sim .010$ and get $\cos\theta_c^{(\beta)} = .968$. The value of $\cos\theta_c$ determined from Ke3 decay is[30]

$$\cos\theta_c^{(k)} \approx .975$$

so that the agreement is still better than 1%.

In any case, the origin of the logarithmic divergence lies in the assumption of a local current algebra, and just as the nature of the states (if any) which saturate the high-q^2 sum rules is very obscure, so also are the states responsible for this divergent correction. It is a challenge to both theoretical and experimental physics to improve this situation.

PROBLEMS

1. Given the form for weak and electromagnetic interactions in the lecture notes, compute the differential cross section for inelastic electron scattering in terms of σ_1 and σ_2.

2. Using current conservation $q_\mu j^{\mu\nu} = 0$, show that as $q^2 \longrightarrow 0$

$$\sigma_1\left(q^2, \nu\right) \longrightarrow - q^2 \frac{\sigma_\gamma(\nu)}{\pi\nu} + \delta \left(\nu + \frac{q^2}{2M}\right)$$

and therefore that the inelastic electron-scattering cross section is proportional to the photo absorption cross section $\sigma_\gamma(\nu)$.

3. (a) Extract the vector (as opposed to axial) $\Delta S = 0$ piece of the Adler neutrino sum rule.

(b) Evaluate the elastic contribution to σ_{1v} proportional to $\delta\left(\nu + \frac{q^2}{2M}\right)$ in in terms of isovector electromagnetic form factors, defined as follows

$$\frac{P_o}{M} \langle N|J_\mu(o)|P\rangle = \bar{u}(N) \left[\gamma_\mu F_{1v}\left(q^2\right) + \frac{\left(\kappa_p - \kappa_n\right)}{4M} \sigma_{\mu\nu} q^\nu F_{2v}\left(q^2\right)\right] u(P)$$

(c) By considering the limit $q^2 \longrightarrow 0$, derive the Cabibbo-Radicati sum rule

$$\frac{1}{3}\langle r^2\rangle_{F_1} - \frac{\left(\kappa_p - \kappa_n\right)^2}{4M^2} = \frac{1}{\pi}\int_o^\infty \frac{d\nu}{\nu}\left[\sigma^{\gamma-}(\nu) - \sigma^{\gamma+}(\nu)\right]$$

with $\sigma^{\gamma\pm}(\nu)$ defined as in (b).

4. Consider Coulomb scattering of a relativistic electron from a system of identical point charges (non-relativistic) bound by a static potential $V\left(r_i - r_j\right)$. The

charge density operator is

$$j_o(\underset{\sim}{x}) = \sum_{i=1}^{Z} \delta\left(\underset{\sim}{x} - \underset{\sim}{x}_i\right)$$

Derive the sum rule

$$\int_0^\infty dq_o \frac{d\sigma}{d\underset{\sim}{q}^2 \, dq_o} = \left[Z + Z(Z-1) f_c\left(q^2\right)\right] \frac{4\pi\alpha^2}{\underset{\sim}{q}^4} \cos^2 \frac{\theta}{2}$$

and determine $f_c\left(q^2\right)$ in terms of the ground state wave function of the Z-particle system.

5. Consider a single particle in a one-dimensional harmonic oscillator potential and discuss how the sum rule in problem 4 is satisfied. In particular what are the important states at high q^2 ??

REFERENCES

1. S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).
2. N. Cabibbo and L. Radicati, Phys. Letters 12, 697 (1965).
3. S. Fubini, G. Furlan and A. Rossetti, Nuovo Cimento 40, 1171 (1965) and Nuovo Cimento 43A, 161 (1966).
4. C. Callan and S. Treiman, Phys. Rev. Letters 16, 153 (1966).
5. For example, see Ref. 1, and R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966).
6. M. Gell-Mann, Phys. Rev. 125, 1062 (1962); Physics 1, 63 (1964).
7. S. Adler, Phys. Rev. 143, 1144 (1966).
8. J. Bjorken, Phys. Rev. Letters 16, 408 (1966).
9. For a review, see T. deForest and J. D. Walecka, Advances in Physics 15, 1 (1966).
10. See,e.g., L. I. Schiff, Phys. Rev. Letters 17, 714 (1966) or M. Gell-Mann, Proceedings of the 13th International Conference on High Energy Physics, (University of California Press, Berkeley, California 1966).
11. A. Cone, W. Chen, J. Dunning, G. Hartwig, N. Ramsey, J. Walker and R. Wilson, Phys. Rev. Letters 14, 326 (1965).
12. S. Berman and S. Drell, Phys. Rev. 133, B791 (1964) have discussed this model for photoproduction.
13. J. Bjorken and J. D. Walecka, Annals of Physics 38, 35 (1966).
14. J. Bjorken, Phys. Rev. (to be published).
15. T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).
16. R. Feynman, M. Gell-Mann and G. Zweig, Phys. Rev. Letters 13, 678 (1964).
17. K. Gottfried, Phys. Rev. Letters 18, 1174 (1967).

18. F. Gilman and H. Schnitzer, Phys. Rev. 150, 1362 (1966).

19. S. Adler and F. Gilman, Phys. Rev. 156, 1598 (1967).

20. P. Jain and P. McNulty, Phys. Rev. Letters 14, 611 (1965).

21. For a review see Proceedings of the International Conference on Weak Interactions, Argonne National Laboratory, Argonne, Illinois (1965).

22. H. Quinn, Ph.D. Thesis, Stanford University, Stanford, California (1967); J. Bjorken and H. Quinn, (to be published).

23. J. Bjorken, Phys. Rev. 148, 1467 (1966).

24. E. Abers, R. Norton and D. Dicus, Phys. Rev. Letters 18, 676 (1967).

25. E. Abers, R. Norton, D. Dicus and H. Quinn, Phys. Rev. (to be published).

26. K. Johnson, F. Low and H. Suura, Phys. Rev. Letters 18, 1224 (1967).

27. N. Cabibbo, L. Maiani and R. Preparata (to be published).

28. S. Berman and A. Sirlin, Ann. Phys. (N.Y.) 20, 20 (1962).

29. J. Freeman in reference 21, Print 66-1435, PRL 16, 872, 20/25837; A. Sirlin, Phys. Rev. Letters 16, 872 (1966).

30. N. Cabibbo, Proceedings of the 13th International Conference on β-Decay High Energy Physics; PCAC, Nambu-Shrauner: 128, 862; B. L. Idfe, JETP 4, 254 (1966); PCAC and Electroproduction: Adler and Gilman, PR 152, 1460.