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Wave propagation in axion electrodynamics

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Abstract In this paper, the axion contribution to the electromagnetic wave propagation is studied. First we show how the axion electrodynamics model can be embedded into a premetric formalism of Maxwell electrodynamics. In this formalism, the axion field is not an arbitrary added Chern–Simon term of the Lagrangian, but emerges in a natural way as an irreducible part of a general constitutive tensor. We show that in order to represent the axion contribution to the wave propagation it is necessary to go beyond the geometric approximation, which is usually used in the premetric formalism. We derive a covariant dispersion relation for the axion modified electrodynamics. The wave propagation in this model is studied for an axion field with timelike, spacelike and null derivative covectors. The birefringence effect emerges in all these classes as a signal of Lorentz violation. This effect is however completely different from the ordinary birefringence appearing in classical optics and in premetric electrodynamics. The axion field does not simply double the ordinary light cone structure. In fact, it modifies the global topological structure of light cones surfaces. In CFJ-electrodynamics, such a modification results in violation of causality. In addition, the optical metrics in axion electrodynamics are not pseudo-Riemannian. In fact, for all types of the axion field, they are even non-Finslerian.

Keywords Axion electrodynamics, Wave propagation, Birefringence

1 Introduction

Axion electrodynamics, i.e., the standard electrodynamics modified by an additional axion field [1], is a subject of a growing theoretical and experimental interest. In particular, such a modification provides a theoretical framework for a possible violation of parity and Lorentz invariance—the Carroll–Field–Jackiw (CFJ)

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model [2; 3]. For recent developments of this model, see [4] and the reference given therein. Also the non-abelian extensions of the axion modified electrodynamics for the Standard Model [5] and gravity [6] were worked out.

Although Lorentz invariance is a basic assumption of the standard classical and quantum field theory, in quantum gravity and string theory this invariance is probably violated. One believes that the low-energy manifestation of Lorentz violation can be observed in experiments with the electromagnetic waves. It justifies the importance of examination of the theoretical aspects of axion contributions to the light propagation effects. Although this problem was investigated intensively, we apply here an alternative approach based on a metric-free (premetric) formulation of electrodynamics, see [7] and the reference given therein. A characteristic feature of such a construction is that the axion field is not involved by hand (merely as an additional term in the Lagrangian). Alternatively, in the premetric formalism, axion emerges as a necessary and natural partner of the standard photon variable.

In fact, there is a certain contradiction between the premetric electrodynamics and the CFJ-model. On one hand, in the premetric construction, one usually states that the axion component does not alternate the wave propagation at all [7]. On the other hand, the Lorentz violation in the CFJ-model is explicitly manifested in a modification of the standard light cone. In fact, this contradiction comes from the specific geometric optics approximation which is usually applied in the premetric electrodynamics. A constant axion field indeed does not contribute to the wave propagation. To describe the wave propagation in the CFJ-model one has to go beyond the geometric optics approximation and take into account the first order derivatives of the axion field [8; 9]. With this modification, two constructions yield the equivalent results.

The original CFJ-model is based on the standard special relativistic electrodynamics Lagrangian modified by an additional Chern–Simon term [2]

$$L = -\frac{1}{4}F_{ij}F^{ij} - \frac{1}{4}p_i\epsilon^{ijkl}A_j\partial_k A_l + A_i j^i, \quad (1)$$

where p_i is a constant time-like vector. In fact, a special parametrization $p_i = (\mu, 0, 0, 0)$ is usually used. Note that, in such a construction, the Lorentz violation is involved by hand. The spatial $SO(3)$ -invariance and gauge invariance are preserved however. It was already pointed out in [2] that the model can be equivalently reformulated in an explicitly gauge invariant form

$$L = -\frac{1}{4}F_{ij}F^{ij} + \psi\epsilon^{ijkl}F_{ij}F_{kl} + A_i j^i, \quad (2)$$

where ψ is a pseudoscalar axion field. It is related to the vector p_i as

$$\psi_{,i} = p_i, \quad \text{or} \quad \psi = \mu t. \quad (3)$$

In the current paper, we use a Lagrangian similar to (2). We will not restrict, however, to an axion field with constant first order derivatives and will not require it to be timelike. Moreover, we will consider a model on a curved pseudo-Riemannian spacetime. Our approximation will be based, however, on an assumption that the

gravitational field changes much more slowly than the axion field. This restriction considerably simplifies the calculations and, hopefully, does not change the results, at least, qualitatively.

The current paper is devoted to the 60th birthday of Professor Bahram Mashhoon. His permanent interest and considerable contribution to the study of wave propagation effects are well known, see, for instance [10; 11]. The methods developed by Mashhoon will be useful also in the axion modified electrodynamics.

2 The axion modified electrodynamics model

Let us briefly describe how the axion modified electrodynamics is embedded in the framework of the premetric approach. We start with two independent antisymmetric tensor fields, the field strength tensor F_{ij} and the excitation field \mathcal{H}^{ij} . The latter field is a pseudotensor density of weight (+1). The flux conservation law (the first Maxwell equation) is postulated,

$$\varepsilon^{ijkl} F_{ij,k} = 0. \quad (4)$$

Here the Roman indices range from 0 to 3, ε^{ijkl} and ε_{ijkl} are the Levi-Civita permutation tensors normalized with $\varepsilon^{0123} = -\varepsilon_{0123} = 1$, the commas denote the ordinary partial derivatives. Due to (4), the field strength tensor is expressed in term of the standard vector potentials A_i ,

$$F_{ij} = \frac{1}{2} (A_{i,j} - A_{j,i}). \quad (5)$$

A local linear homogeneous constitutive relation between the fields F_{ij} and \mathcal{H}^{ij} ,

$$\mathcal{H}^{ij} = \frac{1}{2} \chi^{ijkl} F_{kl}, \quad (6)$$

is assumed. By definition, the constitutive pseudotensor χ^{ijkl} must respect the symmetries of the fields F_{ij} and \mathcal{H}^{ij} ,

$$\chi^{ijkl} = \chi^{[ij]kl} = \chi^{ij[kl]}. \quad (7)$$

Hence it has, in general, 36 independent components. The irreducible decomposition of this tensor under the group of linear transformations involves three independent pieces. One of these three pieces is the axion field, which is a subject of our interest.

The high number of components of χ^{ijkl} allows to describe a wide range of physical effects. The axion field, however, adds only one addition component to the standard electrodynamics. So, in order to embed the axion electrodynamics into the premetric approach, one has to restrict the number of the independent components. The first restriction comes from the Lagrangian formulation of the model. We assume the action integral to be of the standard form

$$\mathcal{A} = \int_M (F_{ij} \mathcal{H}^{ij} + A_i \mathcal{J}^i) d^4x. \quad (8)$$

When (6) is substituted, the action takes the form

$$\mathcal{A} = \int_M \left(\frac{1}{2} \chi^{ijkl} F_{ij} F_{kl} + A_i \mathcal{J}^i \right) d^4x. \quad (9)$$

Note, however, that, in contrast to the ordinary textbooks formulation, the electromagnetic current \mathcal{J}^i and the excitation field \mathcal{H}^{ij} are pseudotensorial densities of weight +1 (see Appendix for definition and details). Thus the Lagrangian (8) is general covariant and admits arbitrary transformations of coordinates. The constitutive pseudotensor involved in (8) respects an additional symmetry

$$\chi^{ijkl} = \chi^{klij}. \quad (10)$$

This condition removes 15 independent components of χ^{ijkl} which compose the so-called skewon part of the constitutive tensor. The remaining quantity of 21 independent components is decomposed irreducibly to a principle part of 20 components plus one component that represents the axion field. In contrast to the Lagrangian (2), the premetric formulation (8) does not involve an additional axion term in the Lagrangian. In fact, such a term is hidden in the constitutive tensor.

The variation of (8) with respect to the vector potentials A_i yields the second Maxwell equation

$$\mathcal{H}^{ij}, \quad j = \mathcal{J}^i, \quad \text{or} \quad \frac{1}{2} \left(\chi^{ijkl} F_{kl} \right)_{,j} = \mathcal{J}^i. \quad (11)$$

Observe that this general construction, is explicitly gauge invariant. As usually, the charge conservation law $\partial_i \mathcal{J}^i = 0$ is a consequence of the field equation (11).

The standard electrodynamics is reinstated in this formalism by a special Maxwell constitutive tensor

$$^{(\text{Max})} \chi^{ijkl} = \left(g^{ik} g^{jl} - g^{il} g^{jk} \right) \sqrt{-g}. \quad (12)$$

Here g^{ij} is a metric tensor with the signature $\{+, -, -, -\}$ and with the determinant g . To involve the axion field contribution to the standard electromagnetic Lagrangian, it is enough now to consider a slightly modified constitutive tensor of the following form

$$\chi^{ijkl} = ^{(\text{Max})} \chi^{ijkl} + \psi \epsilon^{ijkl}. \quad (13)$$

Here the axion ψ is a pseudoscalar field. It is invariant under transformations of coordinates with positive determinant and changes its sign under transformations with negative determinant.

The constitutive tensor (13) is not merely a modification of the standard expression (12). In fact, it is an irreducible decomposition of a general constitutive tensor χ^{ijkl} . The Lagrangian formulation removes the skewon part of it. An additional requirement of closeness, see [7], restricts the principle part of 20 independent components to a pure metrical expression. So the axion part appears in this formalism as a natural and necessary ingredient of a general construction. To simplify our consideration we will treat the axion field only phenomenologically. An additional dynamical axion Lagrangian can be readily added to (8).

3 Wave solution

To study the wave propagation in the axion modified model, we assume the electromagnetic current to be equal to zero. Substitute (5) into the second field equation (11) to rewrite it as

$$\left(\chi^{ijkl}A_{k,l}\right)_{,j} = 0, \quad (14)$$

or, equivalently, as

$$\chi^{ijkl}A_{k,lj} + \chi^{ijkl}_{,j}A_{k,l} = 0. \quad (15)$$

In this paper, we will apply the following approximation

$$\chi^{ijkl}_{,j} = \psi_{,j}\epsilon^{ijkl}. \quad (16)$$

In other words, we restrict to the spacetime regions where the gravitational field varies slowly as compared to the change of the pseudoscalar field. In particular, our analysis will be exact for an axion field considered on a flat Minkowski space.

Substituting (13) and (16) into (15) we get

$$\left(g^{ik}g^{jl} - g^{il}g^{jk}\right)A_{k,lj}\sqrt{-g} + \psi_{,j}\epsilon^{ijkl}A_{k,l} = 0. \quad (17)$$

We are looking for a plane monochromatic wave solution of the Eq. (17). Write it as

$$A_k = \text{Re}\left(a_k e^{iq_m x^m}\right). \quad (18)$$

Here the amplitude a_k and the wave covector q_m do not depend on a point. Both quantities can be complex, the physical solution A_k is equal to the real part of the corresponding complex expression. Since (17) is a linear field equation with real coefficients, it is possible to deal, as usual, with the complex valued expression $A_k = a_k e^{iq_m x^m}$ itself. Substituting this Ansatz in (17) we have

$$\left(g^{ik}g^{jl} - g^{il}g^{jk}\right)\sqrt{-g}q_j q_l a_k - i\psi_{,j}q_l \epsilon^{ijkl}a_k = 0. \quad (19)$$

This tensorial equation is represented by a linear system of four equation for four components of the covector a_k . Write it briefly as

$$M^{ij}a_j = 0, \quad (20)$$

where

$$\begin{aligned} M^{ij} &= \left(g^{ij}g^{kl} - g^{il}g^{jk}\right)\sqrt{-g}q_k q_l + i\psi_{,k}q_l \epsilon^{ijkl} \\ &= \left(g^{ij}q^2 - q^i q^j\right)\sqrt{-g} + i\psi_{,k}q_l \epsilon^{ijkl}. \end{aligned} \quad (21)$$

Observe two evident relations that hold due to the definition of the matrix M^{ik}

$$M^{ij}q_i = 0, \quad M^{ij}q_j = 0. \quad (22)$$

These relations can be given a pure physical sense: The former one represents the charge conservation law, while the latter relation represents the gauge invariance of the field equation.

The linear system (20) has a non-zero solution if and only if its determinant equal to zero. For the system (20), this condition holds identically, which can be seen even without explicit calculation of the determinant. Indeed, the identities (22) express linear relation between the rows (and between the columns) of the matrix M^{ik} . So this matrix is singular. Moreover, (22_b) also means that the linear system (20) has a non-zero solution of the form

$$a_j = Cq_j \tag{23}$$

with an arbitrary constant C . This solution is evidently unphysical since it corresponds to the gauge invariance of the field equations. To describe an observable physically meaningful situation we must have an additional linear independent solution of (20).

4 Dispersion relation

The linear system (19) has two linear independent solutions (one for gauge and one for physics) if and only if its matrix M^{ij} is of rank 2 or less. An algebraic expression of this requirement is

$$A_{ij} = 0, \quad (24)$$

where A_{ij} is the adjoint matrix. This matrix is obtained by removing the i -th row and the i -th column from the original matrix. The determinants of the retaining 3×3 matrices are calculated and assembled in a new matrix A_{ij} . The entries of the adjoint matrix are expressed via the entries of the matrix M^{ij} as

$$A_{ij} = \frac{1}{3!} \epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} M^{i_1 j_1} M^{i_2 j_2} M^{i_3 j_3}. \quad (25)$$

Note that for a covariant tensor of a rank $(2,0)$, the adjoint matrix is a contravariant tensor of a rank $(0,2)$. Since the adjoint matrix has, in general, 16 independent components it seems that we have to require 16 independent conditions. The following algebraic fact [12] shows that the situation is rather simpler.

Proposition *If a square $n \times n$ matrix M^{ij} satisfies the relations*

$$M^{ij} q_i = 0, \quad M^{ij} q_j = 0 \quad (26)$$

for some nonzero vector q_i , its adjoint matrix A_{ij} is represented by

$$A_{ij} = \lambda(q) q_i q_j. \quad (27)$$

Consequently, instead of 16 conditions (24), we have to require only one condition

$$\lambda(q) = 0. \quad (28)$$

Recall that this condition is necessary for existence a physically meaningful solution of the generalized wave equation, so it is a generalized dispersion relation.

We calculate now the adjoint matrix for the axion modified electrodynamics model. Substituting (21) into (25) we get

$$A_{ij} = \frac{1}{3!} \epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \left(\mathcal{M}^{i_1 j_1} + i \psi_{,k_1} q_{l_1} \epsilon^{i_1 j_1 k_1 l_1} \right) \times \left(\mathcal{M}^{i_2 j_2} + i \psi_{,k_2} q_{l_2} \epsilon^{i_2 j_2 k_2 l_2} \right) \left(\mathcal{M}^{i_3 j_3} + i \psi_{,k_3} q_{l_3} \epsilon^{i_3 j_3 k_3 l_3} \right), \quad (29)$$

where a notation

$$\mathcal{M}^{ij} = (g^{ij} q^2 - q^i q^j) \sqrt{-g} \quad (30)$$

for the pure Maxwell part of the matrix M^{ij} is involved.

Calculate now in turn the entries of (29) as the powers of the imaginary unit. The calculations are considerable simplified when we take into account that due to the Proposition above the result has to be a symmetric matrix. It means that all antisymmetric contributions to A_{ij} are canceled. The free in i term, i.e., the pure Maxwell term, takes the form

$$\frac{1}{3!} \epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \mathcal{M}^{i_1 j_1} \mathcal{M}^{i_2 j_2} \mathcal{M}^{i_3 j_3} = -\sqrt{-g} q^4 q_i q_j. \quad (31)$$

The i -term takes the form

$$i(1/2) \epsilon_{ii_1 i_2 i_3} \epsilon_{jj_1 j_2 j_3} \mathcal{M}^{i_1 j_1} \mathcal{M}^{i_2 j_2} \psi_{,k_3} q_{l_3} \epsilon^{i_3 j_3 k_3 l_3}. \quad (32)$$

Since the Maxwell matrix \mathcal{M}^{ij} is symmetric, this expression is antisymmetric in the indices i, j and does not give a contribution to the adjoint matrix. In fact, explicit calculations show that the expression (32) vanishes identically. The i^2 -term is given by

$$\begin{aligned} & i^2 (1/2) \varepsilon_{ii_1 i_2 i_3} \varepsilon_{jj_1 j_2 j_3} \mathcal{M}^{i_1 j_1} \psi_{,k_2} q_{l_2} \varepsilon^{i_2 j_2 k_2 l_2} \psi_{,k_3} q_{l_3} \varepsilon^{i_3 j_3 k_3 l_3} \\ & = -\sqrt{-g} \left[(\psi_{,m} \psi^{,m}) q^2 - (\psi_{,m} q^m)^2 \right] q_i q_j. \end{aligned} \quad (33)$$

The i^3 -term takes the form

$$(i)^3 (1/2) \varepsilon_{ii_1 i_2 i_3} \varepsilon_{jj_1 j_2 j_3} \psi_{,k_1} \psi_{,k_2} \psi_{,k_3} q_{l_1} q_{l_2} q_{l_3} \varepsilon^{i_1 j_1 k_1 l_1} \varepsilon^{i_2 j_2 k_2 l_2} \varepsilon^{i_3 j_3 k_3 l_3} \quad (34)$$

This expression is evidently antisymmetric in the indices i, j so it does not give a contribution to the symmetric matrix A_{ij} . In fact, it is equal to zero.

Consequently we have the adjoint matrix in the following form

$$A_{ij} = -\sqrt{-g} \left[q^4 + (\psi_{,m} \psi^{,m}) q^2 - (\psi_{,m} q^m)^2 \right] q_i q_j. \quad (35)$$

Thus the dispersion relation for the electromagnetic waves in the axion electrodynamics is expressed as

$$q^4 + (\psi_{,m} \psi^{,m}) q^2 - (\psi_{,m} q^m)^2 = 0. \quad (36)$$

Observe some characteristic features of this equation.

1. In Minkowski space, for an axion field with a constant covector of derivatives, it coincides with the dispersion relation expression given in [2].
2. The axion dispersion relation (36) is essentially different from the general covariant dispersion relation appearing in the premetric electrodynamics. The premetric dispersion relation is a quartic homogeneous polynomial in the wave covector variable q . Its general form is

$$\mathcal{G}^{ijkl} q_i q_j q_k q_l = 0. \quad (37)$$

Certainly the homogeneity is originated in the geometric approximation used for its derivation. The quartic polynomial of the axion electrodynamics (36) is not homogeneous, so it provides some additional types of light cones structure. In particular, the birefringence effect here comes from the derivatives of the media parameters. On the contrary, in the premetric electrodynamics as well as in the classical crystal optics, the birefringence effect comes from the tensorial nature of the parameters of the media.

3. Another interesting feature of the dispersion relation (36) is that it is, in fact, a general covariant expression. Indeed it is invariant under general pointwise transformations of coordinates (with positive or negative determinant). This is in spite of the fact that we used for its derivation the ansatz (18) which is only special relativistic.
4. The relation (36) is invariant under the transformation $q^i \rightarrow -q^i$. Thus the light cones structure has to be PT invariant.
5. The relation (36) is invariant under the transformation of the field $\psi^i \rightarrow -\psi^i$. Thus the light cones have to be similarly oriented relative to the vector ψ^i .

6. A quartic expression dispersion relation (36) is sometimes factored to a product of two second order polynomials. It is easy to see that the relation (36) cannot be factored in covariant way. Such factorization is possible, however, in special coordinates. Since (36) is not homogeneous, at least one of these quadratic factors is not homogeneous.
7. For ordinary electrodynamics in vacuum, the factors are homogeneous and coincide. This unique homogeneous quadratic factor corresponds to unique Minkowski metric. For general anisotropic media, in (geometric approximation) two factors are homogeneous but different one from another. This case is formulated in term of two optical metrics, both pseudo-Riemannian. In axion electrodynamics, at least one factor is necessary inhomogeneous. This factor cannot be reformulated in term of an ordinary pseudo-Riemannian metric. In fact, it does not even correspond to a general Finslerian metric, which in general appear in premetric electrodynamics [13; 14].

5 Special axion fields

5.1 Axion field on Minkowski spacetime

In this section, we restrict for simplicity to the Minkowski spacetime. In the Cartesian coordinates, $g^{ij} = \text{diag}(1, -1, -1, -1)$ with $g = -1$, so the dispersion relation (36) takes the form

$$q^4 + (\psi_m \psi^m) q^2 - (\psi_m q^m)^2 = 0. \quad (38)$$

Note that now all scalar products are taken with respect to the constant Minkowski metric. We apply the (1 + 3) splitting and denote

$$q = (w, \mathbf{k}), \quad k = |\mathbf{k}|, \quad (39)$$

and

$$\psi_i = (\mu, \mathbf{m}), \quad m = |\mathbf{m}|. \quad (40)$$

Denote by α the angle between the vectors \mathbf{m} and \mathbf{k} . For a complex vector q^i , the usual analytic extension for α is assumed. Due to the symmetry of (38) under the reflection $\psi_i \rightarrow -\psi_i$, we can deal locally with the case $\mu > 0$.

In the (1 + 3) notation, the dispersion relation (38) takes the form

$$(w^2 - k^2)^2 + (\mu^2 - m^2)(w^2 - k^2) - (w\mu + mk \cos \alpha)^2 = 0. \quad (41)$$

It is useful to express the dispersion relation in term of the phase velocity $v_p = w/k$

$$v_p^4 - v_p^2 \left(2 + \frac{m^2}{k^2} \right) - 2v_p \frac{\mu m}{k^2} \cos \alpha + \left(1 - \frac{\mu^2}{k^2} + \frac{m^2}{k^2} \sin^2 \alpha \right) = 0. \quad (42)$$

We observe now that, in the left hand side of (42), the dependence on the angle α can be removed only if the vector \mathbf{m} can be taken equal to zero. Such coordinates can be chosen if and only if the 4-covector ψ_i is timelike. In this case, the $SO(3)$

invariance is preserved. Alternatively, for an arbitrary null or spacelike covector $\psi_{,i}$, the rotational symmetry is violated.

Additionally, let the term linear in v_p cannot be removed, i.e., the parameters m or μ cannot be chosen be equal to zero. Consequently, v_p and $(-v_p)$ cannot satisfy simultaneously the same dispersion relation. It means that the future and the past light cones are not identical. Thus the time inversion symmetry (T -invariance) is violated. Since the whole Eq. (38) is PT -invariant, the parity invariance (P -invariance) is also violated.

The axion field and the covector of its derivatives are assumed to change smoothly in the whole spacetime. Thus the spacetime itself is separated to distinct regions with different norms of the covector ψ_m . In every specific region, special coordinates can be chosen in order to simplify the parametrization of the covector field ψ_m .

5.2 Axion field with a timelike derivative

This model and its physics consequences was studied in the original version of axion modified electrodynamics—the Carroll–Field–Jackiw model [2]. Consider a spacetime region, where the derivatives of the axion field compose a timelike covector

$$\psi_{,i}\psi^{,i} > 0. \quad (43)$$

Choose in this region the time coordinate axis to be directed along the covector $\psi_{,i}$. Consequently, this covector is parameterized now as

$$\psi_{,i} = (\mu, 0, 0, 0). \quad (44)$$

Due to the symmetries of the dispersion relation (36), we can, without lose of generality, require $\mu > 0$. In the original CFJ-model, the axion field was given as $\psi = \mu t$ with a constant parameter μ . Observe, however, that only the first order derivatives of the axion field are involved in the dispersion relation (36). So, in fact, we can deal with a more general case, where μ is an arbitrary function of a point.

Substituting (44) into (41) we get

$$(w^2 - k^2 - \mu k)(w^2 - k^2 + \mu k) = 0. \quad (45)$$

The solutions of this equation are

$$^{(1)}w = \sqrt{k^2 + \mu k}, \quad ^{(2)}w = -\sqrt{k^2 + \mu k}, \quad (46)$$

$$^{(3)}w = \sqrt{k^2 - \mu k}, \quad ^{(4)}w = -\sqrt{k^2 - \mu k}. \quad (47)$$

Thus, for $k > \mu$, we have four distinct real solutions: two positive and two negative. For $k = \mu$, there are two real solutions of opposite signs and one double solution equal to zero. For $k < \mu$, two solutions are real and two are pure imaginary.

Geometrically, the solutions (46, 47) define two distinct double hypersurfaces

$$w^2 - k_1^2 - k_2^2 - k_3^2 - \mu \sqrt{k_1^2 + k_2^2 + k_3^2} = 0. \quad (48)$$

Fig. 1 The graphs represent two hypersurfaces corresponding to CFJ modified electrodynamics

and

$$w^2 - k_1^2 - k_2^2 - k_3^2 + \mu \sqrt{k_1^2 + k_2^2 + k_3^2} = 0. \quad (49)$$

We depict on Fig. 1 these hypersurfaces in the coordinates (k_1, k_2, w) (the third coordinate k_3 is suppressed, as usual). The first picture corresponds to the solution (46). It is topologically equivalent to the ordinary light cone. In particular, two cones are joined by a unique point ($w = 0, \mathbf{k} = 0$). When this point is removed the interior region bounded by the hypersurface is separated to two disjoint parts—the future and the past spacetime regions. The second hypersurface corresponding to (47) is topologically different from the ordinary light cone structure. It consists of two pieces which are joined by a sphere ($k = \mu, w = 0$) and by a point ($k = 0, w = 0$). Consequently the interior region cannot be separated into two disjoint parts even if the origin is removed. It means that the future (the upper region) and the past (the downer region) always connected and cannot be disjoint. In other words, the causality on this branch is violated.

Since the Eqs. (48,49) involve a term linear in the covector k the corresponding light hypersurfaces cannot be associated with some pseudo-Riemannian optical metrics.

The phase velocities of the waves are given by the expressions

$$^{(1)}v_p = \sqrt{1 + \frac{\mu}{k}}, \quad ^{(2)}v_p = \sqrt{1 - \frac{\mu}{k}}, \quad (50)$$

which coincide with the corresponding formulas of [2]. One of the phase velocities is greater than the speed of light in a vacuum. It increases monotonically when the factor μ/k increases, i.e., when the parameter k tends to zero. The second phase velocity is less than the speed of light in a vacuum and monotonically decreases to zero when the parameter k tends to zero.

Well known that the phase velocity does not completely characterize the energy propagation. Another useful characteristic is the group velocity which is usually thought of as the velocity at which energy is propagated along a wave. This quantity is defined by the derivative

$$v_g = \frac{\partial w}{\partial k}. \quad (51)$$

From (50) we have

$$^{(1)}v_g = \frac{k + \mu/2}{\sqrt{k^2 + \mu k}}, \quad ^{(2)}v_g = \frac{k - \mu/2}{\sqrt{k^2 - \mu k}}. \quad (52)$$

Consequently, both group velocities are superluminal and monotonically increase when the parameter k tends to zero. Observe that on the second branch, both velocities are defined only for $0 \leq \mu/k \leq 1$. For $k \rightarrow \mu$, the phase velocity tends to zero while the group velocity goes to infinity. This behavior corresponds to transmission of energy with infinite velocity, i.e., indicates the runaway modes, see [2; 15].

Fig. 2 The graphs represent two double light hypersurfaces of the spacelike axion modified electrodynamics

Consequently, the waves with $k > \mu$ propagate along two distinct future light hypersurfaces. This behavior corresponds to the birefringence effect known from classical optics. The topological type of the light hypersurfaces is different, however, from the ordinary light cones. For $k < \mu$, the runaway modes emerge.

5.3 Axion field with a spacelike derivative

Let us consider a spacetime region where the axion field has a spacelike covector of derivative

$$\psi_{,m}\psi^{,m} < 0. \quad (53)$$

By transformation of the coordinates, we can choose in the whole region a parametrization

$$\psi_{,i} = (0, m, 0, 0). \quad (54)$$

Also here the parameter m can be considered as a function of a point. We can restrict to $m > 0$. The dispersion relation (41) takes now the form

$$w^2 = k^2 + \frac{m^2}{2} \pm \sqrt{\frac{m^4}{4} + m^2 k^2 \cos^2 \alpha}. \quad (55)$$

Observe that an inequality $w^2 \geq k^2 \geq 0$ holds for (55). Hence this equation for w has four real solutions for every values of parameters. Consequently the runaway solutions are absent.

The corresponding light hypersurfaces are given by

$$w^2 - k_1^2 - k_2^2 - k_3^2 - \frac{m^2}{2} \left(1 + \sqrt{1 + \frac{4k_1^2}{m^2}} \right) = 0. \quad (56)$$

and

$$w^2 - k_1^2 - k_2^2 - k_3^2 - \frac{m^2}{2} \left(1 - \sqrt{1 + \frac{4k_1^2}{m^2}} \right) = 0. \quad (57)$$

These hypersurfaces are depicted on Fig. 2. The first structure is topologically different from the standard one. Indeed, (56) has not real solutions for $w = 0$. Consequently, two branch given by (56) are represented by two disjoint surfaces. In other words there is not, for this lightlike hypersurfaces, a way from the past to the future. The second Eq. (57), for $w = 0$, has a unique solution $k_1 = k_2 = k_3 = 0$. Thus we have on this branch the standard light cone topology with one point joining the past and the future spacetime regions. However, an ordinary definition of the causality is not applicable in this case. Indeed, due to the inhomogeneity of

the dispersion relation, a type of a light trajectory depends on the parametrization. The expressions (56, 57) contain non-polynomial terms. Thus also in this case, the optical metrics cannot be represented in a pseudo-Riemannian form.

In term of the phase velocity (42) is rewritten as

$$v_p^4 - v_p^2 \left(2 + \frac{m^2}{k^2} \right) + \left(1 + \frac{m^2}{k^2} \sin^2 \alpha \right) = 0. \quad (58)$$

Thus the phase velocities are expressed as

$$^{(1)}v_p = \sqrt{1 + \frac{m^2}{2k^2} + \sqrt{\frac{m^2}{k^2} \left(\frac{m^2}{4k^2} + \cos^2 \alpha \right)}} \quad (59)$$

and

$$^{(2)}v_p = \sqrt{1 + \frac{m^2}{2k^2} - \sqrt{\frac{m^2}{k^2} \left(\frac{m^2}{4k^2} + \cos^2 \alpha \right)}}. \quad (60)$$

These expressions depend explicitly on the angle α , so the $SO(3)$ invariance is violated. However, due to the fact that the expression is invariant under the change $v_p \rightarrow -v_p$, the future and the past cone are the same. Thus the T -invariance and consequently the P -invariance are preserved. For small k the first phase velocity (59) goes to infinity while the second phase velocity (60) tends to zero. In the transversal direction, $\alpha = \pi/2$, one of the phase velocities is greater and one is equal to the speed of light in a vacuum.

The group velocities are expressed as

$$v_g = \frac{1}{v_p} \left(1 \pm \frac{2 \cos^2 \alpha}{\sqrt{1 + \frac{4k^2}{m^2} \cos^2 \alpha}} \right). \quad (61)$$

Both expressions tend to zero for small values of k . In transversal direction we have

$$v_g = \frac{1}{v_p}, \quad (62)$$

i.e., one of the group velocities is less and one is equal to the speed of light in vacuum.

5.4 Axion field with a lightlike derivative

Consider a spacetime region, where the derivatives of the axion field compose a null covector

$$\psi_{,m} \psi^{,m} = 0. \quad (63)$$

Fig. 3 The graphs represent two light hypersurfaces of the lightlike axion modified electrodynamics

In this case, the general covariant dispersion relation (36) takes the form

$$q^4 - (\psi_{,m} q^m)^2 = 0. \quad (64)$$

This expression is readily factored as

$$(q^2 - \psi_{,m} q^m)(q^2 + \psi_{,m} q^m) = 0. \quad (65)$$

The expression in the left hand side does not have a defined sign for an arbitrary non zero covector q^m . Thus the non-birefringence condition [16] is explicitly violated. Consequently, the birefringence effect emerges for arbitrary varying null axion fields.

We can choose a parametrization

$$\psi_{,i} = (m, m, 0, 0) \quad m > 0. \quad (66)$$

On a flat Minkowski manifold, the dispersion relation takes now the form

$$(w^2 - k^2)^2 - m^2 (w + k \cos \alpha)^2 = 0, \quad (67)$$

or,

$$w^2 = k^2 \pm m(w + k \cos \alpha), \quad (68)$$

Consequently the light hypersurfaces are expressed as

$$w^2 - k_1^2 - k_2^2 - k_3^2 + m(w + k_1) = 0, \quad (69)$$

and

$$w^2 - k_1^2 - k_2^2 - k_3^2 - m(w + k_1) = 0. \quad (70)$$

The linear terms indicate that these expressions cannot be represented by pseudo-Riemannian metrics.

The Eqs. (68) have four solutions

$$^{(1)}w = \frac{m}{2} + \sqrt{\frac{m^2}{4} + k^2 + mk \cos \alpha} \quad ^{(2)}w = \frac{m}{2} - \sqrt{\frac{m^2}{4} + k^2 + mk \cos \alpha}, \quad (71)$$

and

$$^{(3)}w = -\frac{m}{2} + \sqrt{\frac{m^2}{4} + k^2 - mk \cos \alpha} \quad ^{(4)}w = -\frac{m}{2} - \sqrt{\frac{m^2}{4} + k^2 - mk \cos \alpha}. \quad (72)$$

These expressions are real for every values of the parameters. Consequently, for the axion field with lightlike derivative, the runaway solutions are absent.

The light hypersurfaces are depicted on Fig. 3. The first picture corresponds to (69), while the second one is for (70). The future and the past surfaces are contacted at points

$$w = \pm \frac{m}{2}, \quad k_1 = \pm \frac{m}{2}, \quad k_2 = 0, \quad k_3 = 0. \quad (73)$$

This topological structure is also different from the standard light cone structure.

The electromagnetic waves propagate with two different phase velocities

$$^{(1)}v_p = \frac{m}{2k} + \sqrt{\frac{m^2}{4k^2} + \frac{m}{k} \cos \alpha + 1}, \quad (74)$$

and

$$^{(2)}v_p = -\frac{m}{2k} + \sqrt{\frac{m^2}{4k^2} - \frac{m}{k} \cos \alpha + 1}. \quad (75)$$

When the dimensionless parameter m/k increases, i.e., for small k , the first velocity (74) goes to infinity, while the second one (75) tends to zero.

The group velocities are expressed as

$$^{(1)}v_g = \frac{m \cos \alpha + 2k}{\sqrt{m^2 + 4km \cos \alpha + 4k^2}}, \quad (76)$$

and

$$^{(2)}v_g = \frac{m \cos \alpha - 2k}{\sqrt{m^2 + 4km \cos \alpha - 4k^2}}. \quad (77)$$

For small k they tend to the same value $\cos \alpha \leq 1$.

6 Conclusions

We have considered a general phenomenological model of axion modified electrodynamics. It is shown that the axion modified electrodynamics can be treated as a special case of premetric electrodynamics. In this formalism, the axion field is not involved as an additional Chern–Simon term. Alternatively, it emerges as an irreducible part of a general constitutive tensor. We have derived a covariant dispersion relation of axion electrodynamics. For a varying axion field, it yields a modification of light cone. The birefringence effect indicates violation of the Lorentz invariance for timelike, spacelike and null covector of axion field derivatives. This effect, however, completely different from the ordinary birefringence appearing in classical optics and in the premetric electrodynamics. It can be explicitly seen from the fact that the topological structure of the light hypersurface is different from the ordinary light cone structure. In addition, the optical metrics are not homogeneous in the wave covector so they are even non-Finslerian.

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Appendix A. Tensors and pseudotensors

Since the axion field possesses a special transformational behavior, it is useful to have a precise meaning of all quantities involved in the formalism. Although the notion of weighted tensors and pseudotensors is a classical subject, see [17] and also [7] for a modern treatment, some confusions in the basic definitions can be found in literature. We will characterize the geometrical quantities accordingly to the transformation properties of their components relative to transformations of coordinates.

Let be given a smooth transformation

$$x^i \rightarrow x^{i'} = f^{i'}(x^i) \quad (78)$$

with a transformation matrix

$$L^{i'}_i = \frac{\partial f^{i'}(x^i)}{\partial x^i} \quad (79)$$

and its inverse $L_i^{i'}$. Denote the determinant of the transformation matrix by J . It is expressed by

$$J = \det(L^{i'}_i) = \frac{1}{4!} \varepsilon_{i'j'k'l'} \varepsilon^{ijkl} L^{i'}_i L^{j'}_j L^{k'}_k L^{l'}_l, \quad (80)$$

where ε^{ijkl} and ε_{ijkl} are the permutation symbols. They take the constant values $0, \pm 1$ in all coordinate systems. The relation (80) is equivalent to

$$\varepsilon^{i'j'k'l'} J = \varepsilon^{ijkl} L^{i'}_i L^{j'}_j L^{k'}_k L^{l'}_l, \quad (81)$$

and

$$\varepsilon_{i'j'k'l'} = J \varepsilon_{ijkl} L^{i'}_i L^{j'}_j L^{k'}_k L^{l'}_l. \quad (82)$$

Recall, see [17], the definitions of the extended tensorial objects.

Ordinary tensor has a set of components which are transformed as

$$T^{i' \dots j' \dots} \rightarrow L^{i'}_i \dots L^{j'}_j \dots T^{i \dots j \dots} \quad (83)$$

Tensor density of weight k is a set of components which are transformed as

$$T^{i' \dots j' \dots} \rightarrow \frac{1}{J^k} L^{i'}_i \dots L^{j'}_j \dots T^{i \dots j \dots} \quad (84)$$

So the ordinary tensors are tensor densities of zero weight.

Pseudotensor has a set of components which are transformed as

$$T^{i' \dots j' \dots} \rightarrow (\text{sgn} J) L^{i'}_i \dots L^{j'}_j \dots T^{i \dots j \dots} \quad (85)$$

Pseudotensor density of weight k is a set of components which are transformed as

$$T^{i' \dots j' \dots} \rightarrow \frac{\text{sgn} J}{J^k} L^{i'}_{i'} \dots L^{j'}_{j'} \dots T^{i \dots j \dots} \quad (86)$$

The ordinary pseudotensors are of zero weight.

Let us start with an action

$$\mathcal{A} = \int_M L \text{vol}, \quad (87)$$

which is an ordinary real number. Its integrand, $L \text{vol}$, has to be a scalar valued invariant volume element, i.e., a pseudoscalar density. In the formalism of differential forms, it is an odd (twisted) 4-form, see for instance [18]. Here L is a scalar valued function (Lagrangian) while vol is a special invariant volume element defined from the geometric quantities. On a $4D$ pseudo-Riemannian manifold, the standard invariant volume element is defined by the determinant of the metric tensor $g = \det(g_{ij})$

$$^{(Riem)}\text{vol} = \sqrt{-g} d^4x, \quad d^4x = dx^0 dx^1 dx^2 dx^3. \quad (88)$$

In a pure tensorial form, it is equivalently rewritten as

$$^{(Riem)}\text{vol} = \sqrt{-g} \left(\frac{1}{4!} \varepsilon_{ijkl} dx^i dx^j dx^k dx^l \right). \quad (89)$$

Due to (81), the permutation symbol ε^{ijkl} is a tensor density of weight $(+1)$. Its “inverse” ε_{ijkl} is a tensor density of weight (-1) . Consequently d^4x is a scalar density of weight (-1) .

From the ordinary transformation law for the metric tensor

$$g_{i'j'} = g_{ij} L^i_{i'} L^j_{j'}, \quad (90)$$

one readily has the transformation law for the determinant and its square root

$$g' = \frac{1}{J^2} g, \quad \sqrt{-g'} = \frac{1}{|J|} \sqrt{-g} = \frac{\text{sgn}(J)}{J} \sqrt{-g}. \quad (91)$$

Thus g is a scalar density of weight $(+2)$ while $\sqrt{-g}$ is a pseudoscalar density of weight $(+1)$.

One builds from these two objects a quantity $\tilde{\varepsilon}_{ijkl} = \sqrt{-g} \varepsilon_{ijkl}$, which is a (non-weighted) pseudotensor. Observe that the integrand of the action $L \text{vol}$ has a proper transformation behavior, it is a pseudoscalar of zero weight. Since the expression $L \sqrt{-g}$ is the subject of variation, this pseudoscalar density of weight $(+1)$ is often referred to as the Lagrangian density.

Let us return now to the premetric electrodynamics action

$$\mathcal{A} = \int_M (F_{ij} \mathcal{H}^{ij} + A_i \mathcal{J}^i) d^4x, \quad (92)$$

and, equivalently,

$$\mathcal{A} = \int_M \left(\frac{1}{2} \chi^{ijkl} F_{ij} F_{kl} + A_i \mathcal{J}^i \right) d^4x. \quad (93)$$

Since d^4x is a scalar density of weight (-1) , the scalar integrand (the expression in the parenthesis) has to be treated as a pseudoscalar density of weight $(+1)$. It is constructed from the ordinary tensors F_{ij} and A_i and the pseudotensor densities of weight $(+1)$ — \mathcal{H}^{ij} and \mathcal{J}^i . On a pseudo-Euclidean manifold, the ordinary tensors H^{ij} and J^i are extracted from them by $\mathcal{H}^{ij} = H^{ij} \sqrt{-g}$ and $\mathcal{J}^i = J^i \sqrt{-g}$. Consequently, χ^{ijkl} , which is defined by $\mathcal{H}^{ij} = \chi^{ijkl} F_{kl}$, is a pseudotensor density of weight $(+1)$.

In axion electrodynamics, the constitutive tensor is given by

$$^{(\text{Max})} \chi^{ijkl} = \left(g^{ik} g^{jl} - g^{il} g^{jk} \right) \sqrt{-g} + \psi \epsilon^{ijkl}. \quad (94)$$

Since $\sqrt{-g}$ is a pseudoscalar density of weight $(+1)$, the first term is a pseudotensor density of weight $(+1)$. In the second term, ϵ^{ijkl} is a tensor density of weight $(+1)$. Consequently, ψ is a pseudoscalar. In physics literature, it is called axion.

The matrix $M^{ij} = \chi^{ijkl} q_k q_l$ is a pseudotensor density of weight $(+1)$. Its determinant is a scalar density of weight $(+2)$. Consequently the adjoint matrix A_{ij} is a pseudotensor density of weight $(+1)$ while the function $\lambda(q)$ is a pseudoscalar density of weight $(+1)$. The photon propagator derived for premetric electrodynamics in [12] is a pseudotensor of weight (-1) .

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