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Phenomenology of SO(10) Grand Unified Theories

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Abstract

Although the Standard Model (SM) of particle physics describes observations well, there are several shortcomings of it. The most crucial of these are that the SM cannot explain the origin of neutrino masses and the existence of dark matter. Furthermore, there are several aspects of it that are seemingly *ad hoc*, such as the choice of gauge group and the cancellation of gauge anomalies.

These shortcomings point to a theory beyond the SM. Although there are many proposed models for physics beyond the SM, in this thesis, we focus on grand unified theories based on the SO(10) gauge group. It predicts that the three gauge groups in the SM unify at a higher energy into one, which contains the SM as a subgroup. We focus on the Yukawa sector of these models and investigate the extent to which the observables such as fermion masses and mixing parameters can be accommodated into different models based on the SO(10) gauge group. Neutrino masses and leptonic mixing parameters are particularly interesting, since SO(10) models naturally embed the seesaw mechanism.

The difference in energy scale between the electroweak scale and the scale of unification spans around 14 orders of magnitude. Therefore, one must relate the parameters of the SO(10) model to those of the SM through renormalization group equations. We investigate this for several different models by performing fits of SO(10) models to fermion masses and mixing parameters, taking into account thresholds at which heavy right-handed neutrinos are integrated out of the theory. Although the results are in general dependent on the particular model under consideration, there are some general results that appear to hold true. The observables of the Yukawa sector can in general be accommodated into SO(10) models only if the neutrino masses are normally ordered and that inverted ordering is strongly disfavored. We find that the observable that provides the most tension in the fits is the leptonic mixing angle θ_{23}^{ℓ}, whose value is consistently favored to be lower in the fits than the actual value. Furthermore, we find that numerical fits to the data favor type-I seesaw over type-II seesaw for the generation of neutrino masses.

Key words: Grand unified theories, renormalization group equations, neutrino masses, gauge coupling unification.

Sammanfattning

Trots att partikelfysikens standardmodell beskriver observationer väl så har den ett antal brister. De mest framträdande av dessa är att standardmodellen inte kan beskriva ursprunget av neutrinernas massa samt existensen av mörk materia. Dessutom verkar ett antal av dess aspekter *ad hoc*, så som valet av gaugegrupp och att gaugeanomalierna oväntat tar ut varandra

Dessa brister pekar mot en teori bortom standardmodellen. Trots att det finns många förslag till modeller för fysik bortom standardmodellen så fokuserar vi på storförenade teorier baserade på gaugegruppen SO(10). Den förutspår att de tre gaugegrupperna i standardmodellen förenas vid en hög energiskala. Denna förenade gaugegrupp innehåller standardmodellen som en delgrupp. Vi fokuserar särskilt på Yukawasektorn hos dessa modeller och undersöker i vilken omfattning som observabler såsom fermionmassor och blandningsparametrar kan rymmas i olika modeller baserade på SO(10)-gaugegruppen. Neutrinomassor och leptonska blandningsparametrar är särskilt intressanta eftersom SO(10)-modeller innehåller gungbrädemekanismen.

Skillnaden i energiskala mellan den elektrosvaga skalan och föreningsskalan sträcker sig över 14 storleksordningar. Därför bör man relatera parametrarna i SO(10)-modellen till de i standardmodellen genom renormeringsgruppsekvationer. Vi undersöker detta för ett antal olika modeller genom att utföra numeriska anpassningar av SO(10)-modeller till fermionmassor och blandningsparametrar och tar hänsyn till trösklar vid vilka de tunga högerhänta neutrinerna integreras ut ur teorin. Trots att resultaten i regel beror på den särskilda modellen som studeras så finner vi ett antal resultat som verkar gälla allmänt. Observablerna i Yukawasektorn kan allmänt sett rymmas i SO(10)-modeller endast om neutrinomassorna är normalt ordnade och inverterad ordning är starkt missgynnad. Vi finner att den observabel som är svårast att anpassa till är blandningsvinkeln θ_{23}^{ℓ} vars värde från anpassningarna konsekvent är lägre än det faktiska värdet. Dessutom finner vi att numeriska anpassningar till data gynnar typ-I-gungbrädemekanismen över typ-II-gungbrädemekanismen för genereringen av neutrinomassor.

Nyckelord: Storförenade teorier, renormeringsgruppsekvationer, neutrinomassor, gaugekopplingsförening.

Preface

This thesis is the result of my research at the Department of Physics from August 2017 to October 2019. The first part of the thesis presents an introduction to the subjects relevant to the scientific work of this thesis. These include the Standard Model and its shortcomings, grand unification and SO(10), as well as renormalization group equations and the numerical methods used. The second part contains the three papers that my research has resulted in.

List of papers

The scientific papers included in this thesis are:

Paper (I) [1]
T. Ohlsson and *M. Pernow*
Running of fermion observables in non-supersymmetric SO(10) models
J. High Energy Phys. **11**, 028 (2018)
`arXiv:1804.04560`

Paper (II) [2]
S. M. Boucenna, T. Ohlsson and *M. Pernow*
A minimal non-supersymmetric SO(10) model with Peccei–Quinn symmetry
Phys. Lett. B **792**, 251 (2019)
`arXiv:1812.10548`

Paper (III) [3]
T. Ohlsson and *M. Pernow*
Fits to non-supersymmetric SO(10) models with type I and II seesaw mechanisms using renormalization group evolution
J. High Energy Phys. **06**, 085 (2019)
`arXiv:1903.08241`

The thesis author's contribution to the papers

I participated in the scientific work as well as in the writing of all papers included in this thesis. I am also the corresponding author of all three papers.

1. I modified large parts of the code and ran all numerical computations. All figures were produced by me and I wrote the majority of the paper.

2. All calculations were performed by me and I wrote the code for the numerical computations. I produced all figures and wrote large parts of the paper.

3. I performed all numerical computations, produced all figures, and wrote most of the paper.

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I would like to thank my family for their support during my studies. Thank you to my parents for always supporting and encouraging me and to my brother Jonathan for always sharing his many innovative ideas with me. Thank you to my grandparents for always being supportive and especially to my grandfather for inspiring me to study physics in the first place. Last, but certainly not least, thank you Lara for all your love and support. This thesis is dedicated to you.

Contents

Abstract iii
Sammanfattning iv

Preface **v**
List of papers v
The thesis author's contribution to the papers vi
Acknowledgements vii

Contents **ix**

I Introduction and background material **1**

1 Introduction **3**
1.1 Outline 4

2 The Standard Model **5**
2.1 Overview of the Standard Model 5
2.1.1 Gauge sector 6
2.1.2 Fermionic particle content 6
2.1.3 The Higgs mechanism 8
2.1.4 Fermion masses 10
2.1.5 Parameters of the Standard Model 12
2.2 Open questions and some proposed solutions 12
2.2.1 Observational problems 12
2.2.2 Aesthetic problems 16

3 Grand unified theories and SO(10) **19**
3.1 Gauge coupling unification 20
3.2 Candidate gauge groups 22
3.2.1 Viable gauge groups 24
3.2.2 The SU(5) group 24
3.2.3 The Pati–Salam group 26

3.2.4 The SO(10) group 27
3.3 Aspects of SO(10) model building 28
3.3.1 Symmetry breaking 28
3.3.2 Yukawa sector 30
3.4 Proton decay 33
3.5 Phenomenology in SO(10) models 35
3.6 Current status of SO(10) models 35

4 Renormalization group running and numerical methods 37
4.1 Renormalization group running 37
4.1.1 Regularization 38
4.1.2 Renormalization 39
4.2 Intermediate masses 41
4.3 Numerical procedure 42

5 Summary and conclusions 47

A Group theory 51
A.1 Basics of Lie groups, Lie algebras, and representations 51
A.2 Decompositions of some SO(10) representations 52
A.3 Spinorial representations 59
A.4 Anomalies 61

B Renormalization group equations 63

Bibliography 65

II Scientific papers 83

Part I

Introduction and background material

Chapter 1

Introduction

Particle physics is the branch of physics that deals with the description of reality at the smallest scales and highest energies. The concept of symmetries has been important in particle physics, and in physics in general. Indeed, symmetry aspects are central to the theories of particles physics.

A successful theory of particle physics should describe the particles that we observe and their interactions. The interactions are intimately linked to symmetry aspects of the theory through the concept of gauge invariance. As we impose invariance under local gauge transformations upon the theory, we are led necessarily to interactions of the matter particles via gauge bosons (colloquially known as force carriers).

The centrality of the concept of symmetry in particle physics is an example of mathematical elegance as a guiding principle in the search for the laws of physics. A related manifestation of elegance in physical theories is the unification of phenomena. This has been a motivation since the early days of physics, starting from Isaac Newton who unified the celestial mechanics with the mechanics of everyday objects that we observe on Earth, such as apples. Famously, James Maxwell unified the phenomena of electricity and magnetism into one theory of electromagnetism. Maxwell's theory in turn motivated Albert Einstein to develop his theory of relativity, which unifies space and time into spacetime.

The three forces described by theories of particle physics are the electromagnetic force, the weak nuclear force, and the strong nuclear force. Currently, the best theory of particle physics that we have is the Standard Model (SM). Central to its success is the unification of the electromagnetic force and the weak nuclear force in the electroweak theory.

Predictions that led to the discoveries of new particles provide evidence for the success of the SM and its underlying principles. An example is the discovery of the charm quark from the J/ψ resonance in 1974 by both SLAC and BNL [4, 5]. The strong force was further supported by the discovery of the gluon at DESY in 1979 [6, 7]. In 1995, the top quark was finally discovered at Tevatron [8, 9], thus completing the list of quarks. In the electroweak sector, the W and Z bosons were discovered at CERN in 1983 [10–13]. The most recent discovery, which provided the final piece of the SM, was the Higgs boson, discovered by ATLAS and CMS at CERN in 2012 [14, 15].

To take this concept of unification seriously would mean to unify the electroweak theory and the strong nuclear force in one Grand Unified Theory (GUT), which is the subject of this thesis. Unifying the three forces into one would be a step towards a more fundamental theory of Nature, and would imply that all forces but gravity are in fact different manifestations of the same force. This would extend the SM and provide us with phenomena beyond those predicted by the SM. The fact that gravity is not included, however, means that grand unification does not claim to provide a final theory, but it is a step towards a more fundamental theory of particle physics.

Except for the purely aesthetic motivations for unifying the SM into a GUT, there are several more pragmatic motivations for extending our theories beyond the SM. Despite its successes, it has several shortcomings and open questions. Some of these can be addressed naturally within the framework of GUTs. To ensure that such a model is viable, it must be checked whether it can accommodate the known parameters of the SM. Hence, the work that this thesis is based on has used numerical fits of GUT models to observables of the Yukawa sector of the SM to investigate the viability of different versions such models.

1.1 Outline

This thesis is divided into two parts. Part I contains a description of the theory related to the research work as well as an introduction to the papers. This is divided into five chapters. In Ch. 2, we introduce the main features of the SM as well as some of its shortcomings and open questions. Next, in Ch. 3, we discuss the framework of GUTs and motivate them as a natural extension of the SM. Ch. 4 contains a description of the methods used to investigate the viability of GUTs in the research papers. Finally, in Ch. 5, we conclude this thesis. Following this introduction to the subject, Part II contains the papers upon which this thesis is based.

Chapter 2

The Standard Model

The SM is the description currently used for particle physics. It has so far been extraordinarily powerful in predicting experimental results at the LHC and other particle experiments. However, there are many open questions associated with it. This chapter will summarize the features of the SM relevant for the work in this thesis. Further, we will outline some of the open problems that suggest physics beyond the SM.

2.1 Overview of the Standard Model

The SM is a quantum field theory (QFT) that describes the particles and interactions observed in terms of interactions of quantum fields. It is thus formulated in terms of a Lagrangian density $\mathcal{L}_{\mathrm{SM}}$ which is a function of the fields and their derivatives. Being based on Yang–Mills (YM) theory [16], the SM Lagrangian density satisfies invariance under local non-Abelian gauge transformations belonging to its Lie group, which is[1]

$$\mathcal{G}_{\mathrm{SM}} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y. \tag{2.1}$$

The labels C ("color"), L ("left"), and Y (hypercharge) describe the quantum numbers relevant to each of these gauge groups. The $\mathrm{SU}(3)_{\mathrm{C}}$ factor is responsible for the strong nuclear force through the theory of quantum chromodynamics (QCD) [19–21]. The remaining $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y$ factor is the electroweak theory [22–25].

To further specify the model, one must determine the particle content and their transformation properties under $\mathcal{G}_{\mathrm{SM}}$. Then, the conditions of Lorentz invariance and renormalizability of the theory determines the theory by dictating the allowed terms in the Lagrangian density.

[1]The more general gauge group of the SM is $\mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_Y/\mathcal{Z}$, where $\mathcal{Z}$ is one of the subgroups of $\mathbb{Z}_6$, namely $\mathbb{Z}_6$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, or $\mathbb{1}$ [17]. This does not influence results of typical particle interactions, which only rely on the local structure, but is relevant for topological defects such as monopoles, which depend on the global group structure [18].

2.1.1 Gauge sector

Being a YM theory, the SM gauge group defines the spin-1 vector bosons in the theory. More specifically, each generator of the gauge group corresponds to a gauge boson. These are the force mediators through which the SM gauge interactions occur.

There is one vector field B_μ corresponding to $\mathrm{U}(1)_Y$, since its generator acting on a field is simply the hypercharge of that field. There are three weak gauge bosons W^i_μ, where $i \in \{1, 2, 3\}$, corresponding to the three generators $t^i = \frac{1}{2}\sigma^i$ of $\mathrm{SU}(2)_\mathrm{L}$, where σ^i are the Pauli matrices. In QCD, there are eight gauge bosons, the gluons G^a_μ, where $a \in \{1, \ldots, 8\}$, corresponding to the eight generators $t^a = \frac{1}{2}\lambda^a$, where λ^a are the Gell-Mann matrices. Note that the normalization of the generators used here implies that

$$\mathrm{Tr}[t^m t^n] = \frac{1}{2}\delta^{mn} \tag{2.2}$$

for both SU(2) and SU(3).

From these vector fields, we form the gauge invariant field strength tensors[2]

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.3}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \epsilon^{ijk} W^j_\mu W^k_\nu, \tag{2.4}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_3 f^{abc} G^b_\mu G^c_\nu, \tag{2.5}$$

where g_2 and g_3 are the $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(3)_\mathrm{C}$ gauge coupling constants, respectively, and ϵ^{ijk} and f^{abc} are the structure constants of SU(2) and SU(3), respectively. The kinetic terms in the Lagrangian density for the gauge fields are

$$\mathcal{L}_\mathrm{gauge} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i,\mu\nu} - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}. \tag{2.6}$$

From this and the definitions of the field strength tensors in Eqs. (2.3)–(2.5), it is evident that, in contrast to the hypercharge gauge boson, the gluons and the weak gauge bosons have self-interactions via cubic and quartic couplings. This is due to the non-Abelian nature of these gauge groups and the fact that the gauge bosons carry the corresponding charges.

2.1.2 Fermionic particle content

Matter is composed of spin-1/2 fermions, which are the quarks and leptons in the SM. They can be organized into three generations, which have identical properties except for their masses.

[2]We employ the slight abuse of notation that the two-index tensor $A_{\mu\nu}$ is the field strength tensor corresponding to the vector field A_μ denoted by the same symbol. These two should not be confused.

The SM is a chiral theory, meaning that the interactions of the fermions depend on their chirality. Any spinor ψ may be decomposed into left-handed and right-handed components using the projection operators

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}, \tag{2.7}$$

where $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, in which the γ^μ are the Dirac matrices satisfying the Clifford algebra defined by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$. The left- and right-handed chiral components $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$ of the fermions in the SM transform differently under $\mathcal{G}_{\mathrm{SM}}$.

The quarks carry quantum numbers under all three of the gauge groups of the SM. Specifically, the first generation contains the u and d quarks ("up" and "down") in left-handed and right-handed chiralities. The quarks come in three different color charges, called red, green, and blue, and transform as the **3** representation of $\mathrm{SU}(3)_\mathrm{C}$. Under $\mathrm{SU}(2)_\mathrm{L}$ transformations, the left-handed u_L and d_L quarks form a doublet, $Q_L = (u_L \quad d_L)^T$ while the two right-handed u_R and d_R are singlets. Summarizing, the quarks in the first generation are organized as

$$Q_L = \begin{pmatrix} u_L^r & u_L^g & u_L^b \\ d_L^r & d_L^g & d_L^b \end{pmatrix} \sim (\mathbf{3}, \mathbf{2})_{1/6},$$
$$(u_R^r \quad u_R^g \quad u_R^b) \sim (\mathbf{3}, \mathbf{1})_{2/3}, \quad (d_R^r \quad d_R^g \quad d_R^b) \sim (\mathbf{3}, \mathbf{1})_{-1/3},$$

where the parentheses give their representations under $\mathrm{SU}(3)_\mathrm{C}$ and $\mathrm{SU}(2)_\mathrm{L}$, and the hypercharge is given in the subscript. The next generation contains the c and s ("charm" and "strange") quarks which are identical to u and d except heavier. The third and final known generation similarly contains the heavier t and b ("top" and "bottom") quarks.

The leptons have a similar structure, except that they are all neutral under $\mathrm{SU}(3)_\mathrm{C}$ and there are no right-handed neutrinos in the SM. The first generation contains the electron e and its corresponding electron neutrino ν_e. The left-handed parts belong to an $\mathrm{SU}(2)_\mathrm{L}$ doublet, while the right-handed electron is an $\mathrm{SU}(2)_\mathrm{L}$ singlet. Hence they are organized in a similar way as the quarks,

$$L_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \sim (\mathbf{1}, \mathbf{2})_{-1/2},$$
$$e_R \sim (\mathbf{1}, \mathbf{1})_{-1}.$$

The second generation contains the muon μ and its corresponding muon neutrino ν_μ, which are again heavier copies of the members of the first generation. The heaviest and third generation contain the tau τ and the tau neutrino ν_τ.

Thus, each generation of fermions in the SM contains 15 Weyl spinors. Note that the right-handed fermions all transform as singlets under $\mathrm{SU}(2)_\mathrm{L}$. In other words, the weak interaction only couples to the left-handed components of the fermions. This explains the meaning of the subscript L for "left".

For any spinor field ψ, the kinetic term in the Lagrangian density is

$$\mathcal{L}_{\text{Dirac}} = \mathrm{i}\overline{\psi}\gamma^\mu \partial_\mu \psi, \tag{2.8}$$

where ∂_μ is a spacetime derivative. To make gauge invariance explicit, we write the fermion fields as multiplets under $\mathcal{G}_{\text{SM}}$ and denote them as Ψ. Then, to maintain gauge invariance, the term in the Lagrangian density becomes

$$\mathcal{L}_{\text{Dirac}} = \mathrm{i}\overline{\Psi}\gamma^\mu D_\mu \Psi, \tag{2.9}$$

where D_μ is the covariant derivative, defined as

$$D_\mu \equiv \partial_\mu - \mathrm{i}g_3 G^a_\mu t^a - \mathrm{i}g_2 W^i_\mu t^i - \mathrm{i}g_1 Y B_\mu, \tag{2.10}$$

where the coefficients g_3, g_2, and g_1 are the coupling constants corresponding to their respective gauge fields. Thus, the gauge interactions of the fermions are uniquely determined from the requirement of gauge invariance.

2.1.3 The Higgs mechanism[3]

The final piece of the SM was confirmed in 2012 by the discovery of the Higgs boson by the ATLAS [14] and CMS [15] collaborations. To allow the fermions and electroweak gauge bosons to obtain their experimentally measured masses, the $\mathrm{SU}(2)_\mathrm{L}$ symmetry must be broken. Accomplishing this requires the Higgs field [27–30], which is a complex scalar $\mathrm{SU}(2)_\mathrm{L}$ doublet,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \sim (\mathbf{1}, \mathbf{2})_{1/2}.$$

It has a kinetic and potential term in the Lagrangian density,

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi), \tag{2.11}$$

where the Higgs potential is

$$V(\Phi) = -\mu^2(\Phi^\dagger\Phi) + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2. \tag{2.12}$$

Minimizing this potential gives

$$(\Phi^\dagger\Phi)_{\text{min}} = \frac{2\mu^2}{\lambda}. \tag{2.13}$$

Thus, the Higgs field acquires a vacuum expectation value (vev) of $\langle|\Phi|\rangle = \sqrt{2\mu^2/\lambda} \equiv v/\sqrt{2}$ with $v \approx 246\,\mathrm{GeV}$ [31]. Perturbation expansions must then be performed with

[3] There are several names for this mechanism. It is sometimes known as the BEH-mechanism (for Brout, Englert, and Higgs), the EBHGHK-mechanism (to also credit Guralnik, Hagen, and Kibble) or the ABEGHHK'tH-mechanism (to also credit Anderson and 't Hooft) [26]. We use the commonly used name "Higgs mechanism" without implying any priority.

this value taking the place of the vacuum. Expanding about the vev, the Higgs field may be written as

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^+ \\ v + h + \mathrm{i}\sigma \end{pmatrix}, \tag{2.14}$$

up to $\mathrm{SU}(2)_\mathrm{L}$ transformations. Here, h and σ are real scalar fields, which are excitations around the (also real) vev v.

Since the vacuum is now at a point in field space that is not invariant under $\mathrm{SU}(2)_\mathrm{L}$ transformations, the symmetry is spontaneously broken as

$$\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y \to \mathrm{SU}(3)_\mathrm{C} \times \mathrm{U}(1)_Q, \tag{2.15}$$

where Q is the electric charge given by the Gell-Mann–Nihshijima formula [32, 33]

$$Q = T_3 + Y. \tag{2.16}$$

Here, T_3 is the third generator of $\mathrm{SU}(2)_\mathrm{L}$.

Out of the four generators of the electroweak gauge group, three are broken and one combination remains unbroken, corresponding to electromagnetic $\mathrm{U}(1)_Q$ symmetry. Thus, there are three Nambu-Goldstone bosons [34–36], which are the two components of the complex ϕ^+ and ϕ in Eq. (2.14). They get absorbed ("eaten") by the gauge bosons corresponding to the broken generators and become their longitudinal polarization degrees of freedom [30]. This is the mechanism through which the gauge bosons corresponding to the broken generators acquire mass terms.

Since we are now expanding the Higgs field about its vev as in Eq. (2.14), the covariant derivative term in the Higgs Lagrangian density Eq. (2.11) contains the terms

$$\mathcal{L}_\mathrm{Higgs} \supset \frac{v^2}{8} \left[g_2^2 W_\mu^1 W^{1,\mu} + g_2^2 W_\mu^2 W^{2,\mu} + \left(g_2 W_\mu^3 - g_1 B_\mu\right)\left(g_2 W^{3,\mu} - g_1 B^\mu\right)\right]. \tag{2.17}$$

These mass terms can be diagonalized to give the physical vector bosons and their masses. The three massive gauge bosons corresponding to the W_μ^+, W_μ^-, and Z_μ^0 (where the superscripts refer to their electric charge) are

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp W_\mu^2\right), \quad Z_\mu^0 = \frac{1}{\sqrt{g_1^2 + g_2^2}}\left(g_2 W_\mu^3 - g_1 B_\mu\right), \tag{2.18}$$

with tree-level masses

$$M_W = \frac{g_2 v}{2}, \quad M_Z = \sqrt{g_1^2 + g_2^2}\,\frac{v}{2}. \tag{2.19}$$

The fourth gauge boson is the one that is orthogonal to Z^0, namely

$$A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}\left(g_1 W_\mu^3 + g_2 B_\mu\right), \tag{2.20}$$

which remains massless and corresponds to the gauge boson of $\mathrm{U}(1)_Q$, namely the photon.

It is customary to define the mixing angle between W_μ^3 and B_μ, called the Weinberg angle, as

$$\tan\theta_W = \frac{g_1}{g_2}. \tag{2.21}$$

Then, the physical electrically neutral states can be written as

$$Z_\mu^0 = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \tag{2.22}$$

$$A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu. \tag{2.23}$$

The ratio of masses between the heavy gauge bosons in terms of the Weinberg angle is given by

$$\frac{M_W}{M_Z} = \cos\theta_W. \tag{2.24}$$

2.1.4 Fermion masses

With the introduction of the Higgs boson, we now also have the Yukawa terms which couple the Higgs boson to the fermions. As the symmetry is spontaneously broken and the Higgs field acquires a vev, it will generate mass terms for the fermions.

With a slight abuse of notation, let $i \in \{1,2,3\}$ label the generation, and let Q_L^i be the left-handed quark doublets, u_R^i the vector of right-handed up, charm, and top quarks, let d_R^i the vector of right-handed down, strange, and bottom quarks. Similarly for the leptons, let L_L^i be the lepton doublets and ℓ_R^i the vector of right-handed electrons, muons, and taus.

Then, the Lagrangian density for the Yukawa interactions is

$$\mathcal{L}_{\text{Yuk}} = -Y_d^{ij}\overline{Q_L^i}\Phi d_R^j - Y_u^{ij}\overline{Q_L^i}\widetilde{\Phi} u_R^j - Y_\ell^{ij}\overline{L_L^i}\Phi \ell_R + \text{h.c.}, \tag{2.25}$$

where Y_d^{ij}, Y_u^{ij}, and Y_ℓ^{ij} are dimensionless components of general complex Yukawa coupling matrices and $\widetilde{\Phi} = \mathrm{i}\sigma_2\Phi$.

As the Higgs field acquires a vev, the fermions are given masses proportional to the Higgs vev and the Yukawa couplings. The mass matrices are

$$M_d = Y_d\frac{v}{\sqrt{2}}, \quad M_u = Y_u\frac{v}{\sqrt{2}}, \quad M_\ell = Y_\ell\frac{v}{\sqrt{2}}. \tag{2.26}$$

In order to extract physical masses and mass states of the fermions, we need to diagonalize the mass matrices. Any complex matrix may be diagonalized by a bi-unitary transformation.

$$M_d = U_{d_L}^\dagger \widetilde{M_d} U_{d_R}, \quad M_u = U_{u_L}^\dagger \widetilde{M_u} U_{u_R}, \quad M_\ell = U_{\ell_L}^\dagger \widetilde{M_\ell} U_{\ell_R}, \tag{2.27}$$

in which U_{d_L}, U_{d_R}, U_{u_L}, U_{u_R}, U_{ℓ_L}, and U_{ℓ_R} are unitary matrices in generation space and $\widetilde{M_d}$, $\widetilde{M_u}$, and $\widetilde{M_\ell}$ are diagonal mass matrices. We correspondingly rotate the fermion fields as

$$(d'_L)^i = (U_{d_L})^{ij}(d_L)^j, \quad (d'_R)^i = (U_{d_R})^{ij}(d_R)^j, \tag{2.28}$$

$$(u'_L)^i = (U_{u_L})^{ij}(u_L)^j, \quad (u'_R)^i = (U_{u_R})^{ij}(u_R)^j, \tag{2.29}$$

$$(L'_L)^i = (U_{\ell_L})^{ij}(L_L)^j, \quad (\ell'_R)^i = (U_{\ell_R})^{ij}(\ell_R)^j. \tag{2.30}$$

Note that since the left-handed quark doublet Q_L appears in two of the terms of Eq. 2.25, we must rotate its two components u_L and d_L separately in generation space, while the lepton doublet L_L only appears once, so we can rotate the whole doublet together in generation space. Hence, the same rotation in generation space is applied to both the ν_L and the ℓ_L components of the L_L doublet.

These rotations in generation space have a physical effect on the interactions with the W boson. The interaction current transforms under these rotations as

$$\begin{aligned} j^\mu_W &= \frac{1}{\sqrt{2}}(\overline{\nu_L}\gamma^\mu \ell_L + \overline{u_L}\gamma^\mu d_L) + \text{h.c.} \\ &= \frac{1}{\sqrt{2}}(\overline{\nu'_L}U_{\ell_L}\gamma^\mu U^\dagger_{\ell_L}\ell_L + \overline{u_L}U_{u_L}\gamma^\mu U^\dagger_{d_L}d_L) + \text{h.c.} \\ &= \frac{1}{\sqrt{2}}(\overline{\nu'_L}\gamma^\mu \ell_L + \overline{u_L}V_{\text{CKM}}\gamma^\mu d_L) + \text{h.c.}, \end{aligned} \tag{2.31}$$

where we have arrived at the Cabibbo–Kobayashi–Maskawa (CKM) matrix [37, 38], defined as

$$V_{\text{CKM}} = U_{u_L}U^\dagger_{d_L}. \tag{2.32}$$

This matrix relates the mass eigenstates of the quarks to their eigenstates in the weak interaction basis and gives rise to the phenomenon of quark mixing in weak interactions. Note that no such mixing matrix appeared in the leptonic interaction current, precisely because the left-handed neutrinos and charged leptons were rotated by the same unitary matrix.

The CKM matrix is a unitary 3×3 matrix. It therefore has nine real parameters, three of which are mixing angles and six of which are complex phases. However, we have the freedom of absorbing five of these phases into the six quark fields u^i_L and d^i_L. The last phase cannot be removed in this way, since it can be taken to be a global redefinition of the phases of the quark fields. We are thus left with one complex phase in the CKM matrix and three real mixing angles. There are different ways of parametrizing such a matrix, but we will use the standard parametrization [31]

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{2.33}$$

where $s_{ij} \equiv \sin\theta^q_{ij}$ and $c_{ij} \equiv \cos\theta^q_{ij}$ for the three mixing angles θ^q_{12}, θ^q_{13}, and θ^q_{23}, and δ is the complex phase. This phase gives rise to $\mathcal{CP}$ violation in hadronic processes, such as different decay rates for B^0 and $\overline{B^0}$ mesons or mixing in the $K^0 - \overline{K^0}$ system.

2.1.5 Parameters of the Standard Model

The complete Lagrangian density may now be constructed from the different parts discussed above,[4] *i.e.*

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yuk}}, \tag{2.34}$$

where $\mathcal{L}_{\text{gauge}}$ can be found in Eq. (2.6), $\mathcal{L}_{\text{Dirac}}$ in Eq. (2.9), $\mathcal{L}_{\text{Higgs}}$ in Eq. (2.11), and $\mathcal{L}_{\text{Yuk}}$ in Eq. (2.25).

There are in total 18 parameters in this Lagrangian.[5] These are: the three gauge couplings, the Higgs vev and quartic coupling λ, the nine masses of the quark and leptons, the three CKM mixing angles, and the $\mathcal{CP}$-violating phase of the CKM matrix. Their values are summarized in Tab. 2.1. Finally, we summarize the particle content of the SM in Tab. 2.2.

2.2 Open questions and some proposed solutions

Although the SM is a highly successful theory of particles and their interactions, there are still a number of open questions and shortcomings of the SM.[6] They can be divided into those that have an experimental or observational basis, namely situations in which the predictions from the SM do not agree with observations, and problems of an aesthetic nature, in which the question is why Nature is the way the SM describes it.

2.2.1 Observational problems

There are a number of shortcomings of the SM that stem from observational discrepancies between its predictions and experiments. We review some of them below.

Neutrino masses

One of the most well-known of these is the question of massive neutrinos. Due to the absence of right-handed neutrinos in the SM, there is no term in the Yukawa Lagrangian of Eq. (2.25) that gives the neutrinos masses. However, the phenomenon of neutrino oscillations [41] between flavor eigenstates during their propagation requires that neutrinos be massive. These oscillations can be understood by noting that the mass eigenstates are linear combinations of flavor eigenstates, related

[4] We neglect the terms relating to gauge-fixing and ghosts, since these are not relevant for the rest of this thesis.

[5] Neglecting the QCD $\mathcal{CP}$-violating parameter θ.

[6] We neglect the fact that the SM says nothing about gravitational interactions.

Parameter	Value
g_1	0.461
g_2	0.652
g_3	1.21
v	246 GeV
λ	0.516
y_e	2.79×10^{-6}
y_μ	5.90×10^{-4}
y_τ	1.00×10^{-2}
y_u	7.80×10^{-6}
y_c	3.65×10^{-3}
y_t	0.990
y_d	1.66×10^{-5}
y_s	3.10×10^{-4}
y_b	1.65×10^{-2}
θ^q_{12}	0.227
θ^q_{13}	3.71×10^{-3}
θ^q_{23}	4.18×10^{-2}
δ	1.14

Table 2.1: Parameters of the SM, evaluated at the energy scale M_Z in the $\overline{\text{MS}}$ scheme, taken from Ref. [39], except for v which is taken from Ref. [31] and the quark mixing parameters which were taken from the 2018 updated of Ref. [40]. All other parameters, *e.g.* θ_W or the gauge boson masses may be derived from these.

through the leptonic mixing matrix [41, 42], similar to the CKM matrix for the quarks. Neutrino oscillations have been confirmed by a series of experiments [43–45] and resulted in measurements of the relevant parameters, displayed in Tab. 2.3.

Since neutrino masses cannot be generated in the SM, we must turn to physics beyond the SM to explain them. Perhaps the seemingly most simple way to extend the SM is to include right-handed neutrinos and generate mass terms in the same way as the charged fermions. However, due to the smallness of neutrino masses, the Yukawa couplings would be unnaturally small. Instead, there are several mechanisms to generate such small neutrino masses.

	Field	Representation
Gauge bosons	B_μ	$(\mathbf{1},\mathbf{1})_0$
	W_μ	$(\mathbf{1},\mathbf{3})_0$
	G_μ	$(\mathbf{8},\mathbf{1})_0$
Fermions	L_L^i	$(\mathbf{1},\mathbf{2})_{-1/2}$
	ℓ_R^i	$(\mathbf{1},\mathbf{1})_{-1}$
	Q_L^i	$(\mathbf{3},\mathbf{2})_{1/6}$
	u_R^i	$(\mathbf{3},\mathbf{1})_{2/3}$
	d_R^i	$(\mathbf{3},\mathbf{1})_{-1/3}$
Scalar bosons	Φ	$(\mathbf{1},\mathbf{2})_{1/2}$

Table 2.2: Particle content of the SM. The index $i \in \{1,2,3\}$ labels the generation of the fermion.

Parameter	Value
Δm_{21}^2	$7.55 \times 10^{-5}\,\mathrm{eV}^2$
Δm_{31}^2 (NO)	$2.50 \times 10^{-3}\,\mathrm{eV}^2$
Δm_{32}^2 (IO)	$-2.42 \times 10^{-3}\,\mathrm{eV}^2$
$\sin\theta_{12}^\ell$	0.320
$\sin\theta_{13}^\ell$ (NO)	0.0216
$\sin\theta_{13}^\ell$ (IO)	0.0222
$\sin\theta_{23}^\ell$ (NO)	0.547
$\sin\theta_{23}^\ell$ (IO)	0.551
δ^ℓ (NO)	3.80
δ^ℓ (IO)	4.90

Table 2.3: Central values of the neutrino parameters, derived from a global fit [46]. The abbreviations NO and IO stand for normal and inverted neutrino mass ordering, respectively. We note that the $\mathcal{CP}$-violating phase δ^ℓ has still not been directly measured, and that its value is still largely unknown at 3σ.

One of the most popular types of mechanism for generating neutrino masses is the seesaw mechanisms, in which new heavy degrees of freedom induce light neutrino masses. The type-I seesaw mechanism [47–51] introduces heavy right-handed neutrinos N_R with a Majorana mass term. The Lagrangian density is

$$\mathcal{L}_{\text{seesaw}} = -\frac{1}{2} M_R^{ij} \overline{N_R^{Ci}} N_R^j - Y_\nu^{ij} \overline{L_L^i} \widetilde{\Phi} N_R^j + \text{h.c.} \tag{2.35}$$

After diagonalizing the mass term for the neutrinos, one obtains the Majorana mass matrix

$$m_\nu = -\frac{1}{2} v^2 Y_\nu M_R^{-1} Y_\nu^T \tag{2.36}$$

for the left-handed neutrinos. Hence, the smallness of the left-handed neutrino masses is a direct result of the large M_R.

In the type-II seesaw mechanism [52–54], a heavy scalar triplet is introduced, which allows for a Yukawa coupling of the lepton doublet with itself. A vev is induced for the triplet, inversely proportional to its mass. This results in the left-handed neutrinos obtaining a Majorana mass term that is inversely proportional to the heavy scalar triplet mass. The type-III seesaw mechanism [55] adds a fermion triplet, which couples to the lepton doublet. When integrated out of the theory, a Majorana mass is induced for the left-handed neutrinos, which is inversely proportional to the mass of the fermion triplet.

Another type of mechanisms for generating small neutrino masses is the radiative mechanisms, in which the neutrino mass matrix is generated at loop order. The perturbation expansion suppresses loop-order contributions relative to tree-level, which naturally leads to the observation that the neutrinos are lighter than the charged leptons. Some examples of these radiative mechanisms are the Zee model [56], the Zee-Babu model [57, 58], and the scotogenic model [59]. For reviews and classifications of radiative neutrino mass models, see for example Refs. [60, 61].

The seesaw and radiative mechanisms for neutrino masses require that the neutrinos are Majorana fermions, meaning that they are their own antiparticles. This is possible for the neutrinos since they are electrically neutral. It is therefore relevant to test the Majorana or Dirac nature of the neutrinos. This can be done by searching for neutrinoless double-beta decay, in which two neutrons are converted to two protons, releasing two electrons but no neutrinos. For a review, see *e.g.* Ref. [62]. The non-observation of neutrinoless double-beta decay places an upper bound on the effective mass parameter

$$\langle m_{\beta\beta} \rangle = |U_{ei}^2 m_i|, \tag{2.37}$$

where U is the leptonic mixing matrix and m_i are the light neutrino masses. Currently, the limit is $\langle m_{\beta\beta} \rangle \lesssim 0.2\,\text{eV}$ [62]. Experiments that are searching for this include GERDA [63], NEMO-3 [64], CUORE [65], and KamLAND-Zen [66].

Dark matter

Another problem is that there is no candidate for dark matter (DM) in the SM, while its existence has been verified through several astrophysical and cosmological observations. These include galactic rotation curves [67–70], gravitational lensing [71], and the cosmic microwave background [72, 73]. The current best fit suggests a DM density of [73]

$$\Omega_{\rm DM} h^2 \approx 0.12, \tag{2.38}$$

where h is the Hubble constant in units of 100 km/s/Mpc. There are a multitude of different candidates for DM, which are all beyond the SM. Some examples include sterile neutrinos [74], axions [75–77] (see also the strong $\mathcal{CP}$ problem below), and the lightest supersymmetric partner [78].

Baryon asymmetry

Also of a cosmological origin is the baryon asymmetry of the Universe (BAU). The asymmetry between the number density of baryons and antibaryons, normalized to the photon number density, has been measured to be [73]

$$\eta_{\rm B} \equiv \frac{n_{\rm B} - n_{\overline{\rm B}}}{n_\gamma} \approx 6.14 \times 10^{-10}. \tag{2.39}$$

To produce such an asymmetry, there needs to exist processes that satisfy the three so-called Sakharov conditions [79], namely that there should be baryon number violation, $\mathcal{C}$- and $\mathcal{CP}$-violation, as well as out-of-equilibrium interactions. In particular, the second of these is not sufficiently provided by the SM [80], making the BAU a phenomenon of physics beyond the SM. One proposed mechanism to produce the baryon asymmetry is through leptogenesis, in which the asymmetry is generated in the lepton sector and then transferred to the baryon sector [81].

Other hints

Further, there is a clear discrepancy between SM predictions and measurements of the muon anomalous magnetic moment [82], which could be solved by physics beyond the SM. There are also a number of hints for physics beyond the SM in the flavor sector, particularly some hadronic decay observables [83].

2.2.2 Aesthetic problems

Aside from the observational problems in the SM, there are certain questions of a more aesthetic nature which are intriguing and may be taken as hints that the SM is not a complete theory. Although the SM is built on elegant mathematical principles, there is a certain sense of arbitrariness to it.

Structure of the SM

Firstly, we may ask why this particular gauge group is the one that describes Nature. Secondly, why are there three generations of fermions with identical quantum numbers? Why do they have the charges that they do, and why is the hypercharge (and hence electric charge) quantized?

Anomaly cancellation

Since the SM is a gauge theory, the gauge anomalies [84, 85] need to vanish in order for it to be consistent, which puts constraints on the allowed charges. With this in mind, the choice of hypercharges seem to conspire such that the anomalies cancel exactly and it is intriguing to consider a reason behind this. Although anomaly cancellation in the SM can be related to hypercharge quantization [86–91], that is not an explanation of it.

Naturalness

One may also wonder about the parameter values in the SM. For example, the masses of the three generations are strongly hierarchical. Indeed, this question becomes even more difficult to ignore once we include massive neutrinos which have masses several orders of magnitude lighter than the other massive fermions. Another unsatisfactory feature of the SM is the apparent fine-tuning of the Higgs mass. The problem is that quantum corrections due to physics at higher scales in the theory, such as the Planck scale, should place the Higgs mass at that higher scale, unless very precise cancellations between contributions occur. This is the well-known hierarchy problem [92–94]. A proposed solution to this is supersymmetry (SUSY), in which each particle has a superpartner, such that fermions have bosonic partners and bosons have fermionic partners. For an introduction, see for example Ref. [95].The divergent contributions to the Higgs mass are then canceled by opposite contributions from the supersymmetric partners.

Another problem of fine-tuning is the strong $\mathcal{CP}$ problem, which notes that the SM Lagrangian density in general contains the term [96, 97]

$$\mathcal{L}_{\rm SM} \supset \theta \frac{g_3^2}{32\pi^2} \widetilde{G}^{a,\mu\nu} G^a_{\mu\nu}, \tag{2.40}$$

where $\widetilde{G}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$. This term violates $\mathcal{CP}$ and would induce an electric dipole moment for the neutron. However, measured upper limits on the neutron electric dipole moment imply that $|\theta| \lesssim 10^{-10}$ [98–100]. The problem is thus one of fine-tuning: is there any reason for this parameter to be so small? A solution to the strong $\mathcal{CP}$ problem is to introduce a Peccei–Quinn symmetry, $\mathrm{U}(1)_{\rm PQ}$, which replaces the angle θ by the axion field which is the pseudo-Goldstone boson associated with the spontaneous symmetry breaking of $\mathrm{U}(1)_{\rm PQ}$ [101–104].

Chapter 3

Grand unified theories and SO(10)

As noted in Ch. 2, there are several questions left unanswered in the SM. One of the most outstanding questions is regarding the choice of gauge group. Further, considering the success of the electroweak unification, it is highly appealing to attempt a unification of all three interactions in the SM, leading to the idea of a Grand Unified Theory (GUT).

The idea behind grand unification is that at some higher energy, the gauge group factors of $\mathcal{G}_{\text{SM}} = \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_Y$ unify and we have only one gauge group. This gauge group corresponds to a unification of the three forces described by the SM. Since our observations are mostly in agreement with a model based on the gauge group $\mathcal{G}_{\text{SM}}$, we require that the unified gauge group has $\mathcal{G}_{\text{SM}}$ as a subgroup. Similarly to the Higgs mechanism, we then assume that the unified gauge group spontaneously breaks to $\mathcal{G}_{\text{SM}}$ at some higher energy scale. Cosmologically, higher energies correspond to earlier times, since the Universe was hotter earlier. Thus, a unified theory is more fundamental in the sense that the SM is a result of it.

Further, the unified gauge group should allow for the fermion representations of the SM and its chiral structure. It is conventional to write the fermion representations as all left-handed spinors using the charge-conjugation matrix. In that language, we have[1]

$$\begin{aligned} Q_L \sim (\mathbf{3},\mathbf{2})_{1/6}, \quad (u_R)^c \sim (\overline{\mathbf{3}},\mathbf{1})_{-2/3}, \quad (d_R)^c \sim (\overline{\mathbf{3}},\mathbf{1})_{1/3}, \\ L_L \sim (\mathbf{1},\mathbf{2})_{-1/2}, \quad (e_R)^c \sim (\mathbf{1},\mathbf{1})_{1}. \end{aligned} \tag{3.1}$$

Here, we have defined $\psi^c \equiv C\overline{\psi}^T$, where the matrix C is the charge conjugation matrix such that $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. Charge conjugation affects the chirality of the

[1] We will in the future drop the "L" and "R" subscripts since all spinors are left-handed.

spinor such that $(\psi_R)^c = (\psi^c)_L$, meaning that all spinors in Eq. (3.1) have left-handed chirality.

3.1 Gauge coupling unification

Phenomenologically, one implication of a gauge group is that it has an associated gauge coupling constant which parametrizes the strength of the interaction. The value of this constant depends on the energy scale involved in the interaction through the renormalization group equation (RGE). They are thus said to "run" or "evolve". For the three gauge group factors in $\mathcal{G}_{\text{SM}}$ to unify into one, we require that the three gauge couplings of the SM meet at the energy scale at which the unified gauge group spontaneously breaks to its subgroup. Hence, the scale at which the coupling constants unify defines the scale at which the GUT is the appropriate theory [105]. Details on renormalization can be found in Sec. 4.1.

Before we discuss the renormalization group running of the gauge couplings, we first note that the hypercharges of the SM fermions are arbitrarily normalized, and that they may be normalized in any way, as long as it is done consistently. Thus, when embedding $\text{U}(1)_Y$ into a larger group, we must make sure that the hypercharge normalization is consistent with this embedding. For any representation, there will be a diagonal generator—one of the Cartan generators—that gives the hypercharge of each state in that representation. Since hypercharge is embedded together with $\text{SU}(3)_\text{C}$ and $\text{SU}(2)_\text{L}$, the normalization of the hypercharge generator must correspond to the normalization of the Cartan generators of $\text{SU}(3)_\text{C}$ and $\text{SU}(2)_\text{L}$. Of course, this will depend on how the fermions are embedded into the unified gauge group. Consider that all SM fermions are embedded into one representation (which, as we will see, will be the case for SO(10)-based models). Then the trace of the T_3 operator squared should give the same as the trace of the properly normalized hypercharge operator squared,

$$\operatorname{Tr} T_3^2 = \left(\frac{1}{2}\right)^2 \cdot 3 + \left(\frac{-1}{2}\right)^2 \cdot 3 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 2 \tag{3.2}$$

$$\operatorname{Tr} Y^2 = \left(\frac{1}{6}\right)^2 \cdot 6 + \left(\frac{-2}{3}\right)^2 \cdot 3 + \left(\frac{1}{3}\right)^2 \cdot 3 + \left(\frac{-1}{2}\right)^2 \cdot 2 + (1)^2 = \frac{10}{3}. \tag{3.3}$$

In order to have hypercharge normalized in the same way as $\text{SU}(2)_\text{L}$, we must then define the GUT-normalized hypercharge, related to the SM hypercharge through

$$Y_{\text{GUT}} = \sqrt{\frac{3}{5}} Y_{\text{SM}}. \tag{3.4}$$

Since the hypercharge always appears in the covariant derivative together with the gauge coupling in the combination $g_1 Y$, we must retain the value of this product.

Thus, we also need to redefine the gauge coupling as

$$g_{1,\mathrm{GUT}} = \sqrt{\frac{5}{3}} g_{1,\mathrm{SM}}. \tag{3.5}$$

In what follows, all references to the $\mathrm{U}(1)_Y$ gauge coupling are to be understood to be GUT-normalized. For clarity, we will continue to list the hypercharges themselves with the familiar SM normalization and it is to be understood that they should be normalized appropriately before being used in calculations. One can check that the same normalization factor is, in fact, obtained when this is done for either of the fermion representations in SU(5).

The RGE for the coupling constant g_i evaluated at one-loop order in perturbation theory is

$$\frac{dg_i}{d\ln\mu} = -\frac{b_i}{16\pi^2} g_i^3, \tag{3.6}$$

where b_i is a coefficient determined from the particle content of the theory. The solution to this differential equation is conventionally given in terms of the parameter α_i, defined as

$$\alpha_i \equiv \frac{g_i^2}{4\pi}, \tag{3.7}$$

or its inverse, α_i^{-1}. Then, the solution to the RGE relating α_i at the scale $\mu = M_2$ to the scale $\mu = M_1$ is

$$\alpha_i^{-1}(M_2) = \alpha_i^{-1}(M_1) - \frac{b_i}{2\pi} \ln\left(\frac{M_2}{M_1}\right). \tag{3.8}$$

At the electroweak scale, $M_Z = 91.1876\,\mathrm{GeV}$, the values of the gauge couplings are [31]

$$g_1(M_Z) = 0.461, \quad g_2(M_Z) = 0.652, \quad g_3(M_Z) = 1.22. \tag{3.9}$$

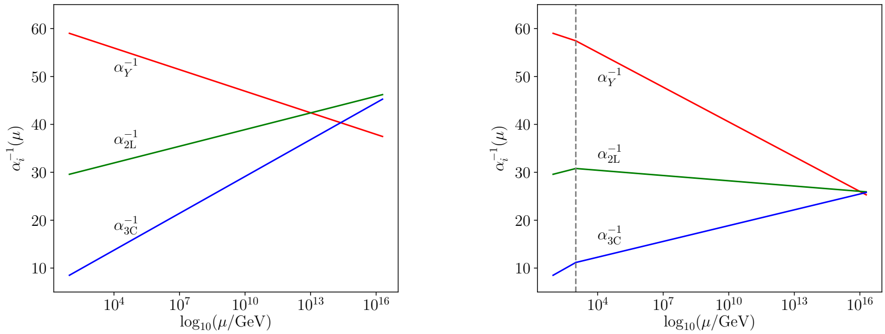
Now we need to find the coefficients b_i for our model. The general formula can be found in Sec. 4.1. In the SM, we have the coefficients

$$b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7. \tag{3.10}$$

In the left panel of Fig. 3.1, the renormalization group running is shown in the SM. As can be seen, the gauge couplings are close to unify, but do not quite do so in the SM. They are much closer to unify in SUSY models, which is why these are often considered together with grand unification. In the Minimal Supersymmetric Standard Model (MSSM), the coefficients b_i are

$$b_1 = \frac{33}{5}, \quad b_2 = 1, \quad b_3 = -3. \tag{3.11}$$

The renormalization group (RG) running in the MSSM with a SUSY-breaking scale of 1 TeV is shown in the right panel of Fig. 3.1.



(a) SM **(b)** MSSM

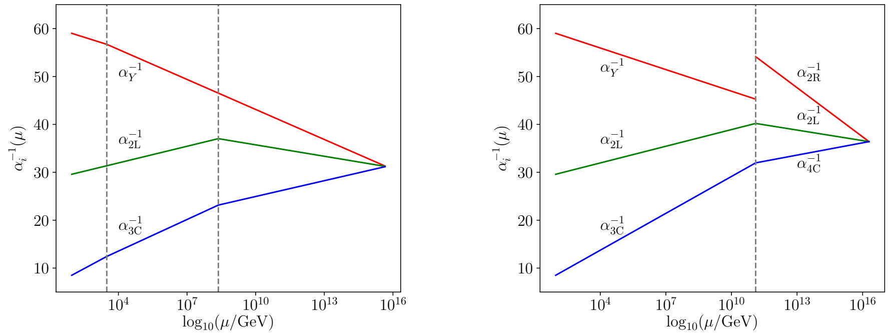
Figure 3.1: Renormalization group running of the gauge couplings in the SM (left) and the MSSM with a SUSY-breaking scale at 1 TeV (right).

The near unification is a further reason to take the idea of unification seriously. In the rest of this thesis, we will focus on non-SUSY models. There are several ways in which to achieve precise unification of gauge couplings without SUSY. One may imagine that in the vast so-called "desert" between M_Z and M_{GUT} spanning 14 orders of magnitude, some new physics enters. For example, this could be particles with masses somewhere in that interval which influence the RG running of the gauge couplings such that precision unification is achieved, as considered in Paper (II). One may also consider a chain of symmetry breaking between the GUT and the SM which also modifies the RG running and produces precision unification, as considered in Paper (I). See Sec. 3.3.1 for more details. These two scenarios are shown in Fig. 3.2 with intermediate scale particles shown on the left and an intermediate symmetry breaking step shown on the right. In the left panel, scalars transforming as $(\mathbf{8},\mathbf{1})_1$ and $(\mathbf{8},\mathbf{3})_0$ at masses 3.1 TeV and 2.34×10^8 GeV, respectively. The right panel has an intermediate Pati–Salam (PS) breaking scale at 1.28×10^{11} GeV.

The solutions to the RGEs are in general altered by increasing the order in perturbation theory to which they are computed. That should be accompanied by matching at a higher order, which entails the inclusion of threshold effects [106, 107]. These can considerably affect the scale at which the gauge coupling unify, and hence the related phenomenology [108–112].

3.2 Candidate gauge groups

In order to be a viable candidate for a unified gauge group, there are several requirements that need to be fulfilled [113].



(a) Intermediate particles **(b)** PS intermediate symmetry

Figure 3.2: Gauge coupling unification using two intermediate-scale fields (left) or an intermediate Pati–Salam gauge symmetry in the breaking chain (right).

Rank

A unified gauge group needs to be large enough to contain $\mathcal{G}_{\mathrm{SM}}$ as a subgroup. Therefore, its rank needs to be at least four.

SM as a subgroup

To reproduce the low-energy phenomenology of the SM that has been rigorously tested, the unified gauge group needs to contain $\mathcal{G}_{\mathrm{SM}}$ as a subgroup.

Chirality

Furthermore, to reproduce the chiral nature of the SM, the GUT group needs to allow for complex representations. The reason for this is that, using the charge conjugation matrix, the left- and right-handed fermions may be related as

$$\psi_R = C\overline{\psi_L^{\mathrm{c}}}^T. \tag{3.12}$$

Hence, if the left-handed fermions transform according to a (possibly reducible) representation $\mathbf{F_L}$, then the right-handed counterparts will transform according to a representation $\mathbf{F_R} = \overline{\mathbf{F_L}}$. Our knowledge of the SM implies that we must have $\overline{\mathbf{F_L}} \neq \mathbf{F_L}$. To be concrete, in the SM we have the left-handed fermions in Eq. (3.1) in a reducible representation

$$\mathbf{F_L} = (\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{1})_{1}. \tag{3.13}$$

Thus, the right-handed counterparts transform according to $\mathbf{F_R} = \overline{\mathbf{F_L}} \neq \mathbf{F_L}$, and hence, the SM is chiral. This must also hold for any grand unified model.

3.2.1 Viable gauge groups

The above list of requirements narrows down the list of possible simple gauge groups considerably [48, 114–116]. At rank 4, the only possibility satisfying the constraints is SU(5), which was the original GUT proposed by Georgi and Glashow [117]. Going to rank 5, the only possibility satisfying all conditions is SO(10), which was first proposed by Fritzsch and Minkowski [118] and independently by Georgi [119, 120]. Turning to rank 6, we find the exceptional group $E(6)$ [121–123] as the only possibility.

There are also some semisimple gauge groups for partial unification into a product of simple gauge groups. There are no semisimple gauge groups of rank 4 that reproduce the SM hypercharge. At rank 5, there are several possibilities. One possibility is to enlarge the SU(5) group to $\mathrm{SU}(5) \times \mathrm{U}(1)$, which allows for a so-called "flipped" embedding of hypercharge [124–128]. There is also the left-right symmetric models $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_{B-L}$ [129–132], in which the left-handed weak interaction is mirrored by a right-handed one, and the Pati–Salam group $\mathcal{G}_\mathrm{PS} = \mathrm{SU}(4)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$ [133], in which the leptons are unified with the quarks as a fourth color.

Below follows some comments regarding the three most important gauge groups for unified model building, before we focus on models based on the SO(10) gauge group.

3.2.2 The SU(5) group

Being the unique embedding of the SM at rank 4, the SU(5) group is the simplest possible candidate for a GUT [117]. The fermions are embedded in $\overline{\mathbf{5}}$ and $\mathbf{10}$ per generation, since under decomposition to $\mathcal{G}_\mathrm{SM}$, they become

$$\begin{aligned} \overline{\mathbf{5}}_F &\to (\overline{\mathbf{3}}, \mathbf{1})_{1/3} \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \\ &= d^c \oplus L \end{aligned} \tag{3.14}$$

$$\begin{aligned} \mathbf{10}_F &\to (\overline{\mathbf{3}}, \mathbf{1})_{-2/3} \oplus (\mathbf{3}, \mathbf{2})_{1/6} \oplus (\mathbf{1}, \mathbf{1})_{1} \\ &= u^c \oplus Q \oplus e^c. \end{aligned} \tag{3.15}$$

Since the adjoint representation is $\mathbf{24}$, there are 24 gauge bosons in the model. The structure of the gauge sector of the Lagrangian follows in the same manner as for SU(3) or SU(2) discussed in Ch. 2. The 24 gauge bosons may be written in matrix form using the generators t^a, where $a \in \{1, \ldots, 24\}$, of SU(5) as

$$A_\mu = A_\mu^a t^a. \tag{3.16}$$

After symmetry breaking, eight of the 24 gauge bosons are identified with the gluons G_μ^a, three are identified with the SU(2) gauge bosons W_μ^i, and one is identified with the hypercharge gauge boson B_μ. There remain twelve gauge bosons which acquire masses of the symmetry breaking scale.

To break the SU(5) symmetry down to $\mathcal{G}_{\text{SM}}$, we employ a mechanism based on the same principle as the Higgs mechanism in the SM. This is achieved by introducing scalar fields transforming as the adjoint, **24**. Using the adjoint to spontaneously break a group conserves its rank, since the vev may be diagonalized by the group generators, and hence commutes with the Cartan generators. Thus, the number of unbroken Cartan generators remains the same [134].

We therefore have 24 scalar fields ϕ^a, which can be written in matrix form

$$\mathbf{24}_H = \phi^a t^a \tag{3.17}$$

and the scalar potential is

$$V(\mathbf{24}_H) = -\mu^2 \operatorname{Tr}(\mathbf{24}_H^2) + \lambda_1 \operatorname{Tr}(\mathbf{24}_H^2)^2 + \lambda_2 \operatorname{Tr}(\mathbf{24}_H^2), \tag{3.18}$$

neglecting the cubic term by imposing a discrete symmetry $\mathbf{24}_H \to -\mathbf{24}_H$. One can set the values of the parameters such that $\mathbf{24}_H$ takes the vev

$$\langle \mathbf{24}_H \rangle = v \operatorname{diag}(2, 2, 2, -3, -3), \tag{3.19}$$

where the scale v is given by

$$v = \frac{\mu^2}{2(30\lambda_1 + 7\lambda_2)}. \tag{3.20}$$

This achieves the desired breaking to the SM.

The scalar sector also needs to include an SU(2) doublet which will provide the Higgs mechanism of the electroweak theory. The simplest choice is $\mathbf{5}_H$, which decomposes as

$$\mathbf{5} \to (\mathbf{3}, \mathbf{1})_{-1/3} \oplus (\mathbf{1}, \mathbf{2})_{1/2}. \tag{3.21}$$

The latter of these is the SM Higgs, while the first one is an extra SU(3) triplet which should have a large mass in order not to affect low-energy phenomenology. This in general requires fine-tuning of the parameters in the scalar potential $V(\mathbf{24}_H, \mathbf{5}_H) = V_{24}(\mathbf{24}_H) + V_5(\mathbf{5}_H) + V_{\text{mix}}(\mathbf{24}_H, \mathbf{5}_H)$ [135] and is known as the "doublet-triplet splitting" problem. See Ref. [136] for a discussion regarding the amount of fine-tuning necessary for this type of splitting in various models.

Fermion masses are a result of Yukawa couplings. In particular, we require the fermions to couple to the $\mathbf{5}_H$ and not the $\mathbf{24}_H$, since the SM Higgs resides in $\mathbf{5}_H$. It happens that Yukawa couplings with the $\mathbf{24}_H$ are forbidden by SU(5) symmetry, as required. In a compact notation (suppressing Lorentz, family, and SU(5) indices), the Yukawa terms of the Lagrangian density is given by

$$\mathcal{L}_{\text{Yuk}} = -Y_5 \overline{\mathbf{5}}_F \mathbf{10}_F \mathbf{5}_H^* - \frac{1}{8} Y_{10} \epsilon_5 \mathbf{10}_F \mathbf{10}_F \mathbf{5}_H + \text{h.c.}, \tag{3.22}$$

where Y_5 and Y_{10} are the Yukawa matrices in family space and ϵ_5 is the 5-index completely antisymmetric tensor. Writing out the charge-conjugation matrix and spinor transposition explicitly, as well as the family indices $(i,j) \in \{1,2,3\}$ and SU(5) indices $(\alpha,\beta,\gamma,\delta,\epsilon) \in \{1,\ldots,5\}$, we obtain

$$\mathcal{L}_{\text{Yuk}} = -(Y_5)^{ij}(\overline{\mathbf{5}}_F^T)^i_\alpha C(\mathbf{10}_F)^j_{\alpha\beta}(\mathbf{5}_H^*)_\beta - \frac{1}{8}(Y_{10})^{ij}\epsilon_{\alpha\beta\gamma\delta\epsilon}(\mathbf{10}_F^T)^i_{\alpha\beta}C(\mathbf{10}_F)^j_{\gamma\delta}(\mathbf{5}_H)_\epsilon + \text{h.c.} \quad (3.23)$$

Expanding out the SU(5) multiplication to write it in terms of the SM fields, the first term gives

$$-Y_5(d^c)^T CQH^* - Y_5 L(e^c)^T CH^* \quad (3.24)$$

and the second term gives

$$-\frac{1}{2}(Y_{10} - Y_{10}^T)(u^c)^T CQH. \quad (3.25)$$

Therefore, the SU(5) symmetry predicts the mass relations

$$M_\ell = M_d^T, \quad M_u = M_u^T, \quad (3.26)$$

the first of which is clearly false [117] and is not entirely corrected by renormalization effects [135]. They can, however, be corrected by adding an additional Higgs field in the **45** representation [137] or taking into account Planck-scale suppressed non-renormalizable operators [138].

3.2.3 The Pati–Salam group

The Pati–Salam model [133] of partial unification is based on the gauge group $\mathcal{G}_{\text{PS}} = \text{SU}(4)_\text{C} \times \text{SU}(2)_\text{L} \times \text{SU}(2)_\text{R}$ and unifies the quarks and leptons by treating the leptons as a fourth color. Often, a discrete left-right parity symmetry called D-parity is also invoked [129, 131, 139–141]. Each family of fermions is embedded as

$$F_L = \begin{pmatrix} u^r & u^g & u^b & \nu \\ d^r & d^g & d^b & e \end{pmatrix} \sim (\mathbf{4},\mathbf{2},\mathbf{1}), \quad (3.27)$$

$$F_R = \begin{pmatrix} d^{r\,c} & d^{g\,c} & d^{b\,c} & e^c \\ -u^{r\,c} & -u^{g\,c} & -u^{b\,c} & -\nu^c \end{pmatrix} \sim (\overline{\mathbf{4}},\mathbf{1},\mathbf{2}). \quad (3.28)$$

This fermion embedding makes evident a symmetry between the left-handed and right-handed fermions which the SM lacks. A result of this symmetry is the introduction of a right-handed neutrino ν^c.

The gauge bosons of $\text{SU}(4)_\text{C}$ reside in the adjoint $(\mathbf{15},\mathbf{1},\mathbf{1})$, which contains the gluons after breaking to the SM. The three gauge bosons of $\text{SU}(2)_\text{L}$ in $(\mathbf{1},\mathbf{3},\mathbf{1})$ are the same as those in the SM. Finally, $\text{SU}(2)_\text{R}$ has its gauge bosons in $(\mathbf{1},\mathbf{1},\mathbf{3})$. The hypercharge gauge boson is a linear combination of the two $(\mathbf{1},\mathbf{1})_0$ components in $(\mathbf{15},\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{3})$ such that the hypercharge is embedded as

$$Y = \frac{B-L}{2} - T_{R,3}, \tag{3.29}$$

where $B-L$ is the (suggestively named) unbroken generator of $\mathrm{SU}(4)_\mathrm{C}$ that is not in $\mathrm{SU}(3)_\mathrm{C}$ and $T_{R,3}$ is the third generator of $\mathrm{SU}(2)_\mathrm{R}$.

The Pati–Salam symmetry can be broken directly to the SM by including scalar bosons transforming according to $(\overline{\mathbf{10}},\mathbf{1},\mathbf{3})$ and arranging the scalar potential such that it takes a vev in the direction that is a singlet under the SM gauge group. However, there are other breaking chains possible that break the symmetry to the SM in several steps. A popular such route is via the left-right symmetric group $\mathrm{SU}(3)_\mathrm{C}\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}\times\mathrm{U}(1)_{B-L}$ [129–132], which is a subgroup of $\mathcal{G}_\mathrm{PS}$ and in turn contains $\mathcal{G}_\mathrm{SM}$ as a subgroup. This can be achieved by including scalar bosons in $(\mathbf{15},\mathbf{1},\mathbf{1})$ and assigning a vev in the appropriate direction of it.

Finally, we need to include a representation of scalar bosons which contains the SM Higgs in order to achieve the Higgs mechanism and give fermions masses. This can be done with $\Phi\sim(\mathbf{1},\mathbf{2},\mathbf{2})$, which contains the SM Higgs doublet. However, this would enable only one Yukawa term, *i.e.*

$$\mathcal{L}_\mathrm{Yuk} = -Y\overline{F_L}\Phi F_R + \mathrm{h.c.}, \tag{3.30}$$

which clearly gives the wrong mass relations, just like in the $\mathrm{SU}(5)$ model. Again, this may be remedied by introducing multiple scalars that each take vevs and couple to the fermions [133], for example more copies of $(\mathbf{1},\mathbf{2},\mathbf{2})$ representations or $(\mathbf{15},\mathbf{2},\mathbf{2})$ representations.

3.2.4 The SO(10) group

The SO(10) group [118–120] is a popular candidate for unification for several reasons. Firstly, it contains both the Pati–Salam group and $\mathrm{SU}(5)\times\mathrm{U}(1)$ (and hence also $\mathrm{SU}(5)$) as subgroups and is therefore more unified in a sense. It also embeds all SM fermions of a generation, plus a right-handed neutrino, into one single representation, the $\mathbf{16}_F$.

Orthogonal groups $\mathrm{SO}(N)$ would *a priori* be thought to contain only real representations, and hence be unsuitable for embedding the SM. However, they also contain spinorial representations which are complex for even N and are therefore suitable for embedding the SM fermions [48, 115, 116, 142, 143].

Under decomposition to $\mathrm{SU}(5)$, the $\mathbf{16}$ becomes

$$\mathbf{16}\to\mathbf{10}\oplus\overline{\mathbf{5}}\oplus\mathbf{1}, \tag{3.31}$$

while in the PS model, it becomes

$$\mathbf{16}\to(\mathbf{4},\mathbf{2},\mathbf{1})\oplus(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}). \tag{3.32}$$

It is thus clear that since it reproduces the fermion sector of the $\mathrm{SU}(5)$ model and of $\mathcal{G}_\mathrm{PS}$, it also reproduces the SM fermion sector with an additional singlet.

The adjoint representation of SO(10) is the **45**, so there are 45 gauge bosons in SO(10) models. As with the previous versions of GUTs discussed, when the symmetry is spontaneously broken, only the gauge bosons corresponding to the generators that are left unbroken remain massless. The rest of the gauge bosons acquire masses of the order of the symmetry breaking scale M_{GUT}.

The scalar sector of SO(10) models is very rich due to the number of possible ways of breaking it to $\mathcal{G}_{\text{SM}}$. It may be broken either through the $\mathcal{G}_{\text{PS}}$ route or through the SU(5) route with many possibilities for intermediate gauge groups before reaching $\mathcal{G}_{\text{SM}}$. There are also three different possibilities for embedding the Higgs doublet into SO(10), namely $\mathbf{10}_H$, $\overline{\mathbf{126}}_H$, and $\mathbf{120}_H$, which produce different mass relations. This will be discussed in more details in Sec. 3.3.

3.3 Aspects of SO(10) model building

3.3.1 Symmetry breaking

Since SO(10) has rank 5, which is one more than $\mathcal{G}_{\text{SM}}$, there are several possibilities for symmetry breaking. On the one hand, this produces more rich features, but on the other hand, it introduces arbitrariness into the model in terms of the choice of scalar sector and potential. Furthermore, since the scalar and intermediate symmetry breaking steps affect the RG running, the breaking chain chosen can have an effect on the unification scale and hence the related phenomenology [144–147].

To analyze the symmetry breaking chains, we must enumerate the decompositions of different multiplets to be used in the breaking under the intermediate gauge groups. The symmetry is broken by assigning a vev to a component of a multiplet and the resulting symmetry group is the largest group under which that component is a singlet. Such a decomposition can be found in Tabs. A.1 and A.2 in App. A. Note that this is in complete analogy with the Higgs mechanism in the SM. A schematic diagram of the subgroups and breaking chains is shown in Fig. 3.3. The different breaking chains have been analyzed in *e.g.* Refs. [140, 148–156].

One can first look for patterns of symmetry breaking that occur in one step. It turns out that this can be achieved with scalars transforming under the **144** representation [158], since it contains a singlet under $\mathcal{G}_{\text{SM}}$ but not under any of the other intermediate subgroups of SO(10).

As can be seen in Fig. 3.3, the spindle of breaking chains separates into two separate sectors, namely the SU(5) path and the PS path. Starting with the SU(5) path, we see that we have singlets under $\text{SU}(5) \times \text{U}(1)$ in **45** and **210**. From there, we can either break the symmetry directly to $\mathcal{G}_{\text{SM}}$ using **16** or **126**, or via SU(5) using the same and then using **45**, **54**, or **210** to $\mathcal{G}_{\text{SM}}$, depending on if we have the flipped or standard embedding. Alternatively, we can bypass $\text{SU}(5) \times \text{U}(1)$ and go directly to SU(5) using **16** or **126**, since the SU(5) singlets therein have a U(1) charge, and then break it to $\mathcal{G}_{\text{SM}}$ as before.

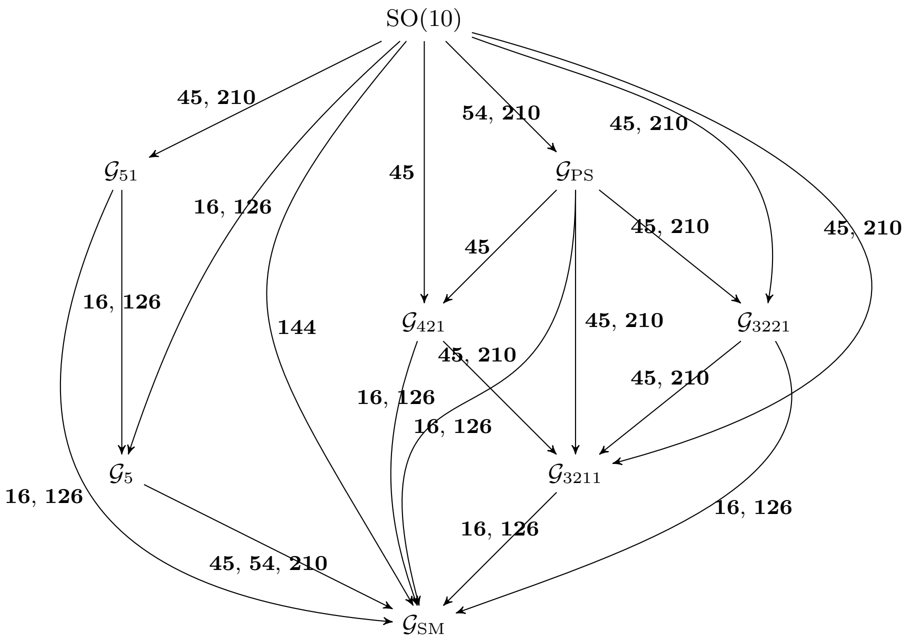


Figure 3.3: Possible breaking chains of SO(10) to $\mathcal{G}_{\text{SM}}$. The representations written next to each arrow are the representation of SO(10) which contain multiplets that can achieve that particular breaking. We introduce the shorthand notation for the different groups, such that $\mathcal{G}_{51} = \text{SU}(5) \times \text{U}(1)$, $\mathcal{G}_5 = \text{SU}(5)$, $\mathcal{G}_{\text{PS}} = \text{SU}(4)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$, $\mathcal{G}_{421} = \text{SU}(4)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_{B-L}$, $\mathcal{G}_{3221} = \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}} \times \text{U}(1)_{B-L}$, and $\mathcal{G}_{3211} = \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_{\text{R}} \times \text{U}(1)_{B-L}$. Figure based on Refs. [140, 157].

Note that if the motivation for using an intermediate symmetry is to allow for successful gauge coupling unification, then it does not make sense to have SU(5) as an intermediate gauge group. The reason is that it would require the three gauge couplings to unify at the intermediate breaking scale, and therefore does not solve the problem. However, this is not required for flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$, so having it as an intermediate gauge group can help achieve gauge coupling unification.

For the Pati–Salam breaking chain, we can break SO(10) to $\mathcal{G}_{\mathrm{PS}}$ by assigning a vev to the appropriate submultiplet of **54** or **210**. To break $\mathcal{G}_{\mathrm{PS}}$, or to break SO(10) directly to any of the subgroups of $\mathcal{G}_{\mathrm{PS}}$, we can assign a vev to the appropriate directions of **45** or **210**. At the end of the breaking chain to $\mathcal{G}_{\mathrm{SM}}$, either **16** or **126** need to be assigned a vev. This is because those multiplets contain multiplets that break the $\mathrm{SU}(2)_{\mathrm{R}}$ or $\mathrm{U}(1)_{\mathrm{R}}$ symmetry.

Any of the symmetry breaking chains may be achieved by assigning a vev to the relevant component of the relevant multiplet. In order to do this, the scalar potential needs to be constructed in such a way as to create a vev in the appropriate direction. This may, in general include some level of fine-tuning. Note also that it is in general possible to break the symmetry at one scale by letting several breaking steps to occur at the same scale.

3.3.2 Yukawa sector

For the Higgs mechanism of the SM, we require not only that we have an $\mathrm{SU}(2)_{\mathrm{L}}$ doublet embedded in a representation of SO(10), but also that it couples to the fermions, which are in $\mathbf{16}_F$. Hence, we need one or more representations $\mathbf{R}$ such that the coupling $\mathbf{16} \cdot \mathbf{R} \cdot \mathbf{16}$ is invariant under SO(10) transformations. There are only three possibilities for $\mathbf{R}$, namely $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$ [159]. Therefore, the Yukawa sector will contain scalars in these three representations. In a compact notation, the Yukawa terms of the Lagrangian density can then be written as

$$\mathcal{L}_{\mathrm{Yuk}} = -\mathbf{16}_F(Y_{10}\mathbf{10}_H + Y_{120}\mathbf{120}_H + Y_{126}\overline{\mathbf{126}}_H)\mathbf{16}_F + \mathrm{h.c.}, \tag{3.33}$$

where Y_{10}, Y_{120}, and Y_{126} are 3×3 Yukawa matrices in family space. The SO(10) structure of this coupling dictates that Y_{10} and Y_{126} are symmetric while Y_{120} is antisymmetric. See Sec. A.3 for more details.

To write out the SO(10) structure of this multiplications explicitly, we need to keep in mind some facts about the spinor structure of the $\mathbf{16}_F$ representation. These can be found in Sec. A.3. The Lorentz spinor structure is the same as in the SM, with the transposition and charge conjugation matrix C acting in the Lorentz spinor space. Secondly, we have the 16-dimensional space corresponding to the spinor representations of SO(10). This will be labeled by indices $(a, b, c, d, e, f, g) \in \{1, \ldots, 16\}$. The 10-dimensional space of the tensorial representations of SO(10) will be labeled by indices $(\alpha, \beta, \gamma, \delta, \epsilon) \in \{1, \ldots, 10\}$. Family indices are labeled by $(i, j) \in \{1, 2, 3\}$. Finally, we need the equivalent of the charge conjugation matrix and the Dirac γ matrices acting in the space of the spinorial SO(10) representation.

These will be denoted by B and Γ_α, respectively, and are 16×16 matrices. Then, the SO(10) structure of the Yukawa coupling is

$$\begin{aligned}\mathcal{L}_{\text{Yuk}} = -(\mathbf{16}_F^T)^{ia} C B^{ab} \Big[&(Y_{10})^{ij} (\Gamma_\alpha)^{bg} (\mathbf{10}_H)_\alpha \\ &+ (Y_{120})_{ij} (\Gamma_\alpha)^{bc} (\Gamma_\beta)^{cd} (\Gamma_\gamma)^{dg} (\mathbf{120}_H)_{\alpha\beta\gamma} \\ + (Y_{126}) ij (\Gamma_\alpha)^{bc} (\Gamma_\beta)^{cd} (\Gamma_\gamma)^{de} (\Gamma_\delta)^{ef} (\Gamma_\epsilon)^{fg} &(\overline{\mathbf{126}}_H)_{\alpha\beta\gamma\delta\epsilon} \Big] (\mathbf{16}_F)^{jg} + \text{h.c.} \end{aligned} \tag{3.34}$$

Since we have three different representations with Higgs doublets, one may ask which one is *the* Higgs doublet. The relevant components under the $\mathcal{G}_{\text{SM}}$ decomposition of the three representations for fermion masses are

$$\mathbf{10}_H \supset (\mathbf{1}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \equiv \Phi_{10}^d \oplus \Phi_{10}^u \tag{3.35}$$

$$\begin{aligned}\mathbf{120}_H &\supset (\mathbf{1}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \oplus (\mathbf{1}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \\ &\equiv \Phi_{120}^d \oplus \Phi_{120}^u \oplus \Sigma_{120}^d \oplus \Sigma_{120}^u \end{aligned} \tag{3.36}$$

$$\begin{aligned}\overline{\mathbf{126}}_H &\supset (\mathbf{1}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2} \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_1 \\ &\equiv \Sigma_{126}^d \oplus \Sigma_{126}^u \oplus \Delta_R \oplus \Delta_L. \end{aligned} \tag{3.37}$$

We thus have eight different $\text{SU}(2)_\text{L}$ doublets that play the roles of Φ and $\widetilde{\Phi}$ in the SM Yukawa Lagrangian Eq. (2.25). They can all develop vevs, depending on the scalar potential of the model, and thereby provide masses to the fermions. The light Higgs boson, which has been observed at the LHC, is in general a linear combination of the $\text{SU}(2)_\text{L}$ doublets above. Note that this may require some fine-tuning of the parameters of the scalar potential.

When we decompose the Yukawa Lagrangian Eq. (3.34), we find terms that produce the fermion masses as well as some additional terms involving the additional fields found in $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$. These mediate exotic processes such as proton decay, which will be discussed in Sec. 3.4. The terms that provide fermion masses are identical to those of the SM Yukawa Lagrangian Eq. (2.25), except that the mass matrices are now given in terms of the Yukawa matrices Y_{10}, Y_{120}, and Y_{126} as well as the vevs of the fields in Eqs. (3.35)–(3.37). The matching conditions between the SO(10) parameters and the mass matrices in the SM are

$$M_u = v_{10}^u Y_{10} + v_{126}^u Y_{126} + (v_{120_1}^u + v_{120_2}^u) Y_{120}, \tag{3.38}$$

$$M_d = v_{10}^d Y_{10} + v_{126}^d Y_{126} + (v_{120_1}^d + v_{120_2}^d) Y_{120}, \tag{3.39}$$

$$M_\nu = v_{10}^u Y_{10} - 3 v_{126}^u Y_{126} + (v_{120_1}^u - 3 v_{120_2}^u) Y_{120}, \tag{3.40}$$

$$M_\ell = v_{10}^d Y_{10} - 3 v_{126}^d Y_{126} + (v_{120_1}^d - 3 v_{120_2}^d) Y_{120}, \tag{3.41}$$

$$M_R = v_R Y_{126}, \tag{3.42}$$

$$M_L = v_L Y_{126}, \tag{3.43}$$

where the vevs are defined as

$$v_{10}^{u,d} = \langle \Phi_{10}^{u,d} \rangle, \tag{3.44}$$

$$v_{120_1}^{u,d} = \langle \Phi_{120}^{u,d} \rangle, \quad v_{120_2}^{u,d} = \langle \Sigma_{120}^{u,d} \rangle, \tag{3.45}$$

$$v_{126}^{u,d} = \langle \Sigma_{126}^{u,d} \rangle, \quad v_R = \langle \Delta_R \rangle, \quad v_L = \langle \Delta_L \rangle. \tag{3.46}$$

The factors of 3 and relative signs of the terms are due to Clebsch-Gordan coefficients. The mass matrix M_R is the mass matrix of the heavy right-handed neutrinos and M_L is the mass matrix contribution from type-II seesaw.

Since all fields charged under $\mathrm{SU(2)_L}$ couple to the $\mathrm{SU(2)_L}$ gauge bosons, their vevs contribute to the gauge boson masses. These are measured to a high degree of accuracy, meaning that there is a constraint on the sum of the vevs, *i.e.*

$$|v_{10}^u|^2 + |v_{10}^d|^2 + |v_{120_1}^u|^2 + |v_{120_1}^d|^2 + |v_{120_2}^u|^2 + |v_{120_2}^d|^2 + |v_{126}^u|^2 + |v_{126}^d|^2 + 2|v_L|^2 = v_{\mathrm{SM}}^2. \tag{3.47}$$

The factor of 2 in front of $|v_L|^2$ is due to a Clebsch-Gordan coefficient from the coupling of the triplet to the gauge bosons. This constraint is dependent on the model in the sense that it depends on which $\mathrm{SU(2)_L}$ multiplets are included in the model. Thus, if any of the above-mentioned multiplets are excluded from the model, they should be removed from Eq. (3.47). Likewise, if the model contains further multiplets charged under $\mathrm{SU(2)_L}$, they should be added to Eq. (3.47).

The constraint Eq. (3.47) allows for most of the vevs to be many orders of magnitude smaller than v_{SM} and one being of the order of v_{SM}. However, since the vevs of the $\mathrm{SU(2)_L}$ doublets are versions of the SM Higgs doublet, one would expect them to be of a similar order of magnitude. The vev v_L is expected to be very small, since it is the induced vev involved in type-II seesaw. Finally, the vev v_R is expected to be large, since it is involved in the breaking chain of SO(10) to $\mathcal{G}_{\mathrm{SM}}$. It is thus of the order of either M_{GUT} or an intermediate breaking scale.

Neutrino masses are generated through a combination of type-I and type-II seesaw mechanisms, from the mass matrices M_ν, M_R, and M_L. In the basis (ν, ν^c), the neutrino mass matrix takes the form

$$\mathcal{M}_\nu = \begin{pmatrix} M_L & M_\nu \\ M_\nu^T & M_R \end{pmatrix}. \tag{3.48}$$

When diagonalized, this mass matrix results in both Type-I and Type-II seesaw mechanisms, resulting in the light neutrino masses

$$\begin{aligned} m_\nu &= m_\nu^{\mathrm{II}} + m_\nu^{\mathrm{I}} \\ &= M_L - M_\nu^T M_R^{-1} M_\nu. \end{aligned} \tag{3.49}$$

The smallness of neutrino masses are thus a result of the smallness of M_L (through the smallness of v_L) and the largeness of M_R (through the largeness of v_R).

In specifying a model, one must determine which of the three possible representations of scalars to include for fermion masses. A guiding principle in model building is minimality. First, we note that we need at least two different multiplets, since otherwise the mass matrices in Eqs. (3.38)–(3.43) will all be proportional, which contradicts the observation of mixing. We would therefore like to have a Yukawa sector consisting of only two out of $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$. One of these should be the $\overline{\mathbf{126}}_H$ in order to produce the seesaw mechanisms. Minimality would then dictate that we should choose the $\mathbf{10}_H$ as the second one. This is the model considered in Papers (II) and (III).

Since the $\mathbf{10}_H$ is a real representation of SO(10), we find that $v_{10}^u = v_{10}^d$, which results in wrong predictions for the mass relations [160, 161]. It can, however, be complexified by adding a second $\mathbf{10}_H$ and forming the combination $\mathbf{10}_H = \mathbf{10}_{H,1} + \mathrm{i}\mathbf{10}_{H,2}$, so that we can use $\mathbf{10}_H$ and $\mathbf{10}_H^*$. Although this solves the problem of wrong mass relations, it decreases the minimality and predictivity of the model, since $\mathbf{10}_H$ and $\mathbf{10}_H^*$ in general have two independent Yukawa coupling matrices. This can, however, be solved by introducing a global Peccei–Quinn symmetry [101, 102] with charge assignment

$$\mathbf{16}_F \to e^{\mathrm{i}\alpha}\mathbf{16}_F, \quad \mathbf{10}_H \to e^{-2\mathrm{i}\alpha}\mathbf{10}_H, \quad \overline{\mathbf{126}}_H \to e^{-2\mathrm{i}\alpha}\overline{\mathbf{126}}_H \tag{3.50}$$

for some real parameter α. This forbids $\mathbf{10}_H^*$ from coupling to the fermion bilinear, meaning that we can have two separate vevs v_{10}^u and v_{10}^d, while only having one Yukawa matrix for the coupling to $\mathbf{10}_H$. The $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry at the same time solves the strong $\mathcal{CP}$ problem and can provide axion dark matter, as discussed in Sec. 2.2. For more details on axions in SO(10) models, see Ref. [162].

One can also consider more extended models in which we include a $\mathbf{120}_H$, as was done in Paper (I). This can allow for better fits to the measured parameter values of the Yukawa sector, but it decreases the minimality of the model. Other Yukawa sectors are of course also possible, such as $\overline{\mathbf{126}}_H \oplus \mathbf{120}_H$, or several copies of each of the three representations. Models without $\overline{\mathbf{126}}_H$ are also possible if one adds scalars in a $\overline{\mathbf{16}}_H$ representation. Then, the neutrino mass can be generated by a non-renormalizable interaction $\mathbf{16}_F \cdot \overline{\mathbf{16}}_H \cdot \overline{\mathbf{16}}_H \cdot \mathbf{16}_F$ via the Witten mechanism [163], since the product of two $\overline{\mathbf{16}}$s contain $\overline{\mathbf{126}}$.

3.4 Proton decay

Grand unified theories generically predict exotic interactions via their additional gauge bosons and scalars, which can mediate proton decay [115, 164, 165]. The gauge boson-mediated proton decay can be seen in the covariant derivative of the fermions, in which gauge bosons in the **45** of SO(10) couple to both quarks and leptons. They are thus called “leptoquark” gauge bosons. In the Yukawa sector, there are scalars in $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$ that do not contribute to the fermion masses which are leptoquark scalars and can mediate proton decay. This is illustrated in Fig. 3.4.

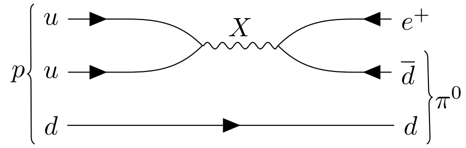


Figure 3.4: Example of a process leading to proton decay through a leptoquark gauge boson X.

These leptoquark gauge and scalar bosons violate baryon number. They have masses around the unification scale in order to suppress the proton decay rate. Hence, the non-observation of proton decay places a lower bound on the scale M_{GUT}. For low-energy phenomenology, one can integrate them out and find effective operators of dimension six that describe proton decay [166–169].

Note that although proton decay is a generic prediction of GUTs, there is some model dependence in the allowed effective operators, depending on which couplings are present in the full theory [124]. Furthermore, there is a difference in models with or without SUSY, since SUSY allows also for dimension-four and -five effective operators which may lead to faster proton decay [170, 171].

An order of magnitude estimate of the proton decay width with a GUT-scale gauge mediator is

$$\Gamma \sim \alpha_{\mathrm{GUT}}^2 \frac{m_p^5}{M_{\mathrm{GUT}}^4}, \tag{3.51}$$

where $\alpha_{\mathrm{GUT}} = g_{\mathrm{GUT}}^2/4\pi$, with g_{GUT} being the gauge coupling at the scale M_{GUT}, and m_p is the proton mass. More precise calculations depend on the particular decay channel. For example, the decay to a pion and a positron, which is the most relevant one, has approximate decay width given by [164, 172]

$$\Gamma(p \to e^+\pi^0) \simeq \frac{m_p}{64\pi f_\pi^2} \frac{g_{\mathrm{GUT}}^4}{M_{\mathrm{GUT}}^4} A_L^2 \alpha_H^2 F_q, \tag{3.52}$$

where $f_\pi \simeq 139\,\mathrm{MeV}$ is the pion decay constant, $A_L \simeq 2.726$ is a renormalization factor, $\alpha_H \simeq 0.012\,\mathrm{GeV}^3$ is the hadronic matrix element, and $F_q \simeq 7.6$ is a quark-mixing factor. This gives an estimate for the proton lifetime[2]

$$\tau(p \to e^+\pi^0) \simeq (7.47 \times 10^{35}\,\mathrm{yr}) \left(\frac{M_{\mathrm{GUT}}}{10^{16}\,\mathrm{GeV}}\right)^4 \left(\frac{0.03}{\alpha_{\mathrm{GUT}}}\right)^2. \tag{3.53}$$

If proton decay were observed in different channels, then it would be useful to consider the decay rates to the different channels in order to determine details of the underlying GUT model. However, the non-observation of proton decay to date puts a lower bound on the proton lifetime. The current best lower bound is from

[2]Note that an incorrect version of this was given in Eq. (13) in Paper (II). An erratum has been submitted to the journal and the version on `arXiv` has been updated.

Super-Kamiokande [173–175], which puts the bounds $\tau(p \to e^+\pi^0) > 1.67 \times 10^{34}$ yr, $\tau(p \to \mu^+\pi^0) > 7.78 \times 10^{33}$ yr, and $\tau(p \to \nu K^+) > 6.61 \times 10^{33}$ yr at 90 % confidence level. The projected Hyper-Kamiokande is expected to increase these bounds to $\tau(p \to e^+\pi^0) > 5.5 \times 10^{34}$ yr and $\tau(p \to \nu K^+) > 1.8 \times 10^{34}$ yr after five years of collecting data [176]. With a coupling $\alpha_{\rm GUT} \approx 0.03$, the current bound on the proton lifetime implies $M_{\rm GUT} \gtrsim 4 \times 10^{15}$ GeV.

3.5 Phenomenology in SO(10) models

A large range of phenomenology can be embedded within SO(10) models. For example, several different dark matter candidates can be accommodated. One such example is the possibility of having axion dark matter due to the $\mathrm{U}(1)_{\rm PQ}$ symmetry, as mentioned in Sec. 3.3.2. Dark matter may also be in the form of scalar or fermion, which is stabilized due to an inherent parity symmetry inside the SO(10) group structure [157, 177–185].

Since SO(10) naturally embeds a heavy right-handed neutrino, it provides the necessary ingredients for generating the baryon asymmetry through the mechanism of leptogenesis [53, 183, 186–193]. Furthermore, due to the additional scalar and vector leptoquarks, both SO(10) and its PS subgroup are interesting from the point of view of the B-physics anomalies [194–197].

3.6 Current status of SO(10) models

Models based on the SO(10) gauge group are still viable models for physics beyond the SM. There are essentially only two ways in which such models could be ruled out. The first is if they predict wrong mass relations. This is the case for the most minimal models in which only one Higgs representation is used. However, it is easy to generalize the model to include another Higgs representation in order to save the mass relation. Thus, only very specific model details may be ruled out on this ground. In fact, SO(10) models with Higgs fields in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations are not only able to accommodate the correct mass relations, but also the neutrino masses through the seesaw mechanism.

The second way in which SO(10) models and GUT models in general may be ruled out is by proton decay. As of the writing of this thesis, proton decay has not yet been observed. This non-observation keeps increasing the lower bound on the proton lifetime. This, in effect, increases the minimum value of $M_{\rm GUT}$ allowed. Constraints on $M_{\rm GUT}$ can rule out the simplest models, but more freedom can always be introduced by adding intermediate symmetry breaking scales or extra fields that alter the RG running. Due to the freedom that is in model building, grand unification is still a viable source for physics beyond the SM.

Chapter 4

Renormalization group running and numerical methods

It is non-trivial to link physics at the unification scale to observations at the scale of experiments. The difference in energy spans around 14 order of magnitude. Therefore, there are significant renormalization group effects between the two scales, as demonstrated for gauge coupling unification in Sec. 3.1. Further, if some particles have their masses between those two scales, the corresponding mass thresholds need to be appropriately taken into account.

An important viability test of GUT models is whether they can accommodate the measured observables of the SM, namely masses and mixing parameters. To test this, one must have a way of relating the GUT-scale parameters to the observables measured in experiments, as well as a way to explore the parameter space of the model under consideration.

This chapter deals with the methods involved in fitting the Yukawa sector of SO(10) models to the measured values of the observables (masses and mixing parameters) in the SM. We start with a general description of renormalization. Then, we describe the process of integrating out intermediate particles such as the heavy right-handed neutrinos (RHNs) between M_{GUT} and M_Z. Finally, we describe the algorithm used in performing the numerical fits.

4.1 Renormalization group running

A key feature of QFTs is the renormalization of parameters of the theory. Calculations performed in perturbation theory suffer from infinities which arise from the calculation of loop-level Feynman diagrams. In order to perform meaningful computations, one has to regularize these divergences, which can be done in a number of ways. In the process, one introduces a dependence on some arbitrary energy scale. Since the physics of the original theory must be invariant under changes of that energy scale, the parameters of the Lagrangian must change accordingly, so as to counter the effect of changes in scale. This gives the renormalized parameters of the theory, which depend on the energy scale through the RGEs. More details on this process may be found in numerous resources, for example the textbooks in Refs. [198–200].

4.1.1 Regularization

There are several ways to regularize the integrals that arise from loops in Feynman diagrams. As an example, consider the integral over a scalar propagator in a loop, *i.e.*

$$\int \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{4.1}$$

where the integral runs over the whole range of four-momentum k^μ. Evidently, the contributions from arbitrarily large four-momentum will cause the integral to diverge. Regularization is the procedure in which we extract a meaningful answer from integrals such as Eq. (4.1).

One such procedure is to introduce a cutoff, meaning that the integral is only evaluated up to some arbitrary scale Λ,

$$\int_0^\infty \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon} \to \int_0^\Lambda \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{4.2}$$

which is finite. The parameter Λ parametrizes the divergence and the original integral is recovered in the limit $\Lambda \to \infty$.

Another procedure of regularization is the Pauli-Villars procedure [201], which involves modifying the integrand as

$$\frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} \to \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} - \frac{1}{k^2 - \Lambda^2 + \mathrm{i}\epsilon}. \tag{4.3}$$

Here, Λ is the parametrization of the divergence and the original expression is again recovered in the limit $\Lambda \to \infty$. The second term may be interpreted as the contribution of a fictitious particle with the wrong sign of the propagator.

The most widely used procedure of regularization is dimensional regularization [202], in which the number of spacetime dimensions is altered from $d = 4$ to $d = 4 - \epsilon$, such that the original divergent expression is recovered in the limit $\epsilon \to 0$. We must similarly modify all expressions that are dependent on the spacetime dimension, such as spinor algebra and surface integrals, to their equivalents in non-integer spacetime dimensions.

Requiring that the action

$$S = \int \mathrm{d}^d x \, \mathcal{L} \tag{4.4}$$

be dimensionless and that the mass parameters have dimensions of energy, we find that the dimensions of scalar, vector, and spinorial fields are

$$[\phi] = [A_\mu] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}. \tag{4.5}$$

In order to keep the coupling constants dimensionless, we redefine them by extracting the relevant mass dimension. For the gauge, Yukawa, and scalar quartic couplings, we find

$$g \to \mu^{\epsilon/2} g, \quad Y \to \mu^{\epsilon/2} Y, \quad \lambda \to \mu^{\epsilon} \lambda, \tag{4.6}$$

where μ is some arbitrary parameter of mass dimension 1.

Within this scheme, the loop corrections to quantities may be calculated and the divergences are encapsulated in poles as $\epsilon \to 0$. The scale μ was introduced to make the mass dimensions correct and physical quantities should be independent of it.

4.1.2 Renormalization

After regularizing the loop integrals, the next step is renormalization, which corresponds to removing the divergent parts of the result so that we are left with a meaningful physical quantity. This is done by adding to the original (called "bare") Lagrangian some counterterms which act to cancel out the divergences in calculated physical quantities,

$$\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}, \tag{4.7}$$

where $\mathcal{L}_0$ is the bare Lagrangian, $\delta\mathcal{L}$ contains the counterterms, and $\mathcal{L}$ is known as the renormalized Lagrangian, from which finite quantities may be derived. The counterterms are divergent such that they will exactly cancel the divergences that arise in quantities of the bare Lagrangian.

In order to define the counterterms, one must choose a renormalization scheme to work in. Here, we use the "modified minimal subtraction" scheme, $\overline{\text{MS}}$ [203], which subtracts the pole $1/\epsilon$ as well as extra constants that often arise, namely $\ln 4\pi - \gamma_{\text{E}}$ where $\gamma_{\text{E}} \approx 0.577$ is the Euler–Mascheroni constant. This is in comparison to the minimal subtraction scheme, MS [204, 205], which subtracts only the $1/\epsilon$ pole.

In the process of dimensional regularization, the energy scale μ was introduced. The counterterms will therefore involve this scale, but the physical results should not. Thus, any dependence of physical quantities on the renormalization scale μ is canceled by an opposite dependence of the coupling constants on μ. Consider a

Green's function involving n fermion fields ψ and m scalar fields ϕ with a coupling y. The bare and renormalized Green's functions will be related via a field rescaling,

$$G_0^{(n,m)}(\{x_i\}, y_0) = Z_\psi^{n/2} Z_\phi^{m/2} G^{(n,m)}(\{x_i\}, y, \mu), \tag{4.8}$$

where y_0 is the bare coupling. The bare Green's function must be independent of μ, since this is an artifact of the renormalization process and not part of the theory itself. In other words,

$$\begin{aligned} 0 &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_0^{(n,m)}(\{x_i\}, y_0, \mu) \\ &= Z_\psi^{n/2} Z_\phi^{m/2} \left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \frac{\mu}{Z_\psi} \frac{\partial Z_\psi}{\partial \mu} + \frac{m}{2} \frac{\mu}{Z_\phi} \frac{\partial Z_\phi}{\partial \mu} + \mu \frac{\partial y}{\partial \mu} \frac{\partial}{\partial y} \right) G^{(n,m)}(\{x_i\}, y, \mu). \end{aligned} \tag{4.9}$$

This is the Callan–Symanzik equation [206–208], from which we define the β-function for the coupling y,

$$\beta_y \equiv \mu \frac{\partial y}{\partial \mu}. \tag{4.10}$$

By computing the Green's functions and counterterms perturbatively, one can solve the Callan–Symanzik equation for the β-function. This gives the RGE for the coupling y which determines its running behavior, meaning that its value depends on the center-of-mass energy of the interaction. Although they will in general vary depending on the QFT under consideration, there are standard formulae for the β-functions of gauge couplings in non-Abelian gauge theories as well as Yukawa couplings and scalar quartic couplings [209–215].

The gauge coupling g_i of a non-Abelian theory with group $\mathcal{G}_i$ has β-function given by [19, 20]

$$\beta g_i = -\frac{g_i^3}{(4\pi)^2} \left[\frac{11}{3} C_2(\mathbf{G}_i) - \frac{2}{3} S_2(\mathbf{F}_i) - \frac{1}{6} S_2(\mathbf{S}_i) \right] \tag{4.11}$$

where $C_2(\mathbf{G}_i)$ is the quadratic Casimir of the adjoint representation of the gauge group $\mathcal{G}_i$ and $S_2(\mathbf{F}_i)$ and $S_2(\mathbf{S}_i)$ are the Dynkin indices of the fermion and scalar representations under the gauge group $\mathcal{G}_i$, respectively, summed over all degrees of freedom in the model. For the fermions, this sum is taken over all Weyl spinor degrees of freedom.

Inserting the SM field content into Eq. (4.11), we find the coefficients given in Eq. (3.10), which were used to investigate the gauge coupling unification. The general formulae for the beta functions of the Yukawa and scalar quartic couplings are somewhat more complicated, and we refer the reader to Refs. [213–217]. They are also implemented in several software packages, such as `PyR@TE` [216] and `SARAH` [218].

The total set of RGEs that are needed to perform the RG running from M_{GUT} to M_Z are given in App. B. These are the β-functions for the gauge couplings, the Yukawa coupling matrices, the right-handed neutrino mass matrix, and the Higgs quartic self-coupling.

4.2 Intermediate masses

The RGEs of a model depend on the fields that are present in the model. As embodied in the Appelquist–Carazzone theorem [219], massive fields decouple from the theory at energies much lower than their masses, resulting in an effective field theory. This applies in mass-dependent renormalization schemes, in which the RGEs depended on the mass parameter μ. This is not the case in the $\overline{\mathrm{MS}}$ scheme, which is mass-independent. Instead, one must put in the scale at which the particle decouples by hand and integrate it out to give the low-energy effective field theory [220]. Thus, at energies above the mass of a particle, one solves the RGEs for the full theory which includes that particle. Below its mass, one solves the RGEs of the effective field theory that results from integrating it out. At its mass scale, one matches the two theories.

An example of this procedure is shown in the leftmost panel of Fig. 3.2. Below the mass threshold of the intermediate-scale field, the β-functions are calculated in a model without that field. There, the RG running was performed at the one-loop level and the matching between the two theories was performed at tree level.

Another important application of effective field theories in the RG running between M_Z and M_{GUT} is the integrating out of the heavy RHNs, which are relevant for generating neutrino masses through the type-I seesaw mechanism. When integrating them out, they are no longer part of the theory, but they contribute to the effective dimension-five operator [167], which generates neutrino masses. Therefore, as they are integrated out, their couplings to the light neutrinos are removed and their contribution to the neutrino masses is encoded by altering the coefficient of the effective operator.

The relevant quantities for this procedure are the Dirac neutrino Yukawa matrix Y_ν, the right-handed Majorana mass matrix M_R, and the effective neutrino mass matrix κ. In the full theory, they are all 3×3 matrices with κ being all zeroes. At the threshold corresponding to the heaviest RHN N_3, the relevant coefficients are removed from Y_ν and M_R and added to κ, following the process of Refs. [221, 222]. Thus, the last row of Y_ν is removed and it becomes a 2×3 matrix. The last row and column of M_R are removed and it becomes a 2×2 matrix. The effective neutrino mass matrix κ is updated to become

$$\kappa \to \kappa + \frac{2}{M_3} \left(Y_\nu^{(3)}\right)^T \left(Y_\nu^{(3)}\right), \tag{4.12}$$

where $Y_\nu^{(3)}$ is the last row of Y_ν, which was removed, and M_3 is the mass of N_3. On the right-hand side, κ is typically zero, but it may be non-zero if it received some other contribution at an energy above this. The procedure of removing rows or columns from matrices is clearly basis-dependent. It is therefore important to transform to a basis in which M_R is diagonal and apply the corresponding basis transformation to Y_ν before applying this procedure.

The matching procedure at the second heaviest RHN N_2 follows the same method. The last row is removed from Y_ν such that it becomes a 1×3 matrix and the last row and column of M_R is removed such that it becomes a 1×1 matrix. The effective neutrino mass matrix is updated according to

$$\kappa \to \kappa + \frac{2}{M_2}\left(Y_\nu^{(2)}\right)^T\left(Y_\nu^{(2)}\right). \tag{4.13}$$

At the last threshold, the lightest RHN N_1 is integrated out and matching is

$$\kappa \to \kappa + \frac{2}{M_1}\left(Y_\nu^{(1)}\right)^T\left(Y_\nu^{(1)}\right). \tag{4.14}$$

The difference now is that all entries in the matrices Y_ν and M_R have been removed and these quantities are no longer a part of the theory.

If there is also a scalar triplet causing the type-II seesaw mechanism, it needs to be integrated out at its mass scale. This is considerably more straightforward than integrating out the RHNs, since there is only one mass threshold. After solving the RGEs down to the mass scale of the scalar triplet, it is integrated out from the theory and the effective neutrino mass matrix is updated as [223]

$$\kappa \to \kappa - 4\frac{v_L}{v_{\mathrm{SM}}^2}Y_L, \tag{4.15}$$

where Y_L is the Yukawa matrix of the coupling between the neutrinos and the scalar triplet, which in the case of and SO(10) model is Y_{126}.

4.3 Numerical procedure

To numerically fit the parameters of the SO(10) model to the measured observables of the SM and the neutrino sector, the RG running must be performed, taking into account the RHN mass thresholds. Since it is not possible to uniquely reverse the matching conditions Eqs. (4.12)–(4.14), one must solve the system of RGEs from M_{GUT} down to M_Z, as opposed to the more obvious approach of extrapolating the observables up to M_{GUT}. This latter procedure ignores the effects of integrating out the RHNs in a consistent way and therefore involves an approximation, which can substantially change the predicted values for the neutrino masses and leptonic mixing parameters [221]. Nevertheless, it provides a good indication of whether an SO(10) model is viable and has therefore been used extensively in numerical fits [111, 191, 224–231].

In the work that comprises this thesis we perform the fitting from the high-energy theory down to the experimentally accessible energy at M_Z, as this enables a consistent analysis of the effects of RG running and matching at RHN thresholds [232] and symmetry breaking scales [233, 234]. For concreteness, consider a model with a Higgs sector consisting of a complexified $\mathbf{10}_H$ and a $\overline{\mathbf{126}}_H$ with a PQ symmetry as discussed in Sec. 3.3.2. Assume that the neutrino masses are generated purely by the type-I seesaw mechanism such that we can ignore any contributions from the scalar triplet that leads to the type-II seesaw mechanism.

We first sample the parameters of the SO(10) model, which are the elements of the Yukawa coupling matrices Y_{10} and Y_{126}, as well as the vevs $v_{10,126}^{u,d}$ and v_σ. It is convenient to rescale the parameters such that [224, 232]

$$\begin{aligned} H &\equiv \frac{v_{10}^d}{v_{\rm SM}} Y_{10}, \quad F \equiv \frac{v_{126}^d}{v_{\rm SM}} Y_{126}, \quad r \equiv \frac{v_{10}^u}{v_{10}^d}, \\ s &\equiv \frac{1}{r} \frac{v_{126}^u}{v_{126}^d} = \frac{v_{10}^d}{v_{10}^u} \frac{v_{126}^u}{v_{126}^d}, \quad r_R \equiv v_\sigma \frac{v_{\rm SM}}{v_{126}^d}, \end{aligned} \tag{4.16}$$

and sample these instead. Note that here we use $v_{\rm SM} = v/\sqrt{2}$, with v being the vev as given in Sec. 2.1.3. The matching conditions for this model to the SM Yukawa matrices Eqs. (3.38)–(3.42) may thus be written as

$$Y_u = r(H + sF), \tag{4.17}$$

$$Y_d = H + F, \tag{4.18}$$

$$Y_\nu = r(H - 3sF), \tag{4.19}$$

$$Y_\ell = H - 3F, \tag{4.20}$$

$$M_R = r_R F. \tag{4.21}$$

The Yukawa matrices Y_{10} and Y_{126} are in general complex symmetric 3×3 matrices. One may, however, choose a basis in which Y_{10} (and hence H) is real and diagonal. This means that there are three parameters in H and twelve parameters in F. Since r and r_R are overall multiplicative factors, their complex phases will have no effect on the fermion observables, so they can be taken to be real. Finally, the complex phase of s will have an effect and thus it remains complex. The total number of parameters in this model is thus $3(H)+12(F)+1(r)+2(s)+1(r_R) = 19$.

The fitting procedure is performed by first sampling these 19 parameters according to some priors that reflect the expected orders of magnitude of these parameters. For parameters that can vary over several orders of magnitude, it is reasonable to assume a logarithmic prior. Rewriting Eq. (3.47) in terms of the new parameters introduced, we find the constraint (neglecting v_L and the vevs from $\mathbf{120}_H$)

$$\left(\frac{v_{10}^d}{v_{\rm SM}} \right)^2 (1 + r^2) + \left(\frac{v_{126}^d}{v_{\rm SM}} \right)^2 (1 + r^2 s^2) = 1. \tag{4.22}$$

Since the sampled parameters are r and s, we have some freedom in choosing v_{10}^d and v_{126}^d such that the constraint is satisfied. The only lower bound on the vevs is that the Yukawa couplings $Y_{10} = v_{\rm SM} H / v_{10}^d$ and $Y_{126} = v_{\rm SM} F / v_{126}^d$ remain perturbative. This is usually automatically satisfied in the fits, but should be checked explicitly.

After the parameter values have been sampled, they are transformed to the Yukawa couplings of the SM using the matching conditions Eqs. (4.17)–(4.21). After this, the RGEs are solved from $M_{\rm GUT}$ down to M_3. There, N_3 is integrated out following the procedure outlined in Sec. 4.2. This modifies Y_ν and M_R and

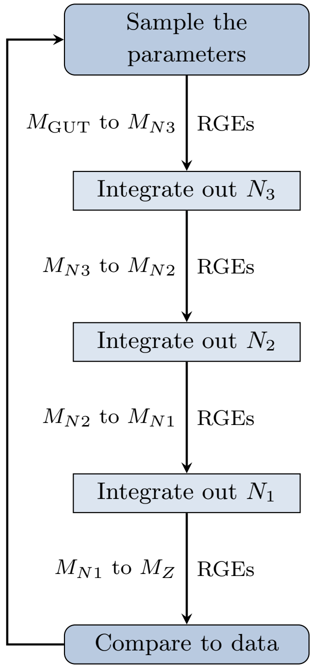


Figure 4.1: Flowchart showing the procedure used in performing the numerical fits for a model with the type-I seesaw mechanism. The type-II seesaw mechanism may be added by introducing another threshold at which the scalar triplet is integrated out. The procedure is coupled to a sampling algorithm, which samples new sets of parameter values based on the output χ^2.

introduces κ into the theory. Thereafter, the RGEs are solved from M_3 down to M_2, where N_2 is integrated out. From there, the RGEs are solved from M_2 to M_1 where N_1 is integrated out. Then, Y_ν and M_R are no longer part of the theory, so they have no corresponding RGEs. Finally, the RGEs are solved from M_1 down to M_Z. This procedure is depicted in Fig. 4.1. In order to include the type-II seesaw mechanism, one just has to include an extra step in the procedure, by integrating out the scalar triplet at the appropriate energy scale. Similarly, the procedure has to be appropriately modified if there are other intermediate particles or an intermediate symmetry breaking step.

At this scale, the parameters are transformed fermion masses and mixing parameters. These are compared to the data through a χ^2 goodness-of-fit function, defined as

$$\chi^2 = \sum_{i=1}^{N} \frac{(x_i - X_i)^2}{\sigma_i^2}, \tag{4.23}$$

where x_i denotes the predicted value of the ith observable obtained from the RG running and X_i denotes the actual value with corresponding error σ_i. Note that we do not assign any statistical meaning to the χ^2 function and merely use it as a measure of the goodness of fit.

The program that takes as input the set of parameter values, performs the RG running and outputs the χ^2 value is linked to a sampling algorithm which performs the minimization of the χ^2 function and thereby produces an optimal set of parameter values. There are many possible optimization algorithms and packages available to choose from. One is `MultiNest` [235–237], which is a nested sampling algorithm with capabilities to perform Bayesian inference. This was used in Paper (I). Another one is `Diver` [238], which is part of the ScannerBit package from the GAMBIT collaboration. It is a differential evolution algorithm and has been shown to outperform `MultiNest` in high-dimensional parameter spaces such as the ones for our problem. This was used in Papers (II) and (III).

The sampling algorithm was run on a computational cluster, utilizing up to 240 CPU cores. It was launched several times in order to increase confidence that the minimum was a global one. After reasonable convergence, another local minimization procedure was used, such as the basin-hopping algorithm [239] from the `Scipy` library [240], which perturbs the point around the starting point in parameter space, and a Nelder–Mead simplex algorithm [241], which further improves the best-fit point by traversing the χ^2 manifold downhill.

Chapter 5

Summary and conclusions

Part I of this thesis introduced the background material relevant to the research presented in Part II of this thesis. We started in Ch. 2 with the SM of particle physics and the reasons for investigating theories beyond it. These included shortcomings such as the massless nature of neutrinos in the SM, the lack of a dark matter candidate, and the unknown mechanism for producing a matter-antimatter asymmetry in the Universe. Certain aesthetic shortcomings also pointed to a theory beyond the SM. These included the origin of the gauge group of the SM, the underlying reasons for anomaly cancellation and charge quantization, as well as problems related to naturalness.

In Ch. 3, we introduced GUTs as an extension of the SM and a possible solution to some of the shortcomings discussed in the preceding chapter. This proceeded by the tantalizing clues of approximate gauge coupling unification at a high scale, followed by a discussion on some general features of GUTs, as well as specific features of popular GUT models. These included the Pati–Salam model, SU(5)-based models, and SO(10)-based models on which we focused in more detail.

Finally, in Ch. 4, we discussed the method used to perform numerical fits to SO(10)-based models. The central concept is renormalization and the solution of RGEs. Crucially, the difference between the method used in our investigations and most other similar investigations is that we appropriately take into account the effects of integrating out the heavy right-handed neutrinos at their thresholds. This can have a large effect on the RG running of the parameters, particularly those related to neutrinos.

In Paper (I), we investigated the RG running of fermion observables in an SO(10) model with intermediate symmetry breaking via the PS group. In this paper, we made the assumption that the heavy RHNs all have masses around the intermediate scale such that they are all integrated out at that scale. We compared two different models, namely the minimal model which has a Yukawa sector consisting of scalars in only the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations, and the extended model which additionally has scalars in the $\mathbf{120}_H$ representation. It was found that it is difficult to find a good fit with the minimal model, while the extended model was able to accommodate the fermion observables. Fits to neutrino masses were done both for normal and inverted ordering and it was found that it was only possible to accommodate normal neutrino mass ordering. The difficulty in finding a good fit was found to come from the value of the leptonic mixing angle θ_{23}^{ℓ}, for which values smaller than the actual value was consistently favored.

In Paper (II), we considered a model with a one-step breaking of the SO(10) symmetry to the SM. Gauge coupling unification was achieved by including extra scalars originating in the $\mathbf{210}_H$ representation with masses between M_Z and M_{GUT}, as shown in the left panel of Fig. 3.2. An analysis of gauge coupling unification resulted in a correlation between the masses of these intermediate scalars and the proton lifetime. If proton decay is not observed at Hyper-Kamiokande within five years, the resulting bound on the proton lifetime would, together with LHC bounds on the mass of the extra scalars, rule out the model. The Yukawa sector contains scalars in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations only. It was found that the model could accommodate the fermion observables well, and also that the presence of the intermediate-scale scalars helped to stabilize the vacuum. Again, the largest source of difficulty in fitting the fermion observables was a too low predicted value of θ_{23}^{ℓ}.

Finally, in Paper (III), we did not consider any specific model, but considered a generic set of models with one-step symmetry breaking. Since no specific model was imposed, we did not require gauge coupling unification, but instead tested the sensitivity of the results to changes in M_{GUT}. The Yukawa sector contained scalars in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations. We considered neutrino masses arising from the type-I or type-II seesaw mechanisms, or a combination of both. We found that a pure type-II seesaw mechanism was disfavored by the fits and that when both were combined, the type-I mechanism was the dominant contributor to neutrino masses. Neutrino masses could only be accommodated with normal ordering and not with inverted ordering. It was seen that the fit results were fairly insensitive to changes in M_{GUT} within an order of magnitude. Once again, it was consistently observed that the most difficulty in finding a good fit came from the favored value of θ_{23}^{ℓ} being too low compared to the actual value.

The work in this thesis has focused on the Yukawa sector of non-SUSY SO(10)-based GUT models. The question that has led this research is: is SO(10) a viable model for the fermion masses and mixing parameters, taking into account the RG running between M_{GUT} and M_Z? The answer is: yes, depending on the details of the model. Since a proper treatment of the RGEs and thresholds of heavy RHNs have a large impact on the neutrino observables, it is important to consider these.

SO(10) and grand unification in general continue to provide a promising framework for physics beyond the SM. In addition to providing more structure behind some of the arbitrary aspects of the SM, it can incorporate a wide range of phenomenology to solve some of the current open questions in particle physics. The criticism against grand unification is primarily that the unification scale is much higher than will be reachable in the near future and that it is thus not testable.

While it is true that the unification scale will most likely not be reachable in colliders in the foreseeable future, there are still some signatures of grand unification that can be observed. The most notable is proton decay, the detection of which may still be possible in future experiments. The arguments in favor of grand unification are still very much the same as those given in the original proposal by Georgi and Glashow is 1974: "[...] the uniqueness and simplicity of our scheme are reasons enough that it be taken seriously" [117]. Although the current state of GUTs are somewhat more complicated than the original proposals, they still have the advantage of simplicity over the SM in some aspects of their structure.

Appendix A

Group theory

This appendix briefly summarizes some aspects of group and representation theory that are relevant to the study of GUTs. Further details about group theory can be found in textbooks such as Ref. [242] or the review Ref. [116], which contains numerous tables that are useful for model building. There are several software packages that are capable of performing group theoretic calculations, such as `Susyno` [243] or `LieART` [244].

A.1 Basics of Lie groups, Lie algebras, and representations

A Lie group is a group in which the elements depend on a set of continuous parameters. Elements can be written in terms of the generators t^a of the group via the exponential map,

$$g(\alpha) = \exp(i\alpha^a t^a). \tag{A.1}$$

The generators form a Lie algebra with the Lie bracket

$$[t^a, t^b] = i f^{abc} t^c, \tag{A.2}$$

where f^{abc} are the structure constants of the Lie algebra.

We may define a representation of the group as a map from the group elements to square matrices such that the elements of the group representation act on elements of a vector space. The representation of the group is also a representation of the algebra, in the sense that the representation matrices of the algebra generate the representation matrices of the group via the exponential map. The dimension of the representation is the number of dimensions of the vector space on which the representation matrices act. Thus, a representation with 3×3 representation matrices is 3-dimensional.

We are usually interested in irreducible representations, which are representations with no invariant subspaces. In contrast, reducible representations contain invariant subspaces and may therefore be decomposed into direct products of irreducible representations.

The Cartan generators of a representation are a set of mutually commuting Hermitian generators. They are useful because they can be simultaneously diagonalized. Hence, they can be used to assign quantum numbers to states within a representation, such as the T_3 generator of $\mathrm{SU}(2)_\mathrm{L}$ in the Gell-Mann–Nishijima formula Eq. (2.16). This becomes particularly relevant in symmetry breaking, since the Abelian charges will be combinations of these quantum numbers. The number of Cartan generators of an algebra is known as its rank.

A common group in particle physics is $\mathrm{SU}(N)$, which is the group defined by $N \times N$ special unitary matrices. That is, its elements in the defining representation are $N \times N$ unitary matrices with determinant 1. There are $N^2 - 1$ such matrices, meaning that there are $N^2 - 1$ generators of $\mathrm{SU}(N)$. These are $N \times N$ traceless Hermitian matrices. In other representations, the generators are of different dimension but with the same structure constants.

Relevant for GUT model building are also the $\mathrm{SO}(N)$ groups. The defining representation of these are the set of $N \times N$ orthogonal matrices with determinant 1, of which there are $N(N-1)/2$. Hence the generators in this representation are antisymmetric traceless $N \times N$ matrices.

Two useful constants for calculations of β-functions are the quadratic Casimir and the Dynkin index. The quadratic Casimir $C_2(\mathbf{r})$ is defined for a representation $\mathbf{r}$ of a Lie algebra as

$$t^a_\mathbf{r} t^a_\mathbf{r} = C_2(\mathbf{r}) \mathbb{1}, \tag{A.3}$$

where the index a is summed over the generators of that representation. The Dynkin index $S_2(\mathbf{r})$ is defined as

$$\mathrm{Tr}(t^a_\mathbf{r} t^b_\mathbf{r}) = S_2(\mathbf{r}) \delta^{ab}. \tag{A.4}$$

The two are related by the relation

$$S_2(\mathbf{r}) = \frac{\dim(\mathbf{r})}{\dim(\mathbf{Adj})} C_2(\mathbf{r}), \tag{A.5}$$

where $\dim(\mathbf{r})$ is the dimension of representation $\mathbf{r}$ and $\mathbf{Adj}$ denotes the adjoint representation.

A.2 Decompositions of some SO(10) representations

To construct a model beyond the SM, we need to make sure that it can reproduce the SM. Thus, the GUT symmetry needs to be broken down to $\mathcal{G}_\mathrm{SM}$. Therefore, it is useful to know how the different representations of SO(10) decompose under its different subgroups. This is necessary both for the study of how to produce each breaking chain by looking for the singlets under the different subgroups and also for tracking how a given representation traverses the breaking chain down to $\mathcal{G}_{\mathrm{SM}}$. Thus, we give the decompositions of the representations up to dimension 210 in the SU(5) breaking chain in Tab. A.1 and the PS breaking chain in Tab. A.2.

SO(10)	$\mathcal{G}_{51}$	$\mathcal{G}_{\mathrm{SM}}$
$\mathbf{10}$	$\mathbf{5}_{-2}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{2}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
$\mathbf{16}$	$\mathbf{10}_{1}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{5}}_{-3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{1}_{5}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{45}$	$\mathbf{24}_{0}$	$(\mathbf{8},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{1},\mathbf{1})_{0}$
	$\mathbf{10}_{-4}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{10}}_{4}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{1}_{0}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{54}$	$\mathbf{24}_{0}$	$(\mathbf{8},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{1},\mathbf{1})_{0}$
	$\mathbf{15}_{-4}$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{3})_{1}$
	$\overline{\mathbf{15}}_{4}$	$(\mathbf{6},\mathbf{1})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{1},\mathbf{3})_{-1}$
$\mathbf{120}$	$\mathbf{45}_{-2}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6}$ $\oplus(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{45}}_{2}$	$(\mathbf{8},\mathbf{2})_{-1/2} \oplus (\overline{\mathbf{6}},\mathbf{1})_{1/3} \oplus (\overline{\mathbf{3}},\mathbf{3})_{1/3} \oplus (\mathbf{3},\mathbf{2})_{7/6}$ $\oplus(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{3},\mathbf{1})_{-4/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{10}_{6}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{10}}_{-6}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{5}_{-2}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{2}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$

$\overline{\mathbf{126}}$	$\overline{\mathbf{50}}_2$	$(\mathbf{8},\mathbf{2})_{-1/2} \oplus (\overline{\mathbf{6}},\mathbf{3})_{1/3} \oplus (\mathbf{6},\mathbf{1})_{-4/3} \oplus (\mathbf{3},\mathbf{2})_{7/6}$ $\oplus(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{1})_2$
	$\mathbf{45}_{-2}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6}$ $\oplus(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{15}}_{-6}$	$(\mathbf{6},\mathbf{1})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{1},\mathbf{3})_{-1}$
	$\mathbf{10}_6$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\overline{\mathbf{5}}_2$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\mathbf{1}_{10}$	$(\mathbf{1},\mathbf{1})_0$
$\mathbf{144}$	$\overline{\mathbf{45}}_{-3}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
	$\mathbf{40}_1$	$(\mathbf{6},\mathbf{2})_{1/6} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\overline{\mathbf{3}},\mathbf{3})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6}$ $\oplus(\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{2})_{-3/2}$
	$\mathbf{24}_5$	$(\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{15}_1$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{3})_1$
	$\mathbf{10}_1$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\mathbf{5}_{-7}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{-3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
$\mathbf{210}$	$\mathbf{75}_0$	$(\mathbf{8},\mathbf{3})_0 \oplus (\mathbf{6},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{6}},\mathbf{2})_{5/6} \oplus (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6}$ $\oplus(\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{3},\mathbf{1})_{5/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-5/3} \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{40}_{-4}$	$(\mathbf{6},\mathbf{2})_{1/6} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\overline{\mathbf{3}},\mathbf{3})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6}$
	$\overline{\mathbf{40}}_4$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6} \oplus (\mathbf{8},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{3})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
	$\mathbf{24}_0$	$(\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{10}_{-4}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\overline{\mathbf{10}}_4$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{5}_8$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{-8}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{1}_0$	$(\mathbf{1},\mathbf{1})_0$

Table A.1: Decompositions of representations of SO(10) up to 210 dimensions under the SU(5) breaking chain. Here, $\mathcal{G}_{51} = \mathrm{SU}(5) \times \mathrm{U}(1)$. We assume the standard hypercharge embedding in $\mathrm{SU}(5) \times \mathrm{U}(1)$. For the flipped hypercharge embedding, the hypercharge should be $Y = \frac{1}{5}(X - Y')$, where Y' is the Abelian charge from within the SU(5) group and X is the charge of the external Abelian group. For the decomposition under $\mathcal{G}_5$, simply remove the U(1) charge from the $G51$ representations.

SO(10)	$\mathcal{G}_{\rm PS}$	$\mathcal{G}_{3221}$	$\mathcal{G}_{3211}$	$\mathcal{G}_{\rm SM}$
$\mathbf{10}$	$(\mathbf{6},\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{2},\mathbf{2})_{0}$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
$\mathbf{16}$	$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{2},\mathbf{1})_{1/3}$	$(\mathbf{3},\mathbf{2})_{0,1/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{1})_{-1}$	$(\mathbf{1},\mathbf{2})_{0,-1}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-1/2,-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{1/2,-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{2})_{1}$	$(\mathbf{1},\mathbf{1})_{1/2,1}$	$(\mathbf{1},\mathbf{1})_{1}$
			$(\mathbf{1},\mathbf{1})_{-1/2,1}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{45}$	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
	$(\mathbf{1},\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\mathbf{1},\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{1})_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{-1,0}$	$(\mathbf{1},\mathbf{1})_{1}$
$\mathbf{54}$	$(\mathbf{20'},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{6},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{6},\mathbf{1})_{0,4/3}$	$(\mathbf{6},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{6}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3}$
	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{1},\mathbf{3},\mathbf{3})$	$(\mathbf{1},\mathbf{3},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{3})_{1,0}$	$(\mathbf{1},\mathbf{3})_{-1}$
			$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
			$(\mathbf{1},\mathbf{3})_{-1,0}$	$(\mathbf{1},\mathbf{3})_{1}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$

$\mathbf{120}$	$(\mathbf{15},\mathbf{2},\mathbf{2})$	$(\mathbf{8},\mathbf{2},\mathbf{2})_0$	$(\mathbf{8},\mathbf{2})_{1/2,0}$	$(\mathbf{8},\mathbf{2})_{-1/2}$
			$(\mathbf{8},\mathbf{2})_{-1/2,0}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{4/3}$	$(\mathbf{3},\mathbf{2})_{1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{7/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
	$(\mathbf{6},\mathbf{3},\mathbf{1})$	$(\mathbf{3},\mathbf{3},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{3})_{0,-2/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{1/3}$
	$(\mathbf{6},\mathbf{1},\mathbf{3})$	$(\mathbf{3},\mathbf{1},\mathbf{3})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{1,-2/3}$	$(\mathbf{3},\mathbf{1})_{-4/3}$
			$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1,-2/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
	$(\mathbf{10},\mathbf{1},\mathbf{1})$	$(\overline{\mathbf{6}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{1/3}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{-2}$	$(\mathbf{1},\mathbf{1})_{0,-2}$	$(\mathbf{1},\mathbf{1})_{-1}$
	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$	$(\mathbf{6},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{6},\mathbf{1})_{0,-2/3}$	$(\mathbf{6},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{2}$	$(\mathbf{1},\mathbf{1})_{0,2}$	$(\mathbf{1},\mathbf{1})_{1}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
$\overline{\mathbf{126}}$	$(\mathbf{15},\mathbf{2},\mathbf{2})$	$(\mathbf{8},\mathbf{2},\mathbf{2})_0$	$(\mathbf{8},\mathbf{2})_{1/2,0}$	$(\mathbf{8},\mathbf{2})_{-1/2}$
			$(\mathbf{8},\mathbf{2})_{-1/2,0}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{4/3}$	$(\mathbf{3},\mathbf{2})_{1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{7/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$

	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{1},\mathbf{3})_{-2/3}$	$(\mathbf{6},\mathbf{1})_{1,-2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{-4/3}$
			$(\mathbf{6},\mathbf{1})_{0,-2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{-1/3}$
			$(\mathbf{6},\mathbf{1})_{-1,-2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{3})_{2}$	$(\mathbf{1},\mathbf{1})_{1,2}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{0,2}$	$(\mathbf{1},\mathbf{1})_{1}$
			$(\mathbf{1},\mathbf{1})_{-1,2}$	$(\mathbf{1},\mathbf{1})_{2}$
	$(\mathbf{10},\mathbf{3},\mathbf{1})$	$(\overline{\mathbf{6}},\mathbf{3},\mathbf{1})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{3})_{0,2/3}$	$(\mathbf{6},\mathbf{3})_{1/3}$
		$(\mathbf{3},\mathbf{3},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{3})_{0,-2/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{1})_{-2}$	$(\mathbf{1},\mathbf{3})_{0,-2}$	$(\mathbf{1},\mathbf{3})_{-1}$
	$(\mathbf{6},\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
144	$(\mathbf{20},\mathbf{1},\mathbf{2})$	$(\mathbf{8},\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{8},\mathbf{1})_{1/2,-1}$	$(\mathbf{8},\mathbf{1})_{-1}$
			$(\mathbf{8},\mathbf{1})_{-1/2,-1}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{6},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{6},\mathbf{1})_{1/2,1/3}$	$(\mathbf{6},\mathbf{1})_{-1/3}$
			$(\mathbf{6},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{6},\mathbf{1})_{2/3}$
		$(\mathbf{3},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{1})_{1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2})_{5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/2,5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1/2,5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
	$(\overline{\mathbf{20}},\mathbf{2},\mathbf{1})$	$(\mathbf{8},\mathbf{2},\mathbf{1})_{1}$	$(\mathbf{8},\mathbf{2})_{0,1}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\overline{\mathbf{6}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6}$
		$(\mathbf{3},\mathbf{2},\mathbf{1})_{-5/3}$	$(\mathbf{3},\mathbf{2})_{0,-5/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
	$(\mathbf{4},\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{3},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{3})_{1/2,1/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
			$(\mathbf{3},\mathbf{3})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{3})_{2/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{3})_{1/2,-1}$	$(\mathbf{1},\mathbf{3})_{-1}$
			$(\mathbf{1},\mathbf{3})_{-1/2,-1}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\overline{\mathbf{4}},\mathbf{2},\mathbf{3})$	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{3})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{3})_{1}$	$(\mathbf{1},\mathbf{2})_{1,1}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{0,1}$	$(\mathbf{1},\mathbf{2})_{1/2}$
			$(\mathbf{1},\mathbf{2})_{-1,1}$	$(\mathbf{1},\mathbf{2})_{3/2}$
	$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{1})_{1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_{1/2,-1}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{-1/2,-1}$	$(\mathbf{1},\mathbf{1})_{0}$

	$(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{1})_{1}$	$(\mathbf{1},\mathbf{2})_{0,1}$	$(\mathbf{1},\mathbf{2})_{1/2}$
210	$(\mathbf{15},\mathbf{3},\mathbf{1})$	$(\mathbf{8},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{3})_{0,0}$	$(\mathbf{8},\mathbf{3})_{0}$
		$(\mathbf{3},\mathbf{3},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{3})_{0,4/3}$	$(\mathbf{3},\mathbf{3})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{-2/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\mathbf{15},\mathbf{1},\mathbf{3})$	$(\mathbf{8},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{8},\mathbf{1})_{1,0}$	$(\mathbf{8},\mathbf{1})_{-1}$
			$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
			$(\mathbf{8},\mathbf{1})_{-1,0}$	$(\mathbf{8},\mathbf{1})_{1}$
		$(\mathbf{3},\mathbf{1},\mathbf{3})_{4/3}$	$(\mathbf{3},\mathbf{1})_{1,4/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
			$(\mathbf{3},\mathbf{1})_{-1,4/3}$	$(\mathbf{3},\mathbf{1})_{5/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-5/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{1})_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{-1,0}$	$(\mathbf{1},\mathbf{1})_{1}$
	$(\mathbf{10},\mathbf{2},\mathbf{2})$	$(\overline{\mathbf{6}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{6}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{5/6}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_{-2}$	$(\mathbf{1},\mathbf{2})_{1/2,-2}$	$(\mathbf{1},\mathbf{2})_{-3/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,-2}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
	$(\overline{\mathbf{10}},\mathbf{2},\mathbf{2})$	$(\mathbf{6},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{6},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{6},\mathbf{2})_{1/6}$
			$(\mathbf{6},\mathbf{2})_{-(1/2),-2/3}$	$(\mathbf{6},\mathbf{2})_{-5/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_{2}$	$(\mathbf{1},\mathbf{2})_{1/2,2}$	$(\mathbf{1},\mathbf{2})_{1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,2}$	$(\mathbf{1},\mathbf{2})_{3/2}$
	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$

Table A.2: Decompositions of the representations of SO(10) up to 210 dimensions under the PS breaking chain. Note that the decomposition under $\mathcal{G}_{421}$ is not shown since it is easy to obtain from $\mathcal{G}_{\mathrm{PS}}$. The hypercharge related to $B-L$ and the third $\mathrm{SU}(2)_{\mathrm{R}}$ generator by $Y = \frac{B-L}{2} - T_{R,3}$.

A.3 Spinorial representations

A feature of $\mathrm{SO}(N)$ groups is that, in addition to the tensorial representations, they also contain spinorial representations. Spinorial representation differ for even and odd N. We will therefore first discuss the case for $N = 2n$ and then comment on how this construction is modified for the case of $\mathrm{SO}(2n-1)$.

In particular, there exists a representation that is generated by

$$\Sigma_{ij} = \frac{\mathrm{i}}{2}[\Gamma_i, \Gamma_j], \tag{A.6}$$

where the N matrices Γ_i are Hermitian and satisfy the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = \frac{1}{2}\delta_{ij}\mathbb{1}. \tag{A.7}$$

The proof of their existence follows most easily from their explicit construction, starting with two of the Pauli matrices for $n = 1$ and building up higher n iteratively. Concretely, we can take [143]

$$\Gamma_1^{(n=1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2^{(n=1)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{A.8}$$

and build up larger representations according to

$$\begin{aligned} \Gamma_i^{(n+1)} &= \begin{pmatrix} \Gamma_i^{(n)} & 0 \\ 0 & -\Gamma_i^{(n)} \end{pmatrix} \text{ for } i \in \{1, \dots, 2n\}, \\ \Gamma_{2n+1}^{(n+1)} &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \Gamma_{2n+2}^{(n+1)} = \begin{pmatrix} 0 & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & 0 \end{pmatrix}. \end{aligned} \tag{A.9}$$

These can be verified to satisfy the Clifford algebra Eq. (A.7). Hence, we have constructed the $2n+2$ matrices with dimension 2^{n+1} for $\mathrm{SO}(2n+2)$. We thus see that the spinorial representation that is generated by σ_{ij} is 2^n-dimensional. However, it is not an irreducible 2^n-dimensional representation, but splits into two irreducible 2^{n-1}-dimensional representations, in analogy with the chirality of Lorentz spinors. This can be done by defining an additional Γ-matrix by

$$\Gamma_0 = (-\mathrm{i})^n(\Gamma_1\Gamma_2\cdots\Gamma_{2n}). \tag{A.10}$$

This matrix satisfies

$$\Gamma_0^2 = \mathbb{1}, \quad [\Gamma_0, \Sigma_{ij}] = 0, \quad \{\Gamma_0, \Gamma_i\} = 0. \tag{A.11}$$

The existence of this matrix which commutes with all the generators shows that the representation is reducible. Defining the projection operators

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma_0), \tag{A.12}$$

we can split the 2^n-dimensional spinor representation into two 2^{n-1}-dimensional irreducible representations.

Specifying to SO(10), *i.e.* $n = 5$, we find that there are two irreducible $2^{n-1} = 16$-dimensional spinorial representations with opposite eigenvalues of Γ_0. This is the representation in which all fermions of one generation of the SM fit.

In order to write down fermion mass terms, we need a bilinear in $\mathbf{16}_F$ which we can use to couple the fermions to the scalars. The simplest version would then be something like

$$\mathbf{16}_F^T\mathbf{16}_F, \tag{A.13}$$

but since $\mathbf{16}_F^T$ does not transform like a conjugate spinor, this is not invariant [142]. To solve this, we introduce a matrix B, which is analogous to the charge conjugation matrix for Lorentz spinors. If this matrix satisfies

$$B^{-1}\Sigma_{ij}^T B = -\Sigma_{ij}, \tag{A.14}$$

then the bilinear $\mathbf{16}_F^T B\mathbf{16}_F$ is invariant under SO(10) transformations. An explicit construction is to start with

$$B^{(n=1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.15}$$

and iteratively build up the matrix for larger n by

$$B^{(n+1)} = \begin{pmatrix} 0 & B^{(n)} \\ (-1)^{n+1}B^{(n)} & 0 \end{pmatrix}, \tag{A.16}$$

from which one can also deduce the properties

$$B^{-1}\Gamma_i^T B = (-1)^n\Gamma_i, \quad B^{-1}\Gamma_0 B = (-1)^n\Gamma_0. \tag{A.17}$$

Using this, we can build the fermion bilinears. One can show that the bilinear

$$\mathbf{16}_F^T B\Gamma_{i_1}\cdots\Gamma_{i_k}\mathbf{16}_F \tag{A.18}$$

transforms as a k-index tensor under SO(10) transformations. Using the properties of the charge conjugation matrix and the fact that $\mathbf{16}_F$ is an eigenstate of Γ_0, one can show that Eq. (A.18) is zero unless k is odd. This is the case for **10**, **120**, and $\overline{\mathbf{126}}$, which are the ones that have an invariant coupling to two **16**s.

To find whether the Yukawa couplings should be symmetric or antisymmetric, we write down the coupling

$$\mathbf{16}_F^T C B \Gamma_{i_1} \cdots \Gamma_{i_k} \mathbf{16}_F, \tag{A.19}$$

where C is the Lorentz charge conjugation matrix. Taking the transpose of this and using the facts that the fermion fields are anticommuting, $C^T = -C$, and $B^T = -B$, we find

$$-\mathbf{16}_F^T \Gamma_{i_k}^T \cdots \Gamma_{i_1}^T B C \mathbf{16}_F = -\mathbf{16}_F^T B B^{-1} \Gamma_{i_k}^T B \cdots B^{-1} \Gamma_{i_1}^T B C \mathbf{16}_F, \tag{A.20}$$

where in the last equality we have inserted factors of $BB^{-1} = \mathbb{1}$ between each Γ-matrix. Using Eq. (A.17), we can write the sequence of Γ-matrices as

$$B B^{-1} \Gamma_{i_k}^T B \cdots B^{-1} \Gamma_{i_1}^T B = (-1)^k B \Gamma_{i_k} \cdots \Gamma_{i_1}. \tag{A.21}$$

The scalar representations that couple to the bilinear all have antisymmetric indices or just one index in the case of the **10**, meaning that the sequence of Γ-matrices must all be different. Thus, we can use the Clifford algebra to permute the Γ-matrices as

$$\Gamma_{i_k} \cdots \Gamma_{i_1} = (-1)^{k(k-1)/2} \Gamma_{i_1} \cdots \Gamma_{i_k}. \tag{A.22}$$

Overall, we have, with the Yukawa coupling matrix Y,

$$Y \mathbf{16}_F^T C B \Gamma_{i_1} \cdots \Gamma_{i_k} \mathbf{16}_F = (-1)^{k+k(k-1)/2+1} Y^T \mathbf{16}_F^T C B \Gamma_{i_1} \cdots \Gamma_{i_k} \mathbf{16}_F. \tag{A.23}$$

In order for this equality to hold, we must have $Y^T = \pm Y$, with the sign being the same as the factor $(-1)^{k+k(k-1)/2+1}$. For $k = 1$, *i.e.* **10**, we find $Y_{10} = Y_{10}^T$, for $k = 3$, *i.e.* **120**, $Y_{120} = -Y_{120}^T$, and for $k = 5$, i.e. $\overline{\mathbf{126}}$, $Y_{126} = Y_{126}^T$.

Finally, we comment on how the construction of the spinor representation changes for odd $N = 2n-1$. Interestingly, the irreducible spinor representation still has dimension 2^{n-1}, just like $N = 2n$ [143]. The same procedure for constructing the representation applies as for $N = 2n$, but we omit the matrix Γ_{2n}. The matrix Γ_0 can be taken to be the same as for $N = 2n$.

A.4 Anomalies

Computation of the triangle anomalies [84, 85] involves a group-theoretic factor

$$\mathcal{A}^{abc} = \mathrm{Tr}(\{t^a, t^b\} t^c), \tag{A.24}$$

where t^a are the generators of the representation to which the fields in the triangle diagram belong. The question of anomalies of a model therefore lends itself to a group-theoretic consideration [245–247].

For SO(N), we may label the $N(N-1)/2$ generators by two indices $i, j \in \{1, \ldots, N\}$, such that $t^{ij} = -t^{ji}$ are $N \times N$ matrices. That is, the indices i and j label the generator and are not matrix indices. Then, the anomaly factor

$$\mathcal{A}^{ijklmn} = \mathrm{Tr}(\{t^{ij}, t^{kl}\}t^{mn}) \tag{A.25}$$

is a six-index object. The SO(N) algebra implies that it must be antisymmetric under the exchange of indices $i \leftrightarrow j$, $k \leftrightarrow l$, and $m \leftrightarrow n$, while the structure of the anomaly factor implies that it must be symmetric under the exchange of pairs $ij \leftrightarrow kl$, $kl \leftrightarrow mn$, and $ij \leftrightarrow mn$. However, these two requirements are incompatible, since the most general object consistent with the antisymmetry requirement is

$$\begin{aligned}\mathcal{A}^{ijklmn} \propto (&\delta^{ik}\delta^{lm}\delta^{jn} - \delta^{jk}\delta^{lm}\delta^{in} - \delta^{il}\delta^{km}\delta^{jn} + \delta^{jl}\delta^{km}\delta^{in} \\ &- \delta^{ik}\delta^{ln}\delta^{jm} + \delta^{jk}\delta^{ln}\delta^{im} + \delta^{il}\delta^{kn}\delta^{jm} - \delta^{jl}\delta^{kn}\delta^{im}).\end{aligned} \tag{A.26}$$

However, under the interchange $ij \leftrightarrow kl$, this is antisymmetric. Hence, it must vanish and these algebras are identically anomaly-free.

For SO(6), this is not the most general six-index object due to the existence of the antisymmetric tensor ϵ^{ijklmn}, which is invariant under the exchange $ij \leftrightarrow kl$. The anomaly factor may be proportional to this, which means that SO(6) is not automatically anomaly-free. The same applies to SO(N) algebras with smaller N, since their N-index totally antisymmetric tensors ruin this proof. However, SO(5) is anomaly-free since one cannot construct a six-index object from its five-index antisymmetric tensor and Kronecker deltas.

Appendix B

Renormalization group equations

In this appendix, we list the RGEs that are required to perform the RG running from $M_{\rm GUT}$ to M_Z. The relevant parameters that run are: the gauge couplings g_i, the Higgs quartic self-coupling λ, the Yukawa coupling matrices for the up-type quarks Y_u, down-type quarks Y_d, neutrinos Y_ν, charged leptons Y_ℓ, scalar triplet Y_Δ (for type-II seesaw), the right-handed neutrino Majorana mass matrix M_R, and the effective neutrino mass matrix κ. Note that the β-functions for the gauge couplings g_1 and g_2 include contributions from the scalar triplet. If the scalar triplet is not in the model, or below its mass threshold, one should use only the first terms of Eqs. (B.1) and (B.2), as well as removing any contribution from Y_Δ. For detail on the numerical procedure used in solving the RGEs and integrating out heavy right-handed neutrinos and the scalar triplet, see Ch. 4. The complete set of RGEs used are [213–215, 221–223, 232, 248]

$$16\pi^2\beta_{g_1} = \frac{41}{10}g_1^3 + \frac{3}{5}g_1^3 = \frac{47}{10}g_1^3, \tag{B.1}$$

$$16\pi^2\beta_{g_2} = -\frac{19}{6}g_2^3 + \frac{2}{3}g_2^3 = -\frac{5}{2}g_2^3, \tag{B.2}$$

$$16\pi^2\beta_{g_3} = -7g_3^3, \tag{B.3}$$

$$\begin{aligned}16\pi^2\beta_\lambda = {} & 6\lambda^2 - 3\lambda\left(3g_2^2 + \frac{3}{5}g_1^2\right) + 3g_2^4 + \frac{3}{2}\left(\frac{3}{5}g_1^2 + g_2^2\right)^2 \\ & + 4\lambda\mathrm{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right] \\ & - 8\mathrm{Tr}\left[Y_\ell^\dagger Y_\ell Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d Y_d^\dagger Y_d + 3Y_u^\dagger Y_u Y_u^\dagger Y_u\right],\end{aligned} \tag{B.4}$$

$$16\pi^2\beta_{Y_u} = Y_u\Bigg(\frac{3}{2}Y_u^\dagger Y_u - \frac{3}{2}Y_d^\dagger Y_d - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \tag{B.5}$$

$$+ \operatorname{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right]\Bigg), \tag{B.6}$$

$$16\pi^2\beta_{Y_d} = Y_d\Bigg(\frac{3}{2}Y_d^\dagger Y_d - \frac{3}{2}Y_u^\dagger Y_u - \frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \tag{B.7}$$

$$+ \operatorname{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right]\Bigg), \tag{B.8}$$

$$16\pi^2\beta_{Y_\nu} = Y_\nu\Bigg(\frac{3}{2}Y_\nu^\dagger Y_\nu - \frac{3}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta - \frac{9}{20}g_1^2 - \frac{9}{4}g_2^2 \tag{B.9}$$

$$+ \operatorname{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right]\Bigg), \tag{B.10}$$

$$16\pi^2\beta_{Y_\ell} = Y_\ell\Bigg(\frac{3}{2}Y_\ell^\dagger Y_\ell - \frac{3}{2}Y_\nu^\dagger Y_\nu + \frac{3}{2}Y_\Delta^\dagger Y_\Delta - \frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 \tag{B.11}$$

$$+ \operatorname{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right]\Bigg), \tag{B.12}$$

$$16\pi^2\beta_{Y_\Delta} = \left(\frac{1}{2}Y_\nu^\dagger Y_\nu + \frac{1}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta\right)^T Y_\Delta + Y_\Delta\left(\frac{1}{2}Y_\nu^\dagger Y_\nu + \frac{1}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta\right)$$

$$+ \left(-\frac{3}{2}\left[\frac{3}{5}g_1^2 + 3g_2^2\right] + \operatorname{Tr}\left[Y_\Delta^\dagger Y_\Delta\right]\right)Y_\Delta, \tag{B.13}$$

$$16\pi^2\beta_{M_R} = (Y_\nu Y_\nu^\dagger)M_R + M_R(Y_\nu Y_\nu^\dagger)^T, \tag{B.14}$$

$$16\pi^2\beta_\kappa = \frac{1}{2}(Y_\nu^\dagger Y_\nu - 3Y_\ell^\dagger Y_\ell + 3Y_\Delta^\dagger Y_\Delta)^T\kappa + \frac{1}{2}\kappa(Y_\nu^\dagger Y_\nu - 3Y_\ell^\dagger Y_\ell + 3Y_\Delta^\dagger Y_\Delta)$$

$$+ 2\operatorname{Tr}\left[Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right]\kappa - 3g_2^2\kappa + \lambda\kappa. \tag{B.15}$$

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Part II

Scientific papers

Paper I

T. Ohlsson and *M. Pernow*

Running of fermion observables in non-supersymmetric SO(10) models

J. High Energy Phys. **11**, 028 (2018)

`arXiv:1804.04560`

Paper II

S. M. Boucenna, T. Ohlsson and *M. Pernow*

A minimal non-supersymmetric SO(10) model with Peccei–Quinn symmetry

Phys. Lett. B **792**, 251 (2019)
arXiv:1812.10548

Paper III

T. Ohlsson and *M. Pernow*

Fits to non-supersymmetric SO(10) models with type I and II seesaw mechanisms using renormalization group evolution

J. High Energy Phys. **06**, 085 (2019)

arXiv:1903.08241