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**Topical Review** 

# Introduction to localization in quantum field theory\*

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This is the introductory chapter to this issue. We review the main idea of the localization technique and its brief history both in geometry and in QFT. We discuss localization in diverse dimensions and give an overview of the major applications of the localization calculations for supersymmetric theories. We explain the focus of the present issue.

Keywords: quantum field theory, supersymmetric theories., diverse dimensions

#### 1. Main idea and history

Abstract

According to the English dictionary<sup>3</sup> the word *localize* means to make local, fix in or assign or restrict to a particular place, locality. Both in mathematics and physics the word 'localize' has multiple meanings and typically physicists with different backgrounds mean different things by localization. This issue is devoted to the extension of the Atiyah–Bott–Berline–Vergne localization formula (and related statements, e.g. the Duistermaat–Heckman formula and different versions of the fixed-point theorem) in differential geometry to an infinite dimensional situation of path integral, and in particular in the context of supersymmetric quantum field theory. In quantum field theory one says 'supersymmetric localization' to denote such computations. In this issue we concentrate on the development of the supersymmetric localization technique during the last ten years, 2007–2016.



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<sup>3</sup>I.e. www.dictionary.com, based on the Random House Dictionary, © Random House, Inc. 2016.

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In differential geometry the idea of localization can be traced back to 1926 [2], when Lefschetz proved the fixed-point formula which counts fixed points of a continuous map of a topological space to itself by using the graded trace of the induced map on the homology groups of this space. In the 1950s, the Grothendieck–Hirzebruch–Riemann–Roch theorem expressed in the most general form the index of a holomorphic vector bundle (supertrace over graded cohomology space) in terms of certain characteristic classes. In the 1960s, the Atiyah-singer index theorem solved the same problem for an arbitrary elliptic complex.

In 1982 Duistermaat and Heckman [3] proved the following formula

$$\int_{M} \frac{\omega^{n}}{n!} e^{-\mu} = \sum_{i} \frac{e^{-\mu(x_{i})}}{e(x_{i})},$$
(1.1)

where *M* is a symplectic compact manifold of dimension 2*n* with symplectic form  $\omega$  and with a Hamiltonian U(1) action whose moment map is  $\mu$ . Here  $x_i$  are the fixed points of the U(1)action and they are assumed to be isolated, and  $e(x_i)$  is the product of the weights of the U(1)action on the tangent space at  $x_i$ . Later independently in 1982 Berline and Vergne [4] and Atiyah and Bott [5] generalized the Duistermaat–Heckman formula to the case of a general compact manifold *M* with a U(1) action and an integral  $\int \alpha$  of an equivariantly-closed form  $\alpha$ , that is  $(d + \iota_V)\alpha = 0$ , where V(x) is the vector field corresponding to the U(1) action. The Atiyah–Bott–Berline–Vergne formula reads as

$$\int_{M} \alpha = \sum_{i} \frac{\pi^{n} \alpha_{0}(x_{i})}{\sqrt{\det(\partial_{\mu} V^{\nu}(x_{i}))}},$$
(1.2)

where it is assumed that  $x_i$  are isolated fixed points of the U(1) action, and  $\alpha_0$  is the zero-form component of  $\alpha$ . The Atiyah–Bott–Berline–Vergne formula has multiple generalizations, to the case of non-isolated fixed locus, to supermanifolds, to the holomorphic case, etc. The more detailed overview of this formula and its relation to equivariant cohomology is given in [6]. Here we will concentrate on conceptual issues and our discussion is rather schematic.

Let us review the proof of the Atiyah–Bott–Berline–Vergne formula (1.2). We will use the language of supergeometry, since it is easier to generalize to the infinite dimensional setup. Consider the odd tangent bundle  $\Pi TM$  where  $x^{\mu}$  are coordinates on M and  $\psi^{\mu}$  are odd coordinates on the fiber (i.e. they transform as  $dx^{\mu}$ ). Functions  $f(x, \psi)$  correspond to differential forms and the integration measure  $d^n x d^n \psi$  on  $\Pi TM$  is canonically defined. Assume that there is a U(1) action on compact M with the corresponding vector field  $V^{\mu}(x)\partial_{\mu}$ . Define the following 'supersymmetry transformations'

$$\delta x^{\mu} = \psi^{\mu}$$
  
$$\delta \psi^{\mu} = V^{\mu}(x) \tag{1.3}$$

which correspond to the equivariant differential  $d + \iota_V$ . We are interested in computation of the integral

$$Z(0) = \int_{\Pi TM} \alpha(x, \psi) \, \mathrm{d}^n x \, \mathrm{d}^n \psi \tag{1.4}$$

for  $\alpha(x, \psi)$  a 'supersymmetric observable', i.e. an equivariantly closed form  $\delta\alpha(x, \psi) = 0$ . We can deform the integral in the following way

$$Z(t) = \int_{\Pi TM} \alpha(x, \psi) \,\mathrm{e}^{-t\delta W(x,\psi)} \,\mathrm{d}^n x \,\,\mathrm{d}^n \psi, \tag{1.5}$$

where  $W(x, \psi)$  is some function. Using the Stokes theorem, one can show that the integral Z(t) is independent of *t*, provided that  $\delta^2 W = 0$ . For example, we can choose  $W = V^{\mu}g_{\mu\nu}\psi^{\nu}$  with  $g_{\mu\nu}$  being a U(1)-invariant metric. If Z(t) is independent of *t*, then we can calculate the original integral at t = 0 at another value of *t*, in particular we can send *t* to infinity

$$Z(0) = \lim_{t \to \infty} Z(t) = \lim_{t \to \infty} \int_{\Pi TM} \alpha(x, \psi) \, \mathrm{e}^{-t\delta W(x, \psi)} \, \mathrm{d}^n x \, \mathrm{d}^n \psi.$$
(1.6)

Thus using the saddle point approximation for Z(t) we can calculate the exact value of Z(0). If we choose  $W = V^{\mu}g_{\mu\nu}\psi^{\nu}$  with the invariant metric and perform the calculation we arrive at the formula (1.2). Let us outline the main steps of the derivation. In the integral (1.6)

$$\delta W = V^{\mu} g_{\mu\nu} V^{\nu} + \partial_{\rho} (V^{\mu} g_{\mu\nu}) \psi^{\rho} \psi^{\nu} \tag{1.7}$$

and thus in the limit  $t \to \infty$  the critical points  $x_i$  of the U(1) action dominate,  $V(x_i) = 0$ . Let us consider the contribution of one isolated point  $x_i$ , and for the sake of clarity let's assume that  $x_i = 0$ . In the neighbourhood of this critical point 0, we can rescale coordinates as follows

$$\sqrt{t}x = \tilde{x}, \quad \sqrt{t}\psi = \tilde{\psi},$$
(1.8)

so that the integral expression (1.6) becomes

$$Z(0) = \lim_{t \to \infty} \int_{\Pi TM} \alpha \left( \frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}} \right) e^{-t\delta W \left( \frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}} \right)} d^n \tilde{x} d^n \tilde{\psi},$$
(1.9)

where we have used the property of the measure on  $\Pi TM$ ,  $d^n x d^n \psi = d^n \tilde{x} d^n \tilde{\psi}$ . Now in (1.9) we may keep track of only leading terms which are independent of *t*. In the exponent with  $\delta W$  only the quadratic terms are relevant

$$\delta W = H_{\mu\nu} \tilde{x}^{\mu} \tilde{x}^{\nu} + S_{\mu\nu} \tilde{\psi}^{\mu} \tilde{\psi}^{\nu}, \qquad (1.10)$$

where the concrete form of the matrices *H* and *S* is irrelevant. In the limit  $t \to \infty$  the 'super-symmetry transformations' (1.3) are naturally linearized

$$\begin{split} \delta \tilde{x}^{\mu} &= \tilde{\psi}^{\mu} \\ \delta \tilde{\psi}^{\mu} &= \partial_{\nu} V^{\mu}(0) \tilde{x}^{\nu}, \end{split} \tag{1.11}$$

and the condition  $\delta^2 W = 0$  now implies

$$H_{\mu\nu} = S_{\mu\sigma}\partial_{\nu}V^{\sigma}(0). \tag{1.12}$$

Now in the integral (1.9) we have to take the limit  $t \to \infty$  and perform the gaussian integral in even and odd coordinates

$$Z(0) = \alpha(0,0) \frac{\pi^{\dim M/2} \operatorname{Pf}(S)}{\sqrt{\det H}}$$
(1.13)

and using (1.12) we arrive at

$$Z(0) = \alpha(0,0) \frac{\pi^{\dim M/2}}{\sqrt{\det \partial_{\mu} V^{\nu}(0)}}.$$
 (1.14)

If we repeat this calculation for every fixed point, we arrive at the Atiyah–Bott–Berline– Vergne formula (1.2). This is the actual proof for a U(1) action on a compact M. In principle the requirement of a U(1) action can be relaxed to V being Killing vector on a compact M, since in the derivation we only use the invariance of the metric to construct the appropriate W. For non-compact spaces, one can use the Atiyah–Bott–Berline–Vergne formula as a suitable definition of the integral, for example to introduce the notion of equivariant volume etc. There are many generalizations of the above logic, for example one can construct the holomorphic version of the equivariant differential with the property  $\delta^2 = 0$  etc.

This setup can be formally generalized to the case where M is an infinite dimensional manifold.

Indeed, we can regard this as the definition of the infinite dimensional integral, provided that the formal properties are preserved. However, in the infinite dimensional case, the main challenge is to make sure that all steps of the formal proof can be suitably defined, for example the choice of a suitable W may become a non-trivial problem. In the infinite dimensional situation the matrix  $\partial_{\nu}V^{\mu}(0)$  in (1.11) turns into a differential operator and the (super)-determinant of this differential operator should be defined carefully.

The most interesting applications of these ideas come from supersymmetric gauge theories. In this case, one tries to recognise the supersymmetry transformations together with the BRST-symmetry coming from the gauge fixing as some type of equivariant differential (1.3) acting on the space of fields (an infinite dimensional supermanifold).

In the context of the infinite-dimensional path integral, the localization construction was first proposed by Witten in his work on supersymmetric quantum mechanics [7]. In that case the infinite dimensional manifold M is the loop space LX of an ordinary smooth manifold X. In the simplest case, the U(1) action on LX comes from the rotation of the loop. Similar ideas were later applied to two-dimensional topological sigma model [8] and four dimensional topological gauge theory [9]. In the 1990s the ideas of localization were widely used in the setup of cohomological topological field theories, e.g. see [10] for nice applications of these ideas to two-dimensional Yang–Mills theory. Further development on supersymmetric localization is related to the calculation of Nekrasov's partition function, or equivariant Donaldson–Witten theory [11], based on earlier works [12–15].

The focus of this issue is on the developments starting from the work [16], where the exact partition function and the expectation values of Wilson loops for  $\mathcal{N} = 2$  supersymmetric gauge theories on  $S^4$  were calculated. In [16] the 4d  $\mathcal{N} = 2$  theory was placed on  $S^4$ , preserving eight supercharges, and the supersymmetry transformations together with BRST-transformations were recognized as the equivariant differential on the space of fields. The zero modes were carefully treated by Atiyah-singer index theorem, and the final result was written as a finite-dimensional integral over the Cartan algebra of the Lie algebra of the gauge group. Later this calculation was generalized and extended to other types of supersymmetric theories, other dimensions and geometries. These exact results provide a unique laboratory for the study of non-perturbative properties of gauge theories. Some contributions to this volume provide an overview of the actual localization calculation in concrete dimensions, for concrete class of theories, while other contributions look at the applications of the results and discuss their physical and mathematical significance.

#### 2. Localization in diverse dimensions

In order to apply the localization technique to supersymmetric theories one needs to resolve a number of technical and conceptual problems. First of all, one needs to define a rigid supersymmetric theory on curved manifolds and understand what geometrical data goes into the construction. The old idea was that rigid supersymmetry on curved manifolds requires an existence of covariantly constant spinors which would correspond to the parameters in the supersymmetry transformations. The next natural generalization would be if the

supersymmetry parameters satisfy the Killing spinor equations [17]. For example, all spheres admit Killing spinors and thus supersymmetric gauge theories can be constructed on spheres. However, a more systematic view on supersymmetric rigid theories on curved manifolds has been suggested in [18] giving background values to auxiliary fields in the supergravity. (More recently an approach of topological gravity was explored in [19, 20].) This approach allows in principle to analyze rigid supersymmetric theories on curved manifolds, although the analysis appears to be increasingly complicated as we deal with higher dimensions and more supersymmetry. At the moment we know how to place on a curved manifold the supersymmetric theories, which in flat space have four or fewer supercharges, in dimensions two, three and four for both Euclidean and Lorentzian signatures [21-24]. For other cases only partial results are available. For example, in 4d the situation for theories with 8 supercharges remains open, see however [25-28]. Situation is similar in five dimensions, see e.g. [29-31] and in six dimensions [32]; see also [33, 34] in the context of superspace treatment of rigid supergravity. Thus despite the surge in the activity the full classification of supersymmetric theories on curved manifolds remains an open problem. Rigid supersymmetric theories on curved manifolds are discussed in contribution [35].

Moreover, in order to be able to carry the localization calculation explicitly and write the result in closed form, we need manifolds with enough symmetries, for example with a rich toric action. Again we do not know the full classification of curved manifolds that allow both a toric action and a rigid supersymmetric gauge theory. In 3d we know how to localize the theories with 4 supercharges on  $S^3$ , on lens spaces  $L_p$  and on  $S^2 \times S^1$ . In 4d the situation becomes more complicated, we know how to localize the theories with 8 supercharges on  $S^3 \times S^1$ , but the general situation in 4d remains to be understood. In 5d there exists an infinite family of toric Sasaki–Einstein manifold ( $S^5$  is one of them) for which the result up to non-perturbative contributions can be written explicitly for the theories with 8 supercharges. Notice, however, that this is not the most general 5d manifolds which admit the rigid supersymmetry, e.g. a bit separated example is  $S^4 \times S^1$ . In 6d the nearly Kähler manifolds (e.g.  $S^6$ ) will allow the theories with 16 supercharges.

The best studied examples are the supersymmetric gauge theories on spheres  $S^d$ , which we are going to review briefly since they provide a nice illustration for the general results. The first results were obtained for  $S^4$  in [16], for  $S^3$  in [36], for  $S^2$  in [37, 38], for  $S^5$  in [39–41] and finally for  $S^6$  and  $S^7$  in [42]. These calculations were generalized and extended to the squashed  $S^3$  [43, 44], to the squashed  $S^4$  [28, 45], the squashed  $S^5$  [46–48] and the result for the squashed  $S^6$  and  $S^7$  was already suggested in [42]. There is also an attempt in [49] to analytically continue the partition function on  $S^d$  to generic complex values of d.

Let us describe the result for different spheres in a uniform fashion. We consider the general case of squashed spheres.

The odd and even dimensional spheres  $S^{2r-1}$  and  $S^{2r}$  lead to two types of special functions called  $S_r$  and  $\Upsilon_r$  that are used to present the result.

The main building block of these functions is the multiple inverse Gamma function  $\gamma_r(x|\epsilon_1, ..., \epsilon_r)$ , which is a function of a variable *x* on the complex plane  $\mathbb{C}$  and *r* complex parameters  $\epsilon_1, ..., \epsilon_r$ . This function is defined as a  $\zeta$ -regularized product

$$\gamma_r(x|\epsilon_1,...,\epsilon_r) = \prod_{n_1,...,n_r=0}^{\infty} (x+n_1\epsilon_1+\cdots+n_r\epsilon_r).$$
(2.1)

The parameters  $\epsilon_i$  should belong to an open half-plane of  $\mathbb{C}$  bounded by a real line passing trough the origin. The unrefined version of  $\gamma_r$  is defined as

$$\gamma_r(x) = \gamma_r(x|1,...,1) = \prod_{k=0}^{\infty} (x+k)^{\frac{(k+1)(k+2)\dots(k+r-1)}{(r-1)!}}.$$
(2.2)

The  $\Upsilon_r$ -function, obtained from the localization on  $S^{2r}$ , is defined as

$$\Upsilon_r(x|\epsilon_1,...,\epsilon_r) = \gamma_r(x|\epsilon_1,...,\epsilon_r)\gamma_r\left(\sum_{i=1}^r \epsilon_i - x|\epsilon_1,...,\epsilon_r\right)^{(-1)^r}.$$
(2.3)

These functions form a hierarchy with respect to a shift of x by one of  $\epsilon$ -parameters

$$\Upsilon_{r}(x+\epsilon_{i}|\epsilon_{1},...,\epsilon_{i},...,\epsilon_{r})=\Upsilon_{r-1}^{-1}(x|\epsilon_{1},...,\epsilon_{i-1},\epsilon_{i+1},...,\epsilon_{r})\Upsilon_{r}(x|\epsilon_{1},...,\epsilon_{i},...,\epsilon_{r})$$
(2.4)

The unrefined version of  $\Upsilon_r$  is defined as follows

$$\Upsilon_r(x) = \Upsilon_r(x|1,...,1) = \prod_{k \in \mathbb{Z}} (k+x)^{\operatorname{sgn}(k+1)\frac{(k+1)(k+2)\dots(k+r-1)}{(r-1)!}}.$$
(2.5)

The  $S_r$ -function, called multiple sine, obtained from localization on  $S^{2r-1}$ , is defined as

$$S_{r}(x|\epsilon_{1},...,\epsilon_{r}) = \gamma_{r}(x|\epsilon_{1},...,\epsilon_{r})\gamma_{r}\left(\sum_{i=1}^{r}\epsilon_{i}-x|\epsilon_{1},...,\epsilon_{r}\right)^{(-1)^{r-i}}.$$
(2.6)

See [50] for exposition and further references. These functions also form a hierarchy with respect to a shift of x by one of the  $\epsilon$ -parameters

$$S_r(x + \epsilon_i | \epsilon_1, ..., \epsilon_i, ..., \epsilon_r) = S_{r-1}^{-1}(x | \epsilon_1, ..., \epsilon_{i-1}, \epsilon_{i+1}, ..., \epsilon_r) S_r(x | \epsilon_1, ..., \epsilon_i, ..., \epsilon_r).$$
(2.7)

Notice that  $S_1(x|\epsilon) = 2 \sin(\frac{\pi x}{\epsilon})$  is a periodic function. Thus  $S_1$  is periodic by itself,  $S_2$  is periodic up to  $S_1^{-1}$ ,  $S_3$  is periodic up to  $S_2^{-1}$  etc. The unrefined version of multiple sine is defined as

$$S_r(x) = S_r(x|1,...,1) = \prod_{k \in \mathbb{Z}} (k+x)^{\frac{(k+1)(k+2)\dots(k+r-1)}{(r-1)!}}.$$
(2.8)

The result for a vector multiplet with 4,8 and 16 supercharges placed on a sphere  $S^2$ ,  $S^4$  and  $S^6$  respectively is given in terms of  $\Upsilon$ , functions as follows

$$Z_{S^{2r}} = \int_{\mathfrak{t}} \mathrm{d}a \, \prod_{w \in R_{\mathrm{ad}\mathfrak{g}}} \Upsilon_r(iw \cdot a | \epsilon) \, \mathrm{e}^{P_r(a)} + \cdots, \qquad (2.9)$$

where the integral is taken over the Cartan subalgebra of the gauge Lie algebra  $\mathfrak{g}$ , the *w* are weights of the adjoint representation of  $\mathfrak{g}$  and  $P_r(a)$  is the polynomial in *a* of degree *r*,

$$P_r(a) = \alpha_r \operatorname{Tr} (a^r) + \dots + \alpha_2 \operatorname{Tr} (a^2) + \alpha_1 \operatorname{Tr} (a).$$
(2.10)

The polynomial  $P_r(a)$  is coming from the classical action of the theory. The parameters  $\alpha_i$  are related to the Yang–Mills coupling, the theta-parameters that couple to the first, second and third Chern classes, and the FI couplings.

The sphere  $S^{2r}$  admits  $T^r$  action with two fixed points, and the parameters  $\epsilon_1, \ldots, \epsilon_r$  are the squashing parameters for  $S^{2r}$  (at the same time  $\epsilon_1, \ldots, \epsilon_r$  are equivariant parameters for the  $T^r$  action).

For  $S^2$ , the dots are non-perturbative contributions coming from other localization loci with non-trivial magnetic fluxes (review in [51]). For  $S^4$ , the dots correspond to the contributions of

point-like instantons over the north and south poles computed by the Nekrasov instanton partition function (review in Contribution [52]). For the case of  $S^6$  the expression corresponds to maximally supersymmetric theory on  $S^6$ , and the nature of the dots remains to be understood.

The partition function of the vector multiplet with 4, 8, or 16 supercharges on the odd-dimensional spheres  $S^3$ ,  $S^5$  and  $S^7$ , or  $S^{2r-1}$  with r = 2, 3, 4, is given by

$$Z_{S^{2r-1}} = \int_{\mathbf{t}} \mathbf{d}a \prod_{w \in R_{\mathrm{ad}\,\mathfrak{g}}} S_r(\mathrm{i}w \cdot a | \epsilon) \, \mathrm{e}^{P_r(a)} + \cdots, \qquad (2.11)$$

where now  $\epsilon$ -parameters are equivariant parameters of the  $T^r \subset SO(2r)$  toric action on  $S^{2r-1}$ .

For  $S^3$  the dots are absent and the expression (2.11) provides the full results for  $\mathcal{N} = 2$  vector multiplet on  $S^3$  (review in contribution [53]). For  $S^5$  the formula (2.11) provides the result for  $\mathcal{N} = 1$  vector multiplet (review in contribution [54]). The theory on  $S^7$  is unique and it corresponds to the maximally supersymmetric Yang–Mills in 7d with 16 supercharges.

For the case of  $S^5$  and  $S^7$  the dots are there and they correspond to the contributions around non-trivial connection satisfying certain non-linear PDEs. There are some natural guesses about these corrections, but there are no systematic derivation and no understanding of them, especially for the case of  $S^7$ .

Dim	Multiplet	#super	Special function	References	Derivation
$S^2$	$\mathcal{N} = 2$ vector	4	$\Upsilon_{l}(x \epsilon)$	[37, 38]	Contribution [51]
$S^3$	$\mathcal{N} = 2$ vector	4	$S_2(x \epsilon)$	[36]	Contribution [53]
$S^4$	$\mathcal{N} = 2$ vector	8	$\Upsilon_2(x \epsilon)$	[16]	Contribution [52]
<i>S</i> <sup>5</sup>	$\mathcal{N} = 1$ vector	8	$S_3(x \epsilon)$	[39–41]	Contribution [54]
<i>S</i> <sup>6</sup>	$\mathcal{N} = 2$ vector	16	$\Upsilon_3(x \epsilon)$	[42]	
$S^7$	$\mathcal{N} = 1$ vector	16	$S_4(x \epsilon)$	[42]	

Our present discussion can be summarized in the following table:

The contribution of matter multiplet (chiral multiplet for theories with 4 supercharges and hypermultiplets for theories with 8 supercharges) can be expressed in terms of the same special functions, see next section.

The detailed discussion of the localization calculation on the spheres and other manifolds can be found in different contributions in this volume, 2d is discussed in contribution [51], 3d in contribution [53], 4d in contribution [52], 5d in contribution [54].

Next we can schematically explain the above result.

#### 2.1. Topological Yang-Mills

We recall that  $\mathcal{N} = 1$  super Yang–Mills theory is defined in dimension d = 3, 4, 6, 10 and that the algebraic structure of supersymmetry transformations is related to an isomorphism that one can establish between  $\mathbb{R}^{d-2}$  and the famous four division algebras:

SYM	Algebra	S	dimS	Top SYM	Equations	#equations	Quotient
3d	$\mathbb{R}$	S	2	1d	_	0	_
4d	$\mathbb{C}$	S	4	2d	F = 0	1	Kahler
6d	$\mathbb{H}$	$S^+\!\otimes \mathbb{C}^2$	8	4d	$F = - \star F$	3	HyperKahler
10d	$\mathbb{O}$	$S^+$	16	8d	$F = - \star (F \land \Omega)$	7	Octonionic

In this table *S* denotes the  $2^{\lfloor d/2 \rfloor}$ -dimensional Dirac spinor representation of Spin(*d*) group. The *S*<sup>+</sup> denotes the chiral (Weyl) spinor representation of Spin(*d*). In all cases, one uses Majorana spinors in Lorenzian signature, or *holomorphic Dirac*<sup>4</sup> spinors in Euclidean signature. Notice the peculiarity of the 6d case where one uses chiral *Sp*(1)-doublet spinors with  $\mathbb{C}^2$  being the fundamental representation of the *Sp*(1)  $\simeq$  SU(2) R-symmetry, and that in the 10d case one uses a single copy of the chiral spinor representation. The number dim**S** is often referred as the *the number of the supercharges in the theory*.

Also, it is well known that the  $\mathcal{N} = 1$  under the dimensional reduction to the dimension d - 2 produces the 'topological' SYM which localizes to the solutions of certain first order (BPS type) elliptic equations on the gauge field strength of the curvature listed in the table. The 1d topYM is, of course, the trivial theory, with empty equations, since there is no room for the curvature 2-form in a 1-dimensional theory. The equation for 2d topYM is the equation of zero curvature, for 4d topYM it is the instanton equation of self-dual curvature (defined by the conformal structure on the 4d manifold), and for 8d topYM it is the equation of the octonionic instanton (defined by the Hodge star  $\star$  operator and the Cayley 4-form  $\Omega$  on a Spin(7)-holonomy 8d manifold).

The corresponding linearized complexes are

topYM	linearized complex	fiber dimensions
$\mathbb{R}: 1d$	$\Omega^0 \to \Omega^1$	$1 \rightarrow 1$
$\mathbb{C}: 2d$	$\Omega^0 \to \Omega^1 \to \Omega^2$	$1 \to 2 \to 1$
$\mathbb{H}:4d$	$\Omega^0 \to \Omega^1 \to \Omega^2_+$	$1 \to 4 \to 3$
$\mathbb{O}: 8d$	$\Omega^0 \to \Omega^1 \to \Omega_{\rm oct}^2$	$1 \rightarrow 8 \rightarrow 7$

Here  $\Omega^p$  is a shorthand for  $\Omega^p(X) \otimes \text{ad } \mathfrak{g}$ , that is the space of  $\mathfrak{g}$ -valued differential *p*-forms on *X*, where  $\mathfrak{g}$  is the Lie algebra of the gauge group and *X* is the space-time manifold. In the 4d theory the space  $\Omega^2_+$  denotes the space of self-dual 2-forms that satisfy the instanton equation  $F = -\star F$ , and in the 8d theory the space  $\Omega^2_{\text{oct}}$  is the space of 2-forms that satisfy the octonionic instanton equation  $F = -\star (F \wedge \Omega)$ .

In these complexes, the first term  $\Omega^0$  describes the tangent space to the infinite-dimensional group of gauge transformations on *X*, the second term  $\Omega^1$  describes the tangent space to the affine space of gauge connections on *X*, and the last term ( $\Omega^2$  for 2d,  $\Omega^2_+$  for 4d,  $\Omega^2_{oct}$  for 8d) describes that space where the equations are valued.

If space-time X is invariant under an isometry group T, the topological YM can be treated equivariantly with respect to the T-action. The prototypical case is the equivariant Donaldson– Witten theory, or 4d topYM in  $\Omega$ -background defined on  $\mathbb{R}^4$  equivariantly with respect to T = SO(4), generating the Nekrasov partition function [11]. Special functions, like the  $\Upsilon$ -function defined by infinite products like (2.3) are infinite-dimensional versions of the equivariant

<sup>&</sup>lt;sup>4</sup> This means that the spinors  $\psi$  in Euclidean signature are taken to be complex, but algebraically speaking, only  $\psi$  appears in the theory but not its complex conjugate  $\bar{\psi}$ .

Euler class of the tangent bundle to the space of all fields appearing after localization of the path integral by Atiyah–Bott–Berline–Vergne fixed point formula (see [6] section 8.1 for more details). The equivariant Euler class can be determined by computing first the equivariant Chern class (index) of the linearized complex describing the tangent space of the topological YM theory. The *T*-equivariant Chern class (index) for the equation elliptic complex

$$D: \dots \to \Gamma(E_k, X) \to \Gamma(E_{k+1}, X) \to \dots$$
(2.13)

on space X made from sections of vector bundles  $E_{\bullet}$ , can be conveniently computed by the Atiyah-singer index theorem

$$\operatorname{ind}_{T}(D) = \sum_{x \in X^{T}} \frac{\sum_{k} (-1)^{k} \operatorname{ch}_{T}(E_{k})|_{x}}{\det_{T,X} (1 - t^{-1})}$$
(2.14)

where  $X^T$  is the fixed point set of T on X (see [6] section 11.1 for details).

#### 2.2. Even dimensions

First we will apply the Atiyah-singer index theorem (review in [6] section 11.1) for the *complexified* complexes (2.12) on  $X = \mathbb{R}^d$  for d = 2, 4, 8 topological YM with respect to the natural *SO*(*d*) equivariant action on  $\mathbb{R}^d$  with fixed point x = 0.

For d = 2r and r = 1,2,4 we pick the Cartan torus  $T^r = U(1)^r$  in the SO(2r) with parameters  $(t_1, ..., t_r) \in U(1)^r$ . The denominator in the Atiyah-singer index theorem is

$$\det(1-t^{-1})\big|_{\mathbb{R}^{2r}} = \prod_{s=1}^{n} (1-t_s)(1-t_s^{-1})$$
(2.15)

The numerator is obtained by computing the graded trace over the fiber of the equation complex at the fixed point x = 0.

For equivariant 2d topYM on  $\mathbb{R}^2$  (coming from SYM with 4 supercharges):

$$\operatorname{ind}_{T}(D, \mathbb{R}^{2}, \Omega_{\mathbb{C}}^{0} \to \Omega_{\mathbb{C}}^{1} \to \Omega_{\mathbb{C}}^{2})_{2d} = \frac{1 - (t_{1} + t_{1}^{-1}) + 1}{(1 - t_{1})(1 - t_{1}^{-1})} = \frac{1}{1 - t_{1}} + \frac{1}{1 - t_{1}^{-1}}$$
(2.16)

For equivariant 4d topYM on  $\mathbb{R}^4$  (coming from SYM with 8 supercharges):

$$\operatorname{ind}_{T}(D, \mathbb{R}^{4}, \Omega_{\mathbb{C}}^{0} \to \Omega_{\mathbb{C}}^{1} \to \Omega_{\mathbb{C}}^{2+})_{4d} = \frac{1 - (t_{1} + t_{1}^{-1} + t_{2} + t_{2}^{-1}) + (1 + t_{1}t_{2} + t_{1}^{-1}t_{2}^{-1})}{(1 - t_{1})(1 - t_{1}^{-1})(1 - t_{2})(1 - t_{2}^{-1})} = \frac{1}{(1 - t_{1})(1 - t_{2})} + \frac{1}{(1 - t_{1}^{-1})(1 - t_{2}^{-1})}$$
(2.17)

For equivariant 8d topYM on  $\mathbb{R}^8$  (coming from SYM with 16 supercharges), to preserve the Cayley form and the octonionic equations coming from the Spin(7) structure, the 4 parameters  $(t_1, t_2, t_3, t_4)$  should satisfy the constraint  $t_1t_2t_3t_4 = 1$ . The weights on 7-dimensional bundle, whose sections are  $\Omega^2_{oct,\mathbb{C}}$ , can be computed from the weights of the chiral spinor bundle  $S^+$  modulo the trivial bundle. The chiral spinor bundle  $S^+$  can be identified (after a choice of complex structure on X) as  $S^+ \simeq (\bigoplus_{p=0}^2 \Lambda^{2p} T_X^{0,1}) \otimes K^{\frac{1}{2}}$  where K is the canonical bundle on  $X = \mathbb{R}^8 \simeq \mathbb{C}^4$  equivariantly trivial with respect to the  $T^3$  action parametrized by  $(t_1, t_2, t_3, t_4)$  with  $t_1t_2t_3t_4 = 1$ . Then

$$\operatorname{ind}_{T}(D, \mathbb{R}^{8}, \Omega_{\mathbb{C}}^{0} \to \Omega_{\mathbb{C}}^{1} \to \Omega_{\operatorname{oct},\mathbb{C}}^{2})_{\mathrm{8d}} = \frac{(1 - (\sum_{s=1}^{4} (t_{s} + t_{s}^{-1})) + (1 + \sum_{1 \leq r < s \leq 4} t_{r}t_{s})}{\prod_{s=1}^{4} (1 - t_{s})(1 - t_{s}^{-1})} = \frac{1}{(1 - t_{1})(1 - t_{2})(1 - t_{3})(1 - t_{4})}, \quad t_{1}t_{2}t_{3}t_{4} = 1.$$
(2.18)

It is also interesting to consider the dimensional reduction of the 8d topYM (coming from the SYM with 16 supercharges) to the 6d theory. The numerator in the index is computed in the same way as (2.18), but the denominator is changed to the 6d determinant, hence we find

$$\operatorname{ind}_{T}(D, \mathbb{R}^{6}, \Omega_{\mathbb{C}}^{0} \to \Omega_{\mathbb{C}}^{1} \to \Omega_{\mathbb{C}}^{2, \operatorname{oct}})_{6d \operatorname{reduction}} = \frac{(1 - (\sum_{s=1}^{4} (t_{s} + t_{s}^{-1})) + (1 + \sum_{1 \leq r < s \leq 4} t_{r} t_{s})}{\prod_{s=1}^{3} (1 - t_{s})(1 - t_{s}^{-1})} = \frac{1 - t_{4}^{-1}}{(1 - t_{1})(1 - t_{2})(1 - t_{3})} \xrightarrow{t_{1} t_{2} t_{3} t_{4} = 1} = 1 = \frac{1}{(1 - t_{1})(1 - t_{2})(1 - t_{3})} + \frac{1}{(1 - t_{1}^{-1})(1 - t_{2}^{-1})(1 - t_{3}^{-1})}.$$

$$(2.19)$$

From equations (2.16), (2.17) and (2.19) we see that the index for the complexified vector multiplet of the 2d theory (4 supercharges), 4d theory (8 supercharges) and 6d theory (16 supercharges) on  $\mathbb{R}^{2r}$  can be uniformly written in the form

$$\operatorname{ind}_{T}(D, \mathbb{R}^{2r}, \operatorname{vector}_{\mathbb{C}}) = \frac{1 + (-1)^{r} \prod_{s=1}^{r} t_{s}}{\prod_{s=1}^{r} (1 - t_{s})} = \frac{1}{\prod_{s=1}^{r} (1 - t_{s})} + \frac{1}{\prod_{s=1}^{r} (1 - t_{s}^{-1})} \qquad r = 1, 2, 3.$$
(2.20)

Hence, the equivariant index of the complexified vector multiplet in 2,4 and 6 dimensions on flat space is equivalent to the index of the Dolbeault complex plus its dual, because (see review in [6] section 9)

$$\operatorname{ind}_{T}(\bar{\partial}, \mathbb{C}^{2r}, \Omega^{0, \bullet}) = \frac{1}{\prod_{s=1}^{r} (1 - t_{s}^{-1})}.$$
(2.21)

The vector multiplet is in a real representation of the equivariant group: each non-zero weight eigenspace appears together with its dual. Generally, the index of a real representation has the form

$$f(t_1, \dots, t_r) + f(t_1^{-1}, \dots, t_r^{-1}).$$
(2.22)

The equivariant Euler class in the denominator of the Atiyah–Bott–Berline–Vergne localization formula ([6] sections 8.1 and 12) is defined as the Pfaffian rather then the determinant, hence each pair of terms in the equivariant index, describing a weight space and its dual, corresponds to a single weight factor in the equivariant Euler class. The choice between two opposite weights leads to a sign issue, which depends on the choice of the orientation on the infinite-dimensional space of all field modes. A careful treatment leads to interesting sign factors discussed in details for example in contribution [51].

A natural choice of orientation leads to the holomorphic projection of the vector multiplet index (2.20) in 2, 4 and 6 dimensions by picking only the first term in (2.20) so that

$$\operatorname{ind}_{T}(D, \mathbb{R}^{2r}, \operatorname{vector}_{\mathbb{C}})_{\operatorname{hol}} = \frac{1}{\prod_{s=1}^{r} (1 - t_{s})}$$
  $r = 1, 2, 3.$  (2.23)

The supersymmetric Yang–Mills with 4, 8 and 16 supercharges can be put on the spheres  $S^2$ ,  $S^4$  and  $S^6$  as was done in [16, 37, 38, 42] and reviewed in contribution [51] and contribution [52].

A certain generator  $Q_{\epsilon}$  of the global superconformal group can be used for the localization computation. This generator  $Q_{\epsilon}$  is represented by a conformal Killing spinor  $\epsilon$  on a sphere  $S^{2r}$ , and satisfies  $Q_{\epsilon}^2 = R$  where R is a rotation isometry. There are two fixed points of R on an evendimensional sphere, usually called the north and the south poles. It turns out that the equivariant elliptic complex of equations, describing the equations of the topological YM, is replaced by a certain equivariant transversally elliptic complex of equations. Near the north pole this complex is approximated by the equivariant topological YM theory (theory in  $\Omega$ -background), and near the south pole by its conjugate.

The index of the transversally elliptic operator can be computed by the Atiyah-singer theorem, see for the complete treatement [55], application [16], Contribution [6] or contribution [52]. The result is that the index is contributed by the two fixed point on the sphere  $S^{2r}$ , with a particular choice of the distribution associated to the rational function, in other words with a particular choice of expansion in positive or negative powers of  $t_s$ , denoted by []<sub>+</sub> or []<sub>-</sub> respectively (see contribution [6] section 11.1):

$$\operatorname{ind}_{T}(D, S^{2r}, \operatorname{vector}_{\mathbb{C}})_{\operatorname{hol}} = \left[\frac{1}{\prod_{s=1}^{r}(1-t_{s})}\right]_{+} + \left[\frac{1}{\prod_{s=1}^{r}(1-t_{s})}\right]_{-} \qquad r = 1, 2, 3.$$
(2.24)

So far we have computed only the space-time geometrical part of the index. Now, suppose that the multiplet is tensored with a representation of a group *G* (like the gauge symmetry, R-symmetry or flavour symmetry), and let  $L_{\xi} \simeq \mathbb{C}$  be a complex eigenspace in representation of *G* with eigenweight  $\xi = e^{ix}$ . Then

$$\operatorname{ind}_{T \times G}(D, S^{2r}, \operatorname{vector}_{\mathbb{C}} \otimes L_{\xi})_{\operatorname{hol}} = \left[\frac{\xi}{\prod_{s=1}^{r}(1-t_s)}\right]_{+} + \left[\frac{\xi}{\prod_{s=1}^{r}(1-t_s)}\right]_{-}.$$
(2.25)

Now let  $\epsilon_s$  and x be the Lie algebra parameters associated with the group parameters  $t_s$  and  $\xi$  as

$$t_s = \exp(i\epsilon_s), \qquad \xi = \exp(ix).$$
 (2.26)

By definition, let  $\Upsilon_r(x|\epsilon)$  be the equivariant Euler class (Pfaffian) of the graded vector space of fields of a vector multiplet on  $S^{2r}$  with the character (index) defined by (2.25)

$$\Upsilon_r(x|\epsilon) = \operatorname{eu}_{T \times G}(D, S^{2r}, \operatorname{vector}_{\mathbb{C}} \otimes L_{\xi})_{\operatorname{hol}}|_{t_s = e^{i\epsilon_s}, \xi = e^{ix}}.$$
(2.27)

Explicitly, converting the infinite Taylor sum series of (2.25)

$$\left[\frac{\xi}{\prod_{s=1}^{r}(1-t_s)}\right]_{+} + \left[\frac{\xi}{\prod_{s=1}^{r}(1-t_s)}\right]_{-} = \sum_{n_1=0,\dots,n_r=0}^{\infty} \xi(t_1^{n_1}\dots t_r^{n_r} + (-1)^r t_1^{-1-n_1}\dots t_r^{-1-n_r})$$
(2.28)

into the product of weights we find the infinite-product definition of the  $\Upsilon_r(x|\epsilon)$  function

$$\Upsilon_r(x|s) \stackrel{\text{reg}}{=} \prod_{n_1=0,\dots,n_r=0}^{\infty} \left( x + \sum_{s=1}^r n_s \epsilon_s \right) \left( \epsilon - x + \sum_{s=1}^r n_s \epsilon_s \right)^{(-1)^r}$$
(2.29)

where  $\stackrel{\text{reg}}{=}$  denotes Weierstrass or  $\zeta$ -function regularization and

$$\epsilon = \epsilon_1 + \dots + \epsilon_r. \tag{2.30}$$

The analysis for the scalar multiplet (the chiral multiplet in 2d for the theory with 4 supercharges or the hypermultiplet in 4d for the theory with 8 supercharges) is similar. On equivariant  $\mathbb{R}^{2r}$  the corresponding complex for the scalar multiplet is the Dirac operator  $S^+ \rightarrow S^-$ , which differs from the Dolbeault complex by the twist by the square root of the canonical bundle, hence

$$\operatorname{ind}_{T}(D, \mathbb{R}^{2r}, \operatorname{scalar})_{\operatorname{hol}} = -\frac{\prod_{s=1}^{r} t_{s}^{\frac{1}{2}}}{\prod_{s=1}^{r} (1 - t_{s})}$$
  $r = 1, 2.$  (2.31)

On the sphere  $S^{2r}$ , again, one takes the contribution from the north and the south pole approximated locally by  $\mathbb{R}^{2r}$  with opposite orientations, and gets

$$\operatorname{ind}_{T}(D, S^{2r}, \operatorname{scalar})_{\operatorname{hol}} = -\left\lfloor \frac{\prod_{s=1}^{r} \frac{1}{2}}{\prod_{s=1}^{r} (1-t_{s})} \right\rfloor_{+} - \left\lfloor \frac{\prod_{s=1}^{r} \frac{1}{2}}{\prod_{s=1}^{r} (1-t_{s})} \right\rfloor_{-} \qquad r = 1, 2.$$
(2.32)

Hence, the equivariant Euler class of the graded space of sections of the scalar multiplet is obtained simply by a shift of the argument of the  $\Upsilon$ -function and inversion

$$\operatorname{eu}_{T\times G}(D, S^{2r}, \operatorname{scalar} \otimes L_{\xi})_{\operatorname{hol}}\big|_{t_{s}=\operatorname{e}^{\operatorname{i} \epsilon_{s}}, w=\operatorname{e}^{\operatorname{i} \kappa}} = \Upsilon_{r}\left(x+\frac{\epsilon}{2}\right)^{-1}.$$
(2.33)

As computed in [16, 37, 38, 42] and reviewed in contribution [51] and contribution [52], the localization by the Atiyah–Bott–Berline–Vergne formula brings the partition function of supersymmetric Yang–Mills with 4, 8 and 16 supercharges on the spheres  $S^2$ ,  $S^4$  and  $S^6$  to the form of an integral over the imaginary line contour in the complexified Lie algebra of the Cartan torus of the gauge group (the zero mode of one of the scalar fields in the vector multiplet). The integrand is a product of the classical factor induced from the classical action and the determinant factor (the inverse of the equivariant Euler class of the tangent space to the space of fields) which has been computed above in terms of the  $\Upsilon_r$ -function. Hence, for r = 1, 2, 3 we get perturbatively exact result of the partition function in the form of a finite-dimensional integral over the Cartan subalgebra of the Lie algebra of the gauge group (generalized matrix model)

$$Z_{S^{2r},\text{pert}} = \int_{\mathfrak{t}_G} \mathrm{d}a \frac{\prod_{w \in R_{\mathrm{ad}\mathfrak{g}}} \Upsilon_r(iw \cdot a | \epsilon)}{\prod_{w \in R_{G \times F}} \Upsilon_r(iw \cdot (a, m) + \frac{\epsilon}{2} | \epsilon)} e^{P(a)}.$$
(2.34)

Hence  $Z_{S^{2r},pert}$  is the contribution to the partition function of the trivial localization locus (all fields vanish except the zero mode *a* of one of the scalars of the vector multiplet and some auxliary fields). The  $Z_{S^{2r},pert}$  does not include the non-perturbative contributions. The factor  $e^{P(a)}$  is induced by the classical action evaluated at the localization locus. The product of  $\Upsilon_r$ -functions in the numerator comes from the vector multiplet and it runs over the weights of the adjoint representation. The product of  $\Upsilon_r$ -functions in the denominator comes from the scalar multiplet (chiral or hyper), and it runs over the weights of a complex representation  $R_G$  of the gauge group *G* in which the scalar multiplet transforms. In addition, by taking the matter fields multiplets to be in a representation of a flavor symmetry *F*, the mass parameters  $m \in \mathfrak{t}_F$  can be introduced naturally. For r = 3 the denominator is empty, because the 6d gauge theory with 16 supercharges is formed only from the gauge vector multiplet.

The non-perturbative contributions come from other localization loci, such as magnetic fluxes on  $S^2$ , or instantons on  $S^4$ , and their effect modifies the equivariant Euler classes presented as  $\Upsilon_r$ -factors in (2.34) by certain rational factors. The 4d non-perturbative contributions

are captured by fusion of Nekrasov instanton partition function with its conjugate [11, 16]. See 2d details in contribution [51] and 4d details in contribution [52].

Much before localization results on gauge theory on  $S^4$  were obtained, the  $\Upsilon_2$  function prominently appeared in Zamolodchikov–Zamolodchikov paper [56] on structure functions of 2d Liouville CFT. The coincidence was one of the key observations by Alday–Gaiotto–Tachikawa [57] that led to a remarkable 2d/4d correspondence (AGT) between correlators in Liouville (Toda) theory and gauge theory partition functions on  $S^4$ , see review in contribution [58].

#### 2.3. Odd dimensions

Next we discuss the odd dimensional spheres (in principle, this discussion is applicable for any simply connected Sasaki–Einstein manifold, i.e. the manifold *X* admits at least two Killing spinors). After field redefinitions, which involve the Killing spinors, the integration space for odd dimensional supersymmetric gauge theories with the gauged fixing fields can be represented as the following spaces

$$\begin{aligned} 3d: & \mathcal{A}(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \\ 5d: & \mathcal{A}(X,\mathfrak{g}) \times \Pi \,\Omega_{H}^{2,+}(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \\ 7d: & \mathcal{A}(X,\mathfrak{g}) \times \Omega_{H}^{3,0} \,(X,\mathfrak{g}) \times \Pi \,\Omega_{H}^{2,+}(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \times \Pi \,\Omega^{0} \,(X,\mathfrak{g}) \end{aligned}$$
(2.35)

where in all cases there are common last two factors  $\Pi \Omega^0(X, \mathfrak{g}) \times \Pi \Omega^0(X, \mathfrak{g})$  coming from the gauge fixing. The space  $\mathcal{A}(X, \mathfrak{g})$  is the space of connections on X with the Lie algebra  $\mathfrak{g}$ . The Sasaki–Einstein manifold is a contact manifold and the differential forms can be naturally decomposed into vertical and horizontal forms using the Reeb vector field R and the contact form  $\kappa$ . The horizontal plane admits a complex structure and thus the horizontal forms can be decomposed further into (p, q)-forms. For two forms we define the space  $\Omega_H^{2,+}$  as (2, 0)-forms plus (0,2)-forms plus forms proportional to  $d\kappa$ . Thus for 5d  $\Omega_H^{2,+}$  is the space of standard selfdual forms in four dimensions (rank 3 bundle), and for 7d forms in  $\Omega_H^{2,+}$  obey the hermitian Yang–Mills conditions in six dimensions (rank 7 bundle: 3 complex components and 1 real). By just counting degrees of freedom one can check that the 3d case corresponds to an  $\mathcal{N} = 2$ vector multiplet (4 supercharges), the 5d case to an  $\mathcal{N} = 1$  vector multiplet (8 supercharges) and 7d to  $\mathcal{N} = 1$  maximally supersymmetric theory (16 supercharges). The supersymmetry square  $Q_e^2$ , which acts on this space, is given by the sum of Lie derivative along the Reeb vector field R and constant gauge transformations:  $Q_e^2 = \mathcal{L}_R + ad_a$ . Around the trivial connection, after some cancelations, the problem boils down to the calculation of the following superdeterminant

$$Z_{S^{2r-1}} = \int_{\mathfrak{t}_G} \mathrm{d}a \,\operatorname{sdet}_{\Omega_H^{(\bullet,0)}(X,\mathfrak{g})}(\mathcal{L}_R + ad_a) \,\mathrm{e}^{P_r(a)} + \cdots, \qquad (2.36)$$

and this is a uniform description for Sasaki–Einstein manifolds in 3d, 5d and 7d. In 3d the only simply connected Sasaki–Einstein manifold is  $S^3$ , while in 5d and 7d there are many examples of simply connected Sasaki–Einstein manifolds (there is a rich class of the toric Sasaki–Einstein manifolds). The determinant can be calculated in many alternative ways, and the result depends on *X*.

If X is a sphere  $S^{2r-1}$ , the determinant in (2.36), equivalently, the inverse equivariant Euler class of the normal bundle to the localization locus in the space of all fields, can be computed from the equivariant Chern character, or the index, of a certain transversally elliptic operator  $D = \pi^* \bar{\partial}$  induced from the Dobeault operator  $\bar{\partial}$  by the Hopf fibration projection  $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ .

The index, or equivariant Chern character, is easy to compute by the Aityah–Singer fixed point theorem (see the details in Contribution [6] section 11.2). The result is

$$\operatorname{ind}_{T}(D, S^{2r-1}) = \sum_{n=-\infty}^{\infty} \operatorname{ind}_{T}(\bar{\partial}, \mathbb{CP}^{r-1}, \mathcal{O}(n)) = \left[\frac{1}{\prod_{k=1}^{r}(1-t_{k})}\right]_{+} + \left[\frac{(-1)^{r-1}t_{1}^{-1}\dots t_{r}^{-1}}{\prod_{k=1}^{r}(1-t_{k}^{-1})}\right]_{-}.$$
(2.37)

Converting the additive equivariant Chern character to the multiplicative equivariant Euler character, we find the definition of the multiple sine function

$$S_r(x|\epsilon) = \operatorname{eu}_{T \times G}(S^{2r-1}, D \otimes L_{\xi})_{\operatorname{hol}}|_{t_{\mathfrak{s}} = e^{i\epsilon_{\mathfrak{s}}}, \xi = e^{i\mathfrak{x}}}$$
(2.38)

where  $L_{\xi}$  is a 1-dimensional complex eigenspace with character  $\xi$ . Explicitly

$$S_r(x|\epsilon) \stackrel{\text{reg}}{=} \prod_{n_1=0,\dots,n_r=0}^{\infty} \left(x + \sum_{s=1}^r n_s \epsilon_s\right) \left(\epsilon - x + \sum_{s=1}^r n_s \epsilon_s\right)^{(-1)^{r-1}}$$
(2.39)

and this leads to the formula (2.11) for the perturbative part of the partition function of a vector multiplet on  $S^{2r-1}$ .

For r = 2,3 we can also treat a scalar supermultiplet (a chiral multiplet for the theory with 4 supercharges or a hypermultiplet for the theory with 8 supercharges). The corresponding complex is described by an elliptic operator  $\pi^* \mathcal{D}$  for  $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ , where  $\mathcal{D}$  is the Dirac operator  $S^+ \to S^-$  on  $\mathbb{CP}^{r-1}$ . The Dirac complex is isomorphic to the Dolbeault complex by a twist by a square root of the canonical bundle. Because of the opposite statistics, there is also an overall sign factor like in (2.32).

Finally, the contribution of both vector multiplet in representation  $R_{adg}$  and scalar multiplet in representation  $R_{G\times F}$  to the perturbative part of the partition function is computed by the finite-dimensional integral over the localization locus  $\mathfrak{t}_G$  with the following integrand made of  $S_r$  functions

$$Z_{S^{2r-1},\text{pert}} = \int_{\mathfrak{t}_{G}} \mathrm{d}a \frac{\prod_{w \in R_{\mathrm{ad}\mathfrak{g}}} S_{r}(iw \cdot a|\epsilon)}{\prod_{w \in R_{G \times F}} S_{r}(iw \cdot (a,m) + \frac{\epsilon}{2}|\epsilon)} e^{P(a)}.$$
(2.40)

Here *F* is a possible flavor group of symmetry, and  $m \in \mathfrak{t}_F$  is a mass parameter.

For reviews of 3d localization see contribution [53], contribution [59], contribution [60], contribution [61] and for reviews of 5d localization see contribution [54], contribution [62], contribution [63].

The case of  $S^n \times S^1$  is built from the trigonometric version of  $S^n$ -result.

The trigonometric version of the  $\Upsilon_r$ -function (2.29) is given by

$$H_{r}(x|\epsilon_{1},...,\epsilon_{r}) = \prod_{n_{1},...,n_{r}=0}^{\infty} (1 - e^{2\pi i x} e^{2\pi i \vec{n} \cdot \vec{\epsilon}}) \left(1 - e^{2\pi i (\sum_{i=1}^{r} \epsilon_{i} - x)} e^{2\pi i \vec{n} \cdot \vec{\epsilon}}\right)^{(-1)^{r}}.$$
 (2.41)

The trigonometric version of the multiple sine function  $S_r$  (2.39) is given by the multiple elliptic gamma function

$$G_{r}(x|\epsilon_{1},...,\epsilon_{r}) = \prod_{n_{1},...,n_{r}=0}^{\infty} (1 - e^{2\pi i x} e^{2\pi i \vec{n} \cdot \vec{\epsilon}}) \left(1 - e^{2\pi i (\sum_{i=1}^{r} \epsilon_{i} - x)} e^{2\pi i \vec{n} \cdot \vec{\epsilon}}\right)^{(-1)^{r-1}}.$$
 (2.42)

where  $G_1$  corresponds to the  $\theta$ -function,  $G_2$  corresponds to the elliptic gamma function.

The partition function on  $S^r \times S^1$  has an interpretation as a supersymmetric index, namely a graded trace over the Hilbert space. The review of supersymmetric index in 2d is in contribution [51], in 4d is in contribution [64] and in 6d is in contribution [65].

#### 3. Applications of the localization technique

The localization technique can be applied only to a very restricted set of supersymmetric observables, e.g. partition functions, supersymmetric Wilson loops etc. Unfortunately, the localization technique does not allow us to calculate correlators of generic local operators. However, the supersymmetric localization offers a unique opportunity to study the full non-perturbative answer for these restricted class of observables and this is a powerful tool to inspect interacting quantum field theory. As one can see from the previous section, the localization results are given in terms of complicated finite dimensional integrals. Thus one has to develop techniques to study these integrals and learn how to deduce the relevant physical and mathematical information. Some of the reviews in this volume are dedicated to the study of the localization results (sometimes in various limits) and to the applications of these results in physics and mathematics.

The original motivation of [16] was to prove the Erickson–Semenoff–Zarembo and Drukker–Gross conjecture, which expresses the expectation value of supersymmetric circular Wilson loop operators in  $\mathcal{N} = 4$  supersymmetric Yang–Mills theory in terms of a Gaussian matrix model, see review in contribution [66]. This conjecture was actively used for checks of AdS/CFT correspondence. After more general localization results became available, they were also used for stronger tests of AdS/CFT.

On the AdS side, it is relatively easy to perform the calculation, since it requires only classical supergravity. However, on the gauge theory side, we need the full non-perturbative result in order to be able to compare it with the supergravity calculation. The localization technique offers us a unique opportunity for non-perturbative checks of AdS/CFT correspondence. A number of reviews are devoted to the use of localization for AdS/CFT correspondence: for AdS<sub>4</sub>/CFT<sub>3</sub> see review in contribution [59] and contribution [60], for AdS<sub>5</sub>/CFT<sub>4</sub> see review in contribution [66], for AdS<sub>7</sub>/CFT<sub>6</sub> see review in contribution [62] and contribution [65]. The localization results for spheres (2.34) and (2.40) gave rise to new matrix models which had not been investigated before. One of the main problems is to find out how the free energy (the logarithm of the partition function) scales in the large *N*-limit. In 3d there is an interesting scaling  $N^{3/2}$ , and the analysis of the partition function on  $S^3$  for the ABJM model is related to different subjects such as topological string, see review in contribution [59]. On the other hand, the 5d theory establishes a rather exotic scaling  $N^3$  for the gauge theory, and it supports the relation of the 5d theory to 6d (2,0) superconformal field theory, see review in contribution [65].

Once we start to calculate the partition functions on different manifolds (e.g.  $S^r$  and  $S^{r-1} \times S^1$ ), we start to realize the composite structure of the answer. Namely the answer can be built from basic objects called holomorphic blocks, this is discussed in details for 2d, 3d, 4d and 5d theories in contribution [53] and contribution [63]. Besides, it seems that in odd dimensions the partition function may serve as a good measure for the number of degrees of freedom. This can be made more precise for the partition function on  $S^3$  which measures the number of degrees of freedom of the supersymmetric theory. Thus one can study how it behaves along the RG flow, see [60].

Another interesting application of localization appears in the context of the BPS/CFTcorrespondence [67], in which BPS phenomena of 4d gauge theories are related to 2d conformal field theory or its massive, lattice, or integrable deformation. A beautiful and precise realization of this idea is the Alday–Gaiotto–Tachikawa (AGT) correspondence which relates 4d  $\mathcal{N} = 2$  gauge theory of class  $\mathcal{S}$  to Liouville (Toda) CFT on some Riemann surface *C*. A 4d  $\mathcal{N} = 2$  gauge theory of class  $\mathcal{S}$  is obtained by compactification of 6d (2,0) tensor self-dual theory on *C*. For a review of this topic see contribution [58].

The 3d/3d version of this correspondence is reviewed in contribution [61] and 5d version is reviewed in contribution [63].

The 2d supersymmetric non-linear sigma models play a prominent role in string theory and mathematical physics, but it is hard to perform direct calculations for non-linear sigma model. However some gauged linear sigma models (2d supersymmetric gauge theories) flow to non-linear sigma model. This flow allows to compute some quantities of non-linear sigma models, such as genus 0 Gromov–Witten invariants (counting of holomorphic maps from  $S^2 \simeq \mathbb{C}P^1$  to a Calabi–Yau target) by localization in 2d gauge theories on  $S^2$ . See review in contribution [68] and contribution [51].

Other important applications of localization calculations are explicit checks of QFT dualities. Sometimes QFT theories with different Lagrangians describe the same physical system and have the same physical dynamics, a famous example is Seiberg duality [69]. The dual theories may look very different in the description by gauge group and matter content, but have the same partition functions, provided approriate identification of the parameters. Various checks of the duality using the localization results are reviewed in contribution [51], contribution [53], contribution [60] and contribution [64].

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