Cubic interaction for higher spins in AdS_{d+1} space in the explicit covariant form

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Abstract

We present a slightly modified prescription of the radial pullback formalism proposed previously by R. Manvelyan, R. Mkrtchyan and W. Rühl in 2012, where authors investigated possibility to connect the main term of higher spin interaction in flat $d + 2$ dimensional space to the main term of interaction in AdS_{d+1} space ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. In this paper we succeeded to solve all necessary *recurrence relations* to finalize full radial pullback of the main term of cubic self-interaction for higher spin gauge fields in Fronsdal's formulation from flat to one dimension less AdS_{d+1} space. Nontrivial solutions of recurrence relations lead to the possibility to obtain the full set of AdS_{d+1} dimensional interacting terms with all curvature corrections including trace and divergence terms from any interaction term in $d + 2$ dimensional flat space.

0. Introduction

This rather technical article is devoted to cubic interaction of the higher spin gauge theory in AdS_{d+1} space. So we start this introduction just pointing some interesting and important things

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for us sending readers for recent and not so recent reviews [1–10] on the state of arts in higher spin gauge theory.

Construction of an interacting Higher Spin (HS) gauge theory is a kind of task with some permanent background interest during more than the last thirty years starting from early work [11]. Periodically, one can observe growing interest to this object of investigation mainly realized as some success in the construction of cubic interaction in AdS or flat background and in connection with AdS/CFT and HS gravity in various dimensions. These attempts were always attractive as one more way to relate quantum theory with General Relativity and investigate HS gauge fields on the same shelf with gravity or understand the uniqueness of gravity (spin 2 field) in comparison with other members of HS hierarchy. Because we are focused in this paper on the cubic interaction, it is worth to recall that even though consistent equations of motion [1] for interacting higher spin fields are known for many years, the action principle for these theories remains unknown. The usual method to construct this interacting Lagrangian was to develop Fronsdal metric formalism for free fields [12]. The crucial point here that during perturbative (Noether method) construction of interaction for HS models we came in parallel to perturbative deformation of the free fields gauge transformation and the certain difficulties connected with the locality of the theory beyond cubic order (see [13–18] and references therein). So we see that cubic interaction up to now is the main building object of HS interaction and not all problems are solving in a fast way even on cubic level. For example, the light-cone gauge construction and classification started from the eighties of the last century for four dimensions [19] and continued and finished by Metsaev [20] during the first decade of current century for arbitrary dimension and even with some interesting results during last years [21]. The covariant approach went even slowly: after seminal work of Berends, Baurgers and van Dam in 1985 [22] and then Fradkin and Vasiliev in 1987 [23] the cubic interaction and classification of vertices came to the center of interest again in 2006-2012 [24–37]. This development in particularly brought to interesting and elegant formulation through the generating function [35,37] and connection with String Theory [35,36]. It is worth to mention also that all these activities supplemented with the parallel development of Vasiliev's frame like formalism to cubic interaction in AdS space [38–40]. It is interesting also in these aspects that covariant classification of cubic vertices was done for parity even dimensions $d \geq 4$ in [33] but classification including parity odd vertices for four and three dimensions was completed only recently in [41–43]. The last point we want to mention here is that although cubic interaction in AdS space has formulation developed in ambient space some years ago [44–48] the direct formulation on the language of AdS_{d+1} covariant derivatives is still unknown and realized before in [48] for some simplest part of interaction only. From other side realization of the Noether program directly in AdS space [49] is also extremely difficult due to noncommutativity of covariant derivatives in space with constant curvature. Therefore at the moment, the only way to see this interaction in AdS space directly is to continue the approach defined in [48].

So the main purpose of this article to complete the first part of the program defined in [48] where authors considered a version of the radial reduction (or pullback) formalism to obtain a cubic interaction of higher spin gauge fields in AdS_{d+1} space from the corresponding cubic interaction in a flat $d + 2$ dimensional background. The crucial point in [48] was to write AdS_{d+1} cubic interaction terms directly in $d + 1$ dimensional space using AdS_{d+1} covariant derivatives. This was done there only for main term and AdS_{d+1} curvature corrections without trace terms. The result was enough elegant but expressed only one simplest type of correction terms. Here we complete setup proposed in appendixes of [48] for all correction terms coming from main (in other words transverse and traceless) term in flat space. The key point of this paper is that

we succeeded in formulation and solution of the corresponding recurrence relations to complete radial pullback from $d + 2$ dimensional flat ambient space to AdS_{d+1} in all orders of curvature expansion including all possible trace terms. Another important point of this consideration is that we constructed general pullback prescription for objects with higher derivatives of higher spin gauge fields to realize corresponding reduction for all other terms of cubic interaction pushing this important remaining task of our program in the field of just technical work which can be done in the future without additional difficulties. This we are left for future publication.

In the first section, we presented and applied the correct radial pullback procedure for the free field reconciled with gauge invariance. Our formulation slightly differs from approaches used in [44–48] but completely equivalent them and more suitable for application to cubic interaction. In the second section we considered pullback for the high power of flat derivatives of HS field in $d + 2$ dimensional space to power of covariant derivatives in AdS_{d+1} which is the most important ingredient of cubic interaction. Doing that we solved all necessary recurrence relations arose from noncommutative algebra. In the third section using the result of previous one, we completed pullback of the main term of cubic interaction with all AdS corrections supplemented by corresponding trace terms. Some technical details of calculations and useful information about cubic interaction in flat space we placed in four appendixes.

1. Prescription for radial pullback and free HS gauge fields in AdS

In this section, we present a short review of the radial pullback technique proposed and developed in [50] and applied in detail to the free higher spin case in [51,48]. We start from $d + 2$ dimensional flat space with coordinates X^A and flat $SO(1, d + 1)$ invariant metric

$$X^A \quad A = 1, 2, \dots. d + 2, \tag{1.1}$$

$$ds^2 = \eta_{AB} dX^A dX^B = -(dX^{d+2})^2 + (dX^{d+1})^2 + dX^i dX^j \eta_{ij}. \tag{1.2}$$

To recognize Euclidean AdS_{d+1} hypersphere inside of this Ambient space we should define the following coordinate transformation to a curvilinear coordinate system $X^A \to (u, r, x^i)$:

$$X^{d+2} = \frac{1}{2} e^u [r + \frac{1}{r}(L^2 + x^i x^j \eta_{ij})],$$

$$X^{d+1} = \frac{1}{2} e^u [r - \frac{1}{r}(L^2 - x^i x^j \eta_{ij})],$$

$$X^i = e^u L \frac{x^i}{r}, \tag{1.3}$$

$$-e^{2u} L^2 = -(X^{d+2})^2 + (X^{d+1})^2 + X^i X^j \eta_{ij}, \tag{1.4}$$

$$ds^2 = L^2 e^{2u} [-du^2 + \frac{1}{r^2}(dr^2 + dx^i dx^j \eta_{ij})]. \tag{1.5}$$

The restriction $e^u = 1$ leads instead of coordinate transformations to the usual embedding of the Euclidean AdS_{d+1} hypersphere with local coordinates $x^\mu = (x^0, x^i) = (r, x^i)$ into $d + 2$ dimensional flat space.

In other words, we can define the Jacobian matrix for transformation (1.3) in the following compact form:

$$E_\mu^A(u, x^\nu) = \frac{\partial X^A}{\partial x^\mu} = e^u e_\mu^A(x^\nu), \tag{1.6}$$

$$E_u^A(u, x^\nu) = \frac{\partial X^A}{\partial u} = X^A(u, x^\nu) = e^u Ln^A(x^\nu), \tag{1.7}$$

where due to (1.4) the $d+1$ tangent vectors $\{e_\mu^A(x)\}_{\mu=0}^d$ and one normal vector $n^A(x)$

$$n^A(x)e_\mu^B(x)\eta_{AB} = 0, \tag{1.8}$$

$$n^A(x)n^B(x)\eta_{AB} = -1 \tag{1.9}$$

for embedded AdS_{d+1} space define the standard induced metric $g_{\mu\nu}(x)$ and extrinsic curvature $K_{\mu\nu}(x)$ for our embedded AdS_{d+1} space:

$$g_{\mu\nu}(x) = e_\mu^A(x)e_\nu^B(x)\eta_{AB} = \left(\frac{L}{x^0}\right)^2 \delta_{\mu\nu}, \tag{1.10}$$

and

$$\partial_\mu e_\nu^A(x) = \Gamma_{\mu\nu}^\lambda(g)e_\nu^A(x) + K_{\mu\nu}(x)n^A(x), \tag{1.11}$$

where

$$\Gamma_{\mu\nu}^\lambda(g) = \Gamma_{\mu\nu}^{\lambda(AdS)} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right), \tag{1.12}$$

$$K_{\mu\nu} = \frac{g_{\mu\nu}}{L}. \tag{1.13}$$

So we see that $\Gamma_{\mu\nu}^\lambda(g)$ is usual Christoffel symbol constructed from induced AdS_{d+1} metric and therefore we can introduce AdS_{d+1} covariant derivative ∇_μ and rewrite (1.10) in convenient form:

$$\nabla_\mu e_\nu^A(x) = K_{\mu\nu}(x)n^A(x), \tag{1.14}$$

$$K_{\mu\nu}(x) = e_\nu^A(x)\partial_\mu n_A = -n_A\nabla_\mu e_\nu^A(x). \tag{1.15}$$

Therefor to restrict our flat theory to AdS hypersphere we should first formulate $d+2$ dimensional field theory in the curvilinear coordinates with flat $e^{2u}(AdS_{d+1} \times \mathcal{R}_u)$ metric

$$ds^2 = e^{2u}[-L^2 du^2 + g_{\mu\nu}(x)dx^\mu dx^\nu] = G_{uu}(u)du^2 + G_{\mu\nu}(u, x)dx^\mu dx^\nu, \tag{1.16}$$

where

$$G_{uu}(u) = E_u^A(u, x^\nu)E_u^B(u, x^\nu)\eta_{AB} = X^A X_A = -L^2 e^{2u}, \tag{1.17}$$

$$G_{\mu\nu} = E_\mu^A(u, x^\nu)E_\nu^B(u, x^\nu)\eta_{AB} = e^{2u}g_{\mu\nu}(x), \tag{1.18}$$

and then define the correct prescription to go from theory in flat curvilinear space defined by Jacobian matrix E_u^A, E_μ^A to the theory with negative constant curvature on the level of $d+2 \times d+1$ embedding matrix e_μ^A or induced metric $g_{\mu\nu}(x)$ getting rid of normal components along of n^A. The most simple check of this statement we can obtain calculating Riemann curvature of the embedded hypersphere. To perform this we should first derive differentiation rules for Frenet basis using (1.13)-(1.15):

$$\nabla_\mu e_\nu^A(x) = \frac{g_{\mu\nu}(x)}{L}n^A(x), \tag{1.19}$$

$$\partial_\mu n^A(x) = \frac{1}{L}e_\mu^A(x), \tag{1.20}$$

and then taking commutator:

$$[\nabla_\mu, \nabla_\nu]e_\lambda^A = R_{\mu\nu,\lambda}{}^\rho e_\rho^A = K_{\lambda[\nu}K_{\mu]}^\rho e_\rho^A, \tag{1.21}$$

we get the standard expression for AdS_{d+1} Riemann curvature and Ricci tensors

$$R_{\mu\nu,\lambda}{}^\rho = -\frac{1}{L^2}(g_{\mu\lambda}\delta_\nu^\rho - g_{\nu\lambda}\delta_\mu^\rho), \tag{1.22}$$

$$R_{\mu,\lambda} = -\frac{d}{L^2}g_{\mu\nu}, \quad R = g^{\mu\lambda}R_{\mu\lambda} = -\frac{d(d+1)}{L^2}. \tag{1.23}$$

Turning to higher spins in flat ambient space we should introduce first the following conventions. As usual, we utilize instead of symmetric tensors such as $h_{A_1A_2...A_s}^{(s)}(X)$ polynomials homogeneous in the vector a^A of degree s at the base point X

$$h^{(s)}(X;a) = \sum_{A_i}(\prod_{i=1}^s a^{A_i})h_{A_1A_2...A_s}^{(s)}(X). \tag{1.24}$$

Then we can write the symmetrized gradient, trace, and divergence[1]

$$Grad : h^{(s)}(X;a) \Rightarrow Grad\, h^{(s+1)}(X;a) = a^A\partial_A h^{(s)}(X;a), \tag{1.25}$$

$$Tr : h^{(s)}(X;a) \Rightarrow Tr\, h^{(s-2)}(X;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(X;a), \tag{1.26}$$

$$Div : h^{(s)}(X;a) \Rightarrow Div\, h^{(s-1)}(X;a) = \frac{1}{s}\eta^{AB}\partial_A\partial_{a^B}h^{(s)}(X;a). \tag{1.27}$$

Moreover, we introduce the notation $*_a^s, *_b^s, \dots$ for a contraction in the symmetric spaces of indices a or b

$$*_{a^A}^s = \frac{1}{(s!)^2}\prod_{i=1}^s \overleftarrow{\partial}_{a^{A_i}}\eta^{A_iB_i}\overrightarrow{\partial}_{a^{B_i}}. \tag{1.28}$$

So we should fix two important points to perform correct pullback of higher spin theory from flat ambient to one dimensional less AdS space:

- We should fix the ansatz for $d+2$ dimensional HS field in a way to get from one spin s field exactly one spin s field in AdS_{d+1}. The natural condition here sends to zero all components normal to the embedded hypersphere

$$n^A h_{AA_2...A_s}^{(s)}(u, x^\nu) \sim X^A(u, x^\nu)h_{AA_2...A_s}^{(s)}(u, x^\nu) = 0. \tag{1.29}$$

- Our auxiliary vector a^A is constant in flat space

$$a^A = E_u^A(u, x)a^u(u, x^\nu) + E_\mu^A(u, x)a^\mu(u, x^\nu)$$
$$= e^u\left(Ln^A(x)a^u(u, x) + e_\mu^A(x)a^\mu(u, x)\right), \tag{1.30}$$

$$\partial_B a^A = 0, \tag{1.31}$$

but in curve AdS_{d+1} space there is no possibility to get covariantly constant vectors.

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[1] To distinguish easily between "a" and "X" spaces we introduce the notation ∂_A for space-time derivatives $\frac{\partial}{\partial X^A}$ and ∂_a for derivatives in a space.

This means that ansatz for HS field itself is not enough for getting correct pullback for objects with derivatives contracted with constant vector a^A. From the other side we have in hand curvilinear metric (1.16)-(1.18) which we can invert and then easily invert the Jacobian matrix (1.6)-(1.7)

$$G^{uu}(u, x) = -\frac{e^{-2u}}{L^2}, \tag{1.32}$$

$$G^{\mu\nu}(u, x) = e^{-2u} g^{\mu\nu}(x), \tag{1.33}$$

$$E_A^u(u, x) = E_u^B(u, x) \eta_{AB} G^{uu}(u, x) = -\frac{e^{-u}}{L} n_A(x), \tag{1.34}$$

$$E_A^\mu(u, x) = E_v^B(u, x) \eta_{AB} G^{\mu\nu}(u, x) = e^{-u} e_A^\mu(x), \tag{1.35}$$

where $g^{\mu\nu}(x)$ is inverse AdS_{d+1} metric and $e_A^\mu(x) = e_v^B(x) \eta_{AB} g^{\mu\nu}(x)$.

Then our flat-space derivative in (1.31) after coordinate transformation is:

$$\partial_A = E_A^u(u, x) \partial_u + E_A^\mu(u, x) \partial_{x^\mu} = -\frac{e^{-u}}{L} n_A(x) \partial_u + e^{-u} e_A^\mu(x) \partial_{x^\mu}. \tag{1.36}$$

Substituting this in (1.31) and taking into account (1.30), (1.19) and (1.20) we obtain the following four relations for derivatives of components $a^u(u, x), a^\mu(u, x)$:

$$\partial_u a^u(u, x) + a^u(u, x) = 0, \tag{1.37}$$

$$\partial_u a^\mu(u, x) + a^\mu(u, x) = 0, \tag{1.38}$$

$$\partial_\mu a^u(u, x) + \frac{1}{L^2} a_\mu(u, x) = 0, \tag{1.39}$$

$$\nabla_\mu a^\nu(u, x) + \delta_\mu^\nu a^u(u, x) = 0. \tag{1.40}$$

First two equations we can solve directly:

$$a^u(u, x) = e^{-u} a^u(x), \tag{1.41}$$

$$a^\mu(u, x) = e^{-u} a^\mu(x). \tag{1.42}$$

Substituting these solutions in (1.30) and using restriction (1.29) we see that in curvilinear coordinates our ansatz leads to the following relation:

$$h^{(s)}(X, a^B) = h_{A_1 A_2 \ldots A_s}^{(s)}(X) a^{A_1} a^{A_2} \ldots a^{A_s}|_{X^A = (u, x^\mu), n^A h_{A \ldots}^{(s)} = 0}$$
$$= h_{\mu_1 \mu_2 \ldots \mu_s}^{(s)}(u, x) a^{\mu_1}(x) a^{\mu_2}(x) \ldots a^{\mu_s}(x) = h^{(s)}(u, x, a^\mu(x)), \tag{1.43}$$

where:

$$h_{\mu_1 \mu_2 \ldots \mu_s}^{(s)}(u, x) = h_{A_1 A_2 \ldots A_s}^{(s)}(u, x) e_{\mu_1}^{A_1}(x) e_{\mu_1}^{A_2}(x) \ldots e_{\mu_s}^{A_s}(x). \tag{1.44}$$

This is correct pullback of spin s tensor field from $d + 2$ dimensional flat space to AdS_{d+1} space. The only reminder about flat space we have here is u-dependence of $d + 1$ dimensional field components in (1.44)

The initial gauge variation of order zero in the spin s field is

$$\delta_{(0)} h^{(s)}(X^A; a^A) = s(a^A \partial_A) \epsilon^{(s-1)}(X^A; a^A), \tag{1.45}$$

with the traceless gauge parameter for the double traceless gauge field

$$\Box_{a^A} \epsilon^{(s-1)}(X^A; a^A) = 0, \tag{1.46}$$

$$\Box_{a^A}^2 h^{(s)}(X^A; a^A) = 0. \tag{1.47}$$

Then combining (1.30) and (1.36) we obtain due to (1.42)

$$a^A \partial_A \epsilon^{(s-1)}(X^A; a^A) = e^{-u}\left(a^u(x)\partial_u + a^\mu(x)\partial_{x^\mu}\right)\epsilon^{(s-1)}(u, x; a^\mu(x)), \tag{1.48}$$

where parameter $\epsilon^{(s-1)}(X^A; a^A)$ obeys to the same type ansatz rule as the $h^{(s)}(X^A; a^A)$ in (1.43)

$$\epsilon^{(s-1)}(X^A; a^A) = \epsilon^{(s-1)}(u, x; a^\mu(x)). \tag{1.49}$$

The next important observation is about derivatives $\partial_{x^\mu} \equiv \partial_\mu$ in respect to AdS_{d+1} coordinates x^μ:

- First note that we mapped scalar object in flat space constructed from X-dependent tensor contracted with constant vectors a^A to the scalar object in curve space constructed from x-dependent tensor contracted with x-dependent vectors $a^\mu(x)$. So as a result we obtain in r.h.s. of (1.48) ordinary derivative ∂_{x^μ}
- To see appearance of the AdS_{d+1} covariant derivatives we should use Leibnitz rule in curve space and conditions (1.39), (1.40):

$$\partial_{x^\mu}(T_\nu(x)a^\nu(x)) = \nabla_\mu T_\nu(x)a^\nu(x) + T_\nu(x)\nabla_\mu a^\nu(x)$$

$$= (\nabla_\mu T_\nu(x))a^\nu(x) - T_\mu(x)a^u(x) = (\nabla_\mu T_\nu(x))a^\nu(x) - a^u(x)\frac{\partial}{\partial a^\mu}(T_\nu(x)a^\nu). \tag{1.50}$$

From this example we see that instead of x-dependent vectors we can use formally x-independent vectors a^μ (and component a^u also) and split AdS space from formal a^μ space inserted only for shortening symmetric tensor contractions and symmetrizing procedures just like in the Cartesian case. But at the same time according to (1.50) we should replace the usual derivative with the following operators in Frenet basis:

$$\partial_A => (e^{-u}\partial_u, e^{-u}\partial_\mu), \tag{1.51}$$

$$\partial_\mu => D_\mu = \nabla_\mu - a^u\partial_{a^\mu} - \frac{a_\mu}{L^2}\partial_{a^u}, \tag{1.52}$$

where ∇_μ is AdS covariant derivative constructed from the Christoffel symbols (1.12) with the following action rule:

$$\nabla_\mu h^{(s)}(u, x; a) = \nabla_\mu h_{\mu_1\mu_2...\mu_s}(u, x)a^{\mu_1}a^{\mu_2}\ldots a^{\mu_s}. \tag{1.53}$$

So from now on we have instead of usual differential operator and coordinate dependent auxiliary vector components "constant" objects a^u and a^μ and covariant derivative operator (1.52) working on rank s symmetric tensors as operators working in both x and a spaces.

Then we can write (1.48) in the form:

$$a^A \partial_A \epsilon^{(s-1)}(X^A; a^A) = e^{-u}\left(a^u\partial_u + a^\mu D_\mu\right)\epsilon^{(s-1)}(u, x; a^\mu)$$

$$= e^{-u}\left[a^u(\partial_u - s + 1) + a^\mu \nabla_\mu\right]\epsilon^{(s-1)}(u, x; a^\mu). \tag{1.54}$$

Using this and restricting the dependence on additional "u" coordinates for all fields and gauge parameters in the following (exponential) way

$$h^{(s)}(u, x^\mu; a^\mu) = e^{\Delta_h u} h^{(s)}(x^\mu; a^\mu), \tag{1.55}$$

$$\epsilon^{(s-1)}(u, x^\mu; a^\mu) = e^{\Delta_\epsilon u} \epsilon^{(s-1)}(x^\mu; a^\mu), \tag{1.56}$$

we obtain from the (1.45) the following relation

$$e^{\Delta_h u} \delta h^{(s)}(x^\mu; a^\mu) = e^{(\Delta_\epsilon - 1)u} s \left[a^u (\Delta_\epsilon - s + 1) + a^\mu \nabla_\mu \right] \epsilon^{(s-1)}(x; a^\mu). \tag{1.57}$$

So we see that for getting from gauge transformation in $d + 2$ dimensional flat space (1.45) the correct AdS_{d+1} gauge transformation

$$\delta h^{(s)}(x^\mu; a^\mu) = s a^\mu \nabla_\mu \epsilon^{(s-1)}(x; a^\mu), \tag{1.58}$$

we should fix the last freedom in our ansatz in unique form

$$\Delta_\epsilon = s - 1, \tag{1.59}$$

$$\Delta_h = \Delta_\epsilon - 1 = s - 2, \tag{1.60}$$

which is in agreement with consideration in [44–47].

After all, we can formulate our final prescription for radial pullback in the massless AdS case slightly differs from our reduction formulated in [48] and can be summarized by the following three points.

1. Expand auxiliary vectors a^A using Frenet basis for embedded AdS space (1.30) and take into account u dependents (1.41), (1.42) for components normal and tangential to the embedded hypersphere coming from condition (1.31) and formal x^μ independence explained above. Finally, we have the following embedding rule

$$a^A => Ln^A(x)a^u + e_\mu^A(x)a^\mu. \tag{1.61}$$

2. Replace all derivatives in the following way:

$$\partial_A => e^{-u} \left(-\frac{n_A(x)}{L} \partial_u + e_A^\mu(x) D_\mu \right), \tag{1.62}$$

   where D_μ defined in (1.52).
3. Restrict the dependence on additional "u" coordinates for all fields and gauge parameters in an exponential way with corresponding weights (1.59) (1.60) to preserve gauge invariants during pullback.

Note also that our reduction rules here slightly different from rules, formulated in [48], especially in the area of "u" dependence. This happened because we used direct solutions (1.41), (1.42) and keep derivative ∂_u unchanged. In [48] we removed exponential factor e^{-u} a front of derivatives and all a^u and a^μ vector components, replacing radial derivatives also with operator $\partial_u - a^u \partial_{a^u} - a^\mu \partial_{a^\mu}$ working in both u and a spaces. In that case scaling behavior of field components and parameters are different from our here and in [48].[2]

In any case, the final result is the same: After some straightforward calculation using our reduction rules we can prove that $d + 2$ dimensional gauge invariant Fronsdal tensor

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[2] in [48] we had $\Delta_h = \Delta_\epsilon = 2(s - 1)$.

$$\mathcal{F}^{(s)}(X^A; a^A) = \Box_{d+2} h^{(s)}(X^A; a^A) - a^A \partial_A \left(\partial^B \partial_{a^B} h^{(s)}(X^A; a^A) \right.$$

$$\left. - \frac{1}{2} (a^B \partial_B) \Box_{a^A} h^{(s)}(X^A; a^A) \right), \tag{1.63}$$

reduces to the AdS_{d+1} gauge invariant Fronsdal tensor

$$\mathcal{F}^{(s)}(x; a^\mu) = \Box_{d+1} h^{(s)}(x^\mu; a^\mu)$$

$$- (a^\mu \nabla_\mu) \left[(\nabla^\nu \partial_{a^\nu}) h^{(s)}(x; a^\mu) - \frac{1}{2} (a^\nu \nabla_\nu) \Box_{a^\mu} h^{(s)}(x; a^\mu) \right]$$

$$- \frac{1}{L^2} [s^2 + s(d-5) - 2(d-2)] h^{(s)}(x^\mu; a^\mu)) - \frac{1}{L^2} a^\mu a_\mu \Box_{a^\mu} h^{(s)}(x^\mu; a^\mu), \tag{1.64}$$

in the following way

$$\mathcal{F}^{(s)}(X^A; a^A) = e^{(s-4)u} \mathcal{F}^{(s)}(x; a^\mu). \tag{1.65}$$

Supplementing this with the reductions for field (1.55), (1.60) and for integration volume:

$$\int d^{d+2} X = \int du d^{d+1} x \sqrt{-G} = L \int du d^{d+1} x \sqrt{g} e^{(d+2)u}, \tag{1.66}$$

we obtain the following reduction rule for Fronsdal actions:

$$S_0[h^{(s)}(X^A; a^A)] = \left[L \int du e^{(d+2s-4)u} \right] \times S_0[h^{(s)}(x^\mu; a^\mu)], \tag{1.67}$$

where

$$S_0[h^{(s)}(X^A; a^A)] = \int d^{d+2} X \left[-\frac{1}{2} h^{(s)}(X^A; a^A) *_{a^A} \mathcal{F}^{(s)}(X^A; a^A) \right.$$

$$\left. + \frac{1}{8s(s-1)} \Box_{a^A} h^{(s)}(X^A; a^A) *_{a^A} \Box_{a^A} \mathcal{F}^{(s)}(X^A; a^A) \right], \tag{1.68}$$

$$S_0[h^{(s)}(x^\mu; a^\mu)] = \int d^{d+1} x \sqrt{g} \left[-\frac{1}{2} h^{(s)}(x; a^\mu) *_{a^\mu} \mathcal{F}^{(s)}(x; a^\mu) \right.$$

$$\left. + \frac{1}{8s(s-1)} \Box_{a^\mu} h^{(s)}(x; a^\mu) *_{a^\mu} \Box_{a^\mu} \mathcal{F}^{(s)}(x; a^\mu) \right]. \tag{1.69}$$

The overall infinite factor

$$\left[L \int du e^{(d+2s-4)u} \right], \tag{1.70}$$

here the same as in [48], where we described prescription to get correct additional AdS correction terms from the full "u" derivative part of interaction terms. This additional terms can be found with insertion of the dimensionless delta function in measure (1.66) [44–47]

$$\int d^{d+2} X \delta \left(\frac{\sqrt{-X^2}}{L} - 1 \right). \tag{1.71}$$

Then full derivative terms will survive only for normal u derivatives:

$$\int d^{d+2} X \delta \left(\frac{\sqrt{-X^2}}{L} - 1 \right) \partial^A \mathfrak{L}_A = \int d^{d+2} X \delta^{(1)} \left(\frac{\sqrt{-X^2}}{L} - 1 \right) \frac{X^A}{L^2} E_A^u \mathfrak{L}_u$$

$$= \int du d^{d+1} x \sqrt{g} e^{(d+2)u} \frac{\delta^{(1)}(e^u - 1)}{L} \mathfrak{L}_u. \tag{1.72}$$

So we see that both approaches produce the same additional corrections coming from the differentiation of overall "u" phase a front of full derivatives in the normal direction. Finally, we note that this reduction procedure is more useful for investigation of interaction terms due to the very simple form of the pullback of fields and auxiliary vectors a^A and star contractions:

$$
*_{a^A}^s = \frac{1}{(s!)^2} \prod_{i=1}^{s} \left(- \overleftarrow{\partial}_{a^{u_i}} \overrightarrow{\partial}_{a^{u_i}} + \overleftarrow{\partial}_{a_{\mu_i}} \overrightarrow{\partial}_{a^{\mu_i}} \right)
$$

$$
= \sum_{n=0}^{s} \frac{(-1)^n}{\binom{s}{n}} *_{a^u}^n *_{a^\mu}^{s-n}. \tag{1.73}
$$

2. Pullback for power of derivatives of HS fields from flat to embedded AdS space

In this section, we discuss radial pullback for Cubic interaction for higher spins in a covariant off-shell formulation derived In [33,34]. This result for flat space is in full agreement with light cone gauge results of Metsaev [20]. Moreover this agreement shows that all interactions of higher spin gauge fields with any spin s_1, s_2, s_3 both in flat space and in dS or AdS are unique up to partial integration and field redefinition.[3] The formulation of the cubic interactions for higher spin fields in ambient space was considered in several papers [44–49]. In [48] we investigated the possibility to connect the main term of interaction in flat $d + 2$ dimensional space to the main term of interaction in AdS_{d+1} space one dimension lower ignoring all trace and divergent terms but expressed directly through the AdS covariant derivatives and including some curvature corrections. In this article, we perform one important step forward solving task for flat main term completely and presenting full reduction or pullback including all trace and other related terms coming from main term of cubic interaction in direct AdS_{d+1} covariant form. We put in appendix A short review for the main term of cubic interaction formulated in details in [33,34] and start here from the more convenient for radial pullback form described in [48], where we reformulated the main term of cubic interaction (A.2), (A.3) in the following way:

$$
\mathcal{L}_I^{main}(h^{(s_1)}(X, a^A), h^{(s_2)}(X, b^A), h^{(s_3)}(X, c^A)) =
$$

$$
\sum_{Q_{ij}} C_{Q_{12}, Q_{23}, Q_{31}}^{s_1, s_2, s_3} \int d^{d+2} X *_{c^A}^{Q_{31}+n_3} K^{(s_1)}(Q_{31}, n_3; c^A, a^A; X)
$$

$$
*_{a^A}^{Q_{12}+n_1} K^{(s_2)}(Q_{12}, n_1; a^A, b^A; X) *_{b^A}^{Q_{23}+n_2} K^{(s_3)}(Q_{23}, n_2; b^A, c^A; X), \tag{2.1}
$$

where

$$
K^{(s_1)}(Q_{31}, n_3; c^A, a^A; X) = (c^A \partial_{a^A})^{Q_{12}} (c^B \partial_B)^{n_3} h^{(s_1)}(X; a^C). \tag{2.2}
$$

The most important advantage of this form that here we can express our cubic interaction as a cube of above bitensor function with cyclic index contraction. From now on we put AdS radius $L = 1$ and use for shortness the brackets $(\ldots, \ldots)$ for AdS_{d+1} index summation. In other words

$$
(a, \partial_b) = a^\mu \partial_{b^\mu}, \tag{2.3}
$$

$$
(a, \nabla) = a^\mu \nabla_\mu, \tag{2.4}
$$

---

[3] This was already proven for some low spin cases of both the Fradkin-Vasiliev vertex for $2, s, s$ and the nonabelian vertex for $1, s, s$ in [29].

and

$$(a, D) = a^\mu D_\mu. \tag{2.5}$$

Another important point here is the difference in the definition of the covariant differentiation operator (1.52) in the case of interaction. The minimal object here is a bitensor (2.2) which has two sets of symmetrized indices. In this case, we should define covariant differentiation operators for both sets of indices:

$$D_\mu = \nabla_\mu - a^u \partial_{a^\mu} - a_\mu \partial_{a^u} - b^u \partial_{b^\mu} - b_\mu \partial_{b^u}. \tag{2.6}$$

and in a similar way for other sets of indices. Now we have all ingredients to start analyzing the "u"-dependence of interaction Lagrangian (2.1) in curvilinear coordinates (1.3). First of all we note that in the new frame only the measure and derivatives create additional u phase (1.66) and (1.62) in addition to the three similar phase (1.60) coming from reduced fields. Finally, we get

$$d + 2 + \sum_{i=1}^{3}(\Delta_{h^{(s_i)}} - n_i) = \sum_{i=1}^{3}(s_i) - \Delta + d - 4, \tag{2.7}$$

where Δ is the number of derivatives in interaction. Then inserting minimal number of derivatives from (A.8) we see that our interaction rescales as[4]

$$\sum_{i=1}^{3} s_i - \Delta_{min} + d - 4 = d + 2s_3 - 4, \tag{2.8}$$

with the obvious limit $d + 2s - 4$ in the self-interacting case $s_1 = s_2 = s_3 = s$. So we see that the cubic interaction in the case of the minimal number of derivatives is relevant for the radial reduction procedure described in the previous section. Therefore it should produce the right curvature corrections for the main term of the cubic interaction in AdS_{d+1}.

In this section, we consider a possible radial pullback scheme for the main object of cubic interaction (2.1): the bitensorial function

$$K^{(s)}(Q, n; a^A, b^A; X) = (a^A \partial_{b^A})^Q (a^B \partial_B)^n h^{(s)}(X; b^C). \tag{2.9}$$

This term should generate all AdS curvature corrections coming from main term. For that we study these operators in a representation that act on pullback HS field

$$h^{(s)}(X; b^A)|_{X=X(u,x)} = h^{(s)}(u, x^\mu; b^\mu) = e^{(s-2)u} h^{(s)}(x^\mu; b^\mu). \tag{2.10}$$

Then we can obtain these AdS corrections expanding all flat $d + 2$ dimensional objects in Frenet basis or in other words in term of $d + 1$ dimensional AdS space derivatives and vectors and normal components surviving after applying our ansatz rules:

$$(a^B \partial_B)^n|_{X=X(u,x)} = \left[e^{-u}(a^u \partial_u + a^\mu D_\mu) \right]^n, \tag{2.11}$$

$$a^\mu D_\mu = (a, D) = (a, \nabla) - a^u(a, \partial_a) - b^u(a, \partial_b) - a^2 \partial_{a^u} - (a, b)\partial_{b^u}, \tag{2.12}$$

$$\text{where} \qquad a^2 = (a, a) = a^\mu a^\nu g_{\mu\nu}(x),$$

and contracting over all a^u, b^u, c^u.

---

[4] In the case of three spins ordered as $s_1 \geq s_2 \geq s_3$.

2.1. Noncommutative algebra and a^u stripping

Now we must deal with the $d+1$ dimensional expansion for the n'th power of $d+2$ dimensional derivatives (2.11), where the operator

$$a^u \partial_u + a^\mu D_\mu = a^\mu \hat{\nabla}_\mu(g) - R, \tag{2.13}$$

$$\hat{\nabla}_\mu = \nabla_\mu - b^u \partial_{b^\mu} - b_\mu \partial_{b^u}, \tag{2.14}$$

$$R = a^u[(a\partial_a) - \partial_u] + a^2 \partial_{a^u}, \tag{2.15}$$

act on ground states (2.10). These ground states can be characterized by the total symmetry in the argument and by the fact that they are annihilated by the following operators:

$$|0> = e^{(s-2)u} h^{(s)}(x^\mu; b^\mu), \tag{2.16}$$

$$\partial_{a^\mu} |0> = \partial_{a^u} |0> = \partial_{b^u} |0> = 0, \tag{2.17}$$

$$R|0> = (2-s)a^u |0>. \tag{2.18}$$

The operator of interest is

$$\left[e^{-u}(a, \hat{\nabla}) - e^{-u} R \right]^n, \tag{2.19}$$

where in the sequel it is advantageous to write the operator R in the following way

$$R = a^u[(a\partial_a) + a^u \partial_{a^u} - \partial_u] + (a^2 - (a^u)^2)\partial_{a^u} \tag{2.20}$$

with the following important algebraic relations:

$$[(a\partial_a) + a^u \partial_{a^u}, R] = R, \tag{2.21}$$

$$[(a\partial_a) + a^u \partial_{a^u}, (a, \hat{\nabla})] = (a, \hat{\nabla}), \tag{2.22}$$

$$[R, e^{-u}(a, \hat{\nabla})] = 2e^{-u} a^u(a, \hat{\nabla}). \tag{2.23}$$

We have to evaluate (2.19) on the ground state (2.16). Hiding all technical details in Appendix B we present here the result of this straightforward manipulations with noncommutative algebra

$$[(a, e^{-u}\hat{\nabla}) - e^{-u} R]^n |0> = e^{(s-2-n)u} \sum_{p=0}^n (-1)^p (a, \hat{\nabla})^{n-p} V^{p+1}(i_{p+1}) h^{(s)}(x^\mu; b^\mu), \tag{2.24}$$

where[5]

$$V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k (a^u)^{p-2k}. \tag{2.25}$$

Then from recurrent relations (B.12) for $\xi_k^{p+1}(i_{p+1})$ and from solutions of latter equations obtained by direct calculation of V^{p+1} using (B.8) for $p = 1, 2, 3, 4, \ldots$ in Appendix B we can finally present solution[6] for general p:

---

[5] See derivation in Appendix B.

[6] Our definition of Pochhammer symbols $(a)_n$ (rising factorial) can be found in (B.19) of Appendix B.

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(2k+i+1)_{p-2k}(2k+2+i-s)_{p-2k}\mathcal{P}_k(i), \tag{2.26}$$

where $\mathcal{P}_k(i)$ is p-independent polynomials and satisfies corresponding simple equation (B.22) with the following solution in the form of multiple sums:

$$\mathcal{P}_k(i) = \sum_{i \ge i_k \ge i_{k-1} \ge i_{k-2} \ldots \ge i_1 \ge 0} \prod_{n=1}^{k}(i_n + 2n - 1)(i_n + 2n - s). \tag{2.27}$$

This solution described in Appendix B also in the form of generating function (B.24) obtained from solution of a differential equation.

2.2. Noncommutative algebra and b^u stripping

To extract exact dependence from b^u and obtain final expressions written directly through the AdS_{d+1} covariant derivatives ∇ we have to evaluate the remaining factors

$$
\begin{aligned}
(a, \hat{\nabla})^{n-p} &= [(a, \nabla) - b^u(a, \partial_b) - (a, b)\partial_{b^u}]^{n-p} \\
&= \sum_{\tilde{p}=0}^{n-p}(-1)^{\tilde{p}}\binom{n-p}{\tilde{p}}(a, \nabla)^{n-p-\tilde{p}}(L^+ + L^-)^{\tilde{p}},
\end{aligned} \tag{2.28}
$$

where L^+, L^- generate a Lie algebra

$$L^+ = b^u(a, \partial_b), \quad L^- = (a, b)\partial_{b^u}, \tag{2.29}$$

$$[L^+, L^-] = H = a^2 b^u \partial_{b^u} - (a, b)(a, \partial_b), \tag{2.30}$$

$$[H, L^{\pm}] = \pm 2a^2 L^{\pm}. \tag{2.31}$$

Representations of this Lie algebra are created from an $(s+1)$-dimensional vector space of "null vectors" $\{\Phi_n(a; b)\}|_{n=0}^{s}$ of "level" n

$$\Phi_n(a; b) = h_{\mu_1, \mu_2, \ldots \mu_s}^{(s)} a^{\mu_1} a^{\mu_2} \ldots a^{\mu_n} b^{\mu_{n+1}} b^{\mu_{n+2}} \ldots b^{\mu_s}, \qquad L^- \Phi_n(a; b) = 0, \tag{2.32}$$

for any fixed tensor function h^s. From (2.29)-(2.31) follows that starting from

$$\Phi_0(a; b) = \Phi_0(b), \tag{2.33}$$

all $\Phi_n(a; b)$ can be produced by application of H[7]

$$\Phi_n(a; b) = H^n \Phi_0(b) = (-1)^n \sum_{r=1}^{n} A_r^{(n)}[s]_r (a^2)^{n-r}(a, b)^r \Phi_r(a; b). \tag{2.34}$$

The ansatz[8] (2.34) leads to the recurrence relation

$$A_{r-1}^{(n)} + r A_r^{(n)} = A_r^{(n+1)}, \tag{2.35}$$

$$A_r^{(n)} = 0 \quad \text{for} \quad r > n. \tag{2.36}$$

The boundary conditions $A_{-1}^{(n)} = 0$ and $A_0^{(0)} = 1$ are assumed.

---

[7] For falling factorial we use in this paper notation $[s]_n = s(s-1)\ldots(s-n+1)$.

[8] To guess this ansatz we used direct calculations of the first three powers of H acting on ground state (2.33) presented in Appendix C.

We can solve recurrent equation (2.35) in common way using generating function method. Shifting in Appendix C all details of this procedure we present here the final solution in the form:

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A_r^{(n)} \frac{x^r t^n}{n!}. \tag{2.37}$$

It is not difficult to get a simple combinatorial formula for $A_r^{(n)}$. Let us denote by $\mathcal{P}(n,r)$ the set of partitions of n into r nonzero parts. The partitions are in one to one correspondence with Young diagrams with n boxes and r rows. An arbitrary partition λ may be represented as $\lambda = 1^{k_1} 2^{k_2} 3^{k_2} \cdots$, where the nonnegative integer k_i indicates the number of rows with length i. For example the partition $8 = 1 + 1 + 3 + 3$ is represented as $1^2 2^0 3^2$, hence $\{k_1, k_2, k_3\} = \{2, 0, 2\}$ and $k_4 = k_5 = \cdots = 0$. The corresponding Young diagram consists of two rows of length 3 and two rows of length 1. For a diagram $\lambda \in \mathcal{P}(n,r)$ let us arbitrarily distribute the integers $1, 2, \cdots n$ among boxes. Let us identify two configurations which differ from each other by permutations of numbers along rows or by permutation of entire rows of same lengths. Evidently, the number of non-equivalent distributions is given by

$$S(\lambda) = \frac{n!}{\prod_{i \geq 1} k_i! (i!)^{k_i}}. \tag{2.38}$$

Now comparing (C.10), (2.37) with (2.38) one easily gets

$$A_n^{(r)} = \sum_{\lambda \in \mathcal{P}(n,r)} S(\lambda). \tag{2.39}$$

With the help of the basis $\{\Phi_n(a; b)\}_{n=0}^s$ of null vectors the representation of the Lie algebra (2.29)-(2.31) can be constructed as follows (see for details Appendix C).

$$(L^+ + L^-)^{\tilde{p}} \Phi_0(b) = \sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]} (b^u)^{\tilde{p}-2\tilde{k}} (-1)^{\tilde{k}} (a, \partial_b)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2, H) \Phi_0(b), \tag{2.40}$$

where

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) \Phi_0(b)$$
$$= \sum_{i_{\tilde{k}+1} \geq i_{\tilde{k}} \geq i_{\tilde{k}-1} \geq i_{\tilde{k}-2} \cdots \geq i_2 \geq i_1 \geq 1} \psi_{i_{\tilde{k}}-\tilde{k}+1} \psi_{i_{\tilde{k}-1}-\tilde{k}+2} \psi_{i_{\tilde{k}-2}-\tilde{k}+3} \cdots \psi_{i_2-1} \psi_{i_1} \Phi_0(b), \tag{2.41}$$

and

$$\psi_i = iH + [i]_2 a^2. \tag{2.42}$$

The sum is a homogeneous polynomial of H and a^2 of degree $\tilde{k}$[9]:

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) = \sum_{m=0}^{\tilde{k}} \eta_{\tilde{k}}^m (i_{\tilde{k}+1}) (a^2)^m H^{\tilde{k}-m}. \tag{2.43}$$

---

[9] Remember that H is second order in a as well.

So we see that representation (2.40) extracts b^u dependence and we can calculate coefficients $\eta_{\tilde{k}}^m(i_{\tilde{k}+1})$ from (2.41) directly. comparing (2.43) with (2.41) and taking into account (2.42) we see that it is possible to write

$$\eta_{\tilde{k}}^m(\tilde{p} - \tilde{k}) = \eta_{\tilde{k}}^m(i_{\tilde{k}+1})|_{i_{\tilde{k}+1}=\tilde{p}-\tilde{k}}, \tag{2.44}$$

in the following form:

$$\eta_{\tilde{k}}^m(\tilde{p} - \tilde{k}) = \sum_{\tilde{p}-\tilde{k} \geq i_{\tilde{k}} \geq i_{\tilde{k}-1} \geq i_{\tilde{k}-2} \cdots \geq i_2 \geq i_1 \geq 1} \quad \sum_{\tilde{k} \geq n_m \geq n_{m-1} \geq n_{m-2} \cdots \geq n_2 \geq n_1 \geq 1}$$

$$\prod_{l_m=n_m+1}^{\tilde{k}} (i_{l_m} - l_k + 1)[i_{n_m} - n_m + 1]_2 \prod_{l_{m-1}=n_{m-1}+1}^{n_m-1} (i_{l_{m-1}} - l_{m-1} + 1)[i_{n_{m-1}} - n_{m-1} + 1]_2 \ldots$$

$$\cdots \prod_{l_2=n_2+1}^{n_3-1} (i_{l_2} - l_2 + 1)[i_{n_2} - n_2 + 1]_2 \prod_{l_1=n_1+1}^{n_2-1} (i_{l_1} - l_1 + 1)[i_{n_1} - n_1 + 1]_2 \prod_{l=1}^{n_1-1} (i_l - l + 1). \tag{2.45}$$

This formula means that we should inside of expression for $\eta_{\tilde{k}}^0(\tilde{p} - \tilde{k})$:

$$\eta_{\tilde{k}}^0(\tilde{p} - \tilde{k}) = \sum_{\tilde{p}-\tilde{k} \geq i_{\tilde{k}} \geq i_{\tilde{k}-1} \geq i_{\tilde{k}-2} \cdots \geq i_2 \geq i_1 \geq 1} \prod_{l=1}^{\tilde{k}} (i_l - l + 1), \tag{2.46}$$

replace m brackets $(i_{n_r} - n_r + 1)|_{r=1}^m$ with the m Pochhammers $\{[i_{n_r} - n_r + 1]_2\}|_{r=1}^m$ in all possible ways and then take sums.

3. Pullback of the main term of cubic self-interaction

Now we start to collect things together and present all terms of cubic interaction produced from the main term in one dimension more flat space. First, we look at the main term in the case of a cubic self-interaction. This can be obtained from the general expressions (A.4)-(A.6) taking

$$s_1 = s_2 = s_3 = s, \tag{3.1}$$

$$v_1 = v_2 = v_3 = 0, \tag{3.2}$$

$$Q_{23} = n_1 = \alpha, \tag{3.3}$$

$$Q_{31} = n_2 = \beta, \tag{3.4}$$

$$Q_{12} = n_3 = \gamma. \tag{3.5}$$

Then (2.1), (2.2) transform to the following nice cyclic (in $(a)\alpha$, $(b)\beta$, $(c)\gamma$) expression with trinomial coefficients:

$$\mathcal{L}_I^{main} = \sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha, \beta, \gamma} \int d^{d+2} X$$

$$*_a^{\gamma+\alpha} (a^A \partial_{b^A})^\gamma (a^B \partial_B)^\alpha h^{(s)}(X; b^C)$$

$$*_b^{\alpha+\beta} (b^D \partial_{c^D})^\alpha (b^E \partial_E)^\beta h^{(s)}(X; c^F)$$

$$*_c^{\beta+\gamma} (c^G \partial_{a^G})^\beta (c^H \partial_H)^\gamma h^{(s)}(X; a^K). \tag{3.6}$$

The main result of the previous section is that we can expand each line of (3.6) and extract a^u, b^u, c^u dependence to contract with expansion of star product and write exact expression in the term of AdS_{d+1} dimensional covariant derivatives and curvature corrections. Combining (2.24), (2.25) and (2.28), (2.40) we can write[10]

$$
(a^B \partial_B)^\alpha h^{(s)}(X; b^C) = e^{(s-2-\alpha)u} \sum_{p_1=0}^{\alpha} \sum_{k_1=0}^{[\frac{p_1}{2}]} \sum_{\tilde{p}_1=0}^{\alpha-p_1} \sum_{\tilde{k}_1=1}^{[\frac{\tilde{p}_1}{2}]} (-1)^{p_1+\tilde{p}_1+\tilde{k}_1} (a^u)^{p_1-2k_1} (b^u)^{\tilde{p}_1-2\tilde{k}_1}
$$

$$
(a, \nabla)^{\alpha-p_1-\tilde{p}_1} \xi_{k_1}^{p_1+1} (\alpha - p_1) \binom{\alpha - p_1}{\tilde{p}_1} (a^2)^{k_1} (a, \partial_b)^{\tilde{p}_1-2\tilde{k}_1} W^{\tilde{k}_1} (a^2, H_1) h^{(s)}(x^\mu; b^\mu).
$$

(3.7)

Then expanding:

$$
(a^A \partial_{b^A})^\gamma = \sum_{m=0}^{\gamma} \binom{\gamma}{m_1} (a^u \partial_{b^u})^{m_1} (a, \partial_b)^{\gamma-m_1},
$$

(3.8)

we obtain

$$
(a^A \partial_{b^A})^\gamma (a^B \partial_B)^\alpha h^{(s)}(X; b^C) = e^{(s-2-\alpha)u} \sum_{\substack{m_1=0 \\ m_1, p_1, k_1, \tilde{p}_1, \tilde{k}_1}}^{\gamma} \sum^{\gamma, \alpha, [\frac{p_1}{2}], \alpha-p_1, [\frac{\tilde{p}_1}{2}]}
$$

$$
(a^u)^{p_1-2k_1+m_1} (b^u)^{\tilde{p}_1-2\tilde{k}_1-m_1} (a, \partial_b)^{\gamma+\tilde{p}_1-2\tilde{k}_1-m_1} (a, \nabla)^{\alpha-p_1-\tilde{p}_1}
$$

$$
\Theta[\gamma, \alpha, m_1, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1] h^{(s)}(b^\mu),
$$

(3.9)

where:

$$
\sum_{m_1, p_1, k_1, \tilde{p}_1, \tilde{k}_1}^{\gamma, \alpha, [\frac{p_1}{2}], \alpha-p_1, [\frac{\tilde{p}_1}{2}]} = \sum_{m_1=0}^{\gamma} \sum_{p_1=0}^{\alpha} \sum_{k_1=0}^{[\frac{p_1}{2}]} \sum_{\tilde{p}_1=0}^{\alpha-p_1} \sum_{\tilde{k}_1=1}^{[\frac{\tilde{p}_1}{2}]},
$$

(3.10)

and

$$
\Theta[\gamma, \alpha, m_1, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1]
$$
$$
= (-1)^{p_1+\tilde{p}_1+\tilde{k}_1} [\tilde{p}_1 - 2\tilde{k}_1]_{m_1} \binom{\gamma}{m_1} \xi_{k_1}^{p_1+1} (\alpha - p_1) \binom{\alpha - p_1}{\tilde{p}_1} (a^2)^{k_1} W^{\tilde{k}_1} (a^2, H_1).
$$

(3.11)

Then we can write expression for the whole main interaction term

$$
\mathcal{L}_I^{main} = \int du e^{(d+2s-4)u} d^{d+1}x \sqrt{g}
$$

$$
\sum_{\substack{\alpha, \beta, \gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha, \beta, \gamma} \sum^{\gamma, \alpha, [\frac{p_1}{2}], \alpha-p_1, [\frac{\tilde{p}_1}{2}]}_{m_1, p_1, k_1, \tilde{p}_1, \tilde{k}_1} \sum^{\alpha, \beta, [\frac{p_2}{2}], \beta-p_2, [\frac{\tilde{p}_2}{2}]}_{m_2, p_2, k_2, \tilde{p}_2, \tilde{k}_2} \sum^{\beta, \gamma, [\frac{p_3}{2}], \gamma-p_3, [\frac{\tilde{p}_3}{2}]}_{m_3, p_3, k_3, \tilde{p}_3, \tilde{k}_3}
$$

---

[10] For shortening notation we introduce instead of $H(a, b)$ from (2.30) H_1 and then $H_2 = H(b, c)$ and $H_3 = H(c, a)$ correspondingly.

$$\sum_{n_1,n_2,n_3=0}^{\gamma+\alpha,\alpha+\beta,\beta+\gamma} \frac{(-1)^{n_1+n_2+n_3}}{\binom{\gamma+\alpha}{n_1}\binom{\alpha+\beta}{n_2}\binom{\beta+\gamma}{n_3}} *_{a^u}^{n_1} *_{b^u}^{n_2} *_{c^u}^{n_3} *_{a^\mu}^{\gamma+\alpha-n_1} *_{b^\mu}^{\alpha+\beta-n_2} *_{c^\mu}^{\beta+\gamma-n_3}$$

$$(a^u)^{p_1-2k_1+m_1}(b^u)^{\tilde{p}_1-2\tilde{k}_1-m_1}(a,\partial_b)^{\gamma+\tilde{p}_1-2\tilde{k}_1-m_1}(a,\nabla)^{\alpha-p_1-\tilde{p}_1}$$

$$\Theta[\gamma,\alpha,m_1,p_1,k_1,\tilde{p}_1,\tilde{k}_1,a^2,H_1]h^{(s)}(b^\mu)$$

$$(b^u)^{p_2-2k_2+m_2}(c^u)^{\tilde{p}_2-2\tilde{k}_2-m_2}(b,\partial_c)^{\alpha+\tilde{p}_2-2\tilde{k}_2-m_2}(b,\nabla)^{\beta-p_2-\tilde{p}_2}$$

$$\Theta[\alpha,\beta,m_2,p_2,k_2,\tilde{p}_2,\tilde{k}_2,b^2,H_2]h^{(s)}(c^\mu)$$

$$(c^u)^{p_3-2k_3+m_3}(a^u)^{\tilde{p}_3-2\tilde{k}_3-m_3}(c,\partial_a)^{\beta+\tilde{p}_3-2\tilde{k}_3-m_3}(c,\nabla)^{\gamma-p_3-\tilde{p}_3}$$

$$\Theta[\beta,\gamma,m_3,p_3,k_3,\tilde{p}_3,\tilde{k}_3,c^2,H_3]h^{(s)}(a^\mu). \tag{3.12}$$

Now we can contract all non AdS_{d+1} components a^u, b^u, c^u using corresponding "u"-stars from second line of (3.12). This leads to the following constraints for summation indices:

$$p_1-2k_1+m_1 = \tilde{p}_3-2\tilde{k}_3-m_3 = n_1, \tag{3.13}$$
$$p_2-2k_2+m_2 = \tilde{p}_1-2\tilde{k}_1-m_1 = n_2, \tag{3.14}$$
$$p_3-2k_3+m_3 = \tilde{p}_2-2\tilde{k}_2-m_2 = n_3. \tag{3.15}$$

So we can take summation over $m_i, i = 1, 2, 3$ with remaining constraints on other variables:

$$p_1+\tilde{p}_1 = n_1+n_2+2(k_1+\tilde{k}_1), \tag{3.16}$$
$$p_2+\tilde{p}_2 = n_2+n_3+2(k_2+\tilde{k}_2), \tag{3.17}$$
$$p_3+\tilde{p}_3 = n_3+n_1+2(k_3+\tilde{k}_3). \tag{3.18}$$

Relations (3.13)-(3.15) restrict also summation ranges for n_1, n_2, n_3 from zero to α, β, γ. Then we have

$$\mathcal{L}_I^{main} = \int du\, e^{(d+2s-4)u} d^{d+1}x \sqrt{g}$$

$$\sum_{\substack{\alpha,\beta,\gamma \\ \alpha+\beta+\gamma=s}} \binom{s}{\alpha,\beta,\gamma} \sum_{p_1,k_1,\tilde{p}_1,\tilde{k}_1}^{\alpha,[\frac{p_1}{2}],\alpha-p_1,[\frac{\tilde{p}_1}{2}]} \sum_{p_2,k_2,\tilde{p}_2,\tilde{k}_2}^{\beta,[\frac{p_2}{2}],\beta-p_2,[\frac{\tilde{p}_2}{2}]} \sum_{p_3,k_3,\tilde{p}_3,\tilde{k}_3}^{\gamma,[\frac{p_3}{2}],\gamma-p_3,[\frac{\tilde{p}_3}{2}]}$$

$$\sum_{n_1,n_2,n_3=0}^{\alpha,\beta,\gamma} \frac{(-1)^{n_1+n_2+n_3}}{\binom{\gamma+\alpha}{n_1}\binom{\alpha+\beta}{n_2}\binom{\beta+\gamma}{n_3}} *_{a^\mu}^{\gamma+\alpha-n_1} *_{b^\mu}^{\alpha+\beta-n_2} *_{c^\mu}^{\beta+\gamma-n_3}$$

$$(a,\partial_b)^{\gamma+n_2}(a,\nabla)^{\alpha-n_1-n_2-2(k_1+\tilde{k}_1)}\tilde{\Theta}[\gamma,\alpha,n_2,p_1,k_1,\tilde{p}_1,\tilde{k}_1,a^2,H_1]h^{(s)}(b^\mu)$$

$$(b,\partial_c)^{\alpha+n_3}(b,\nabla)^{\beta-n_2-n_3-2(k_2+\tilde{k}_2)}\tilde{\Theta}[\alpha,\beta,n_3,p_2,k_2,\tilde{p}_2,\tilde{k}_2,b^2,H_2]h^{(s)}(c^\mu)$$

$$(c,\partial_a)^{\beta+n_1}(c,\nabla)^{\gamma-n_3-n_1-2(k_3+\tilde{k}_3)}\tilde{\Theta}[\beta,\gamma,n_1,p_3,k_3,\tilde{p}_3,\tilde{k}_3,c^2,H_3]h^{(s)}(a^\mu), \tag{3.19}$$

where

$$\tilde{\Theta}[\gamma,\alpha,n_2,p_1,k_1,\tilde{p}_1,\tilde{k}_1,a^2,H_1] = \Theta[\gamma,\alpha,m_1=\tilde{p}_1-2\tilde{k}_1-n_2,p_1,k_1,\tilde{p}_1,\tilde{k}_1,a^2,H_1], \tag{3.20}$$

$$\tilde{\Theta}[\alpha,\beta,n_3,p_2,k_2,\tilde{p}_2,\tilde{k}_2,b^2,H_2] = \Theta[\alpha,\beta,m_2=\tilde{p}_2-2\tilde{k}_2-n_3,p_2,k_2,\tilde{p}_2,\tilde{k}_2,b^2,H_2], \tag{3.21}$$

$$\tilde{\Theta}[\beta, \gamma, n_1, p_3, k_3, \tilde{p}_3, \tilde{k}_3, c^2, H_3] = \Theta[\beta, \gamma, m_3 = \tilde{p}_3 - 2\tilde{k}_3 - n_1, \tilde{p}_3, p_3, k_3, \tilde{k}_3, c^2, H_3].$$

$$(3.22)$$

Taking into account that $\Theta[\ldots, a^2, H_1] \sim (a^2)^{k_1 + \tilde{k}_1}$ we see that our star products in (3.19) contract correctly all auxiliary vectors a^μ, b^ν, c^λ.

Then to understand better the structure of the derivatives of interaction we can take into account constraints (3.16)-(3.18) and rearrange the summations coming from (3.19) in the following way

$$\sum_{n_3 \geq 0} \sum_{n_2 \geq 0} \sum_{n_1 \geq 0} (-1)^{n_1 + n_2 + n_3} = \sum_{N \geq 0} (-1)^N \sum_{\substack{n_1, n_2, n_3 \\ \sum n_i = N}}, \tag{3.23}$$

$$\sum_{\substack{\{p_i, k_i, \tilde{p}_i, \tilde{k}_i\}_{i=1,2,3} \\ p_i + \tilde{p}_i = n_i + n_{i+1} + 2(k_i + \tilde{k}_i)}} = \sum_{K \geq 0} \sum_{\substack{\{P_i, K_i\}_{i=1,2,3} \\ P_i = n_i + n_{i+1} + 2K_i \\ \sum K_i = K}} \sum_{\substack{\{p_i, k_i, \tilde{p}_i, \tilde{k}_i\}_{i=1,2,3} \\ p_i + \tilde{p}_i = P_i; k_i + \tilde{k}_i = K_i}}, \tag{3.24}$$

where in last equation $\{n_i\} = n_1, n_2, n_3$ are with cyclic property $n_4 = n_1$

After that we should introduce instead of α, β, γ new summation variables

$$\tilde{\alpha} = \alpha - n_1 - n_2 - 2K_1 = \alpha - P_1, \tag{3.25}$$
$$\tilde{\beta} = \beta - n_2 - n_3 - 2K_2 = \beta - P_2, \tag{3.26}$$
$$\tilde{\gamma} = \gamma - n_3 - n_1 - 2K_3 = \gamma - P_3, \tag{3.27}$$

with corresponding summation limits and constraints

$$0 \leq \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \leq s - 2(N + K), \tag{3.28}$$
$$\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = s - 2(N + K), \tag{3.29}$$
$$N = \sum_i n_i; \quad K = \sum_i K_i = \sum_i (k_i + \tilde{k}_i). \tag{3.30}$$

These transformations lead to the following formula:

$$\mathcal{L}_I^{main} = \int du \, e^{(d+2s-4)u} d^{d+1}x \sqrt{g} \sum_{N \geq 0} \sum_{K \geq 0} \frac{(-1)^N s!}{(s - 2(N+K))!}$$

$$\sum_{\substack{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \\ \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = s - 2(N+K)}} \binom{s - 2(N + K)}{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}} \sum_{\substack{\{n_i\}_{i=1,2,3} \\ \sum n_i = N}} \sum_{\substack{\{P_i, K_i\}_{i=1,2,3} \\ P_i = n_i + n_{i+1} + 2K_i \\ \sum K_i = K}}$$

$$\sum_{\substack{\{p_i, k_i, \tilde{p}_i, \tilde{k}_i\}_{i=1,2,3} \\ p_i + \tilde{p}_i = P_i; k_i + \tilde{k}_i = K_i}} \frac{*_{a^\mu}^{\tilde{\gamma} + \tilde{\alpha} + N + 2(K_3 + K_1)} *_{b^\mu}^{\tilde{\alpha} + \tilde{\beta} + N + 2(K_1 + K_2)} *_{c^\mu}^{\tilde{\beta} + \tilde{\gamma} + N + 2(K_2 + K_3)}}{\binom{\tilde{\gamma} + \tilde{\alpha} + N + 2(K_3 + K_1) + n_1}{n_1} \binom{\tilde{\alpha} + \tilde{\beta} + N + 2(K_1 + K_2) + n_2}{n_2} \binom{\tilde{\beta} + \tilde{\gamma} + N + 2(K_2 + K_3) + n_3}{n_3}}$$

$$(a, \partial_b)^{\tilde{\gamma} + N + 2K_3} (a, \nabla)^{\tilde{\alpha}} \, \Xi^{2K_1}[\tilde{\gamma}, \tilde{\alpha}, n_2, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1] h^{(s)}(b^\mu)$$
$$(b, \partial_c)^{\tilde{\alpha} + N + 2K_1} (b, \nabla)^{\tilde{\beta}} \, \Xi^{2K_2}[\tilde{\alpha}, \tilde{\beta}, n_3, p_2, k_2, \tilde{p}_2, \tilde{k}_2, b^2, H_2] h^{(s)}(c^\mu)$$
$$(c, \partial_a)^{\tilde{\beta} + N + 2K_2} (c, \nabla)^{\tilde{\gamma}} \, \Xi^{2K_3}[\tilde{\beta}, \tilde{\gamma}, n_1, p_3, k_3, \tilde{p}_3, \tilde{k}_3, c^2, H_3] h^{(s)}(a^\mu), \tag{3.31}$$

where

$$\tilde{\Theta}[\gamma, \alpha, n_2, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1] = \frac{\gamma!}{\tilde{\alpha}!} \, \Xi^{2K_1}[\tilde{\gamma}, \tilde{\alpha}, n_2, P_3, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1], \quad (3.32)$$

$$\tilde{\Theta}[\alpha, \beta, n_3, p_2, k_2, \tilde{p}_2, \tilde{k}_2, b^2, H_2] = \frac{\alpha!}{\tilde{\beta}!} \, \Xi^{2K_2}[\tilde{\alpha}, \tilde{\beta}, n_3, P_1, p_2, k_2, \tilde{p}_2, \tilde{k}_2, b^2, H_2], \quad (3.33)$$

$$\tilde{\Theta}[\beta, \gamma, n_1, p_3, k_3, \tilde{p}_3, \tilde{k}_3, c^2, H_3] = \frac{\beta!}{\tilde{\gamma}!} \, \Xi^{2K_3}[\tilde{\beta}, \tilde{\gamma}, n_1, P_2, p_3, k_3, \tilde{p}_3, \tilde{k}_3, c^2, H_3], \quad (3.34)$$

and

$$\Xi^{2K_1}[\tilde{\gamma}, \tilde{\alpha}, n_2, P_3, p_1, k_1, \tilde{p}_1, \tilde{k}_1, a^2, H_1]$$
$$= \frac{(\tilde{\alpha} + \tilde{p}_1)!(a^2)^{k_1}}{(\tilde{\gamma} + P_3 - \tilde{p}_1 + 2\tilde{k}_1 + n_2)!} \binom{\tilde{p}_1 - 2\tilde{k}_1}{n_2} \xi_{k_1}^{p_1+1}(\tilde{\alpha} + \tilde{p}_1) W^{\tilde{k}_1}(a^2, H_1), \quad (3.35)$$

$$\Xi^{2K_2}[\tilde{\alpha}, \tilde{\beta}, n_3, P_1, p_2, k_2, \tilde{p}_2, \tilde{k}_2, b^2, H_2]$$
$$= \frac{(\tilde{\beta} + \tilde{p}_2)!(a^2)^{k_2}}{(\tilde{\alpha} + P_1 - \tilde{p}_2 + 2\tilde{k}_2 + n_3)!} \binom{\tilde{p}_2 - 2\tilde{k}_2}{n_3} \xi_{k_2}^{p_2+1}(\tilde{\beta} + \tilde{p}_2) W^{\tilde{k}_2}(b^2, H_2), \quad (3.36)$$

$$\Xi^{2K_3}[\tilde{\beta}, \tilde{\gamma}, n_1, P_2, p_3, k_3, \tilde{p}_3, \tilde{k}_3, c^2, H_3]$$
$$= \frac{(\tilde{\gamma} + \tilde{p}_3)!(a^2)^{k_3}}{(\tilde{\beta} + P_2 - \tilde{p}_3 + 2\tilde{k}_3 + n_1)!} \binom{\tilde{p}_3 - 2\tilde{k}_3}{n_1} \xi_{k_3}^{p_3+1}(\tilde{\gamma} + \tilde{p}_3) W^{\tilde{k}_3}(c^2, H_3). \quad (3.37)$$

Finalizing our consideration we can write direct (a^2), $(b)^2$, $(c)^2$ expansion of corresponding Ξ^{2K_i} terms using (2.34) and (2.43)

$$(a^2)^{k_1} W^{\tilde{k}_1}(a^2, H_1) h^{(s)}(b^\mu)$$
$$= \sum_{t_1=0}^{\tilde{k}_1} (-1)^{t_1} \sum_{r_1=1}^{\tilde{k}_1-t_1} \eta_{\tilde{k}_1}^{t_1}(\tilde{p}_1 - \tilde{k}_1) A_{r_1}^{\tilde{k}_1-t_1}[s]_{r_1} (a^2)^{K_1-r_1}(a, b)^{r_1} \Phi_{r_1}(a, b), \quad (3.38)$$

$$(b^2)^{k_2} W^{\tilde{k}_2}(b^2, H_2) h^{(s)}(c^\mu)$$
$$= \sum_{t_2=0}^{\tilde{k}_2} (-1)^{t_2} \sum_{r_2=1}^{\tilde{k}_2-t_2} \eta_{\tilde{k}_2}^{t_2}(\tilde{p}_2 - \tilde{k}_2) A_{r_2}^{\tilde{k}_2-t_2}[s]_{r_2} (b^2)^{K_2-r_2}(b, c)^{r_2} \Phi_{r_2}(b, c), \quad (3.39)$$

$$(c^2)^{k_3} W^{\tilde{k}_3}(a^3, H_3) h^{(s)}(c^\mu)$$
$$= \sum_{t_3=0}^{\tilde{k}_3} (-1)^{t_3} \sum_{r_3=1}^{\tilde{k}_3-t_3} \eta_{\tilde{k}_3}^{t_3}(\tilde{p}_3 - \tilde{k}_3) A_{r_3}^{\tilde{k}_3-t_3}[s]_{r_3} (c^2)^{K_3-r_3}(c, a)^{r_3} \Phi_{r_3}(c, a). \quad (3.40)$$

So we see that Ξ^{2K_i} in (3.32)-(3.34) really behave like a^{2K_1}, b^{2K_2}, c^{2K_3} as they should for correct contractions of indices.

4. Conclusion

We have constructed all AdS corrections including trace and divergence terms to the main term of the cubic self-interaction by a slightly modified method of radial pullback (reduction) proposed in [48], where all quantum fields are carried by a real AdS space and corresponding interaction terms expressed through the covariant AdS derivatives. For given spin s and $\Delta_{min} = s$ we derived all curvature correction terms (3.31) in the form of series of terms with numbers

$s - 2(N + K)$ of derivatives, where $0 \leq N + K \leq \frac{s}{2}$. The latter is the number of seized pair of derivatives replaced by corresponding power of $1/L^2$ and K is the sum of power of a^2, b^2, c^2 terms connected with trace and divergent correction terms produced from the main term of interaction after pullback. Correction terms appear with coefficients that are polynomials in the dimension $d + 1$ and spin number s with rational coefficients. Now we can expect that the same method can be used for the derivation of the AdS corrections to traces and de Donder terms connected with the main term by Noether's procedure derived for the flat case in [33] and [34].

Declaration of competing interest

Acknowledgements

Appendix A. Main term of cubic interaction in flat space

In this Appendix we repeat the general formula for a covariant cubic interaction of higher spin gauge fields in a flat background as presented in [33] and [34]. The main result of [33, 34] is the following. The gauge invariance fixes in a unique way the cubic interaction if the main cyclic ansatz term without divergences and traces is given. Accordingly in this article we consider only the main term of the cubic interaction postponing the proof for all other terms to a future publication, and understanding intuitively that gauge invariance is going to regulate in a correct fashion the radial reduction for all other terms presented in [33,34] and classified in corresponding tables there.

In [33,34] we considered three potentials $h^{(s_1)}(X_1; a^A)$, $h^{(s_2)}(X_2; b^A)$, $h^{(s_3)}(X_3; c^A)$ of $d + 2$ dimensional flat theory with ordered spins s_i

$$s_1 \geq s_2 \geq s_3, \tag{A.1}$$

and with the cyclic ansatz for the interaction

$$
\begin{aligned}
\mathcal{L}_I^{main} & (h^{(s_1)}(X_1, a^A), h^{(s_2)}(X_2, b^A), h^{(s_3)}(X_3 c^A)) \\
& = \sum_{n_i} C_{n_1, n_2, n_3}^{s_1, s_2, s_3} \int d^{d+2} X_1 d^{d+2} X_2 d^{d+2} X_3 \delta(X_3 - X_1) \delta(X_2 - X_1) \\
& \times \tilde{T}(Q_{12}, Q_{23}, Q_{31} | n_1, n_2, n_3) h^{(s_1)}(X_1; a^A) h^{(s_2)}(X_2; b^B) h^{(s_3)}(X_3; c^C),
\end{aligned}
\tag{A.2}
$$

where

$$
\begin{aligned}
\tilde{T} & (Q_{12}, Q_{23}, Q_{31} | n_1, n_2, n_3) \\
& = (\partial_{a^A} \partial_{b_A})^{Q_{12}} (\partial_{b^B} \partial_{c_B})^{Q_{23}} (\partial_{c^C} \partial_{a_C})^{Q_{31}} (\partial_{a^D} \tilde{\nabla}_2^D)^{n_1} (\partial_{b^E} \tilde{\nabla}_3^E)^{n_2} (\partial_{c^F} \tilde{\nabla}_1^F)^{n_3},
\end{aligned}
\tag{A.3}
$$

and the notation "*main*" as a superscript means that it is an ansatz for terms without $Divh^{(s_i-1)}$ and $Trh^{(s_i-2)}$. Denoting the number of derivatives by Δ we have

$$n_1 + n_2 + n_3 = \Delta. \tag{A.4}$$

We shall later determine and then use the minimal possible Δ. As balance equations we have

$$\begin{aligned} n_1 + Q_{12} + Q_{31} &= s_1, \\ n_2 + Q_{23} + Q_{12} &= s_2, \\ n_3 + Q_{31} + Q_{23} &= s_3. \end{aligned} \tag{A.5}$$

These equations are solved by

$$\begin{aligned} Q_{12} &= n_3 - v_3, \\ Q_{23} &= n_1 - v_1, \\ Q_{31} &= n_2 - v_2. \end{aligned} \tag{A.6}$$

Since the l.h.s. cannot be negative, we have

$$n_i \geq v_i.$$

The v_i are determined to be

$$v_i = 1/2(\Delta + s_i - s_j - s_k), \quad i, j, k \quad \text{are all different.} \tag{A.7}$$

It follows that the minimally possible Δ is expressed by Metsaev's [20] (using the ordering of the s_i).

$$\Delta_{min} = \max[s_i + s_j - s_k] = s_1 + s_2 - s_3. \tag{A.8}$$

Another result of [33,34] is the trinomial expression for the coefficients in (A.2) fixed by Noether's procedure. Taking into account (A.5)-(A.8) we can write it in the following elegant form

$$C^{s_1,s_2,s_3}_{n_1,n_2,n_3} = C^{s_1,s_2,s_3}_{Q_{12},Q_{23},Q_{31}} = const \begin{pmatrix} s_{min} \\ Q_{12}, Q_{23}, Q_{31} \end{pmatrix}. \tag{A.9}$$

Appendix B. Solution for ξ_k^{p+1} for general p

To evaluate (2.19) on the ground state (2.16) we have to expand this operator power (2.19) into a noncommutative binomial series:

$$[(a, e^{-u}\hat{\nabla}) - e^{-u}R]^n \, |0> = \sum_{p=0}^{n} (-1)^p$$

$$\sum_{n-p \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} (a, e^{-u}\hat{\nabla})^{n-p-i_p} e^{-u} R(a, e^{-u}\hat{\nabla})^{i_p - i_{p-1}} \ldots$$

$$e^{-u} R(a, e^{-u}\hat{\nabla})^{i_1} \, |0>. \tag{B.1}$$

Then using relation

$$[R, (a, e^{-u}\hat{\nabla})^{i_k}] = 2i_k e^{-i_k u} a^u (a, \hat{\nabla})^{i_k}, \tag{B.2}$$

we can rewrite (B.1) in the following form

$$[(a, e^{-u}\hat{\nabla}) - e^{-u}R]^n \mid 0 >= \sum_{p=0}^{n} (-1)^p (a, \hat{\nabla})^{n-p} e^{(p-n)u}$$

$$\sum_{n-p \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} e^{-u}(2i_p a^u + R) e^{-u}(2i_{p-1}a^u + R) \ldots$$

$$e^{-u}(2i_1 a^u + R) e^{(s-2)u} h^{(s)}(x^\mu; b^\mu). \tag{B.3}$$

Then introducing the new objects

$$\phi_{i_k} = 2i_k a^u + R = a^u[2i_k + (a, \partial_a) + a^u \partial_{a^u} - \partial_u] + [a^2 - (a^u)^2]\partial_{a^u}, \tag{B.4}$$

and taking into account that

$$[(a, \partial_a) + a^u \partial_{a^u} - \partial_u] e^{-nu} f^{(m)}(a^\mu, a^u) = (m+n) e^{-nu} f^{(m)}(a^\mu, a^u), \tag{B.5}$$

we obtain

$$[(a, e^{-u}\hat{\nabla}) - e^{-u}R]^n \mid 0 >= e^{(s-2-n)u} \sum_{p=0}^{n} (-1)^p (a, \hat{\nabla})^{n-p}$$

$$\sum_{n-p \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} \phi_{i_p} \phi_{i_{p-1}} \ldots \phi_{i_2} \phi_{i_1} h^{(s)}(x^\mu; b^\mu), \tag{B.6}$$

where we have ϕ_{i_k} as a very simple "creation" operators

$$\phi_{i_k} = a^u[2(i_k + k) - s] + [a^2 - (a^u)^2]\partial_{a^u}. \tag{B.7}$$

Now we show how to perform summation in (2.24) and obtain wanted expansion on the power of a^u to contract after. Introducing notation

$$V^{p+1}(i_{p+1}) h^{(s)}(x^\mu; b^\mu) = \sum_{i_{p+1} \geq i_p \geq i_{p-1} \geq i_{p-2} \ldots \geq i_1 \geq 0} \phi_{i_p} \phi_{i_{p-1}} \ldots \phi_{i_2} \phi_{i_1} h^{(s)}(x^\mu; b^\mu), \quad \text{(B.8)}$$

and performing summation over the labels $\{i_k\}|_{k=1}^{p}$ we should obtain a polynomial in a^u and (a^2) of the form[11]

$$V^{p+1}(i_{p+1}) = \sum_{k=0}^{[\frac{p}{2}]} \xi_k^{p+1}(i_{p+1})(a^2)^k (a^u)^{p-2k}. \tag{B.9}$$

Considering the last expression as an ansatz for equation

$$V^{p+1}(i_{p+1}) = \sum_{i_p=0}^{i_{p+1}} \phi_{i_p} V^p(i_p), \tag{B.10}$$

and using (B.7) we obtain the following recurrence relation for $2p - k$ order polynomials coefficients $\xi_k^{p+1}(i_{p+1}) \sim (i_{p+1})^{2p-k} + \ldots$

---

[11] Note that $[p/2]$ is integer part of $p/2$ and at the end we have to insert $i_{p+1} = n - p$.

$$\xi_k^{p+1}(j) = \sum_{i=0}^{j} (2i + p + 1 + 2k - s)\xi_k^p(i) + \sum_{i=0}^{j} (p + 1 - 2k)\xi_{k-1}^p(i). \tag{B.11}$$

This equation is easier to consider in "differential" form

$$\xi_k^{p+1}(i) - \xi_k^{p+1}(i-1) = (2i + p + 1 + 2k - s)\xi_k^p(i) + (p + 1 - 2k)\xi_{k-1}^p(i). \tag{B.12}$$

To get general solution of (B.12) we present first solutions of latter equation obtained by direct calculation of V^{p+1}. So using (B.10) for $p = 1, 2, 3, 4, \ldots$ and expanding the results using (2.25) we search for the form of first ξ_k^{p+1} coefficients.

$$V^2(i_2) = \sum_{i_1=0}^{i_2} \phi_{i_1} |0> = (1 + i_2)(2 - s + i_2) a^u |0>, \tag{B.13}$$

$$V^3(i_3) = \sum_{i_1=0}^{i_2} \phi_{i_2} V^2(i_2) = \frac{1}{6} a^2 (1 + i_3)(2 + i_3)(6 - 3s + 2i_3) |0>$$
$$+ \frac{1}{2}(1 + i_3)(2 + i_3)(2 - s + i_3)(3 - s + i_3) (a^u)^2 |0>, \tag{B.14}$$

$$V^4(i_4) = \sum_{i_3=0}^{i_4} \phi_{i_3} V^3(i_3)$$
$$= \frac{1}{6} a^2 (1 + i_4)(2 + i_4)(3 + i_4)(4 - s + i_4)(6 - 3s + 2i_4) a^u |0>$$
$$+ \frac{1}{6}(1 + i_4)(2 + i_4)(3 + i_4)(2 - s + i_4)(3 - s + i_4)(4 - s + i_4) (a^u)^3 |0>, \tag{B.15}$$

$$V^5(i_5) = \sum_{i_4=0}^{i_5} \phi_{i_4} V^4(i_4)$$
$$= \frac{1}{360} a^4 (1 + i_5)(2 + i_5)(3 + i_5)(4 + i_5)$$
$$\left(360 - 270s + 45s^2 + 172i_5 - 60si_5 + 20i_5^2\right) |0>$$
$$+ \frac{1}{12} a^2 (1 + i_5)(2 + i_5)(3 + i_5)(4 + i_5)(4 - s + i_5)$$
$$(5 - s + i_5)(6 - 3s + 2i_5) (a^u)^2 |0>$$
$$+ \frac{1}{24}(1 + i_5)(2 + i_5)(3 + i_5)(4 + i_5)(2 - s + i_5)(3 - s + i_5)$$
$$(4 - s + i_5)(5 - s + i_5) (a^u)^4 |0> . \tag{B.16}$$

After investigation of the structures of first ξ_k^{p+1} coefficients we note that they all have the following general factor

$$\frac{1}{(p - 2k)!}(i + 1)_p (2k + 2 + i - s)_{p-2k}. \tag{B.17}$$

Using this information we arrive to the following important ansatz for $\xi_k^{p+1}(i)$

$$\xi_k^{p+1}(i) = \frac{1}{(p-2k)!}(i+1)_p(2k+2+i-s)_{p-2k}P_k(i),\tag{B.18}$$

where $P_k(i) \sim i^k + \dots$ is now *p-independent polynomial of order k* and we introduced Pochhammer symbols

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1).\tag{B.19}$$

Inserting (B.18) in equation (B.12) we obtain equation for $P_k(i)$:

$$(i+2k)P_k(i) - iP_k(i-1) = (i+2k-s)P_{k-1}(i).\tag{B.20}$$

Then after more convenient normalization of our polynomials with additional $2k$ order factor:

$$\mathcal{P}_k(i) \equiv (i+1)_{2k}P_k(i),\tag{B.21}$$

we arrive to the following simple equation with boundary condition:

$$\mathcal{P}_k(i) - \mathcal{P}_k(i-1) = (i+2k-1)(i+2k-s)\mathcal{P}_{k-1}(i),\tag{B.22}$$

$$\mathcal{P}_0(i) = P_0(i) = 1.\tag{B.23}$$

This we can solve in two way: first in the form of multiple sums (2.27), or solving differential equation for generating function

$$\mathcal{P}_k(y) \equiv \sum_{i=0}^{\infty} \mathcal{P}_k(i)y^i,\tag{B.24}$$

where we introduced formal variable y with $|y| < 1$ for production of the boundary condition:

$$\mathcal{P}_0(y) = \sum_{i=0}^{\infty} y^i = \frac{1}{1-y}.\tag{B.25}$$

For this generation function, we obtain from recurrence relation (B.22) the equation

$$(1-y)\mathcal{P}_k(y) = (y\frac{d}{dy} + 2k - 1)(y\frac{d}{dy} + 2k - s)\mathcal{P}_{k-1}(y).\tag{B.26}$$

Solving recursively and using (B.25) we can write the solution in the form:

$$\mathcal{P}_k(y) = y^{-(2k+1)}\left[\frac{y^4}{1-y}\frac{d}{dy}y^s\frac{d}{dy}y^{-s}\right]^k\frac{y^2}{1-y}.\tag{B.27}$$

Finally, we can write (B.18) in term of $\mathcal{P}_k(i)$ as (2.26).

Appendix C. The structure of the polynomial coefficients and the iterative approach of finding solutions

To obtain the ansatz (2.34) we use direct calculations of the first three powers of H acting on ground state $\Phi_0(b)$

$$H\Phi_0(b) = -s(a,b)\Phi_1(a,b),\tag{C.1}$$

$$H^2\Phi_0(b) = [s]_2(a,b)^2\Phi_2(a;b) + sa^2(a,b)\Phi_1(a;b),\tag{C.2}$$

$$H^3\Phi_0(b) = -\{[s]_3(a,b)^3\Phi_3(a;b) + 3[s]_2a^2(a,b)^2\Phi_2(a;b) + s(a^2)^2(a,b)\Phi_1(a;b)\}.\tag{C.3}$$

Then the way of solution of the recurrent relation (2.35) is obvious: We should multiply $A_r^{(n)}$ by x^r and introduce

$$P_n(x) = \sum_{r=0}^{\infty} A_r^{(n)} x^r. \tag{C.4}$$

Then we obtain a simple differential equation

$$x \frac{d}{dx} \left(e^x P_n(x) \right) = e^x P_{n+1}(x), \tag{C.5}$$

which we can easily solve since $P_0(x) = 1$.

Iterating n times we find

$$e^x P_n(x) = \left(x \frac{d}{dx} \right)^n e^x, \tag{C.6}$$

or

$$P_n(x) = e^{-x} \left(x \frac{d}{dx} \right)^n e^x. \tag{C.7}$$

Evidently, $P_n(x)$ is a polynomial of order n, which means that $A_r^{(n)} = 0$ for $r > n$.

Finally, we can find a "double" generating function. Introducing

$$Q(x,t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}. \tag{C.8}$$

So we see that

$$Q(x,t) = e^{-x} e^{tx \frac{d}{dx}} e^x = e^{x(e^t - 1)}, \tag{C.9}$$

where we have explored the fact that the operator $e^{tx \frac{d}{dx}}$, rescales the variable x by the factor e^t. Expanding (C.9) in x and t we get

$$e^{x(e^t - 1)} = \prod_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \frac{x^{k_i} t^{i k_i}}{k_i! (i!)^{k_i}}. \tag{C.10}$$

This can be used to derive (2.39).

To obtain the important representation (2.40) we start from

$$(L^+ + L^-)^{\tilde{p}} \Phi_0(b) = \sum_{\tilde{k}=0}^{\tilde{p}} \sum_{\tilde{p} - \tilde{k} \geq i_{\tilde{k}} \geq i_{\tilde{k}-1} \geq i_{\tilde{k}-2} \cdots \geq i_1 \geq 1}$$

$$(L^+)^{\tilde{p} - \tilde{k} - i_{\tilde{k}}} L^- (L^+)^{i_{\tilde{k}} - i_{\tilde{k}-1}} L^- (L^+)^{i_{\tilde{k}-1} - i_{\tilde{k}-2}} L^- \ldots (L^+)^{i_2 - i_1} L^- (L^+)^{i_1} \Phi_0(b). \tag{C.11}$$

Only commutators of L^- with powers of L^+ arise

$$[L^-, (L^+)^i] = -\sum_{j=0}^{i-1} (L^+)^{i-j-1} H (L^+)^j =$$

$$-\sum_{j=0}^{i-1} (L^+)^{i-1} (H + 2ja^2) = -(L^+)^{i-1} (iH + [i]_2 a^2). \tag{C.12}$$

Here we recognize that the whole basis $\{\Phi_n(a; b)\}$ of null vectors is produced from $\Phi_0(b)$ by the action of H. With the shorthand

$$\psi_i = iH + [i]_2 a^2, \tag{C.13}$$

the result is

$$\sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]} (-1)^{\tilde{k}} (L^+)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2, H) \Phi_0(b) = \sum_{\tilde{k}=1}^{[\frac{\tilde{p}}{2}]} (b^u)^{\tilde{p}-2\tilde{k}} (-1)^{\tilde{k}} (a, \partial_b)^{\tilde{p}-2\tilde{k}} W^{\tilde{k}}(a^2, H) \Phi_0(b), \tag{C.14}$$

where

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) \Phi_0(b)$$
$$= \sum_{i_{\tilde{k}+1} \geq i_{\tilde{k}} \geq i_{\tilde{k}-1} \geq i_{\tilde{k}-2} \cdots \geq i_2 \geq i_1 \geq 1} \psi_{i_{\tilde{k}}-\tilde{k}+1} \psi_{i_{\tilde{k}-1}-\tilde{k}+2} \psi_{i_{\tilde{k}-2}-\tilde{k}+3} \cdots \psi_{i_2-1} \psi_{i_1} \Phi_0(b). \tag{C.15}$$

The sum is a homogeneous polynomial of H and a^2 of degree $\tilde{k}$[12]:

$$W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}) = \sum_{m=0}^{\tilde{k}} \eta_{\tilde{k}}^m(i_{\tilde{k}+1})(a^2)^m H^{\tilde{k}-m}. \tag{C.16}$$

Using this ansatz and doing in the way similar to (2.25) we derive from

$$W^{\tilde{k}+1}(a^2, H, i_{\tilde{k}+2}) = \sum_{i_{\tilde{k}+1}=1}^{i_{\tilde{k}+2}} \psi_{i_{\tilde{k}+1}-\tilde{k}} W^{\tilde{k}}(a^2, H, i_{\tilde{k}+1}), \tag{C.17}$$

the following recurrence relation

$$\eta_{\tilde{k}+1}^m(j) = \sum_{i=1}^{j} \left[(i - \tilde{k}) \eta_{\tilde{k}}^m(i) + (i - \tilde{k})(i - \tilde{k} - 1) \eta_{\tilde{k}}^{m-1}(i) \right], \tag{C.18}$$

or without summation:

$$\eta_{\tilde{k}+1}^m(i) - \eta_{\tilde{k}+1}^m(i-1) = (i - \tilde{k}) \eta_{\tilde{k}}^m(i) + (i - \tilde{k})(i - \tilde{k} - 1) \eta_{\tilde{k}}^{m-1}(i). \tag{C.19}$$

To investigate the structure of these coefficients and gain more information about polynomials we compute the (2.43) for initial values of $\tilde{k} = 1, 2, 3, 4 \dots$:

$$W^1(a^2, H, i_2) = \sum_{i_1=1}^{i_2} \psi_{i_1} \Phi_0(b) = \frac{1}{2} H i_2 (1 + i_2) \Phi_0 + \frac{1}{3} a^2 (-1 + i_2) i_2 (1 + i_2) \Phi_0, \tag{C.20}$$

---

[12] Remember that H is second order in a as well.

$$W^2(a^2, H, i_3) = \sum_{i_2=1}^{i_3} \psi_{i_2-1} W^1(a^2, H, i_2) = \frac{1}{8} H^2 (-1 + i_3) i_3 (1 + i_3) (2 + i_3) \Phi_0$$

$$+ \frac{1}{12} a^2 H (-1 + i_3) i_3 (1 + i_3) (2 + i_3) (-3 + 2i_3) \Phi_0$$

$$+ \frac{1}{90} a^4 (-2 + i_3) (-1 + i_3) i_3 (1 + i_3) (2 + i_3) (-3 + 5i_3) \Phi_0, \tag{C.21}$$

$$W^3(a^2, H, i_4) = \sum_{i_3=1}^{i_4} \psi_{i_3-2} W^2(a^2, H, i_3) =$$

$$\frac{1}{48} H^3 (-2 + i_4) (-1 + i_4) i_4 (1 + i_4) (2 + i_4) (3 + i_4) \Phi_0$$

$$+ \frac{1}{24} a^2 H^2 (-2 + i_4)^2 (-1 + i_4) i_4 (1 + i_4) (2 + i_4) (3 + i_4) \Phi_0$$

$$+ \frac{1}{180} a^4 H (-2 + i_4) (-1 + i_4)^2 i_4 (1 + i_4) (2 + i_4) (3 + i_4) (-13 + 5i_4) \Phi_0$$

$$+ \frac{a^6 (-3 + i_4) (-2 + i_4) (-1 + i_4) i_4 (1 + i_4) (2 + i_4) (3 + i_4) \left(-2 - 63i_4 + 35i_4^2\right) \Phi_0}{5670}, \tag{C.22}$$

$$W^4(a^2, H, i_5) = \sum_{i_4=1}^{i_5} \psi_{i_4-3} W^3(a^2, H, i_4)$$

$$= \frac{1}{384} H^4 (-3 + i_5) (-2 + i_5) (-1 + i_5) i_5 (1 + i_5) (2 + i_5) (3 + i_5) (4 + i_5) \Phi_0$$

$$+ \frac{1}{288} a^2 H^3 (-3 + i_5) (-2 + i_5) (-1 + i_5) i_5 (1 + i_5) (2 + i_5)$$

$$(3 + i_5) (4 + i_5) (-5 + 2i_5) \Phi_0$$

$$+ \frac{1}{1440} a^4 H^2 (-3 + i_5) (-2 + i_5) (-1 + i_5) i_5 (1 + i_5) (2 + i_5) (3 + i_5) (4 + i_5)$$

$$\left(45 - 46i_5 + 10i_5^2\right) \Phi_0$$

$$+ \frac{1}{22680} a^6 H (-3 + i_5) (-2 + i_5) (-1 + i_5) i_5 (1 + i_5) (2 + i_5) (3 + i_5) (4 + i_5)$$

$$\left(-195 + 731i_5 - 441i_5^2 + 70i_5^3\right) \Phi_0$$

$$+ \frac{1}{340200} a^8 (-4 + i_5) (-3 + i_5) (-2 + i_5) (-1 + i_5) i_5 (1 + i_5) (2 + i_5) (3 + i_5) (4 + i_5)$$

$$\left(570 + 149i_5 - 630i_5^2 + 175i_5^3\right) \Phi_0. \tag{C.23}$$

From these direct calculations it is easy to see that we can factorize again $i^{2\tilde{k}}$ terms and write $\eta_{\tilde{k}}^m(i)$ in this form:

$$\eta_{\tilde{k}}^m(i) = \frac{2^{m-\tilde{k}} 3^{-m}}{(\tilde{k} - m)! m!} (i - \tilde{k} + 1)_{2\tilde{k}} P_m(i, \tilde{k}), \quad P_0(i, \tilde{k}) = 1, \tag{C.24}$$

where the polynomials $P_m(i, \tilde{k}) \sim (i - \frac{\tilde{k}}{2})^m + \dots$ is mth orders in i and $\tilde{k}$ with binomial leading term and satisfy the equation

$$(i + \tilde{k} + 1) P_m(i, \tilde{k} + 1) - (i - \tilde{k} - 1) P_m(i - 1, \tilde{k} + 1)$$
$$= 2(\tilde{k} - m + 1) P_m(i, \tilde{k}) + 3m(i - \tilde{k} - 1) P_{m-1}(i, \tilde{k}), \tag{C.25}$$

with the same level of difficulty to solve as (C.19). The solutions of this recurrent equation can be calculated step by step from the (C.20)-(C.23) for each k:

$$P_0(i, p) = 1, \tag{C.26}$$

$$P_1(i, p) = i - \left(\frac{p}{2} + \frac{1}{2}\right), \tag{C.27}$$

$$P_2(i, p) = i^2 - 2i\left(\frac{p}{2} + \frac{3}{10}\right) + \left(\frac{p^2}{4} + \frac{3p}{20} - \frac{1}{10}\right), \tag{C.28}$$

$$P_3(i, p) = i^3 - 3i^2\left(\frac{p}{2} + \frac{1}{10}\right) + 3i\left(\frac{p^2}{4} - \frac{p}{20} - \frac{67}{210}\right) - \left(\frac{p^3}{8} - \frac{3p^2}{20} - \frac{173p}{280} - \frac{12}{35}\right), \tag{C.29}$$

$$P_4(i, p) = i^4 - 4i^3\left(\frac{p}{2} - \frac{1}{10}\right) + 6i^2\left(\frac{p^2}{4} - \frac{p}{4} - \frac{481}{1050}\right)$$
$$- 4i\left(\frac{p^3}{8} - \frac{3p^2}{10} - \frac{1031p}{1400} - \frac{38}{175}\right) + \frac{p^4}{16} - \frac{11p^3}{40} - \frac{2011p^2}{2800} - \frac{89p}{1400} + \frac{111}{350}. \tag{C.30}$$

From the solutions above we can see that the general ansatz for $P_k(i, p)$ has the following form

$$P_k(i, p) = \sum_{n=0}^{k} i^{k-n}(-1)^n \binom{k}{n} B_k^n(p).$$

From the solutions above for different $P_k(i, p)$ it is possible to find the solutions for $B_k^n(p)$ as follows[13]

$$B^1{}_k(p) = \frac{p}{2} - \frac{k}{5} + \frac{7}{10}, \tag{C.31}$$

$$B^2{}_k(p) = \frac{p^2}{4} + p\left(\frac{11}{20} - \frac{k}{5}\right) + \frac{k^2}{25} - \frac{44k}{105} + \frac{607}{1050}, \tag{C.32}$$

$$B^3{}_k(p) = \frac{p^3}{8} + p^2\left(\frac{3}{10} - \frac{3k}{20}\right) + p\left(\frac{3k^2}{50} - \frac{377k}{700} + \frac{641}{1400}\right)$$
$$- \frac{k^3}{125} + \frac{293k^2}{1750} - \frac{1313k}{1750} + \frac{108}{175}, \tag{C.33}$$

---

[13] In order to compute the $B_k^n(p)$ using this iterative approach one should compute and know the expressions of $P_m(i, p)$ for up to $m = 2k$.

$$
\begin{aligned}
B^4{}_k(p) = {} & \frac{p^4}{16} + p^3\left(\frac{1}{8} - \frac{k}{10}\right) + p^2\left(\frac{3k^2}{50} - \frac{157k}{350} + \frac{13}{112}\right) \\
& + p\left(-\frac{2k^3}{125} + \frac{523k^2}{1750} - \frac{131k}{125} + \frac{519}{1400}\right) + \frac{k^4}{625} - \frac{244k^3}{4375} \\
& + \frac{47728k^2}{91875} - \frac{460722k}{336875} + \frac{256957}{404250}.
\end{aligned}
\tag{C.34}
$$

The final form of $\eta^m_{\tilde{k}}(i)$ coefficients will be

$$
\eta^m_{\tilde{k}}(i) = \frac{2^{m-\tilde{k}}3^{-m}}{(\tilde{k}-m)!m!}(i-\tilde{k}+1)_{2\tilde{k}}\sum_{n=0}^{m} i^{m-n}(-1)^n\binom{m}{n}B^n_m(\tilde{k}).
\tag{C.35}
$$

Appendix D. Mapping operator $(a, \partial_b)^p$ to the product of H and a^2

This is the final exercise to get more freedom in writing of our cubic interaction after our "stripping" for u components of auxiliary vectors. Investigating (C.14) and first operator in (2.9):

$$
(a^A\partial_{b^A})^Q = \sum_{q=0}^{Q}\binom{Q}{q}(a^u\partial_{b^u})^{Q-q}(a, \partial_b)^q,
\tag{D.1}
$$

we see that last thing to do is transform the power of (a, ∂_b) to H and a^2 to write interaction without (a, ∂_b), hiding them then in $\Phi(a, b)$. Note that starting from (C.1) operator H effectively worked with only its second part:

$$
H \equiv H(a, b) = -(a, b)(a, \partial_b),
\tag{D.2}
$$

due to the separation of all b^u dependence to the left from H dependent part in (C.14). Therefore we can write simple relation

$$
(a, \partial_b)^p = (-1)^p\left(\frac{1}{(a, b)}H\right)^p.
\tag{D.3}
$$

Then introducing ansatz for ordered power:

$$
\left(\frac{1}{(a, b)}H\right)^p = \frac{1}{(a, b)^p}\sum_{k=0}^{p-1}\rho_k(p)H^{p-k}(a^2)^k,
\tag{D.4}
$$

and taking into account commutator

$$
[H, (a, b)^{-k}] = \frac{ka^2}{(a, b)^k},
\tag{D.5}
$$

we arrive to the following simple triangular recurrence relation for polynomials $\rho_k(p)$

$$
\rho_k(p+1) = \rho_k(p) + p\rho_{k-1}(p),
\tag{D.6}
$$

with boundary conditions:

$$
\rho_0(p) = 1, \qquad \rho_{p-1}(p) = (p-1)!.
\tag{D.7}
$$

Recurrence relation (D.6) we can easily solve using generation function. Introducing formal variable z with $|z| < 1$

$$\rho_k(z) = \sum_{p=0}^{\infty} z^p \rho_k(p), \tag{D.8}$$

we obtain recursive equation:

$$\rho_k(z) = \frac{z^2}{1-z} \frac{d}{dz} \rho_{k-1}(z), \tag{D.9}$$

with the simple solution due to boundary value $\rho_0(z) = (1-z)^{-1}$:

$$\rho_k(z) = \left[\frac{z^2}{1-z} \frac{d}{dz} \right]^k \frac{1}{1-z}. \tag{D.10}$$

In another way we can write the same solution of (D.6) in the form of multiple sums:

$$\rho_k(p) = \sum_{i_k=k}^{p-1} i_k \sum_{i_{k-1}=k-1}^{i_k-1} i_{k-1} \cdots \sum_{i_2=2}^{i_3-1} i_2 \sum_{i_1=1}^{i_2-1} i_1. \tag{D.11}$$

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