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SYMMETRIES OF THE AdS/CFT S-MATRIX*

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In this paper we will review the symmetries of the AdS/CFT S-matrix, in particular, the Hopf Algebra and Yangian symmetries.

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1. Introduction

In the last couple of years huge progress has been done in understanding the AdS/CFT correspondence by exploiting its integrability properties. Originally, integrability was only discovered in certain subsectors [1] and at one-loop gauge theory [2], and arguments were soon afterwards presented that integrability also persists at all orders and in all sectors of the AdS/CFT correspondence [3]. Not long thereafter, long-range Bethe equations have been conjectured [4] which are supposed to describe the spectrum of all long operators if one uses the correct dressing phase obtained in [5,6], see also [7] for an interesting discussion regarding its origin. These Bethe equations can be obtained from an S-matrix scattering fundamental magnons. The S-matrix can be derived by using invariance under the correct symmetry algebra [8]. The global symmetry algebra of $\mathcal{N} = 4$ Super Yang–Mills theory is the superconformal algebra $\mathfrak{psu}(2,2|4)$, which is also the isometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$. Choosing a vacuum state on the level of the corresponding spin chain this symmetry gets broken to $\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \ltimes \mathfrak{u}(1)$. In principle, one should expect the S-matrix to be invariant under this residual symmetry algebra.

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However, it turns out that this algebra is too restrictive as one of the $\mathfrak{u}(1)$ generators is identified with the spin chains energy, and is fixed to $\frac{1}{2}$ on the fundamental representation, in opposition to the required continuous values. It turns out that the correct S-matrix is invariant under two copies of the universal central extension of $\mathfrak{psu}(2|2)$, which is $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [8]. Due to this product structure, we will henceforth only talk about the S-matrix invariant under one copy of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$, whereas the full S-matrix is a tensor product of two such $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ S-matrices. It turns out that this central extension is crucial in several aspects: If one links those extra central charges correctly with the momentum as well as some additional braiding element [9, 10] it allows the energy to be continuous on the fundamental representation and it fixes the S-matrix uniquely up to a scalar prefactor. This is quite unusual as usually S-matrices in integrable systems are fixed not only by Lie algebra symmetries but by higher symmetries such as quantum affine algebras or Yangians. The reason for this is the unusual tensor product behaviour of this algebra, the tensor product of two fundamental representations is generically irreducible [11]. The triple tensor product of fundamentals also has a comparatively simple structure, which can be used to show that the crucial Yang–Baxter equation holds.

Interestingly, even though no higher symmetry was needed for the derivation of this S-matrix, it was shown [12] that it is additionally invariant under a Yangian. This is expected as there is lots of indication that the full symmetry algebra $\mathfrak{psu}(2,2|4)$ is actually enhanced to the Yangian of $\mathfrak{psu}(2,2|4)$ [13–21]. Furthermore, for the derivation of bound state S-matrices the existence of the Yangian symmetry is actually crucial [22].

A remaining question is the construction of the universal R-matrix which should lead automatically to all S-matrices on all representations. From the underlying symmetry one should expect that it is the universal R-matrix of the Yangian of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. However, because of the degeneracy of the Killing form due to the existence of additional central charges, this universal R-matrix cannot exist. On the classical level it was shown [23] that the two extra central charges and the braiding can be removed as independent degrees of freedom and incorporated at different degrees of the remaining central charge in the loop algebra of $\mathfrak{su}(2|2)$. A universal classical R-matrix then exists if one adds an outer automorphism and all its higher degrees to this algebra, making it the loop algebra $\mathfrak{u}(2|2)[u, u^{-1}]$ which has slightly deformed commutation relations as a remaining effect of the additional central charges which now do not appear explicitly. This classical R-matrix is valid at strong coupling. The corresponding form for weak coupling looks the same but the $\mathfrak{u}(2|2)[u, u^{-1}]$ commutation relations remain undeformed. This makes it possible to find the so called mathematical quantisation, which is the double Yangian $\mathcal{DY}(\mathfrak{u}(2|2))$, and its universal R-matrix has been found in [24]. However, this does not yet lead to the coupling dependent all-loop S-matrix incorporating the effects of the additional central charges, and finding its corresponding universal R-matrix is an outstanding problem. In [25] the second realization of the Yangian needed for the construction of the universal R-matrix was studied. What remains to be done is to remove the gauge central charges in a similar way as on the classical level.

We begin by discussing the $\mathfrak{su}(2|2)$ symmetric spin chain and its particular features involving length changing effects and the appearance of the central extension. In Section 3 we show how to derive the fundamental S-matrix, and finally we discuss its Yangian symmetry in Section 4.

2. The $\mathfrak{su}(2|2)$ symmetric spin chain

We start with a $\mathfrak{psu}(2,2|4)$ symmetric spin chain, where the spin degrees of freedom are equivalent to the fundamental operators of $\mathcal{N}=4$ SYM theory inserted into a single trace local operator. As discussed in [1, 3] the anomalous dimension of such operator corresponds to the energy of a spin chain with this symmetry. One finds this energy with the help of the Bethe Ansatz. This works by choosing a vacuum state, which is a protected BPS operator $\mathcal{Z}^J$, breaking the $\mathfrak{psu}(2,2|4)$ symmetry to $\mathfrak{u}(1) \ltimes \mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2) \rtimes \mathfrak{u}(1)$. The remaining eight bosonic and eight fermionic operators transforming under this residual symmetry are now interpreted as magnons in the spin chain language, they look like

$$|\mathcal{X}\rangle = \sum_n e^{ipn} |\ldots \mathcal{Z}\mathcal{X}\mathcal{Z}\ldots\rangle\,. \tag{1}$$

Let us from now on consider magnons which transform in the fundamental representation of $\mathfrak{su}(2|2)$, *i.e.* the magnons span a four dimensional vector space consisting of two bosons $|\phi^a\rangle$ and two fermions $|\psi^\alpha\rangle$. The 16 original magnons of the full spin chain are then given by the product of two $\mathfrak{su}(2|2)$ magnons of the two $\mathfrak{su}(2|2)$'s in the residual symmetry algebra.

2.1. Central extension and Hopf algebra

We shall now come to two important related features of the $\mathfrak{su}(2|2)$ spin chain considered here. The first one is that the $\mathfrak{u}(1)$ charge $\mathfrak{C}$ of $\mathfrak{su}(2|2)$ is to be considered as the energy eigenvalue of the spin chain. However, representation theory forces this value to be fixed to $\pm\frac{1}{2}$, which is not compatible with the requirement of having a continuously varying energy. This apparent puzzle can be resolved by extending the algebra to its universal central extension $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [8], where, if we call the additional central charges $\mathfrak{P}$ and $\mathfrak{K}$, only the combination

$$\mathfrak{C}^2 - \mathfrak{K}\mathfrak{P} = \tfrac{1}{4} \tag{2}$$

invariant under outer automorphism is fixed on the fundamental representation to 1/4.

Another important feature of this chain is that, as was shown in [26], the length of this spin chain actually fluctuates. This is manifested in the action of the supersymmetry generators on the magnons, which can insert or remove vacuum fields $\mathcal{Z}$ in the spin chain. These additional or missing vacuum fields have no effects on single magnons but only on multi-magnon states. As was shown in [9, 10] this results in a modified Hopf algebra structure. If we call the eight supersymmetry generators $\mathfrak{Q}^\alpha_a$, $\mathfrak{S}^a_\alpha$, $a, \alpha = 1, 2$ then one gets cocommutation relations or coproducts

$$\begin{aligned} \Delta\mathfrak{Q}^\alpha_a &= \mathfrak{Q}^\alpha_a \otimes 1 + \mathfrak{U} \otimes \mathfrak{Q}^\alpha_a \,, \\ \Delta\mathfrak{S}^a_\alpha &= \mathfrak{S}^a_\alpha \otimes 1 + \mathfrak{U}^{-1} \otimes \mathfrak{S}^a_\alpha \,, \end{aligned} \tag{3}$$

where $\mathfrak{U}$ is a new central operator taking the eigenvalue $e^{ip/2}$ on the fundamental representation. The essence of the coproduct is that it tells us how to act on tensor products. We see that unlike what we know from ordinary Lie algebras, in our case the generators do not act on each tensor product factor in the same way. This is actually crucial for the derivation of the fundamental S-matrix. We finally add that the bosonic generators of the two $\mathfrak{su}(2)$'s of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ as well as the central charge $\mathfrak{C}$ have the standard action on tensor products, whereas the relations for the additional central elements $\mathfrak{P}, \mathfrak{K}$ follow from the fact that they arise as commutators

$$\mathfrak{P} = \{\mathfrak{Q}^1_1, \mathfrak{Q}^2_2\}\,, \qquad \mathfrak{K} = \{\mathfrak{S}^1_1, \mathfrak{S}^2_2\}\,, \tag{4}$$

to be

$$\begin{aligned} \Delta\mathfrak{P} &= \mathfrak{P} \otimes 1 + \mathfrak{U}^2 \otimes \mathfrak{P}\,, \\ \Delta\mathfrak{K} &= \mathfrak{K} \otimes 1 + \mathfrak{U}^{-2} \otimes \mathfrak{K}\,. \end{aligned} \tag{5}$$

3. The fundamental S-matrix

3.1. Derivation of the S-matrix

The S-matrix we are talking about is the S-matrix scattering two magnons without particle production. As a symmetry operation should not alter the outcome of a physical process the S-matrix should commute with the action of the symmetry generators on the tensor product of two fundamental representations, which is, as discussed before, defined by the coproduct, *i.e.* we should have

$$[\Delta\mathfrak{J}, \mathcal{S}] = 0\,, \qquad \mathfrak{J} \in \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3\,. \tag{6}$$

Note that as the S-matrix changes the two factors of the tensor product the individual tensor product factors in the coproduct get interchanged if one scatters first and then performs a symmetry transformation. One first observation is the following: as the action of the additional central charges (5) is not symmetric, and all the generators $\mathfrak{P}, \mathfrak{K}, \mathfrak{U}$ involved are central and drop out of the commutation relation with the S-matrix, the coproduct of $\mathfrak{P}, \mathfrak{K}$ needs to be symmetric which is only possible if one identifies

$$\mathfrak{P} = g\alpha(1-\mathfrak{U}^2)\,, \qquad \mathfrak{P} = \frac{g}{\alpha}(1-\mathfrak{U}^{-2})\,, \tag{7}$$

with some *a priori* unidentified constants of proportionality. However, note that on the fundamental representation, where we had $\mathfrak{U} = e^{ip/2}$, this automatically gives the correct dispersion relation from the constraint between the central elements (2),

$$2\mathfrak{C} = \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)}\,, \tag{8}$$

if one chooses the right proportionality factor as above.

One is now in a position to solve (6) for the S-matrix, which is surprisingly fully fixed by the $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ symmetry up to a prefactor. We get the result as spelled out in [11] upon identifying $\xi = e^{ip/2}$ in the S-matrix. Importantly, this S-matrix satisfies the Yang–Baxter equation [11] and leads to the Bethe equations describing the spectrum of long operators in $\mathcal{N} = 4$ SYM theory previously conjectured in [4]. In this paper, however, we want to focus on its additional symmetries and algebraic properties.

4. Yangian symmetry

Despite being fixed by the centrally extended $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ Lie algebra one might ask if there are additional symmetries of the S-matrix. One reason to believe that these symmetries might exist is that integrable systems usually have infinitely many symmetries, and on the level of the full symmetry algebra $\mathfrak{psu}(2,2|4)$, some Yangian enhancement was found. Indeed, in [12] it was shown that the S-matrix is invariant under the Yangian of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. The Yangian is defined as follows: Let $\mathfrak{g}$ be any finite dimensional Lie algebra with non-degenerate Killing form κ^{AB} generated by $\mathfrak{J}^A$ satisfying commutation relations $[\mathfrak{J}^A, \mathfrak{J}^B] = f_C^{AB}\mathfrak{J}^C$. The Yangian $\mathcal{Y}(g)$ is a deformation of the algebra $\mathfrak{g}[u]$ of polynomials with values in $\mathfrak{g}$ defined by the following commutation relations between the level zero generators $\mathfrak{J}^a$ forming $\mathfrak{g}$ and the level one generators $\widehat{\mathfrak{J}}^a$:

$$[\mathfrak{J}^A, \mathfrak{J}^B] \;=\; f_C^{AB}\mathfrak{J}^C\,, \tag{9}$$

$$[\mathfrak{J}^A, \widehat{\mathfrak{J}}^B] \;=\; f_C^{AB}\widehat{\mathfrak{J}}^C\,. \tag{10}$$

The generators of higher levels are derived by demanding compatibility with the Serre relation (for algebras other than $\mathfrak{su}(2)$)

$$[\widehat{\mathfrak{J}}^A,[\widehat{\mathfrak{J}}^B,\mathfrak{J}^C]]+[\widehat{\mathfrak{J}}^B,[\widehat{\mathfrak{J}}^C,\mathfrak{J}^A]]+[\widehat{\mathfrak{J}}^C,[\widehat{\mathfrak{J}}^A,\mathfrak{J}^B]] \\ =\frac{1}{4}f^{AG}_D f^{BH}_E f^{CK}_F f_{GHK}\mathfrak{J}^{\{D}\mathfrak{J}^E\mathfrak{J}^{F\}}\,. \tag{11}$$

Importantly, the Yangian is a Hopf algebra. The Lie generators have undeformed cocommutation relations

$$\Delta\mathfrak{J}^A=\mathfrak{J}^A\otimes 1+1\otimes\mathfrak{J}^A\,, \tag{12}$$

whereas the new generators $\widehat{\mathfrak{J}}^A$ have nontrivial coproduct

$$\Delta\widehat{\mathfrak{J}}^A=\widehat{\mathfrak{J}}^A\otimes 1+1\otimes\widehat{\mathfrak{J}}^A+\frac{1}{2}f^A_{BC}\mathfrak{J}^B\otimes\mathfrak{J}^C\,. \tag{13}$$

If $\mathfrak{g}$ is of type $\mathfrak{u}(n|m)$ and the Yangian generators are represented as $\widehat{\mathfrak{J}}^A = iu\mathfrak{J}^A$ on the fundamental evaluation representation, this leads to standard rational S-matrices $\mathcal{S}=u/(u+i)-i/(u+i)\mathcal{P}$, where $\mathcal{P}$ is the graded permutation operator.

We now face the problem that $\mathfrak{psl}(2|2)\ltimes\mathbb{C}^3$ has no non-degenerate Killing form. Luckily, we can overcome this degeneracy by adjoining the three automorphisms which do not appear on the right-hand side of commutation relations. They also do not appear in the coproduct of the Yangian generators of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ which is necessary as the automorphisms have no fundamental representation and hence the Yangian could not be a symmetry of the fundamental S-matrix. Hence one can use the automorphisms to raise and lower indices on structure constants in (13) without them appearing explicitly. Additionally, in the case of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ we have to twist the coproduct now not only of the Lie generators but also of the Yangian generators, as done in Section 2.1. Now a basic requirement for being a symmetry of the S-matrix is that all central elements are cocommutative, *i.e.* symmetric on the tensor products. This actually links the spectral parameter u appearing in $\widehat{\mathfrak{J}}^A=iu\mathfrak{J}^A$ to the eigenvalues of the central charges and hence to the momentum, as required from its physical interpretation as a rapidity. The final outcome is that the S-matrix is indeed invariant under the Yangian of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$.

4.1. Universal R matrices

It would be desirable to have a universal, representation independent form of the S- or related R-matrix $\mathcal{S}=\mathcal{P}\mathcal{R}$, as this should yield all S-matrices existing on representations. It should be related to the Yangian of $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$.

However, the degeneracy of the Killing form makes the finding of the universal R-matrix a difficult task. If one takes the classical limit $g \to 0$ one remains with a $\mathfrak{u}(2|2)$ symmetry which has non-degenerate Killing form and a universal R-matrix can be constructed [24]. In the strong coupling limit $\frac{1}{g} \to 0$ under appropriate scaling of the momentum as $p \propto \frac{1}{g}$ one finds that all the three central elements of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ reduce effectively to one,

$$\mathfrak{P} = -2i\alpha\frac{\mathfrak{C}}{u}\,, \qquad \mathfrak{K} = \frac{2i}{\alpha}\frac{\mathfrak{C}}{u}\,, \tag{14}$$

and the braiding becomes 1. Hence, when working with the loop algebra of $\mathfrak{su}(2|2)$ the generators $\mathfrak{P}, \mathfrak{K}$ are identified with

$$\mathfrak{P} = -2\alpha\mathfrak{C}_{-1}\,, \qquad \mathfrak{K} = \frac{2}{\alpha}\mathfrak{C}_{-1}\,. \tag{15}$$

Adjoining an outer automorphism one completes the loop algebra of $\mathfrak{su}(2|2)$ to the loop algebra $\mathfrak{u}(2|2)[u, u^{-1}]$. This has a non-degenerate Killing form and a simple classical universal R-matrix

$$r = \frac{\mathcal{T}}{iu_1 - iu_2}\,, \tag{16}$$

where $\mathcal{T}$ is the quadratic Casimir of $\mathfrak{u}(2|2)$. If one incorporates a proper twist as a residual effect coming from the braiding this classical R-matrix reduces to the strong coupling limit of the S-matrix $\mathcal{PS} = 1 + (1/g)r + \ldots$.

5. Outlook

We have discussed the symmetries of the S-matrix of AdS/CFT, which involve standard Yangian structures as well as unusual Hopf algebra symmetries and a central extension unique to the underlying Lie algebra $\mathfrak{psu}(2|2)$. Even though these symmetries seem now to be complete there are several remaining question. It would be desirable to construct a universal R-matrix as done for $\mathfrak{u}(2|2)$ in [24] now for all values of g in order to have a representation independent form which should also be automatically crossing invariant. Furthermore, one should study the algebra behind generalisations of this S-matrix, in particular, its q deformed version considered in [27].

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