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Letter

Chirality of Dirac Spinors Revisited

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Received: 16 March 2020; Accepted: 7 April 2020; Published: 14 April 2020

Abstract: We emphasize the differences between the chirality concept applied to relativistic fermions and the ususal chirality concept in Euclidean spaces. We introduce the gamma groups and we use them to classify as direct or indirect the symmetry operators encountered in the context of Dirac algebra. Then we show how a recent general mathematical definition of chirality unifies the chirality concepts and resolve conflicting conclusions about symmetry operators, and particularly about the so-called chirality operator. The proofs are based on group theory rather than on Clifford algebras. The results are independent on the representations of Dirac gamma matrices, and stand for higher dimensional ones.

Keywords: chirality; symmetry; special relativity; quantum fields; Dirac spinors; gamma groups; gamma matrices

PACS: 11.30.Rd; 03.30.+p

MSC: 15A66; 20C35; 83A05

1. Introduction

When the term *chirality* was introduced by Lord Kelvin [1,2], visibly, its definition targeted the case of the Euclidean space rather than the one of the spacetime, even in its classical version. Quoting Lord Kelvin: *I call any geometrical figure, or group of points, chiral, and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself*.This definition, which is still in use today, has a clear merit: it can be easily understood by a vast majority of people. However, it is not suitable for mathematical purposes, since the terms *figure*, *ideally realized*, *brought* and *coincide* have only an intuitive meaning.

Later, in order to calculate the wave function of the relativistic electron, Dirac [3] introduced the matrices γ_μ, $\mu = 1,2,3,4$ (each γ matrix has four lines an four columns), in the expression of what is called Dirac equation. In this equation, the unknown is no more a single complex function: it has four complex components, and solving the equation gave four wave functions, two for the electron, with respective spins +1/2 and $-1/2$, and two for the positron, with respective spins +1/2 and $-1/2$. The γ matrices act on the space of Dirac spinors (also called bi-spinors), these latter having four components. This space must not be confused with the Minkowski spacetime (i.e., the space of special relativity), in which the elements, called 4-vectors, have also four components: three for the space, one for the time.

The set of the four gamma matrices was completed by a fifth one by Eddington [4], but in fact the author redirected to an older paper from him [5]. In the literature this fifth matrix was later denoted γ_5, while the four gamma matrices were renumbered γ_μ, $\mu = 0,1,2,3$. Then, Eddington [6] used the word *chirality* in reference to Kelvin's term, but without a clear definition. An unambiguous use of the term was done by Watanabe [7], who cited both Kelvin and the edition of 1949 of Eddington's book.

Watanabe considered γ_5 as a *chirality operator*, and generalized it so that it can be applied not only to fermions, but also to bosons. This meaning of γ_5 seems to have been retained by many authors in the relativistic quantum mechanics literature. However, it was considered that *chirality* is a bad name in the context of spinors [8]. We show that the properties of γ_5 are coherent with the definition of chirality of Petitjean [9,10], which recovers the one of Lord Kelvin, and which is based on an unifying definition of symmetry [11].

2. Motivation of the Work

This section is intended to summarize the goal of the current work for readers which are not experts of symmetry in quantum field theory. Chirality is an ubiquitous concept. Its definition, which goes back to 1894, targeted the Euclidean space, and it remains in use today. After the introduction of chirality in quantum field theory, physical requirements lead to retain the spinorial metric γ^0 for the Dirac field. It led to conclusions about chirality which can be considered to be conflicting. Such conflicts have no physical impact, they are only terminological. The resulting ambiguous use of the chirality concept was propagated in the literature, and it is still part of several advanced physics courses. We exemplify it in the case of Dirac fermions, and we show how a recent general mathematical definition of chirality resolve the ambiguities.

Strictly speaking, we do not solve here a physics problem. Rather, we point out a confusion about symmetry (and thus about chirality), which appeared in the physics literature. Let us look at the following example in the real plane. The rotation of $\pi/4$ around the origin is an isometry for the standard metric. Now, we consider the metric $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.The rotation of $\pi/4$ around the origin is no more an isometry. To see this, consider the distance to the origin from the point $(1,0)$. Before rotation, this distance is 1. After rotation, it is 1 for the standard metric, but it is $\sqrt{3/2}$ for the metric defined above. It means that the set of symmetry operators depends on the metric. And this is exactly the problem we outline further with several operators built as products of the gamma matrices. On one side the metric γ^0 is needed for physical reasons (see Section 6), and on an other side, the distance induced by γ^0 is not preserved for all operators generated by the gamma matrices, whichever concrete representation is used for these gamma matrices. And it is the case for the chiral operator γ^5, which can be even considered to be not a symmetry operator for the metric γ^0. Thus we evidence (i) an abuse of language about symmetry (and thus about chirality), and (ii) we attract the attention of the readers about the risk of erroneous claims about symmetries. Our results hereafter lead to a by-product: we show that the main properties of the gamma matrices can be derived only from their group properties. In other words, we did not need the artillery of Clifford algebras: thus, fewer assumptions means stronger results. Our results apply to higher dimensional gamma matrices, and whichever concrete representation they have. The next section contains the proofs of several results issued from the group structure attached to gamma matrices.

3. Gamma Groups

We consider an abstract set $\Gamma_{p,q}$, endowed with a binary operation, i.e., a function: $\Gamma_{p,q} \times \Gamma_{p,q} \mapsto \Gamma_{p,q}$, where p and q are two non-negative integers. We call this operation *multiplication*, and since there is no ambiguity we do not need to denote it by a symbol. The generators of $\Gamma_{p,q}$ are an element denoted $\mathbf{i}$ and the $n = p + q$ elements of a set $\Gamma^*_{p,q}$. These n elements are denoted γ^μ, where μ is an index taking values in a finite set $\mathcal{N}$ of n integers. Thus $\Gamma_{p,q}$ has $n+1$ generators. We assume that they are distinct. The rules defining the multiplication table of $\Gamma_{p,q}$ are as follows.

(R1) The multiplication is associative.
(R2) $\mathbf{i}$ commutes with all elements of $\Gamma_{p,q}$.
(R3) The neutral element of $\Gamma_{p,q}$ is $\mathbf{i}^4$ and we denote it also $\mathbf{1}$.
Conventionally, any element of $\Gamma_{p,q}$ elevated at power 0 is equal to $\mathbf{1}$.

(R4) $\forall \mu \in \mathcal{N}$, either $(\gamma^\mu)^2 = \mathbf{1}$ or $(\gamma^\mu)^2 = \mathbf{i}^2$.
According to increasing values of $\mu \in \mathcal{N}$, the p first of these squares take the value $\mathbf{1}$, $0 \le p \le n$, and the q remaining ones take the value $\mathbf{i}^2$.
(R5) $\forall \mu \in \mathcal{N}, \forall \nu \in \mathcal{N}, \nu \neq \mu, \gamma^\mu \gamma^\nu = \mathbf{i}^2 \gamma^\nu \gamma^\mu$.
We call this rule *anticommutation*.

Definition 1. *$\Gamma_{p,q}$ is called a gamma group.*

The group structure of $\Gamma_{p,q}$ is proved in Theorem 2.

We define the following subset Γ_h of $\Gamma_{p,q}$: $\Gamma_h = \{\mathbf{i}, \mathbf{i}^2, \mathbf{i}^3, \mathbf{i}^4\}$.

Definition 2. *Let x be an element of $\Gamma_{p,q}$, written as follows. The expression of x can contain a heading element taken in Γ_h. If it is $\mathbf{1}$, it must be alone. If it is not $\mathbf{1}$, either the heading term is lacking and a trailing term must exist, or the heading term is a single element of Γ_h, followed or not by a trailing term. This trailing term is either a single element of $\Gamma^*_{p,q}$ or a product of elements of $\Gamma^*_{p,q}$ sorted in strictly increasing order of their indices in $\mathcal{N}$. We call **canonical** such an expression of x.*

*When there are at least two elements in the trailing term, and they are sorted in strictly decreasing order of their indices in $\mathcal{N}$, we call the expression of x **anticanonical**.*

Lemma 1. *Any element of $\Gamma_{p,q}$ can be expressed in canonical form.*

Proof. All generators of $\Gamma_{p,q}$ commute or anticommute. Consider an element x of $\Gamma_{p,q}$ expressed as a product of powers of elements of $\Gamma_{p,q}$. Using rule (R2) for this product, we can shift together in a heading term all powers of $\mathbf{i}$, the trailing term being a product of powers of elements of $\Gamma^*_{p,q}$. Using the anticommutation rule, we reorder the elements of the trailing term according to non-decreasing values of their indices in $\mathcal{N}$. From rule (R4), the even powers of elements of $\Gamma^*_{p,q}$ reduce to powers of $\mathbf{i}^2$, which we can shift in the heading term, and the odd powers of elements of $\Gamma^*_{p,q}$ reduce to the elements of $\Gamma^*_{p,q}$ themselves, ordered in increasing values of their indices in $\mathcal{N}$, while the elements $\mathbf{i}^2$ appeared due to reordering are shifted from the trailing term to the heading term. The heading term reduces to one element of Γ_h. It is either alone or followed by a product of elements of $\Gamma^*_{p,q}$, ordered in increasing values of their indices in $\mathcal{N}$. When the heading term is $\mathbf{1}$ it can be removed unless the ordered product of elements of $\Gamma^*_{p,q}$ is void. □

Theorem 1. *Let x be the product of k distinct elements of $\Gamma^*_{p,q}$, $k \ge 1$, the product being not necessarily sorted in canonical order. The product and its expression in reversed order differ from a factor $(\mathbf{i}^2)^{(k(k-1))/2}$.*

Proof. The reversed expression of x can be generated through $(k-1)+(k-2)+...+2+1$ anticommutations of the k elements of $\Gamma^*_{p,q}$. Each of these $(k(k-1))/2$ anticommutations generates one factor $\mathbf{i}^2$. □

Corollary 1. *Let x be any element of $\Gamma_{p,q}$, containing k elements of $\Gamma^*_{p,q}$ in its canonical expression, $k \ge 0$. The anticanonical expression of x and its canonical one differ from a factor $(\mathbf{i}^2)^{(k(k-1))/2}$.*

Proof. The corollary stands when $k = 0$. When $k > 0$, commute the headers and apply Theorem 1. □

Lemma 2. *Any element $x \in \Gamma_{p,q}$ satisfies to $x^4 = \mathbf{1}$. The elements x and x^3 are inverses.*

Proof. We put x in canonical form. When existing, we call y the product of the elements of $\Gamma^*_{p,q}$ in the canonical expression of x. We try to express x^4 in canonical form. When existing, the heading term in x appears four times in x^4, thus its fourth power is $\mathbf{1}$. We put it as a heading term of x^4. The trailing term of x^4 is y^4. Let k be the number of elements of $\Gamma^*_{p,q}$ in the expression of y, $0 \leq k \leq n$. If $k = 0$ the Lemma is proved. If $k = 1$, from rule (R4) the fourth power of an element of $\Gamma^*_{p,q}$ is $\mathbf{1}$, then the Lemma is proved. If $k > 1$, we decompose $y^4 = y^2 y^2$. In each square y^2, the term y on the left is reordered in decreasing values of the indices of its elements of $\Gamma^*_{p,q}$. From the anticommutation rule, $\mathbf{i}^2$ appears $k-1$ times in each of the two terms y^2, so that $(\mathbf{i}^2)^{2k-2} = \mathbf{1}$, and the neutral element can be put in the heading term of x^4. Now, we write each of the two squares y^2 as a product of elements of $\Gamma^*_{p,q}$ in increasing values of their indices followed by the same product in the reversed order of their indices, together with terms $\mathbf{i}^2$ due to the anticommutations. Applying k times rule (R4) in y^2, these two products taken together reduce to a power of $\mathbf{i}^2$. But there are two squares y^2 in y^4, and the square of a power of $\mathbf{i}^2$ is again the neutral element, thus completing the proof. □

Theorem 2. *$\Gamma_{p,q}$ is a finite group. It contains at most 2^{n+2} elements.*

Proof. From Lemma 1, any element of $\Gamma_{p,q}$ can be written as a product of at most $n+1$ terms, so $\Gamma_{p,q}$is a finite set. In canonical form, either the heading term is $\mathbf{1}$ alone, or it is one of the other three elements of Γ_h, followed by 2^n possible products of elements of $\Gamma^*_{p,q}$, or the heading term is lacking in which case there are $2^n - 1$ possible non empty products of elements of $\Gamma^*_{p,q}$. Total of possible canonical expressions: $1 + 3(2^n) + 2^n - 1 = 2^{n+2}$. From Lemma 2, any element x of $\Gamma_{p,q}$ has indeed an inverse, which is x^3. It completes the proof that $\Gamma_{p,q}$ is a finite group. □

Corollary 2. *The cyclic group Γ_h is a normal subgroup of $\Gamma_{p,q}$.*

Proof. From rule (R3), Γ_h is obviously a cyclic group, and it is isomorphic to $(\mathbb{Z}_4, +)$. Let h be an element of Γ_h, g an element of $\Gamma_{p,q}$, and consider the product $y = ghg^{-1}$. h commutes with all elements of $\Gamma_{p,q}$, thus $y = h$ and y is in Γ_h. □

Theorem 3. *Except the neutral element and $\mathbf{i}^2$, no element of $\Gamma_{p,q}$, written in its canonical expression, can be expressed as a square of an other element of $\Gamma_{p,q}$, written in its canonical expression or not.*

Proof. Consider an element x of $\Gamma_{p,q}$ containing in its canonical expression at least one element of $\Gamma^*_{p,q}$, and written as the square of some element y of $\Gamma_{p,q}$, i.e., $x = y^2$. We try to express y^2 in canonical form. Using rule (R2) we can shift together in a heading term all powers of the elements of Γ_h. Using the anticommutation rule, we can reorder in the trailing term the elements of $\Gamma^*_{p,q}$; thus each of these latter occurs twice or an even number of times in y^2. In the trailing term it remains only powers of $\mathbf{i}^2$ (due to reordering), and powers of $\mathbf{i}^2$ or of $\mathbf{i}^4$, due to rule (R4). We shift all powers of $\mathbf{i}$ in the heading term, which reduces to one element of Γ_h, and there is no more element of $\Gamma^*_{p,q}$ in the canonical expression of y^2, a contradiction. The result still works in the case where x is reduced to its heading element. □

Theorem 4. *Let x and y be two elements of $\Gamma_{p,q}$. Their canonical expressions contain respectively k_x and k_y elements of $\Gamma^*_{p,q}$, and they have k of them in common. The products xy and yx are such that $yx = (\mathbf{i}^2)^{k_x k_y - k} xy$.*

Proof. The headers of xy and yx always commute. Then, observe that commuting successively each element of $\Gamma^*_{p,q}$ in the trailing term of either xy or of yx, needs $k_x k_y$ anticommutations, minus k of them which are common to xy and yx. □

Definition 3. *We define $\omega_{p,q}$ as the non empty product of all n distinct elements of $\Gamma^*_{p,q}$, the product being sorted in increasing values of the indices in $\mathcal{N}$ of these n elements.*

Corollary 3. *Let y be any given element of $\Gamma^*_{p,q}$. The product $\omega_{p,q} y$ commutes with $y\omega_{p,q}$ when n is odd, and $\omega_{p,q} y$ anticommutes with $y\omega_{p,q}$ when n is even.*

Proof. From Theorem 4, $y\omega_{p,q} = (\mathbf{i}^2)^{n-1}\omega_{p,q}y$. □

Theorem 5. *Let $z \in \Gamma_{p,q}$ having m factors in its canonical expression, r of them squaring to $\mathbf{i}^2$, $0 \le r \le m$. $z^2 = (\mathbf{i}^2)^{r+(m(m-1))/2}$.*

Proof. We assume first that z is canonically expressed as the product of m elements of $\Gamma^*_{p,q}$, $0 \le r \le m$, without a heading term. We apply Theorem 1 to write z^2 with the product of its expression by its reversed one, then we replace each square of the m elements of $\Gamma^*_{p,q}$ by its value defined in rule (R4). $z^2 = (\mathbf{i}^2)^{(m(m-1))/2}(\mathbf{i}^2)^r$.

Assuming now that the header of the canonical expression of z is not empty, it can be checked that the result stands again, and it still works in the case where z is reduced to its header. □

Corollary 4. $\omega^2_{p,q} = (\mathbf{i}^2)^{q+(n(n-1))/2}$.

Proof. Set $m = n$ and $r = q$ in Theorem 5. □

4. Comments on Gamma Groups

We did not enumerate the words of $\Gamma_{p,q}$ because additional assumptions are needed to define its full multiplication table. However these additional assumptions are not required to establish the results of Section 5. We show further that these results make sense for Dirac algebra. We intentionnally denoted the elements of $\Gamma^*_{p,q}$ similarly to the contravariant notation of gamma matrices as used by most quantum physicists, so that it helps the reader to understand why gamma groups are relevant. The results of Section 3 are based only on the group properties of $\Gamma_{p,q}$. We did not define an addition operator, nor its neutral element *zero*, nor the minus sign, which usually denotes an inverse in the group defined by the addition. We did not define a set of scalars nor the product of an element of $\Gamma_{p,q}$ by a scalar. However, we outline that the anticommutation is a cornerstone both for gamma groups and Clifford algebras.

While it seems unusual to build a group including the powers of $\mathbf{i}$ together with the gamma matrices, a similar approach exists in the case of the Pauli group [12]. This latter is generated by a set of three 2×2 matrices $\mathcal{P} = \{\sigma_1, \sigma_2, \sigma_3\}$, which were introduced by Pauli [13]. Denoting by i the pure imaginary complex number of square -1, the Pauli matrices are:

$$\mathcal{P} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \quad (1)$$

We define $\mathbf{i}_2 = \sigma_1\sigma_2\sigma_3$ and $\mathbf{1}_2 = \mathbf{i}_2^4$, which is the 2×2 identity matrix. Then we observe that $\{\mathbf{i}_2\} \cup \mathcal{P}$ generates a group $\Gamma_{3,0}$, whereas the Pauli group has only 3 generators. The Pauli group is unambigously defined. However, whether or not it is a representation of $\Gamma_{3,0}$ cannot be decided without more assumptions about $\Gamma_{3,0}$.

As shown in Section 6, Theorem 3 is the key one to see whether or not the concept of chirality introduced for Dirac fermions is coherent with the more general definition of chirality in [10]. This latter is based on the direct vs. indirect classification of isometries, i.e., direct isometries can be written as a product of squared isometries, while indirect isometries cannot. This classification was introduced in classical mechanics [14] and in special relativity [15], but it was unknown in the literature on relativistic quantum physics. We show in Section 6 that it can be done with the help of Theorem 3 for several operators of interest, including P (parity inversion) and T (time reversal). Before that we need to exhibit more properties of gamma matrices.

5. Gamma Matrices and Dirac Spinors

Dirac spinors are defined in the context of special relativity, the 4×4 metric tensor of the Minkowski spacetime is diagonal, and there are $n = 4$ gamma matrices: one associated to the time component and three associated to the spatial components. According to [16], the Dirac matrices are a set of 4×4 irreducible matrices of complex elements, which satisfy to anticommutation rules (the size 4×4 of the gamma matrices is related to algebraic considerations, not to the dimensionality of the spacetime). The squares of the four gamma matrices are equal either to the identity matrix or to its opposite, depending on the diagonal elements of the metric tensor. We retain (+,−,−,−) for the signature of the metric tensor, which corresponds to $p = 1$, because it is an usual choice in special relativity. We set $\mathcal{N} = \{0, 1, 2, 3\}$, as it is usual in most textbooks dealing with gamma matrices. The set of the four gamma matrices is $\widehat{\Gamma}^*_{1,3} = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$. We denote by $\mathbf{1}$ the 4×4 identity matrix and we define the matrix $\mathbf{i} = i\mathbf{1}$. Thus, $(\gamma^0)^2 = \mathbf{1}$, $(\gamma^1) = \mathbf{i}^2$, $(\gamma^2)^2 = \mathbf{i}^2$, and $(\gamma^3)^2 = \mathbf{i}^2$.

Theorem 6. *The set $\widehat{\Gamma}_{1,3} = \{\mathbf{i}\} \cup \widehat{\Gamma}^*_{1,3}$ generates a gamma group $\Gamma_{1,3}$ for the matrix multiplication.*

Proof. Consider the gamma matrices and write their squares and their anticommutations properties, respectively as in rule (R4) and (R5) in Section 3. The rest of the proof is obvious. □

Theorem 6 does not specify which basis is used to write the concrete expressions of the elements of $\widehat{\Gamma}_{1,3}$. An instance of this group, containing 64 elements (see Theorem 2), was mentioned by Salingaros [17] in the case of the *E-symbols* of Eddington [4].

There are three usual basis of gamma matrices mentioned in the literature: Dirac, Weyl, and Majorana (original papers: see [3,18–20]; see [21] for a summary). Their respective expressions are explicited in function of Pauli matrices in Equations (2)–(4). These sets of gamma matrices are frequently used in the literature (e.g., see [22–30]).

$$[\widehat{\Gamma}^*_{1,3}]_D = \left\{ \begin{bmatrix} \mathbf{1}_2 & 0 \\ & \mathbf{i}_2^2 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_1 \\ \mathbf{i}_2^2\sigma_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_2 \\ \mathbf{i}_2^2\sigma_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_3 \\ \mathbf{i}_2^2\sigma_3 & 0 \end{bmatrix} \right\} \tag{2}$$

$$[\widehat{\Gamma}^*_{1,3}]_W = \left\{ \begin{bmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_1 \\ \mathbf{i}_2^2\sigma_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_2 \\ \mathbf{i}_2^2\sigma_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_3 \\ \mathbf{i}_2^2\sigma_3 & 0 \end{bmatrix} \right\} \tag{3}$$

$$[\widehat{\Gamma}^*_{1,3}]_M = \left\{ \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{i}_2\sigma_3 & 0 \\ 0 & \mathbf{i}_2\sigma_3 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i}_2^2\sigma_2 \\ \sigma_2 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{i}_2^3\sigma_1 & 0 \\ 0 & \mathbf{i}_2^3\sigma_1 \end{bmatrix} \right\} \tag{4}$$

Given any set of gamma matrices, it is always possible to build other ones: each set is related to another by a similarity transformation, i.e., s being an arbitrary invertible matrix, each gamma matrix γ^μ becomes $s\gamma^\mu s^{-1}$, $\mu \in \mathcal{N}$ [22]. Given two irreducible sets of gamma matrices, this similarity matrix is unique, except for an arbitrary multiplicative factor [31]. Examples of s matrices are any of the gamma matrices (it generates a change of the sign of the other gamma matrices), or permutation matrices, etc. When existing, the Hermiticity property is not ensured after this transormation, except when S is unitary [21].

Hermiticity and unitary properties are related through Theorem 7. We denote by a dagger the conjugate transpose.

Theorem 7. *The following four propositions are equivalent:*
(a) The gamma matrices are either Hermitian or anti-Hermitian.
(b) The gamma matrices are either unitary or anti-unitary.
(c) All elements of $\widehat{\Gamma}_{1,3}$ are either Hermitian or anti-Hermitian.
(d) All elements of $\widehat{\Gamma}_{1,3}$ are either unitary or anti-unitary.

Proof. Obviously (c) $\implies$ (a) and (d) $\implies$ (b).

Assume that (a) stands and notice that $\mathbf{i}$ is anti-Hermitian and commutes with all elements of $\widehat{\Gamma}_{1,3}$. Then, consider an element $x \in \widehat{\Gamma}_{1,3}$, write its canonical expression, and use the anticommutation rules to see that x is either Hermitian or anti-Hermitian.

Thus, (a) $\implies$ (c).

Similarly, assume that (b) stands and notice that $\mathbf{i}$ is unitary and commutes with all elements of $\widehat{\Gamma}_{1,3}$. Then, consider an element $x \in \widehat{\Gamma}_{1,3}$, write its canonical expression, expand the product $x^\dagger x$ and use the anticommutation rules to see that either $x^\dagger x = \mathbf{i}^2$ or $x^\dagger x = \mathbf{1}$, and proceed similarly for $xx^\dagger$.

Thus, (b) $\implies$ (d).

Assume that (a) stands and consider an element $\gamma \in \Gamma^*_{1,3}$. Compute γ^2, and see that either $\gamma^\dagger\gamma = \gamma\gamma^\dagger = \mathbf{1}$ or $\gamma^\dagger\gamma = \gamma\gamma^\dagger = \mathbf{i}^2$.

Thus (a) $\implies$ (b).

Assume that (b) stands and consider an element $\gamma \in \Gamma^*_{1,3}$. From (b), see that either $\gamma^\dagger(\gamma)^3 = \mathbf{1}$ or $\gamma^\dagger(\gamma)^3 = \mathbf{i}^2$, then use Lemma 2 to deduce the value of $\gamma^\dagger\gamma^{-1}$ and conclude to the Hermiticity or anti-Hermiticity of γ.

Thus (b) $\implies$ (a). □

Theorem 8. *We consider an element $z \in \widehat{\Gamma}_{1,3}$ and we assume that at least one of the four propositions of Theorem 7 stands. Either (a) $z^2 = \mathbf{1}$, or (b) $z^2 = \mathbf{i}^2$.*
(a) $\implies$ (a1) and (a2). (a1) z Hermitian $\iff$ z unitary. (a2) z anti-Hermitian $\iff$ z anti-unitary.
(b) $\implies$ (b1) and (b2). (b1) z Hermitian $\iff$ z is anti-unitary. (b2) z anti-Hermitian $\iff$ z unitary.

Proof. z^2 is either equal to $\mathbf{1}$ or to $\mathbf{i}^2$: see Theorem 5.

Case (a): if $z^\dagger = z$ then $z^\dagger z = zz^\dagger = \mathbf{1}$, and if $z^\dagger z = \mathbf{1}$ then $z^\dagger = z^3 = z$ (see Lemma 2); if $z^\dagger = \mathbf{i}^2 z$ then $z^\dagger z = zz^\dagger = \mathbf{i}^2$, and if $z^\dagger z = \mathbf{i}^2$ then $z^\dagger = \mathbf{i}^2 z^3 = \mathbf{i}^2 z$.

Case (b): if $z^\dagger = z$ then $z^\dagger z = zz^\dagger = \mathbf{i}^2$, and if $z^\dagger z = \mathbf{i}^2$ then $z^\dagger = \mathbf{i}^2 z^3 = z$; if $z^\dagger = \mathbf{i}^2 z$ then $z^\dagger z = zz^\dagger = \mathbf{1}$, and if $z^\dagger z = \mathbf{1}$ then $z^\dagger = z^3 = \mathbf{i}^2 z$. □

Remark 1. *The signature of the metric was not useful to establish the proofs of Theorems 7 and 8: these latter can be extended to higher dimensional gamma matrices.*

Assuming that the gamma matrices are either Hermitian or anti-Hermitian is required for physical applications [22]. This assumption is necessary due to the Hermiticity of the Hamiltonian in Dirac equation. In quantum mechanics, the Hamiltonian is Hermitian because the energy has to be real. It is why Assumption 1 is needed [32,33]:

Assumption 1. *γ^0 is Hermitian. γ^1, γ^2 and γ^3 are anti-Hermitian.*

We consider that Assumption 1 is valid further in the text. It ensures that Theorems 7 and 8 apply.

Corollary 5. *All elements of $\widehat{\Gamma}_{1,3}$ are unitary.*

Proof. The generator $\mathbf{i}$ is unitary. Apply Theorem 8 to see that the gamma matrices of are unitary. Then consider any product of the generators of $\widehat{\Gamma}_{1,3}$ and proceed recursively. □

6. Symmetry Operators; Isometries

First, we outline that in our present approach, the gamma groups include the generator $\mathbf{i}$, although in the literature it was used the product of gamma matrices by the scalar i (see [22,23]), thus requiring to define an operation differing from the group one. Let S_D be the space of Dirac spinors. The group $\widehat{\Gamma}_{1,3}$ acts on S_D. In order to identify the symmetry operators of $\widehat{\Gamma}_{1,3}$, we need to see if the action of $\widehat{\Gamma}_{1,3}$ preserves some distance in S_D, i.e., we need a spinorial metric M. We denote by $<,>$ the inner product over S_D relative to M, and we denote by a dagger the conjugate transpose. The bilinear form between two spinors ψ_1 and ψ_2 is:

$$< \psi_1, \psi_2 >= \psi_1^\dagger M \psi_2 \tag{5}$$

The standard inner product is inadequate because the bilinear $\psi_1^\dagger \psi_2$ is not Lorentz covariant. From physical considerations (invariant length and current density), Crawford outlined the need of normalizing the bilinear forms defining the spinorial metrics [34–36]. He built metrics between Dirac spinors and their generalization in higher dimensional spaces [35–37]. We retain the metric $M = \gamma^0$ in Equation (5) because it was shown that M generates bilinears satisfying to Lorentz invariance for length and current density [25,34–36,38–42].

The operator $\widetilde{\psi} = \psi^\dagger M$ is sometimes called the Dirac adjoint of ψ (see [25,40,42,43]). M is Hermitian and unitary (see Theorem 8). It was noticed that we can always choose a basis of gamma matrices so that M is diagonal [34,36] (this is the case in the Dirac basis: see Equation (2)), but there is no particular reason to retain the Dirac basis to ensure Lorentz covariance of the bilinears built from M. The required properties of the metric is that it is Hermitian and that it render the gamma matrices self-adjoint [37].

Strictly speaking, M is not a true metric because it is not positive definite. This constraint is inherent to special relativity. The same constraint exists in the case of the Minkowski metric, for which isometries preserve inter-event intervals. It does not preclude the analysis of symmetries in terms of direct and indirect symmetries [15].

Theorem 9. *(a) $\widehat{\Gamma}^S_{1,3}$ is the subgroup of $\widehat{\Gamma}_{1,3}$ containing the elements of $\widehat{\Gamma}_{1,3}$ commuting with γ^0.*
(b) This subgroup is the group of symmetry operators acting on S_D, which preserves the distance induced by the metric $M = \gamma^0$.

Proof. (a) Consider two elements x and y of $\widehat{\Gamma}^S_{1,3}$ which commute with γ^0. Obviously xy commutes also with γ^0 and thus $\widehat{\Gamma}^S_{1,3}$ is a subgroup of $\widehat{\Gamma}_{1,3}$.
(b) We consider an element $z \in \widehat{\Gamma}_{1,3}$, and we look at the bilinear $< z\psi_1, z\psi_2 >= \psi_1^\dagger z^\dagger M z \psi_2$. The distance induced by M is preserved when $z^\dagger M z = M$. This occurs under the condition that either z commutes with γ^0 and is unitary or z anticommutes with γ^0 and is anti-unitary. We know that z is always unitary (see Corollary 5), thus the condition reduces to the commutation of z with γ^0. If this condition is not satisfied, the sign of the bilinear is changed. □

7. Direct and Indirect Symmetry; Chirality

We can enumerate the 32 elements of $\widehat{\Gamma}^S_{1,3}$ with the help of Theorem 9. We define:

$$\widehat{\Gamma}^{S_0}_{1,3} = \{\mathbf{1}, \gamma^0, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3, \gamma^0\gamma^1\gamma^2, \gamma^0\gamma^1\gamma^3, \gamma^0\gamma^2\gamma^3\}.$$

$\widehat{\Gamma}^{S_0}_{1,3}$ contains 8 elements. We define also $\widehat{\Gamma}^{S_1}_{1,3}$, which contains the eight products by $\mathbf{i}$ of the elements of $\widehat{\Gamma}^{S_0}_{1,3}$. Similarly, we define $\widehat{\Gamma}^{S_2}_{1,3}$, which contains the eight products by $\mathbf{i}^2$ of the elements of $\widehat{\Gamma}^{S_0}_{1,3}$, and we define $\widehat{\Gamma}^{S_3}_{1,3}$, which contains the eight products by $\mathbf{i}^3$ of the elements of $\widehat{\Gamma}^{S_0}_{1,3}$.

$$\widehat{\Gamma}^S_{1,3} = \widehat{\Gamma}^{S_0}_{1,3} \cup \widehat{\Gamma}^{S_1}_{1,3} \cup \widehat{\Gamma}^{S_2}_{1,3} \cup \widehat{\Gamma}^{S_3}_{1,3} \tag{6}$$

Theorem 10. *Except the neutral element and $\mathbf{i}^2$, the elements of $\widehat{\Gamma}^S_{1,3}$ are indirect symmetry operators.*

Proof. Apply Theorem 3. It is in agreement with the general definition of indirect symmetry in [10], which states that a symmetry operator is direct if and only if it can be expressed as a product of squared isometries. □

The difference between the action on S_D of $\widehat{\Gamma}_{1,3}^{S_0}$ and the action of either $\widehat{\Gamma}_{1,3}^{S_1}$ or $\widehat{\Gamma}_{1,3}^{S_2}$ or $\widehat{\Gamma}_{1,3}^{S_3}$, is just due to a normalizing factor of the spinors, which, in scalar notation, is either i or -1 or $-i$. So, this difference is meaningless in the context of symmetry operators. That could explain why some authors retain for symmetry operators expressions differing by a multiplicative factor such as i or -1 or $-i$.

Parity inversion is defined by $P = \gamma^0$ (see [21,28,44–47]), or by $P = \mathbf{i}^2\gamma^0$ (see [46]). In both cases, $P^2 = \mathbf{1}$, so P is a mirror (as defined in [10]: a mirror is an indirect symmetry which is an involution). Time reversal is defined by $T = \mathbf{i}\gamma^1\gamma^3$ (see [21,28,44,45]), the factor $\mathbf{i}$ being arbitrary [45]. When $\mathbf{i}$ is part of the expression of T, $T^2 = \mathbf{1}$, and T is a mirror. Whichever of the expressions of P and T are retained, the product PT commutes (see Theorem 4), and PT is an indirect symmetry operator. When $\mathbf{i}$ is part of the expression of T, $(PT)^2 = \mathbf{1}$, so PT is a mirror: this is in agreement with [15], and PT is sometimes called a *full reflection* [48].

Charge conjugation is defined by $C = \mathbf{i}\gamma^0\gamma^2$ (see [49]) or $C = \mathbf{i}^3\gamma^0\gamma^2$ (see [40,45,50]), or by $C = \mathbf{i}\gamma^2$ (see [28,51]) or $C = \mathbf{i}^3\gamma^2$ (see [44]), or by $C = \gamma^0\gamma^2$ (see [21]). It was even set $C = \mathbf{1}$ in Majorana basis [51]. Apart the latter, whichever of these expressions is retained, C is not considered to be a symmetry operator for the metric $M = \gamma^0$, and the same conclusion apply to CP, CT and CPT. This may be seen as an unusual conclusion, but it was noticed that, in contrast with P and T, C is not a spacetime discrete symmetry, and that its nature is strongly different from other discrete symmetries [48].

The *chiral operator*, denoted by γ^5 (or γ_5 in old papers), appears many times in the literature. It is defined by $\gamma^5 = \mathbf{i}\omega_{1,3}$, i.e., $\gamma^5 = \mathbf{i}\gamma^0\gamma^1\gamma^2\gamma^3$ (see [21,27,30,40,44,46,47,49–54]), or by $\gamma^5 = \mathbf{i}^3\gamma^0\gamma^1\gamma^2\gamma^3$ (see [33,54,55]). From Corollary 3, γ^5 anticommutes with any of the other gamma matrices, and from Corollary 4, $(\gamma^5)^2 = \mathbf{1}$. From Assumption 1, it can be deduced that γ^5 is Hermitian. It appears that γ^5 is not considered to be a symmetry operator for the metric $M = \gamma^0$. This seemingly shocking conclusion about γ^5, which is representation independent, can be easily explained.

First, the name *chiral operator* is due to the following facts. γ^5 is splitted into two operators, which are, in matrix notation, $(\mathbf{1} + \gamma^5)/2$ and $(\mathbf{1} - \gamma^5)/2$. In Weyl basis (Equation (3)), each of these two operators projects the spinor ψ respectively on its so-called *right-handed* part ψ_R and *left-handed* part ψ_L, i.e., ψ_R and ψ_L are the eigenvectors of γ^5, with respective eigenvalues $+1$ and -1 (Equations (7)–(9)), whence the names *right-handed* and *left-handed* and whence the name *chiral operator*, and whence the name of *chiral basis* for the Weyl basis.

$$\gamma^5_W = \begin{bmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{bmatrix}; \quad \psi_R = \begin{bmatrix} 0 \\ \phi_R \end{bmatrix}; \quad \psi_L = \begin{bmatrix} \phi_L \\ 0 \end{bmatrix} \tag{7}$$

$$\gamma^5_W \left[\begin{array}{c|c} \psi_R & \psi_L \end{array} \right] = \left[\begin{array}{c|c} \psi_R & -\psi_L \end{array} \right] \tag{8}$$

$$\psi = \psi_R + \psi_L; \quad \frac{1}{2}(\mathbf{1} + \gamma^5_W)\psi = \psi_R\,; \quad \frac{1}{2}(\mathbf{1} - \gamma^5_W)\psi = \psi_L \tag{9}$$

Now, the story becomes clear: the transformation of ψ_L into its opposite while ψ_R is unchanged was attributed in 1946 to a chiral property, in Kelvin's sense [6], then this terminology was propagated in the quantum field literature. But a change of sign does not suffice to conclude to chirality, because a symmetry operator must be defined relatively to a space and a metric. Here, the trouble comes from that γ^5 is a symmetry operator when defined from the metric induced by the standard inner product (in which case it would be a mirror, as a consequence of Theorem 3, and because $(\gamma^5)^2 = 1$), but it is no more a symmetry operator for the quadratic form induced by the spinorial metric $M = \gamma^0$.

However, if we relax the sign preservation condition for this quadratic form (see part (b) of Theorem 9 and its proof), then all elements of $\widehat{\Gamma}_{1,3}$ are symmetry operators, and γ^5 is a mirror (it is a consequence of Thorem 3). Strictly speaking, a symmetry operator should be distance preserving, or, when the metric is not positive definite, it should preserve the value of the quadratic form induced by the metric. Due to the customs in quantum field theory, the change of sign of the quadratic form induced by $M = \gamma^0$ upon the action of γ^5 is ignored, and γ^5 is called a chiral operator in the literature. Thus it can be considered that there is an abuse of language about chirality.

The metric γ^0 was shown to be relevant several decades after 1946 (at least, in 1990 [34]). It is not the first time that an ambiguity about the metric induced opposite conclusions: a controversy happened in classical mechanics about rotating molecules, which was recently solved [14]. In fact, it even happened that chirality needed to be clarified in the Euclidean case [56]. Our own conclusion is that the chirality concept cannot be clearly understood as long as ambiguities remain in its use.

Funding: This research received no external funding.

Acknowledgments: I am grateful to the reviewers of this paper. Their criticisms were of great help to improve the quality of the manuscript.

Conflicts of Interest: The author declares no conflict of interest.

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