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DETERMINING THE PION REFERENCE MOMENTUM FOR NUSTORM INJECTION DESIGN

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Abstract

The stochastic injection scenario used by nuSTORM features the pion decay and secondary muon acceptance in the storage ring's long decay straight [1,2]. The designed momentum acceptance of the nuSTORM decay ring is centered at 3.8 GeV/c, based on neutrino detector performance, with a ±10% bin. In order to design the injection section, and to obtain as many useful muons from pion decay as possible, the center momentum of the pion beam being injected needs to be carefully chosen. This paper describes in detail the determination of the center momentum of the pion beam, with simulation from G4Beamline [3].

Key words. nuSTORM, Stochastic injection, Pion Decay, Neutrino factory

1 Introduction

The injection scenario used in design and simulation of the nuSTORM facility requires a special beam combination section (BCS) and a special treatment in designing the FODO cells as a decay straight section [1]. Because the BCS and decay straight are designed to incorporate both the injected pion beam and the circulating muon beam, the need of choosing the right injection momentum is self-evident. The initial designed momentum acceptance of the nuSTORM decay ring is 3.8±10% GeV/c. The transport line of the pion beam is designed to have a P_0±10% momentum acceptance. In order to provide a flat distribution of muon momenta over the 3.8±10% GeV/c range, the center momentum P_0 can not be too large. Nevertheless, a higher P_0 can limit the phase space of the muon beam, and reduce the requirement on the BCS, which creates a large dispersion for orbit separation of the injected reference pion and circulating muon. This trade-off can be balanced by noticing that P_0 could be chosen larger as long as the muon momentum distribution does not fall off on its edges.

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In this paper, we will demonstrate the balancing criteria by showing the theoretical derivations and the simulation results from G4Beamline [3], which utilizes the Monte Carlo simulations in Geant4.

2 Muon momentum distribution over 3.8±10% GeV/c

2.1 Muon momentum distribution from decay of pions at a certain energy

The pion decay to muon is well understood by simple relativistic mechanics. In the rest frame of the injected pions, using the conservation of energy and momenta we have:

$$E'_\pi = E'_{\nu_\mu} + E'_\mu; E'_{\nu_\mu} = cp'_{\nu_\mu}; E'_\mu = \sqrt{(cp'_\mu)^2 + (m_\mu c^2)^2} \tag{2.1}$$

$$p'_{\nu_\mu} = p'_\mu \quad (Magnitude) \tag{2.2}$$

where the prime stands for the value in the rest frame of the pion. The values for p'_μ and E'_μ are found from the above equations:

$$p'_\mu = \frac{{E'_\pi}^2 - m_\mu^2 c^4}{2E'_\pi \cdot c} = 29.81 MeV/c \tag{2.3}$$

$$E'_\mu = \sqrt{(29.76)^2 + (105.7)^2} = 109.8 MeV \tag{2.4}$$

Denote the direction of the moving pion in the lab frame as **z**, and the angle between the direction of muon and **z** as θ, or θ' under the rest frame of the pion, we write the Lorentz transformation of a 4-momentum vector as,

$$p_\mu \cos\theta = \gamma \left(p'_\mu \cos\theta' + \beta \frac{E'_\mu}{c} \right) \tag{2.5}$$

$$p_\mu \sin\theta = p'_\mu \sin\theta' \tag{2.6}$$

which simply implies that,

$$\begin{aligned} p_\mu^2 &= \gamma^2 \left(p'_\mu \cos\theta' + \frac{\beta E'_\mu}{c} \right)^2 + p'^2_\mu \sin^2\theta' \\ &= (\gamma^2 - 1) p'^2_\mu \cos^2\theta' + 2\gamma^2 \frac{\beta E'_\mu}{c} p'_\mu \cos\theta' + p'^2_\mu + \gamma^2 \frac{\beta^2 E'^2_\mu}{c^2} \end{aligned} \tag{2.7}$$

Here the β and γ are relativistic factors rather than the optics β and γ. Now remember the fact that the distribution of muons under the rest frame of decayed pion is flat with respect to the solid angle $d\Omega' = d(\cos\theta')d\phi'$, namely with respect to $\cos\theta'$ and ϕ'. In order to obtain the probability density function, $f_p(p)$, first denote the cumulative density function as $F_p(p)$, and regard $\cos\theta'$ as the independent variable so that $f_X(x) = f(\cos\theta') = \frac{1}{2}$ =const. We have the expression for $F_p(p_\mu)$ that:

$$F_p(p_\mu) = \Pr(P_\mu \le p_\mu) = \Pr(\sqrt{(\gamma^2-1)p'^2_\mu\cos^2\theta' + 2\gamma^2\frac{\beta E'_\mu}{c}p'_\mu\cos\theta' + p'^2_\mu + \gamma^2\frac{\beta E'^2_\mu}{c^2}} \le p_\mu)$$

$$= \Pr(0 \le [(\gamma^2-1)p'^2_\mu\cos^2\theta' + 2\gamma^2\frac{\beta E'_\mu}{c}p'_\mu\cos\theta' + p'^2_\mu + \gamma^2\frac{\beta E'^2_\mu}{c^2}] \le p^2_\mu)$$

$$= \Pr(\cos\theta'_- \le \cos\theta' \le \cos\theta'_+) \tag{2.8}$$

$$= \Pr(-1 \le \cos\theta' \le \cos\theta'_+) \tag{2.9}$$

where $\cos\theta'_-(p_\mu)$ and $\cos\theta'_+(p_\mu)$ are the two roots of Equation 2.7. Considering that Equation 2.7 is monotonically increasing in [-1,1], the probability of Equation 2.8 and Equation 2.9 is the same. According to the definition of cumulative probability function, Equation 2.9 is the same with,

$$F_p(p_\mu) = F_X(\cos\theta'_+) - F_X(-1) \tag{2.10}$$

$$f_p(p_\mu) = f_X(\cos\theta'_+)\frac{d\cos\theta'_+}{dp_\mu} - 0 \tag{2.11}$$

When $\gamma \gg 1$, we have,

$$\cos\theta'_+ \approx \left[-\frac{2\gamma^2 E'_\mu}{c}p'_\mu + \sqrt{\frac{4\gamma^4 E'^2_\mu}{c^2}p'^2_\mu - 4\gamma^2 p'^2_\mu(p'^2_\mu + \gamma^2\frac{E'^2_\mu}{c^2} - p^2_\mu)}\right]/2\gamma^2 p'^2_\mu$$

$$= \left[-\frac{2\gamma^2 E'_\mu}{c}p'_\mu + \sqrt{4\gamma^2 p'^2_\mu p^2_\mu - 4\gamma^2 p'^2_\mu}\right]/2\gamma^2 p'^2_\mu$$

$$\approx \left[-\frac{2\gamma^2 E'_\mu}{c}p'_\mu + 2\gamma p'_\mu p_\mu\right]/2\gamma^2 p'^2_\mu \tag{2.12}$$

since $p_\mu \gg p'^2_\mu$. Then we obtain,

$$\frac{d\cos\theta'_+}{dp_\mu} \approx \frac{1}{\gamma p'_\mu} \tag{2.13}$$

According to Equation 2.11, we have

$$f(p_\mu) = f(\cos\theta'_+)\frac{d\cos\theta'_+}{dp_\mu} = \frac{1}{2\gamma p'_\mu}, \qquad p_{\mu,min} < p_\mu < p_{\mu,max} \tag{2.14}$$

where

$$p_{\mu,max} = \sqrt{\gamma^2\left(p'_\mu + \frac{\beta E'_\mu}{c}\right)^2} = \gamma\left(p'_\mu + \frac{\beta E'_\mu}{c}\right) \tag{2.15}$$

$$p_{\mu,min} = \sqrt{\gamma^2\left(p'_\mu \times (-1) + \frac{\beta E'_\mu}{c}\right)^2} = \gamma\left(-p'_\mu + \frac{\beta E'_\mu}{c}\right) \tag{2.16}$$

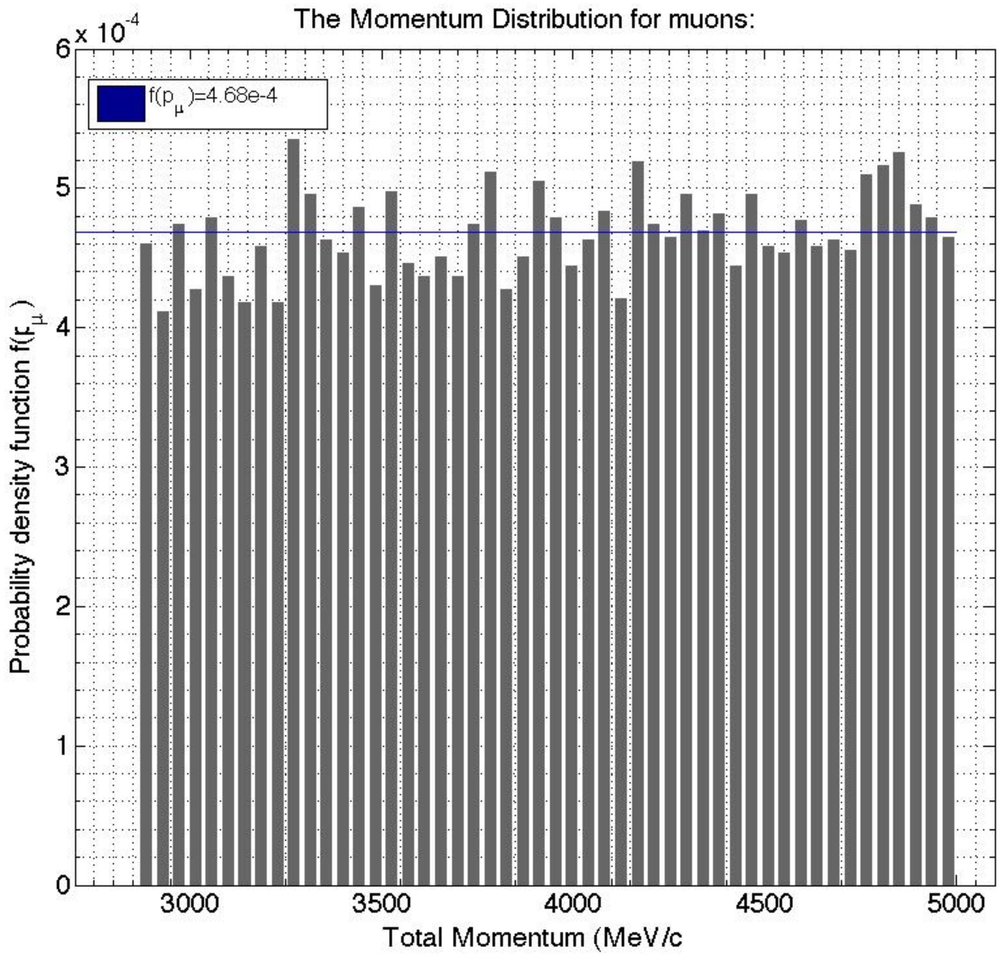

Figure 2.1: Comparison between theoretically calculated f(p_μ) and G4Beamline simulated f(p_μ).

So, we proved when the relativity factor γ is much larger than 1, the distribution over $[p_{\mu,min},p_{\mu,max}]$ is flat with a probability density function f(p_μ)=$\frac{1}{2\gamma p'_\mu}$, and the decayed muons has a momentum band width $2\gamma p'_\mu$.

This result is checked in G4Beamline, where we put 10,000 π^+ with the same momentum 5 GeV/c, and the same moving direction **z**. All the π^+ are forced to decay at the same time by a **particlefilter** at the vergy beginning of the motion, to produce μ^+. A **virtualdetector** is placed right after the particlefilter to record the information of these muons. The figure 2.1 shows the comparison of theoretical and simulated probability density function f(p_μ). In this case, γ $\approx$35.83, so theoretical result gives f(p_μ)$\approx$4.6812$\times$$10^4$. The simulation results corresponds well with the above derivation.

2.2 Muon angles from decay of pions at a certain energy

Because of the angle θ between decayed muon's direction and its parental pion's direction, the phase space area of the muon beam from pion decay will be enlarged. The tangent of θ can be concluded from Equation 2.5 and Equation 2.6:

$$\tan\theta = \frac{p'_\mu \sin\theta'}{\gamma\left(p'_\mu \cos\theta' + \beta\frac{E'_\mu}{c}\right)} \tag{2.17}$$

The maximum of $\tan\theta$ can be obtained by the extremum condition $d\theta/d\theta' = 0$, we then have,

$$\cos\theta' = -\frac{p'_\mu c}{\beta E'_\mu} = -\frac{29.78}{109.8 \times 0.9996} = 0.2713 \Rightarrow \qquad \theta' = 105.7^\circ \tag{2.18}$$

thus we also have,

$$\tan\theta_{max} = \frac{p'_\mu \sin\theta'}{\gamma(p'_\mu \cos\theta' + \beta E'_\mu/c)} = \frac{28.6921}{\gamma(8.0875 + 109.8\beta)} \xrightarrow{\gamma\gg 1} \frac{28.6921}{117.8875\gamma} \tag{2.19}$$

Clearly the phase space area can be less magnified when γ is larger. When $\gamma = 35.83$ for 5 GeV/c pion, $\tan\theta_{max} = 0.0079$. The histogram of θ is shown in Figure 2.2 This clarifies that we need larger γ to reduce phase space area.

2.3 Muon momentum distribution from decay of pions within momentum bin $P_0 \pm 0.1P_0$

Now suppose we have a pion beam which has a uniform momentum distribution over range $[0.9P_0, 1.1P_0]$, we want to check the probability density function of P_μ, $f(P_\mu)$, which is the probability density that a muon from decay has momentum P_μ.

First we find that the probability that the momentum of a pion falls in between $P_\pi \sim P_\pi + dP_\pi$ where $0.9P_0 \leq P_\pi \leq 1.1P_0$ is simply $\frac{1}{(0.2P_0)}dP_\pi$. From Equation 2.14, we know that the probability density of having a muon with momentum P_μ from a pion with momentum P_π is $1/(2\gamma P'_\mu)$, where the relativistic γ is evaluated at P_π. Here we require that $\gamma\left(-p'_\mu + \frac{\beta E'_\mu}{c}\right) \leq P\mu \leq \gamma\left(p'_\mu + \frac{\beta E'_\mu}{c}\right)$, otherwise the probability is 0.

This suggests that the averaged probability density function of having a muon with momentum P_μ from this pion beam is

$$\begin{aligned} f(P_\mu) &= \int_{P_{lower}}^{P_{upper}} \frac{1}{0.2P_0} \cdot \frac{1}{2\gamma P'_\mu}\, dP_\pi \\ &= \int_{P_{lower}}^{P_{upper}} \frac{1}{0.2P_0} \cdot \frac{1}{\frac{2P_\pi P'_\mu}{m'_\pi c}}\, dP_\pi \end{aligned} \tag{2.20}$$

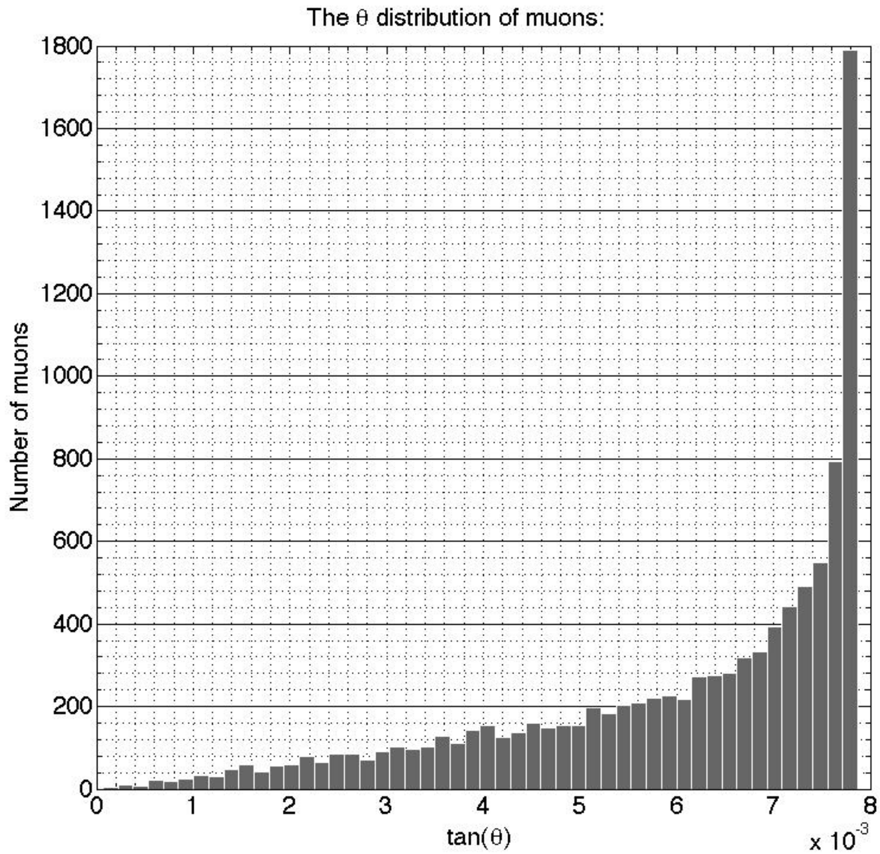

Figure 2.2: Histogram of θ for muons from decay of pions at 5 GeV/c. Simulated in G4Beamline.

where β is omitted for $\gamma \gg 1$. The integration upper and lower limits are determined by the following conditions:

- If $P_{\pi,min} \leq 0.9P_0$ **and** $P_{\pi,max} \leq 1.1P_0$, $\quad P_{lower} = 0.9P_0,\ P_{upper} = P_{\pi,max}$;
- If $P_{\pi,min} \geq 0.9P_0$ **and** $P_{\pi,max} \geq 1.1P_0$, $\quad P_{lower} = P_{\pi,min},\ P_{upper} = 1.1P_0$;
- If $P_{\pi,min} \geq 0.9P_0$ **and** $P_{\pi,max} \leq 1.1P_0$, $\quad P_{lower} = P_{\pi,min},\ P_{upper} = P_{\pi,max}$;
- If $P_{\pi,min} \leq 0.9P_0$ **and** $P_{\pi,max} \geq 1.1P_0$, $\quad P_{lower} = 0.9P_0,\ P_{upper} = 1.1P_0$;
- Otherwise, $f(P_\mu) = 0$

where $P_{\pi,min}$ and $P_{\pi,max}$ are the minimum and maximum momentum of a pion that can decay to a muon at P_μ. They can be determined by the following equations:

$$P_\mu = \gamma_{min}\left(p'_\mu + \frac{E'_\mu}{c}\right) = \frac{P_{\pi,min}}{m'_\pi c}\left(p'_\mu + \frac{\beta E'_\mu}{c}\right) \tag{2.21}$$

$$= \gamma_{max}\left(-p'_\mu + \frac{\beta E'_\mu}{c}\right) = \frac{P_{\pi,max}}{m'_\pi c}\left(-p'_\mu + \frac{\beta E'_\mu}{c}\right) \tag{2.22}$$

or in another form,

$$P_{\pi,min} = \frac{m'_\pi c\, P_\mu}{p'_\mu + \frac{\beta E'_\mu}{c}} \tag{2.23}$$

$$P_{\pi,max} = \frac{m'_\pi c\, P_\mu}{-p'_\mu + \frac{\beta E'_\mu}{c}} \tag{2.24}$$

where m'_π is the rest energy of a pion. Plug in the values for m'_π, p'_μ, and E'_μ, we have a piecewise function for $f(P_\mu)$:

$$f(P_\mu) = \begin{cases} 0 & P_\mu \le 0.5158P_0 \\ \int_{0.9P_0}^{1.7452P_\mu} \frac{1}{0.2P_0}\,\frac{m'_\pi c}{2P_\pi P'_\mu}\,dP_\pi = \frac{11.7075}{P_0}\ln\left(\frac{1.94P_\mu}{P_0}\right) & 0.5158P_0 \le P_\mu \le 0.6303P_0 \\ \int_{0.9P_0}^{1.1P_0} \frac{1}{0.2P_0}\,\frac{m'_\pi c}{2P_\pi P'_\mu}\,dP_\pi = \frac{2.3494}{P_0} & 0.6303P_0 \le P_\mu \le 0.9001P_0 \\ \int_{P_\mu}^{1.1P_0} \frac{1}{0.2P_0}\,\frac{m'_\pi c}{2P_\pi P'_\mu}\,dP_\pi = \frac{11.7075}{P_0}\ln\left(\frac{1.1P_0}{P_\mu}\right) & 0.9001P_0 \le P_\mu \le 1.1001P_0 \\ 0 & P_\mu \ge 1.1001P_0 \end{cases} \tag{2.25}$$

This result is verified by comparing the above equations with the simulation results from G4Beamline. The plots of the probability density function from calculated and simulated results are shown in Figure 2.3. They consist with each other very well.

2.4 Determination of P_0

As a result of Equation 2.24, it is important to realize that the momentum distribution of muons over 3.8±10% GeV/c is flat, as long as $0.9001P_0 \ge 3.8{+}0.38$ and $0.6303P_0 \le 3.8{-}0.38$, or namely $P_0 \in$[4.6439, 5.4260] GeV/c. Notice the fact that the flat top of $f(P_0)$ is smaller when P_0 is larger. There are advantages of having a lower P_0, which can give more muons within 3.8±10% GeV/c and smaller pion beam size after injection by the BCS. However there are also advantages of having a higher P_0, which can reduce the requirement on dispersion D_x created by the BCS and can reduce the emittance growth from decay of pions to muons. The Figure 2.4 shows the trade-off by plotting the number of muons and $\sigma_{x'}$ along with $\sigma_{y'}$. Balancing these criteria, we finally fixed P_0 at a balanced value 5 GeV/c.

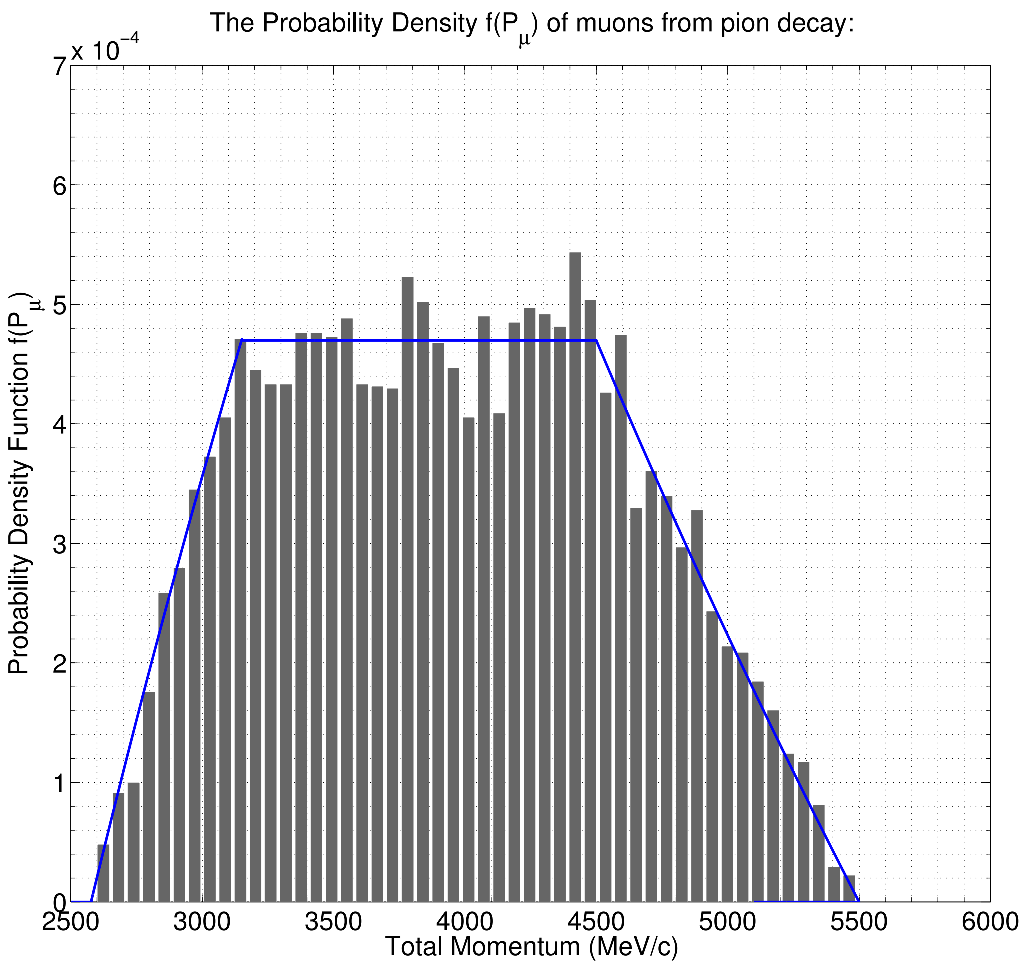

Figure 2.3: Probability density function from calculated (curve) and simulated results (histogram).

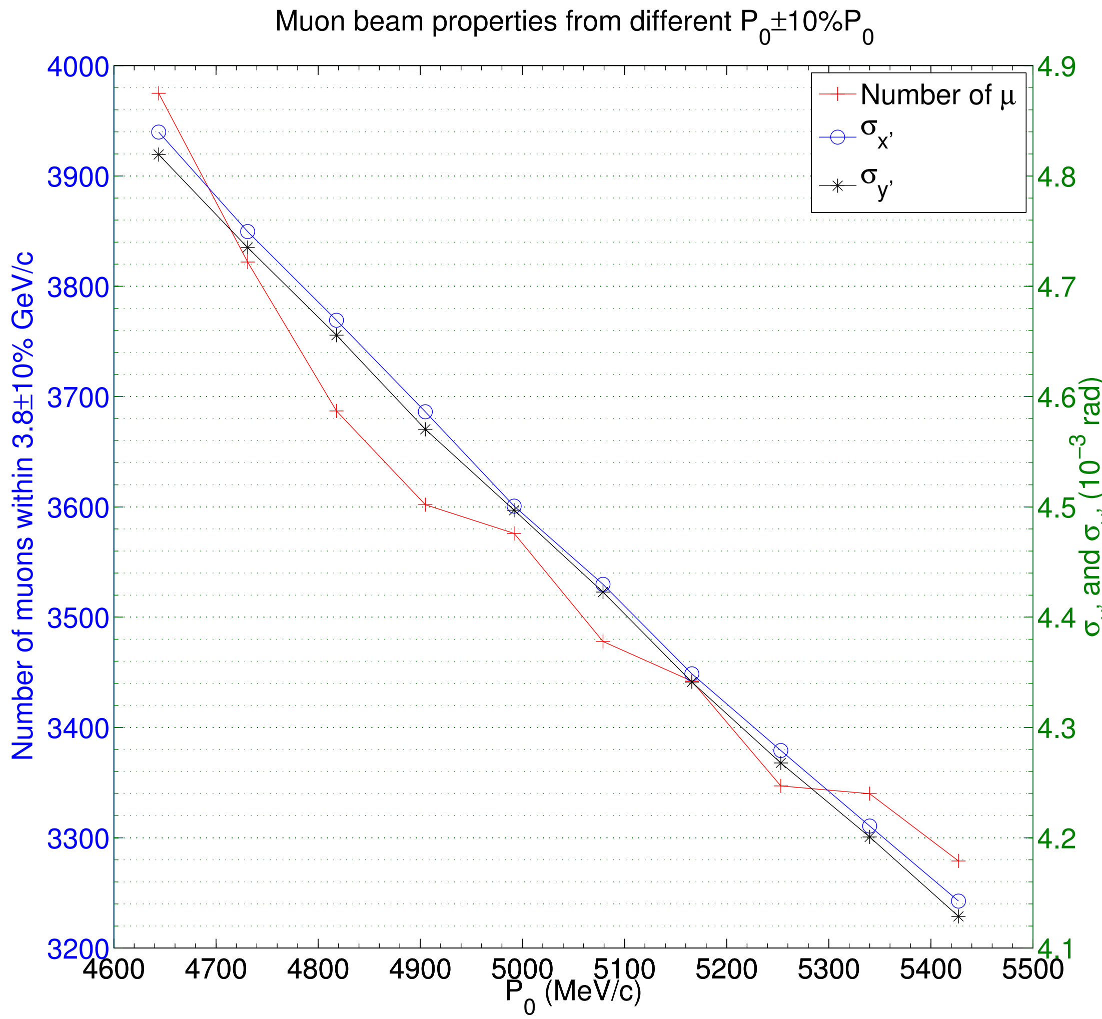

Figure 2.4: Number of muons within $3.8 \pm 10\%$ GeV/c (left vertical axis), $\sigma_{x'}$ and $\sigma_{y'}$ (right vertical axis).

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