CASIMIR OPERATORS OF SUBALGEBRAS OF THE POINCARE LIE ALGEBRA AND OF REAL LIE ALGEBRAS OF LOW DIMENSION†

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1. Introduction

The Poincaré group generated by three space rotations L_1, L_2, L_3, three Lorentz boosts K_1, K_2, K_3, and four translations P_0, P_1, P_2, P_3 is the fundamental transformation group of special relativity. Many of its properties have been studied during the last 70 years by both mathematicians and physicists. Only recently, however, a list of its continuous subgroups has been obtained[1], thus providing an exhaustive description of all possible "subsymmetries" of continuous type.

For many applications in mathematical physics it is essential to know the (Casimir) operators which commute with all generators of each subalgebra S and are themselves elements of the enveloping algebra of S. Thus, for instance, these operators provide a convenient way of labelling representations of S and defining bases for the representations of the entire group, their eigenvalues are measurable physical quantities (quantum numbers), etc.[1].

The purpose of this contribution is to present the list of subalgebras of the Poincaré Lie algebra P (nonconjugate under the connected part of the Poincaré group) which have Casimir operators or other invariant operators, as explained below. A physical interpretation and further use of these operators is postponed to subsequent publications.

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Since there are many subalgebras of dimensions 3 and 4 quite a few among them being isomorphic to each other, we found advantageous first to classify the subalgebras of dimension ≤ 4 into isomorphism classes, then to find the invariant operators for each isomorphic class, and then only to "interpret" the operators using generators of each nonconjugate subalgebra of P. For that we use the list of nonisomorphic real Lie algebras of low dimension as given by Mubarakzyanov[2]. We do not follow, however, the notations of Ref. 2 where algebras of different physical significance are often denoted by the same symbol.

The method for finding the invariant operators is briefly described in Sec. II together with the types of operators obtained. In Sec. III the Tab. I is described. It containes the real nonisomorphic Lie algebras of dimension ≤ 4, their Casimir and other invariant operators. In Sec. IV the isomorphisms between subalgebras of P are presented. Invariant operators of subalgebras of dimension 3 and 4 are given in Tab. II, and for subalgebras of dimension exceeding 4 they are in Tab. III.

II. A Method for Finding Invariants of Lie Algebras

The number of Casimir operators of a semisimple Lie algebra is equal to its rank. Therefore our problem is in the nonsemisimple Lie algebras for which no easy criterion is known to decide readily whether or not they possess a Casimir operator. (For other invariant operators see Ref. 3.)

Consider a Lie algebra $\mathcal{L}$ generated by $A_1,\ldots,A_n$ satisfying

$$[A_i,A_k] = f_{ik}^{\ell} A_{\ell}. \tag{1}$$

We represent the generators A_i as differential operators acting on a space of functions $F(a_1,\ldots,a_n)$ by putting

$$A_i = \sum_{k,\ell=1}^{n} f_{ik}^{\ell} a_{\ell} \frac{\partial}{\partial a_k} \qquad (i = 1,2,\ldots,n). \tag{2}$$

In order to find an operator valued function $P(A_1,\ldots,A_n)$ such that equation

$$[A_i,P(A_1,\ldots,A_n)] = 0 \quad \text{for} \quad i = 1,2,\ldots,n, \tag{3}$$

holds, we find first a numerical function $P(a_1,\ldots,a_n)$ annihilated by all operators A_i in (2). Thus we get a system of n linear homogeneous differential equations

$$\sum_{k,\ell=1}^{n} f_{ik}^{\ell} a_{\ell} \frac{\partial}{\partial a_k} P(a_1,\ldots,a_n) = 0, \qquad (i = 1,2,\ldots,n) \tag{4}$$

where some of the equations may be trivial, depending on actual values of the structure constants f_{ik}^{ℓ} of each algebra. Having found solutions $P(a_1,\ldots,a_n)$ of the system (4), we replace the variables a_i by A_i, symmetrize the solutions with respect to A_i's, and arrive thus at the invariant operators we were searching for.

Generally, three situations may arise:

(i) The system (4) does not have a solution different from zero. Then the algebra $\mathcal{L}$ does not have an invariant operator.

(ii) One or several solutions of (4) exist and are of polynomial form. Then we choose a convenient integrity basis for them (i.e. a minimal set of invariant operators such that any other invariant operator can be expressed through them as a polynomial) and call it the Casimir operators of the algebra (1).

(iii) Solutions of (4) are of nonpolynomial form. Then we have "generalized Casimir operators" which still commute with the algebra in the sense (4) and thus have fixed numerical values within each irreducible representation of $\mathcal{L}$. These invariants do not belong to the enveloping algebra of $\mathcal{L}$. A case of special interest is when these invariant operators are rational functions of the generators.

III. Isomorphic Classes of Real Lie Algebras of Dimension ≤ 4

All one-dimensional Lie algebras are isomorphic to each other. We denote them by A_1. The only generator is obviously also a Casimir operator of A_1.

An algebra of dimension 2 is isomorphic either to a direct sum A_1+A_1 which we denote by $2A_1$ (its two generators are again Casimir operators), or to a non-Abelian algebra generated by e_1, e_2:

$$[e_1 e_2] = e_1. \tag{5}$$

This algebra, denoted by A_2, does not have any invariant operator.

An algebra of dimension 3 which is a direct sum is isomorphic to one of the two:

$$3A_1 \equiv A_1+A_1+A_1 \quad \text{or } A_1+A_2. \tag{6}$$

The algebra $3A_1$ has all its generators as Casimir operators, for A_1+A_2 only the generator of A_1 is a Casimir operator. The remaining 3-dimensional algebras do not decompose into a direct sum of lower algebras. They are described by the entities $A_{3,1}, A_{3,2}, \ldots, A_{3,9}$ in Tab. I, which contains also their invariant operators.

Let us point out the $A^a_{3,5}$ and $A^a_{3,7}$ each represent a continuum of non-isomorphic algebras corresponding to different values of the parameter a; this fact is underlined by the appearance of the upper index a.

A 4-dimensional real Lie algebra is isomorphic either to one of the following direct sums

$$4A_1,\ 2A_1+A_2,\ 2A_2,\ A_1+A_{3,i} \qquad (i = 1,2,\ldots,9) \tag{7}$$

or to one of the algebras $A_{4,1}, A^a_{4,2}, \ldots, A_{4,12}$ of Tab. I. The invariants of algebras (7) are obviously given, by the Casimir operator of each A_1 and the corresponding operators of $A_{3,i}$. For $A_{4,i}$ $(i = 1,\ldots,12)$ the operators are listed in the Tab. I, whenever they exist.

The entities $A^a_{4,2}$, $A^{a,b}_{4,5}$, $A^{a,b}_{4,6}$, $A^b_{4,9}$, $A^a_{4,11}$ are again infinite families of non-isomorphic algebras for each value of the parameters whithin the specified range.

IV. Invariants of the Subalgebras of the Poincaré Lie Algebra

Representatives of conjugacy classes of subalgebras of the Poincaré Lie algebra P are denoted as in Tab. 3 and 4 of Ref. 1, i.e. either by $P_{i,k}$ or by $\tilde{P}_{m,n}$. Their generators are also found in those tables.

All 1-dimensional subalgebras of P are isomorphic to A_1 and their generators are also their Casimir operators. These algebras are:
$P_{11,6}$, $P_{12,10}$, $P_{13,9}$, $P_{14,9}$, $P_{15,8}$, $P_{15,9}$, $P_{15,10}$, $\tilde{P}_{12,23} \div \tilde{P}_{12,26}$, $\tilde{P}_{13,15}$, $\tilde{P}_{14,24}$, $\tilde{P}_{14,25}$, and $\tilde{P}_{14,26}$.

The subalgebras of dimension 2 which are not Abelian are all isomorphic to A_2 and they do not have an invariant operator. These algebras are: $P_{8,9}$, $P_{11,5}$, $P_{13,7}$, $\tilde{P}_{8,17}$, $\tilde{P}_{13,13}$.

The Abelian 2-dimensional subalgebras are all isomorphic to $2A_1$ and all their generators are also their Casimir operators. These algebras are:
$P_{9,6}$, $P_{10,5}$, $P_{12,7} \div P_{12,9}$, $P_{13,8}$, $P_{14,7}$, $P_{14,8}$, $P_{15,5} \div P_{15,7}$, $\tilde{P}_{10,6}$, $\tilde{P}_{12,19} \div \tilde{P}_{12,22}$, $\tilde{P}_{13,14}$, $\tilde{P}_{14,20} \div P_{14,23}$.

The subalgebras of P of dimension 3 and 4 are given in Tab. II together with their generators and invariants whenever these exist. Isomorphic classes are indicated.

The subalgebras of dimension ≥ 5 are given in Tab. III with their generators and invariants. None of these subalgebras are isomorphic to each other.

References

1. J. Patera, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 16, 1597 (1975) and references therein.
2. G.M. Mubarakzyanov, Izv. Vyssh. Uch. Zav. 32, 114 (1963) (in Russian).
3. E.G. Beltrametti and A. Blasi, Phys. Lett. 20, 62 (1966).

TABLE I. Real Lie algebras of dimensions 3 and 4 which are not direct summs of Lie algebras of lower dimensions.

Name	Non-zero commutation relations	Invariants	Comments
$A_{3,1}$	$[e_2e_3] = e_1$	e_1	nilpotent (Algebra of Weyl group)
$A_{3,2}$	$[e_1e_3] = e_1$, $[e_2e_3] = e_1+e_2$	$e_1 \exp\left(-\frac{e_2}{e_1}\right)$	solvable
$A_{3,3}$	$[e_1e_3] = e_1$, $[e_2e_3] = e_2$	$\frac{e_2}{e_1}$	solvable, $D \cong T_2$
$A_{3,4}$	$[e_1e_3] = e_1$, $[e_2e_3] = -e_2$	e_1e_2	solvable, E(1,1)
$A^a_{3,5}$	$[e_1e_3] = e_1$, $[e_2e_3] = ae_2$ $(0<\lvert a\rvert<1)$	$e_2e_1^{-a}$	solvable
$A_{3,6}$	$[e_1e_3] = -e_2$, $[e_2e_3] = e_1$	$e_1^2+e_2^2$	solvable, E(2)
$A^a_{3,7}$	$[e_1e_3] = ae_1-e_2$, $[e_2e_3] = e_1+ae_2$ $(a>0)$	$(e_1^2+e_2^2)\left(\frac{e_1+ie_2}{e_1-ie_2}\right)^{ia}$	solvable,
$A_{3,8}$	$[e_1e_3] = -2e_2$. $[e_1e_2] = e_1$, $[e_2e_3] = e_3$	$2e_2^2+e_1e_3+e_3e_1$	semisimple, SU(1,1
$A_{3,9}$	$[e_1e_2] = e_3$, $[e_2e_3] = e_1$, $[e_3e_1] = e_2$	$e_1^2+e_2^2+e_3^2$	simisimple, SU(2)
$A_{4,1}$	$[e_2e_4] = e_1$, $[e_3e_4] = e_2$	e_1, $e_2^2-2e_1e_3$	nilpotent
$A^a_{4,2}$	$[e_1e_4] = ae_1$, $[e_2e_4] = e_2$, $[e_3e_4] = e_2+e_3$ $(a\neq0)$	$\frac{e_2^a}{e_1}$, $e_2 \exp\left(-\frac{e_3}{e_2}\right)$	solvable, derived algebra ~ $3A_1$
$A_{4,3}$	$[e_1e_4] = e_1$, $[e_3e_4] = e_2$	e_2, $e_1 \exp\left(-\frac{e_3}{e_2}\right)$	solvable, derived algebra ~ $3A_1$

Name	Non-zero commutation relations	Invariants	Comments
$A_{4,4}$	$[e_1e_4] = e_1,\ [e_2e_4] = e_1+e_2,\ [e_3e_4] = e_2+e_3$	$e_1 \exp\left(-\frac{e_2}{e_1}\right),\ \frac{2e_1e_3-e_2^2}{e_1^2}$	solvable, derived algebra ~ $3A_1$
$A^{ab}_{4,5}$	$[e_1e_4] = e_1,\ [e_2e_4] = ae_2,\ [e_3e_4] = be_3$ $(ab\neq 0,\ -1\leq b\leq a\leq 1)$	$\frac{e_1^a}{e_2},\ \frac{e_1^b}{e_3}$	solvable, derived algebra ~ $3A_1$
$A^{ab}_{4,6}$	$[e_1e_4] = ae_1,\ [e_2e_4] = be_2-e_3,\ [e_3e_4] = e_2+be_3$ $(a\neq 0, b\geq 0)$	$\frac{e_1^{\frac{2b}{a}}}{e_2^2+e_3^2},\ e_1^{\frac{i-b}{a}}(e_2-ie_3)+e_1^{\frac{-b-i}{a}}(e_2+ie_3)$	solvable, derived algebra ~ $3A_1$
$A_{4,7}$	$[e_2e_3] = e_1,\ [e_1e_4] = 2e_1,\ [e_2e_4] = e_2,$ $[e_3e_4] = e_2+e_3$	none	solvable, derived algebra ~ $A_{3,1}$
$A_{4,8}$	$[e_2e_3] = e_1,\ [e_2e_4] = e_2,\ [e_3e_4] = -e_3$	$e_1,\ e_2e_3+e_3e_2-2e_1e_4$	solvable, derived algebra ~ $A_{3,1}$
$A^{b}_{4,9}$	$[e_2e_3] = e_1,\ [e_1e_4] = (1+b)e_1,\ [e_2e_4] = e_2,$ $[e_3e_4] = be_3$ $(-1<b\leq 1)$	none	solvable, derived algebra ~ $A_{3,1}$
$A_{4,10}$	$[e_2e_3] = e_1,\ [e_2e_4] = -e_3,\ [e_3e_4] = e_2$	$e_1,\ 2e_1e_4+e_2^2+e_3^2$	solvable, derived algebra ~ $A_{3,1}$
$A^{a}_{4,11}$	$[e_2e_3] = e_1,\ [e_1e_4] = 2ae_1,\ [e_2e_4] = ae_2-e_3$ $[e_3e_4] = e_2+ae_3$ $(a>0)$	none	solvable, derived algebra ~ $A_{3,1}$
$A_{4.12}$	$[e_1e_3] = e_1,\ [e_2e_3] = e_2,\ [e_1e_4] = -e_2,\ [e_2e_4] = e_1$	none	solvable, derived algebra ~ $2A_1$

TABLE II. Invariant operators of 3- and 4-dimensional subalgebras of the Poincaré Lie algebra

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$3A_1$	$P_{10,4}$	$L_2+K_1,\ L_1-K_2,\ P_0-P_3$	all generators	Abelian
	$P_{12,5}$	$L_3,\ P_0,\ P_3$		
	$P_{13,6}$	$K_3,\ P_1,\ P_2$		
	$P_{14,4}$	$L_2+K_1,\ P_0-P_3,\ P_2$		
	$P_{15,2}$	$P_0-P_3,\ P_1,\ P_2$		
	$P_{15,3}$	$P_1,\ P_2,\ P_3$		
	$P_{15,4}$	$P_0,\ P_1,\ P_2$		
	$\tilde{P}_{10,11}$	$L_2+K_1,\ L_1-K_2+P_2,\ P_0-P_3$		
	$\tilde{P}_{14,13}$	$L_2+K_1-\frac{1}{2}(P_0+P_3),\ P_0-P_3,\ P_2$		
A_2+A_1	$P_{8,8}$	$L_2+K_1,\ K_3,\ P_2$	P_2	solvable
	$P_{9,5}$	$K_3,\ P_0-P_3,\ L_3$	L_3	
	$P_{13,5}$	$K_3,\ P_0-P_3,\ P_2$	P_2	
	$\tilde{P}_{13,5}$	$K_3+aP_2,\ P_0-P_3,\ P_1\quad (a>0)$	P_1	
$A_{3,1}$	$P_{14,5}$	$L_2+K_1,\ P_1,\ P_0-P_3$	P_0-P_3	nilpotent
	$P_{14,6}$	$L_2+K_1,\ P_2+bX_3,\ P_0-P_3\quad (b\neq 0)$		
	$\tilde{P}_{10,12}$	$L_2+K_1-P_2,\ L_1-K_2-P_1+aP_2,\ P_0-P_3\quad (a>0)$		

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A_{3,1}$	$\tilde{P}_{10,13}$	$L_2+K_1+P_2,\ L_1-K_2+P_1+aP_2,\ P_0-P_3$ $(a>0)$	P_0-P_3	nilpotent
	$\tilde{P}_{10,14}$	$L_2+K_1-P_2,\ L_1-K_2-P_1,\ P_0-P_3$		
	$\tilde{P}_{10,15}$	$L_2+K_1+P_2,\ L_1-K_2+P_1,\ P_0-P_3$		
	$\tilde{P}_{14,14}$	$L_2+K_1-\frac{1}{2}(P_0+P_3),\ P_1,\ P_0-P_3$		
	$\tilde{P}_{14,15}$	$L_2+K_1-P_2,\ P_1,\ P_0-P_3$		
	$\tilde{P}_{14,16}$	$L_2+K_1+P_2,\ P_1,\ P_0-P_3$		
	$\tilde{P}_{14,17}$	$L_2+K_1-\frac{1}{2}(P_0+P_3),\ P_2+bP_1,\ P_0-P_3$ $(b\neq 0)$		
	$\tilde{P}_{14,18}$	$L_2+K_1-P_2,\ P_2+bP_1,\ P_0-P_3$ $(b\neq 0)$		
	$\tilde{P}_{14,19}$	$L_2+K_1+P_2,\ P_2+bP_1,\ P_0-P_3$ $(b\neq 0)$		
$A_{3,2}$	$\tilde{P}_{8,14}$	$L_2+K_1,\ K_3+aP_1,\ P_0-P_3$ $(a>0)$	$(P_0-P_3)\exp\left(-\frac{1}{a}\frac{L_2+K_1}{P_0-P_3}\right)$	solvable
	$\tilde{P}_{8,16}$	$L_2+K_1,\ K_3+bP_1+aP_2,\ P_0-P_3$ $(a>0,b\neq 0)$	$(P_0-P_3)\exp\left(-\frac{1}{b}\frac{L_2+K_1}{P_0-P_3}\right)$	
$A_{3,3}$	$P_{7,5}$	$K_3,\ L_2+K_1,\ L_1-K_2$	$\frac{L_1-K_2}{L_2+K_1}$	solvable, $D \cong T_2$
	$P_{8,7}$	$K_3,\ L_2+K_1,\ P_0-P_3$	$\frac{P_0-P_3}{L_2+K_1}$	

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A_{3,3}$	$\tilde{P}_{8,15}$	$K_3+aP_2,\ L_2+K_1,\ P_0-P_3\quad (a>0)$	$\frac{P_0-P_3}{L_2+K_1}$	solvable, $D \boxtimes T_2$
$A_{3,4}$	$P_{11,4}$	$\cos c L_3+\sin c K_3,\ P_0,\ P_3\quad (0<c<\pi, c\neq\pi/2)$	$P_0^2-P_3^2$	solvable, E(1,1)
	$P_{13,4}$	$K_3,\ P_0,\ P_3$		
	$\tilde{P}_{13,12}$	$K_3+aP_2,\ P_0,\ P_3\quad (a>0)$		
$A_{3,6}$	$P_{6,4}$	$L_3,\ L_2+K_1,\ L_1-K_2$	$(L_2+K_1)^2+(L_1-K_2)^2$	solvable, E(2)
	$\tilde{P}_{6,9}$	$L_3+\frac{1}{4}(P_0-P_3),\ L_2+K_1,\ L_1-K_1$		
	$\tilde{P}_{6,10}$	$L_3-\frac{1}{4}(P_0-P_3),\ L_2+K_1,\ L_1-K_1$		
	$P_{11,3}$	$\cos c L_3+\sin c K_3,\ P_1,\ P_2\quad (0<c<\pi, c\neq\pi/2)$	$P_1^2+P_2^2$	
	$\tilde{P}_{12,15}$	$L_3+\frac{1}{4}(P_0+P_3),\ P_1,\ P_2$		
	$\tilde{P}_{12,16}$	$L_3-\frac{1}{4}(P_0+P_3),\ P_1,\ P_2$		
	$\tilde{P}_{12,17}$	$L_3+aP_0,\ P_1,\ P_2\quad (a>0)$		
	$\tilde{P}_{12,18}$	$L_3+bP_3,\ P_1,\ P_2\quad (b\neq 0)$		
$A_{3,7}$	$P_{5,4}$	$\cos c L_3+\sin c K_3,\ L_2+K_1,\ L_1-K_2\quad (0<c<\pi, c\neq\pi/2)$	$[(L_2+K_1)^2+(L_1-K_2)^2]\left\{\frac{L_2+K_1+i(L_1-K_2)}{L_2+K_1-i(L_1-K_2)}\right\}^{i\tan c}$	solvable, S(3)

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A_{3,8}$	$P_{4,4}$	L_3, K_1, K_2	$L_3^2-K_1^2-K_2^2$	semisimple, SU(1,1)
$A_{3,9}$	$P_{3,4}$	L_1, L_2, L_3	$L_1^2+L_2^2+L_3^2$	semisimple, SU(2)
$4A_1$	$P_{15,1}$	P_0, P_1, P_2, P_3	all generators	Abelian, T_4
A_2+2A_1	$P_{13,2}$	K_3, P_0-P_3, P_1, P_2	P_1, P_2	
$A_{3,1}+A_1$	$P_{10,3}$	L_2+K_1, L_1-K_2, P_0-P_3, P_2	L_2+K_1, P_0-P_3	
	$P_{14,2}$	L_2+K_1, P_0-P_3, P_1, P_2	P_0-P_3, P_2	
	$\tilde{P}_{10,9}$	L_2+K_1, $L_1-K_2-P_1$, P_0-P_3, P_2	L_2+K_1, P_0-P_3	
	$\tilde{P}_{10,10}$	L_2+K_1, $L_1-K_2+P_1$, P_0-P_3, P_2	L_2+K_1, P_0-P_3	
	$\tilde{P}_{14,10}$	$L_2+K_1-\frac{1}{2}(P_0+P_3)$, P_0-P_3, P_1, P_2	P_0-P_3, P_2	
$A_{3,2}+A_1$	$\tilde{P}_{8,11}$	K_3+aP_2, L_2+K_1, P_0-P_3, P_2 (a>0)	P_2, $(P_0-P_3)\exp\left(\frac{1}{a}\frac{L_2+K_1}{P_0-P_3}\right)$	
$A_{3,3}+A_1$	$P_{8,4}$	K_3, L_2+K_1, P_0-P_3, P_2	P_2, $\frac{L_2+K_1}{P_0-P_3}$	
$A_{3,4}+A_1$	$P_{9,4}$	L_3, K_3, P_0, P_3	L_3, $P_0^2-P_3^2$	
	$P_{13,3}$	K_3, P_0, P_1, P_3	P_1, $P_0^2-P_3^2$	
	$\tilde{P}_{13,10}$	K_3+aP_2, P_0, P_1, P_3 (a>0)	P_1, $P_0^2-P_3^2$	

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A_{3,6}+A_1$	$P_{6,3}$	$L_3,\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$P_0-P_3,\ (L_2+K_1)^2+(L_1-K_2)^2$	
	$P_{9,3}$	$L_3,\ K_3,\ P_1,\ P_2$	$K_3,\ P_1^2+P_2^2$	
	$P_{12,2}$	$L_3,\ P_0-P_3,\ P_1,\ P_2$	$P_0-P_3,\ P_1^2+P_2^2$	
	$P_{12,3}$	$L_3,\ P_1,\ P_2,\ P_3$	$P_3,\ P_1^2+P_2^2$	
	$P_{12,4}$	$L_3,\ P_0,\ P_1,\ P_2$	$P_0,\ P_1^2+P_2^2$	
	$\tilde{P}_{12,11}$	$L_3+\frac{1}{4}(P_0+P_3),\ P_0-P_3,\ P_1,\ P_2$	$P_0-P_3,\ P_1^2+P_2^2$	
	$\tilde{P}_{12,12}$	$L_3-\frac{1}{4}(P_0+P_3),\ P_0-P_3,\ P_1,\ P_2$	$P_0-P_3,\ P_1^2+P_2^2$	
	$\tilde{P}_{12,13}$	$L_3+aP_0,\ P_1,\ P_2,\ P_3\quad (a>0)$	$P_3,\ P_1^2+P_2^2$	
	$\tilde{P}_{12,14}$	$L_3+bP_3,\ P_0,\ P_1,\ P_2\quad (b\neq 0)$	$P_0,\ P_1^2+P_2^2$	
$A_{3,8}+A_1$	$P_{4,3}$	$L_3,\ K_1,\ K_2,\ P_3$	$P_3,\ L_3^2-K_1^2-K_2^2$	
$A_{3,9}+A_1$	$P_{3,3}$	$L_1,\ L_2,\ L_3,\ P_0$	$P_0,\ L_1^2+L_2^2+L_3^2$	
$A_{4,1}$	$P_{14,3}$	$L_2+K_1,\ P_0,\ P_1,\ P_3$	$P_0-P_3,\ P_0^2-P_1^2-P_3^2$	
	$\tilde{P}_{10,7}$	$L_2+K_1-\frac{1}{2}(P_0+P_3),\ L_1-K_2+\frac{b}{2}P_1,$ $P_0-P_3,\ P_2\quad (b\neq 0)$	$P_0-P_3,\ P_2^2+P_3^2-P_0^2+2(P_0-P_3)(L_2+K_1+bP_2)$	

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A_{4,1}$	$\tilde{P}_{10,8}$	$L_2+K_1-\frac{1}{2}(P_0+P_3)$, L_1-K_2, P_0-P_3, P_2	P_0-P_3, $P_2^2+P_3^2-P_0^2+2(P_0-P_3)(L_2+K_1)$	
	$\tilde{P}_{14,11}$	$L_2+K_1-\frac{1}{2}P_2$, P_0, P_1, P_3	P_0-P_3, $P_1^2+P_3^2-P_0^2$	
	$\tilde{P}_{14,12}$	$L_2+K_1+\frac{1}{2}P_2$, P_0, P_1, P_3	P_0-P_3, $P_1^2+P_3^2-P_0^2$	
$A^1_{4,2}$	$\tilde{P}_{7,7}$	K_3+aP_1, L_2+K_1, L_1-K_2, P_0-P_3 $(a>0)$	$\frac{L_1-K_2}{P_0-P_3}$, $(L_1-K_2)\exp\frac{B_3}{a(P_0-P_3)}$	
$A^{1,1}_{4,5}$	$P_{7,4}$	K_3, L_2+K_1, L_1-K_2, P_0-P_3	$\frac{L_2+K_1}{P_0-P_3}$, $\frac{L_1-K_2}{P_0-P_3}$	
$A^{\tan c,0}_{4,6}$	$P_{11,2}$	$\cos c L_3+\sin K_3$, P_0-P_3, P_1, P_2 $(0<c<\pi, c\neq\pi/2)$	$P_1^2+P_2^2$, $(P_0-P_3)^{i\cot c}(P_1-iP_2)+(P_0-P_3)^{-i\cot c}(P_1+iP_2)$	
$A^{a,a}_{4,6}$ $(a \equiv \tan c)$	$P_{5,3}$	$\cos c L_3+\sin c K_3$, L_2+K_1, L_1-K_2, P_0-P_3 $(0<c<\pi, c\neq\pi/2)$	$[(L_2+K_1)^2+(L_1-K_2)^2](P_0-P_3)^{-2}$, $[L_1-K_2-i(L_2+K_1)](P_0-P_3)^{i\cot c-1}$ + $+ [L_1-K_2+i(L_2+K_1)](P_0-P_3)^{-1-i\cot c}$	
$A_{4,10}$	$\tilde{P}_{6,7}$	L_3, $L_2+K_1-P_2$, $L_1-K_2-P_1$, P_0-P_3	P_0-P_3, $L_3(P_0-P_3)+\frac{1}{4}(L_2+K_1-P_2)^2+\frac{1}{4}(L_1-K_2-P_1)^2$	
	$\tilde{P}_{6,8}$	L_3, $L_2+K_1+P_2$, $L_1-K_2+P_1$, P_0-P_3	P_0-P_3, $L_3(P_0-P_3)+\frac{1}{4}(L_2+K_1+P_2)^2+\frac{1}{4}(L_1-K_2+P_1)^2$	
$A^0_{4,9}$	$P_{8,5}$	K_3, L_2+K_1, P_0-P_3, P_1	none	
	$P_{8,6}$	K_3, L_2+K_1, P_0-P_3, P_2+bP_1 $(b\neq0)$	none	

TABLE II. (cont.)

Class of isomorphism	Notation Ref. 1	Generators	Invariants	Comments
$A^0_{4,9}$	$\tilde{P}_{8,12}$	K_3+aP_2, L_2+K_1, P_0-P_3, P_1 $(a>0)$	none	
	$\tilde{P}_{8,13}$	K_3+aP_2, L_2+K_1, P_0-P_3, P_2+bP_1 $(a>0, b\neq 0)$	none	
$A_{4,12}$	$P_{2,4}$	L_3, K_3; L_2+K_1, L_1-K_2	none	

TABLE III. Invariants of subalgebras of dim ≥ 5 of the Poincaré Lie algebra.

Subalgebra	Dim	Generators	Invariants	Comments
P_1	10	$L_1, L_2, L_3, K_k, K_2, K_3, P_0, P_1, P_2, P_3$	$m^2 \equiv P_0^2-\vec{P}^2$, $S^{*)}$	Poincaré
$P_{2,1}$	8	$L_1-K_2, L_2+K_1, L_3, K_3, P_0, P_1, P_2; P_3$	m^2, $L_3 - \frac{P_2}{P_0-P_3}(L_2+K_1) - \frac{P_1}{P_0-P_3}(L_1-K_2)$	maximal solvable
$P_{2,2}$	7	$L_1-K_2, L_2+K_1, L_3, K_3, P_0-P_3, P_1, P_2$	$L_3 - \frac{P_2}{P_0-P_3}(L_2+K_1) - \frac{P_1}{P_0-P_3}(L_1-K_2)$	
$P_{3,1}$	7	$L_1, L_2, L_3, P_0, P_1, P_2, P_3$	$m^2, P_0, \vec{L}\cdot\vec{P}$	$E(3) \boxtimes P_0$
$P_{4,1}$	7	$L_3, K_1, K_2, P_0, P_1, P_2, P_3$	$m^2, P_3, L_3P_0-K_1P_2+K_2P_1$	$E(2,1) \boxtimes P_3$
$P_{5,1}$	7	$\cos cL_3+\sin cK_3, L_1-K_2, L_2+K_1, P_0, P_1, P_2, P_3$ $(0<c<\pi, c\neq\pi/2)$	m^2	$S(3) \boxtimes T_4$
$P_{6,1}$	7	$L_3, L_1-K_2, L_2+K_1, P_0, P_1, P_2, P_3$	$m^2, P_0-P_3, \vec{L}\cdot\vec{P}+K_1P_2-K_2P_1-L_3P_0$	$E(2) \boxtimes T_4$
$P_{7,1}$	7	$K_3, L_1-K_2, L_2+K_1, P_0, P_1, P_2, P_3$	m^2	
$P_{1,2}$	6	$L_1, L_2, L_3, K_1, K_2, K_3$	$\vec{L}\cdot\vec{K}, \vec{L}^2-\vec{K}^2$	$SL(2,C) \sim O(3,1)$
$P_{3,2}$	6	$L_1, L_2, L_3, P_1, P_2, P_3$	$\vec{P}^2, \vec{L}\cdot\vec{P}$	$E(3)$
$P_{4,2}$	6	$L_3, K_1, K_2, P_0, P_1, P_2$	$P_0^2-P_1^2-P_2^2, L_3P_0+K_2P_1-K_1P_2$	$E(2,1)$
$P_{5,2}$	6	$\cos cL_3+\sin cK_3, L_1-K_2, L_2+K_1, P_0-P_3, P_1, P_2$	none	
$P_{6,2}$	6	$L_3, L_1-K_2, L_2+K_1, P_0-P_3, P_1, P_2$	$P_0-P_3, (P_0-P_3)L_3-P_1(L_1-K_2)-P_2(L_2+K_1)$	

*) S denotes the square of the Pauli-Lubanski spin operator.

TABLE III. (cont.)

Subalgebra	Dim	Generators	Invariants	Comments
$P_{7,2}$	6	K_3, L_1-K_2, L_2+K_1, P_0-P_3, P_1, P_2	none	
$P_{8,1}$	6	K_3, L_2+K_1, P_0, P_1, P_2, P_3	m^2, P_2	
$P_{9,1}$	6	L_3, K_3, P_0, P_1, P_2, P_3	m^2, $P_0^2-P_3^2$	
$P_{10,1}$	6	L_1-K_2, L_2+K_1, P_0, P_1, P_2, P_3	m^2, P_0-P_3	
$\tilde{P}_{6,5}$ $(\varepsilon=1)$ $\tilde{P}_{6,6}$ $(\varepsilon=-1)$	6	$L_3+\frac{\varepsilon}{4}(P_0+P_3)$, L_1-K_2, L_2+K_1, P_1, P_2, P_0-P_3	$m^2+4\varepsilon(P_0L_3-P_2K_1+P_1K_2-\vec{L}\cdot\vec{P})$, P_0-P_3	
$P_{2,3}$	5	L_1-K_2, L_2+K_1, L_3, K_3, P_0-P_3	$\frac{(L_1-K_2)^2+(L_2+K_1)^2}{(P_0-P_3)^2}$	
$P_{7,3}$	5	K_3, L_1-K_2, L_2+K_1, P_0-P_3, P_2	$\frac{L_2+K_1}{P_0-P_3}$	
$P_{8,2}$	5	K_3, L_2+K_1, P_0-P_3, P_1, P_2	P_2	
$P_{8,3}$	5	K_3, L_2+K_1, P_0, P_1, P_3	$P_0^2-P_1^2-P_3^2$	
$P_{9,2}$	5	L_3, K_3, P_0-P_3, P_1, P_2	$P_1^2+P_2^2$	
$P_{10,2}$	5	L_1-K_2, L_2+K_1, P_0-P_3, P_1, P_2	P_0-P_3	
$P_{11,1}$	5	$\cos cL_3+\sin cK_3$, P_0, P_1, P_2, P_3 $(c\neq\pi/2, 0<c<\pi)$	m^2, $P_0^2-P_3^2$, $(P_0-P_3)^{2\cot c}\left(\frac{P_2+iP_1}{P_2-iP_1}\right)^i$	

TABLE III. (cont.)

Subalgebra	Dim	Generators	Invariants	Comments
$P_{12,1}$	5	L_3, P_0, P_1, P_2, P_3	m^2, P_0, P_3	
$P_{13,1}$	5	K_3, P_0, P_1, P_2, P_3	m^2, P_1, P_2	
$P_{14,1}$	5	$L_2+K_1, P_0, P_1, P_2, P_3$	m^2, P_0-P_3, P_2	
$\tilde{P}_{7,6}$	5	$K_3+aP_1, L_1-K_2, L_2+K_3, P_0-P_3, P_2$ $(a>0)$	$(P_0-P_3)^a \exp\left(-2\,\frac{L_2+K_1}{P_0-P_3}\right)$	
$\tilde{P}_{8,10}$	5	$K_3+aP_1, L_2+K_1, P_0, P_1, P_3$ $(a>0)$	$P_0^2-P_1^2-P_3^2$	
$\tilde{P}_{10,6}$	5	$L_3+K_1+\frac{1}{2}(P_0+P_3), L_1-K_2, P_0-P_3, P_1, P_2$	P_0-P_3	