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EFFECTS OF RADIATIVE DIFFUSION ON THE SPIN-FLIP IN ELECTRON STORAGE RINGS

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A radiative polarization process in storage rings polarizes electrons in a definite direction determined by the location of the particles in the ring and by the structure of the ring. From the experimental point of view, it is desired to change the direction of the electron spins as easily as possible. It is known that in a ring with low beam energy the spin can be flipped by imposing a perturbation resonating with the spin motion. In high-energy rings, however, the effect of the synchrotron radiation becomes important in the above spin-flip process. This effect is quantitatively estimated in the present paper, together with the effect of synchrotron oscillations which exists in proton rings as well. The results are summarized in a few formulae suitable for designing the spin-flipping devices.

1. INTRODUCTION

In electron storage rings, electrons become polarized due to spin-flip synchrotron radiation in a direction which is determined by the structure of the ring and by the location of the electrons in the ring. In the usual planar rings, this direction is vertical all around the ring. Since longitudinal polarization is desired for high-energy physics, ring designs are being studied in some laboratories in which the polarization is longitudinal at the collision points.

From the experimental standpoint, however, not only longitudinal polarization, but also helicities whose sign can easily be changed are desired. Unfortunately, this is very difficult to achieve. For instance, in the present proposal of HERA,[1] some magnets must be moved in order to change the sign of the helicity. In the so-called vertical S-bend scheme it seems impractical to change the helicity because not only the magnets, but also the detectors must be rearranged. Moreover, in either case, the helicities of electron and positron beams are always the same at a given collision point and it is impossible to collide beams with opposite helicities.

The Novosibirsk[2] group has proposed a method in which the spin can be flipped through adiabatic resonance crossing by an artificial perturbation synchronized with the spin-precession frequency. This has been done in VEPP-2M and satisfactory results have been obtained. In this method, experiments with both signs of helicities can be done without moving magnets and detectors. Moreover, it is possible in principle to invert the polarization either of electron or positron beams by using travelling waves. A drawback is that the polarization is restored to its former direction in a time of the order of the polarization time after the spin-flip. But long experimental time can effectively be obtained by repeating this process.

A problem in applying this method to rings of higher energy has also been pointed out by the same group.[2] During the spin-flip process, the coherence of the spin phase will be lost by the stochastic change of particle energy due to synchrotron radiation and the beam may be depolarized. In order to avoid this phenomenon, the spin must be flipped fast enough. Hence a very large perturbation is required to keep the adiabaticity of the resonance crossing.

Their argument may be summarized as follows. The variation of the spin precession phase Φ can be expressed as

$$\frac{d\Phi}{d\theta} = \gamma a = \gamma_0 a(1 + \epsilon), \tag{1.1}$$

where θ is the generalized azimuth of the machine, γ and γ_0 are the Lorentz factors of the relevant particle and the equilibrium particle, ϵ is equal to $(\gamma - \gamma_0)/\gamma_0$, and a is the coefficient of the anomalous magnetic moment. Let $\Delta\epsilon_i$ be the change of ϵ due to a synchrotron radiation at $\theta = \theta_i$. Then the spin phase will be shifted by the amount

$$\gamma_0 a \Delta\epsilon_i \int_{\theta_i}^{\theta} \cos \nu_s(\theta - \theta_i) d\theta = \frac{\gamma_0 a}{\nu_s} \Delta\epsilon_i \sin \nu_s(\theta - \theta_i),$$

at the azimuth θ after the emission. Here the synchrotron-oscillation tune is denoted by ν_s. Summing up all the effects of radiations between $\theta = 0$ and $\theta = \theta$, we have the total change of phase

$$\Delta\Phi(\theta) = \frac{\gamma_0 a}{\nu_s} \sum_{0<\theta_i<\theta} \Delta\epsilon_i \sin \nu_s(\theta - \theta_i). \tag{1.2}$$

Averaging $\Delta\Phi^2(\theta)$ over many radiations, we get

$$\langle \Delta\Phi^2(\theta) \rangle = \left(\frac{\gamma_0 a}{\nu_s}\right)^2 \langle \Delta\epsilon_i^2 \rangle \frac{dN}{d\theta} \int_0^{\theta} d\theta' \sin^2 \nu_s(\theta - \theta'). \tag{1.3}$$

Here $dN/d\theta$ is the mean number of emitted photons during one radian of θ. Ignoring the periodic term after integration, we get

$$\langle \Delta\Phi^2(\theta) \rangle = \left(\frac{\gamma_0 a}{\nu_s}\right)^2 \langle \Delta\epsilon_i^2 \rangle \frac{dN}{d\theta} \cdot \frac{\theta}{2}. \tag{1.4}$$

Using the relation

$$\langle \Delta\epsilon_i^2 \rangle \frac{dN}{d\theta} = \frac{11}{9} \frac{T_{\text{rev}}}{\tau_0} \frac{1}{2\pi}, \tag{1.5}$$

where T_{rev} is the revolution period and τ_0 is the radiative polarization time for the ideal ring, we find the expectation value of the square of the spin phase shift during $n(= \theta/2\pi)$ revolutions

$$\langle \Delta\Phi^2 \rangle = \frac{11}{9} \left(\frac{\gamma_0 a}{\nu_s}\right)^2 \frac{T_{\text{rev}}}{\tau_0} n. \tag{1.6}$$

The right-hand side of this equation has strong dependence on the beam energy (approximately the seventh power) for a given ring. Hence at high energies n must be very small in order to keep $\langle \Delta\Phi^2 \rangle$ below unity, i.e., the resonance must be crossed very quickly.

More quantitiative discussion is required for the actual design of the spin-flipper. In addition, the influence of non-diffusive synchrotron oscillations that the particle had before the resonance crossing may not be negligible because the synchrotron oscillation tune is generally very large in high-energy electron storage rings. The present paper deals with these problems. Effects of non-diffusive synchrotron oscillations are studied in Section 2 and discussions with diffusion are given in Section 3. A summary of the results and numerical examples are given in Section 4. All the mathematical complications are left to an appendix.

2. INFLUENCE OF SYNCHROTRON OSCILLATIONS

In this section the effects of synchrotron oscillations without radiative diffusion are discussed. Hence the results can be applied to the spin-flip of proton beams if the assumed parameters lie in the allowable range.

The equation of the spin motion can generally be written as

$$\frac{d\mathbf{s}}{d\theta} = (\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_F + \boldsymbol{\Omega}_\epsilon) \times \mathbf{s}. \tag{2.1}$$

Here, $\boldsymbol{\Omega}_0$ is the spin precession frequency vector for the equilibrium particle, $\boldsymbol{\Omega}_F$ is the driving force of the spin-flipping and $\boldsymbol{\Omega}_\epsilon$ is the contribution of the relative energy error ϵ. Since $\boldsymbol{\Omega}_0$ is a periodic function of θ with period 2π, the equation for the equilibrium particle

$$\frac{d\mathbf{s}}{d\theta} = \boldsymbol{\Omega}_0 \times \mathbf{s} \tag{2.2}$$

has three right-handed orthonormal solutions $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, one of which, $\mathbf{n}_3$, is periodic. The other solutions, $\mathbf{n}_1$ and $\mathbf{n}_2$, have the periodicity

$$\mathbf{n}_1(\theta + 2\pi) + i\mathbf{n}_2(\theta + 2\pi) = \exp(i2\pi\nu_0)\cdot[\mathbf{n}_1(\theta) + i\mathbf{n}_2(\theta)], \tag{2.3}$$

where ν_0 is the spin-tune for the equilibrium particle. In planar machines, $\mathbf{n}_3$ is directed to the vertical axis.

Let us rewrite Eq. (2.1) using spinor representation. We define two-component spinor $\Psi(\theta)$ by

$$\mathbf{s} = \sum_{j=1}^{3} \Psi^*\sigma_j\Psi\cdot\mathbf{n}_j, \tag{2.4}$$

where σ_j's are Pauli matrices and asterisks denote Hermitian conjugates on spinors and matrices and complex conjugate on scalars. Using the fact that $\mathbf{n}_j$'s satisfy Eq. (2.2), one can easily verify that the equation

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\sum_{j=1}^{3} (\boldsymbol{\Omega}_\epsilon + \boldsymbol{\Omega}_F)\cdot\mathbf{n}_j\sigma_j\Psi \tag{2.5}$$

is equivalent to Eq. (2.1). This equation can be written using the representation of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.6}$$

as

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{bmatrix} (\boldsymbol{\Omega}_\epsilon + \boldsymbol{\Omega}_F)\cdot \mathbf{n}_3 & (\boldsymbol{\Omega}_\epsilon + \boldsymbol{\Omega}_F)\cdot(\mathbf{n}_1 - i\mathbf{n}_2) \\ (\boldsymbol{\Omega}_\epsilon + \boldsymbol{\Omega}_F)\cdot(\mathbf{n}_1 + i\mathbf{n}_2) & -(\boldsymbol{\Omega}_\epsilon + \boldsymbol{\Omega}_F)\cdot \mathbf{n}_3 \end{bmatrix}\Psi. \tag{2.7}$$

We make the following assumptions on $\boldsymbol{\Omega}_\epsilon$. Since $\boldsymbol{\Omega}_\epsilon \cdot \mathbf{n}_1$ and $\boldsymbol{\Omega}_\epsilon \cdot \mathbf{n}_2$ oscillate rapidly, we will neglect them. The term $\boldsymbol{\Omega}_\epsilon \cdot \mathbf{n}_3$ gives the dependence of spin-tune ν on ϵ. It is given by[3]

$$\left(\frac{d\nu}{d\epsilon}\right)_{\epsilon=0} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \boldsymbol{\Omega}_\epsilon}{\partial \epsilon}\cdot \mathbf{n}_3 \, d\theta.$$

Neglecting rapidly oscillating components, we assume

$$\boldsymbol{\Omega}_\epsilon \cdot \mathbf{n}_3 = -\epsilon\left(\frac{d\nu}{d\epsilon}\right)_{\epsilon=0} \tag{2.8}$$

In actual rings the expression

$$\left(\frac{d\nu}{d\epsilon}\right)_{\epsilon=0} = \gamma_0 a \tag{2.9}$$

is a very good approximation. Hence we use this expression, although the results can easily be extended to a general case. From now on, we will abbreviate γ_0 by γ.

Next, we make assumptions for $\boldsymbol{\Omega}_F$. We will ignore $\boldsymbol{\Omega}_F \cdot \mathbf{n}_3$, since it gives merely a weak modulation of the spin phase and does not contribute to spin-flip. As for $\boldsymbol{\Omega}_F \cdot (\mathbf{n}_1 + i\mathbf{n}_2)$, we will keep resonating terms only

$$\boldsymbol{\Omega}_F \cdot (\mathbf{n}_1 + i\mathbf{n}_2) = -\nu_1 \exp i[(\nu_0 + n)\theta + \phi_F(\theta)] \tag{2.10}$$

as is usually done. Here ν_1 is a real positive constant, n is an integer and $\phi_F(\theta)$ is the phase of the spin-flipper, which varies as

$$\phi_F(\theta) = \phi_{F0} + f_0\theta - \frac{1}{2}\alpha\theta^2, \tag{2.11}$$

where $f_0 = -\nu_0 - n$ is the frequency (in units of revolution frequency) of the flipper at the very instant of resonance crossing and α is a constant that gives the speed of crossing. We will assume that α is positive, but the results for negative α can be obtained simply by replacing α with $|\alpha|$ in the final expressions of depolarization.

Under the above assumptions, Eq. (2.7) can be written as

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{bmatrix} -\gamma a\epsilon & -\nu_1 \exp\left(+\frac{i}{2}\alpha\theta^2 - i\phi_{F0}\right) \\ -\nu_1 \exp\left(-\frac{i}{2}\alpha\theta^2 + i\phi_{F0}\right) & \gamma a\epsilon \end{bmatrix}\Psi \quad (2.12)$$

We have derived this equation in a general way since the ring we treat here is not planar in general, but the same equation can be obtained for planar rings. (See, for example, Ref. 4.)

Replacing the independent variable θ with $t = \theta(\alpha)^{1/2}$ and rotating the axis around $\mathbf{n}_3$ as

$$\Psi = \exp\left[+i\left(\frac{1}{4}\alpha\theta^2 - \frac{1}{2}\phi_{F0}\right)\sigma_3\right]\psi, \quad (2.13)$$

we get an equation for ψ

$$\frac{d\psi}{dt} = \frac{i}{2}(H_0 + \Delta H)\psi, \quad (2.14)$$

with

$$H_0 = \begin{pmatrix} -t & b \\ b & t \end{pmatrix}, \quad (2.15)$$

$$\Delta H = \begin{pmatrix} \gamma a\epsilon/\sqrt{\alpha} & 0 \\ 0 & -\gamma a\epsilon/\sqrt{\alpha} \end{pmatrix}, \quad (2.16)$$

and

$$b = \nu_1/\sqrt{\alpha}.$$

Substituting with

$$\epsilon = \epsilon_{\max}\cos(\nu_s\theta + \phi_0) = \epsilon_{\max}\cos(\mu t + \phi_0), \quad (2.17)$$

where $\epsilon_{\max}$ is the amplitude, ν_s the synchrotron-oscillation tune, ϕ_0 the phase at the moment of resonance crossing and μ is defined by $\mu = \nu_s/\sqrt{\alpha}$, we have

$$\Delta H = u\cos(\mu t + \phi_0)\sigma_3, \quad (2.18)$$

with

$$u = \gamma a\epsilon_{\max}/\sqrt{\alpha}.$$

We are going to solve the differential Eq. (2.14) approximately with Eqs. (2.15) and

(2.18). It contains four parameters, b, u, μ and ϕ_0. In the following we employ the approximations $b \gtrsim 1$ and $u \lesssim 1$. The former says that the polarization is inverted almost completely if there is no synchrotron oscillation. Indeed, the exact solution in the absence of ΔH has been given by Froissart and Stora.[5] When the beam is completely polarized along the direction of $\mathbf{n}_3$ at $t = -\infty$, the polarization at $t = +\infty$ is given by

$$P_{FS} = 2e^{(-\pi/2)b^2} - 1. \tag{2.19}$$

The latter condition is necessary because we will employ the perturbation expansion in terms of ΔH.

Now let us denote the two independent solutions to the unperturbed equation

$$\frac{d\psi}{dt} = \frac{i}{2} H_0 \psi \tag{2.20}$$

by $\psi_1(t)$ and $\psi_2(t)$ which correspond to complete polarization in the direction $+\mathbf{n}_3$ and $-\mathbf{n}_3$, respectively, at $t = -\infty$. At $t = +\infty$ these solutions show almost complete polarization in the opposite direction because of the condition $b \gtrsim 1$. They satisfy the orthogonality relation

$$\psi_i^*(t)\psi_j(t) = \delta_{ij} \qquad (i, j = 1, 2) \tag{2.21}$$

at any t, where δ_{ij} is the Kronecker delta. In addition, it can easily be verified that the spin vectors for these solutions are always opposite, i.e.,

$$\psi_1^*(t)\sigma_j\psi_1(t) + \psi_2^*(t)\sigma_i\psi_2(t) = 0 \qquad (j = 1, 2, 3). \tag{2.22}$$

Let us expand the solution $\psi(t)$ to the perturbed Eq. (2.14) in terms of the unperturbed solutions as

$$\psi(t) = C_1(t)\psi_1(t) + C_2(t)\psi_2(t) \tag{2.23}$$

with

$$|C_1(t)|^2 + |C_2(t)|^2 = 1. \tag{2.24}$$

The initial conditions at $t = -\infty$ are

$$C_1(-\infty) = 1 \qquad \text{and} \qquad C_2(-\infty) = 0. \tag{2.25}$$

Because of the synchrotron oscillation, C_1 and C_2 will gradually move away from these values. The spin component along $\mathbf{n}_3$ at t is given by

$$\begin{aligned} \mathbf{s}\cdot\mathbf{n}_3 &= \psi(t)^*\sigma_3\psi(t) = \psi(t)^*\sigma_3\psi(t) \\ &= |C_1|^2\psi_1{}^*\sigma_3\psi_1 + |C_2|^2\psi_2{}^*\sigma_3\psi_2 + 2Re(C_1{}^*C_2\psi_1{}^*\sigma_3\psi_2). \end{aligned} \tag{2.26}$$

We may omit the last term by averaging over many particles. Then, using Eqs. (2.22) and (2.24), we get the beam polarization at $t = +\infty$ as

$$P = (1 - \Delta P)\psi_1{}^*\sigma_3\psi_1$$

with

$$\Delta P = 2\cdot \lim_{t\to+\infty} \langle|C_2(t)|^2\rangle. \tag{2.27}$$

Here $\psi_1{}^*\sigma_3\psi_1$ is just the final polarization P_{FS} given by Froissart and Stora. Hence we have

$$P = (1 - \Delta P)P_{FS}. \tag{2.28}$$

Since we consider only the case where the spin flips almost completely in the absence of synchrotron oscillation, i.e., $P_{FS} \approx -1$, we may think that the depolarization due to synchrotron oscillation is given by ΔP. Now what we have to know is $C_2(\infty)$.

The equation that C_1 and C_2 must satisfy is

$$\frac{d}{dt}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} \psi_1{}^*\Delta H\psi_1 & \psi_1{}^*\Delta H\psi_2 \\ \psi_2{}^*\Delta H\psi_1 & \psi_2{}^*\Delta H\psi_2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{2.29}$$

Taking the first-order perturbation of ΔH and substituting for C_1 and C_2 on the right-hand side with the unperturbed solutions $C_1(t) = 1$ and $C_2(t) = 0$, we get

$$\begin{aligned} C_2(t) &= \frac{i}{2}\int_{-\infty}^{t} \psi_2{}^*(t)\,\Delta H(t)\,\psi_1(t)dt \\ &= \frac{i}{2}u\int_{-\infty}^{t} \cos(\mu t + \phi_0)(\psi_2{}^*\sigma_3\psi_1)dt, \end{aligned}$$

and, therefore,

$$C_2(\infty) = \frac{i}{2}u\int_{-\infty}^{\infty} \cos(\mu t + \phi_0)(\psi_2{}^*\sigma_3\psi_1)dt. \tag{2.30}$$

Now let us introduce the Fourier transform $G(\omega)$ of $\psi_2{}^*\sigma_3\psi_1$ defined by

$$\psi_2{}^*(t)\sigma_3\psi_1(t) = \exp(ic_0)\int_{-\infty}^{\infty} G(\omega)\exp(i\omega t)d\omega$$

and

$$G(\omega) = \frac{1}{2\pi}\exp(-ic_0)\int_{-\infty}^{\infty} \psi_2{}^*(t)\sigma_3\psi_1(t)\exp(-i\omega t)dt, \tag{2.31}$$

where c_0 is a real constant added for convenience. We can choose c_0 so that $G(\omega)$ is real when $b \gg 1$. By using $G(\omega)$ we can rewrite Eq. (2.30) as

$$C_2(\infty) = i\frac{\pi u}{2}\exp(ic_0)[\exp(i\phi_0)G(\mu) + \exp(-i\phi_0)G(-\mu)]. \tag{2.32}$$

Averaging the absolute square of this expression over the initial phase of the synchrotron oscillation ϕ_0, we get

$$\Delta P = \frac{(\pi u)^2}{2}(|G(\mu)|^2 + |G(-\mu)|^2). \tag{2.33}$$

Since $G(\omega)$ is very small for $\omega < 0$ as stated below, the following expression is sufficient in practice

$$\Delta P = \frac{(\pi u)^2}{2}|G(\mu)|^2 = \frac{(\pi\gamma a\epsilon_{\max})^2}{2\alpha}\left|G\left(\frac{\nu_s}{\sqrt{\alpha}}\right)\right|^2. \tag{2.34}$$

If we average this expression over the distribution of $\epsilon_{\max}$, we get finally

$$\Delta P = \frac{(\pi\gamma a\epsilon_{\mathrm{rms}})^2}{\alpha} G\left(\frac{\nu_s}{\sqrt{\alpha}}\right)^2, \tag{2.35}$$

where ϵ_{rms} is the r.m.s. relative energy spread.

The calculation of $G(\omega)$ is given in the appendix. It is plotted in Fig. 1 for several values of b. When b is large, $G(\omega)$ for $\omega \lesssim 0$ is very small. (The value $G(0)$ is of the order of $b^{1/3}\exp(-\pi b^2/4)$.) It increases rapidly with ω when $\omega > 0$ and reaches a maximum at $\omega_{\max}$, which approaches b from above as b goes to infinity. For $\omega > \omega_{\max}$, $G(\omega)$ oscillates with the amplitude decreasing as $1/\omega$.

An example is shown in Fig. 2 where ΔP calculated by Eq. (2.34) is plotted in a full line against μ with u fixed to 0.5. The crosses show the results of computer simulation or, in more proper words, numerical integration of the differential Eq. (2.14). They agree with each other quite well except around $\mu = 2$ to 3 where ΔP is very large and, therefore, our perturbation approximation is not very good.

A remarkable fact seen from this figure is that the depolarization ΔP does not monotonically increase with synchrotron-oscillation frequency. This behavior of ΔP can be understood as a kind of resonance between the synchrotron oscillation and the spin precession during the spin-flip. The first peak corresponds to $\mu \approx b$ or, equivalently, $\nu_s \approx \nu_1$. This is due to the fact that the spin precesses around $\mathbf{n}_1$ (or $\mathbf{n}_2$) at frequency ν_1 at the very moment of the resonance crossing. At arbitrary time, however, the instantaneous spin precession frequency $\omega(\theta)$ (in units of the revolution frequency) is equal to $({\nu_1}^2 + \alpha^2\theta^2)^{1/2}$, since the spin rotates at the frequency $\alpha\theta$ around $\mathbf{n}_3$ and at ν_1 around $\mathbf{n}_1$ (or $\mathbf{n}_2$). This spin motion contains frequency components higher than ν_1, which explains the complicated behavior for $\mu \gtrsim b$. As one can see in the appendix, $G(\omega)$ is essentially the Fourier transform of $\exp(i\int\omega(\theta)d\theta)$. A similar resonance-like behavior has been found by R. D. Ruth[8] in the case of weak resonances, i.e., spin-nonflip.

It is the case that ν_s is usually considerably smaller than ν_1 in proton rings. In future high-energy electron rings, however, this effect may not be negligible, as shown in Section 4, because the synchroton-oscillation tune ν_s is generally very high there.

3. INFLUENCE OF RADIATIVE DIFFUSION

In this section we will discuss the depolarization caused by the diffusion of spin phase due to synchrotron radiations during the process of spin-flip.

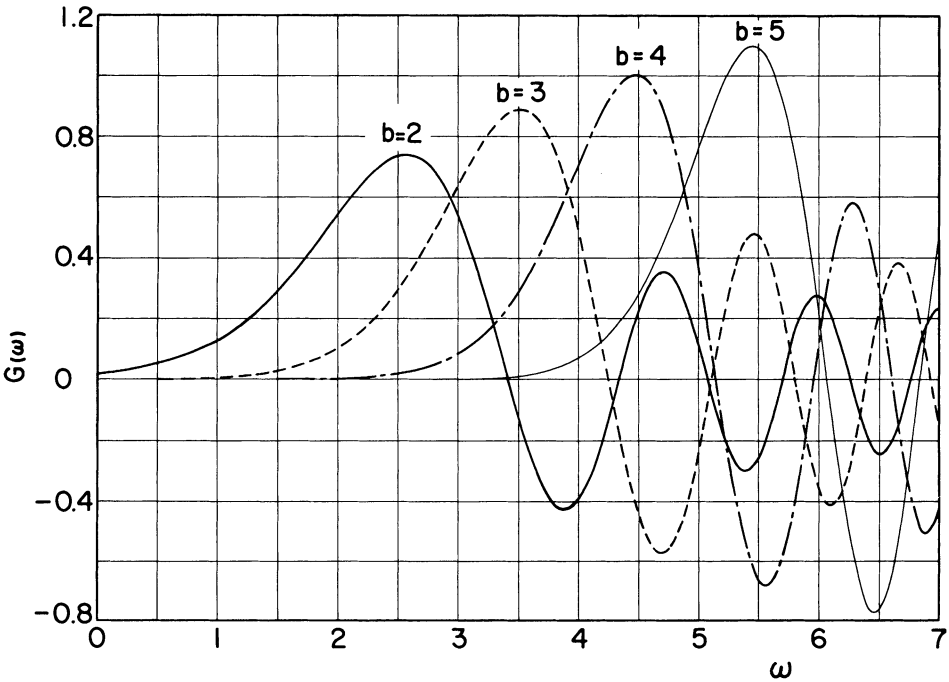

FIGURE 1 Function $G(\omega)$ for several values of b, calculated by numerical integration of Eq. (A.12).

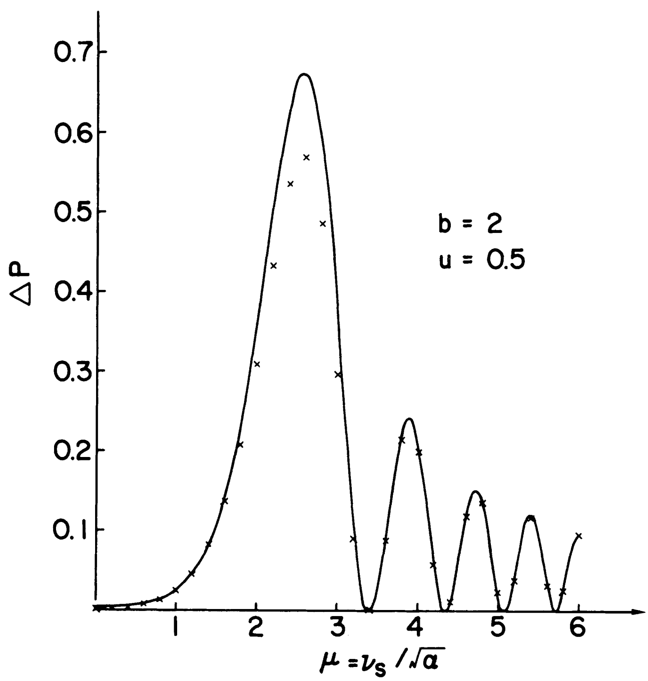

FIGURE 2 An example of the depolarization due to continuous synchrotron oscillations. The full line is plotted using the formula (2.34) and the crosses show the results of the numerical integration of the differential Eq. (2.14). The parameters used are $b = 2$ and $u = 0.5$.

The starting point is Eq. (2.14) with Eqs. (2.15) and (2.16). In the present case, ϵ in Eq. (2.16) can be written as

$$\epsilon(t) = \sum_j \Delta\epsilon_j \cos \mu(t - t_j)\cdot\Theta(t - t_j)\exp[-\lambda(t - t_j)]. \tag{3.1}$$

Here $\Delta\epsilon_j$ is the change of ϵ due to the radiation at $t = t_j$, Θ is the unit step function and λ the damping constant, which is related to the longitudinal damping time τ_ϵ by

$$\tau_\epsilon = \frac{T_{\text{rev}}}{2\pi\lambda\sqrt{\alpha}}. \tag{3.2}$$

In a manner similar to that of the previous section, we find the perturbation of $C_2(\infty)$ up to the first order of ΔH as

$$C_2(\infty) = \frac{i}{2}\frac{\gamma a}{\sqrt{\alpha}}\sum_j \Delta\epsilon_j \int_{-\infty}^{\infty} \Theta(t - t_j)\cos\mu(t - t_j)\exp[-\lambda(t - t_j)]$$
$$\cdot\psi_2^*(t)\sigma_3\psi_1(t)dt. \tag{3.3}$$

Making use of a Fourier-transformation formula

$$\cos \mu t e^{-\lambda t}\Theta(t) = \int_{-\infty}^{\infty} V(\omega)e^{i\omega t}d\omega, \tag{3.4}$$

with

$$V(\omega) = \frac{1}{2\pi}\frac{-i\omega}{\omega^2 - \mu^2 - 2i\lambda\omega}, \tag{3.5}$$

we can rewrite Eq. (3.3) as

$$C_2(\infty) = i\pi\frac{\gamma a}{\sqrt{\alpha}}\sum_j \Delta\epsilon_j \int_{-\infty}^{\infty} d\omega\, V(-\omega)G(\omega)\exp(i\omega t). \tag{3.6}$$

Since there is no correlation among different radiations, the absolute square of the expression (3.6) can be averaged as

$$\begin{aligned}\langle|C_2(\infty)|^2\rangle &= \frac{(\pi\gamma a)^2}{\alpha}\left\langle\sum_j \Delta\epsilon_j^2\left|\int_{-\infty}^{\infty} d\omega\, V(-\omega)G(\omega)\exp(i\omega t_j)\right|^2\right\rangle\\
&= \frac{(\pi\gamma a)^2}{\alpha}\langle\Delta\epsilon_j^2\rangle\frac{dN}{d\theta}\frac{d\theta}{dt}\\
&\quad\times\int_{-\infty}^{\infty} dt_j \iint d\omega d\omega'\, V(-\omega)G(\omega)V^*(-\omega')G^*(\omega')\exp\big(it_j(\omega - \omega')\big)\\
&= \frac{(\pi\gamma a)^2}{\alpha}\langle\Delta\epsilon_j^2\rangle\frac{dN}{d\theta}\frac{2\pi}{\sqrt{\alpha}}\int_{-\infty}^{\infty} d\omega\,|V(-\omega)G(\omega)|^2.\end{aligned}$$

Hence, using Eqs. (1.5), (2.27) and (3.5), we get

$$\Delta P = \frac{11}{18}\cdot\frac{T_{\mathrm{rev}}}{\tau_0}\frac{(\gamma a)^2}{\alpha\sqrt{\alpha}}\int_{-\infty}^{\infty} d\omega\,\frac{\omega^2}{(\omega^2-\mu^2)^2+4\lambda^2\omega^2}|G(\omega)|^2. \tag{3.7}$$

Since $G(\omega)$ is very small for negative ω, as stated in the previous section, the lower limit of this integration can be set to zero.

One sees that the integrand of Eq. (3.7) has a sharp peak near $\omega = \mu$ with a width of the order of λ. First, let us consider the physical meaning of this peak. Since λ is much smaller than μ, we may approximate $G(\omega)$ by $G(\mu)$ around the peak. Then the contribution of this peak ΔP_1 is calculated to be

$$\Delta P_1 = \frac{11}{18}\cdot\frac{T_{\mathrm{rev}}}{\tau_0}\frac{(\gamma a)^2}{\alpha\sqrt{\alpha}}\frac{\pi}{4\lambda}|G(\mu)|^2,$$

which can be rewritten by using Eq. (3.2) as

$$\Delta P_1 = \frac{11\pi^2}{36}\frac{\tau_\epsilon}{\tau_0}\frac{(\gamma a)^2}{\alpha}|G(\mu)|^2. \tag{3.8}$$

Using the expression of the equilibrium energy spread in electron storage rings

$$\epsilon_{\mathrm{rms}}^2 = \frac{11}{36}\frac{\tau_\epsilon}{\tau_0}, \tag{3.9}$$

one finds that Eq. (3.8) exactly coincides with Eq. (2.35). Hence the peak at $\omega = \mu$ is the contribution of the energy spread that the beam had before the spin-flip. Indeed, one finds that such a spread has already been included in the expression (3.1) where the summation over j runs to $t_j = -\infty$.

Next, let us consider the integral (3.7) as a whole. In general, we have to carry out numerical integration, but we may integrate approximately under the assumption $\mu \ll b$ which seems to be a practically important region because of the following reason. For rings with beam energy higher than about 20 GeV, the factor $(\pi\gamma a\epsilon_{\max})^2$ in Eq. (2.35) is at least about 0.02. Therefore, if $G(\mu)$ is of the order of unity, i.e., if $\mu \sim b$, sufficient spin-flip cannot be obtained unless α is more than 0.5 or so. On the other hand, b must be larger than about 2 in order to suppress the depolarization expressed by Froissart and Stora's formula. Hence one has to make $\nu_1 = (\alpha)^{1/2}b \gtrsim 1$. But this means that the spin-flipper has a strength that is enough to rotate the spin by more than 360 degrees during one single passage through the flipper. This seems to be very difficult in practice. Therefore, in order to reduce the value of the expression (2.35), one has to make $|G(\mu)| \ll 1$ which is achieved by making μ considerably smaller than b. (As one can see from Fig. 1., $|G(\mu)|$ has dips where $\mu > b$. This point will be discussed later.)

Now, under this assumption, i.e., $\mu \ll b$, $G(\mu)$ is very small and the integrand of Eq. (3.7) consists of two portions which can clearly be distinguished. They are the sharp peak around $\omega \sim \mu$ and the broad bump around $\omega \sim b$. The former is the contribution of the continuous synchrotron oscillation discussed above and the latter is that of the diffusion, which we denote by ΔP_2. Since near $\omega \sim b$ the factor in front of $|G(\omega)|^2$ can be approximated by ω^{-2}, we have

$$\Delta P_2 = \frac{11}{18}\frac{T_{\mathrm{rev}}}{\tau_0}\frac{(\gamma a)^2}{\alpha\sqrt{\alpha}}K(b), \tag{3.10}$$

with

$$K(b) = \int_{1/b}^{\infty} d\omega \frac{1}{\omega^2} |G(\omega)|^2. \tag{3.11}$$

Here we take the lower limit of integration as $1/b$, because the integrand diverges at $\omega = 0$. This value $1/b$ is near the minimum of the integrand. Since $|G|^2$ is a small quantity of the order of $\exp(-\pi b^2/2)$ near the origin, $K(b)$ is insensitive to the value of the lower integration limit. When b is large, evaluation of this integral gives

$$K(b) = \frac{1}{4b}, \tag{3.12}$$

as shown in the appendix. This is a very good approximation. The error is less than 10 percent even at $b = 2$.

Then finally we get

$$\Delta P_2 = 0.15 \frac{T_{\text{rev}}}{\tau_0} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} \frac{1}{b} = 0.15 \frac{T_{\text{rev}}}{\tau_0} \frac{(\gamma a)^2}{\alpha \nu_1}, \tag{3.13}$$

for $\mu \ll b$ or, equivalently, $\nu_s \ll \nu_1$.

A comment should be added on the fact that this formula does not depend on ν_s as long as $\nu_s \ll \nu_1$. The value of $\langle \Delta\Phi^2 \rangle$ estimated in the Introduction is proportional to ν_s^{-2}, which means that a slow synchrotron oscillation gives large diffusion of the spin phase. We have to note, however, that we have dropped the periodic term between Eqs. (1.3) and (1.4) in the course of the estimation of $\langle \Delta\Phi^2 \rangle$. This is allowed only if the relevant time interval is long enough to satisfy $\nu_s\theta \gg 2\pi$. In the opposite case, $\nu_s\theta \ll 2\pi$, $\langle \Delta\Phi^2 \rangle$ does not depend on ν_s. In the actual case, θ should be taken to be the typical time scale of the resonance crossing.

Now, let us consider the possibility of making use of the dips of $|G(\omega)|^2$. When μ is exactly equal to the n-th zero ω_n of $G(\omega)$, ΔP_1 vanishes and by using Eq. (3.7), we get

$$\Delta P_2 = \frac{11}{18} \frac{T_{\text{rev}}}{\tau_0} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} Q_n(b) \qquad (\mu = \omega_n), \tag{3.14}$$

with

$$Q_n(b) = \int_{-\infty}^{\infty} \left(\frac{\omega}{\omega^2 - \omega_n^2} \right)^2 |G(\omega)|^2 \, d\omega. \tag{3.15}$$

In particular, for the first zero, the following formulae derived in the appendix give very good approximations.

$$\omega_1 = b + 1.866\, b^{-1/3} \tag{3.16}$$

$$Q_1(b) = \frac{b}{2} \frac{1}{1 + 1.866\, b^{-4/3}}. \tag{3.17}$$

4. SUMMARY AND NUMERICAL EXAMPLES

Our aim is to find the optimum values of the parameters of the flipper, ν_1 and α, which give a sufficient spin-flip under the given parameters of the storage ring.

Firstly, the depolarization given by Froissart and Stora is

$$\Delta P_{FS} = 2\exp(-\pi b^2/2), \qquad b = \frac{\nu_1}{\sqrt{\alpha}}. \tag{4.1}$$

Secondly, the effect of the continuous synchrotron oscillation is

$$\Delta P_1 = \frac{(\pi\gamma a\epsilon_{\rm rms})^2}{\alpha}|G(\nu_s/\sqrt{\alpha})|^2. \tag{4.2}$$

Here, G is a function of $\mu = \nu_s/\sqrt{\alpha}$ and b. Its coarse values can be read from Fig. 1 and more accurate values can be calculated by Eq. (A.17).

Finally, the contribution of the radiative diffusion is given by

$$\Delta P_2 = 0.15\frac{T_{\rm rev}}{\tau_0}\frac{(\gamma a)^2}{\alpha\nu_1} \quad \text{for} \quad \nu_s \ll \nu_1, \tag{4.3}$$

or by

$$\Delta P_2 = \frac{11}{18}\frac{T_{\rm rev}}{\tau_0}\frac{(\gamma a)^2}{\alpha\sqrt{\alpha}}Q_1(b) \quad \text{for} \quad \frac{\nu_s}{\sqrt{\alpha}} = \omega_1, \tag{4.4}$$

where ω_1 and Q_1 are given by Eqs. (3.16) and (3.17). For other values of ν_s we have to evaluate Eq. (3.7) numerically to get $\Delta P_1 + \Delta P_2$.

As an illustration we take an example of the parameters of the TRISTAN ring[6] in which

$$R = 480\ \text{m},\ \rho = 246\ \text{m} \quad \text{and} \quad \nu_s = 0.1.$$

The polarization time and the energy spread are

$$\tau_0 = 294\left(\frac{25}{E}\right)^5 \text{sec},$$

and

$$\epsilon_{\rm rms} = 1.37 \times 10^{-3}\frac{E}{25}$$

where E is the beam energy in units of GeV.

We adopt $b = 2$ so that ΔP_{FS} is small. Hence what is to be decided is just one parameter. First, consider the case $\nu_s \ll \nu_1$. Equations (4.2) and (4.3) give

$$\Delta P_1 = 5.9\left(\frac{E}{25}\right)^4(\mu G(\mu))^2 \tag{4.5}$$

and

$$\Delta P_2 = 0.83 \times 10^{-2}\left(\frac{E}{25}\right)^7 \mu^3. \tag{4.6}$$

By using Eq. (A.14) for $G(\mu)$ as a very coarse approximation, we get the rough solutions $\mu = \mu_1$ and μ_2 to Eqs. (4.5) and (4.6), respectively, as

$$\mu_2 \sim 0.72\left(\frac{25}{E}\right)^{1.0}\left(\frac{\Delta P_1}{0.02}\right)^{0.25}. \tag{4.7}$$

and

$$\mu_2 \sim 1.34\left(\frac{25}{E}\right)^{7/3}\left(\frac{\Delta P_2}{0.02}\right)^{1/3}. \tag{4.8}$$

The corresponding α's and ν_1's are

$$\alpha_1 \sim 0.020\left(\frac{E}{25}\right)^{2.0}\left(\frac{0.02}{\Delta P_1}\right)^{0.50},$$

$$\alpha_2 \sim 0.0056\left(\frac{E}{25}\right)^{14/3}\left(\frac{0.02}{\Delta P_2}\right)^{2/3},$$

$$\nu_{11} \sim 0.28\left(\frac{E}{25}\right)^{1.0}\left(\frac{0.02}{\Delta P_1}\right)^{0.25},$$

and

$$\nu_{12} \sim 0.15\left(\frac{E}{25}\right)^{7/3}\left(\frac{0.02}{\Delta P_2}\right)^{2/3}.$$

Hence a stronger flipper field ν_1 is required in order to suppress ΔP_1 than to suppress ΔP_2, for our parameters, i.e. the effect of continuous synchrotron oscillations is more serious than that of diffusion. A large value of $\nu_1 \sim 0.3$ is necessary to give the depolarization less than a few percents.

Next, consider the case $\mu = \omega_1$. As in the above case one easily sees that the larger the value of b, the worse is the situation. Hence, we again assume $b = 2$. Then all parameters are fixed as $\alpha = 8 \times 10^{-4}$ and $\nu_1 = 0.057$. In this case ν_1 is small but the depolarization is found to be a very large value

$$\Delta P_2 \sim 1.6 \times \left(\frac{E}{25}\right)^7.$$

Therefore, unless E is less than 15 GeV, this case is far from practical.

We have given a prescription to find optimum values of spin-flipper parameters. Finally, we would like to pay attention to a point which may be important if the resulting value of ν_1 is large. All our results are based on the differential Eq. (2.14), which has been derived by eliminating all Fourier components except the resonating one. If, however, the flipper field is localized at a single point on the ring and is very large, this procedure may not be justified. In that case one has to solve a difference equation rather than a differential equation. This effect can be qualitatively checked by solving the differential equation numerically with wide integration steps. If the depolarization happens to be large even when the values predicted above are small, we have to divide the flipper into many pieces all around the ring. But it is beyond the scope of the present paper to answer generally how many are enough.

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APPENDIX

In this appendix we investigate the properties of Froissart and Stora's solutions, ψ_1 and ψ_2 and the Fourier transform $G(\omega)$.

Among the solutions to the Eq. (2.14) with (2.15), the solutions which show the polarization in the direction $\pm\mathbf{n}_3$ at $t = +\infty$ are given by

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -g^*(t) \\ f^*(t) \end{pmatrix},$$

respectively. Here, $f(t)$ and $g(t)$ are defined by using the parabolic cyclinder function $\cup$ as

$$f(t) = \exp\left(-\frac{\pi}{16}b^2\right)\cup\left(-\frac{1}{2} + \frac{i}{4}b^2, t\exp\left(\frac{\pi}{4}i\right)\right)$$

and (A.1)

$$g(t) = -\frac{b}{2}\exp\left(-\frac{\pi}{16}b^2 + \frac{\pi}{4}i\right)\cup\left(+\frac{1}{2} + \frac{i}{4}b^2, t\exp\left(\frac{\pi}{4}i\right)\right).$$

See Ref. 7 for the definition of $\cup$.

The same functions $f(t)$ and $g(t)$ can be used to express the solutions which show polarization in the direction $\pm \mathbf{n}_3$ at $t = -\infty$;

$$\psi_1(t) = \begin{pmatrix} f(-t) \\ -g(-t) \end{pmatrix}, \qquad \psi_2(t) = \begin{pmatrix} g^*(-t) \\ f^*(-t) \end{pmatrix}. \tag{A.2}$$

These functions have the asymptotic forms at $t \to +\infty$

$$f(t) \sim \exp\left[-\frac{i}{4}(t^2 + b^2 \log t) \right] \times \left(1 + 0(t^{-2})\right)$$

and (A.3)

$$g(t) \sim -\frac{b}{2t} \exp\left[-\frac{i}{4}(t^2 + b^2 \log t) \right] \times \left(1 + 0(t^{-2})\right).$$

Time-inverted functions $f(-t)$ and $g(-t)$ can be expressed by linear combinations of $f(t)$, $g(t)$ and their complex conjugates as

$$\begin{aligned} f(-t) &= C_{11} f(t) + C_{12} g^*(t) \\ g(-t) &= C_{21} f^*(t) + C_{22} g(t) \end{aligned} \tag{A.4}$$

with

$$C_{11} = -C_{22} = \exp\left(-\frac{\pi}{4} b^2 \right)$$

and

$$C_{21} = C_{12} = -i \frac{2\sqrt{2\pi}}{b\Gamma(ib^2/4)} \exp\left(-\frac{\pi}{8} b^2 + \frac{\pi}{4} i \right).$$

This coefficient C_{11} gives the famous formula of Froissart and Stora;

$$P_{FS} = 2|C_{11}|^2 - 1.$$

The asymptotic forms of $f(t)$ and $g(t)$ at $t = -\infty$ can easily be derived by (A.3) and (A.4)

So far, all formulae are exact. But since the case of almost complete spin-flip is enough for our purpose, let us derive approximate formulae which are easier to handle. Substituting $\psi = \left(f(t), g(t)\right)$ into Eq. (2.20) with (2.15), we get

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -t & b \\ b & t \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}. \tag{A.5}$$

Elimination of g from this equation gives

$$\frac{d^2 f}{dt^2} + \left(\frac{i}{2} + \frac{b^2}{4} + \frac{t^2}{4} \right) f = 0. \tag{A.6}$$

Now, under the assumption $b \gtrsim 1$ which means almost complete spin-flip, we employ the WKB approximation which is, in our case, an asymptotic expansion uniform in t for large b. Then two independent solutions to Eq. (A.6) are

$$\left(\frac{i}{2} + \frac{b^2}{4} + \frac{t^2}{4}\right)^{-1/4} \exp\left[\pm i \int_0^t \left(\frac{t^2}{4} + \frac{b^2}{4} + \frac{i}{2}\right)^{1/2} dt\right].$$

Multiplying a constant factor by this expression so that it has the same behavior for $t = \infty$ as Eq. (A.3), we find

$$f(t) = \left(\frac{t + \sqrt{t^2 + b^2}}{b}\right)^{1/2} \exp\left[\frac{i}{8} b^2 \left(1 - \log\frac{b^2}{4}\right)\right] \frac{1}{\sqrt{2}} F^*(t)(1 + 0(b^{-2})), \quad \text{(A.7)}$$

with

$$\begin{aligned} F(t) &= \left(\frac{b^2}{t^2 + b^2}\right)^{1/4} \exp\left[\frac{i}{2} \int_0^t \sqrt{t^2 + b^2}\, dt\right] \\ &= \left(\frac{b^2}{t^2 + b^2}\right)^{1/4} \exp\left[\frac{i}{4}\left(t\sqrt{t^2 + b^2} + b^2 \sinh^{-1}\frac{t}{b}\right)\right], \end{aligned} \quad \text{(A.8)}$$

which satisfies

$$F^*(t) = F(-t).$$

Similarly, for $g(t)$, we get

$$g(t) = -\left(\frac{b}{t + \sqrt{t^2 + b^2}}\right)^{1/2} \exp\left[\frac{i}{8} b^2 \left(1 - \log\frac{b^2}{4}\right)\right] \frac{1}{\sqrt{2}} F^*(t)(1 + 0(b^{-2})). \quad \text{(A.9)}$$

Now what we want to know is $G(\omega)$. Using the relation

$$\psi_2{}^*\sigma_3\psi_1 = 2f(-t)g(-t) = \exp(ic_0)(F(t))^2$$

with

$$c_0 = \frac{1}{4} b^2 \left(1 - \log\frac{b^2}{4}\right) + \pi,$$

we have

$$\begin{aligned} G(\omega) &= \frac{\exp(-ic_0)}{2\pi} \int_{-\infty}^{\infty} \psi_2{}^*\sigma_3\psi_1 \exp(-i\omega t)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (F(t))^2 \exp(-i\omega t)\, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{b^2}{t^2 + b^2}\right)^{1/2} \exp\left[\frac{i}{2}\left(t\sqrt{t^2 + b^2} + b^2 \sinh^{-1}\frac{t}{b}\right) - i\omega t\right] dt. \end{aligned} \quad \text{(A.10)}$$

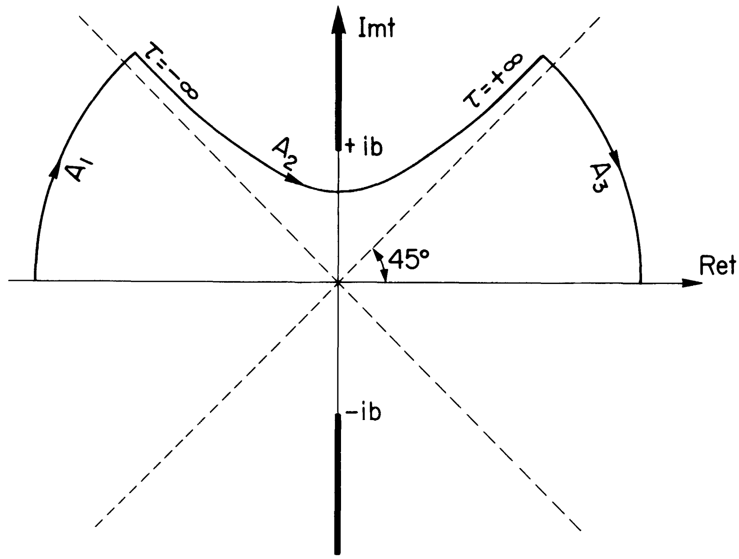

FIGURE 3 The integration contour of Eq. (A.10). The integration along the real axis is replaced with that along $A_1 + A_2 + A_3$. The path A_2 is defined by Eq. (A.11).

This expression is not suitable for numerical evaluation because the phase of the integrand varies rapidly with t. Hence, we deform the integration contour as shown in Fig. 3.

$$\int_{-\infty}^{\infty} = \int_{A1} + \int_{A2} + \int_{A3}.$$

The path A2 approaches $\arg(t) = \pi/4$ and $3\pi/4$ at large $|t|$ and crosses the imaginary axis between the two branch points $\pm ib$ of the integrand. Since the contributions of the arcs A1 and A3 vanish if the radius of the circle is infinitely large, we have

$$\int_{-\infty}^{\infty} = \int_{A2}.$$

The expression becomes simple when we take A2 as

$$t = \exp\left(\frac{\pi}{2} i\right)\left(\frac{1}{2} b^2(1 - i\tau)\right)^{1/2}, \qquad -\infty < \tau < +\infty. \tag{A11}$$

Here and hereafter, the branches of square roots are chosen so that their arguments are larger than $-\pi/2$ and smaller than or equal to $\pi/2$. Then we have

$$G(\omega) = \frac{b}{4\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{1 + \tau^2}} \exp\left(\frac{\omega b}{\sqrt{2}} \sqrt{1 - i\tau} - \frac{\pi b^2}{8} - \frac{b^2}{4} \sqrt{1 + \tau^2} + \frac{ib^2}{4} \sinh^{-1}\tau\right). \tag{A.12}$$

Results of the numerical integration of this expression are plotted in Fig. 1 for several values of b.

The general shape of $G(\omega)$ can be divided into three regions. In the first region $\omega \gtrsim b$, $G(\omega)$ oscillates with ω rapidly. Estimation of Eq. (A.12) by the saddle-point method gives

$$G(\omega) \approx \left(\frac{2}{\pi}\right)^{1/2} \frac{b}{(\omega\sqrt{\omega^2 - b^2})^{1/2}} \cos\frac{1}{2}\left(\omega\sqrt{\omega^2 - b^2} - b^2 \cosh^{-1}\frac{\omega}{b} - \frac{\pi}{2}\right), \quad \text{(A.13)}$$
$$\text{for} \quad \omega \gg b.$$

In the second region $0 \lesssim \omega \lesssim b$, $G(\omega)$ increases exponentially with ω from a small value to a value of order unity. Again the saddle-point method gives

$$G(\omega) \approx \frac{1}{\sqrt{2\pi}} \frac{b}{(\omega\sqrt{b^2 - \omega^2})^{1/2}} \exp\frac{1}{2}\left(\omega\sqrt{b^2 - \omega^2} - b^2 \cos^{-1}\frac{\omega}{b}\right) \quad \text{(A.14)}$$
$$\text{for} \quad 0 \ll \omega \ll b.$$

In the third region $\omega \lesssim 0$, $G(\omega)$ is very small as long as $b \gtrsim 1$. In the vicinity of the origin it can be approximated by

$$G(\omega) \approx \mathrm{Ai}(0) b^{1/3} \exp\left(-\frac{\pi}{4} b^2 + b\omega\right) \times (1 + 0(b^{-4/3})) \quad \text{(A15)}$$
$$\text{for} \quad |\omega| \lesssim 1/b.$$

Here Ai(0) is the value of the Airy function at the origin given by

$$\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3) = 0.35503.$$

In particular, since $G(0)$ is given by

$$G(0) \approx \mathrm{Ai}(0) b^{1/3} \exp\left(-\frac{\pi}{4} b^2\right), \quad \text{(A.16)}$$

$|G(0)|^2$ is of the order of the depolarization of Froissart and Stora's formula. In the present approximation, namely the asymptotic expansion for large b, this value should be taken to be zero.

Here, it may be necessary to make a comment on the physical meaning of $G(0)$. As one can see from the results of Section 2 [Eq. (2.33)], $G(0)$ is related to the depolarization of an off-energy particle with infinitely slow synchrotron oscillation. However, such a particle does not experience any depolarization because its effect is merely to shift the time of resonance crossing. Indeed, one finds that the differential Eq. (2.14) with $\mu = 0$ is equivalent to the unperturbed Eq. (2.20), by shifting the origin of t. In spite of this property of our starting equation, the resulting $G(0)$ is not exactly zero. The cause is not the WKB approximation. In fact, we can express $G(\omega)$ exactly from the expressions (A.1), using confluent hypergeometric functions of the second kind (those which have logarithmic singularity at the origin). But in that expression not only $G(0)$ is non-zero but $G(\omega)$ diverges at the origin as $1/\omega$, although the residue is very small. What is wrong in our approach is the perturbation expansion in terms of ΔH. If we make the synchrotron oscillation slower with its amplitude fixed, the shift of the spin phase due to energy deviation during the half period of synchrotron oscillation becomes larger, and the solution goes away from the unperturbed one. Hence, in order to treat such a limiting case, we have to change the decomposition into H_0 and ΔH. However, in that case the depolarization is small any way as long as u is small, which we have already assumed. Therefore, in our perturbation expansion, the absolute error of the resulting depolarization formula is very small even though the relative error might be large. Moreover, $G(\omega)$ of WKB approximation is easier to handle than the exact expression using a confluent hypergeometric function not only because the former is simpler mathematically, but also because it does not diverge at the origin.

Now, among the three regions stated above, the first and the second regions together with their transition region are covered by the single formula

$$G(\omega) = 2b\left[\frac{\xi}{4\omega^2(\omega^2 - b^2)}\right]^{1/4}\cdot[\mathrm{Ai}(-\xi) + 0(b^{-4/3})] \qquad \text{for} \quad \omega \gtrsim 1/b. \tag{A.17}$$

Here, ξ is defined by

$$\begin{aligned}\xi &= \left[\frac{3}{2}\int_b^\omega \sqrt{\omega^2 - b^2}\, d\omega\right]^{2/3} \\ &= \left[\frac{3}{4}\left(\omega\sqrt{\omega^2 - b^2} - b^2\cosh^{-1}\frac{\omega}{b}\right)\right]^{2/3} \qquad (\omega > b) \\ &= -\left[\frac{3}{2}\int_\omega^b \sqrt{b^2 - \omega^2}\, d\omega\right]^{2/3} \\ &= -\left[\frac{3}{4}\left(-\omega\sqrt{b^2 - \omega^2} + b^2\cos^{-1}\frac{\omega}{b}\right)\right]^{2/3} \qquad (\omega < b),\end{aligned} \tag{A.18}$$

and the factor $\xi/(\omega^2 - b^2)$ is always positive. The function Ai is the Airy function, which is related to the Bessel functions by

$$\begin{aligned}\mathrm{Ai}(-\xi) &= \frac{\sqrt{\xi}}{3}\left[J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right)\right] \qquad (\xi \geqq 0) \\ &= \frac{1}{\pi}\sqrt{-\xi/3}\, K_{1/3}\left(\frac{2}{3}|\xi|^{3/2}\right), \qquad (\xi < 0)\end{aligned}$$

and is tabulated, for instance, in Ref. 7. From the expression (A.17) one finds $\omega_{\max}$ which gives the maximum of $G(\omega)$;

$$\begin{aligned}\omega_{\max} &= b + (2b)^{-1/3}a_0 + 0(b^{-5/3}), \\ G(\omega_{\max}) &= (2b)^{1/3}[\mathrm{Ai}(-a_0) + 0(b^{-4/3})],\end{aligned} \tag{A.19}$$

where $-a_0$ is the maximum point of Ai and its value is[7]

$$\begin{aligned} a_0 &= 1.0188, \\ \mathrm{Ai}(-a_0) &= 0.53566. \end{aligned} \tag{A.20}$$

In addition, the n-th zero of $G(\omega)$ is given by

$$\omega_n = b + (2b)^{-1/3} a_n + 0(b^{-5/3}), \tag{A.21}$$

where $-a_n$ is the n-th zero of Ai;

$$a_1 = 2.3381, \quad a_2 = 4.0879 \ldots.. \tag{A.22}$$

Next, let us consider some integrals appearing in the text. First, we evaluate $K(b)$ defined in Eq. (3.11). The range of the integration can be divided into two regions, $(1/b, b)$ and (b, ∞). One finds that the contribution of the first region is of the order of $b^{-5/3}$ for large b, by using the approximate expression (A.14). If we use (A.13) for the integration of the second region, we obtain

$$\int_b^\infty d\omega \frac{|G(\omega)|^2}{\omega^2} = \int_b^\infty d\omega \frac{2b^2}{\pi\omega^3\sqrt{\omega^2 - b^2}} \cos^2 \times \left[\frac{1}{2}\left(\omega\sqrt{\omega^2 - b^2} - b^2 \cosh^{-1}\frac{\omega}{b} - \frac{\pi}{2}\right)\right].$$

When b is large, the square of the cosine oscillates very rapidly and can be replaced by the average value 1/2. Hence we have

$$\int_b^\infty d\omega \frac{|G(\omega)|^2}{\omega^2} \approx \frac{1}{4b}$$

and

$$K(b) \approx \frac{1}{4b}. \tag{A.23}$$

Though the derivation is very rough, this expression gives a fairly good approximation of $K(b)$. Comparing with the results of numerical integration using Eq. (A.12), one finds that the error of Eq. (A.23) is only 9 percent even for $b = 2$.

Next, let us estimate $Q_n(b)$ defined in Eq. (2.15). Using the representation (A.17) of $G(\omega)$ and choosing ξ as the integration variable instead of ω, we have

$$Q_n(b) = 2b^2 \int_{\xi_0}^\infty \frac{\omega}{(\omega^2 - \omega_n{}^2)^2} \frac{\xi}{\omega^2 - b^2} \mathrm{Ai}^2(-\xi)\, d\xi.$$

Since the integrand is very small for negative ω, the lower limit of integration ξ_0, which corresponds to $\omega = 0$, can be replaced with $-\infty$. By using Eq. (A.18) we can expand ω as

$$\omega = b + (2b)^{-1/3}\xi - \frac{1}{5}(2b)^{-5/3}\xi^2 + \ldots..$$

Then we get

$$Q_n(b) = \frac{b}{2}\int_{-\infty}^{\infty} d\xi \left(\frac{\mathrm{Ai}(-\xi)}{\xi - a_n}\right)^2 \left[1 - 2(2b)^{-4/3}a_n - \frac{2}{5}(2b)^{-4/3}(\xi - a_n) + 0(b^{-8/3})\right],$$

where a_n is the n-th zero of $\mathrm{Ai}(-\xi)$. With the help of the formulae

$$\int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(-\xi)}{(\xi - a_n)^2} d\xi = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(-\xi)}{\xi - a_n} d\xi = 0 \tag{A.24}$$

which will be proved below, we get

$$Q_n(b) = \frac{b}{2}[1 - 2a_n(2b)^{-4/3} + \ldots.].$$

Comparing with the results of numerical integration, one sees that the following formula gives much better approximation than the above one for not so large values of b, although it is still poor for large n.

$$Q_n(b) = \frac{b}{2}\frac{1}{1 + 2a_n(2b)^{-4/3}}. \tag{A.25}$$

The relations (A.24) can be proved as follows. First let us consider a function $M(y)$ of a complex argument y defined by

$$M(y) = \int_M \frac{\mathrm{Ai}^2(x)}{x - y} dx, \tag{A.26}$$

where the integration contour denoted by M runs from $-\infty$ to $+\infty$, passing below the pole at y which is the only singularity of the integrand. By using the differential equation for $\mathrm{Ai}(x)$

$$\mathrm{Ai}''(x) - x\mathrm{Ai}(x) = 0 \tag{A.27}$$

and by the repeated use of partial integration, one can easily show that $M(y)$ obeys the third order differential equation

$$M'''(y) - 4yM'(y) - 2M(y) = 0. \tag{A.28}$$

But this is exactly the same equation which is satisfied by the products of any two solutions to the Airy's equation (A.27). Hence, $M(y)$ is a linear combination of $\mathrm{Ai}^2(y)$, $\mathrm{Ai}(y)\mathrm{Bi}(y)$ and $\mathrm{Bi}^2(y)$, where $\mathrm{Bi}(y)$ is the other Airy function, which increases exponentially as $y \to +\infty$. The coefficients of the combination are common in the whole complex y-plane because $M(y)$ is an entire function owing to the choice of the contour. One sees from the definition of $M(y)$ that $M(y)$ is bounded under $y \to +\infty$. Therefore, the term $\mathrm{Bi}^2(y)$ is not contained in $M(y)$. Moreover, $M(y)$ is bounded under $|y| \to \infty$ in the upper half plane because in this case the contour can be taken to be the real axis. But the only combination of Ai^2 and AiBi which has this property is $\mathrm{Ai}(\mathrm{Ai} + i\mathrm{Bi})$ times a constant, as can be found by using the asymptotic forms of

Ai(y) and Bi(y) for $|y| \to \infty$.[7] Hence we have only to find the overall factor. When y *is* real, one finds

$$\operatorname{Im} M(y) = \operatorname{Im} \int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(x)}{x - y - i0}\, dx = \pi \mathrm{Ai}^2(y), \tag{A.29}$$

which fixes the unknown factor to be πi. Hence, we find

$$\int_M \frac{\mathrm{Ai}^2(x)}{x - y}\, dx = \pi[i\mathrm{Ai}^2(y) - \mathrm{Ai}(y)\mathrm{Bi}(y)]. \tag{A.30}$$

When y is equal to a zero x_n ($= -a_n$) of Ai(x), the contour can be taken to be the real axis because in this case the integrand is free from singularities. Then we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(x)}{x - x_n}\, dx = 0.$$

Similarly, putting $y = x_n$ after differentiation with respect to y, we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(x)}{(x - x_n)^2}\, dx = -\pi \mathrm{Ai}'(x_n)\mathrm{Bi}(x_n).$$

Using the Wronskian relation[7]

$$\mathrm{Ai}(y)\mathrm{Bi}'(y) - \mathrm{Ai}'(y)\mathrm{Bi}(y) = 1/\pi. \tag{A.31}$$

we get

$$\int_{-\infty}^{\infty} \frac{\mathrm{Ai}^2(x)}{(x - x_n)^2}\, dx = 1.$$

This ends the proof.