BRST in the exact renormalization group

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Received April 29, 2019; Revised August 6, 2019; Accepted August 6, 2019; Published October 10, 2019

We show, explicitly within perturbation theory, that the quantum master equation and the Wilsonian renormalization group flow equation can be combined such that for the continuum effective action, quantum BRST invariance is not broken by the presence of an effective ultraviolet cutoff Λ , despite the fact that the structure demands quantum corrections that naïvely break the gauge invariance, such as a mass term for a non-Abelian gauge field. Exploiting the derivative expansion, BRST cohomological methods fix the solution up to choice of renormalization conditions, without inputting the form of the classical, or bare, interactions. Legendre transformation results in an equivalent description in terms of solving the modified Slavnov–Taylor identities and the flow of the Legendre effective action under an infrared cutoff Λ (i.e. effective average action). The flow generates a canonical transformation that automatically solves the Slavnov–Taylor identities for the wavefunction renormalization constants. We confirm this structure in detail at tree level and one loop. Under flow of Λ , the standard results are obtained for the beta function, anomalous dimension, and physical amplitudes, up to the choice of the renormalization scheme.

Subject Index B05, B32, B39

1. Introduction

Exact renormalization group (RG) equations [1], such as those for a Wilsonian effective action, which are exact equations for its flow with respect to some effective ultraviolet (UV) cutoff Λ [1,2] (see also Refs. [3–19]), or those for the effective average action, which are exact equations for the flow of the one-particle irreducible (1PI) effective action, equivalently the Legendre effective action with infrared (IR) cutoff Λ [5,20,21] (see also Refs. [4,22–25] and Ref. [5, *Note Added*]), provide a powerful conceptual and practical approach to developing both exact and approximate continuum solutions in quantum field theory (for reviews, see, e.g., Refs. [26–34]).

They are formulated using a cutoff function that suppresses the corresponding region in momentum space of the quantum field. However, for gauge theories, local symmetry transformations, schematically $\phi(x) \mapsto \Omega(x) \phi(x)$, do not respect such a division of the Fourier transform, $\phi(p)$, into highand low-momentum modes. Then one is faced with either generalizing the cutoff and the flow equations such that they can respect the local invariance of the quantum field [7–19,24] (for alternative approaches see Refs. [35–39]) with, however, an attendant increase in complexity, or only recovering gauge invariance in the limit that the cutoff is removed. In the former case computations can be carried out while preserving manifest gauge invariance; however, in the latter case one must first fix a gauge. Then gauge invariance is replaced by BRST invariance [40–43], and the strategy would be to recover BRST invariance in the limit as the cutoff is removed.

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This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted reuse, distribution, and reproduction in any medium, provided the original work is properly cited. Funded by SCOAP³ Even if gauge invariance is preserved, BRST invariance is deformed by quantum corrections. Taking this into account, BRST invariance is most effectively expressed on the quantum fields through the quantum master equation (QME) [44–47], while for the Legendre effective action it is expressed through the Zinn-Justin equation [48–50], which in fact is most elegantly expressed as the classical master equation (CME).

When combined with the requirement of locality, the CME can be solved by the elegant and powerful methods of BRST cohomology [51-54] (for reviews, see Refs. [55-57]). Of course, locality is an important physical requirement, but it is also crucial to realize the power of BRST cohomology. Indeed, in the space of non-local functionals, this structure disappears: BRST cohomology is trivial [53,54], and a non-Abelian BRST algebra can be rewritten as an Abelian one [46] (see also Ref. [58]). Fortunately, the assumption of locality is justified when analyzing the most general possibilities for interacting gauge-invariant systems at the classical level and, in sensible frameworks, when analyzing only the structure of UV divergences in a perturbative expansion in \hbar (thus, e.g., leading to the proof of renormalizability of Yang–Mills theories [48-50]).

At first sight, then, these methods lose their power when analyzing the full quantum corrections, which are inherently non-local. Moreover, the Batalin–Vilkovisky measure operator Δ , part of the QME, is ill defined without further regularization [47,55], and within the framework of exact RG equations the BRST invariance is anyway broken until the cutoff is removed (as discussed above).

Actually, as we will see, all these unpleasant features can be avoided in an especially natural combination of the QME and exact RG [58]. In particular, they can be combined in such a way that they are mutually compatible [59–68]. Then the QME can be satisfied simultaneously with the flow equation. By working with off-shell BRST (thus also including auxiliary fields) the quantum BRST invariance is then not lost in the continuum effective action in the presence of the cutoff, but gets deformed in a calculable way. This is remarkable given that the cutoff then allows a mass term for the gauge field to be generated by quantum corrections [69]. Such a term is, of course, forbidden by the classical BRST algebra (and the CME). However, it not only becomes allowed but in fact demanded by the now well-defined (regularized) measure operator Δ in this framework.

Also, locality is not lost but is relaxed to the requirement of quasi-locality, which is anyway a fundamental requirement of the Wilsonian RG in the continuum [8,70], corresponding to the existence of a sensible Kadanoff blocking [71] (with radius of order $1/\Lambda$). Quasi-locality implies that vertices must have a derivative expansion, corresponding to a Taylor expansion in dimensionless momenta p^{μ}/Λ . In our formulation, Δ is regularized such that it is well defined when acting on arbitrary local functionals, but the rest of the BRST structure remains unmodified to first order [58]. As we will see, quasi-locality is then sufficient to regain the full potential of BRST cohomological methods.

We can therefore say that BRST invariance is not broken by the cutoff, but remains powerfully present in this structure. This paper is devoted to exploiting and making manifest this symmetry structure in (the solution of) continuum non-Abelian gauge theory.

We stress here and throughout the paper that quasi-locality of the effective action at non-vanishing Λ is essential. In particular, when solving for either such effective action directly in the continuum limit, it is this that replaces the requirement that there exists a local bare action, and it is this that ensures a unique solution for the (non-local) effective action at $\Lambda = 0$. Indeed, the physical vertices are obtained from the Legendre effective action in the $\Lambda \rightarrow 0$ limit. They are non-local and (evidently) independent of Λ . Without the constraint that a derivative expansion exists at non-vanishing Λ one could add any Λ -independent (and BRST-invariant) non-local term to the continuum effective action as a Λ -integration constant, so that solutions to the flow equations become under-determined. On the

other hand, the requirement that the effective action has a derivative expansion at non-vanishing Λ replaces the need to insist on a local bare action: indeed, it builds one in since, by dimensions, higher derivatives appear weighted by the appropriate inverse power of Λ , and thus the effective action for finitely varying fields (finite momentum) but as $\Lambda \rightarrow \infty$ tends to a local action (with divergent bare couplings).

The problems that arise from the impossibility of imposing BRST invariance on some local bare action at an initial UV scale $\Lambda = \Lambda_0$ (see, e.g., [59,68]) are avoided by working directly in the continuum limit ($\Lambda_0 \rightarrow \infty$) with a finite (and thus renormalized) solution for the effective action at scale Λ .

For U(1) gauge theories, such as QED, it is possible to give a closed form expression for the resulting renormalized BRST transformations in the presence of a cutoff, in terms of manifestly finite composite operators [64]. In non-Abelian gauge theory it does not appear possible to solve exactly in closed form for the BRST symmetry in terms of manifestly finite quantities.¹ However, such a closed form can be derived within suitable approximations. We demonstrate this in perturbation theory to one loop, using Yang–Mills theory with general gauge group, and in a general gauge. As we will see, the structure allows us to solve directly for the continuum effective action in a particularly clean way, guided and underpinned at crucial stages by quasi-locality and BRST cohomology. Notably, one can work in the so-called minimal gauge-invariant basis, where the antighost and auxiliary field are absent, and the freedom in perturbations is manifestly independent of the choice of gauge fixing.

The functional RG equation for the Wilsonian effective action is related to that for the 1PI effective action (cutoff Legendre effective action) via a Legendre transform identity [5],² such that the UV cutoff in the former is related to the IR cutoff of the latter. Under this transformation one obtains not only the flow equation for this so-called effective average action, but also the so-called modified Slavnov–Taylor identities (mST) [69] which take the form of the CME (a.k.a. Zinn-Justin equation) together with quantum corrections generated by the cutoff. Thus we find that here too one can solve particularly cleanly for the renormalized 1PI effective action, in the minimal gauge-invariant basis, exploiting BRST cohomology and quasi-locality.

We show that for both effective actions the form of the classical interacting part is uniquely determined by classical BRST cohomology. This was shown in Refs. [51-57]. Here we recover this result in the presence of an effective cutoff. In this way, we derive the solution for either effective action without a priori input of either a classical action or a bare action.

It is clear that since the Wilsonian action can be reconstructed from the 1PI effective action, but satisfies the QME, BRST symmetry is still not lost at the quantum level but deformed by the cutoff in a calculable way. Notably, we show that RG flow generates a particularly simple canonical transformation in which the fields and the so-called antifields renormalize in opposite directions. We show that these renormalizations automatically solve for the standard Slavnov–Taylor relations between the wavefunction renormalization factors. In particular, these relations thus remain *unaltered* at the quantum level by the presence of the cutoff. As we will confirm, the coupling parametrizes the only remaining freedom in the solution of the QME/mST.

We verify that the natural renormalization conditions on the effective action correspond to a finite renormalization of standard schemes. We then find that at one loop, the coupling evolves with Λ

¹ The problem is that the corresponding composite operators [64] now require renormalization.

² This powerful relation was also exploited in Refs. [22,72–74]. See also Refs. [23,75–77].

according to the standard asympttically free β -function (despite the generation of terms that naïvely break the gauge invariance).

The structure of the paper is as follows. In Sect. 2 we show explicitly that the QME and Wilsonian flow equation, which can be written as compatible equations for finite renormalized quantities, can also be solved consistently for the Wilsonian effective action in terms of finite quantities. By considering a perturbative expansion together with the derivative expansion, we show in Sects. 2.2 and 2.3 that the resulting structure, even at a general abstract level, is already sufficient to recover in this context much of the standard lore for renormalization of non-Abelian gauge theories. We then go on to develop in full the simultaneous perturbative solution of the flow equation and QME for the case of Yang–Mills with general gauge group and in general gauge. In Sect. 3 we equivalently solve for the 1PI effective action. We derive from the Legendre transform relation both its flow equation [5,20,21] and its mST equation [69]. Although this latter is more awkward than the QME, it is more streamlined to solve at higher orders since we then avoid having to compute tree-diagram corrections at intermediate stages. However, we also see explicitly, via the Legendre transform relations, how these are the same as found for the Wilsonian effective action. These solutions have Λ -integration constants which must themselves have a derivative expansion. They determine the interactions through solving the classical BRST cohomology, while at the quantum level they play the rôle of wavefunction renormalization. In Sect. 4 we show that the mST or QME then enforces that this scaling parametrizes a simple canonical transformation which automatically solves the corresponding Slavnov-Taylor identities. In Sect. 5 we compute these and the natural renormalization of coupling, verifying that this yields the standard result for the β function but here as a flow of the IR cutoff Λ , and that the resulting vertices satisfy the mST. Finally, in Sect. 6 we summarize and draw our conclusions. In particular, we emphasize once again that, in terms of the Wilsonian effective action, the BRST invariance is unbroken despite the appearance of, for example, an effective Λ^2 mass term.

2. The Wilsonian flow equation and the QME

In this section we solve for gauge theory in the presence of a cutoff in terms of a Wilson/Polchinski effective action S [1,2], by solving the Wilsonian flow equation simultaneously with the QME [44–47]. The latter will imply all the Slavnov–Taylor identities as well as the Ward–Takahashi identities. We first consider the issue in general and then specialize to the case of Yang–Mills theory in four spacetime dimensions.

2.1. BRST invariance and the QME in the presence of a cutoff function

We use the notation of Ref. [67] except for some small changes. In particular, we drop Λ subscripts since everything will be evaluated at Λ . However, we follow the formulation given in Ref. [58]. It is based on Ref. [67], except that we work with off-shell BRST so that we can use its idempotency, $s^2 = 0$, which is the crucial property behind BRST cohomology, and we use a change of variables to a new "basis." The relation of this basis to earlier work is briefly reviewed in Appendix A.

We write the free Wilsonian effective action as [58]

$$S_0[\phi, \phi^*] = \frac{1}{2} \phi^A K^{-1} \triangle_{AB}^{-1} \phi^B + \phi_A^* K^{-1} R^A_{\ B} \phi^B, \qquad (2.1)$$

 \triangle_{AB}^{-1} being the differential operators defining the kinetic terms, which are regularized by a smooth (thus Taylor expandable) UV cutoff function $K(p^2/\Lambda^2)$, satisfying the standard requirements K(0) = 1 and $K(u) \to 0$ sufficiently fast as $u \to \infty$ to ensure that all momentum integrals that we encounter

are UV regulated (faster than power fall off is sufficient). In addition, as we explain later, see Eq. (2.35), in general gauge we need the first derivative to vanish,

$$K'(0) = 0, (2.2)$$

in order to ensure that the effective action has a derivative expansion. The propagator \triangle^{AB} is the inverse of \triangle_{AB}^{-1} . In practice we will be interested in IR-regulated propagators,

$$\bar{\Delta}^{AB} = \bar{K} \Delta^{AB},\tag{2.3}$$

where $\bar{K} = 1 - K$. Indeed, from the above relations we see that \bar{K} works as an IR cutoff, satisfying $\bar{K}(0) = 0$, $\bar{K}'(0) = 0$, and $\bar{K}(u) \to 1$ as $u \to \infty$.

The antifields ϕ_A^* are sources for the BRST transformations. Using these, the free action in Eq. (2.1) incorporates the free (and unregularized) BRST transformations:

$$Q_0 \phi^A = R^A_{\ B} \phi^B. \tag{2.4}$$

The full effective action is written $S = S_0 + S_I$, where $S_I[\phi, \phi^*]$ contains the interactions. The BRST transformations in the interacting theory

$$Q\phi^{A} = (\phi^{A}, S) = K \frac{\partial_{l}S}{\partial \phi^{*}_{A}}$$
(2.5)

then follow from a regularized version of the Batalin–Vilkovisky antibracket [44–47]. This latter and the regularized measure operator are given by [58]

$$(X,Y) = \frac{\partial_r X}{\partial \phi^A} K \frac{\partial_l Y}{\partial \phi_A^*} - \frac{\partial_r X}{\partial \phi_A^*} K \frac{\partial_l Y}{\partial \phi^A} \quad \text{and} \quad \Delta X = (-)^{A+1} \frac{\partial_r}{\partial \phi^A} K \frac{\partial_r}{\partial \phi_A^*} X.$$
(2.6)

Here, X and Y are arbitrary bosonic or fermionic functionals, and $(-)^A = (-)^{\epsilon_A}$, where $\epsilon_A = 1$ (0) if ϕ^A is fermionic (bosonic), ϕ_A^* having the opposite Grassmann grading. It is to be understood that K carries the momentum of the DeWitt-contracted (anti)fields in Eq. (2.6). These constructs lead to many powerful identities [44–47], which continue to hold in the presence of this regularization since the identities follow from symmetry and statistics [58]. Notice that the general form in Eq. (2.5) agrees with the free case in Eq. (2.4), since the cutoff factors cancel between Eqs. (2.6) and (2.1) in this case. Also notice that the definition in Eq. (2.5) implies that the BRST transformations are right-acting, as in Refs. [44–47]. (In Ref. [58], left-acting transformations were used, as in Refs. [51–54,78]. In Appendix B we translate between the two.) We will also need the Kozsul–Tate operator, Q^- [58,79–81]. It acts (from the right) on antifields [51–57]:

$$Q^{-}\phi_{A}^{*} = (\phi_{A}^{*}, S) = -K \frac{\partial_{l} S}{\partial \phi^{A}} \implies Q_{0}^{-}\phi_{A}^{*} = -\Delta_{AB}^{-1}\phi^{B} + (-)^{B}\phi_{B}^{*}R_{A}^{B}.$$
 (2.7)

The QME is the statement that the quantum master functional (QMF) vanishes, where the QMF is given by [44]

$$\Sigma = \frac{1}{2}(S,S) - \Delta S, \qquad (2.8)$$

regularized as in Eq. (2.6). Of particular importance are perturbations, $S + \varepsilon O$, that preserve the QME (where ε is infinitesimal, and O a quasi-local operator integrated over spacetime). This deforms

the BRST algebra, allowing us to explore the space of interacting theories whose gauge invariance is smoothly connected to that of the original. Substituting the perturbed action into the QME we have that the operator must be (quantum) BRST invariant,

$$\hat{s} \mathcal{O} = 0, \tag{2.9}$$

a.k.a. closed under \hat{s} , where the *full* (or total) quantum BRST transformation,³

$$\hat{s}\mathcal{O} = (\mathcal{O}, S) - \Delta\mathcal{O} = (Q + Q^{-} - \Delta)\mathcal{O}, \qquad (2.10)$$

can be shown to be nilpotent: $\hat{s}^2 = 0$ [44–47]. An infinitesimal canonical transformation

$$\mathcal{O} = \hat{s} \,\mathcal{K} = (\mathcal{K}, S) - \Delta \mathcal{K} \tag{2.11}$$

is \hat{s} -exact, and thus trivially a solution to Eq. (2.9) and a symmetry of the QME. However, such an operator just corresponds to the infinitesimal field and source redefinitions

$$\delta\phi^A = -\varepsilon K \frac{\partial_r \mathcal{K}}{\partial \phi_A^*}, \qquad \delta\phi_A^* = \varepsilon K \frac{\partial_r \mathcal{K}}{\partial \phi^A}, \tag{2.12}$$

with $-\Delta \mathcal{K}$ corresponding to the Jacobian of the change of variables in the partition function. We are therefore interested only in operators \mathcal{O} that are closed under \hat{s} but not exact, i.e. we want operators that lie in the non-trivial part of the (full) quantum BRST cohomology.

When we develop solutions as an expansion in \hbar , and also when we consider the 1PI framework in Sect. 3, the classical ($\hbar \rightarrow 0$) equivalents to the above become important. They are the same equations with *S* replaced by the classical action, S_{cl} , and with the measure operator switched off. In particular, the CME is the statement that

$$(S_{\rm cl}, S_{\rm cl}) = 0. (2.13)$$

The full classical BRST transformation, s_{cl} , on an operator O is given by the sum of the classical Koszul–Tate operator and the classical BRST operator:

$$s_{\rm cl} \mathcal{O} = (Q_{\rm cl} + Q_{\rm cl}^{-}) \mathcal{O} = (\mathcal{O}, S_{\rm cl}).$$

$$(2.14)$$

Perturbations, $S_{cl} + \varepsilon O$, that preserve the CME are invariant under the full classical BRST transformation: $s_{cl} O = 0$.

2.2. Compatibility with the Wilsonian flow equation and continuum limit

As shown in Ref. [58], in this formulation the flow equation takes the same form as in Ref. [2]:

$$\dot{S}_{I} = -\frac{1}{2}\partial_{A}^{r}S_{I}\dot{\bar{\bigtriangleup}}^{AB}\partial_{B}^{l}S_{I} + \frac{1}{2}(-)^{A}\dot{\bar{\bigtriangleup}}^{AB}\partial_{B}^{l}\partial_{A}^{r}S_{I} = \frac{1}{2}a_{0}[S_{I}, S_{I}] - a_{1}[S_{I}], \qquad (2.15)$$

where $(a_0)a_1$ is the corresponding (bi)linear functional detailed in the middle equation [7]. Here, $\partial_A \equiv \partial/\partial \phi^A$, and the dot means $\partial_t = -\Lambda \partial_\Lambda$, where $t = \ln(\mu/\Lambda)$, μ being the usual arbitrary fixed finite energy scale. With the regularization of Eq. (2.6), the QMF is compatible with the flow equation in Eq. (2.15) [58]:

$$\dot{\Sigma} = a_0[S_I, \Sigma] - a_1[\Sigma]. \tag{2.16}$$

³ This was called *s* in Ref. [58], which we reserve here for $s = Q + Q^{-}$.

This means that if $\Sigma = 0$, then also $\dot{\Sigma} = 0$, and thus that if the QME is satisfied at some generic scale Λ , it remains satisfied on further RG evolution. (In fact, Eq. (2.16) coincides with the flow equation for a (composite) operator [58,67].)

On the other hand, we are interested in forming a continuum limit, which means that in dimensionless variables (and appropriate wavefunction renormalizations) S tends to a fixed-point action in the limit $\Lambda \rightarrow \infty$. Close to the fixed point, S is then given by the fixed-point action plus an expansion to first order over the (marginally) relevant eigenoperators, with conjugate couplings g which (in dimensionful terms) are constant and of positive or vanishing dimension [1,26,74]. These cases are not generic points on the flow. It is not sufficient that the QME is satisfied at the fixed point, or even at first order, to be able to conclude from Eq. (2.16) that the QME is satisfied at all points on the resulting renormalized trajectory. However, developing the solution as a perturbative expansion over the eigenoperators beyond first order, we will see that the compatibility of BRST invariance with the continuum limit is determined only by integration constants associated with the higher-order solutions.⁴ This serves to determine how a cutoff effective action that is invariant under such non-linear BRST transformations must have its (marginally) relevant couplings constrained.

We emphasize that the above procedure can in principle be carried out around a non-perturbative fixed point (using suitably modified equations to take into account anomalous dimensions). The key observation that makes this possible is that close to the fixed point a perturbative expansion over the (marginally) relevant couplings is allowed even in this case, since such couplings must be vanishingly small there. Then the integration constants will be determined in the way we sketched above.

To carry out such a program explicitly, however, requires explicit solutions for a non-perturbative fixed point which also satisfies the QME, and explicit solutions for the eigenoperators. No exact non-perturbative solutions are known. Therefore, we will content ourselves here with demonstrating the structure within perturbation theory about the Gaussian fixed point.

Thus, consider a renormalized trajectory leaving the Gaussian fixed point at $\Lambda = \infty$. The Gaussian fixed point corresponds to the solution $S_I = 0$ of Eq. (2.15). The flow is parametrized by the finite renormalized (marginally) relevant couplings. For simplicity we assume there is only one coupling g. Since we are studying the flow close to the fixed point, we can expand:

$$S = S_0 + gS_1 + g^2 S_2 + \cdots . (2.17)$$

Here, S_1 is the eigenoperator conjugate to g. When we consider quantum corrections we will need to define g more precisely, through some renormalization condition. As we recall later, depending on our choice of renormalization condition, g can either be a function of Λ or independent of it. Here we treat it as Λ independent. Perturbative expansion of Eq. (2.15) then gives

$$\dot{S}_n = \frac{1}{2} \sum_{m=1}^{n-1} a_0[S_{n-m}, S_m] - a_1[S_n].$$
(2.18)

Similarly, one can expand the QMF as

$$\Sigma = \Sigma_0 + g\Sigma_1 + g^2\Sigma_2 + \cdots .$$
 (2.19)

⁴ Interestingly, this leaves open the possibility of extra constraints from non-perturbative corrections. We will not explore this further here.

Now $\Sigma_0 = 0$ follows from the fact that Eq. (2.4) is an invariance of the kinetic term in Eq. (2.1). Then one has, from Eq. (2.16), that

$$\dot{\Sigma}_n = \sum_{m=1}^{n-1} a_0[S_{n-m}, \Sigma_m] - a_1[\Sigma_n].$$
(2.20)

From Eq. (2.8) we have that

$$\Sigma_n = \hat{s}_0 S_n + \frac{1}{2} \sum_{m=1}^{n-1} (S_{n-m}, S_m), \qquad (2.21)$$

where, from Eq. (2.10), $\hat{s}_0 = Q_0 + Q_0^- - \Delta$ is the free full quantum BRST charge.

Now, from Eq. (2.18), the eigenoperator equation is, in this notation,

$$\dot{S}_1 = -a_1[S_1]. \tag{2.22}$$

From Eq. (2.20) we see that Σ_1 also satisfies this equation. From Eq. (2.21), $\Sigma_1 = \hat{s}_0 S_1$. Therefore $\hat{s}_0 S_1$ must also be an expansion over eigenoperators with constant coefficients. Since we require $\Sigma_1 = 0$ for the QME to be satisfied, we thus see that the quantum BRST cohomology is to be defined within the space spanned by the eigenoperators with constant coefficients [74].

If we have already determined that the QME is solved up to $\Sigma_m = 0$ for m < n, then Eq. (2.20) reduces to

$$\dot{\Sigma}_n = -a_1[\Sigma_n]. \tag{2.23}$$

Since this is again the eigenoperator equation, it means that the QME has only the possibility to be violated by a linear combination of eigenoperators with constant coefficients. By Eq. (2.21), this is to be repaired (if possible) by adding to S_n a linear combination of eigenoperators with constant coefficients. We thus see that the only freedom to make the QME compatible with a continuum-limit solution of the flow equation is that contained within the perturbative development of the BRST cohomology.⁵

We stress that this is a very powerful conclusion. It is clear that in this way we are already recovering the crucial steps in the proof of renormalizability of gauge theories [48-50]. We will develop this in the remainder of the paper. However, we emphasize that this structure is seen here to arise in general, for the continuum (and thus we mean already renormalized) effective action. And note especially that this structure is seen to operate even with the momentum cutoff kept in place.

Notice that although the equations above have been derived by expanding perturbatively in g, they are still exact in \hbar , i.e. non-perturbative in the loop expansion. For example, consider the eigenoperator equation in Eq. (2.22). On the right-hand side (RHS), $a_1[S_1]$ attaches a loop to the S_1 vertex. If the result is non-vanishing, solutions S_1 will be functions of \hbar . This dependence may be Taylor expandable in \hbar , or may also be inherently non-perturbative in \hbar . The latter type of eigenoperator is the crucial starting point in a proposal for a genuinely renormalizable quantization of gravity [58,74,104,105]. The other equations, Eqs. (2.18)–(2.23), are similarly exact in \hbar . As we

⁵ In particular, note that from Eqs. (2.18) and (2.20), we are always free to add to S_n an \hat{s}_0 -closed linear combination of eigenoperators with constant coefficients. It is this freedom that needs to be fixed by renormalization conditions.

have seen they are solved by utilizing \hat{s}_0 -cohomology, where \hat{s}_0 has a linear dependence on \hbar , given exactly by the one-loop measure operator, Δ .

Starting in the next section, however, we will specialize to Yang–Mills theory, whose solution may be computed perturbatively in the loop expansion. In fact, in this case the S_1 we need to use is purely classical (i.e. carries no \hbar dependence) since $a_1[S_1]$ turns out to vanish.

2.3. Five consequences for Yang–Mills theory

Let us illustrate the power of the structure above by now specializing to the case of Yang–Mills theory in four spacetime dimensions, and to the loop expansion (i.e. expansion in \hbar). We will need at this stage only that all (anti)fields have at least dimension one (see table 1), the coupling g is dimensionless, Σ (S) has dimension one (zero) overall, and that S_1 contains monomials with (at least) three (anti)fields.

At the classical level, $S = S_{cl}$, we drop a_1 . Then, from Eq. (2.23) the only way the QME (now CME) can be violated is through terms that are independent of Λ . Since $S(\Sigma)$ must have a derivative expansion and since there are no dimensionful parameters other than Λ itself, such terms in $S(\Sigma)$ are spacetime integrals of local operators of dimension four (five). We therefore conclude that:

C1 From Eq. (2.22), which is now $\dot{S}_{1,cl} = 0$, and $\Sigma_{1,cl} = 0$, which is

$$s_0 S_{1,cl} = 0$$
, where $s_0 := Q_0 + Q_0^-$, (2.24)

 $S_{1,cl}$ is a local operator of dimension four, which moreover must be a non-trivial solution of the classical free BRST cohomology (s_0 being the full free classical BRST charge).

- C2 $S_{2,cl}$, being formed of tree-level corrections involving S_1 , has a derivative expansion to all orders; however, $\Sigma_{2,cl}$ must already vanish for all the Λ dependence, leaving at most a dimension-five local operator that we need to cancel (if possible) with a Λ -independent dimension-four local operator addition to $S_{2,cl}$.
- C3 Since the tree expansion, and similarly the antibracket in Eq. (2.21), has the property that $S_{n,cl}$ ($\Sigma_{n,cl}$) is made of vertices with at least n+2 (anti)fields, no Λ -independent local operator is possible in $S_{3,cl}$. Thus, any Λ -independent (and thus dimension-five) part of $\Sigma_{3,cl}$ would represent an obstruction to the classical solution at $O(g^3)$. For all n > 3, neither $S_{n,cl}$ nor $\Sigma_{n,cl}$ can have a Λ -independent local term, and therefore the $S_{n>3,cl}$ must already satisfy the CME.

Now consider the quantum level at some fixed order ℓ of the loop expansion. As we will review, the solution of the flow equation generates the expected momentum integrals, though with propagators that contain an IR cutoff. Assuming the lower loop orders have been solved already, since the RHS generates an extra loop, Eq. (2.23) again reduces to the statement that the only way the QME can be violated is by terms that are independent of Λ , which thus in $S_n^{(\ell)}$ ($\Sigma_n^{(\ell)}$) correspond to dimensionfour (-five) local operators. We see that the problem of solving simultaneously the flow equation and QME is therefore confined to the question of whether such Λ -integration constants can be chosen satisfactorily. We have thus shown that, at the quantum level:

C4 The body of the momentum integrals, corresponding to all the non-trivial Λ dependence, must already automatically satisfy the QME.

The Λ -integration constants play the same rôle as counterterms in a more standard treatment. Indeed, since the measure operator in Eq. (2.21) also generates an extra loop, the Λ -integration constants are parts of $S_n^{(\ell)}$ that are determined only up to s_0 -closed pieces. The freedom in the solution is thus held in pieces that are either proportional to $S_{1,cl}$ itself, i.e. the solution to Eq. (2.24), or to s_0 -closed bilinear terms, which are therefore contained in the Λ -independent ($K \mapsto 1$) parts of Eq. (2.1). The latter thus correspond to the freedom to introduce wavefunction renormalizations Z_i such that the CME remains satisfied. In other words, the Z_i must in fact parametrize a particularly simple form of canonical transformation. As we will show fully in Sect. 4:

C5 The Z_i parametrize a canonical transformation that automatically solves the standard Slavnov– Taylor identities for the wavefunction renormalization constants.

This freedom in the solution is fixed by choosing suitable renormalization conditions.

Although powerful, as we have seen these conclusions are achieved while remaining at an abstract level. In the rest of the paper, we will solve for the effective actions S and Γ for Yang–Mills at treelevel and one-loop level up to $O(g^3)$, thus providing the explicit form that confirm the above results, and allowing the structure to be understood in full in a concrete example. In particular, we will see explicitly that, despite the apparent explicit breaking of gauge invariance by the cutoff, the QME and its implied BRST invariance remain satisfied. They in fact now demand quantum corrections that naïvely break the gauge invariance, such as a mass term for the gauge field, in a way that ensures a smooth limit to the standard results once the cutoff is removed. They also ensure the standard β -function, which can even be recovered from a flow of $g(\Lambda)$ with respect to cutoff Λ itself.

2.4. Free action, gauge-fixed and (minimal) gauge-invariant basis

Before introducing gauge fixing, the free action is in the so-called minimal gauge-invariant basis [44–47]:

$$S_0 = \frac{1}{2} \partial_\mu a_\nu K^{-1} \partial_\mu a_\nu - \frac{1}{2} \partial \cdot a K^{-1} \partial \cdot a + a^*_\mu K^{-1} \partial_\mu c.$$
(2.25)

Here, a_{μ} is the Yang–Mills gauge field, *c* the ghost field, and a_{μ}^{*} is the source (antifield) for the free BRST transformation on a_{μ} . Our notation is such that any expression for an action functional should be understood to appear inside the braces in

$$2 \operatorname{tr} \int_{X} \{\cdots\}.$$
 (2.26)

The (anti)fields depend on x, and unless explicitly stated they are contracted into the generators of the gauge group, e.g. $c = c^a T^a$. The factor of 2 in Eq. (2.26) compensates for the fact that the generators are orthonormalized to tr($T^a T^b$) = $\delta^{ab}/2$, as usual.

We will see shortly that we can in fact work in this minimal gauge-invariant basis, where expressions are simplest, and it is clear the extent to which calculations are independent of gauge fixing. In this basis, the only extra field we will need is c^* , the source for BRST transformations of c that appear at the interacting level.

In order to gauge fix, however, we must first add the auxiliary field b that allows off-shell BRST invariance, and \bar{c}^* which sources BRST transformations of the antighost, \bar{c} . This takes us to the so-called extended (or non-minimal) gauge-invariant basis. The free action is then written as

$$S_{0}|_{gi} = \frac{1}{2}\partial_{\mu}a_{\nu}K^{-1}\partial_{\mu}a_{\nu} - \frac{1}{2}\partial_{\cdot}aK^{-1}\partial_{\cdot}a + a_{\mu}^{*}K^{-1}\partial_{\mu}c + \frac{\xi}{2}bK^{-1}b + \bar{c}^{*}K^{-1}b, \qquad (2.27)$$

where ξ will become the gauge-fixing parameter. Recall that $S_0 \equiv S_{0,\Lambda}$, so everything here is at a physical scale and renormalized; thus ξ is renormalized and not bare, for example.

For the flow equation in Eq. (2.15) we need propagators, which we get by working in a gauge-fixed basis. This is implemented by a (finite) quantum canonical transformation [47,82–86] that affects only the antifields:

$$\phi_A^*|_{\text{gf}} = \phi_A^*|_{\text{gi}} + \partial_A^r \Psi.$$
(2.28)

Unlike the above references we do not then set the antifields to zero, since we will need them to solve for the BRST transformations [58]. We take the gauge fermion to be

$$\Psi = -i\bar{c}\,\partial\cdot a.\tag{2.29}$$

Thus, explicitly,

$$\bar{c}^*|_{gf} = \bar{c}^*|_{gi} - i\partial \cdot a,$$

$$a^*_{\mu}|_{gf} = a^*_{\mu}|_{gi} + i\partial_{\mu}\bar{c}.$$
(2.30)

Then the free action in the gauge-fixed basis is

$$S_0|_{\rm gf} = S_0|_{\rm gi} - i\partial \cdot aK^{-1}b - i\partial_\mu \bar{c}K^{-1}\partial_\mu c.$$
(2.31)

Leaving out the regulator, and inverting the \triangle_{AB}^{-1} in Eq. (2.31), gives the propagators \triangle^{AB} :

$$\langle c^{a}(p) \bar{c}^{b}(-p) \rangle = i\delta^{ab} \Delta(p), \quad \text{where} \quad \Delta(p) := 1/p^{2},$$

$$\langle a^{a}_{\mu}(p) a^{b}_{\nu}(-p) \rangle = \delta^{ab} \Delta_{\mu\nu}(p), \quad \text{where} \quad \Delta_{\mu\nu}(p) := \frac{1}{p^{2}} \left(\delta_{\mu\nu} + (\xi - 1) \frac{p_{\mu}p_{\nu}}{p^{2}} \right),$$

$$\langle b^{a}(p) a^{b}_{\mu}(-p) \rangle = -\langle a^{a}_{\mu}(p) b^{b}(-p) \rangle = \delta^{ab} p_{\mu} \Delta(p),$$

$$\langle b^{a}(p) b^{b}(-p) \rangle = 0. \qquad (2.32)$$

By definition, since Eq. (2.30) is a quantum canonical transformation, the QMF, viz. Eq. (2.8), is invariant. As we saw in Sect. 2.3, the lowest-order interaction $S_1 = S_{1,cl}$ is given by a Λ -independent non-trivial solution of the BRST cohomology. We can thus determine $S_{1,cl}$ in the gauge-invariant minimal basis, which is manifestly independent of choice of gauge and uses the minimum number of (anti)fields. This means in particular that $S_{1,cl}$ can be chosen to be independent of b and \bar{c}^* . To find the higher orders in Eq. (2.17), we then need the flow equation. Since the flow equation is not invariant under the transformation in Eq. (2.30), one strategy would be first to map $S_{1,cl}$ to the gauge-fixed basis. Once we have the result we need from the flow equation, we can map back to the gauge-invariant basis, where expressions are simpler, using Eq. (2.30). This was the strategy used in Ref. [58].

However, we can streamline this procedure, since we can already deduce the effect of mapping between the different bases. First, we note that from Eq. (2.30), $S_{1,cl}$ in the gauge-fixed basis will still not depend on *b* or \bar{c}^* , and depends on \bar{c} only through the combination $a^*_{\mu} - i\partial_{\mu}\bar{c}$. By iteration, using the flow equation in Eq. (2.15), these properties are inherited by all the higher-order interactions. Mapping back to the gauge-invariant basis using Eq. (2.30), we therefore see that S_I will not depend

on b, \bar{c}^* , or \bar{c} . This means in particular that the full S_I remains in the *minimal* gauge-invariant basis. However, from Eq. (2.30), we must replace field derivatives in Eq. (2.15) with

$$\frac{\partial}{\partial a_{\mu}}\Big|_{\rm gf} = \frac{\partial}{\partial a_{\mu}} - i\partial_{\mu}\frac{\partial}{\partial \bar{c}^{*}}, \quad \frac{\partial}{\partial \bar{c}}\Big|_{\rm gf} = \frac{\partial}{\partial \bar{c}} + i\partial_{\mu}\frac{\partial}{\partial a_{\mu}^{*}}, \quad \frac{\partial}{\partial a_{\mu}^{*}}\Big|_{\rm gf} = \frac{\partial}{\partial a_{\mu}^{*}}, \quad \frac{\partial}{\partial \bar{c}^{*}}\Big|_{\rm gf} = \frac{\partial}{\partial \bar{c}^{*}}, \tag{2.33}$$

where the differentials on the RHS are in gauge-invariant basis. Since S_I does not depend on either \bar{c}^* or \bar{c} , acting on S_I these derivatives are effectively unchanged except for the replacement

$$\frac{\partial}{\partial \bar{c}}\Big|_{\rm gf} \equiv i\partial_{\mu} \frac{\partial}{\partial a^*_{\mu}}\Big|_{\rm gi}.$$
(2.34)

This is equivalent, for the purposes of taking a \bar{c} derivative, to temporarily replacing a_{μ}^* by $a_{\mu}^* - i\partial_{\mu}\bar{c}$. Either way, this one replacement ensures that we get the right result from the flow equation when working in the minimal gauge-invariant basis.

Since it simplifies the results, for our calculations we will therefore work in this basis. Although $S_{1,cl}$ is then entirely independent of choice of gauge, the higher-order corrections do depend on it. Indeed, this is clear from the fact that the gauge field propagator in Eq. (2.32) depends on ξ .

As already stated in Eq. (2.3) and is already clear to some extent from Eq. (2.15), the propagators that appear in vertices will be IR regulated by $\bar{K}(p^2/\Lambda^2)$. As recalled in Sect. 1, the effective actions must have the property that they have a derivative expansion for $\Lambda > 0$. Non-trivial solutions to the effective actions will only have a derivative expansion if the propagators also have a derivative expansion. This means the propagators must have a Taylor expansion in p^{μ} . Since, for the gauge fields,

$$\bar{\Delta}_{\mu\nu}(p) = \bar{K}(p^2/\Lambda^2) \,\Delta_{\mu\nu}(p) = -\frac{K'(0)}{\Lambda^2} \left(\delta_{\mu\nu} + (\xi - 1)\frac{p_{\mu}p_{\nu}}{p^2}\right) + \text{Taylor series}, \qquad (2.35)$$

we see that this property is violated for the gauge propagator outside the Feynman gauge unless we impose the condition in Eq. (2.2), K'(0) = 0.

2.5. BRST algebra

In Table 1 we list the (anti)fields together with their mass dimension, ghost number, and Grassmann grading. These are assigned so that S_0 conserves these charges (i.e. is bosonic and has zero dimension and ghost number), as will also be required of S_I . They imply that Δ , Q, and Q^- increase the dimension and ghost number by one. Similarly, the antibracket in Eq. (2.6) adds one to the sum of the dimensions (sum of ghost charges) of X and Y.

We will see that in practice it is the free BRST cohomology that is central to solving for the effective action. The free BRST transformations follow from Eqs. (2.5) or (2.4) and Eq. (2.25). The only non-vanishing action in minimal basis is⁶

$$Q_0 a_\mu = \partial_\mu c, \tag{2.36}$$

while similarly, from Eq. (2.7), the only non-vanishing actions of the free Kozsul-Tate operator are

$$Q_0^- a_\mu^* = \Box a_\mu - \partial_\mu \partial \cdot a = \Box P_{\mu\nu}^T a_\nu, \qquad Q_0^- c^* = -\partial \cdot a^*, \tag{2.37}$$

⁶ In the extended gauge-invariant basis of Eq. (2.27), we would also have $Q_0\bar{c} = b$ and have a $\bar{c}^{(*)}$ part of Δ^- .

Table 1. The various Abelian charges (a.k.a. gradings) carried by the (anti)fields, namely Grassmann grading, ghost number, antighost/antifield number, pure gh # = gh # + ag #, and mass dimension. The first two rows are the minimial set of fields, the next two make it up to the non-minimal set, then the ensuing two rows are the minimal set of antifields, and \bar{c}^* is needed for the non-minimal set.

	ϵ	gh #	ag #	Pure gh #	Dimension
$\overline{a_{\mu}}$	0	0	0	0	1
c	1	1	0	1	1
\bar{c}	1	-1	1	0	1
b	0	0	1	1	2
$a_{\prime\prime}^{*}$	1	-1	1	0	2
c^{μ}	0	-2	2	0	2
\bar{c}^*	0	0	0	0	2

where $\Box = \partial_{\mu}^2$, and $P_{\mu\nu}^T$ is the transverse projector. Following Henneaux et al. [51–57], we can simplify finding solutions of the free cohomology by splitting the problem up (a.k.a. grading) by antighost (or, depending on context, antifield) number. These are also listed in Table 1. They are chosen so that the free BRST charges have a definite antighost number. We anticipated this with our labeling: Q_0 leaves the antighost number unchanged, while Q_0^- lowers it by one. Under this grading, the measure operator splits into two parts that lower the antighost number by one or two respectively:

$$\Delta = \Delta^{-} + \Delta^{=}, \qquad \Delta^{-} = -\frac{\partial}{\partial a^{a}_{\mu}} K \frac{\partial_{r}}{\partial a^{*a}_{\mu}}, \qquad \Delta^{=} = \frac{\partial_{r}}{\partial c^{a}} K \frac{\partial}{\partial c^{*a}}.$$
 (2.38)

The point of this extra grading is that the action *S* does not have a definite anti-field number. We can therefore split it into parts of definite anti-field number $n: S = \sum_{n=0} S^n$. To see why this is helpful, note that the full free BRST charge in Eq. (2.10) can now be written as

$$\hat{s}_0 = Q_0 + Q_0^- - \Delta^- - \Delta^=.$$
(2.39)

This means that an operator $\mathcal{O} = \sum_{m=0}^{n} \mathcal{O}^{m}$ with some maximum antighost number *n* that is annihilated by \hat{s}_0 must satisfy the descent equations:

$$Q_0 \mathcal{O}^n = 0, \quad Q_0 \mathcal{O}^{n-1} = (\Delta^- - Q_0^-) \mathcal{O}^n, \quad Q_0 \mathcal{O}^{n-2} = (\Delta^- - Q_0^-) \mathcal{O}^{n-1} + \Delta^= \mathcal{O}^n, \quad \dots$$
(2.40)

Starting with the top (left-most) equation, these are often easier to analyze than trying to work with $\hat{s}_0 \mathcal{O} = 0$ directly. Grading the square, we also have the useful identities [58]

$$Q_0^2 = 0, \ (Q_0^-)^2 = 0, \ (\Delta^-)^2 = 0, \ (\Delta^=)^2 = 0,$$

$$\{Q_0, Q_0^-\} = 0, \ \{Q_0, \Delta^-\} = 0, \ \{Q_0^-, \Delta^=\} = 0, \ \{\Delta^-, \Delta^=\} = 0,$$

$$\{Q_0^-, \Delta^-\} + \{Q_0, \Delta^=\} = 0.$$
(2.41)

2.6. First order in coupling

Recall from Eq. (2.24) that we know that $S_{1,cl}$ is a local operator of dimension four which is a nontrivial solution of the full classical BRST cohomology. At this stage we could just write it down and verify Eq. (2.24). However, it is more profound to notice, following Refs. [51–57], that the solution is in fact unique up to normalization and addition of a trivial reparametrization $s_0 \mathcal{K}_1$. In fact, for us even this latter term cannot appear since \mathcal{K}_1 would have to be an integrated local operator of ghost number -1, dimension three, and containing at least three (anti)fields. From Table 1, no such operator exists. For the same reasons we can see that the maximum possible antighost number of $S_{1,cl}$ is two. We can therefore write $S_{1,cl} = S_1^0 + S_1^1 + S_1^2$. If it is non-vanishing, this maximal term is unique up to normalization (equivalent to normalizing g):

$$S_1^2 = -ic^* c^2. (2.42)$$

(Any other arrangement of the (anti)fields is equivalent under the cyclicity of the trace.) Now, Eq. (2.24) implies that S_1^2 must satisfy

$$Q_0 S_1^2 = 0, (2.43)$$

since this part alone has antighost number two. Since from Eq. (2.36) this is clearly true, it is a valid part of S_1 provided we can now find its descendants, S_1^1 and S_1^0 . From Eq. (2.24) we must have that

$$Q_0^- S_1^2 + Q_0 S_1^1 = 0. (2.44)$$

Now, by Eq. (2.37),

$$Q_0^- S_1^2 = i\partial \cdot a^* c^2 = -ia_\mu^* \{\partial_\mu c, c\} = -ia_\mu^* \{Q_0 a_\mu, c\} = iQ_0 \left(a_\mu^* [a_\mu, c]\right),$$
(2.45)

where the second equality follows by integration by parts, $\{, \}$ being the anticommutator, and the third equality uses Eq. (2.36). Thus, since cohomologically trivial parts are ruled out,

$$S_1^1 = -ia_\mu^*[a_\mu, c]. (2.46)$$

Finally, we must have $Q_0 S_1^0 = -Q_0^- S_1^1$. Since

$$Q_0\left(-\frac{i}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu)[a_\mu, a_\nu]\right) = -i(\partial_\mu a_\nu - \partial_\nu a_\mu)[a_\mu, \partial_\nu c]$$
$$= i(\partial_\mu \partial_\nu a - \Box a_\mu)[a_\mu, c] = Q_0^-\left(ia_\mu^*[a_\mu, c]\right), \qquad (2.47)$$

where in the second line we have integrated by parts and recognized that the resulting piece containing $[\partial_{\nu}a_{\mu}, c]$ vanishes under the trace, we find that

$$S_1^0 = -i\partial_\mu a_\nu [a_\mu, a_\nu].$$
(2.48)

Therefore, we have verified conclusion C1 of Sect. 2.3. In fact, the entire solution at first order in the coupling is:

$$S_1 = S_{1,cl} = -ic^*c^2 - ia^*_{\mu}[a_{\mu}, c] - i\partial_{\mu}a_{\nu}[a_{\mu}, a_{\nu}], \qquad (2.49)$$

since the quantum parts, being the tadpole quantum correction on the RHS of Eq. (2.22), and the measure operators in Eqs. (2.10) and (2.38), give vanishing contribution, in particular because they would leave behind a single field contracted into tr $T^a = 0$.

2.7. Classical solution at second order in coupling

From Eqs. (2.15), (2.18), and (2.21), we need to solve, at second order at the classical level,

$$s_0 S_{2,cl} = -\frac{1}{2}(S_1, S_1),$$
 (2.50)

$$\dot{S}_{2,cl} = -\frac{1}{2} \partial_A^r S_1 \dot{\bar{\bigtriangleup}}^{AB} \partial_B^l S_1 = -\frac{1}{2} S_1 \overleftarrow{\partial}_A \dot{\bar{\bigtriangleup}}^{AB} \overrightarrow{\partial}_B S_1$$
(2.51)

(where s_0 was given in Eq. (2.24) and we also display a more intuitive notation for left and right derivatives [66,67]).

At first sight we can solve for $S_{2,cl}$ already by using Eq. (2.50). Indeed, from Eq. (2.6):⁷

$$s_0 S_{2,cl} = \left([c^*, c] + [a^*_{\mu}, a_{\mu}] \right) K(c^2) - \{a^*_{\mu}, c\} K[a_{\mu}, c] - \Theta_{\mu} K[a_{\mu}, c],$$
(2.52)

where Θ_{μ} is shorthand for the result of differentiating Eq. (2.48):

$$\Theta_{\mu} := \partial_{\rho}[a_{\rho}, a_{\mu}] + [\partial_{\mu}a_{\rho} - \partial_{\rho}a_{\mu}, a_{\rho}].$$
(2.53)

Counting antighost number, and bearing in mind that the s_0 -closed part has already been solved for, the highest antighost number that appears is $S_2^2 \in S_{2,cl}$. Thus, we know that

$$Q_0 S_2^2 = [c^*, c] K(c^2).$$
(2.54)

If *K* were not there, the RHS would vanish on using the cyclicity of the overall trace. Therefore, only pure derivative pieces survive in its derivative expansion:

$$K(-\partial^{2}/\Lambda^{2}) = 1 + \sum_{n=1}^{\infty} K_{n}(-\partial^{2}/\Lambda^{2})^{n}.$$
 (2.55)

Since any derivative acting on c^2 turns it into a Q_0 -exact piece, and thus the whole of the RHS into a Q_0 -exact piece, we see that Eq. (2.54) has a solution. However, there is no natural canonical choice. Choosing different $\partial_{\mu}c$ to convert into $Q_0 a_{\mu}$ corresponds to the freedom to add a piece that is trivially annihilated by Q_0 , i.e. to add an exact term $Q_0 \mathcal{K}_2^2 \in s_0 \mathcal{K}_2$. We see that now Λ is involved, there are infinitely many solutions if we consider only Eq. (2.50).

On the other hand, given that S_1 is Λ -independent, the solution to Eq. (2.51) is uniquely determined by the requirement that it have a derivative expansion, up to addition of a Λ -independent dimensionfour local operator \mathcal{O}_2 :

$$S_{2,cl} = -\frac{1}{2} S_1 \overleftarrow{\partial}_A \, \bar{\Delta}^{AB} \, \overrightarrow{\partial}_B S_1 + \mathcal{O}_2. \tag{2.56}$$

We recognize that the first part is the one-particle reducible (1PR) contribution:

$$S_{2,\text{red}} = -\frac{1}{2} S_1 \stackrel{\leftarrow}{\partial}_A \bar{\Delta}^{AB} \stackrel{\rightarrow}{\partial}_B S_1, \qquad (2.57)$$

whilst \mathcal{O}_2 should also contain four (anti)fields but provide a 1PI contribution. The flow equation alone does not determine \mathcal{O}_2 , but we can already see from Eq. (2.50), or Eq. (2.52), that if a solution exists for \mathcal{O}_2 it is unique. This is because any change in \mathcal{O}_2 would have to be s_0 -closed. But we have

⁷ It is most straightforward to use $2 \operatorname{tr}(XT^a) \operatorname{tr}(T^aY) = \operatorname{tr}(XY)$ for traceless X, Y.

already seen in Sect. 2.6 that S_1 is the unique s_0 -closed dimension-four operator containing at least three (anti)fields.

Substituting Eq. (2.49) into Eq. (2.57), and using Eqs. (2.53), (2.3), (2.32), and (2.34):

$$S_{2,\text{red}} = \frac{1}{2} \left(\{ a_{\mu}^{*}, c \} + \Theta_{\mu} \right) \bar{\Delta}_{\mu\nu} \left(\{ a_{\nu}^{*}, c \} + \Theta_{\nu} \right) - \left([c^{*}, c] + [a_{\mu}^{*}, a_{\mu}] \right) \bar{\Delta} \partial_{\nu} [a_{\nu}, c].$$
(2.58)

From this, we see that

$$Q_0 S_{2,\text{red}}^2 = -Q_0 [c^*, c] \bar{\Delta} \,\partial_{\nu} [a_{\nu}, c] = [c^*, c] \bar{\Delta} \Box (c^2) = [c^*, c] K(c^2), \tag{2.59}$$

and thus that the 1PR part already satisfies the CME in Eq. (2.52) at the antighost number two level, namely Eq. (2.54).

We also find that $S_{2,red}$ satisfies Eq. (2.52) at antighost number one. Indeed, recalling Eq. (2.37), defining $P_{\mu\nu}^L = p_{\mu}p_{\nu}/p^2$, and noting that

$$Q_0 \Theta_{\mu} = -\Box P_{\mu\nu}^T [a_{\nu}, c] + [\Box P_{\mu\nu}^T a_{\nu}, c], \qquad (2.60)$$

we verify the antighost number one descendent equation:

$$Q_{0}^{-}\left(\frac{1}{2}\{a_{\mu}^{*},c\}\bar{\Delta}_{\mu\nu}\{a_{\nu}^{*},c\}-[c^{*},c]\bar{\Delta}\partial_{\nu}[a_{\nu},c]\right)+Q_{0}\left(\Theta_{\mu}\bar{\Delta}_{\mu\nu}\{a_{\nu}^{*},c\}-[a_{\mu}^{*},a_{\mu}]\bar{\Delta}\partial_{\nu}[a_{\nu},c]\right)$$

$$=-\{a_{\mu}^{*},c\}\bar{\Delta}_{\mu\rho}\Box P_{\rho\nu}^{T}[a_{\nu},c]+\{a_{\mu}^{*},c\}\partial_{\mu}\bar{\Delta}\partial_{\nu}[a_{\nu},c]-[a_{\mu}^{*},a_{\mu}]\bar{\Delta}\Box(c^{2})$$

$$=[a_{\mu}^{*},a_{\mu}]Kc^{2}-\{a_{\mu}^{*},c\}K[a_{\mu},c],$$

(2.61)

where the first term on the second line comes from combining the first Q_0^- and first Q_0 terms, and the next two terms from combining the second Q_0^- and Q_0 terms. The final line coincides with the appropriate terms in Eq. (2.52).

On the other hand, after some similar manipulation one finds the descendent equation at antighost number zero:

$$Q_0^- \left(\Theta_\mu \bar{\Delta}_{\mu\nu} \{a_\nu^*, c\} - [a_\mu^*, a_\mu] \bar{\Delta} \partial_\nu [a_\nu, c]\right) + Q_0 \left(\frac{1}{2} \Theta_\mu \bar{\Delta}_{\mu\nu} \Theta_\nu\right)$$
$$= -\Theta_\mu K[a_\mu, c] + [a_\mu, a_\nu][a_\mu, \partial_\nu c], \qquad (2.62)$$

and thus that $S_{2,red}$ violates the CME in Eq. (2.52) by the last term on the right. At this point we recall that it is $S_{2,cl}$ that should satisfy the CME. From Eq. (2.56), that is therefore possible by choosing

$$\mathcal{O}_2 = -\frac{1}{4} [a_\mu, a_\nu]^2. \tag{2.63}$$

By the argument below Eq. (2.57), this is the unique solution. We recognize that it is the expected g^2 interaction in the classical bare action. And we note that we have now verified conclusion **C2**.

2.8. Classical solution at third order in coupling

From Eqs. (2.15), (2.18), and (2.56), it is straightforward to see that the g^3 part of the classical solution can be integrated exactly. It is then uniquely determined by the requirement that it have a derivative expansion:

$$S_{3,cl} = -S_1 \overleftarrow{\partial}_A \bar{\Delta}^{AB} \overrightarrow{\partial}_B \mathcal{O}_2 + \frac{1}{2} S_1 \overleftarrow{\partial}_A \bar{\Delta}^{AB} \left(\overrightarrow{\partial}_B S_1 \overleftarrow{\partial}_C \right) \bar{\Delta}^{CD} \overrightarrow{\partial}_D S_1, \qquad (2.64)$$

since from the argument of C3, by dimensions there can be no integration constant. In particular, in our case the $S_{n,cl}$ have precisely n+2 (anti)fields. In Sect. 3.1 we will also derive Eq. (2.64) from the Legendre transform identity [5].

Recall from Sect. 2.3 that similarly, thanks to compatibility with the flow equation, the g^3 part of the CME has only the potential to be violated by a Λ -independent local term. Since such a term must have ghost number one and dimension five, it must contain precisely four *a* fields and one *c*. Then, given also that it must be single trace and valued in the Lie algebra (be composed of the product of commutators), one can show that such terms vanish identically. We will see this explicitly at the end of this section.

Therefore we must already find:

$$\Sigma_{3,cl} = s_0 S_{3,cl} + (S_1, S_{2,cl}) = 0.$$
(2.65)

Now, substituting for \mathcal{O}_2 and S_1 in Eq. (2.64) and using Eqs. (2.63) and (2.49), we find that

$$S_{3,cl} = -i\left(\{a_{\mu}^{*}, c\} + \Theta_{\mu}\right)\bar{\Delta}_{\mu\nu}\left[[a_{\nu}, a_{\sigma}], a_{\sigma}\right] \\ -i\left([c^{*}, c] + [a_{\kappa}^{*}, a_{\kappa}]\right)\left(-\left[(\partial\bar{\Delta}) \cdot a\bar{\Delta}, \partial \cdot [a, c]\right] + \left[\left(\partial_{\mu}\bar{\Delta}\right)c\bar{\Delta}_{\mu\nu}, \{a_{\nu}^{*}, c\} + \Theta_{\nu}\right]\right) \\ -i\left(\{a_{\mu}^{*}, c\} + \Theta_{\mu}\right)\left\{\bar{\Delta}_{\mu\nu}a_{\nu}^{*}\bar{\Delta}, \partial \cdot [a, c]\right\} \\ + \frac{i}{2}\left(\{a_{\mu}^{*}, c\} + \Theta_{\mu}\right)\left[\left(\partial_{\rho}\bar{\Delta}_{\mu\nu}\right)a_{\nu}\bar{\Delta}_{\rho\sigma} - \left(\partial\bar{\Delta}_{\mu\nu}\right) \cdot a\bar{\Delta}_{\nu\sigma} \\ + \bar{\Delta}_{\mu\nu}\left(a_{\rho}\left(\partial_{\nu}\bar{\Delta}_{\rho\sigma}\right) - \left(\partial_{\nu}a_{\rho}\right)\bar{\Delta}_{\rho\sigma} - a \cdot \left(\partial\bar{\Delta}_{\nu\sigma}\right) + \left(\partial_{\rho}a_{\nu}\right)\bar{\Delta}_{\rho\sigma}\right), \{a_{\sigma}^{*}, c\} + \Theta_{\sigma}\right] \\ - \frac{i}{2}\partial \cdot [a, c]\left[\bar{\Delta}c^{*}\bar{\Delta}, \partial \cdot [a, c]\right], \tag{2.66}$$

where in the above equation it should be understood that the kernels $(\overline{\Delta} \text{ and } \partial \overline{\Delta})$ do not take part in the (anti)commutators (see Appendix C for more explanation). We see that all terms in $\Sigma_{3,cl}$ except (S_1, \mathcal{O}_2) are 1PR. Even (S_1, \mathcal{O}_2) is non-local, however, since it contains the regulator K. From Eq. (2.58), the antibracket $(S_1, S_{2,red})$ consists of terms with a factor K and one (ghost or gauge) propagator. These terms collect together with parts of $s_0 S_{3,cl}$ in such a way as to eliminate one propagator from $s_0 S_{3,cl}$. Splitting by antighost number, we thus find:

$$\Sigma_{3,cl}^{2} = i[c^{*},c] \Big(\Big\{ c \ \partial_{\mu}\bar{\Delta}, [a_{\mu},c] \Big\} - \Big\{ (\partial_{\mu}\bar{\Delta})c, [a_{\mu},c] \Big\} + \Big[(\partial_{\mu}\bar{\Delta})a_{\mu}, \frac{1}{2}\{c,c\} \Big] \Big) \\ + i[a_{\mu},c] \Big[(\partial_{\mu}\bar{\Delta})c^{*}, \frac{1}{2}\{c,c\} \Big] + i\{a_{\mu}^{*},c\} \Big(\Big[c \ \bar{\Delta}_{\mu\nu}, \{a_{\nu}^{*},c\} \Big] - \Big[\bar{\Delta}_{\mu\nu} a_{\nu}^{*}, \frac{1}{2}\{c,c\} \Big] \Big), \quad (2.67)$$

$$\begin{split} \Sigma_{3,\mathrm{cl}}^{1} &= i[a_{\kappa}^{*}, a_{\kappa}] \big\{ c \,\bar{\bigtriangleup}, \partial \cdot [a, c] \big\} \\ &- i\{a_{\mu}^{*}, c\} \Big(\bar{\bigtriangleup}_{\mu\nu} \big(\big[[\partial_{\nu}c, a_{\sigma}], a_{\sigma} \big] + \big[[a_{\nu}, \partial_{\sigma}c], a_{\sigma} \big] + \big[[a_{\nu}, a_{\sigma}], \partial_{\sigma}c \big] \big) \\ &- \big[\partial_{\sigma} \bar{\bigtriangleup}_{\mu\nu} a_{\nu} - \partial \bar{\bigtriangleup}_{\mu\sigma} \cdot a + \bar{\bigtriangleup}_{\mu\nu} (\partial_{\sigma} a_{\nu} - \partial_{\nu} a_{\sigma}), [a_{\sigma}, c] \big] \\ &- \big[\big(\bar{\bigtriangleup}_{\mu\nu} a_{\sigma} - \bar{\bigtriangleup}_{\mu\sigma} a_{\nu} \big), \partial_{\nu} [a_{\sigma}, c] \big] + \big[a_{\mu} \bar{\bigtriangleup}, \partial \cdot [a, c] \big] \\ &- \big[c \bar{\bigtriangleup}_{\mu\nu}, \Theta_{\nu} \big] + \bar{\bigtriangleup}_{\mu\nu} \big[c, \Theta_{\nu} \big] \Big) \\ &+ i c^{2} \big[a_{\mu}^{*} \bar{\bigtriangleup}_{\mu\nu}, \Theta_{\nu} \big] - i [a_{\mu}, c] \big\{ a_{\mu}^{*} \bar{\bigtriangleup}, \partial \cdot [a, c] \big\}. \end{split}$$

Using trace properties such as the Jacobi identity, we can then show that $\Sigma_{3,cl}^2 = \Sigma_{3,cl}^1 = 0$. Finally, for antighost number 0,

$$\Sigma_{3,cl}^{0} = -i[a_{\mu}, c]K[a_{\nu}, [a_{\nu}, a_{\mu}]] - i\Theta_{\mu}[a_{\mu}\bar{\Delta}, \partial \cdot [a, c]] - i(-\Box P_{\mu\rho}^{T}([a_{\rho}, c]))\bar{\Delta}_{\mu\nu}[[a_{\nu}, a_{\sigma}], a_{\sigma}]$$

$$= -i[a_{\mu}, c]\Big(K[a_{\nu}, [a_{\nu}, a_{\mu}]] + (\delta_{\mu\nu}(1 - K) + \partial_{\mu}\partial_{\nu}\bar{\Delta})[a_{\sigma}, [a_{\sigma}, a_{\nu}]]$$

$$+ \partial_{\mu}\bar{\Delta}[a_{\rho}, \partial_{\kappa}[a_{\kappa}, a_{\rho}] + [\partial_{\rho}a_{\kappa}, a_{\kappa}] + [a_{\kappa}, \partial_{\kappa}a_{\rho}]]\Big)$$

$$= -i[a_{\mu}, c][a_{\nu}, [a_{\nu}, a_{\mu}]] = 0. \qquad (2.68)$$

Note that the first term in the first line of Eq. (2.68) is (S_1, \mathcal{O}_2) . Its *K* part gets eliminated by a similar 1-K piece, leaving behind the local term in the last line which is of the form that we flagged at the beginning of this section, and thus vanishes.⁸ It corresponds to the unregularized version of (S_1, \mathcal{O}_2) (and also coincides with the term that appears in the 1PI formulation in Sect. 3.4), and thus its vanishing is the statement that the bare classical Yang–Mills action has no g^3 interaction.

We have thus, for this example, verified conclusion C3.

2.9. One-loop solution at second order in coupling

We now turn to the one-loop corrections. We have already seen at the end of Sect. 2.6 that S_1 is purely classical. On the other hand, S_2 has a one-loop part, $S_{2,q}$. From Eqs. (2.18) and (2.21) and the definition of s_0 in Eq. (2.24), it satisfies

$$\dot{S}_{2,q} = -a_1[S_{2,cl}]$$
 and $s_0S_{2,q} = \Delta S_{2,cl}$. (2.69)

From Eq. (2.15) and the explicit formulae for $S_{2,cl}$, viz. Eqs. (2.58) and (2.63), and recalling the rule in Eq. (2.34) for computing in the gauge-invariant basis, we see that $S_{2,q}$ is made of the two-point vertices:⁹

$$S_{2,q} = \frac{1}{2} C_A a_\mu \mathcal{A}_{\mu\nu}(\partial) a_\nu + C_A a^*_\mu \mathcal{B}(-\partial^2) \partial_\mu c, \qquad (2.70)$$

where $C_A \mathcal{A}_{\mu\nu}(p)$ and $C_A \mathcal{B}(p^2) p_{\mu}$ are the Λ -integrals of one-loop Feynman diagrams formed from attaching $\dot{\Delta}$ for the ghosts, and also for the gauge-field. For later convenience we have pulled out a factor of C_A , where $C_A \delta^{ab} = f^{acd} f^{bcd}$ defines the Casimir of the adjoint representation, and for \mathcal{B} we have used Lorentz invariance to pull out a factor of the external momentum p_{μ} . The Λ -integrals can in fact be done exactly, as will become especially clear when we consider just the flow of the 1PI part. We therefore reserve their derivation until Sects. 3.5 and 5.1.

From Eq. (2.38) we see that $\Delta S_{2,cl}$ results in one-loop diagrams formed by the action of Δ^- and $\Delta^=$ on the antighost-number one and two parts respectively of Eq. (2.58), and thus we find that

$$\Delta S_{2,\text{cl}} = C_A \, a_\mu \, \mathcal{F}(-\partial^2) \, \partial_\mu c, \qquad (2.71)$$

$$i[a_{\nu}, [a_{\mu}, c]][a_{\nu}, a_{\mu}] = \frac{i}{2}[[a_{\nu}, a_{\mu}], c][a_{\nu}, a_{\mu}] = \frac{i}{4}[[a_{\nu}, a_{\mu}]^{2}, c] = 0$$

⁸ For example, under the trace it is the same as

⁹ Strictly, we should write $\mathcal{A}_{\mu\nu}(i\partial)$ if we regard it as defined by its momentum representation, as we do for \mathcal{B} .

where

$$\mathcal{F}(p^2) = \frac{p_{\mu}}{p^2} \int_q \bar{\Delta}(q) K(p+q) \Big[(3+\xi)q_{\mu} + (2+\xi)p_{\mu} + (1-\xi)\frac{q_{\mu}}{q^2} (p+q) \cdot q \Big]$$
$$= \int_q \bar{\Delta}(q) K(p+q) \left\{ 4\frac{p \cdot q}{p^2} + 2 + \xi + (1-\xi)\frac{(p \cdot q)^2}{p^2 q^2} \right\}.$$
(2.72)

In the first line we exploit the fact that the Feynman integral is proportional to p_{μ} . Our notation is:

$$a_{\mu}(x) = \int_{p} a_{\mu}(p) e^{-ip \cdot x}, \qquad \int_{p} \equiv \int \frac{d^{4}p}{(2\pi)^{4}}.$$
 (2.73)

Now, from Eq. (2.70), since $s_0 = Q_0 + Q_0^-$ where the latter are given in Eqs. (2.36) and (2.37), we see that the second equation in Eq. (2.69) implies no constraint on \mathcal{B} but

$$p_{\mu} \mathcal{A}_{\mu\nu}(p) = \mathcal{F}(p^2) p_{\nu}. \tag{2.74}$$

From C4, this should be already satisfied, up to a possible Λ -independent piece in $\mathcal{A}_{\mu\nu}(p)$. In Sect. 5.1 we show that, for the natural form of $\mathcal{A}_{\mu\nu}(p)$, Eq. (2.74) is in fact automatically satisfied.

3. The Legendre flow equation and the mST

The 1PI parts of the Wilsonian effective action S are in fact the vertices of the Legendre effective action (Γ^{tot}) regulated by the infrared cutoff \bar{K} [5,22,23,75–77]:

$$\Gamma^{\text{tot}}[\Phi, \Phi^*] = \frac{1}{2} \Phi^A \bar{\Delta}_{AB}^{-1} \Phi^B + \Phi_A^* R^A_B \Phi^B + \Gamma_I[\Phi, \Phi^*].$$
(3.1)

In other words, S can be computed through a tree-level expansion, using vertices from Γ_I [5,22]. It is therefore most efficient to solve first for these 1PI parts. In Sects. 3.3–3.5, we recover the previous results this way, and then in Sect. 5 we evaluate the one-loop contribution and also take it further, to $O(g^3)$.

First, we demonstrate by means of the Legendre transformation [5,22,75,76] that Γ_I is governed by the flow of the effective average action [4,5,20–25,75] and the mST identities [69]. In Sects. 3.3–3.5 we also see how the two formulations match together. In Sect. 4 we demonstrate that the freedom in the continuum solution is parametrized by wavefunction renormalization constants that generate a simple canonical transformation and solve the standard Slavnov–Taylor identities.

3.1. Legendre transform of the Wilsonian flow equation and QME

In Eq. (3.1), $\Phi^* = \phi^*$ are the same antifield sources, renamed for aesthetics, $\overline{\Delta}_{AB}^{-1}$ is the inverse of the IR-regulated propagator in Eq. (2.3), and $\Phi^A = \{A, B, C, \overline{C}\}$ are the so-called classical fields. The interaction parts of each effective action are related via the Legendre transform relation [5,22,75,76]:

$$\Gamma_{I}[\Phi, \Phi^{*}] = S_{I}[\phi, \phi^{*}] - \frac{1}{2} (\phi - \Phi)^{A} \bar{\bigtriangleup}_{AB}^{-1} (\phi - \Phi)^{B}.$$
(3.2)

This means in particular that

$$\frac{\partial_r}{\partial \phi^B} S_I[\phi, \phi^*] = (\phi - \Phi)^A \bar{\Delta}_{AB}^{-1} = \frac{\partial_r}{\partial \Phi^B} \Gamma_I[\Phi, \Phi^*], \qquad (3.3)$$

and similarly for left derivatives; thus, substituting back, we can write [5,22]

$$S_{I}[\phi,\phi^{*}] = \Gamma_{I}[\phi - \bar{\Delta}\partial^{l}S_{I},\phi^{*}] + \frac{1}{2}\partial^{r}_{A}S_{I}\bar{\Delta}^{AB}\partial^{l}_{B}S_{I}.$$
(3.4)

Taylor-expanding the Γ_I term, and iteratively substituting for $S_I[\phi, \phi^*]$, then gives the desired tree (1PR) expansion:

$$S_{I} = \Gamma_{I}[\phi, \phi^{*}] - \Gamma_{I}\overleftarrow{\partial}_{A}\bar{\Delta}^{AB}\overrightarrow{\partial}_{B}S_{I} + \frac{1}{2}S_{I}\overleftarrow{\partial}_{A}\bar{\Delta}^{AB}\overrightarrow{\partial}_{B}\Gamma_{I}\overleftarrow{\partial}_{C}\bar{\Delta}^{CD}\overrightarrow{\partial}_{D}S_{I} + \dots + \frac{1}{2}S_{I}\overleftarrow{\partial}_{A}\bar{\Delta}^{AB}\overrightarrow{\partial}_{B}S_{I} = \Gamma_{I}[\phi, \phi^{*}] - \frac{1}{2}\Gamma_{I}\overleftarrow{\partial}_{A}\bar{\Delta}^{AB}\overrightarrow{\partial}_{B}\Gamma_{I} + \frac{1}{2}\Gamma_{I}\overleftarrow{\partial}_{A}\bar{\Delta}^{AB}(\overrightarrow{\partial}_{B}\Gamma_{I}\overleftarrow{\partial}_{C})\bar{\Delta}^{CD}\overrightarrow{\partial}_{D}\Gamma_{I} + O(\Gamma_{I}^{4}).$$
(3.5)

For example, specializing to the purely classical part reproduces the general form of the solutions in Eqs. (2.56) and (2.64), on identifying $\Gamma_{I,cl}$ with the 1PI part of $S_{I,cl}$.

We now use the Legendre transformation to confirm the form of the flow equation and mST identities for Γ_I . Indeed, from Eqs. (3.2) and (3.3),

$$\partial_t|_{\Phi}\Gamma_I = \partial_t|_{\phi}S_I + \frac{1}{2}(\phi - \Phi)^A \left(\bar{\Delta}^{-1}\dot{\bar{\Delta}}\bar{\Delta}^{-1}\right)_{AB}(\phi - \Phi)^B = \dot{S}_I + \frac{1}{2}\frac{\partial_r S_I}{\partial\phi^A}\dot{\bar{\Delta}}^{AB}\frac{\partial_l S_I}{\partial\phi^B}.$$
(3.6)

Using Eq. (2.15), and differentiating Eq. (3.3) once more with respect to the fields, one has

$$\frac{\partial_r \Phi^A}{\partial \phi^B} = \left(\left[1 + \bar{\Delta} \Gamma_I^{(2)} \right]^{-1} \right)^A_{\ B}, \tag{3.7}$$

and thus we derive the flow equation for Γ_I [5]:

$$\dot{\Gamma}_{I} = -\frac{1}{2} \operatorname{Str} \left(\dot{\bar{\Delta}} \bar{\Delta}^{-1} \left[1 + \bar{\Delta} \Gamma_{I}^{(2)} \right]^{-1} \right)$$
(3.8)

(see also Refs. [4,20,22–25]), where Str $\mathcal{M} = (-)^A \mathcal{M}^A_A$, and we have introduced

$$\Gamma_{IAB}^{(2)} = \frac{\overrightarrow{\partial}}{\partial \Phi^A} \Gamma_I \frac{\overleftarrow{\partial}}{\partial \Phi^B}.$$
(3.9)

It is a short step to recast this as the flow for the effective average action [21,75], Γ , but it is Eq. (3.8) that will be useful here. On the other hand, the mST is best expressed in terms of Γ . This 1PI effective action is just the IR-cutoff Legendre effective action after subtracting the infrared cutoff term, where the latter is expressed in additive form:

$$\Gamma^{\text{tot}} = \Gamma + \frac{1}{2} \Phi^A \mathcal{R}_{AB} \Phi^B, \qquad \bar{\Delta}_{AB}^{-1} = \Delta_{AB}^{-1} + \mathcal{R}_{AB}.$$
(3.10)

As implied by Eq. (3.1), Γ is thus expressed in terms of a free part, Γ_0 , that carries no regularization:

$$\Gamma = \Gamma_0 + \Gamma_I, \qquad \Gamma_0 = \frac{1}{2} \Phi^A \triangle_{AB}^{-1} \Phi^B + \Phi_A^* R_B^A \Phi^B.$$
(3.11)

Consistent with this, the antibracket is most naturally expressed without regularization. Thus, for arbitrary functionals of the classical (anti)fields, $\Xi[\Phi, \Phi^*]$ and $\Upsilon[\Phi, \Phi^*]$, we define

$$(\Xi,\Upsilon) = \frac{\partial_r \Xi}{\partial \Phi^A} \frac{\partial_l \Upsilon}{\partial \Phi_A^*} - \frac{\partial_r \Xi}{\partial \Phi_A^*} \frac{\partial_l \Upsilon}{\partial \Phi^A}.$$
(3.12)

To arrive at the mST, we also need from Eq. (3.2) that (and similarly for left derivatives)

$$\frac{\partial_r}{\partial \phi_A^*} \bigg|_{\phi} S_I[\phi, \phi^*] = \frac{\partial_r}{\partial \Phi_A^*} \bigg|_{\Phi} \Gamma_I[\Phi, \Phi^*].$$
(3.13)

Recalling the K regularization in the antibracket of Eq. (2.6), and using Eq. (2.1), we have

$$\frac{1}{2}(S,S) = (S_{I},S_{0}) + \frac{1}{2}(S_{I},S_{I})
= \frac{\partial_{r}S_{I}}{\partial\phi^{A}}R^{A}_{B}\phi^{B} - \frac{\partial_{r}S_{I}}{\partial\phi^{*}_{A}}\left(\Delta^{-1}_{AB}\phi^{B} + (-)^{A}R^{B}_{A}\phi^{*}_{B}\right) + \frac{\partial_{r}S_{I}}{\partial\phi^{*}_{A}}\bar{K}\frac{\partial_{l}S_{I}}{\partial\phi^{A}} - \frac{\partial_{r}S_{I}}{\partial\phi^{*}_{A}}\frac{\partial_{l}S_{I}}{\partial\phi^{A}}
= \frac{\partial_{r}\Gamma_{I}}{\partial\Phi^{A}}R^{A}_{B}\left(\Phi^{B} + \frac{\partial_{r}\Gamma_{I}}{\partial\Phi^{C}}\bar{\Delta}^{CB}\right) - \frac{\partial_{r}\Gamma_{I}}{\partial\Phi^{*}_{A}}\left(\Delta^{-1}_{AB}\Phi^{B} + (-)^{A}R^{B}_{A}\Phi^{*}_{B}\right) + \frac{1}{2}(\Gamma_{I},\Gamma_{I})
= (\Gamma_{I},\Gamma_{0}) + \frac{1}{2}(\Gamma_{I},\Gamma_{I}) = \frac{1}{2}(\Gamma,\Gamma), \qquad (3.14)$$

where in the third line we used Eqs. (3.3) and (3.13), in particular converting $\bar{K}\partial_A^I S_I$ to $\Delta_{AB}^{-1}(\phi - \Phi)^B$, and the antibracket in Eq. (3.12). Then we note that the second term in the first bracket vanishes because

$$\Delta^{CB}R^A_{\ B} + \Delta^{AB}R^C_{\ B} = 0, \tag{3.15}$$

by linearized BRST invariance (see Appendix B), after which we are left with the expanded version of (Γ_I , Γ_0), as one can see by comparing to the analogous terms in the second line. Finally,

$$\Delta S_I = \frac{\partial_r}{\partial \phi^A} K \frac{\partial_l S_I}{\partial \phi^*_A} = \frac{\partial_r}{\partial \Phi^B} \left(K \frac{\partial_l \Gamma_I}{\partial \Phi^*_A} \right) \frac{\partial_r \Phi^B}{\partial \phi^A} = \operatorname{Tr} \left(K \Gamma_{I*}^{(2)} \left[1 + \bar{\Delta} \Gamma_I^{(2)} \right]^{-1} \right), \tag{3.16}$$

where we used Eq. (2.6) and the fact that S_I is bosonic, followed by Eq. (3.13) then Eq. (3.7), set

$$\left(\Gamma_{I*}^{(2)}\right)_{B}^{A} = \frac{\overrightarrow{\partial}}{\partial \Phi_{A}^{*}} \Gamma_{I} \frac{\overleftarrow{\partial}}{\partial \Phi^{B}}, \qquad (3.17)$$

and write Tr $\mathcal{M} = \mathcal{M}_{A}^{A}$. Thus we have shown that the QMF in Eq. (2.8) can be written as

$$\Sigma = \frac{1}{2}(\Gamma, \Gamma) - \operatorname{Tr}\left(K\Gamma_{I*}^{(2)}\left[1 + \bar{\Delta}\Gamma_{I}^{(2)}\right]^{-1}\right).$$
(3.18)

Its vanishing is the QME. We see that it coincides with the mST identities first derived in Ref. [69].

3.2. Discussion

In the limit $\Lambda \to 0$ (holding (anti)fields fixed), we have that $\Gamma \to \Gamma^{\text{tot}}$, the standard Legendre effective action without cutoff, as is clear from Eq. (3.1). The correction terms in Eq. (3.18) are UV regularized by *K* (and IR regularized by \bar{K}). They therefore vanish in this limit. The antibracket in Eq. (3.12) is not regularized and thus the mST tends to the Zinn-Justin equation [48–50,69], a.k.a. CME,

$$0 = \Sigma \to \frac{1}{2}(\Gamma, \Gamma) \qquad \text{as} \qquad \Lambda \to 0, \tag{3.19}$$

as desired. As we will see in Sect. 3.3, the mST and this CME lead to analogous definitions for the BRST charges and Koszul–Tate operator.

Although the antibracket is not regularized, it only introduces trees. Then, as we will see, UV regularization provided in the correction terms in Eq. (3.18), and through Eq. (3.8), is enough to ensure that expressions are well defined. Introducing an overall UV regularization $K_0 = K(p^2/\Lambda_0^2)$, by writing $1 - K \mapsto K_0 - K$ in Eq. (3.2) [5,22], does not change the form of the flow equation in Eq. (3.8). However, the regularized propagator becomes $\overline{\Delta} = \Delta(K_0 - K)$. Following through the derivations, it indeed makes its appearance this way in Eq. (3.18), while the antibracket in Eq. (3.12) is then regularized by K_0 .

The antibracket (,) $_{\phi}$ for quantum fields introduced in Eq. (2.6) implies the canonical structure $(\phi^A, \phi_B^*)_{\phi} = K \delta_B^A$. However, for this antibracket Φ is not canonically conjugate to Φ^* :

$$(\Phi^A, \Phi^*_B)_{\phi} = (\Phi^A, \phi^*_B)_{\phi} = K \frac{\partial_r}{\partial \phi^B} \Phi^A \neq K \delta^A_B.$$
(3.20)

Indeed, the antibracket $(,)_{\Phi}$ introduced in Eq. (3.12) is not canonically related to $(,)_{\phi}$. Although Σ is invariant under quantum canonical transformations $S \mapsto S + \varepsilon \hat{s} \mathcal{K}$, cf. Eq. (2.11), the correction term in the mST of Eq. (3.18) provides an obstruction to canonical transformations of Γ that respect $(,)_{\Phi}$. In particular, the analogous canonical transformation to Eq. (2.30), namely

$$\bar{C}^*|_{gf} = \bar{C}^*|_{gi} - i\partial \cdot A,$$

$$A^*_{\mu}|_{gf} = A^*_{\mu}|_{gi} + i\partial_{\mu}\bar{C},$$
(3.21)

does not leave Eq. (3.18) invariant. However, since Eq. (2.30) does leave Σ invariant, the Legendre transformation in Eq. (3.2) implies that the mST must be invariant under a non-linear transformation, as we show in Appendix D.

On the other hand, as we will confirm in Sect. 3.3, we can still use the arguments at the end of Sect. 2.4 to see that Γ_I does not depend on \bar{C}^* or *B*, and depends on \bar{C} only through the combination $A^*_{\mu} - i\partial_{\mu}\bar{C}$. Then we can work, as we will do from now on, in the minimal gauge-invariant basis, provided we make the replacement (or equivalently temporarily replace A^*_{μ} with $A^*_{\mu} - i\partial_{\mu}\bar{C}$)

$$\frac{\partial}{\partial \bar{C}}\Big|_{\rm gf} \equiv i\partial_{\mu} \frac{\partial}{\partial A^*_{\mu}}\Big|_{\rm gi} \tag{3.22}$$

in both the flow equation, Eq. (3.8), and the mST, Eq. (3.18).

For example, in this basis we can write the free action Γ_0 explicitly as

$$\Gamma_{0} = \frac{1}{2} \left(\partial_{\mu} A_{\nu} \right)^{2} - \frac{1}{2} \left(\partial \cdot A \right)^{2} + A_{\mu}^{*} \partial_{\mu} C, \qquad (3.23)$$

as can be seen by comparing Eqs. (3.11), (2.1), and (2.25). In the next sections, we show how perturbative expansion of the flow in Eq. (3.8) and mST in Eq. (3.18) via

$$\Gamma = \Gamma_0 + g\Gamma_1 + g^2\Gamma_2 + g^3\Gamma_3 + \cdots$$
(3.24)

rephrases the properties and results for the Wilsonian action in Sects. 2.6–2.9.

3.3. Perturbations: First order in coupling

Expanding Eq. (3.18) to first order, we get the same expression for the free full BRST charge in terms of the classical fields:

$$\hat{s}_0 \,\Gamma_1 = (Q_0 + Q_0^- - \Delta^- - \Delta^=) \,\Gamma_1 = 0. \tag{3.25}$$

Analogous to Eqs. (2.5) and (2.7), we have defined the right-acting

$$Q \Phi^A = (\Phi^A, \Gamma) = \frac{\partial_l \Gamma}{\partial \Phi^*_A}$$
 and $Q^- \Phi^*_A = (\Phi^*_A, \Gamma) = -\frac{\partial_l \Gamma}{\partial \Phi^A}$, (3.26)

the full quantum BRST charge, and full classical BRST charge:

$$\hat{s}\mathcal{O} = (Q+Q^{-}-\Delta)\mathcal{O}, \qquad s_{\rm cl}\mathcal{O} = (Q_{\rm cl}+Q_{\rm cl}^{-})\mathcal{O} = (\mathcal{O},\Gamma_{\rm cl}). \tag{3.27}$$

This leads to the same free algebra as Eqs. (2.36) and (2.37), which in particular are again unregulated:

$$Q_0 A_\mu = \partial_\mu C, \qquad Q_0^- A_\mu^* = \Box A_\mu - \partial_\mu \partial \cdot A, \qquad Q_0^- C^* = -\partial \cdot A^*, \tag{3.28}$$

but this time because neither Γ_0 nor the antibracket is regulated. On the other hand, the measure operator appears from the first-order part of Eq. (3.18) as

$$\Delta \Gamma = \operatorname{Tr}(K\Gamma_*^{(2)}), \tag{3.29}$$

and thus $\Delta = \Delta^- + \Delta^=$ also continues to take the same form:

$$\Delta^{-} = \frac{\partial}{\partial A^{a}_{\mu}} K \frac{\partial_{l}}{\partial A^{*a}_{\mu}}, \qquad \Delta^{=} = -\frac{\partial_{l}}{\partial C^{a}} K \frac{\partial}{\partial C^{*a}}; \qquad (3.30)$$

in particular, it continues to be regulated by K.

Since Eq. (3.25) is thus identical in form to the analogous equation for S_1 , and since at the classical level Eq. (3.8) just says that the operator must be Λ independent, we have the same unique (up to normalization) non-trivial solution:

$$\Gamma_1 = \Gamma_{1,cl} = -iC^*C^2 - iA^*_{\mu}[A_{\mu}, C] - i\partial_{\mu}A_{\nu}[A_{\mu}, A_{\nu}].$$
(3.31)

This is trivially a first-order solution at the quantum level also, for the same reasons as before. Indeed, Eq. (3.30) takes the same form, as does Eq. (3.8) to this order,

$$\dot{\Gamma}_1 = \frac{1}{2}\operatorname{Str}(\dot{\Delta}\Gamma_1^{(2)}) = -a_1[\Gamma_1], \qquad (3.32)$$

by Eq. (2.15). Thus we have constructed the cohomologically unique first-order simultaneous solution of the mST and flow equation for the 1PI effective action.

3.4. Higher orders in coupling at the classical level

At the classical level, the flow equation in Eq. (3.8) just determines Λ -integration constants, since it says that Γ_{cl} must be Λ independent. Meanwhile, the mST, Eq. (3.18), reduces to the CME. Thus the simultaneous solution is just the same as obtained in classical BRST cohomology for a local dimension-four operator.

At $O(g^2)$,

$$s_0 \Gamma_{2,cl} = (Q_0 + Q_0^-) \Gamma_{2,cl} = -\frac{1}{2} (\Gamma_1, \Gamma_1) = Q_0 \left(-\frac{1}{4} [A_\mu, A_\nu]^2 \right),$$
(3.33)

where we substituted Eq. (3.31). This provides us directly with the unique classical integration constant:

$$\Gamma_{2,cl} = -\frac{1}{4} [A_{\mu}, A_{\nu}]^2.$$
(3.34)

We note that although this coincides with the (1PI) integration constant in Eq. (2.63) found in the Wilsonian formulation, it arises in a very different (and much simpler) way. In particular, although the starting point looks the same, Eq. (2.50) contains K regularization in the antibracket.

At $O(g^3)$, we would need to solve $s_0 \Gamma_{3,cl} = -(\Gamma_1, \Gamma_{2,cl})$ for some local classical five-point vertex $\Gamma_{3,cl}$. Since that is not possible by dimensions, it better be that $(\Gamma_1, \Gamma_{2,cl})$ already vanishes. This is straightforward to confirm using Eqs. (3.31) and (3.34). In fact, this is nothing but the final term in Eq. (2.68).

We note that we have now confirmed the identification of $\Gamma_{I,cl}$ with the 1PI part of $S_{I,cl}$ as used below Eq. (3.5).

For n > 3, $\Gamma_{n,cl}$ must also vanish by dimensions, and this is consistent with the classical master functional

$$s_0 \Gamma_n + \frac{1}{2} \sum_{m=1}^{n-1} (\Gamma_{n-m}, \Gamma_m),$$
 (3.35)

which now must also vanish by dimensions (cf. C3), as is also true directly by iteration.

3.5. One-loop equations

To second order and one loop, the flow equation, Eq. (3.8), and mST, Eq. (3.18), now read

$$\dot{\Gamma}_{2,q} = \frac{1}{2} \operatorname{Str} \left(- \dot{\bar{\bigtriangleup}} \Gamma_1^{(2)} \bar{\bigtriangleup} \Gamma_1^{(2)} + \dot{\bar{\bigtriangleup}} \Gamma_{2,cl}^{(2)} \right),$$
(3.36)

$$s_0 \Gamma_{2,q} = \Delta \Gamma_{2,cl} - \operatorname{Tr} \left(K \Gamma_{1*}^{(2)} \bar{\Delta} \Gamma_1^{(2)} \right).$$
(3.37)

We recognize that these equations coincide with Eq. (2.69) since the term in brackets in Eq. (3.36) contains the second functional derivative of Eq. (2.56):

$$S_{2,cl}^{(2)} = -S_1^{(2)}\bar{\bigtriangleup}S_1^{(2)} + \mathcal{O}_2^{(2)} + \cdots, \qquad (3.38)$$

where the ellipses refer to terms where S_1 is differentiated three times and which vanish (by tr $T^a = 0$) on contraction with the differentiated propagator $\dot{\Delta}$ to form a_1 as defined in Eq. (2.15). Similarly, the second term in Eq. (3.37) coincides with the action of Δ on Eq. (2.57). The solution is therefore just a renaming of $S_{2,q}$ in Eq. (2.70), i.e.

$$\Gamma_{2,q} = \frac{1}{2} C_A A_\mu \mathcal{A}_{\mu\nu}(\partial) A_\nu + C_A A^*_\mu \mathcal{B}(-\partial^2) \partial_\mu C, \qquad (3.39)$$

where $A_{\mu\nu}$ obeys Eq. (2.74).

Due to the cyclicity of the supertrace, Eq. (3.36) is immediately integrable:

$$\Gamma_{2,q} = \operatorname{Str}\left(-\frac{1}{4}\bar{\Delta}\Gamma_{1}^{(2)}\bar{\Delta}\Gamma_{1}^{(2)} + \frac{1}{2}\bar{\Delta}\Gamma_{2,cl}^{(2)}\right),\tag{3.40}$$

which of course just comes from the expansion of the closed-form one-loop solution to Eq. (3.8):

$$\Gamma_{q} = \frac{1}{2} \operatorname{Str} \ln \left(\bar{\Delta}^{-1} + \Gamma_{cl}^{(2)} \right).$$
(3.41)

Expanding further, we can therefore also write down the solution to the flow equation at third order, and thus we have, together with expanding the mST in Eq. (3.18) to third order and

one loop:

$$\Gamma_{3,q} = \operatorname{Str}\left(-\frac{1}{2}\bar{\Delta}\Gamma_{1}^{(2)}\bar{\Delta}\Gamma_{2,cl}^{(2)} + \frac{1}{6}\bar{\Delta}\Gamma_{1}^{(2)}\bar{\Delta}\Gamma_{1}^{(2)}\bar{\Delta}\Gamma_{1}^{(2)}\right),\tag{3.42}$$

$$s_0 \Gamma_{3,q} = -(\Gamma_1, \Gamma_{2,q}) + \text{Tr} \Big(-K \Gamma_{1*}^{(2)} \bar{\Delta} \Gamma_{2,cl}^{(2)} + K \Gamma_{1*}^{(2)} \bar{\Delta} \Gamma_1^{(2)} \bar{\Delta} \Gamma_1^{(2)} \Big).$$
(3.43)

The expressions in Eqs. (3.40)–(3.42) for the one-loop 1PI effective action are IR regulated as they must be since they must have derivative expansions. They should be understood to be supplied with integration constants, which are thus also derivative expansions but independent of Λ . However, since the only scale is Λ , by dimensions all but a small number of integration constants must actually vanish.

The non-trivial functional dependence on Λ , as specified by the body of the momentum integrals that make up the one-loop vertices, is thus unambiguous. For such momentum integrals the mST identities in Eqs. (3.18), (3.37), and (3.43) must therefore already be satisfied, as we claimed in C4. We will confirm this in Sect. 5.2.

The Λ -integration constants that do not necessarily vanish multiply marginal and relevant operators whose coefficient integrals are typically UV divergent. In these cases the integration constants must be chosen to cancel the divergence so that we get finite solutions as required. Nevertheless, in the resulting finite solution, since by dimensions the relevant operators necessarily have Λ -dependent coefficients, their integration constants are determined uniquely. (We will see examples in Sect. 5.) Only for the marginal operators is there some arbitrariness, where the integration constants must also be adjusted so as to respect the mST identities. This is the subject of the next section.

4. BRST structure and renormalization in the presence of the effective cutoff

Now we note that we are dealing with a theory that is well defined at non-exceptional momenta (i.e. external momenta such that none of their partial sums vanish or are null). Consider, for example, a two-point vertex that depends only on one external momentum p. The theory thus has the property that, properly renormalized, no IR divergences appear as $\Lambda \rightarrow 0$, provided that we keep the Euclidean momentum $p \neq 0$. Then we see that the UV subtractions provided by the integration constants to render the vertices finite at $\Lambda \neq 0$ can be chosen to render the vertices finite at non-exceptional momenta in the limit $\Lambda \rightarrow 0$. We will confirm this in detail in Sect. 5.

For the marginal operators, we will furthermore be forced by dimensions to include $\ln \mu$ dependence from canceling off logarithmic UV divergences. It is this $\ln \mu$ dependence that introduces arbitrariness into the Λ -integration constants and that we need to keep track of by renormalization conditions.

As can be seen from Eqs. (3.37) and (3.43), the parts that have this arbitrariness must be s_0 -invariant. For the two-point vertices we have two solutions,

$$\frac{1}{2}A_{\mu}(-\Box\delta_{\mu\nu}+\partial_{\mu}\partial_{\nu})A_{\nu} \quad \text{and} \quad A_{\mu}^{*}\partial_{\mu}C, \quad (4.1)$$

while for three-point vertices there is only one solution, namely a multiple of Eq. (3.31). The higherpoint vertices have no freedom in the solution (as we already noted in Sects. 2.7, 2.8, and 3.4).

Let us spell out how this implies, despite the presence of a cutoff *K*, the same key steps that one obtains in the proof of renormalizability of gauge theories using the Zinn-Justin equation, a.k.a. CME: $(\Gamma, \Gamma) = 0$. Altering the coefficients in front of Eqs. (4.1) and (3.31) induces changes in the rest of the solution. Indeed, from inspection of Eq. (3.43) we see that addition of an *s*₀-closed perturbation $\delta\Gamma_{2,q}$ requires a change $\delta\Gamma_{3,q}$ such that

$$s_0 \,\delta\Gamma_{3,q} := (\delta\Gamma_{3,q}, \Gamma_0) = -(\Gamma_1, \delta\Gamma_{2,q}). \tag{4.2}$$

Since the classical solutions for Γ_0 and Γ_1 are unchanged at the quantum level, the above is just the g^3 part of the demand that any change to the one-loop Λ -integration constants results in an operator $\delta\Gamma^{(\ell=1)} \equiv \delta\Gamma_q$ that is invariant under the full classical BRST transformation, Eq. (3.27), i.e.

$$s_{\rm cl}\,\delta\Gamma^{(\ell)} := (\delta\Gamma^{(\ell)}, \Gamma_{\rm cl}) = 0. \tag{4.3}$$

For $\ell = 1$, this is clearly true because at one loop the rest of the mST in Eq. (3.18) depends only on the classical effective action. Summing the contributions of Eqs. (3.23), (3.31), and (3.34) (there being no more, cf. Sect. 3.4), the classical effective average action in Eq. (3.11) is given in total by

$$\Gamma_{\rm cl} = \Gamma_0 + g\Gamma_1 + g^2\Gamma_{2,\rm cl} = -igC^*C^2 + A^*_{\mu}D_{\mu}C + \frac{1}{4}F^2_{\mu\nu}, \tag{4.4}$$

i.e. just the usual expression for the classical action complete with antifield sources for the classical BRST transformations¹⁰ $Q_{cl}C$ and $Q_{cl}A_{\mu}$, cf. Eq. (3.27), where the covariant derivative and field strength are defined as:

$$D_{\mu}C = \partial_{\mu}C - ig[A_{\mu}, C]$$
 and $F_{\mu\nu} = \frac{i}{g}[D_{\mu}, D_{\nu}].$ (4.5)

In general, $\delta\Gamma^{(\ell)}$ contains all the (changes in) Λ -integration constants at the ℓ -loop level. As we already noted in Sect. 2.3, these integration constants play the rôle of counterterms in the standard treatment. Indeed, it is clearly true that at ℓ loops, they must also satisfy Eq. (4.3), just as one would deduce from the Zinn-Justin equation [48–50], since the rest of the mST depends only on lower loop orders, either through $\frac{1}{2} \sum_{j=1}^{\ell-1} (\Gamma^{(\ell-j)}, \Gamma^{(j)})$, or because the correction term in Eq. (3.18) contributes one extra loop (equivalently, one extra factor of \hbar).

Although we have derived this for Γ , closely similar arguments apply to the Wilsonian action *S*, as we have already intimated in Sect. 2.3. Indeed, the part $\delta S^{(\ell)}$ containing the Λ -integration constants must, by the QME in Eq. (2.8), satisfy the analogous equation at ℓ -loop order,

$$s_{\rm cl}\,\delta S^{(\ell)} := (\delta S^{(\ell)}, S_{\rm cl}) = 0,$$
(4.6)

since the measure operator Δ also supplies an extra loop. Furthermore, Eq. (3.14) implies that any violations of $\Sigma = 0$ would be equal under the Legendre transform relation:

$$s_{\rm cl}\,\delta S^{(\ell)} := (\delta S^{(\ell)}, S_{\rm cl}) = (\delta \Gamma^{(\ell)}, \Gamma_{\rm cl}) =: s_{\rm cl}\,\delta \Gamma^{(\ell)}. \tag{4.7}$$

A major difference, however, is that S_{cl} is not a closed expression, but rather contains contributions to all orders in g (this being consistent with the fact that its antibracket in Eq. (2.6) carries K regularization).

To bring the kinetic terms back to normalized form, as defined by some renormalization condition, we need to apply wavefunction renormalization, i.e. a rescaling of the (anti)fields. Order by order in the coupling, or loop expansion, the rescaling will then provide us with the corresponding choices for the Λ -integration constants. We have seen that these perturbative contributions must be closed

¹⁰ Q_{cl} preserves antighost number, but this is somewhat accidental. To exploit antighost number with interacting charges generally requires regrading since antighost number is not conserved. For example, from Eq. (3.27), $Q_{cl}^-C^*$ now has a part that does not lower antighost number, while from Eq. (2.58), we see that $Q_{2,cl}a_{\mu}$ has a part that raises it by one.

under the classical BRST transformations. This implies that the wavefunction rescaling must in fact be a (classical) canonical transformation, and thus of the particularly simple form

$$\mathcal{K} = Z_E^{\frac{1}{2}} \Phi_E^* \Phi_{(r)}^E, \tag{4.8}$$

where summation over the repeated indices should still be understood as determined by the (anti)fields, while Z, however, also depends on (anti)field flavor, and where we use the general recipe for a finite classical canonical transformation (see, e.g., [47]):

$$\Phi^{E} = \frac{\partial_{l}}{\partial \Phi^{*}_{E}} \mathcal{K}[\Phi_{(r)}, \Phi^{*}], \qquad \Phi^{*}_{(r)E} = \frac{\partial_{r}}{\partial \Phi^{E}_{(r)}} \mathcal{K}[\Phi_{(r)}, \Phi^{*}], \tag{4.9}$$

the subscript (r) labeling the renormalized (anti)fields. Thus the fields and antifields renormalize in opposite directions:

$$A_{\mu} = Z_{A}^{\frac{1}{2}} A_{(r)\mu}, \quad A_{\mu}^{*} = Z_{A}^{-\frac{1}{2}} A_{(r)\mu}^{*}, \quad C = Z_{C}^{\frac{1}{2}} C_{(r)}, \quad C^{*} = Z_{C}^{-\frac{1}{2}} C_{(r)}^{*}.$$
(4.10)

The freedom in the Λ -independent part of the solution has therefore been parametrized as

$$\frac{1}{2}Z_{A}^{-1}A_{\mu}(-\Box\delta_{\mu\nu}+\partial_{\mu}\partial_{\nu})A_{\nu} + Z_{A}^{\frac{1}{2}}Z_{C}^{-\frac{1}{2}}A_{\mu}^{*}\partial_{\mu}C - igZ_{g}^{-1}Z_{C}^{-\frac{1}{2}}\left(C^{*}C^{2}+A_{\mu}^{*}[A_{\mu},C]\right) - igZ_{g}^{-1}Z_{A}^{-\frac{3}{2}}\partial_{\mu}A_{\nu}[A_{\mu},A_{\nu}] - \frac{1}{4}g^{2}Z_{g}^{-2}Z_{A}^{-2}[A_{\mu},A_{\nu}]^{2}.$$
(4.11)

Here we have also introduced Z_g for renormalizing the coupling, which thus takes care of the separate freedom that appears at the three-point level. The above equation appears with the inverse transformation so that on applying Eq. (4.10) and $g = Z_g g_{(r)}$ the result appears in normalized form in terms of the renormalized fields and coupling.

We note that at one loop we will find that all these factors take the form

$$Z_n = 1 + g^2 z_n, (4.12)$$

where n = A, C, g, and in fact the

$$z_n = \gamma_n \ln(\mu/\Lambda) + z_n^0, \tag{4.13}$$

where the γ_n are computable (as we will see) and the freedom is here overparametrized in the choice of μ and the z_n^0 . We also note that it is straightforward to verify that perturbative contributions to Eq. (4.11) do satisfy Eq. (4.3). For example, from Eq. (4.12) the one-loop $O(g^2)$ part of Eq. (4.11) parametrizes the change in the two-point vertices (kinetic terms) as $\delta\Gamma_{q,2} = s_0 (g^2 \mathcal{K}_2)$, where

$$\mathcal{K}_2 = \frac{1}{2} z_A A_\mu^* A_\mu + \frac{1}{2} z_C C^* C.$$
(4.14)

Again, closely similar arguments apply to the Wilsonian action. Apart from the freedom Z_g to adjust the coefficient of S_1 , we only have the freedom to alter the analogous kinetic terms to Eq. (4.1). Thus these latter Λ -integration constants must also be parametrized by a canonical transformation:

$$\phi^{E} = \frac{\partial_{l}}{\partial \phi_{E}^{*}} \tilde{\mathcal{K}}[\phi_{(r)}, \phi^{*}], \qquad \phi_{(r)E}^{*} = \frac{\partial_{r}}{\partial \phi_{(r)}^{E}} \tilde{\mathcal{K}}[\phi_{(r)}, \phi^{*}]$$
(4.15)

(the antibracket is invariant under this by statistics and thus also with the regularization in Eq. (2.6) [58]), where now

$$\tilde{\mathcal{K}} = Z_E^{\frac{1}{2}} \phi_E^* \phi_{(r)}^E, \tag{4.16}$$

and thus

$$a_{\mu} = Z_{A}^{\frac{1}{2}} a_{(r)\mu}, \quad a_{\mu}^{*} = Z_{A}^{-\frac{1}{2}} a_{(r)\mu}^{*}, \quad c = Z_{C}^{\frac{1}{2}} c_{(r)}, \quad c^{*} = Z_{C}^{-\frac{1}{2}} c_{(r)}^{*}.$$
(4.17)

Note that these Z factors are indeed the same as the ones in the 1PI effective action, as might be expected given that Γ_I provides the 1PI part of S. See Appendix E for a proof.

The changes of variables in Eqs. (4.10) and (4.17) do not leave the cutoffs, and thus neither the flow equations in Eqs. (3.8) and (2.15) nor the Legendre transform identity in Eq. (3.2), invariant. However, as we have been emphasizing (and will see particularly in the remainder of the paper), the flow equations can be solved for directly in terms of finite, and thus already renormalized, quantities. What this lack of invariance means, however, is that we can only enforce renormalization conditions at some fixed (finite) scale, for example at $\Lambda = \mu$. The simplest choice then is to set

$$Z_n = 1 \qquad \text{at} \qquad \Lambda = \mu, \tag{4.18}$$

so that in Eq. (4.13) we have just

$$z_n = \gamma_n \ln(\mu/\Lambda). \tag{4.19}$$

For finite values of $\Lambda \neq \mu$, the effective action is still finite but the kinetic terms appear as in Eq. (4.11) and are not in normalized form. Equations (4.10) then provide us with the further finite renormalization that would be required to bring the kinetic terms back to normalized form.

We acknowledge that one can start instead with altered flow equations, both for the Wilson/Polchinski effective action and for the 1PI effective action, such that they depend on running anomalous dimensions $\gamma_n(\Lambda)$ that are determined by setting renormalization conditions that hold at all values of Λ . However, in this case the map between the two effective actions is no longer as simple as Eq. (3.2) [76,87].

For completeness we also consider the extended gauge-invariant basis and gauge-fixed basis. Although we display only the 1PI formulation, again the analogous equations hold for the Wilsonian action. From Sect. 2.4, we see that in the extended gauge-invariant basis we also include in Γ

$$\frac{\xi}{2}B^2 + \bar{C}^*B;$$
 (4.20)

however, these terms are not generated by quantum corrections so do not need renormalization. In the gauge-fixed basis we make the analogous canonical transformation, $\Phi_A^*|_{gf} = \Phi_A^*|_{gi} + \partial_A^r \Psi$, to Eq. (2.28), where the gauge fermion is the classical field equivalent of Eq. (2.29), namely $\Psi = -i \bar{C} \partial \cdot A$, implying the transformations in Eq. (3.21). Recall that the dependence on \bar{C} is only through the combination $A_{\mu}^* - i \partial_{\mu} \bar{C}$. In order that this is also renormalized, we therefore need \bar{C} to renormalize like A_{μ}^* . By Eq. (4.8), this implies that \bar{C}^* must renormalize like A_{μ} . We see that these choices then also leave the gauge fermion invariant and preserve the form of the transformations in Eq. (3.21). Since Eq. (4.20) receives no corrections, this also determines the renormalization of B and ξ . In summary, we must supplement Eq. (4.10) with

$$\bar{C} = Z_A^{-\frac{1}{2}} \bar{C}_{(r)}, \quad \bar{C}^* = Z_A^{\frac{1}{2}} \bar{C}_{(r)}^*, \quad B = Z_A^{-\frac{1}{2}} B_{(r)}, \quad \xi = Z_A \,\xi_{(r)}.$$
 (4.21)

Since integrating out *B* gives the gauge-fixing term $(\partial \cdot A)^2 / \xi$, we see that the ξ renormalization is consistent with standard methods where one finds that the gauge-fixing term does not renormalize.

These wavefunction renormalization factors in Eqs (4.10) and (4.21) (similarly in Eq. (4.17), etc.) are not the standard ones, because Eq. (4.11) already solves the Slavnov–Taylor wavefunction renormalization identities. Although Z_A is Z_3 in standard notation, both Z_C and Z_A renormalize the (anti)ghost fields. In a standard parametrization (e.g. [88]) we would identify

$$Z_3 = Z_A, \quad \tilde{Z}_3 = Z_C^{\frac{1}{2}} Z_A^{-\frac{1}{2}}, \quad Z_1 = Z_g Z_A^{\frac{3}{2}}, \quad \tilde{Z}_1 = Z_g Z_C^{\frac{1}{2}}, \quad Z_4 = Z_g^2 Z_A^2, \quad (4.22)$$

where \tilde{Z}_3 is the wavefunction renormalization factor for both *C* and \bar{C} , and Z_1 , \tilde{Z}_1 , and Z_4 are the factors for the A^3 , $\bar{C}AC$, and A^4 vertices, respectively. Then the following fractions are all equal to $Z_g Z_A^{\frac{1}{2}}$:

$$\frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_4}{Z_1}.$$
(4.23)

These are indeed the Slavnov–Taylor identities guaranteeing the universality of the gauge coupling. Thus we have verified **C5**.

5. One-loop 1PI solutions

In the previous section we explained how a restricted set of wavefunction renormalization factors Z_n appear naturally in the finite-continuum solution of the flow equations in such a way as to satisfy the QME of Eq. (2.8) in the presence of a cutoff, or equivalently the mST of Eq. (3.18), and thus satisfy the corresponding Slavnov–Taylor identities. In Sect. 3.5 we used the compact form of the 1PI effective action (a.k.a. IR-cutoff Legendre effective action or effective average action), to list up to $O(g^3)$ expressions for the 1PI vertices and the mST identities they must satisfy. In this section we now compute these vertices as momentum integrals, show how the requirement of a derivative expansion ensures a smooth limit to the standard expressions in the limit in which the IR cutoff $\Lambda \rightarrow 0$, and extract the Z_n factors. We use them to verify the standard form of the one-loop Yang–Mills β function, here computed as the flow with respect to Λ .

5.1. Second order in coupling and wavefunction renormalization

From Eqs. (3.39) and (3.40) one obtains¹¹

$$\mathcal{A}_{\mu\nu}(p) = \int_{q} \left\{ \bar{\Delta}(q)\bar{\Delta}(q+p) q_{\mu}(q+p)_{\nu} - \frac{1}{2} \bar{\Delta}_{\alpha\beta}(q)\bar{\Delta}_{\rho\sigma}(q+p) \Theta_{\rho\alpha\mu}\Theta_{\sigma\beta\nu} + \bar{\Delta}_{\alpha\alpha}(q) \delta_{\mu\nu} - \bar{\Delta}_{\mu\nu}(q) \right\},$$
(5.1)

¹¹ We find that the calculations here and later are facilitated by using the vertices constructed in Appendix G.

where, using Eqs. (3.31), (3.22), and (2.32), the first two terms come from the ghost propagator and gauge field propagator contributions to the first term in Eq. (3.40), and from Eq. (2.53) we have written

$$\Theta_{\rho\alpha\mu} = (p-q)_{\sigma}\delta_{\beta\nu} - (2p+q)_{\beta}\delta_{\nu\sigma} + \delta_{\beta\sigma}(p+2q)_{\nu}.$$
(5.2)

The last two terms in Eq. (5.1) come from the second term in Eq. (3.40). As explained at the end of Sect. 3.5, we should understand Eq. (5.1) as supplied with Λ -integration constants which in particular ensure that the result is UV finite. Thanks to the IR regularization, cf. also Eq. (2.35), the propagators have a Taylor expansion in p^{μ} for any q and thus we confirm that Eq. (5.1) also has a Taylor expansion in p^{μ} . Decomposing $\mathcal{A}_{\mu\nu}(p)$ into its longitudinal and transverse parts,

$$\mathcal{A}_{\mu\nu}(p) = \mathcal{A}^{L}(p^{2}) P_{\mu\nu}^{L} + \mathcal{A}^{T}(p^{2}) P_{\mu\nu}^{T}, \text{ where } P_{\mu\nu}^{L} = p_{\mu}p_{\nu}/p^{2} \text{ and } P_{\mu\nu}^{T} = \delta_{\mu\nu} - P_{\mu\nu}^{L}, \quad (5.3)$$

by dimensions and the Taylor expansion property we have that

$$\mathcal{A}^{I}(p^{2}) = \Lambda^{2} \sum_{n=0}^{\infty} \alpha_{n}^{I} \left(\frac{p^{2}}{\Lambda^{2}}\right)^{n}, \qquad I = L, T,$$
(5.4)

where the coefficients $\alpha_n^I = \alpha_n^I(\xi)$ are dimensionless and well defined, and $\alpha_0^L = \alpha_0^T = \alpha_0$ are equal. In particular, this zeroth order provides a Λ -dependent "mass" term in Eq. (3.39):

$$C_A \mathcal{A}_{\mu\nu}(0) = \alpha_0 C_A \Lambda^2 \delta_{\mu\nu}. \tag{5.5}$$

As advertised at the end of Sect. 3.5, we see that except for the marginal α_1^I coefficients, the Λ -integration constants are already determined uniquely, as they must be by dimensions. Indeed, from Eq. (5.1) the irrelevant pieces $(a_{n>1}^I)$ are given by well-defined momentum integrals since these integrals are also UV finite, while α_0 can be defined by subtracting from $\mathcal{A}_{\mu\nu}(0)$ as defined in Eq. (5.1), the same expression evaluated at $\Lambda = 0$. On the other hand, the form of the marginal transverse part has been fixed by the renormalization condition in Eq. (4.18) to

$$C_A \alpha_1^T = -\gamma_A \ln(\mu/\Lambda), \qquad (5.6)$$

as follows from Eqs. (4.11), (4.12), and (4.19).

Concentrating now on the longitudinal part, we use the symmetry of the first two terms in Eq. (5.1) under $q \mapsto -q-p$. (For more details see Appendix H.) Eliminating propagators when the same factor appears in the numerator, including eliminating the dot product in $p \cdot q/(p+q)^2$ by expressing $p \cdot q = \frac{1}{2}(p+q)^2 - \frac{1}{2}(p^2+q^2)$, and converting terms with single propagators of form $1/(p+q)^2$ to ones with $1/q^2$ by using $q \mapsto -q-p$, the end result is thus uniquely expressed as

$$\mathcal{A}^{L}(p^{2}) = \frac{3}{4}(\xi+3)\int_{q}\frac{\bar{K}(q)}{q^{2}} - \int_{q}\frac{\bar{K}(q)\bar{K}(p+q)}{q^{2}}\left((1-\xi)\frac{(p\cdot q)^{2}}{p^{2}q^{2}} + 4\frac{p\cdot q}{p^{2}} + \xi + 2\right).$$
 (5.7)

This expression is still IR regulated, but at first sight it appears to include unregulated UV divergences. However, the *K*-independent part vanishes (this and similar later manipulations need some care; see Appendix F):

$$\left\{\frac{3}{4}\left(\xi+3\right) - \frac{1}{4}(1-\xi) - \xi - 2\right\} \int_{q} \frac{1}{q^{2}} = 0.$$
(5.8)

This means that Eq. (5.7) is equal to

$$\mathcal{A}^{L}(p^{2}) = -\frac{3}{4}\left(\xi + 3\right) \int_{q} \frac{K(q)}{q^{2}} + \int_{q} \left(\bar{\Delta}(q)K(p+q) + \frac{K(q)}{q^{2}}\right) \left((1-\xi)\frac{(p\cdot q)^{2}}{p^{2}q^{2}} + 4\frac{p\cdot q}{p^{2}} + \xi + 2\right).$$
(5.9)

Collecting the $K(q)/q^2$ pieces, they cancel exactly as in Eq. (5.8), and thus we are left only with an expression identical to Eq. (2.72). Since the modified Ward–Takahashi identity in Eq. (2.74) tells us that $\mathcal{A}^L = \mathcal{F}$, we see that it holds identically as a statement about momentum integrals, confirming point **C4** in this example.

Since Eq. (2.72) is both IR and UV regulated, \mathcal{A}^L has now been cast in a form which has a Taylor expansion in p^2 with well-defined coefficients. As examples, we find, for the mass term in Eq. (5.5) and the marginal $p_{\mu}p_{\nu}$ term in $\mathcal{A}_{\mu\nu}(p)$, i.e. the $O(p^2)$ longitudinal part:

$$\alpha_{0} = \frac{1}{4(4\pi)^{2}} \int_{0}^{\infty} du \left\{ (1+3\xi) K(u) - (5+3\xi) K^{2}(u) \right\},$$

$$\alpha_{1}^{L} = \frac{1}{(4\pi)^{2}} \left\{ -\frac{1}{6} - \frac{1}{4}\xi + \frac{5+\xi}{4} \int_{0}^{\infty} du \, u \left[K'(u) \right]^{2} \right\},$$
(5.10)

where prime is differentiation with respect to u, and we recall that K(q) is really $K(u = q^2/\Lambda^2)$. We note that the K-dependent pieces give integrals that are finite but non-universal.

We also note that, due to the finiteness of Eq. (2.72), α_1^L has no logarithmic running, consistent with the general arguments that led to Eq. (4.11). In principle, having shown that the momentum integral in Eq. (5.9) is equal to $\mathcal{F}(p^2)$, we might expect to write $\mathcal{A}^L(p^2) = \mathcal{F}(p^2) + \mathcal{A}_0^L$, where \mathcal{A}_0^L is the Λ -integration constant. Then the modified Ward–Takahashi identity of Eq. (2.74) would just tell us that \mathcal{A}_0^L vanishes. However, since **C4** tells us that the identity holds as a statement about momentum integrals, and the momentum integrals in this case can be cast in a form in which they are well defined, there is clearly no reason to introduce such a Λ -integration constant in this case.

Finally, we note that from Eq. (2.72), if we stay at $p \neq 0$ we can safely take the limit $\Lambda \rightarrow 0$, thus defining the longitudinal part of the one-loop two-point vertex of the standard physical Legendre effective action. For non-zero p, however, the K(p+q) factor in Eq. (2.72) forces the result to vanish in this limit. Thus we find that the physical one-loop two-point vertex is purely transverse, in agreement with standard results.

To evaluate \mathcal{A}^T and \mathcal{B} in Eq. (3.39), it is helpful to provide UV regularization. We will demonstrate this for \mathcal{B} . For \mathcal{A}^T we work instead with its RG time derivative, which we could also directly extract from Eq. (3.36). This is well defined in both the UV and the IR since $\dot{\bar{K}} = -\dot{K}$ provides UV regularization. The $\dot{\mathcal{A}}^T(0)$ piece confirms the mass term α_0 in Eq. (5.10), while the $O(p^2)$ part yields

$$\left(2\xi - \frac{26}{3}\right) \int_0^\infty du \, \bar{K}'(u) \, \bar{K}(u) \tag{5.11}$$

(up to vanishing surface terms that depend on momentum routing) and thus, from Eq. (5.6) and the normalization conditions below Eq. (2.3), we get

$$\gamma_A = \left(\frac{13}{3} - \xi\right) \frac{C_A}{(4\pi)^2}.$$
(5.12)

For B, again using Eqs. (3.31), (3.22), and (2.32), we get, from Eqs. (3.39) and (3.40),

$$p^{2}\mathcal{B}(p^{2}) = -\int_{q} \bar{\Delta}_{\mu\nu}(q)\bar{\Delta}(q+p)p_{\mu}(q+p)_{\nu} = \int_{q} \bar{\Delta}(q)\bar{\Delta}(q+p)\Big[p^{2} + \xi(p \cdot q) + (\xi - 1)\frac{(p \cdot q)^{2}}{q^{2}}\Big].$$
(5.13)

We note that $\mathcal{B}(p^2)$ has a derivative (a.k.a. Taylor) expansion. However, $\mathcal{B}(0)$ has a logarithmic UV divergence, which will thus be canceled by the dimensionless Λ -integration constant \mathcal{B}_0 . Making this explicit, and converting the integrand in the same way as above Eq. (5.7), we get

$$\mathcal{B}(p^2) = \int_q \frac{\bar{K}(q)\,\bar{K}(p+q)}{q^2} \left(-\frac{1}{2(p+q)^2} + \frac{1-\xi}{4} \frac{p^2}{q^2(p+q)^2} + \frac{1-\xi}{2} \frac{q \cdot p}{p^2 q^2} + \frac{\xi - 1}{4 \, q^2} \right) + \mathcal{B}_0.$$
(5.14)

Thus we find that (see Appendix F)

$$\mathcal{B}(0) = \frac{\xi - 3}{4} \int_{q} \frac{\bar{K}^{2}(q)}{q^{4}} + \mathcal{B}_{0} = \frac{\gamma_{A} - \gamma_{C}}{2C_{A}} \ln \frac{\mu}{\Lambda}, \qquad (5.15)$$

where the last equality follows from Eq. (4.11) and the renormalization conditions in Eqs. (4.12) and (4.19). From the RG time derivative it is almost immediate to compute $\gamma_A - \gamma_C$:

$$\dot{\mathcal{B}}(0) = \frac{3-\xi}{2} \int_{q} \frac{\bar{K}(q)\dot{K}(q)}{q^{4}} = \frac{\xi-3}{(4\pi)^{2}} \int_{0}^{\infty} du \,\bar{K}(u)\bar{K}'(u) = \frac{\xi-3}{2(4\pi)^{2}}$$
(5.16)

(evidently the result is effectively the same as in dimensional regularization), and thus

$$\gamma_A - \gamma_C = \frac{\xi - 3}{(4\pi)^2} C_A.$$
 (5.17)

Keeping $p \neq 0$ and letting $\Lambda \rightarrow 0$ we get the physical vertex. We see that the momentum integral in Eq. (5.14) then coincides with its *K*-independent part:¹²

$$\mathcal{B}_{\text{phys}}(p^2) = \int_q \left(\frac{\xi - 3}{4} \frac{1}{q^2(q+p)^2} + \frac{\xi - 1}{2} \frac{q \cdot p}{q^4(q+p)^2}\right) + \mathcal{B}_0.$$
 (5.18)

To evaluate this completely it is helpful to provide some UV regularization. It does not matter what regularization we choose: different regularizations will give a different finite part to the integral, which, however, then implies a different \mathcal{B}_0 such that Eq. (5.15) remains satisfied. We use dimensional regularization, setting spacetime dimension $d = 4 - 2\epsilon$ with $\epsilon > 0$. Computing the integral in Eq. (5.15),

$$\int_{q} \frac{\bar{K}^{2}(q)}{q^{4}} = \frac{1}{\Gamma(2-\epsilon)\Lambda^{2\epsilon}(4\pi)^{2-\epsilon}} \int_{0}^{\infty} du \, \frac{\bar{K}^{2}(u)}{u^{1+\epsilon}}.$$
(5.19)

Splitting the last integral at u = 1 we have

$$\int_{0}^{\infty} du \, \frac{\bar{K}^{2}(u)}{u^{1+\epsilon}} = \int_{0}^{1} du \, \frac{\bar{K}^{2}(u)}{u^{1+\epsilon}} \, + \, \int_{1}^{\infty} du \, \frac{\bar{K}^{2}(u) - 1}{u^{1+\epsilon}} \, + \, \int_{1}^{\infty} \frac{du}{u^{1+\epsilon}}.$$
(5.20)

¹² Here we recognize that the third term in brackets in Eq. (5.14) vanishes, and that the second and fourth terms can be combined to eliminate their respective IR divergences.

In the first and second integrals on the RHS, the $\epsilon \rightarrow 0$ limit can now be safely taken. The final integral can be done exactly. Substituting into Eq. (5.15) we thus find

$$\mathcal{B}_{0} = \frac{3-\xi}{4(4\pi)^{2}} C_{A} \bigg\{ \frac{1}{\epsilon} + 1 + \ln(4\pi) - \gamma_{E} - \ln(\mu^{2}) + \int_{0}^{1} \frac{du}{u} \bar{K}^{2}(u) + \int_{1}^{\infty} \frac{du}{u} (K^{2}(u) - 2K(u)) \bigg\}, \quad (5.21)$$

where $\gamma_{\rm E}$ is Euler's constant, and terms that vanish as $\epsilon \to 0$ are discarded.

Computing the momentum integrals in Eq. (5.18) by dimensional regularization using standard methods, we confirm that Eq. (5.21) cancels the UV divergence, and thus arrive at the final expression for the physical vertex:

$$\mathcal{B}_{\text{phys}}(p^2) = \frac{3-\xi}{4(4\pi)^2} C_A \left\{ \ln \frac{p^2}{\mu^2} - 1 + \int_0^1 \frac{du}{u} \bar{K}^2(u) + \int_1^\infty \frac{du}{u} (K^2(u) - 2K(u)) \right\} + \frac{1-\xi}{2(4\pi)^2} C_A.$$
(5.22)

We note that all but the $\ln(p^2/\mu^2)$ part amounts to a finite (non-universal) wavefunction renormalization. It is the price to pay for the simple expression in Eq. (5.15), following from the renormalization condition in Eq. (4.18). In particular, it is through such a renormalization condition that \mathcal{B}_0 becomes *K*-dependent, despite the fact that the momentum integral defining the physical vertex in Eq. (5.18) is *K*-independent. We could have chosen a renormalization condition such that \mathcal{B}_{phys} would be given only by the $\ln(p^2/\mu^2)$ part or such that it agrees with the \overline{MS} result,

$$\mathcal{B}_{\text{phys}}^{\overline{\text{MS}}}(p^2) = \frac{C_A}{(4\pi)^2} \left\{ \frac{3-\xi}{4} \ln \frac{p^2}{\mu^2} - 1 \right\},$$
(5.23)

for examples. Then the z_n^0 in Eq. (4.13) would have had finite corrections.

Let us note that, when expressed in the standard form of Eq. (4.22), Eqs. (5.12) and (5.17) give the same anomalous dimensions γ_3 and $\tilde{\gamma}_3$ that one obtains by standard techniques. Indeed, this is guaranteed since the Λ dependence determines the μ dependence in Eq. (4.13) by dimensions, and the μ dependence in turn determines the *p* dependence by dimensions as, e.g., in Eq. (5.23). This *p* dependence better come out the same since it is physical (being related, for example, to the splitting functions by unitarity).

5.2. Third order in coupling, the beta function, and mST

The solution in Eq. (3.42) corresponds to the Feynman diagrams shown in Fig. 1. One can see by inspection that they are all at worst logarithmically UV divergent, and that each UV divergence is proportional to one of the three-point vertices in Eq. (4.11). As discussed in Sect. 4, these divergences and the corresponding freedom in the choice of integration constants are constrained to satisfy the parametrization in Eq. (4.11).

The one-loop vertices as a whole have to satisfy the mST identities in Eq. (3.43). Apart from the freedom to choose the integration constants according to Eq. (4.11), there is no further flexibility, and thus the body of a solution such as in Eq. (3.42), as defined by the momentum integrals, must already satisfy these mST identities as we noted in C4 and at the end of Sect. 3.5, and will confirm shortly. We can write the vertices as

$$\Gamma_{3} = -i \int_{p,q,r} A_{\mu}(p) [A_{\nu}(q), A_{\lambda}(r)] \Gamma_{\mu\nu\lambda}^{AAA}(p,q,r) - i \int_{p,q,r} A_{\mu}^{*}(p) [A_{\nu}(q), C(r)] \Gamma_{\mu\nu}^{A^{*}AC}(p,q,r) - i \int_{p,q,r} C^{*}(p) C(q) C(r) \Gamma^{C^{*}CC}(p,q,r),$$
(5.24)



Fig. 1. One-loop vertices in Γ_3 . In the gauge-invariant basis, A^* also plays the rôle of the antighost.

where $(2\pi)^4 \delta(p+q+r)$ is understood to be included in the measure. The correction terms in the mST can similarly be written as

$$\operatorname{Tr}\left(-K\Gamma_{1*}^{(2)}\bar{\Delta}\Gamma_{2,\mathrm{cl}}^{(2)}+K\Gamma_{1*}^{(2)}\bar{\Delta}\Gamma_{1}^{(2)}\bar{\Delta}\Gamma_{1}^{(2)}\right) = \int_{p,q,r} C(p)[A_{\mu}(q),A_{\nu}(r)]\,\Delta_{\mu\nu}^{CAA}(p,q,r) + \int_{p,q,r} A_{\mu}^{*}(p)C(q)C(r)\,\Delta_{\mu}^{A^{*}CC}(p,q,r).$$
(5.25)

We compute that

$$-(\Gamma_1, \Gamma_2) = -iC_A \left\{ \left(\{A^*_\mu, C\} + [\partial_\mu A_\nu - \partial_\nu A_\mu, A_\nu] \right) \mathcal{B} \,\partial_\mu C + \partial_\mu A^*_\mu \,\mathcal{B} \,C^2 - A_\mu \,\mathcal{A}_{\mu\nu}[A_\nu, C] \right\},\tag{5.26}$$

and thus, for example from the A^*CC part of Eq. (3.43), we find one must have

$$p_{\mu}\Gamma^{C^{*}CC}(p,q,r) + 2q_{\nu}\Gamma^{A^{*}AC}_{\mu\nu}(p,q,r) = -C_{A}\left[p_{\mu}\mathcal{B}(p) + r_{\mu}\mathcal{B}(r)\right] + \Delta^{A^{*}CC}_{\mu}(p,q,r).$$
(5.27)

The Λ -integration constants satisfy the left-hand side (LHS) alone and correspond to the freedom to change the normalization of the bracketed pair in Eq. (4.11). The rest of the above equation can be viewed as defining the longitudinal part, $\Gamma_{\mu\alpha}^{A^*AC}(p,q,r)P_{\alpha\nu}^L(q)$ of the A^*AC vertex. Similarly, other mST relations define either longitudinal or transverse parts via Q_0A_{ν} or $Q_0^-A_{\mu}^*$, respectively.

Inspecting Fig. 1, we see that there is only one diagram that contributes to the C^*C^2 piece in Eq. (4.11). This is therefore the easiest way to extract γ_g , which in turn will give us the one-loop β -function. We find:

$$-\left(\gamma_g + \frac{\gamma_C}{2}\right) = \dot{\Gamma}^{C^*CC}(0,0,0) = \frac{C_A}{2} \frac{\partial}{\partial t} \int_q \bar{\Delta}^2(q) q_\mu q_\nu \bar{\Delta}_{\mu\nu}(q) = \frac{\xi C_A}{(4\pi)^2} \int_0^\infty du \, \frac{\partial}{\partial u} \bar{K}^3(u) = \frac{\xi C_A}{(4\pi)^2}.$$
(5.28)

To extract the β -function, we absorb the Z_g^{-1} in Eq. (4.11) into the coupling

$$g(\Lambda) \equiv g_{(r)} = Z_g^{-1}g, \qquad (5.29)$$

which thus runs. To one loop, the β -function is then

$$\beta(g) = \Lambda \partial_{\Lambda} g(\Lambda) = -\dot{g}(\Lambda) = \gamma_g g^3(\Lambda), \qquad (5.30)$$

where from Eqs. (5.28), (5.17), and (5.12) we recover the famous result, here as a flow in Λ ,¹³

$$\gamma_g = \left(\gamma_g + \frac{1}{2}\gamma_C\right) + \frac{1}{2}\left(\gamma_A - \gamma_C\right) - \frac{1}{2}\gamma_A = -\frac{11}{3}\frac{C_A}{(4\pi)^2}.$$
 (5.31)

There are two diagrams that contribute to the A^*AC vertex in Eq. (4.11), but otherwise the computation offers almost as straightforward a route to the β -function:

$$-\left(\gamma_{g}+\frac{\gamma_{C}}{2}\right)\delta_{\mu\nu}=\dot{\Gamma}_{\mu\nu}^{A^{*}AC}(0,0,0)$$

$$=\frac{C_{A}}{2}\frac{\partial}{\partial t}\int_{q}\bar{\Delta}\left[\bar{\Delta}q_{\rho}q_{\nu}\bar{\Delta}_{\mu\rho}+q_{\rho}q_{\sigma}\bar{\Delta}_{\rho\sigma}\bar{\Delta}_{\mu\nu}+q_{\rho}q_{\sigma}\bar{\Delta}_{\mu\rho}\bar{\Delta}_{\nu\sigma}-2q_{\nu}q_{\sigma}\bar{\Delta}_{\mu\rho}\bar{\Delta}_{\rho\sigma}\right]$$

$$=\delta_{\mu\nu}\left(\xi+\xi(\xi+3)+\xi^{2}-2\xi^{2}\right)\frac{C_{A}}{4(4\pi)^{2}}\int_{0}^{\infty}du\frac{\partial}{\partial u}\bar{K}^{3}(u)=\frac{\xi C_{A}}{(4\pi)^{2}}\delta_{\mu\nu},$$
(5.32)

where the propagators are all evaluated at q. The end result agrees with Eq. (5.28) and thus again we get the famous one-loop β -function coefficient of Eq. (5.31). Verification of the wavefunction renormalization dependence of the other vertices in Eq. (4.11) proceeds in a similar, if somewhat longer, fashion. We note that through such relations, cf. Eqs. (5.12), (5.17), (5.28), and (5.32), we can confirm that the Slavnov–Taylor identities in Eq. (4.23) are indeed satisfied.

We finish by confirming explicitly that the bodies of the momentum integrals do automatically satisfy Eq. (5.27). The A^*C vertex in Eq. (3.39), cf. Eq. (5.13), can be written as

$$-C_A \bar{\Delta}_{\mu\nu} A^*_{\nu} \partial_{\mu} \bar{\Delta} C. \tag{5.33}$$

As we have seen, care is needed in defining the integration constant $\mathcal{B}_0 \partial_{\nu}$ in this vertex, but here we will be interested in putting this to one side and demonstrating that the bulk of the integral already satisfies Eq. (5.27). For this we can proceed more formally. Using the notation in Appendix G, the C^*CC and A^*AC vertices in Eq. (5.24) can similarly be written respectively as

$$\frac{i}{4}\partial_{\nu}\bar{\Delta}C^{*}\{\partial_{\mu}\bar{\Delta}C\bar{\Delta}_{\mu\nu},C\},\qquad\frac{i}{2}C_{A}\bar{\Delta}_{\mu\nu}A^{*}_{\nu}[\partial_{\rho}\bar{\Delta}A_{\rho}\partial_{\mu}\bar{\Delta},C]+\frac{i}{2}C_{A}\partial_{\mu}\bar{\Delta}A^{*}_{\nu}[\mathcal{T}_{\nu\mu}(A),C].$$
(5.34)

The correction vertex $\Delta_{\mu}^{A^{*}CC}$ in Eqs. (5.25) and (5.27) takes the form

$$+\frac{i}{2}C_{A}K\left(C\{\bar{\Delta}_{\mu\rho}A_{\rho}^{*}\partial_{\mu}\bar{\Delta},C\}+C\{\bar{\Delta}_{\mu\rho}C\partial_{\rho}\bar{\Delta},A_{\mu}^{*}\}-C\{\partial_{\mu}\bar{\Delta}C\bar{\Delta}_{\mu\nu},A_{\nu}^{*}\}\right).$$
(5.35)

Using Eqs. (5.33) and (3.31), the antibracket part of Eq. (5.27) can be written as

$$-\frac{i}{2}C_{A}C\{\bar{\Delta}_{\mu\nu}A_{\mu}^{*}\partial_{\nu}\bar{\Delta},C\}+iC_{A}C\{\partial_{\nu}C\bar{\Delta}_{\mu\nu}\bar{\Delta},A_{\mu}^{*}\}.$$
(5.36)

Finally, operating on Eq. (5.34), the LHS of Eq. (5.27), namely $s_0 \Gamma_{3,q} = (Q_0 + Q_0^-) \Gamma_{3,q}$ in Eq. (3.43), can be written as

$$-\frac{i}{4}C_{A}\partial_{\nu}\bar{\Delta}\partial\cdot A^{*}\{\partial_{\mu}\bar{\Delta}C\bar{\Delta}_{\mu\nu},C\}-\frac{i}{2}C_{A}\bar{\Delta}_{\mu\nu}A^{*}_{\nu}\{\partial_{\rho}\bar{\Delta}\partial_{\rho}C\partial_{\mu}\bar{\Delta},C\}$$
$$-\frac{i}{2}C_{A}\partial_{\nu}\bar{\Delta}A^{*}_{\mu}\{\mathcal{T}_{\mu\nu}(\partial C),C\}.$$
(5.37)

¹³ The β -function under flow in Λ , specialized to SU(2) and Feynman gauge, was computed in Ref. [89].

Using integration by parts, we find that

$$\mathcal{T}_{\mu\nu}(\partial C) = (1-K)C\bar{\Delta}_{\mu\nu} - \bar{\Delta}_{\mu\nu}C(1-K) + \partial_{\mu}\partial_{\sigma}\bar{\Delta}C\bar{\Delta}_{\sigma\nu} - \bar{\Delta}_{\mu\sigma}C\partial_{\sigma}\partial_{\nu}\bar{\Delta}, \qquad (5.38)$$

and thus we find that Eq. (5.37) becomes

$$\frac{i}{2}C_{A}(1-K)\left(C\{\partial_{\mu}\bar{\bigtriangleup}C\bar{\bigtriangleup}_{\mu\nu},A_{\nu}^{*}\}-C\{\bar{\bigtriangleup}_{\mu\nu}C\partial_{\nu}\bar{\bigtriangleup},A_{\mu}^{*}\}-C\{\bar{\bigtriangleup}_{\mu\nu}A_{\mu}^{*}\partial_{\nu}\bar{\bigtriangleup},C\}\right),$$
(5.39)

where use has been made of properties of the trace. Finally, substituting this, Eq. (5.35), and Eq. (5.36) into Eq. (5.27), we find that this mST identity is indeed satisfied by the body of the momentum integrals. We remark that we have also confirmed the more complicated *AAC* mST identity this way, but do not report the details.

6. Summary and conclusions

We reiterate the main points. The Wilsonian RG flow equation for the effective action S can be combined with the QME of Eq. (2.8) in such a way that they are mutually compatible [59–68]. A particularly natural formulation is given by the flow equation in Eq. (2.15) (which takes the same form as Polchinski's [2,5]), with the QME regularized as in Eq. (2.6) and the free action incorporating the free BRST transformations as in Eq. (2.1) [58]. We have shown in general, and in detail for the example of perturbative Yang–Mills theory with general gauge group and in general gauge, how for the continuum effective action solution, BRST invariance is then not broken by the presence of an effective ultraviolet cutoff but remains powerfully present in this structure, despite the fact that it demands quantum corrections that naïvely break the gauge invariance, such as a mass term for the non-Abelian gauge field.

In particular, we have demonstrated that, combined with the derivative expansion property, BRST cohomological methods retain their potency. Key to this is the fact that the regularization in Eq. (2.6) preserves the algebraic identities satisfied by the QME and its components, the antibracket and measure operator, and in such a way that the charges are then well defined when acting on arbitrary local functionals. Thus, in particular the full quantum BRST charge $\hat{s} = Q + Q^- - \Delta^- - \Delta^=$, cf. Eqs. (2.10) and (2.39), is well defined acting on arbitrary local functionals, and nilpotent off shell.

The reason why terms which naïvely should break the gauge invariance are nevertheless allowed, in fact demanded, is precisely because the action of the Batalin–Vilkovisky measure operator is now non-trivial and well defined (in contrast to the usual treatment [47,55]). Thus, for example the one-loop two-point vertex $S_{2,q}$ satisfies

$$Q_0 S_{2,q} = (\Delta^- + \Delta^=) S_{2,cl}, \tag{6.1}$$

as follows from Eq. (2.69) and $Q_0^-S_{2,q} = 0$, cf. Eqs. (2.70) and (2.37). This means that as well as any contribution that does satisfy linearized gauge invariance as in Eq. (2.36), a part that is not Q_0 -closed is required, which descends via the measure operator from the classical solution S_{cl} .

In practice it is the free full classical BRST charge $s_0 = Q_0 + Q_0^-$ and its non-trivial cohomology in the space of local functionals (on which it closes) that is key to developing the perturbative solution. Given K, such a solution is then unique up to the choice of renormalization conditions, and thus is found without introducing a priori either a classical action or a bare action. Indeed, thanks to the derivative expansion property, the Λ -integration constants we get from solving the flow equation for a finite effective action are guaranteed by dimensions to be local functionals. Under very few assumptions, we can then recover crucial steps in classical BRST cohomology [51–57] and in the proof of renormalizability of gauge theories [48–50]. This led to the four central conclusions C1–C4 in Sect. 2.3, which state that the classical solution is determined by the non-trivial s_0 -cohomology, with additional constraints arising from dimensional analysis. In particular, C3 shows that if the CME is satisfied by such a solution to $O(g^3)$, then the CME is automatically satisfied at higher orders, while C4 concludes that the body of momentum integrals in the quantum corrections must already satisfy the QME. As fully developed in Sect. 4, we also listed the central conclusion C5, namely that the RG flow under change of cutoff Λ then generates a canonical transformation, Eqs. (4.15)–(4.17), which automatically solves the standard Slavnov–Taylor identities in Eq. (4.23) for the wavefunction renormalization constants (despite the presence of the cutoff). The remainder of the paper was devoted to developing and verifying these in detail for Yang–Mills theory with general gauge group and in general gauge. And as we saw in Sect. 5, one then retrieves both the standard anomalous dimensions and the standard one-loop β -function, Eqs. (5.30) and (5.31), even though these are expressed in terms of this cutoff.

Using the Legendre transformation identity of Eq. (3.2) results in an equivalent 1PI description in terms of the simultaneous solution of the mST identities in Eq. (3.18) [69] and the flow in Eq. (3.8) of the 1PI effective action Γ , the Legendre effective action with infrared cutoff $\bar{K} = 1-K$ (a.k.a. effective average action) [5,20,21]. These are written in terms of classical (anti)fields, whose antibracket in Eq. (3.12) is the original one without the UV regularization [44–46]. Despite this, from Eq. (3.25) the free full BRST charge \hat{s}_0 operates in the same way, with the measure operator in Eq. (3.30) still UV regularized by *K*. However, the presence of the quantum correction terms in the mST of Eq. (3.18) means that for $\Lambda > 0$, the analogous interacting full BRST charge is not nilpotent. In fact, as noted in Sect. 3.2, the classical (anti)fields and their antibracket are not canonically related to the quantum fields and respectively Eq. (2.6) (although this implies that Eq. (3.18) must however be subject to the non-linear symmetry relations, cf. Appendix D).

Nevertheless, the Legendre transformation identity guarantees that the perturbative development of the solution for Γ mirrors that for S, as we verified explicitly in Sects. 3.3–3.5. Indeed, the freedom in the solution is again expressed through the Λ -integration constants which are guaranteed by dimensions to be local functionals. Then the key rôle is again played by the non-trivial cohomology of $s_0 = Q_0 + Q_0^-$ in the space of local functionals, where the free charges in Eq. (3.28) generate the same algebra, Eqs. (2.36) and (2.37), as for the quantum fields. This means that the same conclusions C1-C5 hold for this 1PI formalism. In fact, since the antibracket in Eq. (3.12) is not regularized, the classical Λ -integration constants are much more straightforward to determine, and the resulting classical solution in Eq. (4.4) for Γ_{cl} is a closed expression (unlike for S_{cl}) which coincides with the standard form of the classical action and its BRST transformations, although it is derived rather than taken a priori as input. For the same reason, the canonical transformation in Eqs. (4.8)-(4.10) that provides the wavefunction renormalization factors parametrizing the remaining freedom in Eq. (4.11)in the choice of Λ -integration constants is much easier to determine in this formalism. Finally, and crucially, the $\Lambda \rightarrow 0$ limit in Eq. (3.19) recovers the standard Legendre effective action satisfying the Zinn-Justin equation [48-50,69]. As we confirmed in Sect. 5, we thus recover the physical 1PI amplitudes, i.e. the same ones that can be computed by other methods modulo changes induced by differing renormalization schemes.

In the early stage of the development of the subject, Yang–Mills theory was studied with the exact RG method with antifields introduced as sources for the BRST transformation [59]. The perturbative compatibility of the flow equation and QME is discussed in Ref. [67] by studying the asymptotic behaviors of the Wilsonian action and the QME. In the present paper we have derived an explicit

simultaneous solution to the flow equation and QME for a finite cutoff at the one-loop level in perturbation.

While there is already a large literature on finding exact and approximate simultaneous solutions of the Legendre flow equation and mST for non-Abelian gauge theories, see, e.g., Refs. [31,59–69,89–101], typically these approaches either lack control or become rapidly highly involved, depending especially on the extent to which the extra constraint from the mST is respected. We have shown how, through the derivative expansion property, s_0 -cohomology (in the space of local functionals) facilitated by working in the minimal gauge-invariant basis of Eqs. (2.34) and (3.22) and exploiting antighost grading and the descent equations in Eq. (2.40), both ensures the symmetry and allows elegant streamlined derivations of renormalized perturbative solutions directly in the continuum, that simultaneously solve the flow equations and the mST. Moreover, we have seen that the formulation in terms of the Wilsonian effective action *S* is equivalent, and in this latter formulation the quantum BRST symmetry even at the interacting level is exact and well defined in the presence of the cutoff.

On the other hand, the focus of part of the literature is in developing effective non-perturbative approximations [31,92,93,95–102]. In this paper we have only applied these results explicitly to the development of exact perturbative solutions. It would be most interesting to investigate whether the structure allows a general framework for non-perturbative truncations that continue to yield compatible flow equations and mST/QME identities, and thus to allow approximations that yield non-perturbative simultaneous solutions. Even for a simple shift form of modified Ward–Takahashi identity, compatibility can be difficult to achieve in non-perturbative approximations [103]; however, we take heart from the considerable freedom in derivative expansion solutions of the QME alone (see the discussion at the beginning of Sect. 2.7) and in the fact, noted below Eq. (5.27), that the mST may be viewed as a means to eliminate longitudinal and transverse parts of certain vertices [102].

We saw in Sect. 2.2 how the simultaneous solution of the flow equation and QME holds nonperturbatively in \hbar , defining the continuum limit in terms of expansion over the eigenoperators and the quantum \hat{s}_0 -cohomology within this space. This framework is important to progress to higher orders the quantization of gravity proposed by one of us [58,74,104,105], which realizes quantum gravity as a genuine continuum quantum field theory using interactions that are perturbative in Newton's constant but non-perturbative in \hbar . The equivalence under Legendre transformation will also allow the 1PI formalism to be developed in this new quantization scheme, so that the mST identities become satisfied at finite Λ , and the physical amplitudes are recovered in the $\Lambda \rightarrow 0$ limit.

Acknowledgements

Y.I. and K.I. are supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research (C) JP19K03822. T.R.M. acknowledges support during this research from a Japan Society for the Promotion of Science London (JSPS) Bridge Fellowship BR150303, from the Leverhulme Trust and the Royal Society as a Royal Society Leverhulme Trust Senior Research Fellow, and from Science and Technology Facilities Council (STFC) through Consolidated Grants ST/L000296/1 and ST/P000711/1.

Funding

Open Access funding: SCOAP³.

Appendix A. Relation to earlier formulations of QME and exact RG

The basis introduced in Ref. [58] is related to the fields $\phi^A|_{\text{unshifted}}$ used in Ref. [67] by a change of variables to the shifted fields ϕ^A discovered in Refs. [63,64]. These latter fields appear naturally

in the solution of the flow equation. The change of variables is in fact a (finite) quantum canonical transformation [58] (thus leaving the QMF invariant):

$$\phi^A = \phi^A|_{\text{unshifted}} + \bar{\Delta}^{AB} R^C_B \phi^*_C. \tag{A.1}$$

This basis leads to the advantage mentioned in Sect. 1, namely that it provides us with a free quantum BRST algebra that closes on local functionals (since Δ in Eq. (2.6) is well defined when acting on arbitrary local operators, while the free BRST transformations in Eq. (2.4) are unmodified). In fact, the change of variables to shifted variables is forced on us at first order. See Ref. [58, Sect. 2.5] for more details.

Appendix B. Translating between left- and right-acting differentials

Right-acting free BRST and free Kozsul–Tate differentials are encoded into the free action od Eq. (2.1) by its last term. This should be compared to the case where they are left-acting [58], e.g.

$$Q\phi^A = (S, \phi^A),\tag{B.1}$$

where we still require Eq. (2.36) and thus, thanks to the antibracket in Eq. (2.6), we must have [58]

$$S_0 = \frac{1}{2} \phi^A K^{-1} \Delta_{AB}^{-1} \phi^B - R_B^A \phi^B K^{-1} \phi_A^*.$$
(B.2)

Comparing this with Eq. (2.1), we see that the translation corresponds simply to the replacement

$$R^{A}_{\ B}\big|_{\text{left}} = (-)^{A} R^{A}_{\ B}\big|_{\text{right}}.$$
(B.3)

For example, the relation in Eq. (3.15) then follows from translating Ref. [58, Eq. A.9].

Appendix C. Third order in coupling at classical level: Notation

As an example, suppose that parts of the action have the following form:

$$S_1 = [B, C]A \equiv if^{abc} \int_x B^b(x) C^c(x) A^a(x),$$

$$S_2 = D[E, F],$$

$$S_3 = G[H, J],$$
(C.1)

where $A = A^a T^a$ etc. are fields in the adjoint representation which for simplicity we will assume to be Grassmann even. We use the notation in Eq. (2.26), and f^{abc} are the group structure constants. Then at classical level, third order in the coupling, we encounter terms of the following form, where the notation is explained in full on the second line:

$$(\mathcal{S}_{1}\stackrel{\leftarrow}{\partial}_{A})\bar{\Delta}(\stackrel{\rightarrow}{\partial}_{D}\mathcal{S}_{2}\stackrel{\leftarrow}{\partial}_{F})\bar{\Delta}'(\stackrel{\rightarrow}{\partial}_{G}\mathcal{S}_{4}) = [B,C][\bar{\Delta}E\bar{\Delta}',[H,J]]$$

$$\equiv i^{3}\int_{x,y,z}f^{abc}B^{b}(x)C^{c}(x)\bar{\Delta}(x-y)f^{aef}E^{e}(y)\bar{\Delta}'(y-z)f^{fhj}H^{h}(z)J^{j}(z).$$
(C.2)

Appendix D. Non-linear invariance of the mST

The invariance of the QMF in Eq. (2.8) under Eq. (2.30) induces a transformation on the classical (anti)fields $\{\Phi, \Phi^*\}$ that must leave Eq. (3.18) invariant since it is also Σ . Let us write Eq. (2.30) more generally in the form

$$\phi_A^* = \check{\phi}_A^* + \Psi_{AB} \phi^B. \tag{D.1}$$

This is a finite quantum canonical transformation (see Ref. [58, Appendix A]) generated by some generic bilinear "gauge fermion" $\Psi = \frac{1}{2}\phi^A \Psi_{AB}\phi^B$ (Ψ_{AB} being field independent, with $\epsilon_A = \epsilon_B + 1$ and $\Psi_{AB} = \Psi_{BA}$). Using the Legendre transformation identity of Eq. (3.3), Eq. (D.1) can be recast as

$$\check{\Phi}_{A}^{*} = \Phi_{A}^{*} - \Psi_{AB} \Phi^{B} - \Psi_{AB} \bar{\Delta}^{BC} \frac{\partial_{l}}{\partial \Phi^{C}} \Gamma_{I}[\Phi, \Phi^{*}].$$
(D.2)

Although this is a symmetry of Eq. (3.18), which, by choosing Ψ to be Eq. (2.29), would take us to a gauge-invariant basis $\check{\Psi}^*$, to apply it we would need to solve Eq. (D.2) for Φ^* , yielding an infinite series in the form of a tree expansion.

Appendix E. Equality of wavefunction renormalizations in the two effective actions Expanding

$$\Gamma_I[\Phi, \Phi^*] = \frac{1}{2} \Phi^A \sigma_{AB} \Phi^B + \cdots, \qquad (E.1)$$

where the ellipses stand for Φ^* terms and higher-point vertices, one gets, either by solving Eq. (3.4) for the S_I two-point vertex, or from the tree expansion, cf. Eq. (3.5), that

$$S_{I}[\phi,\phi^{*}] = \frac{1}{2} \phi^{A} \left(\sigma \left[1 + \bar{\Delta}\sigma \right]^{-1} \right)_{AB} \phi^{B} + \cdots$$
(E.2)

(Here one must recall Eq. (3.22) or work directly in a gauge-fixed basis.) Adding to this the free part in Eq. (2.1) gives the modified kinetic term

$$S[\phi, \phi^*] = \frac{1}{2} \phi^A K^{-1} \tilde{\triangle}_{AB}^{-1} \phi^B + \cdots,$$
 (E.3)

where

$$\tilde{\Delta}^{-1} = (\Delta^{-1} + \sigma)[1 + \bar{\Delta}\sigma]^{-1}.$$
(E.4)

On adding Eq. (E.1) to Γ_0 , cf. Eq. (3.11), we set

$$\sigma = (Z^{-1} - 1)\Delta^{-1} + \sigma',$$
(E.5)

with $\sigma'(p)$ being $O(p^4)$, so as to pull out the part parametrized by the field wavefunction renormalization constants of Eq. (4.10) (or rather as parametrized in Eq. (4.22)). We thus have

$$\tilde{\Delta}^{-1} = (Z^{-1}\Delta^{-1} + \sigma') \left[1 + (Z^{-1} - 1)\bar{K} + \bar{\Delta}\sigma' \right]^{-1} = Z^{-1}\Delta^{-1} + O(p^4),$$
(E.6)

where the last equality follows since \overline{K} is $O(p^2)$. (In fact, recalling Eq. (2.2), it is even $O(p^4)$.) This establishes that the wavefunction renormalization constants for Γ work in S and take the same value. A similar analysis can be carried through to verify this property also for the ϕ^* terms.

Appendix F. Defining the conditionally convergent parts

The demonstration that the *K*-independent part in Eq. (5.8) vanishes, and similar manipulations, need care because Eq. (5.7) is only conditionally convergent, the convergence arising from cancelation of divergent terms. Consider, for example, the following integral that arises from extracting the *K*-independent part:

$$4\int_{q} \frac{p \cdot q}{p^2 q^2}.$$
 (F.1)

This integral vanishes by Lorentz invariance. However, since it appeared multiplied by $\bar{K}(q)\bar{K}(p+q)$ in Eq. (5.7), we could equally have written it as

$$2\int_{q} \left(\frac{p \cdot q}{p^2 q^2} - \frac{p \cdot (q+p)}{p^2 (q+p)^2}\right) = 2\int_{q} \frac{p \cdot q(2p \cdot q+p^2) - q^2 p^2}{p^2 q^2 (q+p)^2}.$$
 (F.2)

This latter integral now has a non-vanishing quadratic divergence (which one can readily extract by taking the $p \rightarrow 0$ limit). The problem is that Eq. (F.1) is superficially cubically divergent. The non-vanishing contribution in Eq. (F.2) is in fact a surface term, which cannot be dropped since it is not only non-vanishing but quadratically divergent.

However, the final version, which coincides with Eq. (2.72), is well defined. If the intermediate expressions are treated in the same way, the result is also unambiguous. Thus, for example, if we start with the expression in Eq. (5.7) but with the $p \cdot q/p^2q^2$ replaced by the integrand in Eq. (F.2), then we would now need a divergent integration constant chosen to cancel the quadratic divergence. However, when we finish, we would again get Eq. (2.72). Alternatively, we could also supply a UV regularization, as we have to do anyway for \mathcal{B} , for example, in Sect. 5.1, after which the divergent part, and thus the corresponding integration constant, is unambiguous.

Likewise, the third term in brackets in Eq. (5.14) requires a little care in taking the $p \rightarrow 0$ limit. Using $q \mapsto -q - p$ symmetry we have that

$$\frac{q \cdot p}{p^2 q^4} \equiv \frac{1}{2} \left(\frac{q \cdot p}{p^2 q^4} - \frac{q \cdot p + p^2}{p^2 (q+p)^4} \right).$$
(F.3)

Combining over a common denominator, and expanding the numerator, we can neglect all terms that evidently vanish as $p \rightarrow 0$. We are then left with

$$\frac{(q \cdot p)^2}{p^2 q^6} - \frac{1}{4 q^4},\tag{F.4}$$

which clearly provides a vanishing contribution to Eq. (5.15).

Appendix G. Vertices for one-loop contributions

Using the alternative to Eq. (3.22), we write Eq. (3.31) as

$$\Gamma_{1} = f^{abc} \int \left[\partial_{\mu} A^{a}_{\nu} A^{b}_{\mu} A^{c}_{\nu} + \frac{1}{2} C^{*a} C^{b} C^{c} + \left(A^{*}_{\mu} - i \partial_{\mu} \bar{C} \right)^{a} A^{b}_{\mu} C^{c} \right].$$
(G.1)

Defining $\tau_{AB}^C = \overrightarrow{\partial}_A \Gamma_1 \overleftarrow{\partial}_B$, and writing

$$(\tau_{\mu c}^{A^*})^{ab}(x,y) \equiv \frac{\partial}{\partial A^a_{\mu}(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = -f^{acb} A^{*c}_{\mu}(x) \delta(x-y) = (\tau_{c\mu}^{A^*})^{ab}(x,y),$$

$$(\tau_{cc}^{C^*})^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = -f^{acb} C^{*c}(x) \delta(x-y),$$

$$(\tau_{\mu\nu}^A)^{ab}(x,y) = \frac{\partial}{\partial A^a_{\mu}(x)} \frac{\partial}{\partial A^b_{\nu}(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^l}{\partial C^a(x)} \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_{\mu\nu}^{A^c}(x,y),$$

$$(\tau_{cc}^A)^{ab}(x,y) = \frac{\partial^r}{\partial C^b(y)} \Gamma_1 = f^{acb} \tau_1 + f^{acb} \tau_2 + f^{acb}$$

for derivative couplings connected to propagators on either side, we have

$$\Delta (\tau_{\bar{c}c}^{A})^{ab} \Delta = +if^{acb} \partial_{\mu} \Delta A_{\mu}^{c} \Delta, \quad \Delta (\tau_{c\bar{c}}^{A})^{ab} \Delta = -if^{acb} \Delta A_{\mu}^{c} \partial_{\mu} \Delta,$$

$$\Delta (\tau_{\bar{c}\mu}^{C})^{ab} \Delta = -if^{acb} \partial_{\mu} \Delta C^{c} \Delta, \quad \Delta (\tau_{\mu\bar{c}}^{C})^{ab} \Delta = if^{acb} \Delta C^{c} \partial_{\mu} \Delta, \qquad (G.3)$$

with similar expressions for the $\Gamma_{1*}^{(2)}$ vertices. Defining $\mathcal{T}_{\rho\sigma}(A^c) = \Delta_{\rho\mu} \left(\tau_{\mu\nu}^{A^c} \right) \Delta_{\nu\sigma}$, we find that

$$\mathcal{T}_{\rho\sigma}(A^{c}) = \left[2\partial_{\nu} \triangle_{\rho\mu}A^{c}_{\mu} + 2\triangle_{\rho\mu}A^{c}_{\nu}\partial_{\mu} - \triangle_{\rho\mu}A^{c}_{\mu}\partial_{\nu} - \triangle_{\mu\nu}\left(\partial \triangle_{\rho\mu} \cdot A^{c} + \triangle_{\rho\mu}A^{c} \cdot \partial\right) - \xi\partial_{\rho}\Delta A^{c}_{\nu}\right] \triangle_{\nu\sigma}.$$

Appendix H. Gauge field double propagator terms in the AA vertex

Applying the projectors Eq. (5.3), we have

$$\mathcal{A}^{L} = \mathcal{A}_{\mu\nu}P^{L}_{\mu\nu} \quad \text{and} \quad \mathcal{A}^{T} = \frac{1}{3}\mathcal{A}_{\mu\nu}P^{T}_{\mu\nu} = \frac{1}{3}\left(\mathcal{A}_{\mu\mu} - \mathcal{A}^{L}\right). \tag{H.1}$$

The most involved part, the gauge field double propagator piece, of Eq. (5.1),

$$-\frac{1}{2}\,\bar{\Delta}_{\alpha\beta}(q)\bar{\Delta}_{\rho\sigma}(q+p)\,\Theta_{\rho\alpha\mu}\Theta_{\sigma\beta\nu},\tag{H.2}$$

gives, for I = L, T contribution,

$$-\frac{1}{2}\bar{\Delta}(q)\bar{\Delta}(q+p)\left(X^{I}+\xi Y^{I}+\xi^{2}Z^{I}\right),\tag{H.3}$$

where the expressions X^I , Y^I , and Z^I are $q \mapsto -q-p$ symmetric:

$$\begin{split} X^{T} &= 3 \bigg[(p-q)^{2} - \frac{(p^{2}-q^{2})^{2}}{(p+q)^{2}} \bigg] + 3 \bigg[(2p+q)^{2} - \frac{(2p \cdot q+q^{2})^{2}}{q^{2}} \bigg] \\ &+ (p+2q)^{2} \bigg[2 + \frac{(p \cdot q+q^{2})^{2}}{q^{2}(p+q)^{2}} \bigg], \\ Y^{T} &= (p-q)^{2} + 2(p+2q)^{2} - 3q^{2} + 3p \cdot q + (2p+q)^{2} - 3(p+q)^{2} - 3(p^{2}+p \cdot q) \\ &+ 2 \frac{(2p \cdot q+q^{2})^{2}}{q^{2}} + 2 \frac{(p^{2}-q^{2})^{2}}{(p+q)^{2}} + 3 \bigg[\frac{(p \cdot q+q^{2})^{2}}{q^{2}} + \frac{(p \cdot q)(p \cdot q+q^{2})}{q^{2}} \bigg] \\ &+ 3 \bigg[\frac{(p \cdot q+q^{2})^{2}}{(p+q)^{2}} - \frac{(p \cdot q+q^{2})(p^{2}+p \cdot q)}{(p+q)^{2}} \bigg] - 2(p+2q)^{2} \frac{(p \cdot q+q^{2})^{2}}{q^{2}(p+q)^{2}}, \end{split}$$

$$\begin{split} Z^T &= \frac{(p^2 - q^2)^2}{(p+q)^2} + \frac{(2p \cdot q + q^2)^2}{q^2} + (p+2q)^2 \frac{(p \cdot q + q^2)^2}{q^2(p+q)^2} \\ &\quad - 3 \bigg[\frac{(p \cdot q)(p \cdot q + q^2)}{q^2} + (p \cdot q + q^2)^2 \bigg\{ \frac{1}{q^2} + \frac{1}{(p+q)^2} \bigg\} - \frac{(p \cdot q + q^2)(p^2 + p \cdot q)}{(p+q)^2} \bigg], \\ X^L &= (p^2 + 2p \cdot q)^2 \bigg[2 + \frac{(p \cdot q + q^2)^2}{(p+q)^2 q^2} \bigg], \\ Y^L &= (p+q)^2 \bigg\{ p^2 - \frac{(p \cdot q)^2}{q^2} \bigg\} + q^2 \bigg\{ p^2 - \frac{(p^2 + p \cdot q)^2}{(p+q)^2} \bigg\}, \\ Z^L &= 0. \end{split}$$

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