POTENTIAL GROUP IN OPTICS: THE MAXWELL FISH–EYE SYSTEM†

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Abstract: The $(N-1)$-dimensional Maxwell fish-eye is an optical system with an $SO(N-1)$ manifest symmetry and a $SO(N)$ hidden symmetry, but also an $SO(N,1)$ *potential* group and the $SO(N,2)$ group of the $(N-1)$-dimensional Kepler and point rotor systems. The optical Hamiltonian is proportional to the Casimir invariant. We use a stereographic map extended to a *canonical* transformation between the two phase spaces of the rotor and the fish-eye. The groups permit a succint '4π' wavization that shows that the constrained system can support only discrete light colors and that it unavoidably has chromatic dispersion. Elements of the potential *group* relate the fish-eye asymptotically to free propagation in homogenous media. PACS: 02.20.+b, 42.20.–y

1. Introduction

The Maxwell fish-eye is an optical medium that is truly the Hydrogen atom of optics: it possesses a manifest $SO(N-1)$ and hidden $SO(N)$ rotation symmetry groups, an $SO(N,1)$ dynamical and potential group, and an $SO(N,2)$ group, one of whose compact generators is the root of the $SO(N)$ Casimir —very much like the *number* operator in the classical Kepler system.

In a geometric-optics Maxwell fish-eye, the paths of light rays are circles on planes that contain the origin. It was shown by R.K. Luneburg [1] that these are the *stereographic projection* of great circles on a sphere in one higher dimension. (See also Refs. [2] and [3].) *Confer*: the work of Fock [4] and Bargmann [5] on the hidden symmetry of the H-atom. H.A. Buchdahl in [6] mapped the constants of the fish-eye circles onto the angular momentum and Runge-Lenz vector constants of the Kepler orbits. These are generators of an $SO(4)$ group under the Poisson bracket.

The Maxwell fish-eye is considered as an example of a perfect imaging instrument: all light rays issuing from one point in the medium will follow circle arcs that intersect at the point *conjugate* to the first,[1] all with the same the optical length [3]. This is a rare instance of a '4π' optical instrument; it is a system worth studying for its group-theoretical interest, and as an example to callibrate the Lie-Hamilton formulation of geometric optics and its subsequent wavization.

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[1] Two points $\vec{q}_1$ and $\vec{q}_2$ are conjugate when they are antiparallel and their magnitudes relate as $q_1 q_2 = \rho$. They are the stereographic images of a pair of antipodal points on the sphere of radius ρ.

Section 2 presents and discusses the Hamilton equations for inhomogeneous optical media. Section 3 recalls the Hamiltonian formulation of the spherical rotor (point mass constrained to a sphere) and its subsequent stereographic map over a plane. We find the conjugate *momentum* transformation that must accompany this map if the transformation is to be *canonical* in the usual symplectic sense. The dynamical so$(N,2)$ algebra of the rotor is mapped in Section 4 on the fish-eye, and its action integrated to the group is given explicitly for two key one-parameter subgroups. Section 5 presents the essentials of the wavization process through choosing the realization of the dynamical algebra on the Hilbert space of functions on the sphere, stereographically projected.

2. The Hamiltonian in optics

One may show that the two Hamilton equations of optics derive from the first principles of differential geometry and the Snell-Descartes law of refraction, respectively [8], [9], [10], [11], when the evolution parameter is along an optical axis in space. When the evolution parameter is *time*, similar considerations [12], lead to the Hamilton equations

$$\frac{d\mathbf{q}}{dt} = \frac{c}{n^2}\mathbf{p} = \frac{\partial \mathcal{H}^{\rm opt}}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = \frac{c}{n}\nabla n = -\frac{\partial \mathcal{H}^{\rm opt}}{\partial \mathbf{q}}, \tag{2.1a,b}$$

with the *optical* Hamiltonian function[2]

$$\mathcal{H}^{\rm opt}(\mathbf{p},\mathbf{q}) = c\frac{p^2}{2[n(\mathbf{q})]^2} + \text{constant}. \tag{2.2}$$

This holds for general space dimension D, $\mathbf{q} = (q_1, q_2, \ldots, q_D)$, and similarly for $\mathbf{p}$. Since the value of the Hamiltonian function $\mathcal{H}^{\rm opt}$ is preserved along the lines of the flow, it is a constant of the motion. Hence, the momentum vector $\mathbf{p}$ is *constrained* to have squared length

$$p^2 = n(\mathbf{q})^2. \tag{2.3}$$

This is the *Descartes* sphere of ray directions, whose radius depends, in inhomogeneous media, on the point.

Geometric optics models light rays as the paths taken by points indicated by $\mathbf{q}(t) \in \Re^D$, at a time parameter t, whose velocity vector $\mathbf{v} = d\mathbf{q}/dt$ must be of magnitude $|\mathbf{v}| = c/n(\mathbf{q})$ at each point $\mathbf{q}$ of the medium. The velocity is related to the optical *momentum* by $\mathbf{v} = c/n^2\,\mathbf{p}$.

We should compare the general optical Hamiltonian function (2.2) with the prototype of mechanics, $p^2/2m + V(\mathbf{q})$, to see that they are **not** equal. We expect that the usual tools of quantum mechanics based on the Heisenberg-Weyl algebra where $\mathbf{p} \in \Re^D$ must be adapted to 4π optical systems basing them on a group-theoretical formulation that pays attention to the symmetries of the system. The formulation is *by* these simmetries.

3. The isotropic point rotor, stereographically projected

We work with the Poisson bracket formalism and symplectic geometry of phase space. The configuration space of a point mass in N dimensions is the ensemble of position coordinates $\vec{Q} = \{Q_i\}_{i=1}^N \in \Re^N$. It is a point *rotor* when constrained to a sphere of radius ρ,

$$\vec{Q}^2 = \vec{Q}\cdot\vec{Q} = \sum_{i=1}^{N} Q_i^2 = \mathbf{Q}\cdot\mathbf{Q} + Q_N^2 = \rho^2. \tag{3.1}$$

[2] We assume the optical medium is time-independent.

It is usual to indicate by boldface $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_{N-1})$ the first $N-1$ components of $\vec{Q} = (\mathbf{Q}, Q_N)$. The constraint on configuration space is thus

$$Q_N^{(\sigma)} = \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}, \qquad \sigma \in \{+1, 0, -1\}. \tag{3.2}$$

The *sign* σ of $Q_N^{(\pm)}$ labels the two hemispheres and the $Q_N^{(0)} = 0$ equator. (We shall disregard the latter.) The Hamiltonian flow must leave that sphere invariant.

Functions F of phase space $(\vec{P}, \vec{Q})$ whose Poisson bracket with $\vec{Q}^2$ are zero, preserve the sphere and satisfy $\{F, \vec{Q}^2\} = 0$, *i.e.*, $0 = \sum_{i=1}^N 2Q_i\{F, Q_i\} = -2\,\vec{Q}\cdot\frac{\partial F}{\partial \vec{P}}$. So, among the linear and quadratic functions of the basic N-dimensional Heisenberg-Weyl algebra, only 1, Q_i, Q_iQ_j, ..., and $R_{i,j} = Q_iP_j - Q_jP_i$ have this property, and sums and products thereof, *i.e.*, the enveloping algebra of the Euclidean algebra iso(N). The second degree *Casimir* function (of *fourth* degree in Q_i and P_j), is

$$\Phi = \tfrac{1}{2}\textstyle\sum_{i,j} R_{i,j}R_{i,j} = \vec{Q}^2\vec{P}^2 - (\vec{Q}\cdot\vec{P})^2. \tag{3.3}$$

(This is the angular momentum tensor $\vec{Q}\times\vec{P}$ in $N=3$ dimensions.) There is an abelian ideal of Q's of dimension $1+N+\frac{1}{2}N(N+1)+\cdots$. The Lie transformations [13] generated by these functions leave configuration space $\vec{Q}$ invariant. They only affect *momentum* space through $\vec{P} \mapsto \vec{P} + \vec{f}(\vec{Q})$.

The *isotropic* point rotor is defined by its Hamiltonian being a rotation *invariant*, *i.e.*, $\mathcal{H}^{\rm rot}$ can be only a function of Φ in (3.4); quadratic in momentum and constant of the motion is $\mathcal{H}^{\rm rot} = \omega\Phi = E$, with scale constant ω. The fourth-order invariant is $\sum R_{i,j}R_{j,k}R_{k,l}R_{l,i} = 2\Phi^2$.

The *constraint* of the sphere, Eqs. (3.1)–(3.2), leaves us with a lower-dimensional phase space. We expect *another* phase-space constraint to be imposable [14], because Hamiltonian phase spaces are of even dimension. One may show [12] (it is not obvious) that extinction of Q_N as an independent variable is compatible with setting $P_N = 0$ in the Lie algebra generators $R_{i,j}$, Q_i, etc., and using *Dirac* brackets [14*b*,*c*]. The Dirac brackets turn out to be the Poisson brackets in the first $N-1$ coordinates (Q_N is now simply a function of the remaining coordinates). Since we denoted $\vec{Q} = (\mathbf{Q}, Q_N)$; let us similarly denote $\vec{P} = (\mathbf{P}, 0)$ where $\mathbf{P}$ are the first $N-1$ components Under this $|_{\rm rotor}$ map, the so(N) symmetry subalgebra generators become

$$L_{i,j} = R_{i,j}|_{\rm rotor} = Q_iP_j - Q_jP_i, \qquad i,j = 1,\ldots,N-1, \tag{3.17}$$

$$M_i = R_{i,N}|_{\rm rotor} = -\sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}\,P_i, \qquad i = 2.\ldots,N-1, \tag{3.4}$$

The Hamiltonian function becomes, for some constant ω,

$$\mathcal{H}^{\rm rot} = \omega C, \qquad C = \Phi|_{\rm rotor} = \rho^2|\mathbf{P}|^2 - (\mathbf{Q}\cdot\mathbf{P})^2. \tag{3.5a,b}$$

Under the rotor map the Lie-Poisson bracket relations among these functions are the same in this *reduced* $2(N-1)$-dimensional phase space $(\mathbf{Q}, \mathbf{P})$. Free motion of a point rotor in $(\vec{Q}, \vec{P})$ is on arcs of great circles. The restriction to $(\mathbf{Q}, \sigma, \mathbf{P})$ projects the point rotor on two copies of its equatorial plane (distinguished by the hemisphere sign σ), and with a *new* canonically conjugate momentum $\mathbf{P}$. The trajectory jumps between the two values of the sign σ when it crosses the sphere equator. This is the spherical rotor motion projected on the equatorial plane.

The map between the projected sphere coordinates $\mathbf{Q} \in \Re^{N-1}$, $\mathbf{Q} < \rho$, and the plane $\mathbf{q} \in \Re^{N-1}$, given by

$$\mathbf{q} = \frac{2\rho\mathbf{Q}}{\rho - \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}}, \qquad \mathbf{Q} = \frac{4\rho^2\mathbf{q}}{|\mathbf{q}|^2 + 4\rho^2}, \tag{3.6a,b}$$

is the usual stereographic map, that 'opens' the S_{N-1} rotor sphere to the plane that is the Maxwell fish-eye configuration space. Seen as a *canonical* map between the full phase spaces, it must be accompanied by the corresponding *momentum* map

$$\mathbf{p} = \frac{\rho - \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}}{2\rho}\mathbf{P} - \frac{\mathbf{Q}\cdot\mathbf{P}}{2\rho^2}\mathbf{Q}, \qquad \mathbf{P} = \frac{|\mathbf{q}|^2 + 4\rho^2}{4\rho^2}\left(\mathbf{p} + \frac{2\mathbf{q}\cdot\mathbf{p}}{4\rho^2 - |\mathbf{q}|^2}\mathbf{q}\right). \qquad (4.7a,b)$$

This is a map between two $2(N-1)$-dimensional phase spaces; the $(\vec{Q}, \vec{P})$ space of the constrained rotor motion need not be used further. The nature of the map is that of optical *distorsion, i.e., point* transformation of configuration space; as a map in momentum space, it is *comatic* (and *distinct* from the comatic map studied in Ref. [10]).

We may write the so(N) functions (3.4) in $(\mathbf{q}, \mathbf{p})$. They are

$$L_{i,j} = q_i p_j - q_j p_i, \qquad i,j = 1,\ldots,N-1, \qquad (3.8a)$$

$$M_i = \rho\left[\left(1 - \frac{|\mathbf{q}|^2}{4\rho^2}\right)\mathbf{p} + \frac{\mathbf{q}\cdot\mathbf{p}}{2\rho^2}\mathbf{q}\right], \qquad i = 1,\ldots,N-2. \qquad (3.8b)$$

The Casimir and Hamiltonian functions in can be calculated replacing $\mathbf{q}(\mathbf{Q},\mathbf{P})$ and $\mathbf{p}(\mathbf{Q},\mathbf{P})$ in (3.5). They are

$$C = \left(\sum_{i<j=2}^{N-1} L_{i,j}^2 + \sum_{i=1}^{N-1} M_i^2\right) = \rho^2(1 + |\mathbf{q}|^2/4\rho^2)^2\,|\mathbf{p}|^2 = \frac{1}{\omega}\mathcal{H}(\mathbf{q},\mathbf{p}) = \frac{E}{\omega}. \qquad (3.9)$$

We may now compare this projected rotor Hamiltonian $\mathcal{H}$ with the generic *optical* Hamiltonian $\mathcal{H}^{\rm opt} = c|\mathbf{p}|^2/n(\mathbf{q})|^2$ in (2.2). They are equal only when the refractive index of the optical medium is that which characterizes the Maxwell fish-eye:

$$n(\mathbf{q}) = \frac{n_o}{1 + |\mathbf{q}|^2/4\rho^2}, \qquad n_o = n(0) = \frac{1}{\rho}\sqrt{\frac{c}{2\omega}}. \qquad (3.10)$$

4. The fish-eye so(N,2) dynamical algebra

The realization of the so(N) algebra shown in (3.8) is well known in the theory of the hydrogen atom [4–7]. If we label the so(N,M) generators as $A_{i,j}$, $i,j = 1,2,\ldots N, N+1,\ldots,N+M$, their standard Lie (Poisson) bracket relations are

$$\{A_{i,j}, A_{k,l}\} = g_{j,k}A_{l,i} + g_{i,l}A_{k,j} + g_{j,l}A_{i,k} + g_{i,k}A_{j,l}, \qquad (4.1a)$$

where

$$g_{j,k} = \begin{cases} 1, & j = k \le N, \\ -1, & N+1 \le j = k \le N+M, \\ 0, & \text{otherwise.} \end{cases} \qquad (4.1b)$$

The symmetry properties of the indices are $A_{i,j} = -A_{j,i}$ for i,j both in the range $(1,\ldots,N)$ or both in $(N+1,\ldots,N+M)$, and $+A_{j,i}$ otherwise.

We enlarge the so(N) generator set in (3.8), $L_{i,j} = A_{i,j}$ and $M_i = A_{i,N}$ for $i,j = 1,2,\ldots,N-1$, with the following so($N,1$) generators:

$$K_i = A_{i,N+1} = M_i - 2\rho p_i = \rho\left[-\left(1 + \frac{|\mathbf{q}|^2}{4\rho^2}\right)p_i + \frac{\mathbf{q}\cdot\mathbf{p}}{2\rho^2}q_i\right], \qquad (4.2a)$$

$$K_N = A_{N,N+1} = -\mathbf{q}\cdot\mathbf{p}. \qquad (4.2b)$$

These are the 'noncompact' generators because of the minus sign in $\{K_i, K_j\} = -L_{i,j}$. Thus $K_N = -\mathbf{q}\cdot\mathbf{p}$ generates magnifications of configuration space:

$$\exp\beta\{K_N, \circ\} : f(\mathbf{q},\mathbf{p}) \mapsto f(e^{-\beta}\mathbf{q}, e^{\beta}\mathbf{p}), \qquad \beta \in \Re. \tag{4.3}$$

In particular,

$$\exp\beta\{K_N, \circ\} : \mathcal{H}(\rho)^{\text{fish-eye}} \mapsto \mathcal{H}(e^{\beta}\rho)^{\text{fish-eye}}. \tag{4.4}$$

We have thus the line β of noninvariance transformations, where the radius ρ of the rotor sphere dilates to infinity as $\beta \to \infty$. Setting $\omega = \frac{1}{2}cn_o^{-2}\rho^{-2}$, we map the Maxwell fish-eye Hamiltonian $\mathcal{H}(\rho)^{\text{fish-eye}}$ onto the homogeneous medium Hamiltonian $\mathcal{H}(\infty)^{\text{fish-eye}} = n_o$.

We note the coordinate functions $p_i = (M_i - K_i)/2\rho$. Thus a second visible noninvariance transformation of the optical fish-eye Hamiltonian is the ordinary configuration-space translation:

$$\exp\textstyle\sum_i a_i\{M_i - K_i, \circ\} : \mathcal{H}(\mathbf{q},\mathbf{p})^{\text{fish-eye}} \mapsto \mathcal{H}(\mathbf{q} - 2\rho\mathbf{a}, \mathbf{p})^{\text{fish-eye}}. \tag{4.5}$$

We may thus map a Maxwell fish-eye to *another* fish-eye whose center of symmetry is at a instead of the origin, and/or dilated, up to the homogeneous medium. The algebra with the property that it relates the system to the homogeneous medium, is the *potential algebra* of the Maxwell fish-eye. Up to now, potential *algebras* of the family so(M, N) were used in Refs. [15], [16], and others, to relate the Pöschl-Teller potential to the free particle, for example. Here we see that the concept also serves optical systems, and in this case the algebra can be truly integrated to the Lie *group*: (4.4) and (4.5).

The set of generators of so$(N, 1)$ given above is further extended to so$(N, 1)$ when we add

$$H_i = \Lambda_{i,N+2} = q_i p, \qquad i = 1, 2, \ldots, N-1, \tag{4.6a}$$

$$H_N = \Lambda_{N,N+2} = H_{N+1} - 2\rho\, p = -\rho(1 - |\mathbf{q}|^2/4\rho^2)\, p, \tag{4.6b}$$

$$\mathcal{N} = H_{N+1} = \Lambda_{N+1,N+2} = \rho(1 + |\mathbf{q}|^2/4\rho^2)\, p = +\sqrt{C}. \tag{4.6c}$$

Here $\mathcal{N}$ is the root of the so(N) Casimir function C in (3.9), and a *compact* generator of so$(N, 2)$ that is a constant of the motion. In the Hydrogen-atom [17], [18], this is the *number* operator. Note that we have introduced the function $p = \sqrt{\mathbf{p}\cdot\mathbf{p}} = |\mathbf{p}|$.

The integrated group action on phase space generated by $\mathcal{N} = H_{N+1} = \Lambda_{n+1,N+2}$ is

$$\exp s\{\mathcal{N}, \circ\} : p_i = \left[1 - \frac{1-\cos s}{2}\left(1 + \frac{|\mathbf{q}|^2}{4\rho^2}\right)\right] p_i + \left[p \sin s + \frac{1-\cos s}{4\rho^2}\mathbf{q}\cdot\mathbf{p}\right] q_i, \tag{4.7a}$$

$$\exp s\{\mathcal{N}, \circ\} : q_i = \frac{[p\cos s + \mathbf{q}\cdot\mathbf{p}/2\rho \sin s] q_i - \rho(1 + |\mathbf{q}|^2/4\rho^2)\sin s\, p_i}{[\cos s + \frac{1}{2}(1-\cos s)(1 + |\mathbf{q}|^2/4\rho^2)]p - \frac{1}{2}\mathbf{q}\cdot\mathbf{p}/\rho \sin s}. \tag{4.7b}$$

The parameter s measures length along the flow lines. The time evolution generated by the Hamiltonian $H^{\text{fish-eye}} = \omega\mathcal{N}^2$ is thus

$$\exp t\{H^{\text{fish-eye}}, \circ\} = \exp 2\omega\eta t\{\mathcal{N}, \circ\}, \tag{4.8}$$

in termos of (4.7) with $s = 2\omega\eta t$, where η is the constant of the motion $\mathcal{N} = \rho(1 + |\mathbf{q}|^2/4\rho^2)p$.

5. Wavization of the Maxwell fish-eye

We may *'wavize'* the Maxwell fish-eye as a dynamical quantization problem of mechanical systems [19], since we know the symmetry and dynamical groups. This is based on the scalar wave equation for a function $\Phi(\vec{Q},t)$, of a single color ν [so $\Phi^{(\nu)}(\vec{Q},t) = \Phi(\vec{Q})e^{i\nu t}$], restricted to a sphere $\vec{Q} = \rho\vec{\Omega}$ [so $\Phi(\vec{Q}) = \Phi_\rho(\vec{\Omega})$, $\Omega_i = Q_i/|\vec{Q}|$] —the sphere on which the mass point moves, and stereographically projected on the Maxwell fish-eye space of $N-1$ dimensions, whose coordinate $\mathbf{q} \in \Re^{N-1}$ we continue to indicate by boldface. It leads to the solutions of

$$\tilde{C}\Phi(\vec{\Omega}) = \left(\frac{n_o\nu\rho}{c}\right)^2 \Phi(\vec{\Omega}), \tag{5.1}$$

where $\tilde{C}$ is the Casimir operator of $\mathrm{so}(N)$,

$$\tilde{C} = -\tfrac{1}{2}\sum_{j,k}^{N} \tilde{\Lambda}_{j,k}^2, \qquad \tilde{\Lambda}_{j,k} = i\left(Q_j\frac{\partial}{\partial Q_k} - Q_k\frac{\partial}{\partial Q_j}\right). \tag{5.2}$$

This realization of $\mathrm{so}(N)$ is also well known from the theory of quantum angular momentum. It is *different* from the Poisson bracket realization of geometric optics used in the foregoing sections. We surmise this leads to the proper 4π wavization of the Maxwell fish-eye. The operators $\tilde{\Lambda}_{j,k}$ are self-adjoint in the space $\mathcal{L}^2(S_{N-1})$ of Lebesgue square-integrable functions on the sphere S_{N-1} [20]. The wave equation (5.1) is thus reduced to a Sturm-Liouville problem whose proper eigenvalues on the right-hand side are $\lambda = \ell_N(\ell_N + N - 2)$, $\ell_N = 0,1,2,\ldots$ and consequently, the possible colors ν the medium can sustain are:

$$\nu_N = \frac{c}{n_o\rho}\sqrt{\ell_N(\ell_N + N - 2)}, \qquad \ell_N = 0,1,2,\ldots. \tag{5.3}$$

The projection of the Lie algebra of $\tilde{\Lambda}_{j,k}$'s from the sphere onto the Maxwell fish-eye space $\mathbf{q}$, proceeds *as if* we were to use the chain rule to write

$$\frac{\partial}{\partial Q_i} = \left(1 + \frac{|\mathbf{q}|^2}{4\rho^2}\right)\left(\frac{\partial}{\partial q_i} + \frac{2q_i}{4\rho^2 - |\mathbf{q}|^2}\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{q}}\right), \tag{5.4}$$

for $i = 1,2,\ldots N-1$ and, for $i = N$, functions on the ρ-sphere have $Q_N = \sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}$; so $\partial/\partial Q_N$ acts as zero playing no further role[3]

The Hilbert space $\mathcal{L}^2(S_{N-1})$ of functions Φ, Ψ on the sphere is mapped through the stereographic projection on wave functions on $\mathbf{q}(\mathbf{Q},\sigma) \in \Re^{N-1}$. The inner product integral of two functions is

$$(\Phi,\Psi)_{\mathcal{L}^2(S_{N-1})} = \int_{S_{N-1}} d^{N-1}\Omega\,\Phi(\vec{\Omega})^*\Psi(\vec{\Omega}) \tag{5.5a}$$

$$= \sum_{\sigma=\pm1}\frac{1}{\rho^{N-2}}\int_{|\mathbf{Q}|<\rho}\frac{d^{N-1}\mathbf{Q}}{\sigma\sqrt{\rho^2 - |\mathbf{Q}|^2}}\,\bar{\Phi}_\sigma(\mathbf{Q})^*\bar{\Psi}_\sigma(\mathbf{Q})$$

$$= \frac{1}{\rho^{N-1}}\int_{\Re^{N-1}}\frac{d^{N-1}\mathbf{q}}{(1 + |\mathbf{q}|^2/4\rho^2)^{N-1}}\,\Phi(\vec{\Omega}(\mathbf{q}))^*\,\bar{\Psi}(\vec{\Omega}(\mathbf{q}))$$

$$= (\phi,\psi)_{\text{fish-eye}} = \int_{\Re^{N-1}} d^{N-1}\mathbf{q}\,\phi(\mathbf{q})^*\psi(\mathbf{q}), \tag{5.5b}$$

[3] Compare with the geometrical (classical) expression (3.8) using the Schrödinger quantization map on p_i. Keep in mind that operators '$\hat{q}_i$' cannot be self-adjoint in domains of functions whose Fourier transform have compact support, because they are generators of translations that do not leave the domain invariant.

where $\bar{\Phi}_{\sigma=\pm}(\mathbf{Q}) = \Phi(\vec{Q})$ and $\phi(\mathbf{q}) = (1 + |\mathbf{q}|^2/4\rho^2)^{(N-1)/2}\Phi(\vec{\Omega}(\mathbf{q}))$, and similarly for Ψ and ψ. Under this inner product integral, symmetry transformations are unitary and their infinitesimal generators (5.6*b*) are self-adjoint. When we use the customary inner product form $(\phi, \psi)_{\text{fish-eye}}$ in (5.5*b*) for a 'flat' space of measure $d^{N-1}\mathbf{q}$, the so(N) generators $\hat{L}_{i,j}$ and $\hat{M}_i$ are the Schrödinger quantization of the 'classical functions' in (3.8) and K_i, K_N in (4.2).[4] The functions are *linear* in the components of p_i, so there is no ordering ambiguity: any quantization rule gives the same result. The optical fish-eye Hamiltonian (2.2), being neither $p^2/2 + V(q)$ nor of the form $pf(q) + g(q)$, *would* be subject to ordering freedom [21]. As Casimir operator however, it is the well-defined sum of squares of the operators (5.2). The wave Hamiltonian *is* thus unique and independent of the quantization scheme.

As in the Hydrogen atom, there are a finite number of independent Maxwell fish-eye states for each allowed color. They may be labelled by the so(N) representation ℓ_N in (5.3), and the *row* indices of the canonical basis [22], $\{\ell_{N-1}, \ldots, \ell_2\}$, with integer ℓ_j's bound by the usual branching rules. For $N = 3$ (the 2-dimensional Maxwell fish-eye), we have the spherical harmonics on the surface of the ordinary sphere S_2. In this plane optical world, the description of the wave patterns in the Maxwell fish-eye by the usual $\{\ell, m\}$-labels is by projecting $\mathcal{Y}_{\ell,m}(\vec{\Omega})$ to [20]

$$\Upsilon_{\ell,m}(\mathbf{q}, t) = \frac{Y_{\ell,m}\left(\frac{\mathbf{q}}{1+|\mathbf{q}|^2/4\rho^2}\right)}{1 + |\mathbf{q}|^2/4\rho^2} \exp ict\sqrt{\frac{\ell(\ell+1)}{n_o\rho}}, \tag{5.6}$$

vibrating with $\nu(\ell) = (c/n_o\rho)\sqrt{\ell(\ell+1)}$, $\ell = 0, 1, 2, \ldots$. These functions can be visualized as patterns of light with an intensity weighted by an *obliquity* factor $\sim |\Upsilon_{\ell,m}(\mathbf{q})|^2 \tilde{<} (1 + |\mathbf{q}|^2/4\rho^2)^{-2}$.

For $m = \pm\ell$ we have waves travelling around the equator. The nodes of the real part are moving sphere meridians. This rotating light pattern projects on the fish-eye plane as nodes moving as spokes in a rotating, rigid wheel. When the rotation axis is inclined, the belt of maxima projects on an off-center circle and the nodal meridians on circular nodes that cross through the two projected rotation poles. We conjecture that these 'circle-of-light' solutions are the best wave analogues of the geometric light orbits. We note that since the phase velocity is not linear in ℓ, *chromatic dispersion* takes place whenever more light has more than one constituent color. The periods are incommensurable.

The $m = 0$ solutions contain a Legendre polynomial $P_\ell(\cos\beta)$ and are independent of the longitude angle γ. They can be described as standing-wave solutions that have their maxima at two conjugate points. They will be in or out of phase according to the parity of ℓ. A flash at some point of the fish-eye will decompose into $P_\ell(\cos\beta_0)$'s, subject *dispersion* again. Thus signals will loose their shape, even though the optical path between two conjugate points is equal along any circle arc, and wavefronts are well defined [2]. The Maxwell fish-eye is thus *not quite* a perfect imaging device [3] because it cannot forestall the chromatic dispersion.

Other operators in the hidden, potential, or dynamical algebras may be used to define other bases for the polychromatic, wavized fish-eye. We may choose the two commuting operators $\hat{p}_i = \frac{1}{2\rho}(\hat{M}_i - \hat{K}_i) = -i\partial/\partial q_i$, $i = 1, 2$. These yield the plane-wave basis, but since they do not commute with the driving Hamiltonian, the solutions quickly loose their shape. They are in fact the cross-basis representations between the elliptic (ℓ, m) and parabolic (p_1, p_2) subgroups of the SO(3,1) group that we have realized on the $\Re^2$ plane [22]. The conformal so(3,2) algebra adds

[4] *i.e.* through the replacements $q_i \mapsto q_i\cdot$, (multiplication by q_i), and $p_i \mapsto \hat{p}_i = -i\partial/\partial q_i$.

the to the list the generators $H_i = q_i\, p$, $i = 1, 2$, $H_3 = \mathcal{N} - 2\rho\, p$, and $H_4 = \mathcal{N} = \rho(1 + |\mathrm{q}|^2/4\rho^2)\, p$. They are wavized with the integral operator that realizes the formal root of the Laplacian integral operator $p = \sqrt{p_1^2 + p_2^2}$.

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