Generalized Hidden Symmetries and the Kerr-Sen Black Hole

Tsuyoshi Houri[1]

Osaka City University Advanced Mathematical Institute (OCAMI),
3-3-138 Sugimoto, Sumiyoshi, Osaka, 558-8585 JAPAN

Abstract

We study an extension of Killing-Yano symmetry in the presence of totally skew-symmetric torsion, which is called generalized hidden symmetry. Such a symmetry gives rank-2 irreducible Killing tensors which don't in general commute. We further study Kerr-Sen black hole spacetime and its generalizations in hetelotic supergravity theory. It is shown that these spacetimes possess generalized Killing-Yano symmetry and the torsion is identified with 3-form flux naturally.

1 Introduction

Killing–Yano symmetry has been studied as a fundamental hidden symmetry which plays a crucial role in black hole spacetimes. It is known that in the four-dimensional Kerr spacetime [1], all the symmetries necessary for separability of the geodesic, Klein-Gordon and Dirac equations, are described by a Killing–Yano tensor [2]. Higher-dimensional solutions describing rotating black holes have attention in the recent developments of superstring and supergravity theories. It was demonstrated that the vacuum rotating black hole solutions (with spherical horizon topology) [3–5] have Killing–Yano symmetry and generalize separability of Hamilton-Jacobi equation [6–9], Klein-Gordon equation [10, 11] and Dirac equation [12]. In this presentation we discuss a Killing–Yano symmetry in the presence of skew-symmetric torsion. The spacetimes with skew-symmetric torsion occur naturally in supergravity theories, where the torsion may be identified with a 3-form field strength. Black hole spacetimes of such theories are natural candidates to admit the Killing–Yano symmetry with torsion. This generalized symmetry was first introduced by Bochner and Yano [13] from the mathematical point of view and recently rediscovered [14–16] as a hidden symmetry of the Chong–Cvetic–Lü–Pope rotating black hole of $D = 5$ minimal gauged supergravity [17]. Furthermore, this was found in the Kerr–Sen black hole solution [18, 19] of effective string theory and its higher-dimensional generalizations [20]. The discovered generalized symmetry shears almost identical properties with its vacuum cousin; it gives rise symmetries that imply separability of the Hamilton–Jacobi, Klein-Gordon, and Dirac equations in this background [21].

2 Generalized Killing-Yano symmetries

We first recall some notations concerning a connection with totally skew-symmetric torsion. Let T_{abc} be a 3-form and ∇^T_a be a connection defined by

$$\nabla^T_a Y^b = \nabla_a Y^b + \frac{1}{2} T_{cab} Y^c \ , \tag{1}$$

where ∇_a is the Levi-Civita connection. The connection ∇^T_a satisfies a metricity condition $\nabla^T_a g_{bc} = 0$, and preserves the geodesics. For a p-form $\psi_{a_1 \cdots a_p}$ the covariant derivative is calculated as

$$\nabla^T_a \psi_{b_1 \cdots b_p} = \nabla_a \psi_{b_a \cdots b_p} + \frac{1}{2} T_{ca[b_1} \psi^c{}_{b_2 \cdots b_p]} \ . \tag{2}$$

We further define an exterior derivative d^T and a co-exterior derivative δ^T by

$$(d^T \psi)_{a_1 \cdots a_{p+1}} = \frac{1}{p!} \nabla^T_{[a_1} \psi_{a_2 \cdots a_{p+1}]} \ , \quad (\delta^T \psi)_{a_1 \cdots a_{p-1}} = -\nabla^T_c \psi^c{}_{a_1 \cdots a_{p-1}} \ . \tag{3}$$

[1]Email address: houri@sci.osaka-cu.ac.jp

A *generalized conformal Killing-Yano (GCKY) tensor* k was introduced as a p-form satisfying for any vector field X

$$\nabla^T_X k = \frac{1}{p+1} X \lrcorner\, d^T k - \frac{1}{D-p+1} X^* \wedge \delta^T k \ , \tag{4}$$

where $\lrcorner$ and $\wedge$ stand for an inner and a wedge product, respectively. A GCKY p-form f obeying $\delta^T f = 0$ is called a *generalized Killing-Yano (GKY) tensor*, and a GCKY p-form h obeying $d^T h = 0$ is called a *generalized closed conformal Killing-Yano (GCCKY) tensor*.

Proposition 2.1 *GCKY tensors possess the following basic properties:*

1. *A GCKY 1-form is equal to a conformal Killing 1-form.*
2. *The Hodge star $*$ maps GCKY p-forms into GCKY $(D-p)$-forms. In particular, the Hodge star of a GCCKY p-form is a GKY $(D-p)$-form and vice versa.*
3. *When h_1 and h_2 is a GCCKY p-form and q-form, then $h_3 = h_1 \wedge h_2$ is a GCCKY $(p+q)$-form.*
4. *Let k be a GCKY p-form for a metric g and a torsion 3-form T. Then, $\tilde{k} = \Omega^{p+1} k$ is a GCKY p-form for the metric $\tilde{g} = \Omega^2 g$ and the torsion $\tilde{T} = \Omega^2 T$.*
5. *Let k be a GCKY p-form. Then*

$$Q_{ab} \equiv k_{ac_1\cdots c_{p-1}} k_b{}^{c_1\cdots c_{p-1}} \tag{5}$$

is a rank-2 conformal Killing tensor. In particular, Q is a rank-2 Killing tensor if k is a GKY tensor.

We define a $2j$-form $h^{(j)}$ as $h^{(j)} = h \wedge h \wedge \cdots \wedge h$ where the wedge products are taken $j-1$ times such as $h^{(0)} = 1$, $h^{(1)} = h$, $h^{(2)} = h \wedge h$, $\cdots$. If we put the dimension $D = 2n + \varepsilon$, where $\varepsilon = 0$ for even dimensions and $\varepsilon = 1$ for odd dimensions, $h^{(j)}$ are non-trivial only for $j = 0, \cdots, n-1+\varepsilon$, i.e., $h^{(j)} = 0$ for $j > n-1+\varepsilon$. Since the wedge product of two GCCKY tensors is again a GCCKY tensor, $h^{(j)}$ are GCCKY tensors for all j. Moreover the Hodge dual of the GCCKY tensors $h^{(j)}$ gives rise to the GKY tensors $f^{(j)} = *h^{(j)}$. For odd dimensions, since $h^{(n)}$ is a rank-$2n$ GCCKY tensor, $f^{(n)}$ is a Killing vector. Given these GKY tensors $f^{(j)}$ $(j = 0, \ldots, n-1)$, one can construct the rank-2 Killing tensors

$$K^{(j)}_{ab} = \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}_{ac_1\cdots c_{D-2j-1}} f^{(j)c_1\cdots c_{D-2j-1}}_b \ , \tag{6}$$

obeying the equation $\nabla_{(a} K^{(j)}_{bc)} = 0$, and

$$[K^{(j)}, K^{(\ell)}]^T_{abc} \equiv K^{(j)}_{e(a} \nabla^{Te} K^{(\ell)}_{bc)} - K^{(\ell)}_{ea} \nabla^{Te} K^{(j)}_{bc)} = 0 \ . \tag{7}$$

This means that the integrals of motion generated from Killing tensors don't commute with respect to Poisson bracket.

When the torsion is absent, it is shown that δh is a Killing vector. On the other hand, when the torsion is present, neither $\delta^T h$ nor δh are in general Killing vectors. Such a torsion anomaly appears everywhere in considering geometry with the GCCKY 2-form. For instance, it is seen in separability of field equations. Separation of variables in differential equations is deeply related to the existence of symmetry operators, which commute between themselves and whose number is that of dimensions. It is known that such symmetry operators can be generated by a CCKY 2-form in the absence of torsion. In the presence of torsion, however, the commutator between a symmetry operator generated by a Killing tensor and the laplacian don't vanish in general. This means that a GCCKY 2-form no longer generates symmetry operators for Klein-Gordon equation. Similarly, it is known that the GCCKY 2-form doesn't in general generates symmetry operators for Dirac equation, while it is possible for CCKY tensor.

2.1 Kerr–Sen Black Hole

Let us see an example of spacetimes admitting a GCCKY 2-form. Actually, it can be shown that $(2n+\varepsilon)$-dimensional metric

$$g = \sum_{\mu=1}^{n} \frac{U_\mu}{X_\mu} dx_\mu^2 + \sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k - \sum_{\nu=1}^{n} \frac{N_\nu}{HU_\nu} \sum_{k=0}^{n-1} A_\nu^{(k)} d\psi_k \right)^2 + \varepsilon \frac{c}{A^{(n)}} \left(\sum_{k=0}^{n} A^{(k)} d\psi_k - \sum_{\nu=1}^{n} \frac{N_\nu}{HU_\nu} \sum_{k=0}^{n-1} A_\nu^{(k)} d\psi_k \right)^2 , \tag{8}$$

where c is a constant, admits a GCCKY 2-form h. It is convenient to introduce an orthonormal basis $\{e^\mu, e^{\hat\mu}, e^0\}$,

$$e^\mu = \sqrt{\frac{U_\mu}{X_\mu}} dx_\mu , \quad e^{\hat\mu} = \sqrt{\frac{X_\mu}{U_\mu}} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k - \sum_{\nu=1}^{n} \frac{N_\nu}{HU_\nu} \sum_{k=0}^{n-1} A_\nu^{(k)} d\psi_k \right) ,$$
$$e^0 = \frac{c}{A^{(n)}} \left(\sum_{k=0}^{n} A^{(k)} d\psi_k - \sum_{\nu=1}^{n} \frac{N_\nu}{HU_\nu} \sum_{k=0}^{n-1} A_\nu^{(k)} d\psi_k \right) , \tag{9}$$

in which g, h and the torsion T are written as

$$g = \sum_{\mu=1}^{n} \left(e^\mu e^\mu + e^{\hat\mu} e^{\hat\mu} \right) + \varepsilon e^0 e^0 ,$$
$$h = \sum_{\mu=1}^{n} x_\mu \, e^\mu \wedge e^{\hat\mu} , \quad T = -\left(\sum_{\mu=1}^{n} \frac{\partial_\mu H}{H} \, e^\mu \wedge e^{\hat\mu} \right) \wedge \sum_{\nu=1}^{n} \sqrt{\frac{X_\nu}{U_\nu}} \, e^{\hat\nu} . \tag{10}$$

Here the metric functions are given as

$$U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2) , \quad H = 1 + \sum_{\mu=1}^{n} \frac{N_\mu}{U_\mu} ,$$
$$A_\mu^{(k)} = \sum_{\substack{1 \le \nu_1 < \cdots < \nu_k \le n \\ \nu_i \neq \mu}} x_{\nu_1}^2 \cdots x_{\nu_k}^2 , \quad A^{(k)} = \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} x_{\nu_1}^2 \cdots x_{\nu_k}^2 , \quad A_\mu^{(0)} = A^{(0)} = 1 , \tag{11}$$

and the functions X_μ and N_μ depend on the single variable x_μ: $X_\mu(x_\mu)$, $N_\mu(x_\mu)$.

In considering an effective theory of hetelotic supergravity,

$$S = \int e^\phi \Big(*R + *d\phi \wedge d\phi - *F \wedge F - \frac{1}{2} * H \wedge H \Big) , \tag{12}$$

where $F = dA$ and $H = dB - A \wedge dA$, the metric g and the 3-form field strength H identified with the torsion T are required to satisfy the equations of motion

$$R_{ab} - \nabla_a \nabla_b \phi - F_a{}^c F_{bc} - \frac{1}{4} H_a{}^{cd} H_{bcd} = 0 ,$$
$$d\Big(e^\phi * F \Big) = e^\phi * H \wedge F , \quad d\Big(e^\phi * H \Big) = 0 ,$$
$$(\nabla\phi)^2 + 2\nabla^2\phi + \frac{1}{2} F_{ab} F^{ab} + \frac{1}{12} H_{abc} H^{abc} - R = 0 . \tag{13}$$

These equations determine the unknown functions X_μ and N_μ as

$$X_\mu = \sum_{k=0}^{n-1} c_k x_\mu^{2k} + 2m_\mu x_\mu^{1-\varepsilon} + \varepsilon \frac{(-1)^n \tilde c}{x_\mu^2} , \quad N_\mu = 2m_\mu x_\mu^{1-\varepsilon} s^2 , \tag{14}$$

where $s = \sinh\delta$, $c = \cosh\delta$, $c_{n-1} = -1$, and m_μ ($\mu = 1, \cdots, n$), c_k ($k = 0, \cdots, n-2$), $\tilde{c}$ and δ are arbitrary constants. In addition, the Maxwell potential A and the dilaton field ϕ become

$$A = \frac{c}{s} \sum_{\mu=1}^{n} \frac{N_\mu}{HU_\mu} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \ , \quad \phi = \log H \ . \tag{15}$$

When we take the special choices of the constants, the solutions represent charged rotating black hole solutions including the Kerr-Sen black hole [18, 19] and its higher-dimensional generalizations [20]. The torsion anomalies vanish on these black hole spacetimes, and hence one can expect that integrable structures [22–25] are subject to a generalized Killing-Yano symmetry.

3 Conclusion

We have studied an extension of Killing-Yano symmetry in the presence of 3-form torsion. We have demonstrated that, when the torsion is an arbitrary 3-form, one obtains various torsion anomalies and the implications of the existence of the generalized Killing-Yano symmetry are relatively weak compared with ordinary Killing-Yano symmetry. However, in the spacetimes where there is a natural 3-form obeying the appropriate field equations, these anomalies disappear and the concept of generalized Killing-Yano symmetry may become very powerful.

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