



Aspects of Symmetry in Asymptotically Flat Spacetimes

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Aspects of Symmetry in Asymptotically Flat Spacetimes

A DISSERTATION PRESENTED
BY
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ABSTRACT

We explore the nature and implications of a variety of asymptotic symmetry groups associated to gauge theory and quantum gravity in asymptotically flat spacetimes. We re-express Weinberg's soft graviton theorem as a Ward identity for supertranslation invariance of the gravitational \mathcal{S} -matrix, and provide an alternate derivation of these symmetries as nontrivial large diffeomorphisms preserving finite-energy boundary conditions at null infinity. Similarly, we recast Weinberg's soft photon theorem as a Ward identity for infinitely many new symmetries of the massless QED \mathcal{S} -matrix. These symmetries are identified as large gauge transformations with angle-dependent gauge parameters, and lead to a degeneracy of the gauge theory vacuum. We then extend the analysis to incorporate massive charged particles. Transitions among the degenerate vacua are induced in any nontrivial scattering process, but conventional computations of scattering amplitudes in QED ignore this fact and therefore always give zero due to infrared divergences. We demonstrate that if these vacuum transitions are properly accounted for, the resulting amplitudes are nonzero and infrared finite.

We then utilize the subleading soft graviton theorem to demonstrate that the \mathcal{S} -matrix for quantum gravity in four-dimensional Minkowski space has a Virasoro symmetry which acts on the celestial sphere at null infinity. We construct an operator T_{zz} whose insertion in the four-dimensional \mathcal{S} -matrix obeys the Ward identities of the energy-momentum tensor of a CFT_2 . Generalizing to higher dimensions, the $(d+2)$ -dimensional \mathcal{S} -matrix elements are recast as correlation functions

Thesis advisor: Professor Andrew Strominger

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of local operators living on a spacelike cut \mathcal{M}_d of the null momentum cone. The Lorentz group $SO(d+1, 1)$ is nonlinearly realized as the Euclidean conformal group on \mathcal{M}_d . We demonstrate that the leading soft photon operator is the shadow transform of a conserved spin-one primary operator J_a , and the subleading soft graviton operator is the shadow transform of a conserved spin-two symmetric traceless primary operator T_{ab} . The universal form of the soft limits ensures that J_a and T_{ab} obey the Ward identities expected of a conserved current and energy-momentum tensor in a Euclidean CFT _{d} , respectively.

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Citations to previously published work

Parts of this dissertation cover research reported in the following articles:

Chapter 2 is a lightly edited version of the paper

Higher-Dimensional Supertranslations and Weinberg's Soft Graviton Theorem

D. Kapec, V. Lysov, S. Pasterski and A. Strominger

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Asymptotic Symmetries of Massless QED in Even Dimensions

D. Kapec, V. Lysov and A. Strominger

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New Symmetries of QED

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Infrared Divergences in QED, Revisited

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Semiclassical Virasoro Symmetry of the Quantum Gravity S-matrix

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A d -Dimensional Stress Tensor for $Mink_{d+2}$ Gravity

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For Steve and Robin.

1

Introduction and Summary

New and surprising connections have recently emerged between seemingly unrelated infrared phenomena in gauge and gravitational theories in asymptotically flat spacetimes. Soft theorems for scattering amplitudes, infinite-dimensional asymptotic symmetry groups, and physically measurable memory effects, once studied in isolation and regarded as disparate subjects, are now viewed as different manifestations of a single underlying structure. The work described in this dissertation lies at the intersection of these exciting fields, and has played a crucial role in establishing and clarifying the connections between these infrared phenomena. These results and related work have shed new light on a variety of old problems, ranging from infrared divergences in gauge theory to the black hole information paradox, and have led to the discovery of new soft theorems and memory effects. Ultimately, this line of investigation is motivated by the search for, and may well lead to, a holographic description of quantum gravity in asymptotically flat spacetimes.

Asymptotically flat spacetimes model isolated, interacting gravitational systems within the

framework of general relativity. They approximate a wide variety of physical systems, including many of astrophysical interest. Early investigations of these spacetimes produced both exciting results and intriguing puzzles. The pioneering work by Bondi, van der Burg, Metzner and Sachs (BMS) [1–3] successfully identified the class of relevant spacetimes and elucidated many of their expected physical properties. However, in the course of their investigation, they encountered a strikingly novel phenomenon whose physical implications are still being actively explored. Straightforward calculations demonstrated that the group of nontrivial symmetry transformations of asymptotically flat spacetimes, known as the asymptotic symmetry group, was an infinite-dimensional extension of the expected Poincaré group. At the time of its discovery, this surprising structure, now known as the BMS group, clashed with naive expectations. The assumption that general relativity reduced to special relativity in the limit of large distances and low energies was apparently incorrect: gravity leaves an infrared footprint. Today, symmetry enhancement in gravitational theories is a familiar occurrence, and recent work has demonstrated that the BMS group plays a crucial role in the infrared dynamics of gravity in asymptotically flat spacetimes. In fact, although it was only discovered recently, the BMS group has an exact mathematical analog in deconfined gauge theories, where angle-dependent “large gauge transformations” with noncompact support play an equally important role in controlling infrared phenomena.

While the relativity community was grappling with the implications of the BMS group, particle physicists were simultaneously exploring extreme infrared effects in perturbative quantum field theories. In the process of investigating the infrared divergences which plague perturbative scattering amplitudes, a number of physicists were led to investigate so-called soft limits of the \mathcal{S} -matrix [4–8]. Their results, commonly referred to as soft photon theorems and soft graviton theorems, display a striking degree of universality. In the limit in which the momentum of an external photon or graviton becomes much smaller than any scale in the scattering process, the amplitude factorizes into a universal piece that depends very simply on the soft particle, multiplied by a reduced amplitude which is independent of the soft photon or graviton. These soft theorems are crucial for the cancellation of infrared divergences in inclusive cross sections, and have analogues in a variety of

theories. Some receive interesting loop corrections, while others are tree-level exact.

Universal behavior is often indicative of an underlying physical symmetry, and the soft theorems are no exception. As we will see in chapter 2, the leading soft graviton theorem is simply a reflection of the underlying BMS invariance of the gravitational \mathcal{S} -matrix. This connection links two lines of research pursued independently for decades, and offers a new picture for the role of soft emissions in scattering processes. The infinite-dimensional “supertranslation” subgroup of the BMS group is spontaneously broken on the standard perturbative vacuum, leading to a family of degenerate vacua related by supertranslations. The soft graviton is the Goldstone boson, and the supertranslation Ward identity yields precisely the leading soft graviton theorem. Chapter 2 demonstrates this correspondence in all even-dimensional spacetimes, while clarifying the nature of the BMS group in higher dimensions. Perhaps unsurprisingly, similar statements exist relating the soft photon theorem to large gauge transformations which act nontrivially at null infinity. Aside from the constant phase rotation corresponding to global charge conservation, these symmetries are all spontaneously broken on the vacuum and the soft photon is interpreted as the associated Goldstone boson. Chapter 3 establishes this relationship in all even dimensions for tree-level massless quantum electrodynamics (QED). Most of the work on this subject focuses on the scattering of massless particles coupled to gauge theory or gravity. Chapter 4 extends the analysis to include the phenomenologically relevant case of four-dimensional massive QED and makes use of qualitatively new insights into the correspondence.

It turns out that the relations of these asymptotic symmetry groups to infrared phenomena run even deeper than originally suspected. It is straightforward to demonstrate that in four dimensions, generic scattering processes induce transitions between the degenerate vacua in gauge and gravitational theories. Conventional Fock-space calculations of scattering amplitudes in QED fail to take this into account, and the resultant infrared “divergences” set all such amplitudes to zero. Chapter 5 demonstrates that if these vacuum transitions are accounted for, the resulting transition amplitudes are nonzero and infrared finite.

Perhaps the most interesting, and certainly the least understood, part of this story pertains to

a conjectured extension of the four-dimensional BMS group. The BMS group has a natural action at null infinity, where the Lorentz group $SL(2, \mathbb{C})$ is realized as the group of conformal motions of the celestial sphere. The original BMS analysis predated the discovery of the infinite-dimensional symmetry of 2D conformal field theories [9], but today we are familiar with a number of physical systems whose $SL(2, \mathbb{C})$ symmetry is enhanced to an infinite-dimensional Virasoro symmetry. This observation led a number of authors [10–14] to conjecture that a similar phenomenon might occur in four-dimensional quantum gravity. The connection between asymptotic symmetries and soft theorems suggests that a new soft graviton theorem is a necessary condition for the validity of this conjecture. Motivated by this reasoning, Cachazo and Strominger derived a new universal subleading soft graviton theorem [15]. In chapter 6 we demonstrate its tree-level equivalence to the Ward identities of the extended BMS group, offering strong evidence for an enhancement of the global conformal group acting on the celestial sphere.

In chapter 7 we provide further evidence for this conjecture. Scattering amplitudes for massless particles in 4D can be written as correlation functions on the celestial sphere at null infinity: single particle asymptotic states simply correspond to operator insertions on the sphere where the particles enter or exit the spacetime. Lorentz invariance of the \mathcal{S} -matrix then guarantees that these correlation functions transform appropriately under the conformal group. In chapter 7 we use the subleading soft graviton theorem [15] to construct a boundary operator T_{zz} whose insertion into the four-dimensional \mathcal{S} -matrix obeys the Virasoro-Ward identities of a CFT_2 energy-momentum tensor. This result represents a concrete step towards a potential holographic realization of quantum gravity in asymptotically flat spacetimes. Chapter 9 extends this analysis to any number of dimensions, providing the construction of a d -dimensional stress tensor for $(d + 2)$ - dimensional asymptotically flat gravity. The 4D subleading soft graviton theorem receives an interesting one-loop exact renormalization due exclusively to infrared divergences. In chapter 8 we demonstrate that this infrared anomaly can be removed by a universal shift of the energy-momentum tensor T_{zz} , establishing a loop-corrected Virasoro symmetry of 4D quantum gravity.

Another exciting application of this circle of ideas pertains to the black hole information para-

dox in asymptotically flat spacetimes. Hawking, Perry and Strominger [16–18] have argued that asymptotic symmetry groups like the (extended) BMS group also significantly constrain the black hole evaporation process. The implications of this so-called “soft hair” on black holes are still under active investigation. Chapter 10 addresses a related problem. Motivated by possible supertranslation ambiguities in the definition of the Page curve, we consider various area/entropy bounds at null infinity. Our investigations lead to an interesting conjectural “second law of \mathcal{I}^+ ” relating area changes of cross-sectional cuts of \mathcal{I}^+ to the entanglement entropy across the cut. This and related work will help to sharpen our understanding of the black hole information paradox.

In the following sections we briefly review the background material needed to motivate and understand this interconnected web of phenomena. The technical discussion of the results begins in chapter 2.

1.1 Asymptotically Flat Spacetimes

Asymptotically flat spacetimes model isolated gravitational systems within the framework of classical general relativity. Although the precise, rigorous description of this class of spacetimes is rather technical [1–3, 19–23], the physical characterization is extremely simple. On physical grounds one expects that, at large distances far from the complicated nonlinear gravitational dynamics, the spacetime should settle down and the curvature should become small. These conditions are typically enforced through the imposition of boundary conditions on solutions to the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} . \quad (1.1.1)$$

The proper choice of boundary conditions in general relativity is often a subtle problem. Optimal boundary conditions should be weak enough to encompass all physical processes under consideration, but strong enough so that the the resulting class of spacetimes is tractable and amenable to study. For instance, in the case of asymptotically flat spacetimes, any candidate set of boundary conditions must be weak enough to allow for gravitational radiation, but strong enough so that

generic solutions have physically reasonable properties like finiteness of energy. In fact, the pioneering work conducted by BMS [1–3], which established the working definition of asymptotically flat spacetimes as well as the existence of the BMS group, was primarily concerned with finding a non-perturbative, diffeomorphism invariant characterization of gravitational radiation.

To concretely describe the class of spacetimes we have in mind, it is useful to introduce an explicit retarded null coordinate system (u, r, z, \bar{z}) to represent the geometry in the asymptotic region. Asymptotically, these coordinates are related to the standard Cartesian coordinates in flat space

$$u = t - r, \quad r^2 = x^i x_i, \quad x^i = r \hat{x}^i(z, \bar{z}), \quad (1.1.2)$$

where $\hat{x}^i(z, \bar{z})$ is an embedding of the round S^2 in \mathbb{R}^3 . These coordinates are particularly well-suited to describe the conformal boundary of the spacetime. Future null infinity, \mathcal{I}^+ , is given by the null surface $(u, r = \infty, z, \bar{z})$. We denote its future boundary where $u \rightarrow \infty$ by \mathcal{I}_+^+ and its past boundary where $u \rightarrow -\infty$ by \mathcal{I}_-^+ . The null cones \mathcal{N}_u defined by $u = \text{constant}$ intersect \mathcal{I}^+ on round two-spheres. Spatial infinity i^0 is obtained through the limit $r \rightarrow \infty$ with t held finite, while future/past timelike infinity (i^\pm) corresponds to the limit $t \rightarrow \pm\infty$ with r held fixed. To cover the spacetime patch near the past null boundary, we introduce a separate set of advanced null coordinates (v, r, z, \bar{z}) , where the advanced time v is asymptotically related to the usual Cartesian coordinates according to

$$v = t + r. \quad (1.1.3)$$

Past null infinity, \mathcal{I}^- , is given by the null surface $(v, r = \infty, z, \bar{z})$, and its future/past boundaries are denoted \mathcal{I}_\pm^- . The conformal structure of asymptotically flat spacetimes is illustrated in the Penrose diagram in figure 1.1.

Four-dimensional coordinate invariance requires the imposition of four gauge fixing conditions. We will work in Bondi gauge, requiring that

$$g_{rr} = 0, \quad g_{ra} = 0, \quad \det g_{ab} = r^4 \det \gamma_{ab}, \quad (1.1.4)$$

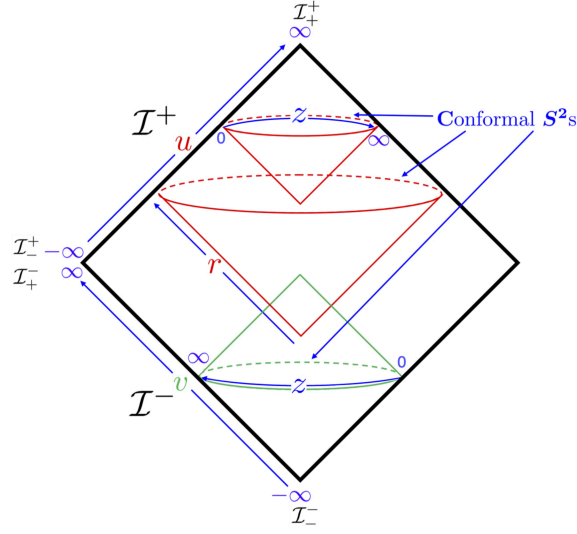


Figure 1.1: Conformal structure of asymptotically flat spacetimes.

where γ_{ab} is the round metric on the unit S^2 . To complete the definition of asymptotically flat spacetimes, we need to impose boundary (large- r) falloff conditions on the remaining components of the metric. In four dimensions, the appropriate conditions are

$$g_{uu} = -1 + \mathcal{O}(r^{-1}) , \quad g_{ur} = -1 + \mathcal{O}(r^{-2}) , \quad g_{ua} = \mathcal{O}(1) , \quad g_{ab} = r^2 \gamma_{ab} + \mathcal{O}(r) . \quad (1.1.5)$$

Assuming these conditions and imposing the Einstein equation order by order in the large- r expansion, one arrives at a family of metrics of the form

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + \frac{2m_B}{r} du^2 + r C_{zz} dz^2 + r C_{\bar{z}\bar{z}} d\bar{z}^2 + \dots , \quad (1.1.6)$$

where the ellipsis refers to subleading metric components which will not figure prominently in the discussion that follows. The Bondi mass aspect $m_B(u, z, \bar{z})$ is a measure of the energy of the spacetime as a function of retarded time. For the Schwarzschild black hole it is constant and is given by the Schwarzschild mass, while for time dependent solutions its integral over a $u = \text{constant}$ slice of \mathcal{I}^+ measures the energy of the spacetime at a particular retarded time.

A primary advantage of working with the Einstein equation at the null boundary is that it

enables one to explicitly solve the constraint equations and identify the free data of the gravitational field. Defining the Bondi news tensor

$$N_{zz} = \partial_u C_{zz} \quad (1.1.7)$$

and expanding the Einstein equation to leading order, one obtains the famous Bondi mass loss formula

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T_{uu}^M - \frac{1}{4} N_{zz} N^{zz} . \quad (1.1.8)$$

Integrating this equation against a $u = \text{constant}$ slice of \mathcal{I}^+ and making mild assumptions on the sign of the matter stress tensor T_{uu}^M , one discovers that the Bondi mass is a monotonically decreasing function along \mathcal{I}^+ . Moreover, it decreases only in the presence of matter flux through \mathcal{I}^+ , represented by nonzero T_{uu}^M , or in the presence of nonzero Bondi news. The symmetric traceless tensor N_{zz} is therefore a nonperturbative diagnostic of gravitational radiation. Before turning to a discussion of the asymptotic symmetry group of this class of spacetimes, we would like to point out one more important feature of equation (1.1.8). The seemingly innocuous term $\frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}}$ in the Bondi mass loss formula, which disappears upon integration over the full two-sphere, will nonetheless play an important role in later chapters. The fact that the term vanishes when integrated over the whole S^2 indicates that it is describing soft radiation: the term only redistributes the angular distribution of energy on the asymptotic sphere, without altering the total energy. Indeed it is through this term that we will connect the BMS group to the soft graviton theorem.

1.2 Asymptotic Symmetry Groups

When studying a physical system, the first step is always to understand the symmetries of the problem. Symmetries control universal phenomena, simplify the analysis and render otherwise intractable calculations feasible. In strongly coupled systems, they are often the only tool at our disposal. However, the notion of symmetry in quantum field theory is somewhat subtle, and an important distinction is typically drawn between “local” gauge symmetries and global symmetries.

The subtleties in this distinction will be crucial in all that follows. A basic requirement of a physical symmetry is that it should act nontrivially on physical states and operators. A transformation law which acts trivially on the entire Hilbert space as well as on the set of local and nonlocal operators has absolutely no operational meaning. In other words, if every state and operator in the theory transforms in the trivial representation of a “symmetry,” then the corresponding transformations merely introduce redundant descriptions of the same physical state and should be quotiented out. In the following subsections we will discuss four distinct examples of symmetry in quantum field theory and quantum gravity, ranging from the well-understood to topics of current research.

1.2.1 Global Symmetries in Quantum Field Theory

The ordinary global symmetries one encounters in the simplest gapped quantum field theories certainly satisfy our physicality criterion. Internal flavor symmetries and rigid spacetime isometries all map physical states onto other distinct physical states, and the charges Q that generate these transformations commute with the Hamiltonian and are represented nontrivially on the Hilbert space. In examples, global charges most often arise as spacelike integrals of local conserved currents $j_\mu(x)$. The condition $\partial^\mu j_\mu(x) = 0$, which holds classically on solutions to the equations of motion and quantum mechanically as an operator equality at non-coincident points inside correlation functions, is equivalent to the requirement that the Hodge dual $*j$ be closed:

$$d * j = 0 . \tag{1.2.1}$$

Given a spacelike Cauchy surface Σ_t asymptoting to spatial infinity i^0 (i.e. $\partial\Sigma_t = i^0$), we can consider the global charge operator $Q(\Sigma_t)$:

$$Q(\Sigma_t) = \int_{\Sigma_t} *j . \tag{1.2.2}$$

It then follows immediately from (1.2.1) that the operator $Q(\Sigma_t)$ is actually independent of the choice of Cauchy surface Σ_t and therefore conserved under time evolution. Indeed, given two such

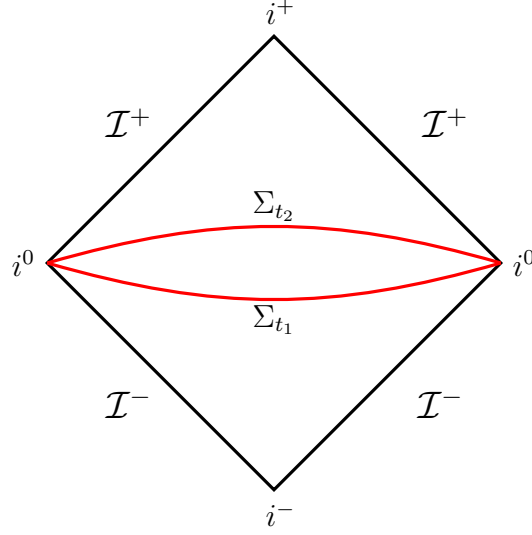


Figure 1.2: The charge Q is conserved under deformations of the Cauchy surface.

Cauchy surfaces Σ_{t_1} and Σ_{t_2} , and denoting the four-volume which they bound by \mathcal{M} , we have

$$0 \equiv \int_{\mathcal{M}} d * j = Q(\Sigma_{t_2}) - Q(\Sigma_{t_1}) , \quad (1.2.3)$$

assuming sufficient falloffs for the current at i^0 . This is illustrated in figure 1.2.

Analyses based on spacelike Cauchy surfaces and the corresponding behavior at spatial infinity are useful for describing gapped systems. However, such descriptions often obscure physical effects in gapless systems, such as the presence of radiation. For instance, if there are massless particles charged under some symmetry, one way to characterize the presence of radiation would be to calculate the charge flux carried away by the radiation through \mathcal{I}^+ . Charge integrals defined on spacelike Cauchy surfaces are obviously inadequate, since the quantities they describe are absolutely conserved: the passing radiation intersects each Cauchy surface on its way to \mathcal{I}^+ . This situation is illustrated in figure 1.3. Instead, one can foliate the spacetime with null cones \mathcal{N}_u intersecting \mathcal{I}^+ on two-spheres S_u^2 of fixed retarded time u . The charges evaluated on these hypersurfaces are not

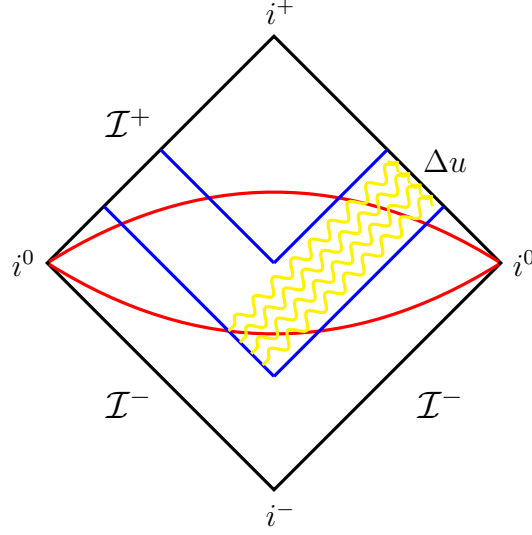


Figure 1.3: The charge carried by the outgoing radiation is registered by any two spacelike Cauchy surfaces. Null cones separated by the null displacement Δu are sensitive to radiation exiting the spacetime.

conserved in the presence of radiation. Instead, they satisfy a balance equation of the form

$$Q(\mathcal{N}_{u_2}) - Q(\mathcal{N}_{u_1}) = - \int_{\mathcal{I}_{\Delta u}^+} *j , \quad (1.2.4)$$

where $\mathcal{I}_{\Delta u}^+$ is the cylindrical segment of \mathcal{I}^+ bounded by the spheres $S_{u_1}^2$ and $S_{u_2}^2$. The gravitational analogue of equation (1.2.4) is of course the celebrated Bondi mass loss formula (1.1.8) in the theory of gravitational radiation, and will play a crucial role in the remainder of this work.

1.2.2 Symmetries in Gauge Theory

Our first example of a nonstandard symmetry comes from abelian gauge theory coupled to charged matter. We can consider electric and/or magnetic charges: the conclusions will apply provided that the system is in the electric and/or magnetic Coulomb phase. In four dimensions, the electro-

magnetic field strength F and its dual $*F$ are both two-forms. The Maxwell equations

$$\begin{aligned} d * F &= *j_E , \\ dF &= *j_M , \end{aligned} \tag{1.2.5}$$

therefore provide two currents $*j_E$ and $*j_M$ which are conserved independently of the equations of motion due to the nilpotency of the exterior derivative:

$$\begin{aligned} d * j_E &= d^2 * F \equiv 0 , \\ d * j_M &= d^2 F \equiv 0 . \end{aligned} \tag{1.2.6}$$

The important feature distinguishing the symmetry of the gauge theory from ordinary global symmetries is the fact that the currents $*j_E$ and $*j_M$ are exact. This is in turn responsible for the mild nonlocal features of the charged sectors of gauge theories. Given the electric and magnetic charges defined on a Cauchy slice Σ , we can use Stokes' theorem to rewrite the charges as boundary integrals at spatial infinity:

$$Q_E(\Sigma) = \int_{\Sigma} *j_E = \int_{\Sigma} d * F = \int_{i^0} *F , \tag{1.2.7}$$

$$Q_M(\Sigma) = \int_{\Sigma} *j_M = \int_{\Sigma} dF = \int_{i^0} F . \tag{1.2.8}$$

In other words, one can determine the presence of a charge deep in the interior of a spacetime by doing measurements far removed from the location of the charge. Both charges are conserved and generate $U(1)$ symmetries of the theory, but the nature of these symmetries is distinct from that of the global symmetries discussed in the previous section. First of all, because the charge can be written as a surface integral at infinity, it is clear that no local operators can be charged under the symmetry: local operators commute at spacelike separation. Only nonlocal operators which somehow “reach” out to spatial infinity can be charged.

These statements are familiar when phrased in terms of the usual Lagrangian description of the theory, although it should be noted that no local Lagrangian is capable of describing a system with

both light electric and magnetic charges. Focusing for the moment on the electric symmetry, one is forced to introduce a redundant description of the physical degrees of freedom in order to describe the resulting dynamics with local fields in a Lorentz invariant manner. The photon's two physical degrees of freedom are embedded into a four-component vector potential $A_\mu(x)$ and coupled to the matter fields $\psi(x)$ of charge Q through an interaction with the conserved current $*j_E$ so that the entire Lagrangian is invariant under gauge transformations of the form

$$A(x) \rightarrow A(x) + d\varepsilon(x) , \quad \psi(x) \rightarrow e^{iQ\varepsilon(x)}\psi(x) . \quad (1.2.9)$$

Most of these transformations correspond to redundant descriptions of the same physical state. They are not physical symmetries, and they are represented trivially on the physical Hilbert space. An important and obvious exception is the case $\varepsilon = 1$, which is represented as a constant phase rotation on charged states and nonlocal charged operators. The corresponding conserved charge is simply the global electric charge, given by (1.2.7). More generally, the gauge transformations with noncompact support which do not die off at infinity act nontrivially on the Hilbert space and can be realized as physical symmetries. Indeed, the preceding discussion has a straightforward generalization which will play an important role in the remainder of this dissertation. The mere existence of the two-form $*F$ allows us to consider a more general class of exactly conserved currents¹

$$*j_\varepsilon = d(\varepsilon * F) \quad (1.2.10)$$

which are again trivially closed irrespective of the equations of motion. The corresponding charge is given by

$$Q_\varepsilon = \int_{i^0} \varepsilon * F \quad (1.2.11)$$

and can be nonzero provided that the system is in the Coulomb phase. Note that for the special

¹For the remainder of this section we will focus on the electrically charged sector of the theory, although the discussion that follows has an immediate generalization to the magnetic sector in four dimensions. One simply replaces $*F$ with F in all formulas.

case $\varepsilon = 1$, this reduces to the ordinary electric charge. For the $\varepsilon = \mathcal{O}(r^{-1})$ which fall off at infinity, the charge is zero and acts trivially on the Hilbert space. These trivial charges correspond to the redundant gauge transformations which have been quotiented out. However, for $\varepsilon = \mathcal{O}(1)$ at i^0 , the charges are generically nonzero and generate physical transformations on charged states. They correspond to gauge transformations with noncompact support which act on boundary data and generate physical symmetries.

It is often useful to rewrite equation (1.2.10) as the sum of two terms

$$*j_\varepsilon = d\varepsilon \wedge *F + \varepsilon d * F = d\varepsilon \wedge *F + \varepsilon * j_E , \quad (1.2.12)$$

where we used the Maxwell equations to obtain the second equality. The term $d\varepsilon \wedge *F$ is linear in the electromagnetic field and creates photons when acting on states, while the term $\varepsilon * j_E$ is a weighted version of the matter charge current. This representation will be particularly useful in making connection with the soft theorems of section 1.3.

These extra symmetry transformations are our first example of an enhanced asymptotic symmetry group. This distinction between unphysical gauge transformations with compact support and the physical symmetry transformations, which arose in this case as local gauge transformations with noncompact support, is even more important in gravitational theories where we must distinguish between diffeomorphisms with compact and noncompact support.

1.2.3 The BMS Group

As we have just seen, the proper determination of the complete set of physical internal symmetries is somewhat subtle in gauge theories. In gravitational theories with a fluctuating metric, the definition of physical “spacetime symmetries” is similarly complicated.

The formulation of local quantum field theory assumes a fixed background metric and corresponding causal structure for the spacetime on which the field theory is defined. The spacetime symmetries are then identified as the isometries of the background metric and are generated by

vector fields ξ satisfying the Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 . \quad (1.2.13)$$

The isometry group of flat spacetime is the Poincaré group, consisting of the four translations, three rotations and three boosts. The corresponding conserved charges are simply the momentum, boost, and angular momentum charges. All local (non-gravitational) quantum field theories come equipped with a local energy-momentum tensor $T_{\mu\nu}$, and the spacetime symmetry charges can be expressed as local integrals of a current

$$j_\mu = \xi^\nu T_{\nu\mu} \quad (1.2.14)$$

formed by contracting the corresponding Killing vector with the energy-momentum tensor. The spacetime charge integrals

$$Q = \int_\Sigma *j \quad (1.2.15)$$

then generate the symmetry transformations on local operators.

This straightforward story changes drastically when we consider theories interacting gravitationally with a dynamical metric. First of all, generic solutions to the Einstein equations have no isometries, and small metric fluctuations about any symmetric spacetime would destroy any accidental isometries. Asking for an exact isometry in a theory with a fluctuating metric is like asking for a global symmetry of a particular excited state in a quantum field theory. Generic states are not invariant under any symmetries. Rather, the symmetries of the theory map states onto different states in a controlled manner, governed by the representations of the symmetry group.

Clearly, exact isometry is not the appropriate notion of symmetry in gravitational theories. Instead, we are interested in the symmetries of the *collection* of asymptotically flat spacetimes defined in section 1.1 and satisfying the boundary conditions (1.1.5) along with the gauge fixing conditions (1.1.4). Diffeomorphisms preserving these conditions that do not vanish at \mathcal{I}^\pm comprise the asymptotic symmetry group of four-dimensional asymptotically flat spacetimes. Straightfor-

ward calculation reveals that all such diffeomorphisms are generated by vector fields with leading order large- r behavior near \mathcal{I}^+ of the form

$$\begin{aligned} \xi = & \left(1 + \frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^z \partial_{\bar{z}} - \frac{1}{2}(u+r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + c.c. \\ & + f \partial_u - \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r . \end{aligned} \quad (1.2.16)$$

The two-dimensional vector field Y^z satisfies the two-dimensional conformal Killing equation. The solutions to this equation which are globally well-defined on S^2 generate the group $SL(2, \mathbb{C})$ of global conformal transformations of the celestial sphere. Asymptotically, the Lorentz group is therefore realized as the group of conformal motions of the celestial sphere. This correspondence begs to be put to use holographically. The second ingredient in the definition of the vector field ξ is the function $f(z, \bar{z})$ on the S^2 . Without loss of generality we can expand $f(z, \bar{z})$ in a basis of spherical harmonics. One finds that the four lowest lying harmonics ($l = 0, 1$) generate spatial and time translations. The higher order harmonics are known as supertranslations. Geometrically, they correspond to angle-dependent retarded time translations at \mathcal{I}^+ .

Just as in gauge theory, we would like to determine the charges associated to these symmetries as well as the physical consequences of charge conservation. In abelian gauge theory, the photon couples to a conserved charge which is expressible as a surface integral at spatial infinity. In gravitational theories, the equivalence principle states that the “charge” coupled to the graviton is energy-momentum itself. Classic results in the field [24] demonstrate that in the presence of a dynamical metric the energy and momentum charges can indeed be written as surface integrals at spatial infinity. This is a very powerful result, since the charge associated to time translations is simply the Hamiltonian of the system. It lies at the heart of the holographic principle, making it possible for a single Hamiltonian to simultaneously describe $(d+1)$ -dimensional quantum gravity and d -dimensional non-gravitational systems. Since the Hamiltonian can be written as a boundary integral, no local operators undergo nontrivial time evolution: there are no interesting local operators in theories of quantum gravity. The supertranslation charges are straightforward generalizations of the ADM mass [24] and are given by the higher moments of the Bondi mass

aspect

$$Q_f = \frac{1}{4\pi G} \int_{i^0} \sqrt{\gamma} d^2z f(z, \bar{z}) m_B(z, \bar{z}) . \quad (1.2.17)$$

1.2.4 The Extended BMS Group

As we remarked earlier, the holomorphic vector fields Y^z that enter into the definition of the BMS generators arise as solutions to the conformal Killing equation on the celestial sphere. The set of globally defined solutions to this equation on S^2 is six-dimensional, and the resulting algebra is that of the Lorentz group. However, it is well known that allowing for locally holomorphic vector fields with isolated singularities dramatically enlarges this symmetry, yielding instead the infinite-dimensional Virasoro algebra. In the study of two-dimensional conformal field theory, the isolated singularities prove innocuous and the enhanced symmetry provides tremendous constraining power on physical predictions [9].

The original analysis of the BMS group predated the discovery of the infinite-dimensional symmetry of two-dimensional conformal field theories, so the authors only considered the conformal Killing vectors which are globally defined on the S^2 . Given that the enhancement from $SL(2, \mathbb{C})$ to Virasoro plays such a crucial role both in AdS_3 quantum gravity [25, 26] and in physically realizable two-dimensional systems [9], several authors [10–14] conjectured that a similar mechanism of symmetry enhancement might occur in four-dimensional asymptotically flat quantum gravity. The transformations generated by the singular solutions to the conformal Killing equation have been termed “superrotations,” and the enlarged asymptotic symmetry group containing both the supertranslations and superrotations is often referred to as the extended BMS group. We will later find evidence that these enhanced symmetry transformations do have a role to play in four-dimensional quantum gravity, although their precise status remains a topic of current research.

1.3 Soft Theorems and Infrared Dynamics

The primary object of interest in any quantum mechanical theory in asymptotically flat spacetime is the \mathcal{S} -matrix. In realistic theories, generic \mathcal{S} -matrix elements describing the scattering of a fixed number of external particles are often extremely complicated and difficult to compute. However, it turns out that a striking simplification occurs when the momentum of an external photon or graviton approaches zero. In this “soft limit,” the photon’s (graviton’s) wavelength becomes larger than any scale in the scattering process, and it is no longer able to resolve the short distance interactions undergone by the other “hard” particles. Indeed the soft photon (graviton) is only sensitive to the conserved quantum numbers of the external hard particles. In this limit, the \mathcal{S} -matrix element with the soft insertion factorizes into the product of a universal “soft operator” acting on the simplified \mathcal{S} -matrix element without the soft insertion. This universal relation is known as a soft theorem, and is illustrated in figure 1.4.

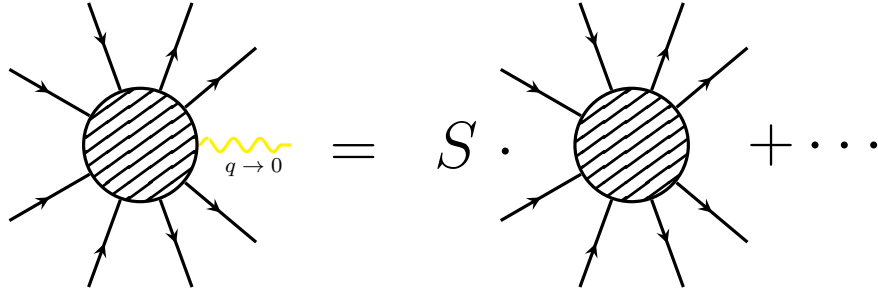


Figure 1.4: Graphical illustration of the soft theorem.

To make the discussion more concrete, we can consider the tree-level scattering amplitude for n massless particles

$$\mathcal{A}_n = \langle \text{out} | \mathcal{S} | \text{in} \rangle, \quad (1.3.1)$$

where the in- and out-states are labeled by the external particles’ momentum and helicity variables²

$$| \text{in} \rangle = | p_1, s_1; \dots; p_m, s_m \rangle, \quad \langle \text{out} | = \langle p_{m+1}, s_{m+1}; \dots; p_n, s_n |. \quad (1.3.2)$$

²In our conventions incoming states are described as CPT conjugate outgoing states with negative p^0 so that momentum conservation implies $\sum_{k=1}^n p_k^\mu = 0$.

The amplitude $\mathcal{A}_{n+1}^{(\pm)}(q)$ on the left hand side of figure 1.4 includes a graviton of momentum q^μ and polarization $\varepsilon_{\mu\nu}^{(\pm)}(q)$ along with n other massless particles:

$$\mathcal{A}_{n+1}^{(\pm)}(q) = \langle \text{out} ; q, \pm 2 | \mathcal{S} | \text{in} \rangle . \quad (1.3.3)$$

The soft graviton theorem states that in an expansion about the limit $q \rightarrow 0$, the amplitude factorizes up to order $\mathcal{O}(q)$ terms:

$$\mathcal{A}_{n+1}^{(\pm)}(q) \rightarrow \left[S_0^{(\pm)} + S_1^{(\pm)} + \mathcal{O}(q) \right] \mathcal{A}_n . \quad (1.3.4)$$

The soft factors $S_0^{(\pm)}$ and $S_1^{(\pm)}$ are given by

$$S_0^{(\pm)} = \frac{\kappa}{2} \sum_{k=1}^n \frac{p_k^\mu p_k^\nu \varepsilon_{\mu\nu}^{(\pm)}}{p_k \cdot q} , \quad S_1^{(\pm)} = -\frac{i\kappa}{2} \sum_{k=1}^n \frac{p_k^\mu q_\lambda \varepsilon_{\mu\nu}^{(\pm)}}{p_k \cdot q} \mathcal{J}_k^{\lambda\nu} , \quad \kappa = \sqrt{32\pi G} , \quad (1.3.5)$$

where $\mathcal{J}_k^{\lambda\nu}$ is the k^{th} particle's total angular momentum operator. Note that $S_0^{(\pm)}$ is of order $\mathcal{O}(q^{-1})$, while $S_1^{(\pm)}$ is of order $\mathcal{O}(q^0)$. Both soft factors are completely universal and model independent, and depend only on the conserved quantum numbers of the external particles. If we replace the external graviton line with an external photon with polarization $\varepsilon_\mu^{(\pm)}(q)$ and take the soft limit, we obtain a similar factorization to order $\mathcal{O}(q^0)$:

$$\mathcal{A}_{n+1}^{(\pm)}(q) \rightarrow \left[S_0^{(\pm)} + \mathcal{O}(q^0) \right] \mathcal{A}_n . \quad (1.3.6)$$

This relation is known as the soft photon theorem, and the soft operator is given by

$$S_0^{(\pm)} = \sum_{k=1}^n Q_k \frac{p_k \cdot \varepsilon^{(\pm)}}{p_k \cdot q} , \quad (1.3.7)$$

where Q_k is the electric charge of the k^{th} particle.

1.4 New Developments

The soft photon and soft graviton theorems demonstrate a striking degree of universality. Recent developments described in this dissertation have revealed that this universality is a direct consequence of a physical symmetry. The argument, which will be made rigorous in the main body of the text, goes roughly as follows. Consider the scattering of massless particles coupled to an abelian gauge field. We have argued in section 1.2.2 that the quantity

$$*j_\varepsilon = d\varepsilon \wedge *F + \varepsilon d * F = d\varepsilon \wedge *F + \varepsilon * j_E , \quad (1.4.1)$$

integrated over a Cauchy surface, is conserved in any scattering process. This is equivalent to the statement that the corresponding charge commutes with the \mathcal{S} -matrix. Since the theory contains only massless particles, both \mathcal{I}^+ and \mathcal{I}^- are Cauchy surfaces. Making the definitions

$$\begin{aligned} Q^+ &= \int_{\mathcal{I}^+} *j_\varepsilon = \int_{\mathcal{I}^+} d\varepsilon \wedge *F + \int_{\mathcal{I}^+} \varepsilon * j_E \equiv Q_S^+ + Q_H^+ , \\ Q^- &= \int_{\mathcal{I}^-} *j_\varepsilon = \int_{\mathcal{I}^-} d\varepsilon \wedge *F + \int_{\mathcal{I}^-} \varepsilon * j_E \equiv Q_S^- + Q_H^- , \end{aligned} \quad (1.4.2)$$

we can rewrite

$$\langle \text{out} | Q^+ \mathcal{S} - \mathcal{S} Q^- | \text{in} \rangle = 0 \quad (1.4.3)$$

as a matrix element identity

$$\langle \text{out} | Q_S^+ \mathcal{S} - \mathcal{S} Q_S^- | \text{in} \rangle = - \langle \text{out} | Q_H^+ \mathcal{S} - \mathcal{S} Q_H^- | \text{in} \rangle . \quad (1.4.4)$$

The soft charge Q_S^\pm is linear in the electromagnetic field and creates a single photon when acting on the scattering states. Moreover, because the integral over \mathcal{I}^\pm involves an integral over the null coordinate u or v , the energy of this photon is vanishingly small. The hard charge Q_H^\pm measures the weighted outgoing/ingoing charge flux through \mathcal{I}^\pm , and acts diagonally on the scattering states. Equation (1.4.4) therefore relates a scattering amplitude with an extra soft photon insertion to the action of an operator on the reduced amplitude without the soft insertion. Further investigation

reveals that this relation is in fact the soft photon theorem.

Similar arguments can be made relating conservation of supertranslation charge to the leading soft graviton theorem. It is only the form of the hard and soft charges that changes. Assuming suitable boundary conditions, one can use the Bondi mass loss formula

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T_{uu}^M - \frac{1}{4} N_{zz} N^{zz} \quad (1.4.5)$$

to rewrite the conserved supertranslation charge

$$Q_f = \frac{1}{4\pi G} \int_{i^0} \sqrt{\gamma} d^2 z f(z, \bar{z}) m_B(z, \bar{z}) \quad (1.4.6)$$

as an integral over \mathcal{I}^+

$$\begin{aligned} Q_f^+ &= -\frac{1}{16\pi G} \int_{\mathcal{I}^+} \sqrt{\gamma} d^2 z f(z, \bar{z}) (D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}) + \frac{1}{8\pi G} \int_{\mathcal{I}^+} \sqrt{\gamma} d^2 z f(z, \bar{z}) \left(T_{uu}^M + \frac{1}{2} N_{zz} N^{zz} \right) \\ &\equiv Q_S^+ + Q_H^+. \end{aligned} \quad (1.4.7)$$

The charge can similarly be written as an integral over \mathcal{I}^- . The soft charge Q_S^\pm is linear in the gravitational field and creates gravitons of vanishingly small energy when acting on the scattering states. The hard charge Q_H^\pm measures energy flux through null infinity and acts diagonally on the scattering states. Just as in electromagnetism, the identity

$$\langle \text{out} | Q_f^+ \mathcal{S} - \mathcal{S} Q_f^- | \text{in} \rangle = 0 \quad (1.4.8)$$

relates scattering amplitudes with an extra soft graviton insertion to the action of an operator on the reduced amplitude without the soft insertion. This relation is simply the leading soft graviton theorem. As we will see in chapter 6, similar manipulations also relate superrotation symmetry to the subleading soft graviton theorem. As we will see, these developments have implications for a wide range of outstanding problems in theoretical physics.

1.5 Outline of This Dissertation

The organization of the remainder of this dissertation is as follows. In chapter 2, we study even-dimensional asymptotically flat spacetimes and their symmetries. We re-express Weinberg's soft graviton theorem as a Ward identity for the gravitational \mathcal{S} -matrix. The corresponding asymptotic symmetries are identified as higher-dimensional generalizations of the supertranslations. We provide an alternate derivation of these asymptotic symmetries as the group of diffeomorphisms preserving finite-energy boundary conditions at null infinity and acting nontrivially on physical data. In all even dimensions, the supertranslation symmetry is spontaneously broken in the conventional vacuum and the soft gravitons are the corresponding Goldstone bosons.

Chapter 3 contains an analysis of abelian gauge theories coupled to massless matter in even-dimensional spacetimes. We recast Weinberg's soft photon theorem as a Ward identity for infinitely many new nontrivial symmetries of the massless QED \mathcal{S} -matrix. We identify these symmetries as large gauge transformations with noncompact support. Almost all of the symmetries are spontaneously broken in the standard vacuum and the soft photons are the corresponding Goldstone bosons. Chapter 4 develops new techniques in order to extend the analysis to include four-dimensional $U(1)$ gauge theories coupled to massive matter, including the phenomenologically relevant case of QED.

In Chapter 5, we revisit the problem of infrared divergences in QED in light of these newly discovered symmetries. The existence of an enhanced asymptotic symmetry group in QED leads to an infinite set of degenerate vacua, each differing according to its soft photon content. Moreover, generic scattering processes induce transitions among the degenerate vacua. Conventional computations of Fock-space scattering amplitudes in QED do not account for this vacuum degeneracy and therefore always give zero due to infrared divergences. We demonstrate that if these vacuum transitions are properly accounted for, the resulting amplitudes are nonzero and infrared finite.

In chapter 6 we begin our investigation of the extended BMS group. We use the four-dimensional subleading soft graviton theorem to demonstrate that the tree-level \mathcal{S} -matrix for quantum gravity in four-dimensional Minkowski space has a Virasoro symmetry which acts on the conformal sphere

at null infinity. Chapter 7 continues with the analysis of the superrotations. We use the subleading soft graviton theorem to construct an operator T_{zz} whose insertion in the four-dimensional tree-level quantum gravity \mathcal{S} -matrix obeys the Virasoro-Ward identities of the energy-momentum tensor of a two-dimensional conformal field theory (CFT_2). The celestial sphere at null infinity plays the role of the Euclidean sphere of the CFT_2 , with the Lorentz group acting as the unbroken $SL(2, \mathbb{C})$ subgroup. The subleading soft graviton theorem, and therefore the tree-level operator T_{zz} , receives an interesting one-loop exact correction due to infrared divergences. In chapter 8, we demonstrate that the effects of the IR divergent part of this “anomaly” can be eliminated by a one-loop renormalization that shifts T_{zz} , establishing a loop-corrected Virasoro symmetry of 4D quantum gravity.

In chapter 9 we generalize the construction of the boundary stress tensor to all even-dimensional asymptotically flat spacetimes. Specifically, we consider the tree-level scattering of massless particles in $(d + 2)$ -dimensional asymptotically flat spacetimes. The \mathcal{S} -matrix elements are recast as correlation functions of local operators living on a spacelike cut \mathcal{M}_d of the null momentum cone. The Lorentz group $SO(d + 1, 1)$ is nonlinearly realized as the Euclidean conformal group on \mathcal{M}_d . Operators of nontrivial spin arise from massless particles transforming in nontrivial representations of the little group $SO(d)$, and distinguished operators arise from the soft-insertions of gauge bosons and gravitons. We demonstrate that the leading soft photon operator is the shadow transform of a conserved spin-one primary operator J_a , and that the subleading soft graviton operator is the shadow transform of a conserved spin-two symmetric traceless primary operator T_{ab} . The universal form of the soft-limits ensures that J_a and T_{ab} obey the Ward identities expected of a conserved current and energy-momentum tensor in a Euclidean CFT_d , respectively.

Finally, in chapter 10 we explore various area/entropy bounds at null infinity in light of the new developments regarding the BMS group. Our investigations lead to an interesting conjectural “second law of \mathcal{I}^+ ” relating area changes of cross-sectional cuts of \mathcal{I}^+ to the entanglement entropy across the cut.

2

Higher-Dimensional Supertranslations and Weinberg's Soft Graviton Theorem

2.1 Introduction

Asymptotic symmetry groups play a vital role in our modern understanding of general relativity. Although the concept originated in the early 1960's, it continues to influence much of the contemporary research on classical and quantum gravity. From holography and black hole thermodynamics to the infrared behavior of the gravitational \mathcal{S} -matrix, asymptotic symmetry groups have provided crucial insights into many of today's most exciting research topics. They will undoubtedly continue to play a critical role in further clarifying the nature of quantum gravity.

The asymptotic symmetry group of asymptotically flat spacetimes is particularly interesting from both theoretical and phenomenological points of view. Early research on this topic by Bondi,

Van der Berg, Metzner and Sachs [1–3] led to a surprising conclusion: the asymptotic symmetry group of four-dimensional asymptotically flat spacetimes is not the finite-dimensional Poincaré group, but an infinite-dimensional group now known as the Bondi-Metzner-Sachs (BMS) group. The BMS group contains the boosts, rotations and translations that comprise the isometry group of flat spacetime. However, it also includes an infinite-dimensional abelian subgroup, known as the supertranslation subgroup, whose existence seems to have troubled the early pioneers of the subject. Repeated attempts to eliminate these extra symmetries proved unsuccessful, and the BMS group ultimately gained acceptance as the physically correct asymptotic symmetry group for four-dimensional asymptotically flat spacetimes.

It was eventually recognized that the BMS supertranslations were related to the infrared behavior of the gravitational theory [27–29]. This relationship has recently been made precise. In [30] it was argued that a certain diagonal subgroup of the product of the past and future BMS groups is a symmetry of both classical and quantum gravitational scattering. In [31] it was further demonstrated that the Ward identity associated to this diagonal supertranslation invariance of the gravitational \mathcal{S} -matrix is equivalent to Weinberg’s soft graviton theorem [4]. Further investigations along these lines have established a robust and detailed correspondence between soft theorems for gauge theory/gravity scattering amplitudes and Ward identities for extended asymptotic symmetry groups [30–44]. Moreover, it has been shown [45] that the gravitational memory effect [46–48], which occurs in the deep infrared, provides direct and measurable consequences of BMS symmetry.

Although the asymptotic symmetry group of asymptotically flat spacetimes is well-studied in four dimensions, the higher-dimensional analog has received limited attention [49–56]. Interestingly, nearly all of these analyses concluded that supertranslations do not exist in higher dimensions. This result seems to be at odds with the soft theorem/asymptotic symmetry correspondence, given that Weinberg’s soft graviton theorem holds in any dimension. The resolution of this discrepancy is the focus of this chapter.

Briefly, the analyses that claim to eliminate supertranslations in higher dimensions do so by placing restrictive boundary conditions on the metric at null infinity. We demonstrate that by slightly

relaxing these boundary conditions to allow for zero-energy, “large diffeomorphism” contributions to the metric, one may recover the full BMS group, including supertranslations, in any even dimension.¹ We corroborate this with an investigation of Weinberg’s soft graviton theorem in arbitrary even-dimensional spacetimes. Manipulation of this universal relation allows us to prove a Ward identity relating \mathcal{S} -matrix elements and derive a corresponding conserved charge. This charge is rewritten, using the constraints and our boundary conditions, as a boundary integral involving the generalized Bondi mass aspect. A set of brackets is proposed for which this charge generates the supertranslations. Hence, our weakened boundary conditions allow a derivation of the Weinberg soft identities between \mathcal{S} -matrix elements, while the imposition of stronger boundary conditions misses this important feature of scattering. The argument can also be turned around, reverse-engineering Weinberg’s soft theorem into a supertranslation symmetry of gravitational scattering.

We defer the study of odd dimensions due to known difficulties in defining null infinity in odd-dimensional spacetimes [57].

The outline of this chapter is as follows. In section 2.2 we define and discuss asymptotically flat spacetimes in even dimensions. We establish our coordinate system and relevant boundary conditions, and then derive the corresponding asymptotic symmetry group. In section 2.3 we briefly discuss the semiclassical gravitational scattering problem as well as the known subtleties in connecting past and future null infinity. In section 2.4 we specialize to six dimensions for ease of presentation, deriving equations needed in the analysis of Weinberg’s soft theorem. In section 2.5 we derive a Ward identity from Weinberg’s soft theorem, and in section 2.6 we demonstrate its equivalence to the supertranslation Ward identity for soft gravitons and hard matter fields. We do *not* verify that the terms in the charge which are quadratic in the metric perturbations correctly generate supertranslations for hard gravitons. We expect this to be the case but an analysis of gauge fixing and Dirac brackets at subleading order would be required. Section 2.7 details the

¹In $d = 4$, the “boundary gravitons” produced by supertranslations and the radiative gravitons both appear at the same order in $\frac{1}{r}$. For higher d , the boundary gravitons appear at lower order in the $\frac{1}{r}$ expansion, so that boundary conditions constraining pure diffeomorphisms at lower order than the radiative modes will kill supertranslations. The situation here is like AdS_3 [25], where the leading allowed excitations are all pure diffeomorphisms.

generalization to arbitrary even-dimensional spacetime. Section 2.8 concludes with a series of open questions.

2.2 General Relativity in $d = 2m + 2$ Dimensions

In this section we study even-dimensional asymptotically flat solutions to Einstein's field equations without a cosmological constant. Working in Bondi gauge, we propose appropriate boundary conditions for asymptotically flat spacetimes and derive their asymptotic symmetry groups. Our definition of asymptotic flatness differs from that used in previous analyses, and we comment on the implications. Finally, we collect a series of useful equations in the linearized theory.

2.2.1 Bondi Gauge

Metric solutions to Einstein's equations in $d = 2m + 2$ dimensions satisfy

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}^M, \quad (2.2.1)$$

where $T_{\mu\nu}^M$ is the matter stress-energy tensor. We choose coordinates u, r, z^a ($a = 1, \dots, 2m$) that asymptote to the usual retarded coordinates on flat spacetime. At large- r and in terms of flat space Cartesian coordinates t, x^i we have

$$u = t - r, \quad r^2 = x^i x_i, \quad x^i = r \hat{x}^i(z^a), \quad (2.2.2)$$

where $\hat{x}^i(z^a)$ defines an embedding of S^{2m} in \mathbb{R}^{2m+1} . Future null infinity \mathcal{I}^+ is given by the null hypersurface ($r = \infty, u, z^a$), with future ($u = \infty$) and past ($u = -\infty$) boundaries denoted by \mathcal{I}_+^+ and \mathcal{I}_-^+ , respectively. In this coordinate system, Bondi gauge is defined by the $2m + 2$ gauge fixing conditions

$$g_{rr} = 0, \quad g_{ra} = 0, \quad \det g_{ab} = r^{4m} \det \gamma_{ab}, \quad (2.2.3)$$

where γ_{ab} is the round metric on the unit S^{2m} with covariant derivative D_a . Such a metric can always be put into the form

$$ds^2 = e^{2\beta} M du^2 - 2e^{2\beta} du dr + g_{ab}(dz^a - U^a du)(dz^b - U^b du) . \quad (2.2.4)$$

For the case of asymptotically flat spacetimes, we will assume β , M , U_a , and g_{ab} admit an expansion near \mathcal{I}^+ of the form:

$$\begin{aligned} \beta &= \sum_{n=2}^{\infty} \frac{\beta^{(n)}(u, z)}{r^n} , & M &= -1 + \sum_{n=1}^{\infty} \frac{M^{(n)}(u, z)}{r^n} , & U_a &= \sum_{n=0}^{\infty} \frac{U_a^{(n)}(u, z)}{r^n} , \\ g_{ab} &= r^2 \gamma_{ab} + \sum_{n=-1}^{\infty} \frac{C_{ab}^{(n)}(u, z)}{r^n} . \end{aligned} \quad (2.2.5)$$

In the vicinity of past null infinity, \mathcal{I}^- , we choose advanced coordinates v, r, z^a asymptotically related to the flat space Cartesian coordinates through the relations

$$v = t + r , \quad r^2 = x^i x_i , \quad x^i = -r \hat{x}^i(z^a) . \quad (2.2.6)$$

Here \hat{x}^i is the same embedding of the S^{2m} in \mathbb{R}^{2m+1} as in (2.2.2). Note in particular that the angular coordinate z^a on \mathcal{I}^- is antipodally related to the angular coordinate on \mathcal{I}^+ , so that null generators of \mathcal{I} passing through spatial infinity (i^0) are labeled by the same numerical value of z^a . \mathcal{I}^- is the $(r = \infty, v, z^a)$ null hypersurface, with future ($v = \infty$) and past ($v = -\infty$) boundaries denoted by \mathcal{I}^-_+ and \mathcal{I}^-_- , respectively. The metric in advanced Bondi gauge takes the form

$$ds^2 = e^{2\beta^-} M^- dv^2 + 2e^{2\beta^-} dv dr + g_{ab}^-(dz^a - W^a dv)(dz^b - W^b dv) , \quad (2.2.7)$$

where β^- , M^- , W_a , and g_{ab}^- admit the large- r expansions

$$\begin{aligned}\beta^- &= \sum_{n=2}^{\infty} \frac{\beta^{-(n)}(v, z)}{r^n}, & M^- &= -1 + \sum_{n=1}^{\infty} \frac{M^{-(n)}(v, z)}{r^n}, & W_a &= \sum_{n=0}^{\infty} \frac{W_a^{(n)}(v, z)}{r^n}, \\ g_{ab}^- &= r^2 \gamma_{ab} + \sum_{n=-1}^{\infty} \frac{D_{ab}^{(n)}(v, z)}{r^n}.\end{aligned}\tag{2.2.8}$$

2.2.2 Asymptotically Flat Spacetimes

Having fixed a coordinate system, we must now choose the boundary conditions that define asymptotic flatness at \mathcal{I}^+ in this gauge. Our conditions on the metric components are the same as those in four dimensions:

$$g_{uu} = -1 + \mathcal{O}(r^{-1}), \quad g_{ur} = -1 + \mathcal{O}(r^{-2}), \quad g_{ua} = \mathcal{O}(1), \quad g_{ab} = r^2 \gamma_{ab} + \mathcal{O}(r). \tag{2.2.9}$$

We also require

$$R_{uu} = \mathcal{O}(r^{-2m}), \quad R_{ur} = \mathcal{O}(r^{-2m-1}), \quad R_{ua} = \mathcal{O}(r^{-2m}), \tag{2.2.10}$$

$$R_{rr} = \mathcal{O}(r^{-2m-2}), \quad R_{ra} = \mathcal{O}(r^{-2m-1}), \quad R_{ab} = \mathcal{O}(r^{-2m}). \tag{2.2.11}$$

The analogous conditions define asymptotic flatness at \mathcal{I}^- . When the theory is coupled to matter sources, we impose the same falloff conditions on the components of $T_{\mu\nu}^M$ as on $R_{\mu\nu}$. It is important to note that the boundary conditions (2.2.9)-(2.2.11) are less restrictive than those typically considered in the literature [49–54]. In particular, the falloff condition on g_{ab} in equation (2.2.9) does not reflect the large- r behavior of generic gravitational radiation in $d = 2m + 2$ dimensions, for which $g_{ab} = r^2 \gamma_{ab} + \mathcal{O}(r^{2-m})$. As we will see, the choice of this boundary condition essentially determines whether or not the corresponding asymptotic symmetry group contains supertranslations. Naively, our less restrictive falloff conditions on the metric components could lead to bad behavior at infinity and divergences in physical quantities. However, the metric itself is not physically observable and the boundary conditions on the Ricci tensor ensure finiteness of energy flux and other gravitational

observables. As we will see, the potentially dangerous pieces of the metric turn out to be pure “large diffeomorphism” for the spacetimes we consider. In the next section we demonstrate that allowing these leading pieces of the metric to be pure diffeomorphism, rather than setting them equal to zero, enlarges the asymptotic symmetry group from $ISO(2m+1, 1)$ to BMS_{2m+2} .

2.2.3 Asymptotic Symmetries

We are now in a position to discuss the asymptotic symmetry group of asymptotically flat $2m+2$ -dimensional spacetimes. We define the asymptotic symmetry group to be the group of all diffeomorphisms preserving Bondi gauge (2.2.3) and the boundary conditions (2.2.9)-(2.2.11), modulo the subgroup of trivial diffeomorphisms.² All such diffeomorphisms are generated by vector fields ξ satisfying the following set of differential equations:

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{ra} = 0, \quad g^{ab} \mathcal{L}_\xi g_{ab} = 0, \quad (2.2.12)$$

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_\xi g_{ua} = \mathcal{O}(1), \quad \mathcal{L}_\xi g_{ab} = \mathcal{O}(r). \quad (2.2.13)$$

The most general vector field satisfying (2.2.12)-(2.2.13) takes the form

$$\xi^u = f(z) + \frac{u}{2m} D_a Y^a(z), \quad (2.2.14)$$

$$\xi^a = Y^a(z) - D_b \xi^u \int_r^\infty e^{-2\beta} g^{ab} dr', \quad (2.2.15)$$

$$\xi^r = -\frac{r}{2m} [D_a \xi^a - U^a D_a \xi^u]. \quad (2.2.16)$$

The $Y^a(z)$ are conformal Killing vectors on the S^{2m} , and generate the $SO(2m+1, 1)$ transformations of the Poincaré group. Transformations with $Y^a = 0$ and an arbitrary function $f(z)$ on the sphere are known as supertranslations. Near \mathcal{I}^+ , they are generated by the vector field

$$\xi = f \partial_u - \frac{1}{r} \gamma^{ab} D_a f \partial_b + \frac{1}{2m} D^2 f \partial_r + \dots \quad (2.2.17)$$

² See [55] for a related derivation of the BMS algebra in higher dimensions.

In the linearized theory, where we ignore transformations homogenous in metric perturbations (such transformations require quadratic terms in the associated charges), the effect of a supertranslation is to shift $C_{ab}^{(-1)}$ according to

$$\delta C_{ab}^{(-1)} = \frac{1}{m} D^2 f \gamma_{ab} - (D_a D_b + D_b D_a) f, \quad (2.2.18)$$

leaving all other $C_{ab}^{(n \geq 0)}$ fixed. Note that $\delta C_{ab}^{(-1)} = 0$ for the $2m + 2$ global translations given by $f(z) \propto 1, \hat{x}^i(z)$. A similar analysis holds for the asymptotic symmetry group at \mathcal{I}^- . There, supertranslation generators take the asymptotic form

$$\xi^- = f^- \partial_v + \frac{1}{r} \gamma^{ab} D_a f^- \partial_b - \frac{1}{2m} D^2 f^- \partial_r + \dots \quad (2.2.19)$$

and generate the transformation

$$\delta D_{ab}^{(-1)} = - \left[\frac{1}{m} D^2 f^- \gamma_{ab} - (D_a D_b + D_b D_a) f^- \right]. \quad (2.2.20)$$

2.2.4 Discussion of the BMS Group in Higher Dimensions

Previous analyses of higher-dimensional asymptotically flat spacetimes have concluded that the BMS group does not exist in higher dimensions and that the appropriate asymptotic symmetry group is the finite-dimensional Poincaré group [49–54]. Notable exceptions include [55, 56], where it was argued that supertranslations do exist in higher dimensions. The source of the apparent discrepancy can be found in the choice of boundary conditions. In $2m + 2$ spacetime dimensions, the radiative degrees of freedom of the gravitational field enter the metric on the sphere at order $\mathcal{O}(r^{2-m})$. As we have seen, supertranslations affect an $\mathcal{O}(r)$ change in the metric on the sphere. For $d = 4$, these two orders agree, and it is impossible to eliminate the supertranslations without simultaneously eliminating radiative solutions. In higher dimensions, one can consistently set $C_{ab}^{(-1)} = 0$ while still allowing for radiative solutions, which have nonzero $C_{ab}^{(m-2)}$. Since the boundary condition $C_{ab}^{(-1)} = 0$ is not supertranslation invariant, it effectively reduces the infinite-dimensional BMS

group to the finite-dimensional Poincaré group.

The definition of an asymptotic symmetry group depends on the boundary conditions of the theory, and the appropriate boundary conditions are often determined by the phenomena under consideration. Therefore, in higher dimensions it is meaningless to discuss the “correct” asymptotic symmetry group, and one should simply choose the group best adapted to the problem at hand. In four dimensions, the extra supertranslation symmetries were intimately related to the infrared behavior of gravitational scattering amplitudes. Therefore it seems reasonable that if one wishes to study the infrared dynamics of higher-dimensional gravity, one should choose the relaxed boundary conditions (2.2.9) which allow for supertranslations. As we will see in sections 2.5 and 2.6, this is indeed the case.

2.2.5 Boundary Conditions and Constraints

In this section we collect a few select formulas which will prove useful for the analysis of Weinberg's soft theorem. The analysis up to this point is completely general and holds in the non-linear theory. In what follows we will focus on the linearized theory for ease of presentation. First, note that linearization effectively eliminates the function β in the metric. In the nonlinear theory,

$$R_{rr} = \frac{4m}{r} \partial_r \beta + \frac{2m}{r^2} - \frac{1}{4} g^{ac} g^{bd} \partial_r g_{ab} \partial_r g_{cd} . \quad (2.2.21)$$

The boundary conditions $R_{rr} = \mathcal{O}(r^{-2m-2})$ and $g_{ur} = \mathcal{O}(r^{-2})$ then imply that β is quadratic in metric perturbations up to order $\mathcal{O}(r^{-2m+1})$. Since these are the only orders of β that could appear in the equations (2.2.28)-(2.2.32) needed for our analysis, β may be consistently set to zero along with all other terms quadratic in metric perturbations. Also note that in the linearized theory, the Bondi gauge determinant condition requires that all $C_{ab}^{(n)}$ be traceless. We then have

$$R_{uu} = -\frac{1}{r^2} \partial_u D^a U_a - \frac{1}{2} r^{-2m} \partial_r (r^{2m} \partial_r M) - \frac{m}{r} \partial_u M - \frac{1}{2r^2} D^2 M , \quad (2.2.22)$$

$$R_{ra} = \frac{1}{2}r^{-2m}\partial_r(r^{2m+2}\partial_r(r^{-2}U_a)) + \frac{1}{2r^2}\partial_r D^b g_{ba} + r\gamma_{ca}D_b g^{bc}, \quad (2.2.23)$$

$$R_{ur} = -\frac{1}{2}r^{-2m}\partial_r(r^{2m}\partial_r M) - \frac{1}{2r^2}\partial_r D^a U_a. \quad (2.2.24)$$

The corresponding equations in advanced Bondi gauge are:

$$R_{vv} = -\frac{1}{r^2}\partial_v D^a W_a - \frac{1}{2}r^{-2m}\partial_r(r^{2m}\partial_r M^-) + \frac{m}{r}\partial_v M^- - \frac{1}{2r^2}D^2 M^-, \quad (2.2.25)$$

$$R_{ra} = -\frac{1}{2}r^{-2m}\partial_r(r^{2m+2}\partial_r(r^{-2}W_a)) + \frac{1}{2r^2}\partial_r D^b g_{ba}^- + r\gamma_{ca}D_b g^{-bc}, \quad (2.2.26)$$

$$R_{vr} = \frac{1}{2}r^{-2m}\partial_r(r^{2m}\partial_r M^-) - \frac{1}{2r^2}\partial_r D^a W_a. \quad (2.2.27)$$

The boundary condition on R_{uu} reads

$$\frac{1}{2}[D^2 + n(n+1-2m)]M^{(n)} + \partial_u D^a U_a^{(n)} + m\partial_u M^{(n+1)} = 0, \quad 0 \leq n \leq 2m-3. \quad (2.2.28)$$

The boundary condition on R_{ur} reads

$$-\frac{n(n+1-2m)}{2}M^{(n)} + \frac{(n-1)}{2}D^a U_a^{(n-1)} = 0, \quad 0 \leq n \leq 2m-2. \quad (2.2.29)$$

The boundary condition on R_{ra} reads

$$\frac{(n+2)(n+1-2m)}{2}U_a^{(n)} - \frac{(n+1)}{2}D^b C_{ba}^{(n-1)} = 0, \quad 0 \leq n \leq 2m-2. \quad (2.2.30)$$

The null normal to \mathcal{I}^+ is given by $n = \partial_u - \frac{1}{2}\partial_r$. The constraint equations take the form

$$n^\mu(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 8\pi G n^\mu T_{\mu\nu}^M. \quad (2.2.31)$$

The boundary conditions (2.2.10)-(2.2.11) ensure that $R = \mathcal{O}(r^{-2m-1})$. In what follows we never need terms of this order, so we drop the trace term from the constraint equation. The first nontrivial order of the u -constraint equation then reads

$$\frac{1}{2}[D^2 - 2(m-1)]M^{(2m-2)} + \partial_u D^a U_a^{(2m-2)} + m\partial_u M^{(2m-1)} + 8\pi G T_{uu}^{M(2m)} = 0. \quad (2.2.32)$$

2.3 The Semi-Classical Scattering Problem

So far, our analysis has treated \mathcal{I}^+ and \mathcal{I}^- separately. In order to discuss the symmetries of the gravitational \mathcal{S} -matrix, we need to define the semiclassical scattering problem in general relativity and determine how to relate symmetry transformations at \mathcal{I}^+ and \mathcal{I}^- . In essence, given initial data for the characteristic Cauchy problem at \mathcal{I}^- , we must determine the corresponding outgoing data on \mathcal{I}^+ . One of the most attractive features of the asymptotic analysis based at \mathcal{I} is the ability to solve this problem without making reference to the interior of the spacetime. However, in order to do so, we need to be able to relate data and symmetry transformations on \mathcal{I}^- to the corresponding data and transformations on \mathcal{I}^+ . Doing so requires us to impose certain regularity conditions at spatial infinity.

2.3.1 CK Constraint in Higher Dimensions

In four dimensions, arbitrary asymptotically flat initial data does not lead to a well-defined scattering problem. In fact, i^0 is generically a singular point of the conformal compactification of asymptotically flat spacetimes. This naively precludes the identification of BMS transformations at \mathcal{I}^+ and \mathcal{I}^- . In four dimensions, the work of Christodoulou and Klainerman (CK) [58] established necessary bounds on initial data in order to allow for smooth identifications at i^0 . To our knowledge, no such analysis has been performed in higher dimensions. In four dimensions, the CK conditions played an essential role in connecting \mathcal{I}^+ to \mathcal{I}^- and matching gravitational data at i^0 . We are thus led to impose a “generalized CK constraint.” Specifically, we require that the

higher-dimensional analog of the magnetic component of the Weyl tensor $C_{\mu\nu\rho\sigma}$ vanishes near the boundaries of \mathcal{I}^+ :

$$C_{urab}|_{\mathcal{I}^+_{\pm}} = \mathcal{O}(r^{-2}) . \quad (2.3.1)$$

The $\mathcal{O}(r^{-1})$ term in this constraint requires that

$$D_a U_b^{(0)} - D_b U_a^{(0)} = 0 \quad (2.3.2)$$

at \mathcal{I}^+_{\pm} . The $\mathcal{O}(1)$ R_{ab} boundary condition implies $\partial_u C_{ab}^{(-1)} = 0$. Combining these two conditions we see that $D^b C_{ba}^{(-1)} = D_a g(z)$ for some function $g(z)$ on the sphere. The most general solution consistent with Bondi gauge is

$$C_{ab}^{(-1)} = \frac{1}{m} \gamma_{ab} D^2 \psi(z) - 2 D_a D_b \psi(z) , \quad (2.3.3)$$

with $\psi(z)$ unconstrained. Note that this is simply the requirement that $C_{ab}^{(-1)}$ be pure supertranslation. Under the action of a supertranslation with parameter $f(z)$, $\psi(z)$ transforms according to $\psi(z) \rightarrow \psi(z) + f(z)$. The function $\psi(z)$ will later be identified as the Goldstone mode for spontaneously broken supertranslation symmetry. The analogous condition at \mathcal{I}^- yields

$$D_{ab}^{(-1)} = -\frac{1}{m} \gamma_{ab} D^2 \psi^-(z) + 2 D_a D_b \psi^-(z) . \quad (2.3.4)$$

In what follows, we are primarily interested in vacuum to vacuum geometries, and we impose the “scattering constraints”

$$M^{(2m-1)}|_{\mathcal{I}^+_{\pm}} = M^{-(2m-1)}|_{\mathcal{I}^-_{\pm}} = 0 , \quad C_{ab}^{(2m-3)}|_{\mathcal{I}^+_{\pm}} = D_{ab}^{(2m-3)}|_{\mathcal{I}^-_{\pm}} = 0 . \quad (2.3.5)$$

2.3.2 Scattering and Matching

In order to connect \mathcal{I}^- to \mathcal{I}^+ we must match data at i^0 . Following the analysis in [30], all fields and functions are taken to be continuous along the null generators of \mathcal{I} passing through i^0 . Due

to the antipodal identification of the angular coordinates on \mathcal{I}^+ and \mathcal{I}^- , the zero-modes $C_{ab}^{(-1)}$ and $D_{ab}^{(-1)}$ are matched according to

$$\psi(z) = \psi^-(z) . \quad (2.3.6)$$

This identification also allows for a canonical identification of BMS^+ and BMS^- transformations according to the rule

$$f(z) = f^-(z) , \quad (2.3.7)$$

yielding a diagonal BMS subgroup that may be identified as a symmetry of the gravitational \mathcal{S} -matrix.

2.4 Six-Dimensional Gravity

In this section we focus on six-dimensional asymptotically flat spacetimes. We identify the free radiative data, and collect a number of equations needed for the analysis of Weinberg's soft theorem.

2.4.1 Boundary Conditions and Constraints

In six dimensions, the boundary conditions for asymptotically flat spacetimes satisfying the scattering constraints and the generalized CK constraint take the form

$$\begin{aligned} g_{uu} &= -1 + \mathcal{O}(r^{-1}) , & g_{ur} &= -1 + \mathcal{O}(r^{-2}) , & g_{ua} &= \mathcal{O}(1) , & g_{ab} &= r^2 \gamma_{ab} + \mathcal{O}(r) , \\ R_{uu} &= \mathcal{O}(r^{-4}) , & R_{ur} &= \mathcal{O}(r^{-5}) , & R_{ua} &= \mathcal{O}(r^{-4}) , & R_{rr} &= \mathcal{O}(r^{-6}) , & R_{ra} &= \mathcal{O}(r^{-5}) , \\ R_{ab} &= \mathcal{O}(r^{-4}) , & C_{ab}^{(1)}|_{\mathcal{I}_{\pm}^+} &= 0 , & C_{urab}|_{\mathcal{I}_{\pm}^+} &= \mathcal{O}(r^{-2}) , & M^{(3)}|_{\mathcal{I}_{\pm}^+} &= 0 . \end{aligned}$$

The R_{uu} boundary conditions take the form

$$\partial_u \left[D^a U_a^{(0)} + 2M^{(1)} \right] = 0 , \quad \frac{1}{2} [D^2 - 2] M^{(1)} + \partial_u \left[D^a U_a^{(1)} + 2M^{(2)} \right] = 0 , \quad (2.4.1)$$

while the R_{ur} boundary conditions reduce to

$$M^{(1)} = 0, \quad M^{(2)} = -\frac{1}{2}D^a U_a^{(1)}. \quad (2.4.2)$$

The R_{ra} boundary conditions read

$$U_a^{(0)} = -\frac{1}{6}D^b C_{ba}^{(-1)}, \quad U_a^{(1)} = -\frac{1}{3}D^b C_{ba}^{(0)}, \quad U_a^{(2)} = -\frac{3}{4}D^b C_{ba}^{(1)}. \quad (2.4.3)$$

The $\mathcal{O}(r^{-4})$ u -constraint equation reads

$$\frac{1}{2} [D^2 - 2] M^{(2)} + \partial_u D^a U_a^{(2)} + 2\partial_u M^{(3)} = -8\pi G T_{uu}^{M(4)}. \quad (2.4.4)$$

$C_{ab}^{(0)}$ is free, unconstrained data. A complete solution of course requires integration of the constraints and equations of motion to all orders.

2.4.2 Mode Expansions

The fluctuations of the gravitational field in an asymptotically flat spacetime are determined by the relation $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa^2 = 32\pi G$ and $\eta_{\mu\nu}$ is the flat metric. We represent the radiative degrees of freedom of the gravitational field by the mode expansion

$$h_{\mu\nu}(x) = \sum_{\alpha} \int \frac{d^5 q}{(2\pi)^5} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{*\alpha} a_{\alpha}(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^{\alpha} a_{\alpha}(\vec{q})^{\dagger} e^{-iq \cdot x} \right]. \quad (2.4.5)$$

Here, $\omega_q = |\vec{q}|$ and $\varepsilon_{\mu\nu}^{\alpha}$ are the polarization tensors of the graviton in six dimensions. The commutation relations are given by

$$[a_{\alpha}(\vec{p}), a_{\beta}(\vec{q})^{\dagger}] = 2\omega_q \delta_{\alpha\beta} (2\pi)^5 \delta^{(5)}(\vec{p} - \vec{q}). \quad (2.4.6)$$

The free radiative data at \mathcal{I}^+ then takes the form

$$C_{ab}^{(0)}(u, z) \equiv \kappa \lim_{r \rightarrow \infty} \partial_a x^\mu \partial_b x^\nu h_{\mu\nu}(u + r, r\hat{x}(z)) . \quad (2.4.7)$$

We evaluate this limit using the large- r saddle point approximation to obtain the expression

$$C_{ab}^{(0)}(u, z^a) = -\frac{2\pi^2\kappa}{(2\pi)^5} \partial_a \hat{x}^i \partial_b \hat{x}^j \sum_\alpha \int \omega_q d\omega_q \left[\varepsilon_{ij}^{*\alpha} a_\alpha(\omega_q \hat{x}) e^{-i\omega_q u} + \varepsilon_{ij}^\alpha a_\alpha(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right] . \quad (2.4.8)$$

The positive and negative frequency modes are then given by

$$\begin{aligned} C_{ab}^{\omega(0)}(z) &= -\frac{\kappa\omega}{8\pi^2} \partial_a \hat{x}^i(z) \partial_b \hat{x}^j(z) \sum_\alpha \varepsilon_{ij}^{*\alpha} a_\alpha(\omega \hat{x}(z)) , \\ C_{ab}^{-\omega(0)}(z) &= -\frac{\kappa\omega}{8\pi^2} \partial_a \hat{x}^i(z) \partial_b \hat{x}^j(z) \sum_\alpha \varepsilon_{ij}^\alpha a_\alpha(\omega \hat{x}(z))^\dagger , \end{aligned} \quad (2.4.9)$$

where $\omega > 0$ in both formulas. The $\omega \rightarrow 0$ limit of these expressions defines a zero-mode operator

$$C_{ab}^{0(0)} = \frac{1}{2} \lim_{\omega \rightarrow 0} \left(C_{ab}^{\omega(0)} + C_{ab}^{-\omega(0)} \right) . \quad (2.4.10)$$

The asymptotic data at \mathcal{I}^- is given by

$$D_{ab}^{(0)}(v, z) = \kappa \lim_{r \rightarrow \infty} \partial_a x^\mu \partial_b x^\nu h_{\mu\nu}(v - r, r\hat{x}^i(z)) , \quad (2.4.11)$$

which may be decomposed into the positive and negative frequency modes

$$\begin{aligned} D_{ab}^{\omega(0)}(z) &= -\frac{\kappa\omega}{8\pi^2} \partial_a \hat{x}^i(z) \partial_b \hat{x}^j(z) \sum_\alpha \varepsilon_{ij}^{*\alpha} a_\alpha(-\omega \hat{x}(z)) , \\ D_{ab}^{-\omega(0)}(z) &= -\frac{\kappa\omega}{8\pi^2} \partial_a \hat{x}^i(z) \partial_b \hat{x}^j(z) \sum_\alpha \varepsilon_{ij}^\alpha a_\alpha(-\omega \hat{x}(z))^\dagger . \end{aligned} \quad (2.4.12)$$

The associated zero-mode creation operator is given by

$$D_{ab}^{0(0)} = \frac{1}{2} \lim_{\omega \rightarrow 0} \left(D_{ab}^{\omega(0)} + D_{ab}^{-\omega(0)} \right) . \quad (2.4.13)$$

2.5 Ward Identity from Weinberg's Soft Theorem

In this section we use Weinberg's six-dimensional soft graviton theorem to derive a Ward identity for a charge operator constructed from the gravitational field. In the following section, we demonstrate the relationship between this charge operator and the supertranslations described in section 2.2.

In six dimensions, Weinberg's soft graviton theorem takes the form

$$\lim_{\omega \rightarrow 0} \omega \langle z_{n+1}, \dots | a_\alpha(q) \mathcal{S} | z_1, \dots \rangle = \frac{\omega \kappa}{2} \left[\sum_{k=n+1}^{n+n'} \frac{\varepsilon_{\mu\nu}^\alpha p_k^\mu p_k^\nu}{p_k \cdot q} - \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^\alpha p_k^\mu p_k^\nu}{p_k \cdot q} \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle . \quad (2.5.1)$$

Here, $a_\alpha(q)$ is a creation operator for an outgoing on-shell graviton with energy ω , polarization $\varepsilon_{\mu\nu}^\alpha$ and momentum q^μ . A null momentum vector in six dimensions is determined by an energy ω and a point z^a on the S^4 , so we parametrize the soft graviton's momentum as

$$q^\mu = \omega [1, \hat{x}^i(z)] . \quad (2.5.2)$$

Here, $\hat{x}^i(z)$ is the embedding of S^4 into \mathbb{R}^5 defined previously. The momenta of the massless external particles are similarly given by

$$p_k^\mu = E_k [1, \hat{x}^i(z_k)] . \quad (2.5.3)$$

Thus the in- and out-states are determined by the energy E_k and \mathcal{I}^+ crossing point z_k for each external particle. For simplicity we suppress internal quantum numbers which are totally decoupled from the analysis. We denote the in- and out-states by

$$|z_1, \dots, z_n\rangle , \quad \langle z_{n+1}, \dots, z_{n+n'}| , \quad (2.5.4)$$

respectively. Motivated by the form of the radiative modes (2.4.9), we define a function

$$\begin{aligned}
 F_{ab}^{out}(z, z_1, \dots, z_{n+n'}) &\equiv \omega \partial_a \hat{x}^i(z) \partial_b \hat{x}^j(z) \sum_{\alpha} \varepsilon_{ij}^{*\alpha} \left[\sum_{k=n+1}^{n+n'} \frac{\varepsilon_{\mu\nu}^{\alpha} p_k^{\mu} p_k^{\nu}}{p_k \cdot q} - \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^{\alpha} p_k^{\mu} p_k^{\nu}}{p_k \cdot q} \right] \\
 &= \sum_{k=n+1}^{n+n'} E_k \partial_b P(z, z_k) \partial_a \log(1 - P(z, z_k)) - \sum_{k=1}^n E_k \partial_b P(z, z_k) \partial_a \log(1 - P(z, z_k)) .
 \end{aligned} \tag{2.5.5}$$

Here, we have used the completeness relation for polarization tensors

$$2 \sum_{\alpha} \varepsilon_{\alpha}^{*ij}(\vec{q}) \varepsilon_{\alpha}^{kl}(\vec{q}) = \pi^{ik} \pi^{jl} + \pi^{il} \pi^{jk} - \frac{1}{2} \pi^{ij} \pi^{kl} , \quad \pi^{ij} = \delta^{ij} - \frac{q^i q^j}{\vec{q}^2} , \tag{2.5.6}$$

energy and momentum conservation

$$\sum_{k=n+1}^{n+n'} E_k - \sum_{k=1}^n E_k = 0 , \quad \sum_{k=n+1}^{n+n'} E_k \hat{x}^i(z_k) - \sum_{k=1}^n E_k \hat{x}^i(z_k) = 0 , \tag{2.5.7}$$

and defined a function³

$$P(z, z_k) \equiv \hat{x}_i(z) \hat{x}^i(z_k) . \tag{2.5.8}$$

We then use $F_{ab}^{out}(z, z_1, \dots, z_{n+n'})$ (abbreviated $F_{ab}^{out}(z; z_k)$) to relate Weinberg's soft theorem to the zero-mode insertion:

$$\langle z_{n+1}, \dots | C_{ab}^{0(0)}(z) \mathcal{S} | z_1, \dots \rangle = -\frac{\kappa^2}{2(4\pi)^2} F_{ab}^{out}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle . \tag{2.5.9}$$

Note that $F_{ab}^{out}(z; z_k)$ obeys the differential equation

$$\sqrt{\gamma} [D^2 - 2] D^a D^b F_{ab}^{out} = 3(4\pi)^2 \left[\sum_{k=n+1}^{n+n'} E_k \delta^{(4)}(z - z_k) - \sum_{k=1}^n E_k \delta^{(4)}(z - z_k) \right] . \tag{2.5.10}$$

³ P is known as the invariant distance on the S^4 , and is related to the cosine of the geodesic distance.

We can similarly consider Weinberg's soft theorem for an incoming soft graviton

$$\lim_{\omega \rightarrow 0} \omega \langle z_{n+1}, \dots | \mathcal{S} a_\alpha(q)^\dagger | z_1, \dots \rangle = -\frac{\omega \kappa}{2} \left[\sum_{k=n+1}^{n+n'} \frac{\varepsilon_{\mu\nu}^{*\alpha} p_k^\mu p_k^\nu}{p_k \cdot q} - \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^{*\alpha} p_k^\mu p_k^\nu}{p_k \cdot q} \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle .$$

This can similarly be rewritten

$$\langle z_{n+1}, \dots | \mathcal{S} D_{ab}^{0(0)}(z) | z_1, \dots \rangle = \frac{\kappa^2}{2(4\pi)^2} F_{ab}^{in}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle , \quad (2.5.11)$$

where

$$\begin{aligned} F_{ab}^{in}(z, z_1, \dots, z_{n+n'}) = \\ - \sum_{k=n+1}^{n+n'} E_k \partial_b P(z, z_k) \partial_a \log(1 + P(z, z_k)) + \sum_{k=1}^n E_k \partial_b P(z, z_k) \partial_a \log(1 + P(z, z_k)) . \end{aligned} \quad (2.5.12)$$

Combining equations (2.5.9), (2.5.10) and (2.5.11), we obtain the identity

$$\begin{aligned} & -\frac{1}{3\kappa^2} \int d^4 z \sqrt{\gamma} f(z) (D^2 - 2) D^a D^b \langle z_{n+1}, \dots | C_{ab}^{0(0)}(z) \mathcal{S} | z_1, \dots \rangle \\ & + \frac{1}{3\kappa^2} \int d^4 z \sqrt{\gamma} f^-(z) (D^2 - 2) D^a D^b \langle z_{n+1}, \dots | \mathcal{S} D_{ab}^{0(0)}(z) | z_1, \dots \rangle \\ & = \left[\sum_{k=n+1}^{n+n'} E_k f(z_k) - \sum_{k=1}^n E_k f(z_k) \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle \end{aligned} \quad (2.5.13)$$

for an arbitrary function $f(z)$ on the sphere. This relation can be rewritten as a Ward identity

$$\langle z_{n+1}, \dots | Q^+ \mathcal{S} - \mathcal{S} Q^- | z_1, \dots \rangle = 0 , \quad (2.5.14)$$

where Q^\pm have been decomposed into hard and soft parts

$$Q^\pm = Q_H^\pm + Q_S^\pm . \quad (2.5.15)$$

The action of the hard charges is defined so that

$$Q_H^- |z_1, \dots\rangle = \sum_{k=1}^n E_k f(z_k) |z_1, \dots\rangle, \quad \langle z_{n+1}, \dots | Q_H^+ = \langle z_{n+1}, \dots | \sum_{k=n+1}^{n+n'} E_k f(z_k), \quad (2.5.16)$$

while the soft charges take the form

$$\begin{aligned} Q_S^+ &= \frac{1}{3\kappa^2} \int d^4z \sqrt{\gamma} f(z) (D^2 - 2) D^a D^b C_{ab}^{0(0)}(z), \\ Q_S^- &= \frac{1}{3\kappa^2} \int d^4z \sqrt{\gamma} f^-(z) (D^2 - 2) D^a D^b D_{ab}^{0(0)}(z). \end{aligned} \quad (2.5.17)$$

2.6 From Ward Identity to BMS Supertranslations

The charges (2.5.15) commute with the \mathcal{S} -matrix and represent a symmetry of the theory. In this section we argue that, given the zero-mode bracket postulated below, the symmetry generated by these charges is none other than the supertranslation symmetry encountered in section 2.2.

2.6.1 Action of the Matter Charges

The form of the hard charges may be deduced from (2.5.16), yielding

$$Q_H^+ = \lim_{r \rightarrow \infty} r^4 \int_{\mathcal{I}^+} du d^4z \sqrt{\gamma} f(z) T_{uu}^M(u, r, z), \quad (2.6.1)$$

$$Q_H^- = \lim_{r \rightarrow \infty} r^4 \int_{\mathcal{I}^-} dv d^4z \sqrt{\gamma} f^-(z) T_{vv}^M(v, r, z). \quad (2.6.2)$$

These expressions can be rewritten in the form

$$Q_H^+ = \lim_{\Sigma \rightarrow \mathcal{I}^+} \int_{\Sigma} d\Sigma \xi^\mu n_\Sigma^\nu T_{\mu\nu}^M, \quad Q_H^- = \lim_{\Sigma \rightarrow \mathcal{I}^-} \int_{\Sigma} d\Sigma \xi^\mu n_\Sigma^\nu T_{\mu\nu}^M. \quad (2.6.3)$$

Here Σ is a spacelike Cauchy surface, n_Σ is a unit normal to Σ , and ξ is the BMS vector field (2.2.17). Written in this form, it is clear that the hard charges generate supertranslations on the asymptotic states. Standard commutation relations for the matter fields confirm that the quantities (2.6.3) generate the large diffeomorphisms for any matter field coupled to gravity.

Note that because we consider the linearized theory, the gravitational field does not appear in the hard charges. In the full nonlinear theory, when we allow for external graviton states with non-zero momentum, we expect new contributions to the hard charge quadratic in the gravitational field. These additional terms characterize the energy and momentum flux carried by gravitational radiation through null infinity, and serve to generate transformations of the metric that are proportional to metric perturbations (transformations which we neglect in the linearized theory).

2.6.2 Action of the Gravitational Charges

Supertranslations are by definition large diffeomorphisms, and general relativity is diffeomorphism invariant if and only if all fields transform under the diffeomorphisms. Therefore, it is intuitively clear that the charges (2.5.15) must generate supertranslations of the gravitational field. We can make this relationship precise by using the boundary conditions and constraints of section 2.4.1 to rewrite the charges as boundary integrals

$$\begin{aligned} Q^+ &= \frac{1}{4\pi G} \int_{\mathcal{I}^+} d^4z \sqrt{\gamma} f(z) M^{(3)}(z) , \\ Q^- &= \frac{1}{4\pi G} \int_{\mathcal{I}^-} d^4z \sqrt{\gamma} f^-(z) M^{-(3)}(z) . \end{aligned} \tag{2.6.4}$$

These expressions resemble the supertranslation generators encountered in the four-dimensional case [31]. In order to claim that they generate supertranslations of the gravitational field, we need to discuss the symplectic form for the gravitational free data.

2.6.3 Brackets for the Free Data

The commutator for the radiative degrees of freedom of the gravitational field is familiar from four dimensions and can be derived from the plane wave mode expansion. It is given by

$$\left[C_{ab}^{(0)}(u, z), \partial_{u'} C_{cd}^{(0)}(u', z') \right] = i \frac{\kappa^2}{4} \frac{\delta(u - u') \delta^{(4)}(z - z')}{\sqrt{\gamma}} \left[\gamma_{ac} \gamma_{bd} + \gamma_{ad} \gamma_{bc} - \frac{1}{2} \gamma_{ab} \gamma_{cd} \right] . \tag{2.6.5}$$

This expression does not determine the zero-mode brackets. We postulate the simple form

$$\left[M^{(3)}(z), \psi(z') \right] = 4\pi i G \frac{\delta^{(4)}(z - z')}{\sqrt{\gamma}} . \quad (2.6.6)$$

This reproduces (2.2.18) and defines a symplectic form on the extended gravitational phase space.

It closely resembles the analogous zero-mode bracket in QED [36].

2.7 Generalization to Arbitrary Even-Dimensional Space-time

The results of the preceding sections generalize to arbitrary even-dimensional asymptotically flat spacetimes. In this section, we outline the derivation of the supertranslation Ward identity for $d = 2m + 2$ - dimensional spacetime. The perturbations of the asymptotically flat gravitational field are defined by $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ as in six dimensions. The plane wave expansion takes the form

$$h_{\mu\nu}(x) = \sum_{\alpha} \int \frac{d^{2m+1}q}{(2\pi)^{2m+1}} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{*\alpha}(\vec{q}) a_{\alpha}(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^{\alpha}(\vec{q}) a_{\alpha}(\vec{q})^{\dagger} e^{-iq \cdot x} \right] . \quad (2.7.1)$$

Here, $\omega_q = |\vec{q}|$ and α labels the polarizations of the graviton. The operator $a_{\alpha}(\vec{q})^{\dagger}$ is a graviton creation operator satisfying the commutation relations

$$\left[a_{\alpha}(\vec{p}), a_{\beta}(\vec{q})^{\dagger} \right] = 2\omega_q \delta_{\alpha\beta} (2\pi)^{2m+1} \delta^{(2m+1)}(\vec{p} - \vec{q}) . \quad (2.7.2)$$

The leading term in the large- r expansion of (2.7.1) yields an expression for the radiative degrees of freedom of the gravitational field near \mathcal{I}^+ . The positive and negative frequency modes take the

form

$$C_{ab}^{\omega(m-2)}(z) = \frac{(-i)^m \omega^{m-1} \kappa}{2(2\pi)^m} \partial_a \hat{x}^j(z) \partial_b \hat{x}^k(z) \sum_{\alpha} \varepsilon_{jk}^{*\alpha} a_{\alpha}(\omega \hat{x}(z)) , \quad (2.7.3)$$

$$C_{ab}^{-\omega(m-2)}(z) = \frac{i^m \omega^{m-1} \kappa}{2(2\pi)^m} \partial_a \hat{x}^j(z) \partial_b \hat{x}^k(z) \sum_{\alpha} \varepsilon_{jk}^{\alpha} a_{\alpha}(\omega \hat{x}(z))^{\dagger} . \quad (2.7.4)$$

The corresponding zero-mode operator is defined to be

$$C_{ab}^{0(m-2)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (i\omega)^{2-m} \left[C_{ab}^{\omega(m-2)} + (-1)^m C_{ab}^{-\omega(m-2)} \right] . \quad (2.7.5)$$

In terms of this zero-mode operator, Weinberg's soft theorem (2.5.1) takes the form

$$\langle z_{n+1}, \dots | C_{ab}^{0(m-2)}(z) \mathcal{S} | z_1, \dots \rangle = -\frac{(-1)^m \kappa^2}{8(2\pi)^m} F_{ab}^{out}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle . \quad (2.7.6)$$

The soft factor

$$\begin{aligned} F_{ab}^{out}(z, z_1, \dots, z_{n+n'}) &\equiv \partial_a \hat{x}^i \partial_b \hat{x}^j \omega \sum_{\alpha} \varepsilon_{ij}^{*\alpha} \left[\sum_{k=n+1}^{n+n'} \frac{\varepsilon_{\mu\nu}^{\alpha} p_k^{\mu} p_k^{\nu}}{p_k \cdot q} - \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^{\alpha} p_k^{\mu} p_k^{\nu}}{p_k \cdot q} \right] \\ &= \sum_{k=n+1}^{n+n'} E_k \partial_b P(z, z_k) \partial_a \log(1 - P(z, z_k)) - \sum_{k=1}^n E_k \partial_b P(z, z_k) \partial_a \log(1 - P(z, z_k)) \end{aligned} \quad (2.7.7)$$

satisfies the dimension-dependent differential equation

$$\begin{aligned} &(-1)^m \sqrt{\gamma} \prod_{l=m+1}^{2m-1} [D^2 - (2m-l)(l-1)] D^a D^b F_{ab}^{out} \\ &= (2m-1) \Gamma(m) 2^m (2\pi)^m \left[\sum_{k=n+1}^{n+n'} E_k \delta^{(2m)}(z - z_k) - \sum_{k=1}^n E_k \delta^{(2m)}(z - z_k) \right] . \end{aligned} \quad (2.7.8)$$

Here, we have used the completeness relation for polarization tensors

$$2 \sum_{\alpha} \varepsilon_{\alpha}^{*ij}(\vec{q}) \varepsilon_{\alpha}^{kl}(\vec{q}) = \pi^{ik} \pi^{jl} + \pi^{il} \pi^{jk} - \frac{1}{m} \pi^{ij} \pi^{kl} , \quad \pi^{ij} = \delta^{ij} - \frac{q^i q^j}{\vec{q}^2} , \quad (2.7.9)$$

along with energy and momentum conservation. The in-modes take the form

$$D_{ab}^{\omega(m-2)}(z) = \frac{i^m \omega^{m-1} \kappa}{2(2\pi)^m} \partial_a \hat{x}^j(z) \partial_b \hat{x}^k(z) \sum_{\alpha} \varepsilon_{jk}^{*\alpha} a_{\alpha}(-\omega \hat{x}(z)) , \quad (2.7.10)$$

$$D_{ab}^{-\omega(m-2)}(z) = \frac{(-i)^m \omega^{m-1} \kappa}{2(2\pi)^m} \partial_a \hat{x}^j(z) \partial_b \hat{x}^k(z) \sum_{\alpha} \varepsilon_{jk}^{\alpha} a_{\alpha}(-\omega \hat{x}(z))^{\dagger} , \quad (2.7.11)$$

and the associated zero-mode operator is

$$D_{ab}^{0(m-2)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (i\omega)^{2-m} \left[D_{ab}^{\omega(m-2)} + (-1)^m D_{ab}^{-\omega(m-2)} \right] . \quad (2.7.12)$$

The soft theorem for an incoming soft graviton may be rewritten as

$$\langle z_{n+1}, \dots | \mathcal{S} D_{ab}^{0(m-2)}(z) | z_1, \dots \rangle = \frac{\kappa^2}{8(2\pi)^m} F_{ab}^{in}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle , \quad (2.7.13)$$

where

$$\begin{aligned} F_{ab}^{in}(z, z_1, \dots, z_{n+n'}) &= - \sum_{k=n+1}^{n+n'} E_k \partial_b P(z, z_k) \partial_a \log(1 + P(z, z_k)) \\ &\quad + \sum_{k=1}^n E_k \partial_b P(z, z_k) \partial_a \log(1 + P(z, z_k)) . \end{aligned} \quad (2.7.14)$$

After applying (2.7.8) to equations (2.7.6) and (2.7.13), we may integrate against an arbitrary function $f(z)$ on the sphere to obtain the Ward identity

$$\langle z_{n+1}, \dots | (Q^+ \mathcal{S} - \mathcal{S} Q^-) | z_1, \dots \rangle = 0 . \quad (2.7.15)$$

The charges $Q^{\pm} = Q_H^{\pm} + Q_S^{\pm}$ commute with the \mathcal{S} -matrix and induce infinitesimal symmetry transformations on \mathcal{I}^{\pm} states. Q_H^{\pm} is defined by its action on the asymptotic states:

$$Q_H^- | z_1, \dots \rangle = \sum_{k=1}^n E_k f(z_k) | z_1, \dots \rangle , \quad \langle z_{n+1}, \dots | Q_H^+ = \langle z_{n+1}, \dots | \sum_{k=n+1}^{n+n'} E_k f(z_k) . \quad (2.7.16)$$

The soft charges are given by

$$Q_S^+ = \frac{1}{(2m-1)\kappa^2} \frac{2^{2-m}}{\Gamma(m)} \int d^{2m}z \sqrt{\gamma} f(z) \prod_{l=m+1}^{2m-1} [D^2 - (2m-l)(l-1)] D^a D^b C_{ab}^{0(m-2)}, \quad (2.7.17)$$

$$Q_S^- = \frac{(-1)^m}{(2m-1)\kappa^2} \frac{2^{2-m}}{\Gamma(m)} \int d^{2m}z \sqrt{\gamma} f^-(z) \prod_{l=m+1}^{2m-1} [D^2 - (2m-l)(l-1)] D^a D^b D_{ab}^{0(m-2)}. \quad (2.7.18)$$

The hard charges Q_H^\pm can be written in terms of the matter stress-energy tensor

$$Q_H^+ = \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^+} du d^{2m}z \sqrt{\gamma} f(z) T_{uu}^M(u, r, z), \quad (2.7.19)$$

$$Q_H^- = \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^-} dv d^{2m}z \sqrt{\gamma} f^-(z) T_{vv}^M(v, r, z). \quad (2.7.20)$$

This operator induces an infinitesimal supertranslation with parameter $f(z)$ when acting on the matter fields. The boundary conditions and constraints of section 2.2.5, combined with the generalized CK constraint and the scattering constraints, allow us to write the total charge $Q^\pm = Q_H^\pm + Q_S^\pm$ as a boundary integral

$$Q^+ = \frac{4m}{\kappa^2} \lim_{r \rightarrow \infty} r^{2m-1} \int_{\mathcal{I}_-^+} d^{2m}z \sqrt{\gamma} f(z) M(z), \quad (2.7.21)$$

$$Q^- = \frac{4m}{\kappa^2} \lim_{r \rightarrow \infty} r^{2m-1} \int_{\mathcal{I}_+^-} d^{2m}z \sqrt{\gamma} f^-(z) M^-(z).$$

A straightforward generalization of the brackets of section 2.6.3 can then be used to demonstrate that (2.7.21) generates supertranslations on the matter and gravitational fields.

2.8 Conclusions and Open Questions

In this chapter we considered the asymptotic symmetry group of even-dimensional asymptotically flat spacetimes. Using less restrictive boundary conditions than those previously considered in the literature, we demonstrated that the BMS group naturally arises as the asymptotic symmetry group of asymptotically flat spacetimes in any even dimension. Motivated by the recently discovered

correspondence between soft theorems and asymptotic symmetry groups, we considered Weinberg's soft graviton theorem in even-dimensional spacetime and used it to derive a Ward identity for a set of symmetry transformations. Reasonable, physically motivated boundary conditions and a natural extension of the symplectic form allowed us to identify these symmetry transformations as supertranslations. This result further solidifies the general correspondence between soft theorems and asymptotic symmetry groups.

It would be worthwhile to tackle the hard metric fluctuations at quadratic order and thereby extend the analysis to the full nonlinear theory. It would also be interesting to consider the odd-dimensional case, where special properties of radiating solutions make the conformal methods usually employed in four dimensions essentially useless. Our analysis is carried out at tree-level, and while we expect that the leading soft factor is not renormalized (as in four dimensions), it would be useful to explicitly verify this. One could also extend the analysis to allow for massive external states.

3

Asymptotic Symmetries of Massless QED in Even Dimensions

3.1 Introduction

Recent work [30–35] has connected long understood soft theorems [4–8] for gauge theory and gravity scattering amplitudes to Ward identities for asymptotic symmetry groups of massless interacting theories coupled to gauge theory and gravity in four dimensions. While several of the soft theorems have been known and understood since the 1960’s, many of their associated asymptotic symmetry groups have only recently drawn attention. Conjectures stemming from the correspondence have led to the discovery and investigation of new soft theorems in four dimensions [15, 44, 59–65], many of which were subsequently identified with asymptotic symmetry groups [33, 35, 39]. The leading and subleading soft theorems have been investigated at loop-level [66–69], in the context of string

and (ambi)-twistor string theory [42, 43, 70–72], and have been shown to hold in higher dimensions [73–76]. Some of the asymptotic symmetry groups associated to these new soft theorems, such as the extended BMS group, were previously conjectured [10–13], while the nature of others [35] remains unknown.

Although much work has been done, many questions remain unanswered. The tree-level leading soft theorem is universal among all theories in arbitrary dimensions. Such striking universality can only be a reflection of an underlying symmetry. Motivated by the established correspondence between soft theorems and asymptotic symmetries in four dimensions and the existence of the soft theorems in higher dimensions, we are led to consider the leading soft theorem in massless QED in even-dimensional Minkowski spacetime. The odd-dimensional case is of course also of interest but has additional subtleties which require a separate investigation. We recast the soft theorem in the form of a Ward identity for a new group of asymptotic symmetries. The asymptotic symmetry group in $d = 2m + 2$ dimensions is the subgroup of the local $U(1)$ large gauge transformations with a gauge parameter given by an unconstrained function on the S^{2m} . Our result generalizes the analysis performed in the four-dimensional case [34], further strengthening the relationship between soft theorems and asymptotic symmetries in all even dimensions.

We work in the semiclassical limit and therefore prove the result only at tree-level. However, given that the leading QED soft factor is not renormalized in four dimensions, the result may be exact. Although massless QED is not renormalizable in dimensions greater than four, we are interested in infrared effects where two derivative theories minimally coupled to matter fields serve as good low energy effective theories. The infrared structure of gauge theories is simpler in $d > 4$ due to the absence of soft divergences, so we hope that studying the soft theorems and associated asymptotic symmetries in higher dimensions will help to clarify the fate of the new symmetries in the presence of loop corrections both in $d = 4$ and $d > 4$.

This chapter is organized as follows. In section 3.2 we review the structure of massless QED in $d = 2m + 2$ dimensions and establish our coordinates and conventions. In section 3.3 we restrict our attention to six dimensions for illustrative purposes. We determine appropriate boundary

conditions and introduce the gauge field mode expansions and the relevant soft photon operators. In section 3.4 we rewrite Weinberg's soft theorem as a Ward identity for local charge operators involving the matter current and the soft photon operators. In section 3.5 we demonstrate that these charges generate a new asymptotic symmetry group, which is a subgroup of the original $U(1)$ gauge group. In section 3.6 we discuss how to generalize our result to the arbitrary even-dimensional case. Section 3.7 concludes with a series of open questions.

3.2 Maxwell's Equations in Even-Dimensional Minkowski Spacetime

Abelian gauge theory in $d = 2m+2$ - dimensional flat Minkowski spacetime is governed by Maxwell's equations

$$\nabla^\mu F_{\mu\nu} = e^2 J_\nu^M, \quad (3.2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, J_ν^M is the matter current density, and e is the coupling constant of the theory. The equations (3.2.1) are invariant under local gauge transformations of the form

$$A_\mu \rightarrow A_\mu + \partial_\mu \varepsilon, \quad \Psi_Q \rightarrow e^{iQ\varepsilon} \Psi_Q, \quad (3.2.2)$$

where Ψ_Q is a matter field with electric charge Q . It is useful to introduce retarded coordinates (u, r, z^a) given by

$$x^0 = u + r, \quad x^i = r \hat{x}^i(z), \quad (3.2.3)$$

where u is retarded time and $\hat{x}^i(z)$ describes an embedding of the unit S^{2m} with coordinates $z^a, a = 1, \dots, 2m$ into \mathbb{R}^{2m+1} with coordinates $x^i, i = 1, \dots, 2m+1$. The flat Minkowski metric then takes the form

$$ds^2 = -(dx^0)^2 + (dx^i)^2 = -du^2 - 2dudr + r^2 \gamma_{ab} dz^a dz^b. \quad (3.2.4)$$

Here γ_{ab} is the metric on the unit radius S^{2m} with covariant derivative D_a . In the conformal compactification of Minkowski spacetime, we can identify future null infinity (\mathcal{I}^+) as the null surface ($r = \infty, u, z^a$). We also employ advanced coordinates

$$x^0 = v - r, \quad x^i = -r\hat{x}^i(z), \quad (3.2.5)$$

so that past null infinity (\mathcal{I}^-) is identified as the surface ($r = \infty, v, z^a$) and

$$ds^2 = -dv^2 + 2dvdr + r^2\gamma_{ab}dz^a dz^b. \quad (3.2.6)$$

The advanced S^{2m} coordinate z is antipodally related to the retarded S^{2m} coordinate z in such a way that null generators of \mathcal{I} passing through spatial infinity are labeled by the same value of z on \mathcal{I}^+ and \mathcal{I}^- . We denote the $u = \pm\infty$ boundaries of \mathcal{I}^+ as \mathcal{I}_\pm^+ , and the $v = \pm\infty$ boundaries of \mathcal{I}^- as \mathcal{I}_\pm^- . Maxwell's equations in retarded coordinates take the form

$$\begin{aligned} r^{-2m}\partial_r(r^{2m}F_{ru}) - \partial_u F_{ru} + r^{-2}D^a F_{au} &= e^2 J_u^M, \\ -r^{-2m}\partial_r(r^{2m}F_{ur}) + r^{-2}D^a F_{ar} &= e^2 J_r^M, \\ r^{-2m+2}\partial_r(r^{2m-2}(F_{ra} - F_{ua})) - \partial_u F_{ra} + r^{-2}D^b F_{ba} &= e^2 J_a^M. \end{aligned} \quad (3.2.7)$$

Similar expressions hold for the advanced coordinates. The constraint equation for the hypersurface at future null infinity is

$$n^\mu \nabla^\nu F_{\nu\mu} = \frac{1}{2}r^{-2m}\partial_r(r^{2m}F_{ru}) - \partial_u F_{ru} + r^{-2}D^a \left(F_{au} - \frac{1}{2}F_{ar} \right) = e^2 n^\mu J_\mu^M, \quad (3.2.8)$$

where the null normal vector is $n = \partial_u - \frac{1}{2}\partial_r$.

3.3 Six-Dimensional Maxwell Primer

In this section we consider six-dimensional abelian gauge theory at null infinity,¹ postponing the discussion of arbitrary even dimensions to section 3.6. We determine appropriate boundary conditions for the gauge fields, determine matching conditions to link \mathcal{I}^- quantities to \mathcal{I}^+ quantities, and isolate the gauge field zero-mode operators appearing in Weinberg's soft theorem.

3.3.1 Asymptotic Analysis at \mathcal{I}^+

We work in retarded radial gauge. The gauge fixing conditions are

$$A_r = 0, \quad A_u|_{\mathcal{I}^+} = 0. \quad (3.3.1)$$

This leaves unfixed a residual large gauge symmetry parameterized by an unconstrained function $\varepsilon(z)$ on the S^4 at \mathcal{I}^+ . Under such a large gauge transformation

$$\delta A_a(z) = \partial_a \varepsilon(z). \quad (3.3.2)$$

In order to analyze the field equations near \mathcal{I}^+ we assume an asymptotic expansion for the gauge field:

$$A_a = \sum_{n=0} \frac{A_a^{(n)}}{r^n}, \quad A_u = \sum_{n=1} \frac{A_u^{(n)}}{r^n}. \quad (3.3.3)$$

The $\mathcal{O}(r^{-2}, r^{-3}, r^{-4})$ orders of the constraint equation are (in the absence of matter currents)

$$\partial_u (A_u^{(1)} - D^a A_a^{(0)}) = 0, \quad (3.3.4)$$

$$(D^2 - 1)A_u^{(1)} + \partial_u (2A_u^{(2)} - D^a A_a^{(1)}) = 0, \quad (3.3.5)$$

$$-A_u^{(2)} - \partial_u F_{ru}^{(4)} + D^a (F_{au}^{(2)} - \frac{1}{2} F_{ar}^{(2)}) = 0. \quad (3.3.6)$$

¹Janis and Newman studied the null Cauchy problem for Maxwell's equations in four dimensions in [77] with similar conclusions.

In six dimensions, a plane wave has transverse field strength behaving as $F_{ab} \sim \frac{1}{r}$. Finiteness of the energy flux at each point on \mathcal{I}^+ and finiteness of the total energy evaluated on a spacelike Cauchy surface requires

$$\begin{aligned} F_{ab}^{(0)} &= \partial_a A_b^{(0)} - \partial_b A_a^{(0)} = 0 , \\ F_{ub}^{(0)} &= \partial_u A_b^{(0)} = 0 , \end{aligned} \tag{3.3.7}$$

which implies

$$A_a^{(0)} = \partial_a \phi(z) . \tag{3.3.8}$$

Here $\phi(z)$ is a free, unconstrained function on S^4 which will later be identified as the Goldstone mode of the spontaneously broken large gauge symmetry.

The subleading term $A_a^{(1)}(u, z)$ represents the free radiative data. Finiteness of the total radiated energy requires that at large values of $|u|$

$$A_a^{(1)}|_{\mathcal{I}^\pm_\pm} = 0 . \tag{3.3.9}$$

Finiteness of the Coulombic energy and integration of (3.3.4) then imply

$$A_u^{(1)} = 0 . \tag{3.3.10}$$

Demanding that the electric field fall off like $\frac{1}{r^4}$ near \mathcal{I}^+_+ together with (3.3.5) then imply

$$A_u^{(2)}|_{\mathcal{I}^+_+} = 0 , \tag{3.3.11}$$

and interior values of $A_u^{(2)}$ are determined by integrating (3.3.5). At the next order we must specify the boundary data for the electric field

$$F_{ru}^{(4)}|_{\mathcal{I}^+_+} = -3A_u^{(3)}|_{\mathcal{I}^+_+} \equiv E_r . \tag{3.3.12}$$

We are interested in scattering processes that revert to the vacuum at $u = \infty$, so we require

$F_{ru}^{(4)}|_{\mathcal{I}_+^+} = 0$. We additionally require $A_a^{(2)}|_{\mathcal{I}_\pm^\pm} = 0$. A full perturbative solution of course requires the equations of motion as well as the constraints.

3.3.2 Asymptotic Analysis at \mathcal{I}^-

Similar analysis can be applied to a Maxwell field B_μ in advanced coordinates near \mathcal{I}^- . We label the corresponding field strength tensor $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and denote $G_{rv}^{(4)}|_{\mathcal{I}_+^-} = E_r^-$. Advanced radial gauge

$$A_r = 0, \quad A_v|_{\mathcal{I}^-} = 0 \quad (3.3.13)$$

leaves unfixed a residual large gauge symmetry parameterized by an unconstrained function $\varepsilon^-(z)$. The various finiteness conditions applied in the previous section lead to a similar set of boundary conditions for B_μ . In particular, we have $B_a^{(0)}(z) = \partial_a \psi(z)$, with $\psi(z)$ an unconstrained function on S^4 .

3.3.3 Scattering

Given asymptotic data A_μ on \mathcal{I}^+ and B_μ on \mathcal{I}^- , we must specify a matching condition for the boundary values of the two gauge fields in order to properly define the scattering problem. In doing so, we also single out a diagonal subgroup of the large gauge transformations acting separately at \mathcal{I}^+ and \mathcal{I}^- , which can then be interpreted as a symmetry of the \mathcal{S} -matrix.

The boundary condition (3.3.9) provides a trivial matching condition for the radiative data. The only nontrivial components of the gauge field strength on the boundaries of \mathcal{I}^+ and \mathcal{I}^- are the quantities E_r and E_r^- . As in four dimensions [34], we impose the matching condition

$$E_r(z) = E_r^-(z) . \quad (3.3.14)$$

Here z labels a null generator, so that the coordinate argument of E_r^- is antipodally related to the

argument of E_r . The corresponding matching condition for the Goldstone modes is

$$\phi(z) = \psi(z) . \quad (3.3.15)$$

The diagonal subgroup of large gauge transformations acting at \mathcal{I}^+ and \mathcal{I}^- is therefore obtained by imposing

$$\varepsilon(z) = \varepsilon^-(z) . \quad (3.3.16)$$

3.3.4 Mode Expansions

The radiative modes of the gauge field in the plane wave basis take the form

$$A_\mu(x) = e \sum_\alpha \int \frac{d^5 q}{(2\pi)^5} \frac{1}{2\omega_q} \left[\varepsilon_\mu^{*\alpha}(\vec{q}) a_\alpha(\vec{q}) e^{iqx} + \varepsilon_\mu^\alpha(\vec{q}) a_\alpha(\vec{q})^\dagger e^{-iqx} \right] , \quad (3.3.17)$$

where $\omega_q = |\vec{q}|$, ε_μ^α are the four independent polarization vectors for the photon in six dimensions, and

$$\left[a_\alpha(\vec{p}), a_\beta(\vec{q})^\dagger \right] = 2\omega_q \delta_{\alpha\beta} (2\pi)^5 \delta^{(5)}(\vec{p} - \vec{q}) . \quad (3.3.18)$$

The free radiative data in this basis is of the form

$$A_a^{(1)}(u, z^a) = -\frac{2\pi^2 e}{(2\pi)^5} \partial_a \hat{x}^i \sum_\alpha \int \omega_q d\omega_q \left[\varepsilon_i^{*\alpha} a_\alpha(\omega_q \hat{x}) e^{-i\omega_q u} + \varepsilon_i^\alpha a_\alpha(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right] . \quad (3.3.19)$$

We can define a Fourier image for the radiative modes

$$\begin{aligned} A_a^{\omega(1)}(z) &= -\frac{e\omega}{8\pi^2} \partial_a \hat{x}^i(z) \sum_\alpha \varepsilon_i^{*\alpha} a_\alpha(\omega \hat{x}(z)) , \\ A_a^{-\omega(1)}(z) &= -\frac{e\omega}{8\pi^2} \partial_a \hat{x}^i(z) \sum_\alpha \varepsilon_i^\alpha a_\alpha(\omega \hat{x}(z))^\dagger , \end{aligned} \quad (3.3.20)$$

with $\omega > 0$ assumed for both expressions. We can define the corresponding zero-mode operator

$$A_a^{0(1)} \equiv \frac{1}{2} \lim_{\omega \rightarrow 0} \left(A_a^{\omega(1)} + A_a^{-\omega(1)} \right) . \quad (3.3.21)$$

In a similar way we can introduce the in-modes

$$B_a^{(1)}(v, z) \equiv \lim_{r \rightarrow \infty} r \partial_a x^i(r, z) A_i(v - r, r \hat{x}^i(z)) \quad (3.3.22)$$

so that

$$\begin{aligned} B_a^{\omega(1)}(z) &= -\frac{e\omega}{8\pi^2} \partial_a \hat{x}^i(z) \sum_{\alpha} \varepsilon_i^{*\alpha} a_{\alpha}(-\omega \hat{x}(z)) , \\ B_a^{-\omega(1)}(z) &= -\frac{e\omega}{8\pi^2} \partial_a \hat{x}^i(z) \sum_{\alpha} \varepsilon_i^{\alpha} a_{\alpha}(-\omega \hat{x}(z))^{\dagger} . \end{aligned} \quad (3.3.23)$$

The corresponding zero-mode operator is

$$B_a^{0(1)} \equiv \frac{1}{2} \lim_{\omega \rightarrow 0} \left(B_a^{\omega(1)} + B_a^{-\omega(1)} \right) . \quad (3.3.24)$$

3.4 Soft Theorem as a Ward Identity

In this section we recast Weinberg's soft theorem as a Ward identity for charges constructed out of the matter and gauge fields. In the following section, we demonstrate that this Ward identity is associated to a new group of asymptotic symmetries of massless QED.

3.4.1 Soft Theorem

Weinberg's soft theorem takes the same form in any dimension:

$$\lim_{\omega \rightarrow 0} \omega \langle z_{n+1}, \dots | a_{\alpha}(q) \mathcal{S} | z_1, \dots \rangle = e\omega \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} - \sum_{k=1}^n \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle . \quad (3.4.1)$$

Here, $a_{\alpha}(q)$ is a creation operator for an outgoing on-shell photon with polarization ε_{α} and momentum q . A null momentum vector in six dimensions is completely characterized by its energy ω and

a point z on the S^4 . This allows us to express the soft photon's momentum as

$$q^\mu = \omega [1, \hat{x}^i(z)] . \quad (3.4.2)$$

Here, $\hat{x}(z)$ is the embedding of the unit S^4 into \mathbb{R}^5 . We use the same parametrization for the momenta of the massless external particles:

$$p_k^\mu = E_k [1, \hat{x}^i(z_k)] . \quad (3.4.3)$$

In-states and out-states are then determined by the energy E_k , electric charge \mathcal{Q}_k , and \mathcal{I} -crossing point z_k for each external particle. We denote the in- and out-states by

$$|z_1, \dots, z_n\rangle , \quad \langle z_{n+1}, \dots, z_{n+n'}| , \quad (3.4.4)$$

respectively. In what follows, we assume that the incoming and outgoing states do not include soft photons.

Motivated by the expression for the radiative modes (3.3.20), we define the function

$$F_a^{out}(z, z_1, \dots, z_{n+n'}) \equiv \partial_a \hat{x}^i(z) \omega \sum_{\alpha} \varepsilon_i^{*\alpha} \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} - \sum_{k=1}^n \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} \right] \quad (3.4.5)$$

$$= \sum_{k=n+1}^{n+n'} \mathcal{Q}_k \partial_a \log(1 - P(z, z_k)) - \sum_{k=1}^n \mathcal{Q}_k \partial_a \log(1 - P(z, z_k)) . \quad (3.4.6)$$

Here we have used the completeness relation for polarization vectors

$$\sum_{\alpha} \varepsilon_{\alpha}^{*i}(\vec{q}) \varepsilon_{\alpha}^j(\vec{q}) = \delta^{ij} - \frac{q^i q^j}{\vec{q}^2} \quad (3.4.7)$$

and defined a function²

$$P(z, z_k) \equiv \hat{x}_i(z) \hat{x}^i(z_k) . \quad (3.4.8)$$

² P is known as the invariant distance on the S^4 , and is related to the cosine of the geodesic distance.

$F_a^{out}(z, z_1, \dots, z_{n+n'})$ (abbreviated $F_a(z; z_k)$) is simply related to the zero-mode insertion:

$$\langle z_{n+1}, \dots | A_a^{0(1)}(z) \mathcal{S} | z_1, \dots \rangle = -\frac{e^2}{(4\pi)^2} F_a^{out}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle. \quad (3.4.9)$$

Straightforward algebra reveals that $F_a^{out}(z; z_k)$ obeys the differential equation

$$\sqrt{\gamma}(D^2 - 2)D^a F_a^{out}(z; z_k) = -(4\pi)^2 \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \delta^{(4)}(z - z_k) - \sum_{k=1}^n \mathcal{Q}_k \delta^{(4)}(z - z_k) \right]. \quad (3.4.10)$$

We may also consider Weinberg's soft theorem for an incoming soft photon, which reads

$$\lim_{\omega \rightarrow 0} \omega \langle z_{n+1}, \dots | \mathcal{S} a_\alpha(q)^\dagger | z_1, \dots \rangle = -e\omega \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} - \sum_{k=1}^n \mathcal{Q}_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle. \quad (3.4.11)$$

We similarly define

$$F_a^{in}(z, z_1, \dots, z_{n+n'}) \equiv \partial_a \hat{x}^i(z) \omega \sum_\alpha \varepsilon_i^\alpha \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} - \sum_{k=1}^n \mathcal{Q}_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} \right] \quad (3.4.12)$$

$$= - \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \partial_a \log(1 + P(z, z_k)) - \sum_{k=1}^n \mathcal{Q}_k \partial_a \log(1 + P(z, z_k)) \right], \quad (3.4.13)$$

which is in turn related to the zero-mode insertion

$$\langle z_{n+1}, \dots | \mathcal{S} B_a^{0(1)}(z) | z_1, \dots \rangle = \frac{e^2}{(4\pi)^2} F_a^{in}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle. \quad (3.4.14)$$

Combining equations (3.3.16), (3.4.9), (3.4.10), and (3.4.14), we obtain the relation

$$\begin{aligned} & \frac{1}{2e^2} \int d^4 z \sqrt{\gamma} \varepsilon(z) (D^2 - 2) D^a \langle z_{n+1}, \dots | A_a^{0(1)}(z) \mathcal{S} | z_1, \dots \rangle \\ & + \frac{1}{2e^2} \int d^4 z \sqrt{\gamma} \varepsilon^-(z) (D^2 - 2) D^a \langle z_{n+1}, \dots | \mathcal{S} B_a^{0(1)}(z) | z_1, \dots \rangle \\ & = \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \varepsilon(z_k) - \sum_{k=1}^n \mathcal{Q}_k \varepsilon(z_k) \right] \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle. \end{aligned} \quad (3.4.15)$$

We can rewrite this expression as a Ward identity

$$\langle z_{n+1}, \dots | (Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-) | z_1, \dots \rangle = 0, \quad (3.4.16)$$

where Q_ε^\pm are charges acting on \mathcal{I}^\pm states. Q_ε^\pm can be decomposed into a hard charge and a soft charge:

$$Q_\varepsilon^\pm = Q_H^\pm + Q_S^\pm. \quad (3.4.17)$$

The hard charges Q_H^\pm are defined so that

$$Q_H^- | z_1, \dots \rangle = \sum_{k=1}^n \mathcal{Q}_k \varepsilon(z_k) | z_1, \dots \rangle, \quad \langle z_{n+1}, \dots | Q_H^+ = \langle z_{n+1}, \dots | \sum_{k=n+1}^{n+n'} \mathcal{Q}_k \varepsilon(z_k). \quad (3.4.18)$$

The soft charges are given by

$$\begin{aligned} Q_S^+ &= -\frac{1}{2e^2} \int d^4z \sqrt{\gamma} \varepsilon(z) [D^2 - 2] D^a A_a^{0(1)}(z), \\ Q_S^- &= \frac{1}{2e^2} \int d^4z \sqrt{\gamma} \varepsilon^-(z) [D^2 - 2] D^a B_a^{0(1)}(z). \end{aligned} \quad (3.4.19)$$

3.5 From Ward Identity to Asymptotic Symmetry

We would now like to interpret the Ward identity (3.4.16) in terms of symmetry transformations on the matter and gauge fields and to identify the asymptotic symmetry group of six-dimensional massless QED.

3.5.1 Action on Matter Fields

Equation (3.4.18) indicates that the charges Q_H^\pm generate a gauge transformation on the matter fields. We can express Q_H^\pm in terms of the gauge current:

$$\begin{aligned} Q_H^+ &= \lim_{r \rightarrow \infty} \int_{\mathcal{I}^+} r^4 \sqrt{\gamma} du d^4z \varepsilon(z) J_u^M(u, r, z), \\ Q_H^- &= \lim_{r \rightarrow \infty} \int_{\mathcal{I}^-} r^4 \sqrt{\gamma} dv d^4z \varepsilon^-(z) J_v^M(v, r, z). \end{aligned} \quad (3.5.1)$$

For a matter field Ψ_Q of charge Q we have

$$[Q_H^+, \Psi_Q(u, r, z)] = \left[\lim_{r \rightarrow \infty} \int_{\mathcal{I}^+} r^4 \sqrt{\gamma} \varepsilon J_u^M, \Psi_Q(u, r, z) \right] = -\varepsilon(z) Q \Psi_Q(u, r, z) . \quad (3.5.2)$$

The soft charges Q_S^\pm commute with Ψ_Q , so we see that the total charges Q_ε^\pm generate gauge transformations on the matter fields with gauge parameter $\varepsilon(z)$.

3.5.2 Action on Gauge Fields

Since the full theory is invariant only under combined gauge transformations of the matter and gauge fields, it is intuitively obvious that Q_S^\pm must generate a large gauge transformation for the gauge fields A_a and B_a . In order to make this relationship precise, we can use the constraint equation (3.2.8) along with the boundary conditions from section 3.3.1 to rewrite the total charge as a boundary integral

$$Q_\varepsilon^+ = \frac{1}{e^2} \lim_{r \rightarrow \infty} \int_{\mathcal{I}_-^+} r^4 \sqrt{\gamma} d^4 z \varepsilon(z) F_{ru}(u, r, z) = \frac{1}{e^2} \int_{\mathcal{I}_-^+} \sqrt{\gamma} d^4 z \varepsilon(z) E_r(z) , \quad (3.5.3)$$

$$Q_\varepsilon^- = \frac{1}{e^2} \lim_{r \rightarrow \infty} \int_{\mathcal{I}_+^-} r^4 \sqrt{\gamma} d^4 z \varepsilon^-(z) G_{rv}(v, r, z) = \frac{1}{e^2} \int_{\mathcal{I}_+^-} \sqrt{\gamma} d^4 z \varepsilon^-(z) E_r^-(z) . \quad (3.5.4)$$

At this point several comments are in order. For $\varepsilon(z) = 1$ these expressions reduce to the familiar expressions for total electric charge at \mathcal{I}_-^+ and \mathcal{I}_+^- . For non-constant $\varepsilon(z)$ they are the natural generalization of the asymptotic symmetry generators in the four-dimensional case [34]. Both charges are written as pure boundary integrals of the free data E_r and E_r^- , allowing for a canonical identification of asymptotic symmetry transformations at \mathcal{I}^+ and \mathcal{I}^- . In the next subsection we demonstrate that it is possible to define a symplectic structure on the phase space of the theory so that the charges do in fact generate large gauge transformations on the gauge fields.

3.5.3 Bracket for the Free Data

In order to claim that the Q_ε^\pm generate gauge transformations we need to define the symplectic structure on the phase space of the theory. The bracket for the radiative modes is unambiguous [78, 79] and can be deduced from the mode expansion:

$$\left[A_a^{(1)}(u, z), \partial_{u'} A_b^{(1)}(u', z') \right] = i \frac{e^2}{2} \gamma_{ab} \delta(u - u') \frac{\delta^{(4)}(z - z')}{\sqrt{\gamma}} . \quad (3.5.5)$$

The bracket for the zero-modes can then be defined so that the charge Q_ε^\pm generates the correct gauge transformation. The correct bracket is given by

$$\left[E_r(z), \phi(z') \right] = i e^2 \frac{\delta^{(4)}(z - z')}{\sqrt{\gamma}} . \quad (3.5.6)$$

The bracket (3.5.6) resembles that of the constant modes in [34]. Similar expressions hold for \mathcal{I}^- quantities. It follows that

$$\left[Q_\varepsilon^+, A_a(z) \right] = i \partial_a \varepsilon(z) , \quad (3.5.7)$$

and we conclude that the charges Q_ε^\pm generate large gauge transformations on the matter fields and gauge fields of the theory.

As we have seen, Q_S^\pm does not annihilate the conventional vacuum of the theory. In fact, when Q_S^\pm acts on the vacuum it creates a soft photon, indicating that the large gauge symmetries are spontaneously broken. Under a large gauge transformation with parameter $\varepsilon(z)$, the free data $\phi(z)$ transforms as a Goldstone boson:

$$\phi(z) \rightarrow \phi(z) + \varepsilon(z) . \quad (3.5.8)$$

3.6 Generalization to Arbitrary Even-Dimensional Spacetime

The results of the preceding sections can be straightforwardly generalized to arbitrary even-dimensional flat spacetimes. In this section, we sketch the derivation of the Ward identity for $d = 2m + 2$ - dimensional spacetime, omitting a detailed discussion of the boundary conditions and symplectic form.

The plane wave expansion of the gauge field in $d = 2m + 2$ dimensions is given by

$$A_\mu(u, r, z) = e \sum_\alpha \int \frac{d^{2m+1}q}{(2\pi)^{2m+1}} \frac{1}{2\omega_q} \left[\varepsilon_\mu^{*\alpha}(\vec{q}) a_\alpha(\vec{q}) e^{iqx} + \varepsilon_\mu^\alpha(\vec{q}) a_\alpha(\vec{q})^\dagger e^{-iqx} \right]. \quad (3.6.1)$$

Here, $\omega_q = |\vec{q}|$ and α labels the $2m$ polarizations of the photon with corresponding polarization vectors $\varepsilon_\mu^\alpha(\vec{q})$. The operator $a_\alpha(\vec{q})^\dagger$ is a photon creation operator normalized so that

$$\left[a_\alpha(\vec{p}), a_\beta(\vec{q})^\dagger \right] = 2\omega_q \delta_{\alpha\beta} (2\pi)^{2m+1} \delta^{(2m+1)}(\vec{p} - \vec{q}). \quad (3.6.2)$$

We can evaluate the leading term in the large- r expansion of (3.6.1) using the saddle point approximation, yielding an expression for the radiative degrees of freedom of the Maxwell field in $d = 2m + 2$ dimensions near \mathcal{I}^+ . The expression for the Fourier image is

$$A_a^{\omega(m-1)}(z) = \frac{(-i)^m \omega^{m-1} e}{2(2\pi)^m} \partial_a \hat{x}^j(z) \sum_\alpha \varepsilon_j^{*\alpha} a_\alpha(\omega \hat{x}(z)), \quad (3.6.3)$$

$$A_a^{-\omega(m-1)}(z) = \frac{i^m \omega^{m-1} e}{2(2\pi)^m} \partial_a \hat{x}^j(z) \sum_\alpha \varepsilon_j^\alpha a_\alpha(\omega \hat{x}(z))^\dagger, \quad (3.6.4)$$

where $\hat{x}^i(z)$ is an embedding of S^{2m} into \mathbb{R}^{2m+1} . We can define a generalized zero-mode operator

$$A_a^{0(m-1)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (i\omega)^{2-m} \left[A_a^{\omega(m-1)} + (-1)^m A_a^{-\omega(m-1)} \right]. \quad (3.6.5)$$

Using the conventions (3.4.2)-(3.4.4), we can rewrite Weinberg's soft theorem (3.4.1) in the form

$$\langle z_{n+1}, \dots | A_a^{0(m-1)}(z) \mathcal{S} | z_1, \dots \rangle = -\frac{(-1)^m e^2}{4(2\pi)^m} F_a^{out}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle . \quad (3.6.6)$$

Here, the soft factor

$$F_a^{out}(z, z_1, \dots, z_{n+n'}) \equiv \partial_a \hat{x}^i(z) \omega \sum_{\alpha} \varepsilon_i^{*\alpha} \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} - \sum_{k=1}^n \mathcal{Q}_k \frac{p_k \cdot \varepsilon_{\alpha}}{p_k \cdot q} \right] \quad (3.6.7)$$

$$= \sum_{k=n+1}^{n+n'} \mathcal{Q}_k \partial_a \log(1 - P(z, z_k)) - \sum_{k=1}^n \mathcal{Q}_k \partial_a \log(1 - P(z, z_k)) \quad (3.6.8)$$

satisfies the differential equation

$$\begin{aligned} & (-1)^{m+1} \sqrt{\gamma} \prod_{l=m+1}^{2m-1} [D^2 - (2m-l)(l-1)] D^a F_a^{out} \\ &= \Gamma(m) 2^m (2\pi)^m \left[\sum_{k=1}^n \mathcal{Q}_k \delta^{(2m)}(z - z_k) - \sum_{k=n+1}^{n+n'} \mathcal{Q}_k \delta^{(2m)}(z - z_k) \right] . \end{aligned} \quad (3.6.9)$$

We can similarly introduce the in-modes

$$B_a^{\omega(m-1)}(z) = \frac{i^m \omega^{m-1} e}{2(2\pi)^m} \partial_a \hat{x}^j(z) \sum_{\alpha} \varepsilon_j^{*\alpha} a_{\alpha}(-\omega \hat{x}(z)) , \quad (3.6.10)$$

$$B_a^{-\omega(m-1)}(z) = \frac{(-i)^m \omega^{m-1} e}{2(2\pi)^m} \partial_a \hat{x}^j(z) \sum_{\alpha} \varepsilon_j^{\alpha} a_{\alpha}(-\omega \hat{x}(z))^{\dagger} , \quad (3.6.11)$$

and the associated zero-mode operator

$$B_a^{0(m-1)} = \frac{1}{2} \lim_{\omega \rightarrow 0} (i\omega)^{2-m} \left[B_a^{\omega(m-1)} + (-1)^m B_a^{-\omega(m-1)} \right] . \quad (3.6.12)$$

We then have

$$\langle z_{n+1}, \dots | \mathcal{S} B_a^{0(m-1)}(z) | z_1, \dots \rangle = \frac{e^2}{4(2\pi)^m} F_a^{in}(z; z_k) \langle z_{n+1}, \dots | \mathcal{S} | z_1, \dots \rangle , \quad (3.6.13)$$

where

$$F_a^{in}(z, z_1, \dots, z_{n+n'}) = - \left[\sum_{k=n+1}^{n+n'} \mathcal{Q}_k \partial_a \log(1 + P(z, z_k)) - \sum_{k=1}^n \mathcal{Q}_k \partial_a \log(1 + P(z, z_k)) \right] . \quad (3.6.14)$$

Combining equations (3.3.16), (3.6.6), (3.6.9), and (3.6.13), we can rewrite the soft theorem as the Ward identity

$$\langle z_{n+1}, \dots | (Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-) | z_1, \dots \rangle = 0 . \quad (3.6.15)$$

The charges $Q_\varepsilon^\pm = Q_H^\pm + Q_S^\pm$ act on \mathcal{I}^\pm states, with the action of Q_H^\pm defined so that

$$Q_H^- | z_1, \dots \rangle = \sum_{k=1}^n \mathcal{Q}_k \varepsilon(z_k) | z_1, \dots \rangle , \quad \langle z_{n+1}, \dots | Q_H^+ = \langle z_{n+1}, \dots | \sum_{k=n+1}^{n+n'} \mathcal{Q}_k \varepsilon(z_k) . \quad (3.6.16)$$

The soft charges take the form

$$Q_S^+ = -\frac{1}{2e^2} \frac{2^{2-m}}{\Gamma(m)} \int d^{2m} z \sqrt{\gamma} \varepsilon(z) \prod_{l=m+1}^{2m-1} (D^2 - (2m-l)(l-1)) D^a A_a^{0(m-1)} , \quad (3.6.17)$$

$$Q_S^- = \frac{(-1)^m}{2e^2} \frac{2^{2-m}}{\Gamma(m)} \int d^{2m} z \sqrt{\gamma} \varepsilon^-(z) \prod_{l=m+1}^{2m-1} (D^2 - (2m-l)(l-1)) D^a B_a^{0(m-1)} . \quad (3.6.18)$$

Note that the careful limiting procedure of section 3.4 may still be applied to A_a^ω near $\omega = 0$. The charges Q_H^\pm can be written in terms of the gauge current

$$Q_H^+ = \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^+} \sqrt{\gamma} \varepsilon(z) J_u^M(u, r, z) , \quad Q_H^- = \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^-} \sqrt{\gamma} \varepsilon^-(z) J_v^M(v, r, z) . \quad (3.6.19)$$

This operator generates a gauge transformation with parameter $\varepsilon(z)$ when acting on the matter fields. Assuming a natural generalization of the boundary conditions from section 3.3 and using Maxwell's equations (3.2.7), we can write the total charge $Q_\varepsilon^\pm = Q_H^\pm + Q_S^\pm$ as a boundary integral

$$\begin{aligned} Q_\varepsilon^+ &= \frac{1}{e^2} \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^+} d^{2m} z \sqrt{\gamma} \varepsilon(z) F_{ru}(u, r, z) , \\ Q_\varepsilon^- &= \frac{1}{e^2} \lim_{r \rightarrow \infty} r^{2m} \int_{\mathcal{I}^-} d^{2m} z \sqrt{\gamma} \varepsilon^-(z) G_{rv}(v, r, z) . \end{aligned} \quad (3.6.20)$$

We can introduce an extended phase space for the modes on \mathcal{I}^+ and \mathcal{I}^- to include $\phi(z)$ and $E_r(z)$. The symplectic form of section 3.5.3 can then be used to demonstrate that (3.6.20) generates large gauge transformations on the matter fields and gauge fields. These large gauge transformations are the asymptotic symmetries for even-dimensional massless QED.

3.7 Open Questions and Relations to Subsequent Work

Our analysis has been restricted to the case of massless matter fields. It would be interesting to extend the analysis to include massive particles, along the lines of [80, 81]. Since this work first appeared, there have been a number of papers which analyze the action of large gauge symmetries in four-dimensional QED in a basis of infrared safe states [82–84]. Since the QED \mathcal{S} -matrix is infrared finite for $D > 4$, it is unclear whether or not such analyses are relevant to the questions considered in this chapter, but the issue deserves further consideration. In four dimensions, the soft theorems and asymptotic symmetries of QED have been related to so-called electromagnetic memory effects [85–87]. The soft theorems, asymptotic symmetries [88] and memory effects [89, 90] have been separately analyzed in the higher-dimensional case. These are outstanding problems for future investigations.

4

New Symmetries of QED

4.1 Introduction

The soft photon theorem [4–7, 91] has played a ubiquitous role in the study of QED and more general abelian gauge theories. For example, it is essential for taming otherwise uncontrollable infrared divergences in the \mathcal{S} -matrix and is central to the analysis of jet substructure. Recent considerations [34, 36–38, 44, 92–95] have demonstrated that, in abelian gauge theories with only massless charged particles, the soft theorem is a Ward identity of an infinite-dimensional symmetry group comprised of certain “large” gauge transformations which do not die off at infinity. These symmetries are spontaneously broken and the soft photons are the Goldstone bosons. This is but one instance of a recently-discovered universal triangle connecting soft theorems, symmetries and memory in gauge and gravitational theories [15, 30–45, 61, 62, 87, 88, 96–99].

Of course in the real world QED has massive, not massless, charged particles. Hence, it is

desirable to extend our results to the massive case. That goal is achieved in this chapter. As seen below, the massive case is rather more subtle than the massless one and requires a careful analysis of timelike infinity.

We hope that the identification given herein of the symmetry which controls the electromagnetic soft behavior of QED and more generally, the Standard Model will have practical utility for organizing and predicting a variety of physical phenomena.

The outline of this chapter is as follows. In section 4.2, we establish conventions and review relevant aspects of abelian gauge theories and their asymptotic symmetries. In section 4.3, we discuss the asymptotic states, derive the Ward identity of the asymptotic symmetries, and demonstrate its equivalence to the soft photon theorem.

A key ingredient of our analysis is that, in physical applications, the electromagnetic field is generically¹ not smooth near spatial infinity i^0 . Rather, it obeys a matching condition near i^0 which identifies its value at the future of past null infinity (\mathcal{I}_+^-) with its value at the antipodal point on the sphere at the past of future null infinity (\mathcal{I}_-^+). In appendix 4.A, we show in detail how this follows from the standard Liénard-Wiechert formulae.

4.2 Abelian Gauge Theory with Massive Matter

We consider the theory of an abelian gauge field A_μ coupled to massive matter fields Ψ_i with charges eQ_i , where Q_i is an integer, in Minkowski space. In retarded coordinates, the Minkowski metric reads

$$ds^2 = -dt^2 + (dx^i)^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad (4.2.1)$$

¹For instance when, as in electron-positron scattering, the dipole moment is not constant in the far past or future.

where u is retarded time and $\gamma_{z\bar{z}}$ is the round metric on the unit radius S^2 with covariant derivative D_z . The S^2 coordinates (z, \bar{z}) are related to standard Cartesian coordinates by

$$r^2 = x_i x^i, \quad u = t - r, \quad x^i = r \hat{x}^i(z, \bar{z}). \quad (4.2.2)$$

In retarded coordinates, future null infinity (\mathcal{I}^+) is the null hypersurface ($r = \infty, u, z, \bar{z}$).

Near past null infinity (\mathcal{I}^-), we work in advanced coordinates (v, r, z, \bar{z}) with line element

$$ds^2 = -dv^2 + 2dvdr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z}. \quad (4.2.3)$$

Advanced coordinates are given by

$$r^2 = x_i x^i, \quad v = t + r, \quad x^i = -r \hat{x}^i(z, \bar{z}), \quad (4.2.4)$$

and \mathcal{I}^- corresponds to the null hypersurface ($r = \infty, v, z, \bar{z}$). Note in particular that the angular coordinates on \mathcal{I}^+ are antipodally related to those on \mathcal{I}^- so that a light ray passing through the interior of Minkowski space reaches the same value of z, \bar{z} at both \mathcal{I}^+ and \mathcal{I}^- . We denote the future (past) boundary of \mathcal{I}^+ by \mathcal{I}_+^+ (\mathcal{I}_-^+), and the future (past) boundary of \mathcal{I}^- by \mathcal{I}_+^- (\mathcal{I}_-^-).

We consider theories with a $U(1)$ gauge field strength $F = dA$ subject to the Maxwell equation

$$\nabla^\mu F_{\mu\nu} = e^2 J_\nu, \quad (4.2.5)$$

where J_ν is the matter charge current. This is invariant under the gauge transformations

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \varepsilon(x), \quad \Psi_i(x) \rightarrow e^{iQ_i \varepsilon(x)} \Psi_i(x), \quad (4.2.6)$$

where $\varepsilon \sim \varepsilon + 2\pi$ and Ψ_i is a wavefunction or field. Gauge transformations that vanish at infinity correspond to redundant descriptions of the same physical state and can be eliminated by a choice of gauge. However, as in the massless case [34], we are interested in certain angle-dependent large gauge transformations which act nontrivially on physical states.

4.2.1 Asymptotics

We now analyze the behavior of the theory near \mathcal{I}^+ in retarded radial gauge

$$A_r = 0, \quad A_u|_{\mathcal{I}^+} = 0. \quad (4.2.7)$$

This gauge choice leaves unfixed a class of residual large gauge transformations parameterized by an arbitrary function $\varepsilon^+(z, \bar{z})$ on S^2 . These gauge transformations change boundary data at \mathcal{I}^+ and are to be regarded as physical symmetries of the theory. Near \mathcal{I}^+ , we assume the asymptotic expansion

$$A_u = \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u, z, \bar{z})}{r^n}, \quad A_z = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, z, \bar{z})}{r^n}. \quad (4.2.8)$$

A similar asymptotic expansion holds for fields near \mathcal{I}^- .

We are interested in scattering processes for which the initial and final states consist of non-interacting massive charges moving at constant velocities. Hence, we require that the only contributions to the electric and magnetic fields at future/past timelike infinity ($i^\pm, t \rightarrow \pm\infty$) are those fields sourced by the constant velocity massive charges, and that the magnetic fields vanish at spatial infinity:

$$F_{z\bar{z}}|_{\mathcal{I}^+} = 0, \quad F_{z\bar{z}}|_{\mathcal{I}^-} = 0. \quad (4.2.9)$$

In retarded coordinates, Maxwell's equations read

$$r^{-2}\partial_r (r^2 F_{ru}) - \partial_u F_{ru} + r^{-2} (D^z F_{zu} + D^{\bar{z}} F_{\bar{z}u}) = e^2 J_u, \quad (4.2.10)$$

$$r^{-2}\partial_r (r^2 F_{ru}) + r^{-2} (D^z F_{zr} + D^{\bar{z}} F_{\bar{z}r}) = e^2 J_r, \quad (4.2.11)$$

$$\partial_r (F_{rz} - F_{uz}) - \partial_u F_{rz} + r^{-2} D^{\bar{z}} F_{\bar{z}z} = e^2 J_z. \quad (4.2.12)$$

Massive particles with finite energy cannot reach \mathcal{I} , so the matter current vanishes at this surface:

$$J_\mu|_{\mathcal{I}} = 0. \quad (4.2.13)$$

The leading order equation for the evolution of the gauge field along \mathcal{I}^+ is then given by

$$\partial_u F_{ru}^{(2)} + \partial_u \left(D^z A_z^{(0)} + D^{\bar{z}} A_{\bar{z}}^{(0)} \right) = 0 . \quad (4.2.14)$$

The free data at this order includes the boundary data $F_{ru}^{(2)}|_{\mathcal{I}_+^+}$ along with the radiative mode $A_z^{(0)}(u, z, \bar{z})$.

In advanced coordinates, we can perform the analogous large- r expansion near \mathcal{I}^- and obtain the leading order equation

$$\partial_v F_{rv}^{(2)} - \partial_v \left(D^z A_z^{(0)} + D^{\bar{z}} A_{\bar{z}}^{(0)} \right) = 0 . \quad (4.2.15)$$

The free data at this boundary includes the field strength boundary data $F_{rv}^{(2)}|_{\mathcal{I}_+^-}$ along with the radiative mode $A_z^{(0)}(v, z, \bar{z})$. The residual large gauge symmetry is parameterized by an arbitrary function $\varepsilon^-(z, \bar{z})$ on S^2 .

4.2.2 Matching Near Spatial Infinity

The above discussion treats the asymptotic dynamics at \mathcal{I}^+ and \mathcal{I}^- separately. However, to study the semiclassical scattering problem, we must first specify how to relate free data and symmetry transformations at \mathcal{I}^+ to their counterparts at \mathcal{I}^- . Generic solutions to the sourced Maxwell equations satisfy²

$$F_{ru}^{(2)}(z, \bar{z})|_{\mathcal{I}_+^+} = F_{rv}^{(2)}(z, \bar{z})|_{\mathcal{I}_+^-} . \quad (4.2.16)$$

Recalling that, according to (4.2.2) and (4.2.4), the points labelled by the same (z, \bar{z}) in retarded and advanced coordinates are antipodally related, this equates the boundary values of past and future fields at antipodal points near spatial infinity i^0 . As discussed in [34, 96], a CPT and

²See appendix 4.A for an expanded discussion of this matching condition.

Lorentz-invariant matching condition for the gauge field is given by

$$A_z(z, \bar{z})|_{\mathcal{I}_-^+} = A_z(z, \bar{z})|_{\mathcal{I}_+^-} . \quad (4.2.17)$$

Requiring that the large gauge transformations preserve this matching condition gives:

$$\varepsilon^+(z, \bar{z}) = \varepsilon^-(z, \bar{z}) . \quad (4.2.18)$$

This matching condition singles out a canonical diagonal subgroup of the large gauge transformations at \mathcal{I}^+ and \mathcal{I}^- . The corresponding gauge parameters are constant along the null generators of \mathcal{I} and generate nontrivial physical symmetries of the \mathcal{S} -matrix.

4.2.3 Mode Expansions

The standard mode expansion for the gauge field in the plane wave basis takes the form

$$A_\mu(u, r, z, \bar{z}) = e \sum_\alpha \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_\mu^{*\alpha}(\vec{q}) a_\alpha(\vec{q}) e^{iq \cdot x} + \varepsilon_\mu^\alpha(\vec{q}) a_\alpha(\vec{q})^\dagger e^{-iq \cdot x} \right] . \quad (4.2.19)$$

The free data is contained in the $\mathcal{O}(r^0)$ term in this expansion, which we may isolate using the saddle point approximation:

$$A_z^{(0)}(u, z, \bar{z}) = -\frac{ie}{2(2\pi)^2} \partial_z \hat{x}^i \sum_\alpha \int_0^\infty d\omega_q \left[\varepsilon_i^{*\alpha} a_\alpha(\omega_q \hat{x}) e^{-i\omega_q u} - \varepsilon_i^\alpha a_\alpha(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right] . \quad (4.2.20)$$

To extract the contribution from the zero-modes, we define the following operator:

$$\begin{aligned} F_{uz}^\omega(z, \bar{z}) &\equiv \int_{-\infty}^\infty du e^{i\omega u} \partial_u A_z^{(0)}(u, z, \bar{z}) \\ &= -\frac{e}{4\pi} \partial_z \hat{x}^i \sum_\alpha \int_0^\infty d\omega_q \omega_q \left[\varepsilon_i^{*\alpha} a_\alpha(\omega_q \hat{x}) \delta(\omega - \omega_q) + \varepsilon_i^\alpha a_\alpha(\omega_q \hat{x})^\dagger \delta(\omega + \omega_q) \right] . \end{aligned} \quad (4.2.21)$$

We can separate this operator into its positive and negative frequency components

$$F_{uz}^\omega(z, \bar{z}) = -\frac{e\omega}{4\pi} \partial_z \hat{x}^i \sum_\alpha \varepsilon_i^{*\alpha} a_\alpha(\omega \hat{x}) , \quad F_{uz}^{-\omega}(z, \bar{z}) = -\frac{e\omega}{4\pi} \partial_z \hat{x}^i \sum_\alpha \varepsilon_i^\alpha a_\alpha(\omega \hat{x})^\dagger , \quad (4.2.22)$$

with $\omega > 0$ in both expressions. The zero-mode is then given by

$$F_{uz}^0(z, \bar{z}) \equiv \frac{1}{2} \lim_{\omega \rightarrow 0} (F_{uz}^\omega + F_{uz}^{-\omega}) = -\frac{e}{8\pi} \partial_z \hat{x}^i \lim_{\omega \rightarrow 0} \sum_\alpha \left[\omega \varepsilon_i^{*\alpha} a_\alpha(\omega \hat{x}) + \omega \varepsilon_i^\alpha a_\alpha(\omega \hat{x})^\dagger \right] , \quad (4.2.23)$$

and creates/annihilates soft photons. An analogous construction holds at \mathcal{I}^- with the incoming soft photon operator given by

$$F_{vz}^0(z, \bar{z}) = \frac{e}{8\pi} \partial_z \hat{x}^i \lim_{\omega \rightarrow 0} \sum_\alpha \left[\omega \varepsilon_i^{*\alpha} a_\alpha(-\omega \hat{x}) + \omega \varepsilon_i^\alpha a_\alpha(-\omega \hat{x})^\dagger \right] . \quad (4.2.24)$$

4.2.4 Liénard-Wiechert Fields

In the analysis that follows, we will need expressions for the electric field due to moving point charges, commonly known as Liénard-Wiechert fields. The radial electric field due to a single particle of charge eQ , moving with constant velocity $\vec{\beta}$ and passing through the origin at $t = 0$ is given by

$$E_r(t, r, z, \bar{z}) = \frac{Qe^2}{4\pi} \frac{\gamma(r - t\hat{x}(z, \bar{z}) \cdot \vec{\beta})}{|\gamma^2[t - r\hat{x}(z, \bar{z}) \cdot \vec{\beta}]^2 - t^2 + r^2|^{3/2}} . \quad (4.2.25)$$

Here, $\hat{x}(z, \bar{z})$ is a unit vector specifying a point on the sphere and $\gamma^{-2} = 1 - \beta^2$.

The Liénard-Wiechert field near \mathcal{I}^+ due to a set of particles, each with charge eQ_k and moving with constant velocity $\vec{\beta}_k$, is derived by taking a superposition of the single-particle fields (4.2.25), writing them in retarded coordinates, and taking the large- r limit with $u = t - r$ held fixed

$$E_r^+(z, \bar{z}) = \sum_k \frac{Q_k e^2}{4\pi \gamma_k^2 r^2} \frac{1}{[1 - \hat{x}(z, \bar{z}) \cdot \vec{\beta}_k]^2} . \quad (4.2.26)$$

Likewise, the field near \mathcal{I}^- is derived by taking the large- r limit of the field (4.2.25) in advanced

coordinates with fixed v

$$E_r^-(z, \bar{z}) = \sum_k \frac{Q_k e^2}{4\pi\gamma_k^2 r^2} \frac{1}{[1 + \hat{x}(z, \bar{z}) \cdot \vec{\beta}_k]^2} . \quad (4.2.27)$$

Importantly, the Liénard-Wiechert formula (4.2.25) implies that the value of E_r near spatial infinity i^0 depends on how it is approached. In particular, E_r^+ and E_r^- at a fixed angle from the origin are not in general equal near i^0 : rather they obey the antipodal matching condition (4.2.16).³

For unaccelerated charges, the asymptotic electric field and the asymptotic magnetic field $\vec{B} = \hat{x} \times \vec{E}$ are time-independent. Since the “hard” radiative photons involved in the scattering process exit/enter \mathcal{I}^\pm at finite values of retarded/advanced time, the electromagnetic fields at \mathcal{I}_+^+ and \mathcal{I}_-^- arise solely from the collection of charged particles long after/before the scattering process occurs and thus are of the form given above.

4.3 Symmetries of the \mathcal{S} -matrix

In this section we determine the phase associated to a large gauge transformation on an asymptotic massive charged particle state, find the \mathcal{S} -matrix Ward identity and finally demonstrate its equivalence to the soft photon theorem.

4.3.1 Gauge Transformations of Asymptotic States

Outgoing massless particles of charge eQ and momentum p , as considered in [34], pierce \mathcal{I}^+ at a definite point $(z(p), \bar{z}(p))$. The associated out-state therefore acquires a phase

$$|p\rangle_{out} \rightarrow e^{iQ\varepsilon(z(p), \bar{z}(p))} |p\rangle_{out} \quad (4.3.1)$$

³The constant-velocity trajectory considered in this section gives rise to a Liénard-Wiechert field that is insensitive to the choice of Green’s function. In appendix 4.A, we consider slightly more complicated trajectories to demonstrate that this matching condition holds for a generic Green’s function.

under a large gauge transformation. Here we are interested in massive particles that never reach \mathcal{I}^+ , so determining the associated phase is more subtle. There is no canonical point on the S^2 associated to a massive particle in the plane wave basis.⁴ Indeed, a massive particle with zero three-momentum is rotationally invariant. In this subsection, we use the Liénard-Wiechert formula to determine the analog of the phase (4.3.1).

The asymptotic states associated to the QED \mathcal{S} -matrix are typically taken to be free photons and “bare” non-interacting charged particles. However, the “bare” electron states are not strictly speaking bare (nor are they non-interacting) in the sense that they source non-vanishing electromagnetic fields. These long-range fields accompanying the scattering states are responsible for the infrared divergences in the loop-level matrix elements: even widely separated electrons experience a nonzero acceleration, causing them to bremsstrahlung radiate infinite numbers of low-energy photons. Physically, it is impossible to separate a charged particle from its electromagnetic field. Mathematically, the ability to do so would violate Gauss’s law, which is a constraint in the quantum theory in physical gauges:

$$\left[\nabla \cdot \vec{E}(x, t) - e^2 \rho(x, t) \right] |phys\rangle = 0 . \quad (4.3.2)$$

This sourced electromagnetic field can be treated as a classical background, while the transverse photons describe excitations of the quantized, unsourced electromagnetic field. Since outgoing charged particle states with momentum $p^\mu = \gamma m [1, \vec{\beta}]$ source Liénard-Wiechert fields, the action of the electric field at \mathcal{I}_+^+ is nontrivial and given by

$$E_r(r, z, \bar{z}) |p\rangle_{out} = \left[\frac{Qe^2}{4\pi\gamma^2 r^2} \frac{1}{[1 - \hat{x}(z, \bar{z}) \cdot \vec{\beta}]^2} \right] |p\rangle_{out} . \quad (4.3.3)$$

Then, since gauge transformations are generated by the electric charge operator, under a large

⁴Since plane waves of massless particles localize to points on the conformal sphere at null infinity, they are related to local operator insertions on that sphere. Likewise, massive particles in boost eigenstates are also associated to local points on the sphere.

gauge transformation, such a state acquires a phase

$$\begin{aligned} |p\rangle_{out} &\rightarrow \exp \left[i \frac{1}{e^2} \int_{\mathcal{I}_+^+} d^2 z \gamma_{z\bar{z}} \varepsilon^+ F_{ru}^{(2)} \right] |p\rangle_{out} \\ &= \exp \left[i \int_{\mathcal{I}_+^+} d^2 z \gamma_{z\bar{z}} \varepsilon^+ \left(\frac{1}{4\pi\gamma^2} \frac{Q}{[1 - \hat{x}(z, \bar{z}) \cdot \vec{\beta}]^2} \right) \right] |p\rangle_{out} . \end{aligned} \quad (4.3.4)$$

Similarly, in-states transform as

$$|p\rangle_{in} \rightarrow \exp \left[i \int_{\mathcal{I}_-^-} d^2 z \gamma_{z\bar{z}} \varepsilon^- \left(\frac{1}{4\pi\gamma^2} \frac{Q}{[1 + \hat{x}(z, \bar{z}) \cdot \vec{\beta}]^2} \right) \right] |p\rangle_{in} . \quad (4.3.5)$$

For an n -particle state, the phase will be a sum of n such terms. This phase replaces the much simpler expression (4.3.1) for massless particles but nevertheless, as will be seen shortly, precisely reproduces the soft factor for massive particles.

4.3.2 Ward Identity

We are now in a position to discuss the symmetries of the \mathcal{S} -matrix. The symmetry transformations (4.2.6) for massless matter fields have already been analyzed in [34], where it was demonstrated that the charge

$$Q_\varepsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}_+^+} d^2 z \gamma_{z\bar{z}} \varepsilon^+(z, \bar{z}) F_{ru}^{(2)}(z, \bar{z}) \quad (4.3.6)$$

generates the correct \mathcal{I}^+ symmetry transformation on the gauge field and matter fields. The form of this charge is essentially fixed by the transformation law for the gauge field. We can use the leading order Maxwell equation (4.2.14) to turn this expression into an integral over \mathcal{I}^+ . As discussed in section 4.2.4, the existence of massive particles generates charge flux through future timelike infinity, so the local charge operator takes the form

$$\begin{aligned} Q_\varepsilon^+ &= \frac{1}{e^2} \int_{\mathcal{I}_+^+} \gamma_{z\bar{z}} du d^2 z \varepsilon^+ \partial_u \left(D^z A_z^{(0)} + D^{\bar{z}} A_{\bar{z}}^{(0)} \right) + \frac{1}{e^2} \int_{\mathcal{I}_+^+} d^2 z \gamma_{z\bar{z}} \varepsilon^+ F_{ru}^{(2)} \\ &\equiv \frac{1}{e^2} \int_{S^2} \gamma_{z\bar{z}} d^2 z \varepsilon^+ \left(D^z F_{uz}^0 + D^{\bar{z}} F_{u\bar{z}}^0 \right) + \frac{1}{e^2} \int_{\mathcal{I}_+^+} d^2 z \gamma_{z\bar{z}} \varepsilon^+ F_{ru}^{(2)} . \end{aligned} \quad (4.3.7)$$

The first piece of the charge is written in terms of the soft photon operator and will be referred to as the soft charge Q_S^+ . If we consider the fixed-angle charge by choosing $\varepsilon(z, \bar{z}) = \delta^{(2)}(z - w)$, then the second term is simply the radial electric field in the direction (w, \bar{w}) resulting from the charged particles in the final state at i^+ . We label this term the hard charge Q_H^+ . It differs from the expression for the hard charge in the massless case which involves an integral over \mathcal{I}^+ .

An analogous computation can be performed at \mathcal{I}^- , where the hard charge encodes the radial electric field of the charged particles in the initial state at i^- . The charge is given by

$$Q_\varepsilon^- = \frac{1}{e^2} \int_{S^2} \gamma_{z\bar{z}} d^2z \varepsilon^- (D^z F_{vz}^0 + D^{\bar{z}} F_{v\bar{z}}^0) + \frac{1}{e^2} \int_{\mathcal{I}_-^+} d^2z \gamma_{z\bar{z}} \varepsilon^- F_{rv}^{(2)} \equiv Q_S^- + Q_H^- . \quad (4.3.8)$$

The statement that the transformations (4.2.6) are symmetries of the \mathcal{S} -matrix is equivalent to the statement that the charges (4.3.7) and (4.3.8) commute with the \mathcal{S} -matrix:

$$\langle \text{out} | (Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-) | \text{in} \rangle = 0 . \quad (4.3.9)$$

In order to facilitate comparison with the soft theorem, we separate the hard and soft contributions and rearrange the Ward identity:

$$\langle \text{out} | (Q_S^+ \mathcal{S} - \mathcal{S} Q_S^-) | \text{in} \rangle = - \langle \text{out} | (Q_H^+ \mathcal{S} - \mathcal{S} Q_H^-) | \text{in} \rangle . \quad (4.3.10)$$

4.3.3 Soft Theorem \rightarrow Ward Identity

The soft photon theorem for the emission of an outgoing photon in a scattering process with m incoming hard particles and $(n - m)$ outgoing hard particles reads

$$\begin{aligned} & \lim_{\omega \rightarrow 0} \omega \langle p_{m+1}, \dots | a_\alpha(q) \mathcal{S} | p_1, \dots \rangle \\ &= e\omega \left[\sum_{k=m+1}^n Q_k \frac{p_k \cdot \varepsilon_\alpha}{p_k \cdot q} - \sum_{k=1}^m Q_k \frac{p_k \cdot \varepsilon_\alpha}{p_k \cdot q} \right] \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle . \end{aligned} \quad (4.3.11)$$

A null momentum vector is uniquely specified by an energy and a point z on the asymptotic sphere, and so we parameterize the photon's momentum as

$$q^\mu = \omega [1, \hat{x}(z, \bar{z})] \equiv \omega \hat{q}^\mu(z, \bar{z}) , \quad (4.3.12)$$

where $\hat{x} : S^2 \rightarrow \mathbb{R}^3$ is an embedding of the sphere into flat three-dimensional space.

We parameterize a massive particle's momentum as

$$p_k^\mu = \gamma_k m_k [1, \vec{\beta}_k] , \quad (4.3.13)$$

where m is the rest mass of the particle, $\vec{\beta}$ is the particle's velocity and γ is the relativistic factor $\gamma^{-2} = 1 - \beta^2$.

We can relate the left-hand side of equation (4.3.11) to the zero-mode operator defined in equation (4.2.23) by taking a weighted sum over polarizations. If we perform the analogous operation on the right-hand side and use the identity

$$\partial_z \hat{x}^i(z, \bar{z}) \sum_\alpha \varepsilon_i^{*\alpha} \frac{p_k \cdot \varepsilon_\alpha}{p_k \cdot \hat{q}(z, \bar{z})} = \partial_z \log(p_k \cdot \hat{q}) , \quad (4.3.14)$$

the soft theorem can be written

$$\begin{aligned} & \langle p_{m+1}, \dots | F_{uz}^0 \mathcal{S} | p_1, \dots \rangle \\ &= -\frac{e^2}{8\pi} \left[\sum_{k=m+1}^n Q_k \partial_z \log(p_k \cdot \hat{q}) - \sum_{k=1}^m Q_k \partial_z \log(p_k \cdot \hat{q}) \right] \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle . \end{aligned} \quad (4.3.15)$$

The soft photon theorem for an incoming soft photon reads

$$\begin{aligned} & \lim_{\omega \rightarrow 0} \omega \langle p_{m+1}, \dots | \mathcal{S} a_\alpha(q)^\dagger | p_1, \dots \rangle \\ &= -e\omega \left[\sum_{k=m+1}^n Q_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} - \sum_{k=1}^m Q_k \frac{p_k \cdot \varepsilon_\alpha^*}{p_k \cdot q} \right] \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle . \end{aligned} \quad (4.3.16)$$

An identical calculation yields

$$\begin{aligned} & \langle p_{m+1}, \dots | \mathcal{S} F_{vz}^0 | p_1, \dots \rangle \\ &= \frac{e^2}{8\pi} \left[\sum_{k=m+1}^n Q_k \partial_z \log(p_k \cdot \hat{q}') - \sum_{k=1}^m Q_k \partial_z \log(p_k \cdot \hat{q}') \right] \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle, \end{aligned} \quad (4.3.17)$$

where $\hat{q}' = [1, -\hat{x}^i(z, \bar{z})]$. Taking the divergence of each equation, using global charge conservation, and integrating against the respective gauge parameter, we find

$$\begin{aligned} & \langle p_{m+1}, \dots | \left(\int_{S^2} d^2 z \gamma_{z\bar{z}} \varepsilon^+ (D^z F_{uz}^0 + D^{\bar{z}} F_{u\bar{z}}^0) \right) \mathcal{S} | p_1, \dots \rangle \\ &= -\frac{1}{2} \int_{S^2} d^2 z \gamma_{z\bar{z}} \varepsilon^+ r^2 \left([E_r^+]_{\text{out}} - [E_r^+]_{\text{in}} \right) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle \end{aligned} \quad (4.3.18)$$

and

$$\begin{aligned} & \langle p_{m+1}, \dots | \mathcal{S} \left(\int_{S^2} d^2 z \gamma_{z\bar{z}} \varepsilon^- (D^z F_{vz}^0 + D^{\bar{z}} F_{v\bar{z}}^0) \right) | p_1, \dots \rangle \\ &= \frac{1}{2} \int_{S^2} d^2 z \gamma_{z\bar{z}} \varepsilon^- r^2 \left([E_r^-]_{\text{out}} - [E_r^-]_{\text{in}} \right) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle. \end{aligned} \quad (4.3.19)$$

Taking the difference and using the matching conditions (4.2.16)-(4.2.18), we obtain

$$\begin{aligned} & \langle p_{m+1}, \dots | Q_S^+ \mathcal{S} - \mathcal{S} Q_S^- | p_1, \dots \rangle \\ &= -\frac{1}{e^2} \int_{S^2} d^2 z \gamma_{z\bar{z}} r^2 \left(\varepsilon^+ E_r|_{\mathcal{I}_+^+} - \varepsilon^- E_r|_{\mathcal{I}_-^-} \right) \langle p_{m+1}, \dots | \mathcal{S} | p_1, \dots \rangle \\ &= -\langle p_{m+1}, \dots | Q_H^+ \mathcal{S} - \mathcal{S} Q_H^- | p_1, \dots \rangle. \end{aligned} \quad (4.3.20)$$

This precisely reproduces the Ward identity (4.3.10).

In conclusion, while the details are more intricate than in the massless case, the soft photon theorem is the Ward identity of an infinite-dimensional asymptotic symmetry group for abelian gauge theories with massive particles. We expect similar conclusions to apply in other contexts such as non-abelian gauge theory and gravity.

4.A Gauge Field Strength Near i^0

In this section, we consider an idealized semiclassical scattering process in which m incoming massive particles with constant velocities $\{\vec{\beta}_1, \dots, \vec{\beta}_m\}$ scatter to $(n-m)$ outgoing massive particles with constant velocities $\{\vec{\beta}_{m+1}, \dots, \vec{\beta}_n\}$. For scattering occurring at the origin at $t = 0$, the semiclassical electromagnetic current is given by

$$j_\mu(x) = \sum_{k=1}^m Q_k \int d\tau U_\mu^k \Theta(-\tau) \delta^{(4)}(x - U^k \tau) + \sum_{k=m+1}^n Q_k \int d\tau U_\mu^k \Theta(\tau) \delta^{(4)}(x - U^k \tau), \quad (4.A.1)$$

where $U_\mu^k = \gamma_k [1, \vec{\beta}_k]$ is the 4-velocity of the k^{th} particle. Ignoring the radiative contributions arising from the infinite acceleration of particles at the origin, the field strength sourced by this current takes the form

$$F_{rt}(x) = \frac{e^2}{4\pi} \sum_{k=1}^m \frac{g(x; \vec{\beta}_k) Q_k \gamma_k (r - t\hat{x} \cdot \vec{\beta}_k)}{|\gamma_k^2 [t - r\hat{x} \cdot \vec{\beta}_k]^2 - t^2 + r^2|^{3/2}} + \frac{e^2}{4\pi} \sum_{k=m+1}^n \frac{h(x; \vec{\beta}_k) Q_k \gamma_k (r - t\hat{x} \cdot \vec{\beta}_k)}{|\gamma_k^2 [t - r\hat{x} \cdot \vec{\beta}_k]^2 - t^2 + r^2|^{3/2}}, \quad (4.A.2)$$

where the functional form of g and h depends on the choice of Green's function.

For the retarded solution, the asymptotic behavior of g and h is given by

$$g(r = \infty, u, \hat{x}; \vec{\beta}_k) = \Theta(-u), \quad h(r = \infty, u, \hat{x}; \vec{\beta}_k) = \Theta(u), \quad (4.A.3)$$

$$g(r = \infty, v, \hat{x}; \vec{\beta}_k) = 1, \quad h(r = \infty, v, \hat{x}; \vec{\beta}_k) = 0. \quad (4.A.4)$$

The electric field at \mathcal{I}_-^+ is obtained by working in retarded coordinates and taking the limit $r \rightarrow \infty$, followed by the limit $u \rightarrow -\infty$. This electric field will be of the form (4.2.26), but only receives contributions from the incoming particles. On the other hand, the electric field at \mathcal{I}_+^- is obtained by working in advanced coordinates and taking the limit $r \rightarrow \infty$, followed by the limit $v \rightarrow +\infty$. The electric field measured at \mathcal{I}_+^- will be of the form (4.2.27), but will also only receive contributions from the incoming particles, thereby satisfying the matching condition (4.2.16).

Likewise, for the advanced solution, the asymptotic behavior of g and h is given by

$$g(r = \infty, u, \hat{x}; \vec{\beta}_k) = 0, \quad h(r = \infty, u, \hat{x}; \vec{\beta}_k) = 1, \quad (4.A.5)$$

$$g(r = \infty, v, \hat{x}; \vec{\beta}_k) = \Theta(-v), \quad h(r = \infty, v, \hat{x}; \vec{\beta}_k) = \Theta(v). \quad (4.A.6)$$

Hence, the advanced solution also obeys the matching condition (4.2.16) near i^0 , but in contrast to the retarded solution, only receives contributions from the outgoing particles. Moreover, linear combinations of the advanced and retarded solutions evaluated near i^0 will obey the matching condition and receive contributions from both outgoing and incoming particles.

Of course, one could always add a homogeneous solution to the free Maxwell equations which does not obey the matching condition. However, we do not know of any physical application in which it is natural to do so: finite energy wave packets die off at i^0 . Hence, we conclude that the antipodal matching condition (4.2.16) holds in generic physical applications.

5

Infrared Divergences in QED, Revisited

5.1 Introduction

Recently, it has been shown [32, 34, 80, 81, 96] (see [100] for a review) that the infrared (IR) sector of all abelian gauge theories, including QED, is governed by an infinite-dimensional symmetry group. The symmetry group is generated by large gauge transformations that approach angle-dependent constants at null infinity. The soft photon theorem is the matrix element of the associated conservation laws. This large gauge symmetry is spontaneously broken, resulting in an infinite vacuum degeneracy.

QED has been tested to 16 decimal places and is the most accurate theory in the history of human thought. The preceding statements have no mathematically new content within QED, and certainly do not imply errors in any previous QED calculations! However, as emphasized herein, they do perhaps provide a physically illuminating new way of describing the IR structure. Moreover,

generalizations of this perspective to other contexts have led to a variety of truly new mathematical relations in both gauge theory and gravity [100].

One of the puzzling features of the IR structure of QED is the appearance of IR divergences.¹ These divergences set all conventional Fock-basis \mathcal{S} -matrix elements to zero. Often they are dealt with by restricting to inclusive cross sections in which physically unmeasurable photons below some IR cutoff are traced over. The trace gives a divergence which offsets the zero and yields a finite result for the physical measurement [4, 103–105]. While this is adequate for most experimental applications, for many purposes it is nice to have an \mathcal{S} -matrix.² For example, precise discussions of unitarity or symmetries require an \mathcal{S} -matrix.

It is natural to ask if the newly-discovered IR symmetries are related to the IR divergences of the \mathcal{S} -matrix. We will see that the answer is yes. The conservation laws imply that every nontrivial scattering process is necessarily accompanied by a transition among the degenerate vacua. Conventional QED \mathcal{S} -matrix analyses tend to assume the vacuum is unique and hence that the initial and final vacua are the same. Since this violates the conservation laws, the Feynman diagrammatics give a vanishing result. This is usually attributed to “IR divergences,” but we feel that this phrase is something of a misnomer. Rather, zero is the correct physical answer. The vanishing of the amplitudes is a penalty for not accounting for the required vacuum transition. In this chapter we allow for vacuum transitions to occur, and find that the resulting amplitudes are perfectly IR finite and generically nonvanishing when the conservation laws are obeyed.

Although we have phrased this result in a way that sounds new, the mathematics behind it is not new. We have merely rediscovered the 1970 formulae [106–110] of Faddeev and Kulish (FK) and others, who showed that certain dressings of charges by clouds of soft photons yield IR finite scattering amplitudes. The FK dressings implicitly generate precisely the required shift between

¹ Originally, these were found by looking at the spectrum and number of photons produced by particles undergoing acceleration. Mott [101] looked at corrections to Rutherford scattering as a result of the emission of photons. Bloch and Nordsieck [102] examined the spectrum of photons produced by a small change in the velocity of an electron.

²It may also be challenging to describe experimental measurements of the electromagnetic memory effect [85–87] in a theory with a finite IR cutoff.

degenerate vacua.

While our formulas are not new,³ our physical interpretation is new. One may hope that the new physical insight will enable a construction of IR finite \mathcal{S} -matrices for deconfined non-abelian gauge theory and also have useful applications to gravity.

Related discussions of the FK construction in the context of large gauge symmetry have appeared in [82, 83, 111, 112].

5.2 Vacuum Selection Rules

In this section we review the derivation of, and formulas for, the vacuum transitions induced by the scattering of charged massless particles. We refer the reader to [32, 34, 37, 96] for further details. The conceptually similar massive case is treated in section 5.5. Incoming states are best described in advanced coordinates in Minkowski space

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} , \quad (5.2.1)$$

while outgoing states employ retarded coordinates

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} . \quad (5.2.2)$$

Here $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the unit round metric on S^2 and $u = t - r$ ($v = t + r$) is the retarded (advanced) time. The z coordinates used in the advanced and retarded coordinate systems differ

³Except for, in section 5.6, a conjectured generalization of the FK IR divergence cancellation mechanism to amplitudes involving some undressed charges but still obeying the conservation laws. An example given there is e^+e^- scattering with no incoming radiation.

by an antipodal map on S^2 . In- and out-states are characterized by the charges⁴

$$\begin{aligned} Q_\varepsilon^- &= \frac{1}{e^2} \int_{\mathcal{I}_+^-} d^2w \gamma_{w\bar{w}} \varepsilon F_{rv} , \\ Q_\varepsilon^+ &= \frac{1}{e^2} \int_{\mathcal{I}_-^+} d^2w \gamma_{w\bar{w}} \varepsilon F_{ru} , \end{aligned} \quad (5.2.3)$$

where F is the electromagnetic field strength, \mathcal{I}_+^- is the future boundary of \mathcal{I}^- , \mathcal{I}_-^+ is the past boundary of \mathcal{I}^+ and ε is any function on S^2 . The conservation law for these charges

$$\langle \text{out} | (Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^-) | \text{in} \rangle = 0 \quad (5.2.4)$$

is implied by the soft photon theorem. In and out soft photon modes are defined as integrals of the radiative part of F over the null generators of past and future null infinity (\mathcal{I}^\pm) according to

$$\int_{-\infty}^{\infty} du F_{uz} \equiv N_z^+ , \quad (5.2.5)$$

$$\int_{-\infty}^{\infty} dv F_{vz} \equiv N_z^- . \quad (5.2.6)$$

Choosing $\varepsilon(w, \bar{w}) = \frac{1}{z-\bar{w}}$, (5.2.4) can be written in the form

$$\langle \text{out} | (N_z^+ \mathcal{S} - \mathcal{S} N_z^-) | \text{in} \rangle = \Omega_z^{\text{soft}} \langle \text{out} | \mathcal{S} | \text{in} \rangle , \quad (5.2.7)$$

where the soft factor is

$$\Omega_z^{\text{soft}} = \Omega_z^{\text{soft}-} - \Omega_z^{\text{soft}+} , \quad (5.2.8)$$

$$\Omega_z^{\text{soft}-} = \frac{e^2}{4\pi} \sum_{k \in \text{in}} \frac{Q_k}{z - z_k} , \quad \Omega_z^{\text{soft}+} = \frac{e^2}{4\pi} \sum_{k \in \text{out}} \frac{Q_k}{z - z_k} . \quad (5.2.9)$$

⁴In these and the following equations, $(F_{rv}, F_{ru}, j_v, j_u)$ denote the coefficient of the leading $\mathcal{O}(r^{-2})$ term of the large- r field expansions, while (F_{vz}, F_{uz}) denote the leading $\mathcal{O}(r^0)$ terms.

Here, Q_k and z_k denote the charges of the asymptotic particles and the angles at which they enter or exit at \mathcal{I}^\pm . Degenerate incoming vacua can be characterized by their N_z^- eigenvalue:

$$N_z^-(z, \bar{z}) | N_z^{\text{in}} \rangle = N_z^{\text{in}}(z, \bar{z}) | N_z^{\text{in}} \rangle . \quad (5.2.10)$$

Let us consider special states denoted $|\text{in}; N_z^{\text{in}}\rangle$ comprised of finite numbers of non-interacting incoming charged particles and hard photons built by acting with asymptotic creation operators on eigenstates (5.2.10) of N_z^- .⁵ Such hard particles do not affect the zero-modes and hence obey

$$N_z^- |\text{in}; N_z^{\text{in}}\rangle = N_z^{\text{in}} |\text{in}; N_z^{\text{in}}\rangle . \quad (5.2.11)$$

Adopting a similar notation for out-states, (5.2.7) becomes

$$(N_z^{\text{out}} - N_z^{\text{in}}) \langle \text{out}; N_z^{\text{out}} | \mathcal{S} | N_z^{\text{in}}; \text{in} \rangle = \Omega_z^{\text{soft}} \langle \text{out}; N_z^{\text{out}} | \mathcal{S} | N_z^{\text{in}}; \text{in} \rangle . \quad (5.2.12)$$

We conclude that either

$$\langle \text{out}; N_z^{\text{out}} | \mathcal{S} | N_z^{\text{in}}; \text{in} \rangle = 0 , \quad (5.2.13)$$

or

$$N_z^{\text{out}} - N_z^{\text{in}} = \Omega_z^{\text{soft}} . \quad (5.2.14)$$

The second relation (5.2.14) expresses conservation of the charges - one for each point on the sphere - associated to large gauge symmetries. The first states that any amplitude violating the conservation law must vanish.

In conventional formulations of QED, the vacuum is presumed to be unique.⁶ In that case, (5.2.14) is not an option, and we conclude that, according to (5.2.13), all \mathcal{S} -matrix elements vanish.

⁵It is assumed here that the Fourier coefficients of the photon creation operators are finite as the frequency $\omega \rightarrow 0$.

⁶Since N_z manifestly carries zero energy, if the vacuum is assumed to be unique it would have to be an N_z eigenstate.

In fact, this result is well-known and attributed to IR divergences. We see here that the IR divergences which set all such amplitudes to zero can be understood as a penalty for neglecting the fact that the in- and out-vacua differ for every nontrivial scattering process. Armed with this insight, we will construct a natural and IR finite set of scattering amplitudes.

The necessity for vacuum transitions in any scattering process follows from the constraint equations on \mathcal{I}^+

$$\partial_u F_{ru} + D^z F_{uz} + D^{\bar{z}} F_{u\bar{z}} + e^2 j_u = 0 , \quad (5.2.15)$$

and \mathcal{I}^-

$$\partial_v F_{rv} - D^z F_{vz} - D^{\bar{z}} F_{v\bar{z}} - e^2 j_v = 0 . \quad (5.2.16)$$

Assuming that the electric field vanishes in the far past and far future and using the matching conditions⁷

$$F_{ru}|_{\mathcal{I}_-^+} = F_{rv}|_{\mathcal{I}_+^-} , \quad (5.2.17)$$

$$A_z|_{\mathcal{I}_-^+} = A_z|_{\mathcal{I}_+^-} , \quad (5.2.18)$$

the divergence of (5.2.14) (and its complex conjugate) is the sum of the integrals of (5.2.15) and (5.2.16).

Let us examine the classical electromagnetic field configuration needed to satisfy the constraints. A single charge Q_0 particle incoming at (v_0, z_0, \bar{z}_0) corresponds to

$$j_v = Q_0 \delta(v - v_0) \gamma^{z\bar{z}} \delta^{(2)}(z - z_0) . \quad (5.2.19)$$

We write the state consisting of one such particle in the $N_z^{\text{in}} = 0$ vacuum as

$$|z_0; 0\rangle , \quad N_z^- |z_0; 0\rangle = 0 . \quad (5.2.20)$$

⁷Here we consider theories with no magnetic charges so that $F_{z\bar{z}}|_{\mathcal{I}_-^+} = 0 = F_{z\bar{z}}|_{\mathcal{I}_+^-}$.

We can solve the constraints for finite z either using the Coulombic modes with⁸

$$F_{rv} = Q_0 e^2 \theta(v - v_0) \gamma^{z\bar{z}} \delta^{(2)}(z - z_0) , \quad (5.2.21)$$

or with the radiative modes⁹

$$A_z = -\frac{Q_0 e^2}{4\pi} \partial_z G(z, z_0) \theta(v - v_0) , \quad F_{vz} = -\frac{Q_0 e^2}{4\pi} \partial_z G(z, z_0) \delta(v - v_0) , \quad (5.2.22)$$

where

$$\partial_z \partial_{\bar{z}} G(z, w) = 2\pi \delta^{(2)}(z - w) . \quad (5.2.23)$$

For finite z the choice

$$G(z, w) = \ln |z - w|^2 \quad (5.2.24)$$

gives simply

$$A_z = -\frac{Q_0 e^2}{4\pi(z - z_0)} \theta(v - v_0) . \quad (5.2.25)$$

The purely Coulombic choice will violate the matching conditions (5.2.17) unless the outgoing state also has Coulomb fields at $z = z_0$, where there may not even be any particles on \mathcal{I}^+ . We first consider the radiative dressing (5.2.25). This potential is pure gauge except at advanced time $v = v_0$ where a radiative shock wave emerges. There is a shift in the flat gauge connection between the boundaries \mathcal{I}_+^- and \mathcal{I}_-^- of \mathcal{I}^- given by $N_z^- = -\frac{Q_0 e^2}{4\pi(z - z_0)}$.

Of course more general solutions of the constraints, which do involve Coulomb fields, are possible and can be obtained by adding to (5.2.25) any solution of the source-free equation. Indeed, the difference between (5.2.21) and (5.2.25) is such a solution. We will return to the more general case in sections 5.5 and 5.6.

The Green function G in (5.2.24) leads to image charges at $z = \infty$. Since there are no physical

⁸ $\theta(v) = 1$ for $v > 0$ and vanishes otherwise.

⁹Note that the “soft charge” N_z^- vanishes for the Coulombic dressing, while for the radiative dressing all Q_ε^- reduce to multiples of the global charge for which $\varepsilon = 1$.

charges presumed at this point and we must preserve the constraints, delta function “wires” of nonzero F_{vr} are added connecting the images at various values of v_k where particles enter. One such wire with net integral $\sum_{k \in \text{in}} Q_k$ will cross to \mathcal{I}^+ . Overall charge conservation guarantees, if a similar construction is used to satisfy the \mathcal{I}^+ constraints, that this will match with the $z = \infty$ wire on \mathcal{I}^+ .

Of course these wires can be smoothed out by adding source-free solutions of the free Maxwell equation.¹⁰ For example we can use

$$G(z, w) = \ln [|z - w|^2 (1 + z\bar{z})^{-1} (1 + w\bar{w})^{-1}] , \quad (5.2.26)$$

which obeys $2\partial_z \partial_{\bar{z}} G = 4\pi \delta^{(2)}(z - w) - \gamma_{z\bar{z}}$. This effectively spreads the image charges, and along with them the F_{vr} flux wires required for their cancellation, evenly over the sphere.

For our purposes we are primarily interested in the structure near $z = z_0$ which has the same singularity for all the G ’s. The choice of G will not be central and we focus on the simplest one (5.2.24).

5.3 Dressed Quantum States

The story of dressed charges began with Dirac [114], who realized that part of the problem with the formulation of quantum electrodynamics was that conventional states for charged particles were not gauge invariant. Suppose one is considering the Dirac field for an electron, $\psi(x)$. Then one usually thinks of the operator $\psi(x)$ as creating an electron at the point x . In classical physics, if one makes a gauge transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu \varepsilon(x) , \quad (5.3.1)$$

¹⁰There may be a preferred Lorentz covariant dressing if the charged particles are taken as conformal primaries rather than plane waves as in [113].

then the gauge transformation of a field with charge Q_0 is

$$\psi(x) \rightarrow e^{iQ_0\epsilon(x)}\psi(x) . \quad (5.3.2)$$

Dirac made a field invariant under gauge transformations which die at infinity by introducing a dressing of the charged particle. One replaces $\psi(x)$ by the gauge invariant $\psi^*(x)$ defined by

$$\psi^*(x) = \psi(x)e^{iQ_0 \int A^\mu(x') C_\mu(x') d^4x'} . \quad (5.3.3)$$

$C_\mu(x)$ is then required to obey the equation

$$\partial_\mu C^\mu = \delta^{(4)}(x - x') . \quad (5.3.4)$$

Solutions to this equation are of course not unique, since we are free to add any solution of the homogeneous equation. Thus Dirac's prescription is not unique, and may involve either radiative or Coulomb modes depending on how C^μ is chosen.

In quantum field theory in the Schrödinger picture, operators are time-independent and so one would replace these expressions by the corresponding non-covariant forms in which only the spatial components of A_μ and C^μ are used. The integral is then taken over a three-dimensional spatial section of spacetime and the four-dimensional delta function is replaced by the three-dimensional delta function. Thus

$$\psi^*(x) = \psi(x)e^{iQ_0 \int A^i(x') C_i(x') d^3x'} \quad (5.3.5)$$

and

$$\partial_i C^i = \delta^{(3)}(x - x') . \quad (5.3.6)$$

This prescription replaces the bare electron by an electron together with an electromagnetic cloud. It is important to note that the operator (5.3.3) is only invariant under small gauge transformations vanishing sufficiently quickly at infinity, since an integration by parts is needed in order to demonstrate invariance. Under large gauge transformations, Dirac's operators transform with a

phase, as charged operators should. Dirac provided an illuminating example of a C_i that satisfies $\partial_i C^i = \delta^{(3)}(x - x')$. This is just

$$C_i = -\partial_i \left\{ \frac{1}{4\pi|x - x'|} \right\}. \quad (5.3.7)$$

C_i is then just the electric Coulomb field of a point charge.

In the full interacting quantum theory, the radiative modes of the electromagnetic field obey the exact \mathcal{I}^- commutator

$$[A_w(v, w, \bar{w}), F_{v'\bar{z}}(v', z, \bar{z})] = \frac{ie^2}{2} \delta(v - v') \delta^{(2)}(w - z). \quad (5.3.8)$$

The commutators of Coulombic modes are then, according to Dirac, whatever they must be in order that the constraints (5.2.16) are satisfied. That is, the operator F_{rv} is defined by

$$F_{rv} \equiv \int_{-\infty}^v dv' (D^z F_{v'z} + D^{\bar{z}} F_{v'\bar{z}} + e^2 j_{v'}) , \quad (5.3.9)$$

where the constant of integration is set (for massless charges only) by demanding that the Coulomb field vanish in the far past. Its commutators are then computed using (5.3.8) along with those for the matter fields appearing in j_v .

The coherent quantum state corresponding to (5.2.25) is, up to a large gauge transformation,

$$\begin{aligned} |z_0; 0\rangle_{\text{dressed}} &\equiv e^{iR_0} |z_0; 0\rangle, \\ R_0 &\equiv \frac{Q_0}{2\pi} \int d^2w \gamma_{w\bar{w}} G(z_0, w) D \cdot A(v_0, w, \bar{w}), \end{aligned} \quad (5.3.10)$$

where $D \cdot A \equiv D^w A_w + D^{\bar{w}} A_{\bar{w}}$. We may describe this as a charged particle surrounded by a cloud of soft photons. It easily follows from (5.3.8) and (5.3.9) that $|z_0; 0\rangle_{\text{dressed}}$ obeys the constraints without any Coulombic F_{vr} wires extending out of the charge, and the matching condition (5.2.17)

is trivially satisfied. The dressing shifts the action of F_{vz} on states by

$$[F_{vz}(v, z, \bar{z}), iR_0] = -\frac{Q_0 e^2 \delta(v - v_0)}{4\pi(z - z_0)} . \quad (5.3.11)$$

We see explicitly that the early and late vacua on \mathcal{I}^- differ by a large gauge transformation and

$$N_z^- |z_0; 0\rangle_{\text{dressed}} = -\frac{Q_0 e^2}{4\pi(z - z_0)} |z_0; 0\rangle_{\text{dressed}} . \quad (5.3.12)$$

Had we started with a vacuum state with nonzero N_z^{in} , the dressing would have simply shifted the eigenvalue.

The dressed single particle state (5.3.10) is easily generalized to a multiparticle state

$$\begin{aligned} |\text{in}; 0\rangle_{\text{dressed}} &\equiv e^{iR} |\text{in}; 0\rangle , \\ R &\equiv \frac{1}{2\pi} \int dv d^2 w \gamma_{w\bar{w}} d^2 z \gamma_{z\bar{z}} j_v(v, z, \bar{z}) G(z, w) D \cdot A(v, w, \bar{w}) . \end{aligned} \quad (5.3.13)$$

Also, for outgoing states

$$\langle \text{out}; 0 |_{\text{dressed}} \equiv \langle \text{out}; 0 | e^{-iR} . \quad (5.3.14)$$

The dressed states accompany the charges with nonzero eigenvalues for the soft photon operator,

$$\begin{aligned} N_z^- |\text{in}; 0\rangle_{\text{dressed}} &= -\sum_{k \in \text{in}} \frac{Q_k e^2}{4\pi(z - z_k)} |\text{in}; 0\rangle_{\text{dressed}} , \\ \langle \text{out}; 0 |_{\text{dressed}} N_z^+ &= -\langle \text{out}; 0 |_{\text{dressed}} \sum_{k \in \text{out}} \frac{Q_k e^2}{4\pi(z - z_k)} . \end{aligned} \quad (5.3.15)$$

In particular, the eigenvalues automatically obey the selection rule (5.2.14)

$$N_z^+ - N_z^- = \Omega_z^{\text{soft}} , \quad (5.3.16)$$

so that generically

$$\langle \text{out}; 0 |_{\text{dressed}} \mathcal{S} | \text{in}; 0 \rangle_{\text{dressed}} \neq 0 . \quad (5.3.17)$$

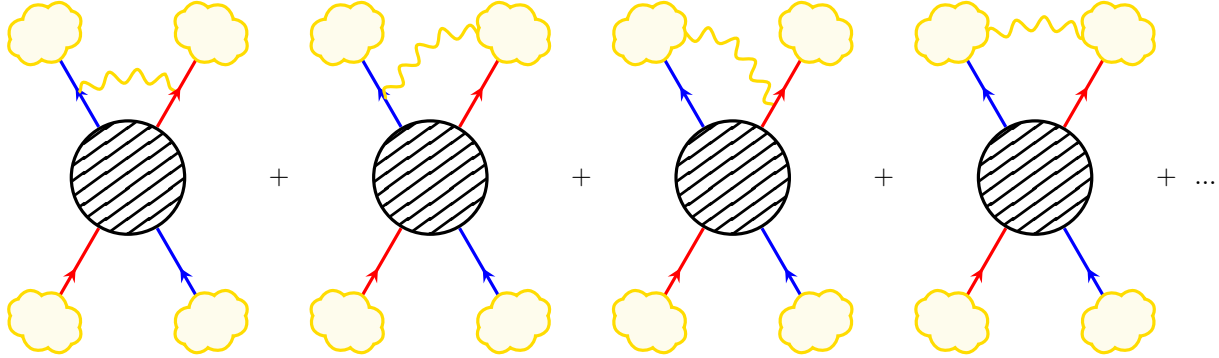


Figure 5.1: IR divergences arise from soft photon exchange between pairs of external charges. When the charges are dressed with appropriately correlated clouds of soft photons, these divergences are pairwise cancelled by exchanges involving the soft clouds.

In fact, the dressed amplitudes are free of IR divergences altogether. The basic mechanism is illustrated in figure 5.1. IR divergences arise from the exchange of soft photons between pairs of external legs. These exponentiate in such a way to cause ordinary Fock-basis amplitudes to vanish. However, when the charged particles are dressed by soft photon clouds, further divergences arise when a soft photon is exchanged between one external leg and the soft cloud surrounding the second external leg or between the pair of soft clouds. FK [110] showed that by a judicious choice of such a soft cloud one can arrange for the IR divergences to cancel and obtain an IR finite \mathcal{S} -matrix. In the next section we show that our dressed states differ from those of FK only by terms which are subleading in the IR, and therefore effect the same IR cancellations.

5.4 Faddeev-Kulish States

Faddeev and Kulish [110], building on Dirac and others [106–109], developed a scheme for dressing charged particles which eliminates IR divergences. Their starting point was to argue that the LSZ procedure for identifying asymptotic states is inapplicable in quantum electrodynamics: since the electromagnetic interaction has infinite range, there can be no isolated interaction region. They

resolved this by observing that the action for charged particles contains a term

$$\int J^\mu A_\mu d^4x , \quad (5.4.1)$$

where J^μ is the electromagnetic current. Since this current is conserved, the action is gauge invariant provided appropriate boundary conditions hold for the gauge transformations. This is a special case of Dirac's treatment which leads to a collection of soft photons accompanying any charged particle.

If one studies the state for a single electron of three-momentum p^i , then in the eikonal approximation the current is the classical current of a single charged particle located at $x^i = p^i t/m$. FK dressed a single particle charged state $|\vec{p}\rangle$ with the associated soft cloud

$$|\vec{p}\rangle_{FK} = \exp \left[-\frac{eQ_0}{(2\pi)^3} \int \frac{d^3q}{2q_0} (f^\mu a_\mu^\dagger(\vec{q}) - f^{*\mu} a_\mu(\vec{q})) \right] |\vec{p}\rangle , \quad f^\mu = \left[\frac{p^\mu}{p \cdot q} - c^\mu \right] e^{i \frac{p \cdot k}{p_0} t} , \quad (5.4.2)$$

where c^μ satisfies $c \cdot q = 1$, $c^2 = 0$. In fact, they demonstrated that in order to cancel infrared divergences, it is sufficient to choose an arbitrary dressing

$$f^\mu = \left[\frac{p^\mu}{p \cdot q} - c^\mu \right] \psi(p, q) , \quad (5.4.3)$$

with the condition that $\psi(p, q) = 1$ in a neighborhood of $q = 0$. For a multi-particle state with zero net total charge and minimal dressing $\psi = 1$, the c^μ terms cancel out of the dressing function and we can deal solely with the dressing factor

$$f^\mu = \frac{p^\mu}{p \cdot q} . \quad (5.4.4)$$

We wish to rewrite (5.4.2) in terms of asymptotic fields at \mathcal{I} . The plane-wave expansion of the radiative modes of the electromagnetic potential is given by

$$A_\nu(x) = e \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_\nu^{*\alpha}(\vec{q}) a_\alpha^{\text{in}}(\vec{q}) e^{iq \cdot x} + \varepsilon_\nu^\alpha(\vec{q}) a_\alpha^{\text{in}}(\vec{q})^\dagger e^{-iq \cdot x} \right] , \quad (5.4.5)$$

where $q^2 = 0$, the two polarization vectors satisfy a normalization condition $\varepsilon_\alpha^\nu \varepsilon_{\beta\nu}^* = \delta_{\alpha\beta}$, and

$$\left[a_\alpha^{\text{in}}(\vec{p}), a_\beta^{\text{in}}(\vec{q})^\dagger \right] = \delta_{\alpha\beta} (2\pi)^3 2\omega_q \delta^{(3)}(\vec{p} - \vec{q}) . \quad (5.4.6)$$

Null momenta can be characterized by a point on the asymptotic S^2 and an energy ω

$$q^\mu = \frac{\omega}{1 + z\bar{z}} (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) = (\omega, q^1, q^2, q^3) . \quad (5.4.7)$$

The expansion for the spatial part of the plane wave is given by

$$e^{i\vec{q} \cdot \vec{x}} = \sum_l i^l (2l + 1) j_l(qr) P_l(\cos \gamma) , \quad (5.4.8)$$

where j_l are the spherical Bessel functions, γ is the angle between \vec{q} and \vec{x} , and $q = |\vec{q}|$. The asymptotic form of $j_l(qr)$, for $qr \gg 1$, is given by

$$j_l(qr) \sim \frac{1}{qr} \sin(qr - \frac{1}{2}\pi l) , \quad (5.4.9)$$

yielding an approximation that localizes the momentum of the gauge field in the optical direction

$$\begin{aligned} A_z^{(0)}(v, z, \bar{z}) &= \lim_{r \rightarrow \infty} A_z(v, r, z, \bar{z}) \\ &= -\frac{i}{8\pi^2} \frac{\sqrt{2}e}{1 + z\bar{z}} \int_0^\infty d\omega \left[a_+^{\text{in}}(\omega \hat{x}) e^{-i\omega v} - a_-^{\text{in}}(\omega \hat{x})^\dagger e^{i\omega v} \right] , \quad \hat{x} = \hat{x}(z, \bar{z}) . \end{aligned} \quad (5.4.10)$$

One then finds

$$|\vec{p}\rangle_{FK} = \exp \left[\frac{i}{2\pi} \int dv d^2w \gamma_{w\bar{w}} d^2z \gamma_{z\bar{z}} j_v(v, z, \bar{z}) G(z, w) D \cdot A(0, w, \bar{w}) \right] |\vec{p}\rangle . \quad (5.4.11)$$

This is almost the same as our dressed state (5.3.10), except that the soft cloud in an FK state is always at $v = 0$, whereas in (5.3.10) it appears at the same advanced time v_0 as the charged

particle.¹¹ This difference is subleading in ω , since leading order quantities at $\omega = 0$ are insensitive to null separations. In fact, with the natural choice

$$\psi = e^{i\omega v_0} , \quad (5.4.12)$$

which clearly satisfies $\psi = 1$ near $\omega = 0$, one obtains precisely the dressing in section 5.3. Since the two dressings differ only by terms higher order in ω , both implement the all-orders cancellation of IR divergences established by FK.

5.5 Massive Particles

In this section the discussion is generalized to massive particles. The soft factor Ω_z^{soft} , which determines the change in the vacuum state, is given for massless particles in (5.2.8) in terms of the points z_k at which they exit or enter the celestial sphere. Such a formula cannot exist for massive particles in eigenstates with momentum p_k^μ as they never reach null infinity. Instead, a massive particle is characterized by a point on the unit 3D hyperboloid H_3 which may be parameterized by

$$\hat{p}^\mu = \frac{p^\mu}{m} , \quad \hat{p}^2 = -1 . \quad (5.5.1)$$

As described in [80, 115], the contribution to Ω_z^{soft} from such a particle is proportional to

$$G_z(\hat{p}_k) = \int d^2w G(w, \bar{w}; \hat{p}_k) \frac{1}{w - z} , \quad (5.5.2)$$

where G here is the bulk-to-boundary propagator on H_3 obeying $\square G = 0$. If we infinitely boost \hat{p} , G reduces to a boundary delta function. Hence, in analogy with (5.3.10), in order to prevent IR

¹¹ The FK states can still satisfy the constraints, at the price of Coulomb fields extending from $v = 0$ to the locations of the charged particles.

divergences from setting the amplitudes to zero, we should dress such massive particle states as

$$\begin{aligned} |\hat{p}_1, \dots; 0\rangle_{\text{dressed}} &\equiv e^{iR_m} |\hat{p}_1, \dots; 0\rangle, \\ R_m &\equiv \sum_{k \in \text{in}} \frac{Q_k}{2\pi} \int d^2z (G_z(\hat{p}_k) A_{\bar{z}}(0, z, \bar{z}) + h.c.) , \end{aligned} \quad (5.5.3)$$

where to avoid separate discussion of the zero-mode we restrict to the special case $\sum_{k \in \text{in}} Q_k = 0$ with zero net charge. It is straightforward to show that this is precisely the FK state given by (5.4.4) and therefore has IR finite scattering amplitudes.

Unlike the massless case studied above, this construction gives Coulomb fields. One finds

$$[D^z F_{vz} + D^{\bar{z}} F_{v\bar{z}}, iR_m] = -\delta(v) \gamma^{z\bar{z}} \sum_{k \in \text{in}} Q_k e^2 G(z, \bar{z}, \hat{p}_k) . \quad (5.5.4)$$

This is a radiative shock wave coming out at $v = 0$. As there are no charged particles incoming at $v = 0$, the constraints then imply that $F_{rv}(z, \bar{z})$ must shift by $-\gamma^{z\bar{z}} \sum_{k \in \text{in}} Q_k e^2 G(z, \bar{z}, \hat{p}_k)$ at $v = 0$. This is precisely the (negative of the) asymptotic incoming Coulomb field in the absence of any radiation, associated with a collection of incoming massive point particles with momentum $m\hat{p}_k$. The constant part of F_{rv} is fixed by demanding that near \mathcal{I}_-^- it equal the Coulomb field sourced by the massive charges entering through past timelike infinity i^- . This gives

$$F_{rv} = \theta(-v) \gamma^{z\bar{z}} \sum_{k \in \text{in}} Q_k e^2 G(z, \bar{z}, \hat{p}_k) . \quad (5.5.5)$$

The effect of the radiative shock wave is to set to zero the Coulomb fields after $v = 0$ (note we are considering zero net charge). Similarly, fixing the integration function for F_{ru} with a boundary condition at \mathcal{I}_+^+ , the corresponding out-state has no Coulomb fields before $u = 0$. This is illustrated in figure 5.2. Since we then have

$$F_{ru}|_{\mathcal{I}_+^+} = 0 = F_{rv}|_{\mathcal{I}_+^-} , \quad (5.5.6)$$

this construction guarantees that the matching condition (5.2.17) is trivially satisfied and the amplitude need not vanish. Had we not restricted to the zero charge sector, the boundary field

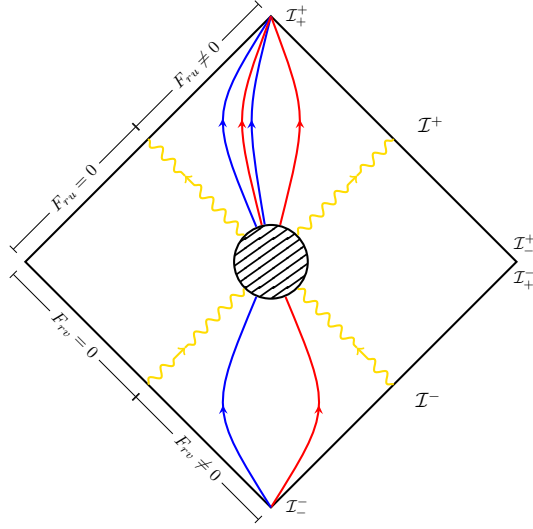


Figure 5.2: Dressed massive particles of zero net charge come in from i^- , scatter and go out to i^+ . The Faddeev-Kulish dressing introduces radiative shock waves at $v = 0$ and $u = 0$ which cancel the asymptotic Coulomb fields of the particles for $v > 0$ and $u < 0$ respectively. For neutral scattering states the Coulomb field will vanish near spatial infinity while for charged ones it will be an angle-independent constant.

strengths (5.5.6) would be angle-independent constants.

Note that for massive charged particles in plane wave states, the soft photon cloud can never be “on top of” the particle. Massive particles go to timelike infinity, while radiative photons always disperse to null infinity. The simplest FK states have them coming out at $u = 0$, but exactly when they come out, or the fact that they come out before the charges themselves, is unimportant since only the leading IR behavior of the cloud is relevant to the cancellation of divergences.

5.6 Charged States

In this section we consider a more general class of physical states in which Coulomb fields persist to \mathcal{I}_-^+ and \mathcal{I}_+^- , the generic charges are nonzero and the matching conditions nontrivially satisfied.

The condition (5.5.6), which is obeyed by all FK states (with zero net charge) implies that the

in- and out-charges Q_ε^\pm vanish. Explicitly, one finds

$$Q^-|_{\varepsilon=\frac{1}{z-w}}|\text{in}; 0\rangle_{\text{dressed}} = \left(\frac{4\pi}{e^2} N_z^- - \sum_{k \in \text{in}} Q_k G_z(\hat{p}_k) \right) |\text{in}; 0\rangle_{\text{dressed}} = 0, \quad (5.6.1)$$

$$\langle \text{out}; 0 |_{\text{dressed}} Q^+|_{\varepsilon=\frac{1}{z-w}} = \langle \text{out}; 0 |_{\text{dressed}} \left(\frac{4\pi}{e^2} N_z^+ - \sum_{k \in \text{out}} Q_k G_z(\hat{p}_k) \right) = 0. \quad (5.6.2)$$

That is, the incoming (outgoing) soft photon cloud shields all of the nonzero-mode charges of the incoming (outgoing) charged particles. Since the charges all commute with $a_\pm^{\text{in}\dagger}(\vec{k})$, adding radiative photons will not change the charges.¹²

In contrast to the situation for FK states, nonzero-mode Q_ε^\pm charges are generically all nonvanishing in the real world. Consider for example e^+e^- scattering in the center-of-mass frame with incoming velocities $\pm\vec{v}$, unaccompanied by incoming radiation. The coefficient of the $\frac{1}{r^2}$ radial electric field is given by the Liénard-Wiechert formula

$$F_{rv} = \frac{e^2(1-\vec{v}^2)}{4\pi(1+\hat{x}\cdot\vec{v})^2} - \frac{e^2(1-\vec{v}^2)}{4\pi(1-\hat{x}\cdot\vec{v})^2} = -\frac{e^2}{\pi} \frac{\hat{x}\cdot\vec{v}(1-\vec{v}^2)}{(1-(\hat{x}\cdot\vec{v})^2)^2}. \quad (5.6.3)$$

The charges constructed from this are nonzero:

$$Q^-|_{\varepsilon=\frac{1}{z-w}} = - \int_{\mathcal{I}_+^-} d^2w \gamma_{w\bar{w}} \frac{1}{z-w} \frac{1}{\pi} \frac{\hat{x}\cdot\vec{v}(1-\vec{v}^2)}{(1-(\hat{x}\cdot\vec{v})^2)^2} = Q^+|_{\varepsilon=\frac{1}{z-w}} = G_z(\hat{p}_-) - G_z(\hat{p}_+), \quad (5.6.4)$$

where \hat{p}_- (\hat{p}_+) is the momentum of the electron (positron). Nonzero Q_ε^\pm charges can be generically sourced by radiative Maxwell fields and arise even in the absence of charged particles. Source free initial data for the Maxwell equation is given by specifying an arbitrary function $F_{vz}(v, z, \bar{z})$ on \mathcal{I}^- . Assuming that F_{rv} vanishes in the far past, one has

$$F_{rv}(z, \bar{z})|_{\mathcal{I}_+^-} = \int_{-\infty}^{\infty} dv \left(D^z F_{vz}(v, z, \bar{z}) + D^{\bar{z}} F_{v\bar{z}}(v, z, \bar{z}) \right). \quad (5.6.5)$$

¹²Assuming the frequency distribution does not have poles or other singularities for $\omega \rightarrow 0$. If it does have such poles, new IR singularities may appear, and the state would not be among those shown by FK to be IR finite.

Demanding that the right hand side vanishes is a nonlocal constraint on the incoming initial data. Such nonlocal constraints are indeed imposed on FK states, which are dressed charged particles plus radiative modes. As mentioned above, the frequency-space coefficients of the field operators A_z are (except for the charge dressings) presumed to be finite for $\omega \rightarrow 0$, which precisely imposes the nonlocal constraint on the field strength that the integral (5.6.5) vanish.

Quantum states which describe these physical situations with nonzero Q_ε^\pm charges certainly exist, even if they are not FK states.¹³ It is natural to ask if such states can ever have IR finite scattering amplitudes. Given our earlier argument that the true role of IR divergences is simply to enforce conservation of all the charges, one might expect it to be possible. We now propose that this is indeed the case.

The basic idea is that the soft photon clouds and charged particles can be separated without affecting the IR cancellation mechanism of FK, even if we move a particle from incoming to outgoing, leaving its cloud intact. However, moving a charged particle from incoming to outgoing will in general take a zero-charge FK state to one with all charges excited.

Let's consider Bhabha scattering as a specific example

$$e^+e^- \rightarrow e^+e^- , \tag{5.6.6}$$

where the incoming and outgoing charges are all given FK dressings. Then there will be an incoming wave of photons shielding the incoming charges at $v = 0$ and an outgoing one at $u = 0$ shielding the outgoing charges. The long range fields will be angle-independent, in contrast to (5.6.3). The scattering is IR finite, with IR divergences from soft photon exchanges between pairs of external charges canceled by divergences from soft photon exchanges between external particles and radiative clouds and pairs of radiative clouds. This was depicted in figure 5.1.

Now, let us move the outgoing positron to an ingoing electron with the same momentum, and

¹³We note that all the nonzero-mode charges can be shielded, classically or quantum mechanically, by a correlated cloud of very soft radiation with arbitrarily small energy at arbitrarily large radius. In this sense, any state can be approximated by an FK state.

add a radiative photon to the out-state to conserve energy and momentum:

$$e^+ e^- e^- \rightarrow e^- + \gamma . \quad (5.6.7)$$

This has no effect on the soft factor, since the motion from out to in and the change in the sign of the charge each contribute a factor minus one. The same $(-1)^2 = 1$ applies to the leading IR divergence of an attached soft photon and ensures that these soft exchanges will continue to cancel. See figure 5.3.

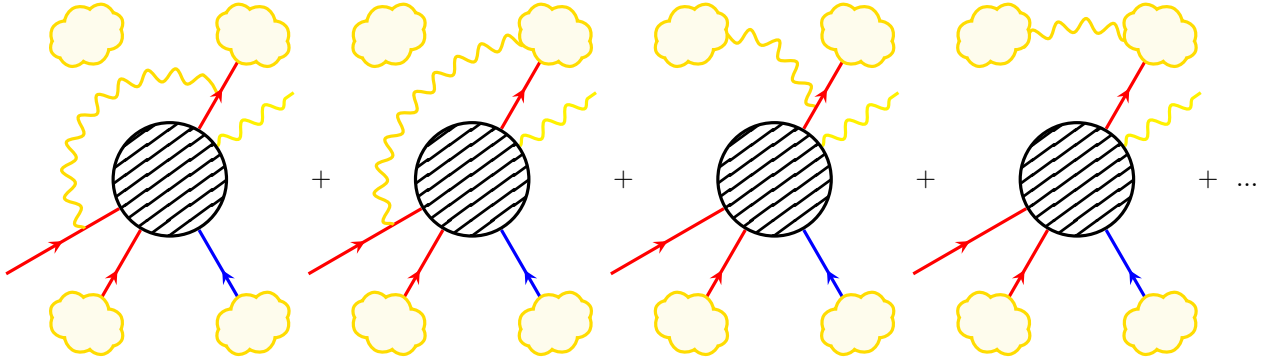


Figure 5.3: In this figure, the outgoing positron in figure 5.1 is crossed to an incoming electron, but its associated soft photon cloud remains as part of the out-state. The leading IR divergences from the depicted soft photon exchange still cancel, even though the in- and out-states carry nontrivial Q_ε^\pm charges and are no longer Faddeev-Kulish states.

Since we have also done nothing to N_z^\pm , the conservation law (5.3.16) remains satisfied. However, we have changed the charges. While they were previously zero, the contribution from the FK shield of the outgoing positron is no longer cancelled, while the new incoming electron does not have an FK shield. These give equal contributions to the incoming and outgoing charges

$$Q^\pm(z) = G_z(\hat{p}) . \quad (5.6.8)$$

Ultimately, one might hope to use crossing symmetry to prove IR finiteness in this context. FK infrared cancellations occur order by order in Feynman diagram perturbation theory, while crossing symmetry also holds in perturbation theory. The action of crossing a single outgoing particle to

an ingoing particle produces general Q_ϵ^\pm charges while changing FK to some more general class of states. We conjecture that scattering amplitudes among these more general charged states are IR finite.

6

Semiclassical Virasoro Symmetry of the Quantum Gravity \mathcal{S} -matrix

6.1 Introduction

BMS^+ transformations [1, 2] comprise a subset of diffeomorphisms which act nontrivially on future null infinity of asymptotically Minkowskian space times, or \mathcal{I}^+ . BMS^- transformations act isomorphically on past null infinity, or \mathcal{I}^- . A particular “diagonal” subgroup of the product group $\text{BMS}^+ \times \text{BMS}^-$ has recently been shown [30] to be a symmetry of gravitational scattering. Ward identities of this diagonal symmetry relate \mathcal{S} -matrix elements with and without soft gravitons. These \mathcal{S} -matrix relations are not new [31]: they comprise Weinberg’s soft graviton theorem [4]. More generally, the connection to soft theorems provides a new perspective on asymptotic symmetries in Minkowski space [32].

Over the decades a number of extensions/modifications to the BMS group have been proposed: e.g. the Newman-Unti group [116], the Spi group [22] and the extended BMS group [10–13]. A criterion is needed to decide whether or not such extensions are “physical.” Here we adopt the pragmatic approach that a Minkowskian asymptotic symmetry is physical if and only if it provides nontrivial relations among \mathcal{S} -matrix elements. We will view these \mathcal{S} -matrix relations as a definition of the symmetry.

In this chapter we will show that, at tree-level, quantum gravity in asymptotically Minkowskian spaces in this sense has a physical Virasoro symmetry. The symmetry is implied by a recently proven soft theorem [15] and acts (diagonally) on the conformal S^2 at \mathcal{I}^+ .

Our story begins with a conjecture of Barnich, Troessaert and Banks (BTB) [10–13]. BMS^+ has an $SL(2, \mathbb{C})$ Lorentz subgroup generated by the six global conformal Killing vectors (CKVs) on the S^2 at \mathcal{I}^+ . Locally, BTB showed that all of the infinitely many CKVs preserve the same asymptotic structure at \mathcal{I}^+ and are hence also candidate asymptotic symmetry generators. This larger set of vector fields was a priori excluded in the original work of BMS, who demanded that they be nonsingular everywhere on S^2 . This restriction cuts the Virasoro group down to a mere $SL(2, \mathbb{C})$. BTB conjectured that the true asymptotic symmetry group of \mathcal{I}^+ is the “extended BMS^+ group” generated by all CKVs. However, it has not been clear if or in what sense the singular CKVs truly generate physical asymptotic symmetries.

Herein we consider, in the spirit of [30], a certain diagonal subgroup of (extended BMS^+) \times (extended BMS^-), denoted \mathcal{X} . Ward identities are derived for a Virasoro subgroup of \mathcal{X} . They are found to involve a soft graviton insertion with the Weinberg pole projected out, leaving the finite subleading term in the soft expansion. These Ward identities are in turn shown to be implied by a conjectured [117] soft relation schematically of the form

$$\lim_{\omega \rightarrow 0} \mathcal{M}_{n+1} = S^{(1)} \mathcal{M}_n . \quad (6.1.1)$$

Here \mathcal{M}_{n+1} is an $n + 1$ -particle amplitude with a certain (pole-projected) energy ω soft graviton insertion, and $S^{(1)}$ involves the soft graviton momentum as well as the energies and angular mo-

menta of the incoming and outgoing particles. Details are given below. The proof [15] of (6.1.1) for tree-level gravity amplitudes then implies a semiclassical Virasoro symmetry for the case of pure gravity. This demonstrates that the singularities in the generic CKVs do not, at least in this context, prevent them from generating physical symmetries.¹

One might also hope to run the argument backwards and see to what extent the Virasoro symmetry of the \mathcal{S} -matrix implies the soft relation (6.1.1). In the case of supertranslations, the argument can be run in both directions [31]. However, here we encounter several obstacles, including the need for a prescription for handling the CKV singularities and some zero-mode issues. We leave this to future investigations.² Hence, at this point the existence of a Virasoro symmetry is potentially a *weaker* condition than the validity of the soft relation (6.1.1).

The analysis of [30, 31] related two structures which have been well-established and thoroughly studied over the last half-century: BMS symmetry and Weinberg’s soft graviton theorem. Here the situation is rather different. We are relating two unestablished and understudied structures: asymptotic Virasoro symmetries and subleading soft graviton theorems. We hope the relation will illuminate both. In any case it is a rather different enterprise!

An important issue which we will not address is the quantum fate of the semiclassical Virasoro symmetry. Here the situation is currently up in the air. In [66, 67] it was shown that, in a standard regulator scheme, (6.1.1) receives IR divergent quantum corrections (at one loop only), which also make the \mathcal{S} -matrix ill-defined in this scheme. However in [68], the factor $S^{(1)}$ in (6.1.1) relating the 5 and 4 point amplitude was found to remain uncorrected at one loop in a scheme with the soft limit taken prior to removing the IR cutoff.³ In the recent work [120, 121] (see also [122]) it

¹ It may alternatively be possible to reach this conclusion without appealing to direct computations, such as in [15], by carefully regulating the singularities and analyzing their effects. We do not attempt such an analysis herein.

²The Virasoro charges constructed in [10–12] may be useful for this purpose.

³ [68] claims a result only for this one special case by direct computation. However, it has been suggested [118] that, using [119], a proof can be constructed in the scheme of [66, 67] that all loop corrections to $S^{(1)}$ in (6.1.1) are linked to discontinuities arising from infrared singularities and hence in the scheme of [68] (with the soft limit taken first) all loop corrections would disappear along with the discontinuities.

was shown that a properly defined \mathcal{S} -matrix utilizing the gravity version of the Kulish-Faddeev construction [110] is free of all IR divergences. This may be the proper context for the discussion, as it is hard to have a symmetry of an \mathcal{S} -matrix without an \mathcal{S} -matrix! Should it ultimately be found that (6.1.1) does receive scheme-independent corrections, one must then determine whether it implies a quantum anomaly in the asymptotic Virasoro symmetry (which is potentially weaker than (6.1.1)), or a quantum deformation in its action on the amplitudes. Clearly highly relevant, but not yet fully incorporated into this discussion, is the low-energy theorem of Gross and Jackiw [8] who use dispersion theory to show that there is no correction to the first three terms⁴ of the Born approximation to soft graviton-scalar scattering. This generalized the classic low-energy theorem for QED by Low [91]. Progress on the gravity version was recently made by White [122]. Clearly, there is much to understand!

The existence of a Virasoro symmetry potentially has far-reaching implications for Minkowski quantum gravity in general. However, at this point there are many basic unresolved points. For example we do not know if the symmetry has quantum anomalies, what kind of representations appear,⁵ the role of IR divergences or the connection to stringy Virasoro symmetries [32, 41]. Very recent developments indicate that these ideas, including the realization of the subleading soft theorem as a Virasoro symmetry, have a natural home in the twistor string [42, 43]. Since the symmetry acts at the boundary, it is likely relevant to any holographic duality as long ago envisioned in [27, 28, 79].

This chapter is organized as follows. Section 6.2 establishes notation and reviews a few salient formulae for asymptotically flat geometries. Section 6.3 describes the conjectured extended BMS^\pm symmetry following [10–12]. In section 6.4 we define the diagonal subgroup \mathcal{X} of (extended BMS^+) \times (extended BMS^-) transformations, review Christodoulou-Klainerman (CK) spaces, and define extended CK spaces by acting with \mathcal{X} . A prescription is given to define classical gravitational scattering from \mathcal{I}^- to \mathcal{I}^+ and shown to be symmetric under \mathcal{X} . In section 6.5 the discussion of

⁴ $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ in the notation of [68].

⁵They may not be the familiar ones from the study of unitary 2D CFT on the sphere.

the quantum theory begins with the the action of extended BMS^\pm generators on in- and out-states. A Ward identity is then derived which is equivalent to infinitesimal \mathcal{X} -invariance of \mathcal{S} . It relates amplitudes with and without a particular soft graviton insertion. Finally in section 6.6 we give the detailed form of the soft relation (6.1.1) and show that it implies the \mathcal{X} Ward identity.

6.2 Asymptotically Flat Geometry

6.2.1 Metrics

A general asymptotically flat metric can be expanded in $\frac{1}{r}$ around \mathcal{I}^+ . In retarded Bondi coordinates it takes the form⁶

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{uz}dudz + 2g_{u\bar{z}}dud\bar{z} + \dots, \quad (6.2.1)$$

where the first line is the flat Minkowski metric, $\gamma_{z\bar{z}}$ (D_z) is the round metric (covariant derivative) on the unit S^2 and

$$g_{uz} = \frac{1}{2}D^z C_{zz} + \frac{1}{6r}C_{zz}D_z C^{zz} + \frac{2}{3r}N_z + \mathcal{O}(r^{-2}). \quad (6.2.2)$$

The Bondi mass aspect m_B , the angular momentum aspect N_z and C_{zz} depend only on (u, z, \bar{z}) and not r . The outgoing news tensor is defined by

$$N_{zz} \equiv \partial_u C_{zz}. \quad (6.2.3)$$

\mathcal{I}^+ is the null surface $(r = \infty, u, z, \bar{z})$. We use the symbol \mathcal{I}_+^+ (\mathcal{I}_-^+) to denote the future (past) boundary of \mathcal{I}^+ at $(r = \infty, u = \infty, z, \bar{z})$ ($(r = \infty, u = -\infty, z, \bar{z})$). This is depicted in figure 6.1.

⁶We largely adopt the notation of [10–12] to which we refer the reader for further details.

There is an analogous construction on \mathcal{I}^- with the metric given by

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{vz}dvdz + 2g_{v\bar{z}}dv d\bar{z} + \dots, \quad (6.2.4)$$

with

$$g_{vz} = -\frac{1}{2}D^z D_{zz} - \frac{1}{6r}D_{zz}D_z D^{zz} - \frac{2}{3r}N_z^- + \mathcal{O}(r^{-2}). \quad (6.2.5)$$

The \mathcal{I}^- coordinate z in (6.2.4) is antipodally related to the \mathcal{I}^+ coordinate z in (6.2.1) in the sense that, for flat Minkowski space, a null geodesic begins and ends at the same value of z . Put another way, in the conformal compactification of asymptotically flat spaces, all of \mathcal{I} is generated by null geodesics which run through spatial infinity i^0 . These generators have the same constant z value on both \mathcal{I}^+ and \mathcal{I}^- . The incoming news tensor is defined by

$$M_{zz} \equiv \partial_v D_{zz}. \quad (6.2.6)$$

When expanding about flat Minkowski space we sometimes employ flat coordinates in which the flat metric takes the form

$$ds_F^2 = \eta_{\mu\nu}dx^\mu dx^\nu. \quad (6.2.7)$$

These are related to Bondi coordinates in flat space by

$$x^0 = u + r = v - r, \quad x^1 + ix^2 = \frac{2rz}{1 + z\bar{z}}, \quad x^3 = \frac{r(1 - z\bar{z})}{1 + z\bar{z}}. \quad (6.2.8)$$

6.2.2 Constraints

The data in (6.2.1) are related by the constraint equations $G_{\mu\nu} = \mathcal{T}_{\mu\nu}^M$, where $\mathcal{T}_{\mu\nu}^M$ is the matter stress tensor and we adopt units in which $8\pi G = 1$. The leading term in the expansion of the G_{uu}

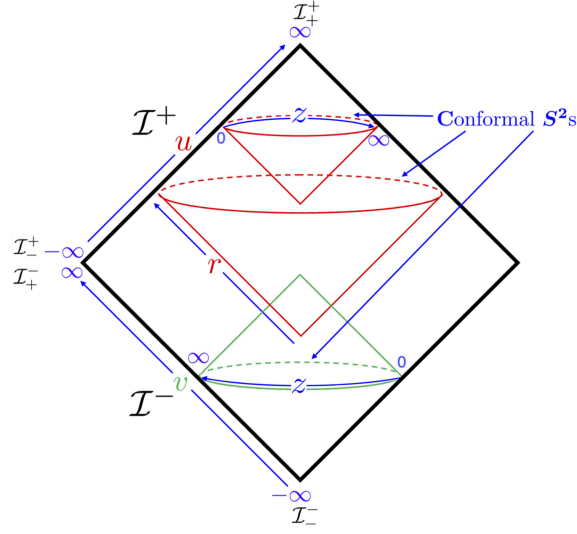


Figure 6.1: Penrose diagram for Minkowski space. Near \mathcal{I}^+ surfaces of constant retarded time u (red) are cone-like and intersect \mathcal{I}^+ in a conformal S^2 parametrized by (z, \bar{z}) . Cone-like surfaces of constant advanced time v (green) intersect \mathcal{I}^- in a conformal S^2 also parametrized by (z, \bar{z}) . The future (past) S^2 boundary of \mathcal{I}^+ is labelled \mathcal{I}_+^+ (\mathcal{I}_+^-), while the future (past) boundary of \mathcal{I}^- is labelled \mathcal{I}_-^+ (\mathcal{I}_-^-).

constraint equation about \mathcal{I}^+ is

$$\partial_u m_B = \frac{1}{4} D_z^2 N^{zz} + \frac{1}{4} D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2} T_{uu}^M - \frac{1}{4} N_{zz} N^{zz} , \quad (6.2.9)$$

where

$$T_{\mu\nu}^M(u, z, \bar{z}) = \lim_{r \rightarrow \infty} r^2 \mathcal{T}_{\mu\nu}^M(u, r, z, \bar{z}) \quad (6.2.10)$$

is the rescaled matter stress tensor which we have assumed falls off like $\frac{1}{r^2}$ near \mathcal{I}^+ . The G_{uz} constraint gives

$$\begin{aligned} \partial_u N_z = & -\frac{1}{4} (D_z D_{\bar{z}}^2 C^{\bar{z}\bar{z}} - D_z^3 C^{zz}) - T_{uz}^M + \partial_z m_B + \frac{1}{16} D_z \partial_u (C_{zz} C^{zz}) \\ & - \frac{1}{4} N^{zz} D_z C_{zz} - \frac{1}{4} N_{zz} D_z C^{zz} - \frac{1}{4} D_z (C^{zz} N_{zz} - N^{zz} C_{zz}) . \end{aligned} \quad (6.2.11)$$

Given the Bondi news, m_B , N_z and C_{zz} are all determined up to u -independent integration constants which are discussed below. The \mathcal{I}^- constraints are

$$\partial_v m_B^- = \frac{1}{4} D_z^2 M^{zz} + \frac{1}{4} D_{\bar{z}}^2 M^{\bar{z}\bar{z}} + \frac{1}{2} T_{vv}^M + \frac{1}{4} M^{zz} M_{zz} , \quad (6.2.12)$$

$$\begin{aligned} \partial_v N_z^- = & \frac{1}{4} (D_z D_{\bar{z}}^2 D^{\bar{z}\bar{z}} - D_{\bar{z}}^3 D^{zz}) - T_{vz}^M - \partial_z m_B^- + \frac{1}{16} D_z \partial_v (D_{zz} D^{zz}) \\ & - \frac{1}{4} M^{zz} D_z D_{zz} - \frac{1}{4} M_{zz} D_z D^{zz} - \frac{1}{4} D_z (D^{zz} M_{zz} - M^{zz} D_{zz}) . \end{aligned} \quad (6.2.13)$$

6.3 Extended BMS $^\pm$ Transformations

The extended BMS $^+$ group has been proposed [10–13] as the asymptotic symmetry group at \mathcal{I}^+ of gravity on asymptotically flat spacetimes. It is generated by vector fields ξ^+ that locally preserve the asymptotic form (6.2.1) of the metric at \mathcal{I}^+

$$\mathcal{L}_{\xi^+} g_{ur} = \mathcal{O}(r^{-2}) , \quad \mathcal{L}_{\xi^+} g_{uz} = \mathcal{O}(1) , \quad \mathcal{L}_{\xi^+} g_{zz} = \mathcal{O}(r) , \quad \mathcal{L}_{\xi^+} g_{uu} = \mathcal{O}(r^{-1}) . \quad (6.3.1)$$

All such vector fields near \mathcal{I}^+ are of the form

$$\begin{aligned} \xi^+ = & \left(1 + \frac{u}{2r}\right) Y^{+z} \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^{+z} \partial_{\bar{z}} - \frac{1}{2} (u + r) D_z Y^{+z} \partial_r + \frac{u}{2} D_z Y^{+z} \partial_u + c.c. \\ & + f^+ \partial_u - \frac{1}{r} (D^z f^+ \partial_z + D^{\bar{z}} f^+ \partial_{\bar{z}}) + D^z D_z f^+ \partial_r , \end{aligned} \quad (6.3.2)$$

where f^+ is an arbitrary function on S^2 and here and elsewhere we suppress (in some cases metric-dependent) terms which are further subleading in $\frac{1}{r}$ and irrelevant to our analysis: see [10–12] for a recent treatment specifying these terms. Y^+ must be a conformal Killing vector on S^2 which obeys the equation

$$\partial_{\bar{z}} Y^{+z} = 0 . \quad (6.3.3)$$

Globally there are six real vector fields in an antisymmetric matrix $Y_{\mu\nu}^z$ obeying (6.3.3):

$$\begin{aligned} Y_{12}^z &= iz, \quad Y_{13}^z = -\frac{1}{2}(1+z^2), \quad Y_{23}^z = -\frac{i}{2}(1-z^2), \\ Y_{03}^z &= z, \quad Y_{01}^z = -\frac{1}{2}(1-z^2), \quad Y_{02}^z = -\frac{i}{2}(1+z^2). \end{aligned} \quad (6.3.4)$$

These generate the six Lorentz boosts and rotations on \mathcal{I}^+ . Locally there are infinitely many solutions of the form $Y^z \sim z^n$ with poles somewhere on the sphere. In their original work [1, 2], BMS excluded these singular vector fields. However, in this chapter we shall explore the conjecture of [10–13] that all of these “superrotations” should be included as part of the asymptotic symmetry group.

The extended BMS^+ group is a semi-direct product of superrotations with supertranslations. The supertranslations were recently analyzed in [30, 31]. For notational brevity we henceforth consider only the superrotation subgroup which has $f^+ = 0$ in (6.3.2) and reduces to

$$\xi^+ = \left(1 + \frac{u}{2r}\right) Y^{+z} \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^{+z} \partial_{\bar{z}} - \frac{1}{2}(u+r) D_z Y^{+z} \partial_r + \frac{u}{2} D_z Y^{+z} \partial_u + c.c. . \quad (6.3.5)$$

This maps \mathcal{I}^+ to itself via

$$\xi^+|_{\mathcal{I}^+} = Y^{+z} \partial_z + \frac{u}{2} D_z Y^{+z} \partial_u + c.c. . \quad (6.3.6)$$

Similarly, on \mathcal{I}^- we have BMS^- vector fields parametrized by Y^-

$$\xi^- = \left(1 - \frac{v}{2r}\right) Y^{-z} \partial_z + \frac{v}{2r} D^{\bar{z}} D_z Y^{-z} \partial_{\bar{z}} - \frac{1}{2}(r-v) D_z Y^{-z} \partial_r + \frac{v}{2} D_z Y^{-z} \partial_v + c.c. . \quad (6.3.7)$$

Infinitesimal BMS^+ transformations act on the Bondi-gauge metric components as

$$\begin{aligned} \delta_{Y^+} C_{zz} &= \frac{u}{2} (D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}) \partial_u C_{zz} + \mathcal{L}_{Y^+} C_{zz} - \frac{1}{2} (D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}) C_{zz} - u D_z^3 Y^{+z}, \\ \delta_{Y^+} N_{zz} &\equiv \partial_u \delta C_{zz} = \frac{u}{2} (D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}) \partial_u N_{zz} + \mathcal{L}_{Y^+} N_{zz} - D_z^3 Y^{+z}. \end{aligned} \quad (6.3.8)$$

Similarly, at \mathcal{I}^-

$$\begin{aligned}\delta_{Y^-} D_{zz} &= \frac{v}{2} (D_z Y^{-z} + D_{\bar{z}} Y^{-\bar{z}}) \partial_v D_{zz} + \mathcal{L}_{Y^-} D_{zz} - \frac{1}{2} (D_z Y^{-z} + D_{\bar{z}} Y^{-\bar{z}}) D_{zz} + v D_z^3 Y^{-z} , \\ \delta_{Y^-} M_{zz} &= \frac{v}{2} (D_z Y^{-z} + D_{\bar{z}} Y^{-\bar{z}}) \partial_v M_{zz} + \mathcal{L}_{Y^-} M_{zz} + D_z^3 Y^{-z} .\end{aligned}\tag{6.3.9}$$

6.4 \mathcal{X} Transformations

BMS $^\pm$ symmetries act on the physical data at \mathcal{I}^\pm while preserving certain asymptotic structures such as the symplectic form [27, 28, 79]. They are not themselves symmetries of gravitational scattering: that is, given some solution (D_{zz}, C_{zz}) of the gravitational scattering problem, we cannot obtain a new one by acting with an element of BMS $^+$ or BMS $^-$. However, for the case of supertranslations, it was argued in [30] that a certain diagonal subgroup of BMS $^+ \times$ BMS $^-$ is a symmetry of gravitational scattering in a suitable neighborhood [58] of flat space. This subgroup is generated by pairs of $SL(2, \mathbb{C})$ Killing vector fields and supertranslations $(Y^+, f^+; Y^-, f^-)$ obeying

$$Y^{+z}(z) = Y^{-z}(z) \equiv Y^z(z) , \quad f^+(z, \bar{z}) = f^-(z, \bar{z}) \equiv f(z, \bar{z}) ,\tag{6.4.1}$$

with the understanding that the coordinate z is constant along null generators of \mathcal{I} as they pass from \mathcal{I}^- to \mathcal{I}^+ through spatial infinity i^0 in the conformal compactification of the spacetime. This means that points labelled by the same value of z on \mathcal{I}^- and \mathcal{I}^+ lie at antipodal angles from the origin. This antipodal identification may sound a little odd at first, but in fact is required in order for the subgroup (6.4.1) to contain the usual global Poincaré transformations.

In this chapter we are interested in *extended* BMS $^+ \times$ BMS $^-$ transformations. We denote by \mathcal{X} the subgroup of these transformations generated by vector fields asymptotic to (ξ^+, ξ^-) on $(\mathcal{I}^+, \mathcal{I}^-)$ subject to (6.4.1), where now Y^z is any of the infinitely many conformal Killing vectors on the sphere. Elements of \mathcal{X} transform a solution (D_{zz}, C_{zz}) of the gravitational scattering problem to a new one (D'_{zz}, C'_{zz}) with different final and initial data. We will argue below that the new data is a new solution of the scattering problem.

6.4.1 Christodoulou-Klainerman Spaces

We are interested in asymptotically flat solutions of the Einstein equation which revert to the vacuum in the far past and future. In particular we want to remain below the threshold for black hole formation. We will adopt the rigorous definition of such spaces given by Christodoulou and Klainerman (CK) [58] who also proved their global existence and analyzed their asymptotic behavior.

CK studied asymptotically flat initial data in the center-of-mass frame on a maximal spacelike slice for which the Bach tensor $\epsilon^{ijk} D_j^{(3)} G_{kl}^{(3)}$ of the induced three-metric decays like $r^{-7/2}$ (or faster) at spatial infinity and the extrinsic curvature like $r^{-5/2}$. This implies that in normal coordinates about infinity the leading part of the three-metric has the (conformally flat) Schwarzschild form, with corrections which decay like $r^{-3/2}$. CK showed that all such initial data which moreover satisfy a global smallness condition give rise to a global, i.e. geodesically complete, solution. We will refer to these solutions as CK spaces.

The smallness condition is satisfied in a finite neighborhood of Minkowski space, so this result established the stability of Minkowski space. Moreover, many asymptotic properties of CK spaces at null infinity were derived in detail. See [48] for a summary. Here we note that the Bondi news N_{zz} vanishes on the boundaries of \mathcal{I}^+ as

$$N_{zz}(u) \sim |u|^{-3/2} \quad (6.4.2)$$

or faster. Similarly on \mathcal{I}^- ,

$$M_{zz}(v) \sim |v|^{-3/2} \quad (6.4.3)$$

or faster. The Weyl curvature component Ψ_2^0 which in coordinates (6.2.1) is given by

$$\Psi_2^0(u, z, \bar{z}) \equiv - \lim_{r \rightarrow \infty} (r C_{u\bar{z}rz} \gamma^{z\bar{z}}) = -m_B - \frac{1}{4} C_{zz} N^{zz} + \frac{1}{4} (D^z D^z C_{zz} - D^{\bar{z}} D^{\bar{z}} C_{\bar{z}\bar{z}}) \quad (6.4.4)$$

obeys

$$\Psi_2^0|_{\mathcal{I}_+^+} = 0 , \quad (6.4.5)$$

while at $u = -\infty$

$$\Psi_2^0|_{\mathcal{I}_-^+} = -GM , \quad (6.4.6)$$

where G is Newton's constant and M is the ADM mass. Similar results pertain to \mathcal{I}^- .

In this chapter we consider generalizations of pure gravity, which include coupling massless matter which dissipates at late (early) times on \mathcal{I}^+ (\mathcal{I}^-) so that the system begins and ends in the vacuum. The CK analysis has not been fully generalized to this case, although there is no obvious reason analogs of (6.4.2)-(6.4.6) might not still pertain to a suitably defined neighborhood of the gravity+matter vacuum. In the absence of such a derivation (6.4.2)-(6.4.6) will simply be imposed, in the matter-coupled case, as restrictions on the solutions under consideration.

6.4.2 Classical Gravitational Scattering

The classical problem of gravitational scattering is to find the outgoing data at \mathcal{I}^+ resulting from the evolution of given data on \mathcal{I}^- . We take the incoming data to be $D_{zz}(v, z, \bar{z})$ and the outgoing data to be $C_{zz}(u, z, \bar{z})$. The remaining metric components on \mathcal{I} are then determined by constraints. We consider the geometries in the neighborhood of flat space defined by CK, which have $m_B = 0$ ($m_B^- = 0$) at \mathcal{I}_+^+ (\mathcal{I}_-^-). In particular, we remain below the threshold for black hole formation.

A CK geometry, as described in (t, r, θ, ϕ) coordinates, does not quite provide a solution to this scattering problem. To find the in (out) data, one must perform a coordinate transformation to ingoing (outgoing) Bondi coordinates and determine D_{zz} (C_{zz}). This procedure is not unique: the coordinate transformations are ambiguous up to extended BMS $^\pm$ transformations on \mathcal{I}^+ or \mathcal{I}^- . D_{zz} and C_{zz} are not invariant under these transformations. Hence, a solution of the scattering problem requires a prescription for fixing this ambiguity. A prescription to fix this ambiguity is to demand that

$$D_{zz}|_{\mathcal{I}_+^-} = C_{zz}|_{\mathcal{I}_-^+} = 0 . \quad (6.4.7)$$

It was shown in [30] that the falloffs (6.4.2)-(6.4.6) imply this is always possible. One may then integrate the constraint equations to determine D_{zz} and C_{zz} , which will not in general vanish at \mathcal{I}_+^+ and \mathcal{I}_-^- .

This prescription does not give all near-flat solutions of the scattering problem. Indeed, all such solutions are in the center-of-mass frame and have vanishing ADM three-momentum. However, given any solution of the scattering problem obeying (6.4.7), a new one with nonzero three-momentum may be obtained simply by acting with the boost element of \mathcal{X} . More generally, our prescription to define gravitational scattering is to take all solutions obtained by doing arbitrary \mathcal{X} transformations on the solutions obeying (6.4.7). We shall refer to such scattering geometries, complete with \mathcal{I}^\pm data, as *extended CK spaces*. Acting with an arbitrary finite conformal transformation $w(z)$ followed by an arbitrary finite supertranslation f on (6.4.7) leads to the asymptotic behaviors for large negative u and positive v ⁷

$$\begin{aligned} C_{ww}(u, w, \bar{w}) &\sim -2u (\partial_w z)^{1/2} \partial_w^2 (\partial_w z)^{-1/2} - 2D_w^2 f + \mathcal{O}(u^{-3/2}) , \\ D_{ww}(v, w, \bar{w}) &\sim 2v (\partial_w z)^{1/2} \partial_w^2 (\partial_w z)^{-1/2} + 2D_w^2 f + \mathcal{O}(v^{-3/2}) . \end{aligned} \quad (6.4.8)$$

We also have the relations at all the boundaries of \mathcal{I}^\pm

$$\begin{aligned} \partial_{\bar{z}} N_{zz}|_{\mathcal{I}_\pm^\pm} &= 0 , \\ \partial_{\bar{z}} M_{zz}|_{\mathcal{I}_\pm^\pm} &= 0 , \\ [D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}}]|_{\mathcal{I}_\pm^\pm} &= 0 , \\ [D_{\bar{z}}^2 D_{zz} - D_z^2 D_{\bar{z}\bar{z}}]|_{\mathcal{I}_\pm^\pm} &= 0 . \end{aligned} \quad (6.4.9)$$

⁷ Interestingly, the news tensor at the boundary \mathcal{I}_-^+ obeys $N_{ww}|_{\mathcal{I}_-^+} = -2(\partial_w z)^{1/2} \partial_w^2 (\partial_w z)^{-1/2}$, which is the transformation law for a 2D CFT stress tensor.

6.5 \mathcal{X} Ward Identity

6.5.1 Quantum States

In the quantum theory, incoming (outgoing) states on \mathcal{I}^- (\mathcal{I}^+) are presumed to form representations of extended BMS $^-$ (BMS $^+$). In this subsection we will describe the action of an infinitesimal Virasoro transformation δ_Y parameterized by Y^z on a generic Fock-basis in-state. For \mathcal{I}^- we define

$$Q^-(Y^-)|\text{in}\rangle = -i\delta_{Y^-}|\text{in}\rangle, \quad (6.5.1)$$

and similarly we define $Q^+(Y^+)$ on \mathcal{I}^+ .⁸ Q^- may be decomposed into a hard and soft part as

$$Q^- = Q_H^- + Q_S^-, \quad (6.5.2)$$

where Q_H^- generates the diffeomorphism $\xi^-(Y^-)$ on the incoming hard particles, and Q_S^- creates a soft graviton. Let us denote an in-state comprised of n particles with energies E_k incoming at points z_k for $k = 1, \dots, n$ on the conformal S^2 by

$$|z_1, z_2, \dots\rangle. \quad (6.5.3)$$

Then the hard action is simply to act with $\xi^{-\mu}\partial_{k\mu}$ on each scalar particle

$$Q_H^-|z_1, z_2, \dots\rangle = -i \sum_k \left(Y^{-z}(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^{-z}(z_k) \partial_{E_k} \right) |z_1, z_2, \dots\rangle, \quad (6.5.4)$$

Here, $-(1 + E_k \partial_{E_k})$ arises from the Fourier transform of $v \partial_v$, and the coefficient of $D_z Y^{-z}$ is shifted by one half as in [32] due to the $r \partial_r$ term in (6.3.7). For spinning particles we must replace

⁸ Explicit expressions for the proper BMS $^\pm$ charges as integrals of fields on \mathcal{I} were worked out in detail in [31] and shown to generate the proper BMS $^\pm$ symmetries. Expressions for the Virasoro charges Q^\pm are given in [10–12], but were not shown to generate the symmetries. In this chapter such explicit expressions will not be needed: transformation laws for the states suffice.

$Y^{-z}(z_k)\partial_{z_k}$ with the Lie derivative $\mathcal{L}_{Y^{-(z_k)}}$.⁹

To determine Q_S^- , note that the inhomogeneous transformation of the incoming Bondi news M_{zz} is

$$\delta_{Y^-} M_{zz}(v, z, \bar{z}) = D_z^3 Y^{-z} . \quad (6.5.5)$$

The action of Q_S^- on a state must implement this shift. It follows that

$$[Q_S^-, M_{zz}] = -i D_z^3 Y^{-z} . \quad (6.5.6)$$

Using the commutator [27, 28, 79]

$$[M_{\bar{z}\bar{z}}(v, z, \bar{z}), M_{ww}(v', w, \bar{w})] = 2i\gamma_{z\bar{z}}\delta^{(2)}(z-w)\partial_v\delta(v-v') , \quad (6.5.7)$$

one concludes that, up to a total derivative commuting with M_{zz} ,

$$Q_S^- = \frac{1}{2} \int_{\mathcal{I}^-} dv d^2z D_z^3 Y^{-z} v M_{\bar{z}}^z . \quad (6.5.8)$$

This reproduces the linear term in the full expression for the charge given in [10–12].¹⁰ Q_S^- is a zero-frequency operator (because of the v integral) linear in the metric fluctuation. Acting on the in-vacuum, it creates a soft graviton with polarization tensor proportional to $D_z^3 Y^{-z}$. The explicit form of the momentum space creation operator will be constructed below in subsection 6.5.3. Altogether then, Q^- maps the n -particle states into themselves plus an n -hard+1-soft state:

$$Q^-|z_1, z_2, \dots\rangle = -i \sum_{k=1}^n \left(Y^{-z}(z_k)\partial_{z_k} - \frac{E_k}{2} D_z Y^{-z}(z_k)\partial_{E_k} \right) |z_1, z_2, \dots\rangle + Q_S^-|z_1, z_2, \dots\rangle . \quad (6.5.9)$$

⁹More explicitly if we have a particle of helicity h , and Rindler energy $-iv\partial_v = E_R$, the parentheses in (6.5.4) are of the form $Y^z\partial_z + Y^{\bar{z}}\partial_{\bar{z}} + h_R D_z Y^z + h_L D_{\bar{z}} Y^{\bar{z}}$ where for helicity h , the “conformal weights” (see e.g. [32]) are $h_R = \frac{h}{2} - \frac{1}{2}E\partial_E = \frac{1}{2}(h+1+iE_R)$, $h_L = -\frac{h}{2} - \frac{1}{2}E\partial_E = \frac{1}{2}(-h+1+iE_R)$.

¹⁰The formula in [10–12] differs by a total derivative which improves the large $|v|$ behavior and may be essential in a more general context. The slightly simpler expression here is sufficient for the present purpose.

Similarly Virasoro transformations on \mathcal{I}^+ are decomposed as

$$Q^+ = Q_H^+ + Q_S^+, \quad (6.5.10)$$

and we denote out-states comprised of m particles with energies E_k outgoing at points z_k for $k = n+1, \dots, n+m$ by

$$\langle z_{n+1}, z_{n+2}, \dots |. \quad (6.5.11)$$

One finds

$$\begin{aligned} \langle z_{n+1}, z_{n+2}, \dots | Q^+ = i \sum_{k=n+1}^{n+m} \left(Y^{+z}(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^{+z}(z_k) \partial_{E_k} \right) \langle z_{n+1}, z_{n+2}, \dots | \\ + \langle z_{n+1}, z_{n+2}, \dots | Q_S^+, \end{aligned} \quad (6.5.12)$$

where

$$Q_S^+ = -\frac{1}{2} \int_{\mathcal{I}^+} du d^2z D_z^3 Y^{+z} u N_{\bar{z}}^z. \quad (6.5.13)$$

6.5.2 \mathcal{X} -Invariance of \mathcal{S}

In this section we derive a quantum Ward identity from the assumption that \mathcal{X} -invariance survives quantization. The quantum version of infinitesimal \mathcal{X} -invariance of classical gravitational scattering is, using (6.4.1)

$$\langle \text{out} | Q^+(Y) \mathcal{S} - \mathcal{S} Q^-(Y) | \text{in} \rangle = 0, \quad (6.5.14)$$

for any pair of in- and out-states ($|\text{in}\rangle, |\text{out}\rangle$). Let us define the normal-ordered soft graviton insertion

$$: Q_S(Y) \mathcal{S} := Q_S^+(Y) \mathcal{S} - \mathcal{S} Q_S^-(Y). \quad (6.5.15)$$

(6.5.14) is then the Ward identity

$$\begin{aligned} \langle z_{n+1}, z_{n+2}, \dots | : Q_S \mathcal{S} : | z_1, z_2, \dots \rangle = \\ -i \sum_{k=1}^{n+m} \left(Y^z(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} \right) \langle z_{n+1}, z_{n+2}, \dots | \mathcal{S} | z_1, z_2, \dots \rangle, \end{aligned} \quad (6.5.16)$$

where the k sum now runs over both in- and out-particles. For spinning particles the Lie derivative replaces the ordinary one on the right hand side. This relates the derivatives of any \mathcal{S} -matrix element to the same \mathcal{S} -matrix element with a particular soft graviton insertion.

6.5.3 Mode Expansions

We wish to express Q_S^\pm in terms of standard momentum space soft graviton creation and annihilation operators. The flat space graviton mode expansion is¹¹

$$h_{\mu\nu}^{\text{out}}(x) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha*}(\vec{q}) a_\alpha^{\text{out}}(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^\alpha(\vec{q}) a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq \cdot x} \right], \quad (6.5.17)$$

where $q^0 = \omega_q = |\vec{q}|$, $\alpha = \pm$ are the two helicities and

$$\left[a_\alpha^{\text{out}}(\vec{p}), a_\beta^{\text{out}}(\vec{q})^\dagger \right] = 2\omega_q \delta_{\alpha\beta} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \quad (6.5.18)$$

The outgoing gravitons with momentum q correspond to final-state insertions of $a_\alpha^{\text{out}}(\vec{q})$. It is convenient to parametrize the graviton four-momentum by (ω_q, w, \bar{w})

$$q^\mu = \frac{\omega_q}{1 + w\bar{w}} (1 + w\bar{w}, w + \bar{w}, i(\bar{w} - w), 1 - w\bar{w}), \quad (6.5.19)$$

¹¹Here, we take $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$.

with polarization tensors

$$\begin{aligned}\varepsilon^{\pm\mu\nu} &= \varepsilon^{\pm\mu} \varepsilon^{\pm\nu} , \\ \varepsilon^{+\mu}(\vec{q}) &= \frac{1}{\sqrt{2}} (\bar{w}, 1, -i, -\bar{w}) , \\ \varepsilon^{-\mu}(\vec{q}) &= \frac{1}{\sqrt{2}} (w, 1, i, -w) .\end{aligned}\tag{6.5.20}$$

These obey $\varepsilon^{\pm\mu} q_\mu = \varepsilon^{\pm\mu}{}_\mu = 0$ and

$$\varepsilon_{\bar{z}}^+(\vec{q}) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^+(\vec{q}) = \frac{\sqrt{2}r(1+z\bar{w})}{(1+z\bar{z})^2} , \quad \varepsilon_{\bar{z}}^-(\vec{q}) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^-(\vec{q}) = \frac{\sqrt{2}rz(w-z)}{(1+z\bar{z})^2} .\tag{6.5.21}$$

In retarded Bondi coordinates

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) = 2 \lim_{r \rightarrow \infty} \frac{1}{r} h_{\bar{z}\bar{z}}^{\text{out}}(r, u, z, \bar{z}) .\tag{6.5.22}$$

Using $h_{\bar{z}\bar{z}}^{\text{out}} = \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu h_{\mu\nu}^{\text{out}}$ and the mode expansion one finds

$$C_{\bar{z}\bar{z}} = 2 \lim_{r \rightarrow \infty} \frac{1}{r} \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha*}(\vec{q}) a_\alpha^{\text{out}}(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} + h.c. \right] ,\tag{6.5.23}$$

where θ is the angle between \vec{x} and \vec{q} . This integral is dominated for large r by the contribution near $\theta=0$:

$$C_{\bar{z}\bar{z}} = -\frac{i}{4\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \int_0^\infty d\omega_q \left[a_-^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_q u} - a_+^{\text{out}}(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right] .\tag{6.5.24}$$

Here, \hat{x} is parameterized by (z, \bar{z})

$$\hat{x} \equiv \frac{\vec{x}}{r} = \frac{1}{1+z\bar{z}} (z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z})\tag{6.5.25}$$

and

$$\hat{\varepsilon}_{\bar{z}\bar{z}}^+ = \frac{\partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu}{r^2} \varepsilon_{\mu\nu}^+ = \frac{2}{(1+z\bar{z})^2} .\tag{6.5.26}$$

Define

$$N_{\bar{z}\bar{z}}^\omega \equiv \int e^{i\omega u} \partial_u C_{\bar{z}\bar{z}} du . \quad (6.5.27)$$

Then from the large- r saddle point expansion of (6.5.23), we have

$$\begin{aligned} N_{\bar{z}\bar{z}}^\omega &= -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_-^{\text{out}}(\omega \hat{x}) , \\ N_{\bar{z}\bar{z}}^{-\omega} &= -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_+^{\text{out}}(\omega \hat{x})^\dagger , \end{aligned} \quad (6.5.28)$$

with $\omega > 0$ in both cases. We define $N_{\bar{z}\bar{z}}^{(1)}$ as

$$\begin{aligned} N_{\bar{z}\bar{z}}^{(1)} &\equiv \int du u N_{\bar{z}\bar{z}} \\ &= -\lim_{\omega \rightarrow 0} \frac{i}{2} (\partial_\omega N_{\bar{z}\bar{z}}^\omega + \partial_{-\omega} N_{\bar{z}\bar{z}}^{-\omega}) \\ &= \frac{i}{4\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left[a_-^{\text{out}}(\omega \hat{x}) - a_+^{\text{out}}(\omega \hat{x})^\dagger \right] . \end{aligned} \quad (6.5.29)$$

A mode expansion analogous to (6.5.24) can be defined for $D_{\bar{z}\bar{z}}$ on \mathcal{I}^-

$$D_{\bar{z}\bar{z}} = -\frac{i}{4\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \int_0^\infty d\omega_q \left[a_-^{\text{in}}(\omega_q \hat{x}) e^{-i\omega_q v} - a_+^{\text{in}}(\omega_q \hat{x})^\dagger e^{i\omega_q v} \right] , \quad (6.5.30)$$

from which we find

$$\begin{aligned} M_{\bar{z}\bar{z}}^\omega &= -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_-^{\text{in}}(\omega \hat{x}) , \\ M_{\bar{z}\bar{z}}^{-\omega} &= -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_+^{\text{in}}(\omega \hat{x})^\dagger , \end{aligned} \quad (6.5.31)$$

and

$$\begin{aligned} M_{\bar{z}\bar{z}}^{(1)} &= -\lim_{\omega \rightarrow 0} \frac{i}{2} (\partial_\omega M_{\bar{z}\bar{z}}^\omega + \partial_{-\omega} M_{\bar{z}\bar{z}}^{-\omega}) \\ &= \frac{i}{4\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left[a_-^{\text{in}}(\omega \hat{x}) - a_+^{\text{in}}(\omega \hat{x})^\dagger \right] . \end{aligned} \quad (6.5.32)$$

We are interested in the matrix element

$$\begin{aligned}
 & \langle \text{out} | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} + \mathcal{S} M_{\bar{z}\bar{z}}^{(1)} | \text{in} \rangle \\
 &= \frac{i}{4\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \langle \text{out} | \left(a_-^{\text{out}}(\omega \hat{x}) - a_+^{\text{out}}(\omega \hat{x})^\dagger \right) \mathcal{S} + \mathcal{S} \left(a_-^{\text{in}}(\omega \hat{x}) - a_+^{\text{in}}(\omega \hat{x})^\dagger \right) | \text{in} \rangle \quad (6.5.33) \\
 &= \frac{i}{4\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \langle \text{out} | a_-^{\text{out}}(\omega \hat{x}) \mathcal{S} - \mathcal{S} a_+^{\text{in}}(\omega \hat{x})^\dagger | \text{in} \rangle ,
 \end{aligned}$$

which is $\langle \text{out} | \mathcal{S} | \text{in} \rangle$ with soft graviton insertions.¹² Such insertions generically have Weinberg poles behaving as $\frac{1}{\omega}$. However the prefactor $1 + \omega \partial_\omega$ projects out this pole, leaving the subleading $\mathcal{O}(\omega^0)$ soft factor. Equation (6.5.33) and its hermitian conjugate are related to the Q_S matrix element by

$$\begin{aligned}
 & \langle \text{out} | : Q_S \mathcal{S} : | \text{in} \rangle \\
 &= -\frac{1}{2} \int d^2 z \gamma^{z\bar{z}} D_z^3 Y^z \langle \text{out} | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} + \mathcal{S} M_{\bar{z}\bar{z}}^{(1)} | \text{in} \rangle \quad (6.5.34) \\
 &= -\frac{i}{8\pi} \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \int d^2 z \gamma^{z\bar{z}} D_z^3 Y^z \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \langle \text{out} | a_-^{\text{out}}(\omega \hat{x}) \mathcal{S} - \mathcal{S} a_+^{\text{in}}(\omega \hat{x})^\dagger | \text{in} \rangle .
 \end{aligned}$$

Given the asymptotic behavior (6.4.8) near i^0 , the boundary relation $N_{\bar{z}\bar{z}}|_{\mathcal{I}_+^+} = -M_{\bar{z}\bar{z}}|_{\mathcal{I}_+^-}$ establishes a correspondence between the in- and out-modes, such that the contributions to the matrix element (6.5.34) from the $a_-^{\text{out}}(\omega \hat{x})$ and $-a_+^{\text{in}}(\omega \hat{x})^\dagger$ insertions are equal.

6.6 From Soft Theorem to Virasoro Symmetry

In this section we begin by assuming the subleading soft relation¹³

$$\lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \langle z_{n+1}, z_{n+2}, \dots | a_-(q) \mathcal{S} | z_1, z_2, \dots \rangle = S^{(1)-} \langle z_{n+1}, z_{n+2}, \dots | \mathcal{S} | z_1, z_2, \dots \rangle , \quad (6.6.1)$$

¹² Here, we assume that the $|\text{in}\rangle$ and $\langle \text{out}|$ states contain no soft gravitons.

¹³ A single soft graviton insertion has the ω expansion

$$\langle z_{n+1}, z_{n+2}, \dots | a_-(q) \mathcal{S} | z_1, z_2, \dots \rangle = \left(S^{(0)-} + S^{(1)-} \right) \langle z_{n+1}, z_{n+2}, \dots | \mathcal{S} | z_1, z_2, \dots \rangle + \mathcal{O}(\omega) .$$

with

$$S^{(1)-} = -i \sum_k \frac{p_{k\mu} \varepsilon^{-\mu\nu} q^\lambda J_{k\lambda\nu}}{p_k \cdot q} . \quad (6.6.2)$$

Here, $J_{k\lambda\nu} \equiv L_{k\lambda\nu} + S_{k\lambda\nu}$ is the total ingoing orbital + spin angular momentum of the k^{th} particle which obeys the global conservation law $\sum J_{k\lambda\nu} = 0$. We note the $(1 + \omega \partial_\omega)$ prefactor on the left hand side projects out the would-be Weinberg pole accompanying a soft insertion. For notational brevity we consider the contribution for negative polarization: the general formula has an $S^{(1)}$ with a general polarization tensor replacing (6.6.2). We will show that (6.6.1) implies the Ward identity (6.5.16), which in turn is equivalent to infinitesimal \mathcal{X} -invariance of the \mathcal{S} -matrix. Although the relation (6.6.1) potentially has wider validity, the only case in which it is known to be a theorem is for tree-level gravitons [15]. Hence, only for this case do we claim the results of this section imply a Virasoro symmetry.

Gauge invariance provides an important check on this formula. Amplitudes must vanish for pure gauge gravitons with polarizations

$$\varepsilon_\Lambda^{\mu\nu} = q^\mu \Lambda^\nu + q^\nu \Lambda^\mu \quad (6.6.3)$$

for any Λ . Inserting this into (6.6.2) we find

$$iS^{(1)}(\varepsilon_\Lambda) = q^\mu \Lambda^\nu \sum_k J_{k\mu\nu} + \sum_k \frac{p_k \cdot \Lambda q^\mu q^\nu J_{k\mu\nu}}{p_k \cdot q} . \quad (6.6.4)$$

The first term vanishes by global angular momentum conservation, while the second vanishes by antisymmetry of $J_{k\mu\nu}$. This is very similar to the gauge invariance of the Weinberg pole, which vanishes due to global energy-momentum conservation or equivalently translational symmetry. The Weinberg soft theorem implies that this global translational symmetry is promoted to a local supertranslational symmetry on the sphere [31], because there is one symmetry for every angle \vec{q} . In this section we will see a parallel story for rotational invariance: the soft relation (6.6.1) implies that rotations are promoted to a local superrotational - equivalently Virasoro - symmetry on the sphere.

The first step is to write the hard particle momenta p_k , the soft graviton momentum q and chosen polarization $\varepsilon^{-\mu\nu} = \varepsilon^{-\mu}\varepsilon^{-\nu}$ in terms of the points z_k and z at which they arrive on the asymptotic S^2 and their energies E_k, ω

$$p_k^\mu = \frac{E_k}{1 + z_k \bar{z}_k} (1 + z_k \bar{z}_k, \bar{z}_k + z_k, i(\bar{z}_k - z_k), 1 - z_k \bar{z}_k) , \quad (6.6.5)$$

$$q^\mu = \frac{\omega}{1 + z \bar{z}} (1 + z \bar{z}, \bar{z} + z, i(\bar{z} - z), 1 - z \bar{z}) , \quad (6.6.6)$$

$$\varepsilon^{-\mu} = \frac{1}{\sqrt{2}}(z, 1, i, -z) . \quad (6.6.7)$$

One then finds for the orbital terms

$$S^{(1)-} = \sum_k \left(\frac{E_k(z - z_k)(1 + z \bar{z}_k)}{(\bar{z}_k - \bar{z})(1 + z_k \bar{z}_k)} \partial_{E_k} + \frac{(z - z_k)^2}{(\bar{z}_k - \bar{z})} \partial_{z_k} \right) . \quad (6.6.8)$$

The spin term will be added in below. This expression obeys

$$\gamma^{z\bar{z}} D_z^3 \left(\hat{\varepsilon}_{\bar{z}\bar{z}}^+ S^{(1)-} \right) = -2\pi \sum_k \left(D_z \delta^{(2)}(z - z_k) E_k \partial_{E_k} + 2\delta^{(2)}(z - z_k) \partial_{z_k} \right) . \quad (6.6.9)$$

Multiplying both sides of (6.6.1) by $D_z^3 Y^z \varepsilon^{+z}_{\bar{z}}$ and integrating over the soft graviton angle z gives

$$\langle z_{n+1}, z_{n+2}, \dots | : Q_S \mathcal{S} : | z_1, z_2, \dots \rangle = \quad (6.6.10)$$

$$-i \sum_k \left(Y^z(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} \right) \langle z_{n+1}, z_{n+2}, \dots | \mathcal{S} | z_1, z_2, \dots \rangle , \quad (6.6.11)$$

which is exactly the Ward identity (6.5.16) arising from an asymptotic Virasoro symmetry, minus the thus-far-omitted spin terms.

The spin contribution comes from evaluating:

$$S_S^{(1)-} = -i \sum_k \frac{p_{k\lambda} \varepsilon^{-\lambda\nu} q^\mu S_{k\mu\nu}}{p_k \cdot q} . \quad (6.6.12)$$

In terms of the helicity h defined by

$$hp_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}S^{\nu\lambda}p^\rho, \quad (6.6.13)$$

one finds

$$S_S^{(1)-} = \sum_k \frac{(z - z_k)(1 + z\bar{z}_k)}{(\bar{z} - \bar{z}_k)(1 + z_k\bar{z}_k)} h_k, \quad (6.6.14)$$

while the third derivative obeys

$$\gamma^{z\bar{z}} D_z^3 \left(\hat{\varepsilon}_{\bar{z}\bar{z}}^+ S_S^{(1)-} \right) = 2\pi \sum_k h_k D_z \delta^{(2)}(z - z_k). \quad (6.6.15)$$

Hence, the spin contribution for the helicity states corrects (6.6.10) to

$$\begin{aligned} \langle z_{n+1}, z_{n+2}, \dots | : Q_S \mathcal{S} : | z_1, z_2, \dots \rangle &= -i \sum_k \left(Y^z(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} + \frac{h_k}{2} D_z Y^z(z_k) \right) \\ &\quad \langle z_{n+1}, z_{n+2}, \dots | \mathcal{S} | z_1, z_2, \dots \rangle, \end{aligned} \quad (6.6.16)$$

in agreement with the spin-corrected version of (6.5.16). In conclusion, the soft relation (6.6.1), whenever valid, implies a Virasoro symmetry of the quantum gravity \mathcal{S} -matrix.

7

A 2D Stress Tensor for 4D Gravity

7.1 Introduction

Any quantum scattering amplitude of massless particles in four-dimensional (4D) asymptotically Minkowskian spacetime can be rewritten as a correlation function on the celestial sphere at null infinity. Asymptotic one-particle states are represented as operator insertions on the sphere at the points where they exit or enter the spacetime. The energy and other flavor or quantum numbers then label distinct operators. The $SL(2, \mathbb{C})$ Lorentz invariance acts as the global conformal group on the celestial sphere and implies that these correlators lie in $SL(2, \mathbb{C})$ representations.

In this chapter we consider the \mathcal{S} -matrix for 4D quantum gravity in asymptotically Minkowskian spacetime. We construct an explicit soft graviton mode, denoted T_{zz} , and prove that its insertions in the tree-level \mathcal{S} -matrix (with no other external soft insertions) obey all the Virasoro-Ward identities of a stress tensor insertion in a CFT_2 correlator on the sphere. Our main tool is the subleading soft

graviton theorem [8, 15, 122, 123]. Our construction refines and extends results and conjectures of [10, 11, 13, 14, 33, 124]. It demonstrates that such quantum gravity scattering amplitudes are in Virasoro representations, as are CFT₂ correlators. This extends, from gauge theory to gravity, earlier work [34, 96] in which soft photon and gluon insertions were shown to obey the Ward identities of a Kac-Moody algebra on the celestial sphere.

The current work has several limitations. We do not consider massive particles, but do expect the extension to the massive case to be possible along the lines of [80, 81, 125]. Qualitatively important issues arise - including a possible central term - when there are multiple soft insertions that are not addressed here. At the one-loop level, corrections to the Ward identity are expected as a consequence of corrections to the soft theorem [66, 67, 72]. We have not analyzed their implications. Finally, although our results imply that certain quantum gravity scattering amplitudes are in Virasoro representations, there is no reason to expect that they are the same kinds of unitary representations appearing in conventional 2D CFTs. We leave the nature of these representations to future work.

7.2 Soft Graviton Limits

In this chapter we consider tree-level scattering amplitudes of massless particles in four dimensions. The single particle states are labeled by $|p, s\rangle$, where p and s denote the four-momentum and helicity of the particle respectively. The particles may carry charges or flavors, but these indices are not relevant and are suppressed. The normalization of these states is given by

$$\langle p, s | p', s' \rangle = (2\pi)^3 (2p^0) \delta_{s,s'} \delta^{(3)}(\vec{p} - \vec{p}') . \quad (7.2.1)$$

The tree-level scattering amplitude involving n massless states is denoted by

$$\mathcal{A}_n = \langle \text{out} | \mathcal{S} | \text{in} \rangle , \quad (7.2.2)$$

where we use the shorthand

$$|\text{in}\rangle = |p_1, s_1; \dots; p_m, s_m\rangle, \quad \langle \text{out}| = \langle p_{m+1}, s_{m+1}; \dots; p_n, s_n|, \quad (7.2.3)$$

and suppress the dependence of \mathcal{A}_n on the momenta p_k . We use a convention in which incoming states are described as CPT conjugate outgoing states with negative p^0 so that momentum conservation implies $\sum_{k=1}^n p_k^\mu = 0$.

Let $\mathcal{A}_{n+1}^{(\pm)}(q)$ be an amplitude involving a graviton of momentum q^μ and polarization $\varepsilon_{\mu\nu}^{(\pm)}(q)$ as well as n other massless asymptotic states

$$\mathcal{A}_{n+1}^{(\pm)}(q) = \langle \text{out}; q, \pm 2 | \mathcal{S} | \text{in} \rangle. \quad (7.2.4)$$

The soft $q^0 \rightarrow 0$ limit of this amplitude is governed by the leading [4] and subleading [8, 15, 122, 123] soft graviton theorems¹

$$\mathcal{A}_{n+1}^{(\pm)}(q) \rightarrow \left[S_0^{(\pm)} + S_1^{(\pm)} + \mathcal{O}(q) \right] \mathcal{A}_n, \quad (7.2.5)$$

where \mathcal{A}_n is the original amplitude without the soft graviton (7.2.2) and

$$S_0^{(\pm)} = \frac{\kappa}{2} \sum_{k=1}^n \frac{p_k^\mu p_k^\nu \varepsilon_{\mu\nu}^{(\pm)}(q)}{p_k \cdot q}, \quad S_1^{(\pm)} = -\frac{i\kappa}{2} \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^{(\pm)}(q) p_k^\mu q_\lambda}{p_k \cdot q} \mathcal{J}_k^{\lambda\nu}, \quad \kappa = \sqrt{32\pi G}. \quad (7.2.6)$$

Here, $\mathcal{J}_{k\mu\nu}$ is the angular momentum operator acting on the k^{th} outgoing state. It is the sum of the orbital angular momentum operator $\mathcal{L}_{k\mu\nu}$ and spin angular momentum $\mathcal{S}_{k\mu\nu}$. Explicitly (see [126]),

$$\begin{aligned} \mathcal{L}_{k\mu\nu} &= -i \left[p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right], \\ \mathcal{S}_{k\mu\nu} &= -is_k \left[\varepsilon_\mu^{(+)}(p_k) \varepsilon_\nu^{(-)}(p_k) - \varepsilon_\nu^{(+)}(p_k) \varepsilon_\mu^{(-)}(p_k) \right] + s_k \varepsilon_\rho^{(+)}(p_k) \mathcal{L}_{k\mu\nu} \varepsilon^{(-)\rho}(p_k). \end{aligned} \quad (7.2.7)$$

¹As shown in [15, 44, 61], tree-level graviton amplitudes are also constrained by a sub-subleading soft graviton theorem.

The $\varepsilon_\mu^{(\pm)}(p)$ are polarization vectors that satisfy²

$$\varepsilon^{(\pm)}(p) \cdot p = 0, \quad \varepsilon^{(\pm)}(p) \cdot \varepsilon^{(\pm)}(p) = 0, \quad \varepsilon^{(\pm)}(p) \cdot \bar{\varepsilon}^{(\pm)}(p) = 1. \quad (7.2.8)$$

Equation (7.2.7) continues to hold for particles of half-integer helicity provided that the little group phase of the wavefunction is chosen consistently. Gauge invariance of the leading and subleading soft limits implies momentum and angular momentum conservation respectively,

$$\sum_{k=1}^n p_k^\mu \mathcal{A}_n = \sum_{k=1}^n \mathcal{J}_{k\mu\nu} \mathcal{A}_n = 0. \quad (7.2.9)$$

Null momenta are characterized by an energy and a direction (equivalently, a point on S^2). To write out the soft factors explicitly, we parameterize the massless particles' momenta and polarization vectors using stereographic coordinates on the sphere³

$$\begin{aligned} p_k^\mu &= \omega_k \left(1, \frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k}, \frac{-i(z_k - \bar{z}_k)}{1 + z_k \bar{z}_k}, \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right), \\ \varepsilon_\mu^{(+)}(p_k) &= \frac{1}{\sqrt{2}} (-\bar{z}_k, 1, -i, -\bar{z}_k), \\ \varepsilon_\mu^{(-)}(p_k) &= \frac{1}{\sqrt{2}} (-z_k, 1, i, -z_k), \quad k = 1, \dots, n. \end{aligned} \quad (7.2.10)$$

For the soft graviton, we write

$$\begin{aligned} q^\mu &= \omega \left(1, \frac{z + \bar{z}}{1 + z \bar{z}}, \frac{-i(z - \bar{z})}{1 + z \bar{z}}, \frac{1 - z \bar{z}}{1 + z \bar{z}} \right), \\ \varepsilon_\mu^{(+)}(q) &= \frac{1}{\sqrt{2}} (-\bar{z}, 1, -i, -\bar{z}), \\ \varepsilon_\mu^{(-)}(q) &= \frac{1}{\sqrt{2}} (-z, 1, i, -z). \end{aligned} \quad (7.2.11)$$

²Note that (7.2.8) is invariant under $\varepsilon_\mu^{(\pm)}(q) \rightarrow e^{i\theta_\pm(q)} \varepsilon_\mu^{(\pm)}(q)$, i.e. (7.2.8) only determines the polarizations up to an overall momentum dependent phase. These correspond to the little group transformations.

³In writing the explicit forms of the polarization vectors in (7.2.10), (7.2.11) we have specified our choice of little group phase.

The graviton polarization is $\varepsilon_{\mu\nu}^{(\pm)}(q) = \varepsilon_\mu^{(\pm)}(q)\varepsilon_\nu^{(\pm)}(q)$. In this parameterization, the soft factors (7.2.6) are given by

$$\begin{aligned} S_1^{(+)} &= \frac{\kappa}{2} \sum_{k=1}^n \frac{(\bar{z} - \bar{z}_k)^2}{z - z_k} \left[\frac{2\bar{h}_k}{\bar{z} - \bar{z}_k} - \Gamma_{\bar{z}_k \bar{z}_k}^{\bar{z}_k} \bar{h}_k - \partial_{\bar{z}_k} + |s_k| \Omega_{\bar{z}_k} \right] , \\ S_1^{(-)} &= \frac{\kappa}{2} \sum_{k=1}^n \frac{(z - z_k)^2}{\bar{z} - \bar{z}_k} \left[\frac{2h_k}{z - z_k} - \Gamma_{z_k z_k}^{z_k} h_k - \partial_{z_k} + |s_k| \Omega_{z_k} \right] . \end{aligned} \quad (7.2.12)$$

Here $\Gamma_{z\bar{z}}^z$ is the connection with respect to the unit round metric $\gamma_{z\bar{z}} = 2(1 + z\bar{z})^{-2}$ on the sphere, $\Omega_z = \frac{1}{2}\Gamma_{z\bar{z}}^z$ is the spin connection,⁴ and we have defined the operators⁵

$$h_k \equiv \frac{1}{2}(s_k - \omega_k \partial_{\omega_k}) , \quad \bar{h}_k \equiv \frac{1}{2}(-s_k - \omega_k \partial_{\omega_k}) . \quad (7.2.13)$$

In this parameterization, equation (7.2.9) takes the form

$$\begin{aligned} \left(\sum_{k=1}^n \omega_k \right) \mathcal{A}_n &= \left(\sum_{k=1}^n \omega_k \frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k} \right) \mathcal{A}_n = -i \left(\sum_{k=1}^n \omega_k \frac{z_k - \bar{z}_k}{1 + z_k \bar{z}_k} \right) \mathcal{A}_n = \left(\sum_{k=1}^n \omega_k \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right) \mathcal{A}_n = 0, \\ -i \sum_{k=1}^n [Y^{z_k} (\partial_{z_k} - |s_k| \Omega_{z_k}) + Y^{\bar{z}_k} (\partial_{\bar{z}_k} - |s_k| \Omega_{\bar{z}_k}) + D_{z_k} Y^{z_k} h_k + D_{\bar{z}_k} Y^{\bar{z}_k} \bar{h}_k] \mathcal{A}_n &= 0 , \end{aligned} \quad (7.2.14)$$

where $Y^z(z) = a + bz + cz^2$ is a global conformal Killing vector and D_z is the covariant derivative on the unit sphere.

⁴The zweibein chosen here is $(e^+, e^-) = \sqrt{2\gamma_{z\bar{z}}}(\mathrm{d}z, \mathrm{d}\bar{z})$ for which $\Omega^\pm_\pm = \pm \frac{1}{2}(\Gamma_{z\bar{z}}^z \mathrm{d}z - \Gamma_{\bar{z}z}^{\bar{z}} \mathrm{d}\bar{z})$. This choice is related to the little group phase chosen in (7.2.10), (7.2.11).

⁵Single particle momentum eigenstates do not diagonalize the dilation operator $h_k + \bar{h}_k$. At tree-level, amplitudes are rational functions of the external momenta and we can formally define Mellin-transformed primary operators $\tilde{\mathcal{O}}(m, z, \bar{z}) = \int_0^\infty d\omega \omega^{m-1} \mathcal{O}(\omega, z, \bar{z})$ with conformal weights $h = \frac{1}{2}(s + m)$, $\bar{h} = \frac{1}{2}(-s + m)$.

7.3 Mode Expansions Near \mathcal{I}^+

Four-dimensional asymptotically flat metrics [1, 2, 10–12, 124] admit an expansion near \mathcal{I}^+ of the form

$$\begin{aligned}
 ds^2 = & -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 \\
 & + D^z C_{zz}dudz + D^{\bar{z}} C_{\bar{z}\bar{z}}dud\bar{z} + \dots .
 \end{aligned} \tag{7.3.1}$$

In these coordinates, \mathcal{I}^+ is the null surface $(u, r = \infty, z, \bar{z})$. The retarded time u parameterizes the null generators of \mathcal{I}^+ , and (z, \bar{z}) are stereographic coordinates on the conformal S^2 . The boundaries of \mathcal{I}^+ are located at $(u = \pm\infty, r = \infty, z, \bar{z})$ and are denoted \mathcal{I}_+^+ and \mathcal{I}_-^+ respectively. The Bondi mass aspect m_B and C_{zz} depend only on (u, z, \bar{z}) and not on r . The news tensor is defined by

$$N_{zz} \equiv \partial_u C_{zz} . \tag{7.3.2}$$

When expanding near flat spacetime, the Bondi coordinates are related to flat Cartesian coordinates by

$$x^0 = u + r , \quad x^i = r\hat{x}^i(z, \bar{z}) , \quad \hat{x}^i(z, \bar{z}) = \frac{1}{1 + z\bar{z}} (z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}) . \tag{7.3.3}$$

The space of asymptotically flat metrics in Bondi gauge with prescribed falloffs [1, 2] admits an infinite-dimensional asymptotic symmetry group, the BMS group, parameterized by vector fields of the form

$$\begin{aligned}
 \xi = & \left(1 + \frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^z \partial_{\bar{z}} - \frac{1}{2}(u + r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + c.c. \\
 & + f \partial_u - \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r + \dots .
 \end{aligned} \tag{7.3.4}$$

Here, $f(z, \bar{z})$ is a free function on the sphere associated to the supertranslation subgroup of the BMS group. The two-dimensional vector field $Y(z, \bar{z})$ is a conformal Killing vector (CKV) which realizes the action of the Lorentz group $SL(2, \mathbb{C})$ on the asymptotic sphere. For a more general CKV, obeying $\partial_{\bar{z}} Y^z = 0$ except at isolated singularities, the Bondi gauge condition is preserved,

but the falloffs imposed on the metric are violated at the singularities. It was conjectured [10, 11, 13, 14, 124] and proven in tree-level perturbation theory [33] that such symmetries nevertheless play an important role.

The flat space outgoing graviton mode expansion is⁶

$$h_{\mu\nu}^{\text{out}}(x^0, \vec{x}) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\bar{\varepsilon}_{\mu\nu}^{(\alpha)}(q) a_{\alpha}^{\text{out}}(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^{(\alpha)}(q) a_{\alpha}^{\text{out}}(\vec{q})^{\dagger} e^{-iq \cdot x} \right],$$

where $\omega_q = |\vec{q}|$ and

$$\left[a_{\alpha}^{\text{out}}(\vec{p}), a_{\beta}^{\text{out}}(\vec{q})^{\dagger} \right] = (2\pi)^3 (2p^0) \delta_{\alpha\beta} \delta^{(3)}(\vec{p} - \vec{q}). \quad (7.3.5)$$

Outgoing gravitons with momentum q and polarization α as in the amplitude (7.2.2) correspond to final-state insertions of $a_{\alpha}^{\text{out}}(\vec{q})$.

In retarded Bondi coordinates

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) = \kappa \lim_{r \rightarrow \infty} \frac{1}{r} \partial_{\bar{z}} x^{\mu} \partial_{\bar{z}} x^{\nu} h_{\mu\nu}^{\text{out}}(u + r, r\hat{x}(z, \bar{z})). \quad (7.3.6)$$

This large- r limit can be computed using the stationary phase approximation [31, 33] and one finds

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) = -\frac{i\kappa}{8\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}} \int_0^{\infty} d\omega_q \left[a_{-}^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_q u} - a_{+}^{\text{out}}(\omega_q \hat{x})^{\dagger} e^{i\omega_q u} \right]. \quad (7.3.7)$$

Here, $\hat{x} \equiv \hat{x}(z, \bar{z})$ and

$$\hat{\varepsilon}_{\bar{z}\bar{z}} = \frac{1}{r^2} \partial_{\bar{z}} x^{\mu} \partial_{\bar{z}} x^{\nu} \varepsilon_{\mu\nu}^{(+)}(\omega_q \hat{x}) = \frac{2}{(1 + z\bar{z})^2}. \quad (7.3.8)$$

We define the first moment of the Bondi news

$$\begin{aligned} N_{\bar{z}\bar{z}}^{(1)} &\equiv \int du u N_{\bar{z}\bar{z}} = -\frac{i}{2} \lim_{\omega \rightarrow 0} \partial_{\omega} \int du (e^{i\omega u} - e^{-i\omega u}) N_{\bar{z}\bar{z}} \\ &= \frac{i\kappa}{8\pi} \hat{\varepsilon}_{\bar{z}\bar{z}} \lim_{\omega \rightarrow 0} (1 + \omega \partial_{\omega}) \left[a_{-}^{\text{out}}(\omega \hat{x}) - a_{+}^{\text{out}}(\omega \hat{x})^{\dagger} \right], \end{aligned} \quad (7.3.9)$$

⁶Here, we take $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ which implies a canonical normalization for the graviton field, $\mathcal{L} \sim \frac{1}{2}(\partial h)^2$.

along with a similar definition for $N_{zz}^{(1)}$. We note that $N_{\bar{z}\bar{z}}^{(1)}$ has the Weinberg pole projected out by the factor $1 + \omega\partial_\omega$. Hence, it has nonzero finite scattering amplitudes.

The insertion of the zero-mode (7.3.9) is then given by (7.2.5) and (7.2.12) with

$$\langle \text{out} | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} | \text{in} \rangle = \frac{4Gi}{(1 + z\bar{z})^2} \sum_{k=1}^n \frac{(z - z_k)^2}{\bar{z} - \bar{z}_k} \left[\frac{2h_k}{z - z_k} - \Gamma_{z_k z_k}^{z_k} h_k - \partial_{z_k} + |s_k| \Omega_{z_k} \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle . \quad (7.3.10)$$

7.4 A 2D Stress Tensor

Massless scattering amplitudes \mathcal{A}_n of any four-dimensional theory may always be recast as two-dimensional correlation functions of local operators on the asymptotic S^2 at null infinity [96],

$$\mathcal{A}_n = \langle \mathcal{O}_1(\omega_1, z_1, \bar{z}_1) \dots \mathcal{O}_n(\omega_n, z_n, \bar{z}_n) \rangle . \quad (7.4.1)$$

The operator \mathcal{O}_k creates a massless single-particle state with momentum and polarization given by (7.2.10). The particle intersects the asymptotic S^2 at the point (z_k, \bar{z}_k) .⁷ The four-dimensional Lorentz group $SL(2, \mathbb{C})$ acts as the global conformal group on the asymptotic S^2 according to⁸

$$z \rightarrow z' = \frac{az + b}{cz + d} , \quad ad - bc = 1 . \quad (7.4.2)$$

This implies that all Minkowskian QFT₄ amplitudes are in representations of the same global conformal group as Euclidean CFT₂ correlators. In this section we will see that (hard) quantum

⁷The same is not true for scattering amplitudes involving massive particles since a massive four-momentum does not localize to a point on \mathcal{I} . However following [80, 81, 125] we expect the analysis of this chapter to have a suitable generalization to the massive case, as the subleading soft theorem [8, 15, 122, 123] remains valid for massive particles.

⁸This also acts on the energy as

$$\tilde{\omega} \rightarrow \tilde{\omega} |cz + d|^2 , \quad \tilde{\omega} = \frac{\omega}{1 + z\bar{z}} .$$

gravity amplitudes are in representations of the full CFT_2 Virasoro group. Indeed, it has already been shown that the leading soft photon and graviton theorems are the Ward identities of abelian Kac-Moody current algebras acting on the asymptotic S^2 [30–32, 34]. A similar Kac-Moody structure for non-abelian gauge theory scattering amplitudes was studied in [127]. The leading soft gluon theorem in a non-abelian gauge theory with gauge group \mathcal{G} was shown in [96] to be equivalent to the Ward identity of a \mathcal{G} Kac-Moody current algebra. In all of these cases, holomorphic Kac-Moody current insertions were related to positive helicity soft insertions. For instance, the soft photon Kac-Moody current is

$$J_z = -\frac{8\pi}{e^2} F_{uz}^{(0)} = \frac{1}{e} \hat{\varepsilon}_z \lim_{\omega \rightarrow 0} \left[\omega a_+^{\text{out}}(\omega \hat{x}) + \omega a_-^{\text{out}}(\omega \hat{x})^\dagger \right], \quad (7.4.3)$$

where $F_{uz}^{(0)}$ is the zero-mode of the photon field strength, $\hat{\varepsilon}_z = \sqrt{\hat{\varepsilon}_{zz}}$, and $a_+^{\text{out}}(\omega \hat{x})$ creates outgoing positive helicity photons. Insertions of this current take the form

$$\langle J_z \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_k \frac{Q_k}{z - z_k} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle, \quad (7.4.4)$$

where eQ_k is the electric charge of the operator \mathcal{O}_k and we have dropped the dependence of the operators on $(\omega_k, z_k, \bar{z}_k)$ for compactness.

In a similar vein, it has been shown [33, 39] that the subleading soft graviton theorem is the Ward identity for the superrotations [10] which generate an infinite-dimensional Virasoro subgroup of the extended BMS group.⁹ In the language of 2D correlators, the current corresponding to these local conformal transformations is the stress tensor. We now turn to an explicit construction of this operator.

Our starting point is (7.3.10), which has a form reminiscent of a stress tensor Ward identity.

⁹The sub-subleading soft graviton theorem has also been recently recast as a symmetry of the \mathcal{S} -matrix (see [128, 129]).

To bring this into the usual form, we define

$$T_{zz} \equiv \frac{i}{8\pi G} \int d^2w \frac{1}{z-w} D_w^2 D_{\bar{w}} N_{\bar{w}\bar{w}}^{(1)}. \quad (7.4.5)$$

This integro-differential operator relating T_{zz} to $N_{\bar{w}\bar{w}}^{(1)}$ can be applied to the matrix element (7.3.10) in order to determine the matrix elements of T_{zz} . One finds

$$\langle T_{zz} \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{k=1}^n \left[\frac{h_k}{(z-z_k)^2} + \frac{\Gamma_{z_k z_k}^{z_k}}{z-z_k} h_k + \frac{1}{z-z_k} (\partial_{z_k} - |s_k| \Omega_{z_k}) \right] \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle, \quad (7.4.6)$$

which is the precise form of the stress tensor correlator in a conformal field theory on a curved background. This can be brought to the more familiar form by dressing the operators with appropriate factors of the zweibein. See [130] for a more detailed discussion.

Define the charge

$$T_{\mathcal{C}}[Y] = \oint_{\mathcal{C}} \frac{dz}{2\pi i} Y^z T_{zz}, \quad (7.4.7)$$

where Y^z is a local CKV obeying $\partial_{\bar{z}} Y^z = 0$ with no singularities inside the contour. Geometrically, these vector fields generate the local conformal transformations of the sphere. Therefore, one expects the operators (7.4.7) to implement the action of the Virasoro algebra quantum-mechanically. Indeed, insertions of (7.4.7) take the form

$$\langle T_{\mathcal{C}}[Y] \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{k \in \mathcal{C}} [D_{z_k} Y^{z_k} h_k + Y^{z_k} (\partial_{z_k} - |s_k| \Omega_{z_k})] \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle. \quad (7.4.8)$$

Thus, $T_{\mathcal{C}}[Y]$ generates a local conformal transformation on all operators inside \mathcal{C} .¹⁰

Now, consider a contour \mathcal{C} that encircles all z_k and a Y^z that is globally defined on the sphere, i.e. $Y^z = a + bz + cz^2$. Since we are on a compact S^2 , insertions of $T_{\mathcal{C}}[Y]$ can be computed by

¹⁰This operator is closely related to the soft part of the superrotation charge defined in [33]. More precisely, if \mathcal{C} is a contour that surrounds all z_k , then

$$Q_S^+ = -\frac{i}{2} T_{\mathcal{C}}[Y].$$

either closing the contour towards $z = z_k$ or away from it. No poles are crossed when the contour is closed away from $z = z_k$ and these insertions must vanish. In other words,

$$\sum_{k=1}^n [D_{z_k} Y^{z_k} h_k + Y^{z_k} (\partial_{z_k} - |s_k| \Omega_{z_k})] \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = 0, \quad Y^z = a + bz + cz^2, \quad (7.4.9)$$

which is the statement of boost/angular momentum conservation (7.2.14). In ordinary two-dimensional conformal field theories, the existence of the full Virasoro symmetry serves as a drastic constraint on allowed dynamics and is responsible for most of the simplifications in 2D CFTs relative to higher-dimensional theories. One hopes that the identification of this symmetry of the four-dimensional gravitational \mathcal{S} -matrix can be exploited to similar effect.

The stress tensor (7.4.5) is non-local on S^2 in the news tensor zero-mode $N_{\bar{z}\bar{z}}^{(1)}$. Nevertheless, we have proven that insertions of T_{zz} are local on the S^2 . In contrast, the construction of the boundary stress tensor in AdS/CFT [131, 132] is local in the bulk fields when written in terms of subleading terms in the metric expansion. Leading and subleading terms in the metric expansion have a gauge-dependent and generally non-local relation on the S^2 enforced by the Einstein equation. We have tried unsuccessfully to find, by rewriting $N_{\bar{z}\bar{z}}^{(1)}$ in terms of subleading metric components, such a local expression in Bondi gauge.¹¹ However, it is possible that such a manifestly local expression exists in some other gauge. On the other hand, the non-locality may indicate that the Virasoro action in 4D quantum gravity has a different character than that in conventional 2D CFT. We leave this question unanswered for now.

Obviously, an anti-holomorphic stress tensor $T_{\bar{z}\bar{z}}$ could be similarly constructed. However, a number of yet unresolved issues arise for multiple soft current insertions, even in the Maxwell case, as discussed in [34, 96]. The result of this chapter is that insertions of a single T_{zz} generate local conformal transformations when all other insertions are hard.

¹¹The $\mathcal{O}(r^0)$ term in g_{zz} is an obvious suspect.

8

Loop-Corrected Virasoro Symmetry of 4D Quantum Gravity

8.1 Introduction

Recently, it has been demonstrated [33] that any theory of gravity in four asymptotically flat dimensions has, at tree-level, a Virasoro or “superrotational” symmetry that acts on the celestial sphere at null infinity. This verified conjectures in [10, 11, 13, 14, 124] and follows from the newly discovered subleading soft graviton theorem [15]. Except for an anomaly arising from one-loop exact infrared (IR) divergences [44, 66, 67, 69, 72], this subleading soft theorem extends to the full quantum theory. However, the implications of this anomaly for the Virasoro symmetry of the full quantum theory are not understood, and the exploration of such implications comprises the subject of the present chapter.

There are several open possibilities. One possibility is that the Virasoro action is defined in the classical but not in the quantum theory. If so, anomalous symmetries still have important quantum constraints that would be interesting to understand. A second possibility is that the Virasoro action acts on the full quantum theory, but that the generators and symmetry action are renormalized at one-loop. This is suggested by the discussion in [16], where it is pointed out that the very definition of a scattering problem in asymptotically flat gravity requires an infinite number of exactly conserved charges and associated symmetries, as well as by [68], which found that the anomaly vanishes with an alternate order of soft limits. A third possibility is that the implications can only be properly formulated in a Faddeev-Kulish [106–110, 133] basis of states (constructed for gravity in [121]), in which case all IR divergences are absent. After all, IR divergences preclude a Fock-basis \mathcal{S} -matrix for quantum gravity and, although we have become accustomed to ignoring this point, it is hard to discuss symmetries of an object which exists only formally!

In this chapter we give evidence which is consistent with, but does not prove, the second hypothesis, which states that the Virasoro action persists to the full quantum theory but requires the generators to have a one-loop correction. We use the recent construction of a 2D energy-momentum tensor T_{zz} found in [134, 135], where z is a coordinate on the celestial sphere, in terms of soft graviton modes. The tree-level subleading soft theorem [15] implies that insertions of T_{zz} in the tree-level \mathcal{S} -matrix infinitesimally generate a Virasoro action on the celestial sphere. At one-loop order, the subleading soft theorem has an IR divergent term with a known universal form. This spoils the Virasoro action generated by T_{zz} insertions. However, we show explicitly that the effects of the IR divergent term can be removed by a certain shift in T_{zz} that is quadratic in the soft graviton modes. The possibility that this could be achieved by a simple shift is far from obvious and requires a number of nontrivial cancellations.

This does not demonstrate that there is a Virasoro action on the full quantum theory generated by a renormalized energy-momentum tensor, as there may also be an IR finite one-loop correction to the subleading soft theorem. At present, little is known about such finite corrections. In all cases which have been analyzed [66, 67], the finite part of the correction vanishes. Yet, there is no

known argument that this should always be the case, and this remains an open issue for us.

The outline of this chapter is as follows. In section 8.2 we fix conventions and recall the construction of the tree-level soft graviton energy-momentum tensor. We then reproduce the derivation of the one-loop exact IR divergent corrections to the subleading soft graviton theorem in section 8.3. Finally, this divergence is rewritten in section 8.4 in terms of the formal matrix element of another quadratic soft graviton operator, effectively renormalizing the tree-level energy-momentum tensor.

8.2 Tree-Level Energy-Momentum Tensor

In this section, we review the derivation of the 2D tree-level energy-momentum tensor living on the celestial sphere at null infinity [134, 135]. Asymptotic one-particle states are denoted by $|p, s\rangle$, where p is the 4-momentum and s is the helicity, and such states are normalized so that

$$\langle q, s' | p, s \rangle = (2\pi)^3 (2p^0) \delta_{ss'} \delta^{(3)}(\vec{p} - \vec{q}) . \quad (8.2.1)$$

An n -particle \mathcal{S} -matrix element is denoted by

$$\mathcal{M}_n \equiv \langle \text{out} | \mathcal{S} | \text{in} \rangle , \quad (8.2.2)$$

where $|\text{in}\rangle \equiv |p_1, s_1; \dots; p_m, s_m\rangle$ and $\langle \text{out} | \equiv \langle p_{m+1}, s_{m+1}; \dots; p_n, s_n |$. Consider the amplitude

$$\mathcal{M}_{n+1}^\pm(q) \equiv \langle \text{out}; q, \pm 2 | \mathcal{S} | \text{in} \rangle , \quad (8.2.3)$$

consisting of n external hard particles along with an additional external graviton that has momentum $p_{n+1} \equiv q$, energy $p_{n+1}^0 \equiv \omega$, and polarization $\varepsilon_{\mu\nu}^\pm$. Denoting the same amplitude without the extra external graviton as \mathcal{M}_n , the tree-level soft graviton theorem states that

$$\lim_{\omega \rightarrow 0} \mathcal{M}_{n+1}^\pm(q) = \left[S_n^{(0)\pm} + S_n^{(1)\pm} + \mathcal{O}(q) \right] \mathcal{M}_n , \quad (8.2.4)$$

where the leading and subleading soft factors are given by

$$S_n^{(0)\pm} = \frac{\kappa}{2} \sum_{k=1}^n \frac{p_k^\mu p_k^\nu \varepsilon_{\mu\nu}^\pm(q)}{p_k \cdot q}, \quad S_n^{(1)\pm} = -\frac{i\kappa}{2} \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^\pm(q) p_k^\mu q_\lambda}{p_k \cdot q} \mathcal{J}_k^{\lambda\nu}, \quad (8.2.5)$$

respectively. Here, $\kappa \equiv \sqrt{32\pi G}$ is the gravitational coupling constant, and $\mathcal{J}_k^{\lambda\nu}$ is the total angular momentum operator for the k^{th} particle.

Asymptotically flat metrics in Bondi gauge take the form

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} + \frac{2m_B}{r} du^2 + r C_{zz} dz^2 + r C_{\bar{z}\bar{z}} d\bar{z}^2 + D^z C_{zz} dudz + D^{\bar{z}} C_{\bar{z}\bar{z}} dud\bar{z} + \dots \quad (8.2.6)$$

Here, $\gamma_{z\bar{z}} \equiv \frac{2}{(1+z\bar{z})^2}$ is the round metric on the S^2 and D_z is the associated covariant derivative. The coordinates (u, r, z, \bar{z}) are asymptotically related to the standard Cartesian coordinates according to

$$x^0 = u + r, \quad x^i = r \hat{x}^i(z, \bar{z}), \quad \hat{x}^i(z, \bar{z}) = \frac{1}{1+z\bar{z}} (z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}). \quad (8.2.7)$$

A massless particle with momentum p_k crosses the celestial sphere at a point (z_k, \bar{z}_k) . In the helicity basis, the particle momentum and polarization can be parameterized by an energy ω_k and this crossing point. It follows from (8.2.7) that

$$\begin{aligned} p_k^\mu &= \omega_k \left(1, \frac{z_k + \bar{z}_k}{1 + z_k \bar{z}_k}, \frac{-i(z_k - \bar{z}_k)}{1 + z_k \bar{z}_k}, \frac{1 - z_k \bar{z}_k}{1 + z_k \bar{z}_k} \right), \quad k = 1, \dots, n, \\ \varepsilon_\mu^+(p_k) &= \frac{1}{\sqrt{2}} (-\bar{z}_k, 1, -i, -\bar{z}_k), \quad \varepsilon_\mu^-(p_k) = \frac{1}{\sqrt{2}} (-z_k, 1, i, -z_k), \\ q^\mu(z) &= \omega \left(1, \frac{z + \bar{z}}{1 + z\bar{z}}, \frac{-i(z - \bar{z})}{1 + z\bar{z}}, \frac{1 - z\bar{z}}{1 + z\bar{z}} \right) \equiv \omega \hat{q}^\mu(z), \\ \varepsilon_\mu^+(q) &= \frac{1}{\sqrt{2}} (-\bar{z}, 1, -i, -\bar{z}), \quad \varepsilon_\mu^-(q) = \frac{1}{\sqrt{2}} (-z, 1, i, -z), \end{aligned} \quad (8.2.8)$$

where the soft graviton polarization tensor is taken to be $\varepsilon_{\mu\nu}^\pm = \varepsilon_\mu^\pm \varepsilon_\nu^\pm$.

Now, the perturbative fluctuations of the gravitational field have a mode expansion given by

$$h_{\mu\nu}^{\text{out}}(x^0, \vec{x}) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\bar{\varepsilon}_{\mu\nu}^\alpha(q) a_\alpha^{\text{out}}(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^\alpha(q) a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq \cdot x} \right], \quad (8.2.9)$$

where $a_\alpha^{\text{out}}(\vec{q})^\dagger$ and $a_\alpha^{\text{out}}(\vec{q})$ are the standard creation and annihilation operators for gravitons obeying the commutation relations

$$\left[a_\alpha^{\text{out}}(\vec{p}), a_\beta^{\text{out}}(\vec{q})^\dagger \right] = (2\pi)^3 (2\omega_p) \delta_{\alpha\beta} \delta^{(3)}(\vec{p} - \vec{q}). \quad (8.2.10)$$

The transverse components of the metric fluctuations near \mathcal{I}^+ are given by

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) \equiv \kappa \lim_{r \rightarrow \infty} \frac{1}{r} \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu h_{\mu\nu}^{\text{out}}(u + r, r\hat{x}(z, \bar{z})). \quad (8.2.11)$$

The large- r saddle-point approximation yields

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) = -\frac{i\kappa}{8\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}} \int_0^\infty d\omega_q \left[a_-^{\text{out}}(\omega_q \hat{x}) e^{-i\omega_q u} - a_+^{\text{out}}(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right], \quad (8.2.12)$$

where

$$\hat{\varepsilon}_{\bar{z}\bar{z}} = \frac{1}{r^2} \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu \varepsilon_{\mu\nu}^+(q) = \frac{2}{(1 + z\bar{z})^2}. \quad (8.2.13)$$

Note that (8.2.12) is an intuitively plausible result since it states that the graviton field operator at a point (z, \bar{z}) on the celestial sphere has an expansion in plane wave modes whose momenta are aimed towards that point.

The Bondi news tensor $N_{zz} = \partial_u C_{zz}$ has Fourier components

$$N_{zz}^\omega \equiv \int du e^{i\omega u} N_{zz}, \quad N_{\bar{z}\bar{z}}^\omega \equiv \int du e^{i\omega u} N_{\bar{z}\bar{z}}. \quad (8.2.14)$$

The zero-mode of the news tensor is defined by

$$N_{\bar{z}\bar{z}}^{(0)} \equiv \frac{1}{2} \lim_{\omega \rightarrow 0} (N_{\bar{z}\bar{z}}^\omega + N_{\bar{z}\bar{z}}^{-\omega}) = -\frac{\kappa}{8\pi} \hat{\varepsilon}_{\bar{z}\bar{z}} \lim_{\omega \rightarrow 0} \left[\omega a_-^{\text{out}}(\omega \hat{x}) + \omega a_+^{\text{out}}(\omega \hat{x})^\dagger \right]. \quad (8.2.15)$$

Similarly, the first zero energy moment of the news is defined by

$$N_{\bar{z}\bar{z}}^{(1)} \equiv -\frac{i}{2} \lim_{\omega \rightarrow 0} \partial_\omega (N_{\bar{z}\bar{z}}^\omega - N_{\bar{z}\bar{z}}^{-\omega}) = \frac{i\kappa}{8\pi} \hat{\varepsilon}_{\bar{z}\bar{z}} \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left[a_-^{\text{out}}(\omega \hat{x}) - a_+^{\text{out}}(\omega \hat{x})^\dagger \right]. \quad (8.2.16)$$

All of these quantities have nonvanishing \mathcal{S} -matrix insertions even as $\omega \rightarrow 0$. The operator $N_{\bar{z}\bar{z}}^{(0)}$ projects onto the leading Weinberg pole in the soft graviton theorem [4], so its matrix elements are tree-level exact and are given by

$$\langle \text{out} | N_{\bar{z}\bar{z}}^{(0)} \mathcal{S} | \text{in} \rangle = -\frac{\kappa}{8\pi} \hat{\varepsilon}_{\bar{z}\bar{z}} \lim_{\omega \rightarrow 0} \omega S_n^{(0)-} \langle \text{out} | \mathcal{S} | \text{in} \rangle, \quad (8.2.17)$$

where

$$S_n^{(0)-} = -\frac{\kappa}{2\omega} (1 + z\bar{z}) \sum_{k=1}^n \frac{\omega_k (z - z_k)}{(\bar{z} - \bar{z}_k)(1 + z_k \bar{z}_k)}. \quad (8.2.18)$$

In a similar fashion, $N_{\bar{z}\bar{z}}^{(1)}$ projects onto the subleading $\mathcal{O}(1)$ term in the soft graviton theorem. At tree-level, its matrix elements are given by

$$\langle \text{out} | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} | \text{in} \rangle = \frac{i\kappa}{8\pi} \hat{\varepsilon}_{\bar{z}\bar{z}} S_n^{(1)-} \langle \text{out} | \mathcal{S} | \text{in} \rangle, \quad (8.2.19)$$

where

$$S_n^{(1)-} = \frac{\kappa}{2} \sum_{k=1}^n \frac{(z - z_k)^2}{\bar{z} - \bar{z}_k} \left[\frac{2h_k}{z - z_k} - \Gamma_{z_k z_k}^{z_k} h_k - \partial_{z_k} + |s_k| \Omega_{z_k} \right]. \quad (8.2.20)$$

In this expression, Γ_{zz}^z is the connection on the asymptotic S^2 , h_k and \bar{h}_k are the conformal weights given by

$$h_k \equiv \frac{1}{2} (s_k - \omega_k \partial_{\omega_k}), \quad \bar{h}_k \equiv \frac{1}{2} (-s_k - \omega_k \partial_{\omega_k}), \quad (8.2.21)$$

and Ω_z is the corresponding spin connection.¹ As was demonstrated in [134], (8.2.19) implies that insertions of the operator

$$T_{zz} \equiv \frac{4i}{\kappa^2} \int d^2 w \frac{\gamma^{w\bar{w}}}{z - w} D_w^3 N_{\bar{w}\bar{w}}^{(1)} \quad (8.2.22)$$

¹The zweibein is $(e^+, e^-) = \sqrt{2\gamma_{z\bar{z}}} (dz, d\bar{z})$ and $\Omega^\pm_\pm = \pm \frac{1}{2} (\Gamma_{zz}^z dz - \Gamma_{\bar{z}\bar{z}}^{\bar{z}} d\bar{z})$.

into the tree-level \mathcal{S} -matrix reproduce the Ward identity for a 2D conformal field theory:

$$\langle \text{out} | T_{zz} \mathcal{S} | \text{in} \rangle = \sum_{k=1}^n \left[\frac{h_k}{(z - z_k)^2} + \frac{h_k}{z - z_k} \Gamma_{z_k z_k}^{z_k} + \frac{1}{z - z_k} (\partial_{z_k} - |s_k| \Omega_{z_k}) \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle . \quad (8.2.23)$$

8.3 One-Loop Correction to the Subleading Soft Graviton Theorem

The matrix element (8.2.17) is exact because the leading Weinberg pole in the soft graviton expansion is uncorrected. On the other hand, the subleading theorem which governs the $\mathcal{O}(1)$ terms in the soft graviton expansion does have quantum corrections that modify the matrix element (8.2.19) [66]. These corrections are known to be one-loop exact, and arise from IR divergences in soft exchanges between external lines. Indeed, they must be present in order to cancel (within suitable inclusive cross-sections) IR divergences that arise from the Weinberg pole. The divergent part of this one-loop correction was derived in [66], which we will now review.

The loop expansion of the n -particle scattering amplitude is

$$\mathcal{M}_n = \sum_{\ell=0}^{\infty} \mathcal{M}_n^{(\ell)} \kappa^{2\ell} , \quad (8.3.1)$$

where we factored out the κ^2 term that comes along with each additional loop.² In dimensional regularization with $d = 4 - \epsilon$, the divergent part of the one-loop n -point graviton scattering amplitude is universally related to the tree-level amplitude according to [136, 137]

$$\mathcal{M}_n^{(1)} \Big|_{\text{div}} = \frac{\sigma_n}{\epsilon} \mathcal{M}_n^{(0)} , \quad (8.3.2)$$

with

$$\sigma_n \equiv -\frac{1}{4(4\pi)^2} \sum_{i,j=1}^n (p_i \cdot p_j) \log \frac{\mu^2}{-2p_i \cdot p_j} . \quad (8.3.3)$$

²In addition, there is a factor of κ^{n-2} in each $\mathcal{M}_n^{(\ell)}$ due to the n external lines.

The $\mathcal{O}(\epsilon^{-1})$ singularity is due exclusively to IR divergences because pure gravity is on-shell one-loop finite in the UV and has no collinear divergences. Using (8.3.2) and applying the tree-level soft theorem involving a negative-helicity soft graviton, we obtain

$$\mathcal{M}_{n+1}^{(1)-}(q) \Big|_{\text{div}} \xrightarrow{q \rightarrow 0} \frac{\sigma_{n+1}}{\epsilon} \left(S_n^{(0)-} + S_n^{(1)-} \right) \mathcal{M}_n^{(0)}. \quad (8.3.4)$$

We would like to expand the above equation in powers of the soft energy ω . To proceed, we separate σ_{n+1} into two terms, one with the soft graviton momentum q and one without:

$$\sigma_{n+1} = \sigma_n + \sigma'_{n+1}, \quad \sigma'_{n+1} \equiv -\frac{1}{2(4\pi)^2} \sum_{i=1}^n (p_i \cdot q) \log \frac{\mu^2}{-2p_i \cdot q}. \quad (8.3.5)$$

Note that $\sigma'_{n+1} = \mathcal{O}(\omega)$ as the $\log \omega$ term vanishes by momentum conservation, while $\sigma_n = \mathcal{O}(\omega^0)$. We then find, up to $\mathcal{O}(\omega^0)$,

$$\mathcal{M}_{n+1}^{(1)-} \Big|_{\text{div}} \xrightarrow{q \rightarrow 0} \left(S_n^{(0)-} + S_n^{(1)-} \right) \mathcal{M}_n^{(1)} \Big|_{\text{div}} + \frac{\sigma'_{n+1}}{\epsilon} S_n^{(0)-} \mathcal{M}_n^{(0)} - \frac{1}{\epsilon} \left(S_n^{(1)-} \sigma_n \right) \mathcal{M}_n^{(0)}, \quad (8.3.6)$$

where $S_n^{(1)-}$ in the last term acts only on the scalar σ_n . The anomalous term consists of the last two terms on the right-hand-side of the above equation and is $\mathcal{O}(\omega^0)$. It is a universal correction to the subleading soft theorem from IR divergences.

Thus far, we have been focusing on the IR divergent part of the one-loop amplitude. However, the one-loop amplitude also has a finite piece:

$$\mathcal{M}_n^{(1)} = \mathcal{M}_n^{(1)} \Big|_{\text{div}} + \mathcal{M}_n^{(1)} \Big|_{\text{fin}}. \quad (8.3.7)$$

It is expected from [66] that

$$\mathcal{M}_{n+1}^{(1)-} \Big|_{\text{fin}} \xrightarrow{q \rightarrow 0} \left(S_n^{(0)-} + S_n^{(1)-} \right) \mathcal{M}_n^{(1)} \Big|_{\text{fin}} + \Delta_{\text{fin}} S_n^{(1)-} \mathcal{M}_n^{(0)}, \quad (8.3.8)$$

where $\Delta_{\text{fin}} S_n^{(1)-}$ is the one-loop finite correction to the negative-helicity subleading soft factor $S_n^{(1)-}$.

Given that the subleading soft graviton theorem is one-loop exact [66], it follows that the all-loop soft graviton theorem is

$$\mathcal{M}_{n+1}^- \xrightarrow{q \rightarrow 0} \left[S_n^{(0)-} + S_n^{(1)-} + \kappa^2 \left(\frac{\sigma'_{n+1}}{\epsilon} S_n^{(0)-} - \frac{1}{\epsilon} \left(S_n^{(1)-} \sigma_n \right) + \Delta_{\text{fin}} S_n^{(1)-} \right) \right] \mathcal{M}_n, \quad (8.3.9)$$

where the terms in square brackets proportional to κ^2 are the IR divergent and finite parts of the anomaly.

Little appears to be currently known about $\Delta_{\text{fin}} S_n^{(1)-}$. In all explicitly checked cases, including all identical helicity amplitudes and certain low-point single negative-helicity amplitudes, it was demonstrated that there are no IR finite corrections to the subleading soft graviton theorem [66, 67], implying $\Delta_{\text{fin}} S_n^{(1)-} = 0$ for these cases. Nevertheless, we are unaware of any argument indicating that this term always vanishes, or on the contrary that its form is universal. In the absence of such information, we will restrict our consideration to the universal divergent correction given in (8.3.6).

8.4 One-Loop Correction to the Energy-Momentum Tensor

The one-loop corrections (8.3.9) to the subleading soft factor are expected to result in corrections to the tree-level Virasoro-Ward identity (8.2.23). In this section, we show that this is indeed the case. Moreover, we find that the effects of the universal divergent correction in (8.3.6) can be eliminated by a corresponding one-loop correction to the energy-momentum tensor. That is, whenever we have $\Delta_{\text{fin}} S_n^{(1)-} = 0$, the shifted energy-momentum tensor obeys the unshifted Virasoro-Ward identity (8.2.23).

The tree-level matrix elements of the operator $N_{\bar{z}\bar{z}}^{(1)}$ are given by (8.2.19). At one-loop level, the matrix elements acquire a divergent correction of the form

$$\langle \text{out} | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} | \text{in} \rangle \Big|_{\text{div}} = \frac{i\kappa^3}{8\pi} \hat{\epsilon}_{\bar{z}\bar{z}} \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left(\frac{\sigma'_{n+1}}{\epsilon} S_n^{(0)-} - \frac{1}{\epsilon} \left(S_n^{(1)-} \sigma_n \right) \right) \langle \text{out} | \mathcal{S} | \text{in} \rangle. \quad (8.4.1)$$

It immediately follows from (8.2.22) that the IR divergent one-loop correction to the T_{zz} Ward identity is given by

$$\begin{aligned} & \langle \text{out} | \Delta T_{zz} \mathcal{S} | \text{in} \rangle \\ &= -\frac{\kappa}{2\pi\epsilon} \int d^2w \frac{\gamma^{w\bar{w}}}{z-w} D_w^3 \left[\hat{\epsilon}_{\bar{w}\bar{w}} \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left(\sigma'_{n+1} S_n^{(0)-} - \left(S_n^{(1)-} \sigma_n \right) \right) \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle , \end{aligned} \quad (8.4.2)$$

where

$$\langle \text{out} | T_{zz} \mathcal{S} | \text{in} \rangle |_{\text{div}} \equiv \langle \text{out} | \Delta T_{zz} \mathcal{S} | \text{in} \rangle . \quad (8.4.3)$$

It is far from obvious, but nevertheless possible, to rewrite this in terms of the zero-modes of the Bondi news. This computation is done explicitly in appendix 8.A, and we find that ΔT_{zz} can be expressed as

$$\Delta T_{zz} = -\frac{2}{\pi\kappa^2\epsilon} \int d^2w \frac{\gamma^{w\bar{w}}}{z-w} \left(2N_{ww}^{(0)} D_w N_{\bar{w}\bar{w}}^{(0)} + D_w \left(N_{ww}^{(0)} N_{\bar{w}\bar{w}}^{(0)} \right) \right) . \quad (8.4.4)$$

Hence, the shifted energy-momentum tensor, given by

$$\tilde{T}_{zz} = T_{zz} - \Delta T_{zz} , \quad (8.4.5)$$

obeys the unshifted Ward identity

$$\langle \text{out} | \tilde{T}_{zz} \mathcal{S} | \text{in} \rangle = \sum_{k=1}^n \left[\frac{h_k}{(z-z_k)^2} + \frac{h_k}{z-z_k} \Gamma_{z_k z_k}^{z_k} + \frac{1}{z-z_k} (\partial_{z_k} - |s_k| \Omega_{z_k}) \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle \quad (8.4.6)$$

to all orders, whenever $\Delta_{\text{fin}} S_n^{(1)-} = 0$.

This result seems interesting for a number of reasons. First of all, note that while the renormalized soft factor contains logarithms and explicit dependence on the renormalization scale, such terms do not appear in the anomalous contribution to the energy-momentum tensor. Furthermore, the fact that the divergence takes the form of a matrix element involving only the local operators $N_{ww}^{(0)}(w)$ and $N_{\bar{w}\bar{w}}^{(0)}(w)$ allows us to perform an “IR renormalization” of the operator T_{zz} by sub-

tracting away the divergent operator. The form of the divergence, when rewritten in terms of the soft graviton operators, is reminiscent of the forward limit of a scattering amplitude. However, it remains to be seen whether or not there are finite corrections to the Ward identity (8.4.6) and, if so, whether or not they can be eliminated by a further finite shift of the energy-momentum tensor.

8.A IR Divergence of One-Loop T_{zz} Correction

In this appendix, we explicitly compute the matrix elements of ΔT_{zz} given in (8.4.2) by

$$\begin{aligned} & \langle \text{out} | \Delta T_{zz} \mathcal{S} | \text{in} \rangle \\ &= -\frac{\kappa}{2\pi\epsilon} \int d^2w \frac{\gamma^{w\bar{w}}}{z-w} D_w^3 \left[\hat{\varepsilon}_{\bar{w}\bar{w}} \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left(\sigma'_{n+1} S_n^{(0)-} - \left(S_n^{(1)-} \sigma_n \right) \right) \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle. \end{aligned} \quad (8.A.1)$$

For completeness, we recall the expressions for the leading and subleading tree-level soft factors:

$$S_n^{(0)\pm} = \frac{\kappa}{2} \sum_{k=1}^n \frac{p_k^\mu p_k^\nu \varepsilon_{\mu\nu}^\pm(q)}{p_k \cdot q}, \quad S_n^{(1)\pm} = -\frac{i\kappa}{2} \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^\pm(q) p_k^\mu q_\lambda}{p_k \cdot q} \mathcal{J}_k^{\lambda\nu}. \quad (8.A.2)$$

Since $S_n^{(1)-}$ acts on a scalar in (8.A.1), the action of $\mathcal{J}_{k\mu\nu}$ is given by

$$\mathcal{J}_{k\mu\nu} \sigma_n = -i \left[p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right] \sigma_n. \quad (8.A.3)$$

Using (8.A.2), (8.A.3), and momentum conservation, it follows that

$$\begin{aligned} \Delta_{\text{div}} S_n^{(1)-} &\equiv \frac{1}{\epsilon} \left[\sigma'_{n+1} S_n^{(0)-} - \left(S_n^{(1)-} \sigma_n \right) \right] \\ &= \frac{\kappa}{4(4\pi)^2 \epsilon} \sum_{i,j=1}^n \left[\frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} (p_j \cdot q) \log \frac{p_j \cdot q}{p_i \cdot p_j} - (p_i \cdot \varepsilon^-) (p_j \cdot \varepsilon^-) \log \frac{\mu^2}{-2p_i \cdot p_j} \right]. \end{aligned} \quad (8.A.4)$$

Momentum conservation implies that (8.A.4) is independent of both the soft energy ω and the renormalization scale μ . It follows that (8.A.1) becomes

$$\langle \text{out} | \Delta T_{zz} \mathcal{S} | \text{in} \rangle = -\frac{\kappa}{2\pi} \int d^2w \frac{\gamma^{w\bar{w}}}{z-w} D_w^3 \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \Delta_{\text{div}} S_n^{(1)-} \right) \langle \text{out} | \mathcal{S} | \text{in} \rangle. \quad (8.A.5)$$

Before proceeding, it is useful to define the quantity

$$\hat{\varepsilon}_{\bar{w}} \equiv \partial_{\bar{w}} \hat{x}^i(w) \varepsilon_i^+(q(w)) = \frac{\sqrt{2}}{1 + w\bar{w}} , \quad (8.A.6)$$

so that $\hat{\varepsilon}_{\bar{w}\bar{w}} = \hat{\varepsilon}_{\bar{w}} \hat{\varepsilon}_{\bar{w}}$. It is then straightforward to show that

$$\begin{aligned} D_w^2 (\hat{\varepsilon}_{\bar{w}} \varepsilon^-) &= 0 , \\ D_w^2 q &= 0 , \\ D^{\bar{w}} D_w \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot \hat{q}} \right) &= -2\pi\omega_i \delta^{(2)}(w - z_i) , \end{aligned} \quad (8.A.7)$$

where $q^\mu = \omega \hat{q}^\mu$. Using the first of the above identities, we have

$$D_w^3 \left(\hat{\varepsilon}_{\bar{w}\bar{w}} (p_i \cdot \varepsilon^-) (p_j \cdot \varepsilon^-) \log \frac{\mu^2}{-2p_i \cdot p_j} \right) = 0 , \quad (8.A.8)$$

which implies

$$D_w^3 \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \Delta_{\text{div}} S_n^{(1)-} \right) = \frac{\kappa}{4(4\pi)^2 \epsilon} \sum_{i,j}^n D_w^3 \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} (p_j \cdot q) \log \frac{p_j \cdot q}{p_i \cdot p_j} \right) . \quad (8.A.9)$$

To evaluate this, we distribute the covariant derivatives via the product rule and first compute the term

$$\begin{aligned} &\sum_{i,j=1}^n D_w^3 \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} (p_j \cdot q) \right) \log \frac{p_j \cdot q}{p_i \cdot p_j} \\ &= -2\pi\gamma_{w\bar{w}} \sum_{i,j=1}^n \omega_i \left[(p_j \cdot \hat{q}) D_w \delta^{(2)}(w - z_i) + 3(p_j \cdot \partial_w \hat{q}) \delta^{(2)}(w - z_i) \right] \log \frac{p_j \cdot \hat{q}}{p_j \cdot \hat{p}_i} , \end{aligned} \quad (8.A.10)$$

where we have used the last two identities of (8.A.7) along with momentum conservation. Similarly, using (8.A.7) and momentum conservation, we find

$$\begin{aligned} 3 \sum_{i,j=1}^n D_w \left[D_w \left(\hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} (p_j \cdot q) \right) D_w \left(\log \frac{p_j \cdot q}{p_i \cdot p_j} \right) \right] \\ = 3 \sum_{i,j=1}^n D_w \left[\hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} \frac{(p_j \cdot \partial_w q)^2}{p_j \cdot q} \right] , \end{aligned} \quad (8.A.11)$$

and

$$\sum_{i,j=1}^n \hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} (p_j \cdot q) D_w^3 \left(\log \frac{p_j \cdot q}{p_i \cdot p_j} \right) = \sum_{i,j=1}^n \hat{\varepsilon}_{\bar{w}\bar{w}} \frac{(p_i \cdot \varepsilon^-)^2}{p_i \cdot q} \frac{2}{(p_j \cdot q)^2} (p_j \cdot \partial_w q)^3 . \quad (8.A.12)$$

Finally, using momentum conservation and the relationship between the soft momenta and polarization vectors [135]

$$\varepsilon_\mu^+ = \partial_w \left(\frac{1}{\sqrt{\gamma_{w\bar{w}}}} \hat{q}_\mu \right) , \quad (8.A.13)$$

we find

$$\sum_{j=1}^n \frac{(p_j \cdot \partial_w \hat{q})^2}{p_j \cdot \hat{q}} = \sum_{j=1}^n \hat{\varepsilon}_{w\bar{w}} \frac{(p_j \cdot \varepsilon^+)^2}{p_j \cdot \hat{q}} . \quad (8.A.14)$$

Substituting (8.A.10), (8.A.11), and (8.A.12) into (8.A.9), and then using (8.2.17) along with (8.A.14), we find

$$\begin{aligned} \langle \text{out} | \Delta T_{zz} \mathcal{S} | \text{in} \rangle \\ = -\frac{2}{\pi \kappa^2 \epsilon} \int d^2 w \frac{\gamma^{w\bar{w}}}{z-w} \langle \text{out} | \left[-2N_{\bar{w}\bar{w}}^{(0)} D_w N_{w\bar{w}}^{(0)} + 3D_w \left(N_{w\bar{w}}^{(0)} N_{\bar{w}\bar{w}}^{(0)} \right) \right] \mathcal{S} | \text{in} \rangle \\ = -\frac{2}{\pi \kappa^2 \epsilon} \int d^2 w \frac{\gamma^{w\bar{w}}}{z-w} \langle \text{out} | \left[2N_{w\bar{w}}^{(0)} D_w N_{\bar{w}\bar{w}}^{(0)} + D_w \left(N_{w\bar{w}}^{(0)} N_{\bar{w}\bar{w}}^{(0)} \right) \right] \mathcal{S} | \text{in} \rangle , \end{aligned} \quad (8.A.15)$$

which is precisely (8.4.4).

9

A d -Dimensional Stress Tensor for Mink_{d+2}

Gravity

9.1 Introduction

The lightlike boundary of asymptotically flat spacetimes, $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$, is a null cone with a (possibly singular) vertex at spatial infinity. Massless excitations propagating in such a spacetime pass through \mathcal{I} at isolated points on the celestial sphere. Guided by the holographic principle, one might hope that the \mathcal{S} -matrix for the scattering of massless particles in asymptotically flat spacetimes in $(d + 2)$ -dimensions might be re-expressed as a collection of correlation functions of local operators on the celestial sphere S^d at null infinity, with operator insertions at the points where the particles enter or exit the spacetime. The Lorentz group would then be realized as the group of conformal motions of the celestial sphere, and the Lorentz covariance of the \mathcal{S} -matrix

would guarantee that the local operators have well-defined transformation laws under the action of the Euclidean conformal group $SO(d+1, 1)$. On these general grounds one expects the massless \mathcal{S} -matrix to display some of the features of a d -dimensional Euclidean conformal field theory (CFT_d).

It has recently become possible to make some of these statements more precise in four dimensions, due in large part to Strominger’s infrared triangle that relates soft theorems, asymptotic symmetry groups and memory effects [30–37, 39, 40, 42, 43, 45, 80, 81, 83, 84, 88, 96, 100, 113, 124, 125, 128, 129, 134, 138–152]. While the specific details of a putative holographic formulation are expected to be model dependent, it should be possible to make robust statements (primarily regarding symmetries) based on universal properties of the \mathcal{S} -matrix. One interesting class of universal statements about the \mathcal{S} -matrix concerns the so-called soft limits [4–8, 91, 102, 123] of scattering amplitudes. In the limit when the wavelength of an external gauge boson or graviton becomes much larger than any scale in the scattering process, the \mathcal{S} -matrix factorizes into a universal soft operator (controlled by the soft particle and the quantum numbers of the hard particles) acting on the amplitude without the soft insertion. This sort of factorization is reminiscent of a Ward identity, and indeed in four dimensions the soft photon, soft gluon, and soft graviton theorems have been recast in the form of Ward identities for conserved operators in a putative CFT_2 [32, 96, 127, 134, 135, 144, 146]. Most importantly for the present work, in [134, 135, 144] an operator was constructed from the subleading soft graviton theorem whose insertion into the four-dimensional \mathcal{S} -matrix reproduces the Virasoro Ward identities of a CFT_2 energy-momentum tensor. The subleading soft graviton theorem holds in all dimensions [15, 61, 66, 73, 153–155], so it should be possible to construct an analogous operator in any dimension. We will see that this is indeed the case, and that the construction is essentially fixed by Lorentz (conformal) invariance.

The organization of this chapter is as follows. In section 9.2 we establish our conventions for massless particle kinematics and describe the map from the $(d+2)$ -dimensional \mathcal{S} -matrix to a set of d -dimensional “celestial correlators” defined on a spacelike cut of the null momentum cone. Section 9.3 describes the realization of the Euclidean conformal group on these correlation functions in terms of the embedding space formalism. Section 9.4 outlines the construction of

conserved currents – namely the conserved $U(1)$ current and the stress tensor – in the boundary theory and their relations to the leading soft photon and subleading soft graviton theorems. Section 9.5 concludes with a series of open questions. In appendix 9.A, we briefly discuss the bulk spacetime interpretation of our results and their relations to previous work.

9.2 Massless Particle Kinematics

The basic observable in asymptotically flat quantum gravity is the \mathcal{S} -matrix element

$$\mathcal{S} = \langle \text{out} \mid \text{in} \rangle \quad (9.2.1)$$

between an incoming state on past null infinity (\mathcal{I}^-) and an outgoing state on future null infinity (\mathcal{I}^+). The perturbative scattering states in asymptotically flat spacetimes are characterized by collections of well separated, non-interacting particles.¹ Each massless particle is characterized by a null momentum p^μ and a representation of the little group $SO(d)$, as well as a collection of other quantum numbers such as charge, flavor, etc. Null momenta are constrained to lie on the future light cone \mathcal{C}^+ of the origin in momentum space $\mathbb{R}^{d+1,1}$,

$$\mathcal{C}^+ = \{p^\mu \in \mathbb{R}^{d+1,1} \mid p^2 = 0, p^0 > 0\} . \quad (9.2.2)$$

A convenient parametrization for the momentum, familiar from the embedding space formalism in conformal field theories [156, 157], is given by

$$p^\mu(\omega, x) = \omega \Omega(x) \hat{p}^\mu(x) , \quad \hat{p}^\mu(x) = \left(\frac{1+x^2}{2}, x^a, \frac{1-x^2}{2} \right) , \quad \omega \geq 0 , \quad x^a \in \mathbb{R}^d , \quad (9.2.3)$$

¹In four dimensions, the probability to scatter into a state with a finite number of gauge bosons or gravitons is zero due to infrared divergences [102]. In higher dimensions, infrared divergences are absent and one can safely consider the usual Fock space basis of scattering states.

where $x^2 = x^a x_a = \delta_{ab} x^a x^b$. The metric on this null cone is degenerate and is given by

$$ds_{\mathcal{C}^+}^2 = dp^\mu dp_\mu = 0 d\omega^2 + \omega^2 \Omega(x)^2 dx^a dx_a . \quad (9.2.4)$$

For a fixed ω , this parametrization specifies a d -dimensional spacelike cut \mathcal{M}_d of the future light cone with a conformally flat metric induced from the flat Lorentzian metric on $\mathbb{R}^{d+1,1}$:

$$ds_{\mathcal{M}_d}^2 = \Omega(x)^2 dx^a dx_a . \quad (9.2.5)$$

$\Omega(x)$ defines the conformal factor on \mathcal{M}_d . In most of what follows we will choose $\Omega(x) = 1$ for computational simplicity, although the generalization to an arbitrary conformally flat Euclidean cut is straightforward.² The $(d+2)$ -dimensional Lorentz-invariant measure takes the form

$$\int \frac{d^{d+1}p}{p^0} = \int d^d x \int d\omega \omega^{d-1} , \quad (9.2.6)$$

while the Lorentzian inner product is given by

$$-2\hat{p}(x_1) \cdot \hat{p}(x_2) = (x_1 - x_2)^2 . \quad (9.2.7)$$

Massless particles of spin s can be described by symmetric traceless fields

$$\Phi_{\mu_1 \dots \mu_s}(X) = \sum_{a_i} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{1}{2\omega_p} \varepsilon_{\mu_1 \dots \mu_s}^{a_1 \dots a_s}(p) \left[\mathcal{O}_{a_1 \dots a_s}(p) e^{ip \cdot X} + \mathcal{O}_{a_1 \dots a_s}^\dagger(p) e^{-ip \cdot X} \right] \quad (9.2.8)$$

satisfying the equations

$$\square_X \Phi_{\mu_1 \dots \mu_s}(X) = 0 , \quad \partial_{\mu_1} \Phi^{\mu_1}_{\mu_2 \dots \mu_s}(X) = 0 . \quad (9.2.9)$$

²Other choices of $\Omega(x)$ have also proved useful in previous analyses. In particular, the authors of [15, 30, 31, 34, 81] choose $\Omega(x) = 2(1+x^2)^{-1}$, yielding the round metric on S^d . For non-constant $\Omega(x)$, d -dimensional partial derivatives are simply promoted to covariant derivatives, and powers of the Laplacian are replaced by their conformally covariant counterparts, the GJMS operators [158].

Under gauge transformations,

$$\Phi_{\mu_1 \dots \mu_s}(X) \rightarrow \Phi_{\mu_1 \dots \mu_s}(X) + \partial_{(\mu_1} \lambda_{\mu_2 \dots \mu_s)}(X) . \quad (9.2.10)$$

We will work in the gauge

$$n^{\mu_1} \Phi_{\mu_1 \dots \mu_s}(X) = 0 , \quad n^\mu = (1, 0^a, -1) . \quad (9.2.11)$$

A natural basis for the vector representation of the little group $SO(d)$ is given in terms of the d polarization vectors

$$\varepsilon_a^\mu(x) \equiv \partial_a \hat{p}^\mu(x) = (x_a, \delta_a^b, -x_a) . \quad (9.2.12)$$

These are orthogonal to both n and \hat{p} and satisfy

$$\varepsilon_a(x) \cdot \varepsilon_b(x) = \delta_{ab} , \quad \varepsilon_\mu^a(x) \varepsilon_a^\nu(x) = \Pi_\mu^\nu(x) \equiv \delta_\mu^\nu + n_\mu \hat{p}^\nu(x) + n^\nu \hat{p}_\mu(x) . \quad (9.2.13)$$

We also note the property

$$\hat{p}(x) \cdot \varepsilon_a(x') = x_a - x'_a . \quad (9.2.14)$$

The polarization tensors for higher spin representations of the little group can be constructed from the spin-1 polarization forms. For instance, the graviton's polarization tensor is given by

$$\varepsilon_{\mu\nu}^{ab}(x) = \frac{1}{2} \left[\varepsilon_\mu^a(x) \varepsilon_\nu^b(x) + \varepsilon_\mu^b(x) \varepsilon_\nu^a(x) \right] - \frac{1}{d} \delta^{ab} \Pi_{\mu\nu}(x) . \quad (9.2.15)$$

The Fock space of massless scattering states is generated by the algebra of single particle creation and annihilation operators satisfying the standard commutation relations

$$\left[\mathcal{O}_a(p), \mathcal{O}_b(p')^\dagger \right] = (2\pi)^{d+1} \delta_{ab} (2p^0) \delta^{(d+1)}(\vec{p} - \vec{p}') . \quad (9.2.16)$$

We can rewrite these relations in terms of our parametrization (9.2.3) of the momentum light cone

$$\left[\mathcal{O}_a(\omega, x), \mathcal{O}_b(\omega', x')^\dagger \right] = 2(2\pi)^{d+1} \delta_{ab} \omega^{1-d} \delta(\omega - \omega') \delta^{(d)}(x - x') . \quad (9.2.17)$$

Note that in terms of the commutation relations of local operators defined on the light cone, the energy direction actually appears spacelike rather than timelike.³ The creation and annihilation operators can be viewed as operators inserted at specific points of \mathcal{M}_d carrying an additional quantum number ω , so that the \mathcal{S} -matrix takes the form of a conformal correlator of primary operators. In other words, the amplitude with m incoming and $n - m$ outgoing particles⁴

$$\mathcal{A}_{n,m} = \langle p_{m+1}, \dots, p_n | p_1, \dots, p_m \rangle \quad (9.2.18)$$

can be equivalently represented as a correlation function on \mathcal{M}_d

$$\mathcal{A}_n = \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle_{\mathcal{M}_d} = \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle_{\mathcal{M}_d} . \quad (9.2.19)$$

In this representation, outgoing states have $\omega > 0$ and ingoing states have $\omega < 0$. In the rest of this chapter, we will freely interchange the notation $p_i \Leftrightarrow (\omega_i, x_i)$ to describe the insertion of local operators.

In appendix 9.A, we demonstrate that the spacelike cut \mathcal{M}_d of the momentum cone is naturally identified with the cross-sectional cuts of \mathcal{I}^+ . We construct the bulk coordinates whose limiting metric on future null infinity is that of a null cone with cross-sectional metric (9.2.5). This provides a holographic interpretation of our construction, recasting scattering amplitudes in asymptotically flat spacetimes in terms of a “boundary” theory that lives on \mathcal{I}^+ .

³This is also the case for the null direction on \mathcal{I} in asymptotic quantization.

⁴Here, we have suppressed all other quantum numbers that label the one-particle states, such as the polarization vectors, flavor indices, or charge quantum numbers.

9.3 Lorentz Transformations and the Conformal Group

In this section, we make explicit the map from $(d+2)$ -dimensional momentum space \mathcal{S} -matrix elements to conformal correlators on the Euclidean manifold \mathcal{M}_d . The setup is mathematically similar to the embedding space formalism, although in this case the $(d+2)$ -dimensional “embedding space” is the physical momentum space rather than merely an auxiliary ambient construction. The Lorentz group $SO(d+1, 1)$ with generators $M_{\mu\nu}$ acts linearly on momentum space vectors $p^\mu \in \mathbb{R}^{d+1,1}$. The isomorphism with the conformal group of \mathcal{M}_d is given by the identifications

$$J_{ab} = M_{ab} , \quad T_a = M_{0,a} - M_{d+1,a} , \quad D = M_{d+1,0} , \quad K_a = M_{0,a} + M_{d+1,a} . \quad (9.3.1)$$

The J_{ab} generate $SO(d)$ rotations, D is the dilation operator, and T_a and K_a are the generators of translations and special conformal transformations, respectively. These operators satisfy the familiar conformal algebra

$$\begin{aligned} [J_{ab}, J_{cd}] &= i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}) , \\ [J_{ab}, T_c] &= i(\delta_{ac}T_b - \delta_{bc}T_a) , \\ [J_{ab}, K_c] &= i(\delta_{ac}K_b - \delta_{bc}K_a) , \\ [D, T_a] &= -iT_a , \\ [D, K_a] &= iK_a , \\ [T_a, K_b] &= -2i(\delta_{ab}D + J_{ab}) . \end{aligned} \quad (9.3.2)$$

Equation (9.3.1) describes the precise map between the linear action of Lorentz transformations on p^μ and nonlinear conformal transformations on (ω, x) . The latter amount to transformations $x \rightarrow x'(x)$ for which

$$\frac{\partial x'^c}{\partial x^a} \frac{\partial x'^d}{\partial x^b} \delta_{cd} = \gamma(x)^2 \delta_{ab} \quad (9.3.3)$$

along with

$$\omega \rightarrow \omega' = \frac{\omega}{\gamma(x)} . \quad (9.3.4)$$

Using (9.3.1), the conformal properties of the operators $\mathcal{O}(\omega, x)$ (we suppress spin labels for convenience) follow from their Lorentz transformations,

$$[\mathcal{O}(p), M_{\mu\nu}] = \mathcal{L}_{\mu\nu} \mathcal{O}(p) + \mathcal{S}_{\mu\nu} \cdot \mathcal{O}(p) , \quad [\mathcal{O}(p), P_\mu] = -p_\mu \mathcal{O}(p) , \quad (9.3.5)$$

where

$$\mathcal{L}_{\mu\nu} = -i \left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu} \right) \quad (9.3.6)$$

and $\mathcal{S}_{\mu\nu}$ denotes the spin- s representation of the Lorentz group. We would like to rewrite these relations in a way that manifests the action on \mathcal{M}_d . First, we note that

$$\frac{\partial \omega}{\partial p^a} = \frac{2x_a}{1+x^2} , \quad \omega \frac{\partial x^a}{\partial p^b} = \delta_b^a - \frac{2x^a x_b}{1+x^2} , \quad \frac{\partial \omega}{\partial p^{d+1}} = \frac{2}{1+x^2} , \quad \omega \frac{\partial x^a}{\partial p^{d+1}} = -\frac{2x^a}{1+x^2} . \quad (9.3.7)$$

It follows that

$$\begin{aligned} \mathcal{L}_{0,a} &= ix_a [\omega \partial_\omega - x^b \partial_b] + \frac{i}{2} (1+x^2) \partial_a , & \mathcal{L}_{0,d+1} &= i [\omega \partial_\omega - x^a \partial_a] , \\ \mathcal{L}_{a,d+1} &= -ix_a [\omega \partial_\omega - x^b \partial_b] + \frac{i}{2} (1-x^2) \partial_a , & \mathcal{L}_{ab} &= -i [x_a \partial_b - x_b \partial_a] . \end{aligned} \quad (9.3.8)$$

The action of the spin matrix $\mathcal{S}_{\mu\nu}$ can be conveniently expressed in terms of the polarization vectors (9.2.12). For instance, the action on a spin-1 state is given by

$$[\mathcal{S}_{\mu\nu}]_a{}^b = -i [\varepsilon_{\mu a} \varepsilon_\nu^b - \varepsilon_{\nu a} \varepsilon_\mu^b] + \varepsilon_a^\rho \mathcal{L}_{\mu\nu} \varepsilon_\rho^b . \quad (9.3.9)$$

In general one finds

$$\mathcal{S}_{0,d+1} = 0 , \quad \mathcal{S}_{0a} = \mathcal{S}_{d+1,a} = x^b \mathcal{S}_{ab} , \quad (9.3.10)$$

where \mathcal{S}_{ab} is the representation of the massless little group $SO(d)$. The action of the conformal

group on the creation and annihilation operators is then given by

$$\begin{aligned}
 [\mathcal{O}(\omega, x), T_a] &= i\partial_a \mathcal{O}(\omega, x) , \\
 [\mathcal{O}(\omega, x), J_{ab}] &= -i(x_a \partial_b - x_b \partial_a) \mathcal{O}(\omega, x) + \mathcal{S}_{ab} \cdot \mathcal{O}(\omega, x) , \\
 [\mathcal{O}(\omega, x), D] &= i(x^a \partial_a - \omega \partial_\omega) \mathcal{O}(\omega, x) , \\
 [\mathcal{O}(\omega, x), K_a] &= i(x^2 \partial_a - 2x_a x^b \partial_b + 2x_a \omega \partial_\omega) \mathcal{O}(\omega, x) + 2x^b \mathcal{S}_{ab} \cdot \mathcal{O}(\omega, x) .
 \end{aligned} \tag{9.3.11}$$

We recognize these commutation relations as the defining properties of a spin- s conformal primary operator, with a non-standard dilation eigenvalue

$$\Delta = -\omega \partial_\omega . \tag{9.3.12}$$

The fact that Δ is realized as a derivative simply reflects the fact that the energy eigenstates do not diagonalize the dilation operator, which simply translates the spacelike cut \mathcal{M}_d of the momentum cone along its null direction.⁵

9.4 Conserved Currents and Soft Theorems

The operator content and correlation functions of the theory living on \mathcal{M}_d are highly dependent on the spectrum and interactions of the $(d+2)$ -dimensional theory under consideration. However, the universal properties of the $(d+2)$ -dimensional \mathcal{S} -matrix are expected to translate into general, model independent features of the “boundary theory.” In this section, we explore the consequences of the universal soft factorization properties of \mathcal{S} -matrix elements. Soft factorization formulas closely resemble Ward identities, and indeed many soft theorems are known to be intimately related to symmetries of the \mathcal{S} -matrix. The existence of the soft theorems should therefore enable one to construct associated conserved currents for the theory living on \mathcal{M}_d . The appropriate currents were constructed for $d=2$ in [32, 96, 134, 135, 144, 146]. Here, we generalize these results to $d > 2$.

⁵It is possible to obtain standard conformal primary operators via a Mellin transform, $\mathcal{O}(\Delta, x) = \int_{\mathcal{C}} d\omega \omega^{\Delta-1} \mathcal{O}(\omega, x)$ for some contour \mathcal{C} in the complex ω plane.

9.4.1 Leading Soft Photon Theorem and the Conserved $U(1)$ Current

The leading soft photon theorem is a universal statement about the behavior of \mathcal{S} -matrix elements in the limit that an external photon's momentum tends to zero. It is model independent, exists in any dimension, and states that

$$\langle \mathcal{O}_a(q) \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle \xrightarrow{q^0 \rightarrow 0} \sum_{k=1}^n Q_k \frac{\varepsilon_a \cdot p_k}{q \cdot p_k} \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle, \quad (9.4.1)$$

where $\mathcal{O}_a(\omega, x)$ creates an outgoing photon of momentum $p(\omega, x)$ and polarization $\varepsilon_a^\mu(x)$, and Q_k is the charge of the k^{th} particle. We will first define the “leading soft photon operator”

$$S_a(x) = \lim_{\omega \rightarrow 0} \omega \mathcal{O}_a(\omega, x). \quad (9.4.2)$$

Insertions of this operator are controlled by the leading soft photon theorem (9.4.1) and take the form

$$\langle S_a(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = \partial_a \sum_{k=1}^n Q_k \log[(x - x_k)^2] \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle. \quad (9.4.3)$$

From (9.4.3), we see that $S_a(x)$ is a conformal primary operator with $(\Delta, s) = (1, 1)$. Note also that $S_a(x)$ satisfies

$$\partial_a S_b - \partial_b S_a = 0 \quad (9.4.4)$$

identically, without contact terms. In even dimensions,⁶ the leading soft photon theorem is known [34, 36, 37, 80, 81] to be completely equivalent to the invariance of the \mathcal{S} -matrix under a group of angle dependent $U(1)$ gauge transformations with noncompact support. In $d = 2$, this symmetry is generated by the action of a holomorphic boundary current J_z satisfying the appropriate Kac-

⁶From this point on, we will consider only the even-dimensional case in order to avoid discussion of fractional powers of the Laplacian.

Moody Ward identities (see [96]). In higher dimensions, one consequently expects to encounter a conformal primary operator $J_a(x)$ with $(\Delta, s) = (d-1, 1)$ satisfying the Ward identity

$$\langle \partial^b J_b(y) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = \sum_{k=1}^n Q_k \delta^{(d)}(y - x_k) \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle . \quad (9.4.5)$$

Our goal is to construct the conserved current $J_a(x)$ from the soft operator $S_a(x)$. The inverse problem – constructing $S_a(x)$ from an operator $J_a(x)$ satisfying (9.4.5) – is easily solved. Multiplying both sides of (9.4.5) by $\int d^d y \partial_a \log[(x - y)^2]$, we find

$$\begin{aligned} \int d^d y \partial_a \log[(x - y)^2] \langle \partial^b J_b(y) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle \\ = \partial_a \sum_{k=1}^n Q_k \log[(x - x_k)^2] \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle \\ = \langle S_a(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle . \end{aligned} \quad (9.4.6)$$

We therefore identify⁷

$$S_a(x) = \int d^d y \partial_a \log[(x - y)^2] \partial^b J_b(y) = 2 \int d^d y \frac{\mathcal{I}_{ab}(x - y)}{(x - y)^2} J^b(y) , \quad (9.4.7)$$

where $\mathcal{I}_{ab}(x - y)$ is the conformally covariant tensor

$$\mathcal{I}_{ab}(x - y) = \delta_{ab} - 2 \frac{(x - y)_a (x - y)_b}{(x - y)^2} . \quad (9.4.8)$$

This nonlocal relationship between the $\Delta = 1$ primary S_a and the $\Delta = d - 1$ primary J_a is known as a shadow transform. For a spin- s operator of dimension Δ , the shadow operator is given by [156]

$$\tilde{\mathcal{O}}_{a_1 \dots a_s}(x) = \delta_{a_1 \dots a_s}^{b_1 \dots b_s} \int d^d y \frac{\mathcal{I}_{b_1 c_1}(x - y) \dots \mathcal{I}_{b_s c_s}(x - y)}{[(x - y)^2]^{d-\Delta}} \mathcal{O}^{c_1 \dots c_s}(y) . \quad (9.4.9)$$

⁷Note that this integral expression is insensitive to improvement terms of the form $J_a \rightarrow J_a + \partial^b K_{[ba]}$ which do not affect the Ward identity (9.4.5).

Here, $\delta_{a_1 \dots a_s}^{b_1 \dots b_s}$ is the invariant identity tensor in the spin- s representation,

$$\delta_{a_1 \dots a_s}^{b_1 \dots b_s} = \delta_{\{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s\}} , \quad (9.4.10)$$

where the notation $\{, \}$ denotes the symmetric traceless projection on the indicated indices. The shadow transform is the unique integral transform that maps conformal primary operators with (Δ, s) onto conformal primary operators with $(d - \Delta, s)$. Given that S_a has $(\Delta, s) = (1, 1)$ while J_a has $(\Delta, s) = (d - 1, 1)$, it seems natural to expect the appearance of the shadow transform.

The shadow transform is, up to normalization [156, 159], its own inverse⁸

$$\tilde{\tilde{\mathcal{O}}}_{a_1 \dots a_s}(x) = c(\Delta, s) \mathcal{O}_{a_1 \dots a_s}(x) , \quad c(\Delta, s) = \frac{\pi^d (\Delta - 1)(d - \Delta - 1) \Gamma(\frac{d}{2} - \Delta) \Gamma(\Delta - \frac{d}{2})}{(\Delta - 1 + s)(d - \Delta - 1 + s) \Gamma(\Delta) \Gamma(d - \Delta)} . \quad (9.4.11)$$

Using this, we can immediately write

$$S_a(x) = 2\tilde{J}_a(x) , \quad J_a(x) = \frac{1}{2c(1, 1)} \tilde{S}_a(x) . \quad (9.4.12)$$

Interestingly, the property (9.4.4) allows one to obtain a local relation between $J_a(x)$ and $S_a(x)$

$$J_a(x) = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} (-\square)^{\frac{d}{2}-1} S_a(x) . \quad (9.4.13)$$

It is straightforward to verify that insertions of $J_a(x)$ are given by

$$\langle J_a(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = \frac{\Gamma(d/2)}{2\pi^{d/2}} \sum_{k=1}^n Q_k \frac{(x - x_k)_a}{|x - x_k|^d} \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle \quad (9.4.14)$$

and satisfy (9.4.5).

In summary, we find that the leading soft photon theorem in any dimension implies the existence of a conserved current $J_a(x)$ on the spatial cut \mathcal{M}_d . This current is constructed as the shadow

⁸The spatial integrals involved here are formally divergent and are regulated by the $i\epsilon$ -prescription.

transform of the soft photon operator $S_a(x)$. This correspondence is reminiscent of a similar construction in AdS/CFT, where again the presence of a massless bulk gauge field produces a dual conserved boundary current. There have been attempts to make this analogy more precise using a so-called “holographic reduction” of Minkowski space [14, 135]. The $(d + 2)$ -dimensional Minkowski space can be foliated by hyperboloids (and a null cone), each of which is invariant under the action of the Lorentz group. Inside the light cone, this amounts to a foliation using a family of Euclidean AdS_{d+1} , all sharing an asymptotic boundary given by the celestial sphere.⁹ Performing a Kaluza-Klein reduction on the (timelike, noncompact) direction transverse to the AdS_{d+1} slices decomposes the gauge field $A(X)$ in Minkowski space into a continuum of AdS_{d+1} gauge fields $A_\omega(x)$ with masses $\sim \omega$ (ω is the so-called *Milne* energy). The $\omega \rightarrow 0$ gauge field – equivalent to the soft limit – is massless in AdS_{d+1} and therefore induces a conserved current on the d -dimensional boundary. The holographic dictionary suggests that in the boundary theory, one has a coupling of the form

$$\int d^d x S^a(x) J_a(x) . \quad (9.4.15)$$

The discussion here suggests that a deeper holographic connection (beyond the simple existence of conserved currents) may exist between the theory on \mathcal{M}_d and dynamics in Minkowski space. While intriguing, much remains to be done in order to elucidate this relationship. The hypothetical boundary theory is expected to have many peculiar properties, one of which we discuss in section 9.4.3.

9.4.2 Subleading Soft Graviton Theorem and the Stress Tensor

In the previous section, we demonstrated that the presence of gauge fields in Minkowski space controls the global symmetry structure of the putative theory on \mathcal{M}_d . As in AdS/CFT, more interesting features arise when we couple the bulk theory to gravity and consider gravitational

⁹This construction seems more natural in momentum space, where it is only the interior of the light cone which is physically relevant, and one never needs to discuss the timelike de Sitter hyperboloids lying outside the light cone.

perturbations. Flat space graviton scattering amplitudes also display universal behavior in the infrared that is model independent and holds in any dimension. Of particular interest here is the subleading soft graviton theorem, which states¹⁰

$$\lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \langle \mathcal{O}_{ab}(q) \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle = -i \sum_{k=1}^n \frac{\varepsilon_{ab}^{\mu\nu} p_{k\mu} q_\rho}{p_k \cdot q} \mathcal{J}_k^{\rho\nu} \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle . \quad (9.4.16)$$

Here, $\mathcal{O}_{ab}(q)$ creates a graviton with momentum q and polarization $\varepsilon_{\mu\nu}^{ab}(q)$, and $\mathcal{J}_{k\rho\nu}$ is the total angular momentum operator for the k^{th} particle. The operator $(1 + \omega \partial_\omega)$ projects out the Weinberg pole [4], yielding a finite $\omega \rightarrow 0$ limit.

Returning to the analogy with $\text{AdS}_{d+1}/\text{CFT}_d$, one might expect that the bulk soft graviton is associated to a boundary stress tensor, just as the bulk soft photon is related to a boundary $U(1)$ current. In a quantum field theory, the stress tensor generates the action of spacetime (conformal) isometries on local operators. As we saw in (9.3.11), the angular momentum operator $\mathcal{J}_{\rho\nu}$ generates these transformations on the local operators on \mathcal{M}_d . Therefore, it is natural to suspect that the bulk subleading soft graviton operator

$$B_{ab}(x) = \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \mathcal{O}_{ab}(\omega, x) \quad (9.4.17)$$

is related to the boundary stress tensor. Such a relationship was derived in four dimensions ($d = 2$) in [134, 135, 144]. In this section, we generalize the construction to $d > 2$.

Insertions of $B_{ab}(x)$ are controlled by the subleading soft graviton theorem (9.4.16) and take the form (see (9.3.8) and (9.3.10) for the explicit forms of the orbital and spin angular momentum operators)

$$\begin{aligned} \langle B_{ab}(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle &= \sum_{k=1}^n \left[\mathcal{P}_{ab}^c(x - x_k) \partial_{x_k^c} + \frac{1}{d} \partial_c \mathcal{P}_{ab}^c(x - x_k) \omega_k \partial_{\omega_k} \right. \\ &\quad \left. - \frac{i}{2} \partial^{[c} \mathcal{P}^{d]}_{ab}(x - x_k) \mathcal{S}_{kcd} \right] \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle , \end{aligned} \quad (9.4.18)$$

¹⁰We work in units such that $\sqrt{8\pi G} = 1$.

where

$$\mathcal{P}^c_{ab}(x) = \frac{1}{2} \left[x_a \delta_b^c + x_b \delta_a^c + \frac{2}{d} x^c \delta_{ab} - \frac{4}{x^2} x^c x_a x_b \right]. \quad (9.4.19)$$

From (9.4.18), we see that $B_{ab}(x)$ is a conformal primary operator with $(\Delta, s) = (0, 2)$. One can also check that

$$\partial_{\{c} \mathcal{P}_{d\}ab}(x) = \mathcal{I}_{\{a\{c}(x) \mathcal{I}_{d\}b\}}(x). \quad (9.4.20)$$

As in section 9.4.1, it is easiest to first determine B_{ab} in terms of T_{ab} . Recall that the Ward identities for the energy-momentum tensor of a CFT_d take the form [160]

$$\langle \partial^d T_{dc}(y) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = - \sum_{k=1}^n \delta^{(d)}(y - x_k) \partial_{x_k^c} \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle, \quad (9.4.21)$$

$$\langle T^c_c(y) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = \sum_{k=1}^n \delta^{(d)}(y - x_k) \omega_k \partial_{\omega_k} \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle, \quad (9.4.22)$$

$$\langle T^{[cd]}(y) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = -\frac{i}{2} \sum_{k=1}^n \delta^{(d)}(y - x_k) \mathcal{S}_k^{cd} \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle. \quad (9.4.23)$$

Multiplying (9.4.21) by $-\int d^d y \mathcal{P}^c_{ab}(x - y)$, (9.4.22) by $\frac{1}{d} \int d^d y \partial_c \mathcal{P}^c_{ab}(x - y)$, (9.4.23) by $\int d^d y \partial^c \mathcal{P}^d_{ab}(x - y)$, and taking the sum, one finds

$$\begin{aligned} B_{ab}(x) &= - \int d^d y \partial_{\{c} \mathcal{P}_{d\}ab}(x - y) T^{cd}(y) \\ &= - \int d^d y \mathcal{I}_{\{a\{c}(x - y) \mathcal{I}_{d\}b\}}(x - y) T^{cd}(y) \\ &= -\tilde{T}_{\{ab\}}(x). \end{aligned} \quad (9.4.24)$$

Once again, the soft operator appears as the shadow transform of a conserved current. The relationship could have been guessed from the outset based on the dimensions of B_{ab} and $T_{\{ab\}}$.¹¹

¹¹Note that only the symmetric traceless part of the stress tensor appears in this dictionary since the graviton lies in the symmetric traceless representation of the little group. The trace term may be related to soft dilaton theorems.

Having derived (9.4.24), we can now invert the shadow transform to find

$$T_{\{ab\}}(x) = -\frac{1}{c(0, 2)} \tilde{B}_{ab}(x) . \quad (9.4.25)$$

The shadow relationship between the soft operator B_{ab} and the energy-momentum tensor is again suggestive of a coupling

$$\int d^d x B^{ab}(x) T_{ab}(x) \quad (9.4.26)$$

in some hypothetical dual formulation of asymptotically flat gravity: the soft graviton creates an infinitesimal change in the boundary metric, sourcing the operator T_{ab} . In [134], it was viewed as a puzzle that the energy-momentum tensor appears non-local when written in terms of the soft modes of the four-dimensional gravitational field. Here we see that this is essentially the consequence of a linear response calculation, and that the non-locality is actually the only one allowed by conformal symmetry.

As in [134], it is possible to derive a local differential equation for $T_{\{ab\}}$ in even dimensions. We first define the following derivative operator

$$\mathbb{D}^a O_{ab} \equiv \frac{1}{2(4\pi)^{d/2} \Gamma(d/2 + 1)} \left[(-\square)^{d/2} \partial^a O_{ab} + \frac{d}{d-1} \partial_b (-\square)^{d/2-1} \partial^e \partial^f O_{ef} \right] . \quad (9.4.27)$$

One can check that

$$\mathbb{D}^a \mathcal{P}^c_{ab} = -\delta_b^c \delta^{(d)}(x) . \quad (9.4.28)$$

Then, acting on the first equation of (9.4.24) with \mathbb{D}^a , we find

$$\mathbb{D}^a B_{ab}(x) = \partial^a T_{\{ab\}}(x) . \quad (9.4.29)$$

9.4.3 Leading Soft Graviton Theorem and Momentum Conservation

In the previous two subsections, we have avoided the discussion of currents related to the leading soft graviton theorem. This soft theorem is associated to spacetime translational invariance (and more generally to BMS supertranslations [30, 31, 88, 125]). For scattering amplitudes in the usual plane wave basis, this symmetry is naturally enforced by a momentum conserving Dirac delta function. In our discussion above, we have chosen to make manifest the Lorentz transformation properties of the scattering amplitudes. Consequently, translation invariance, or global momentum conservation, is unwieldy in our formalism. In fact, it must somehow appear as a non-local constraint on the correlation functions on \mathcal{M}_d , since arbitrary operator insertions corresponding to arbitrary configurations of incoming and outgoing momenta will in general violate momentum conservation. The difficulty can also be seen at the level of the symmetry algebra. Momentum conservation cannot arise simply as a global $\mathbb{R}^{d+1,1}$ symmetry of the CFT_d , since the associated conserved charges do not commute with the conformal (Lorentz) group. In light of this, it is not clear that our construction can really be viewed as a local conformal field theory living on \mathcal{M}_d .¹² We have tried, unsuccessfully, to find a natural set of operators whose shadow reproduces the leading soft graviton theorem¹³

$$\lim_{\omega \rightarrow 0} \omega \langle \mathcal{O}_{ab}(q) \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle = \omega \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}^{ab} p_k^\mu p_k^\nu}{p_k \cdot q} \langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle. \quad (9.4.30)$$

The soft operator

$$G_{ab}(x) = \lim_{\omega \rightarrow 0} \omega \mathcal{O}_{ab}(\omega, x) \quad (9.4.31)$$

¹²However, it is also not clear that we should expect a local QFT dual to asymptotically flat quantum gravity. The flat space Bekenstein-Hawking entropy is always super-Hagedorn in $d \geq 4$. The high energy density of states grows faster than in any local theory.

¹³It has also been suggested [161] that translational invariance of the \mathcal{S} -matrix is realized through null state relations of boundary correlators rather than through local current operators, since the former are typically non-local constraints on CFT correlation functions.

has insertions given by

$$\langle G_{ab}(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = \frac{2}{d} \sum_{k=1}^n \omega_k \mathcal{I}_{ab}^{(d)}(x - x_k) \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle, \quad (9.4.32)$$

where

$$\mathcal{I}_{ab}^{(d)}(x) = \delta_{ab} - d \frac{x_a x_b}{x^2}. \quad (9.4.33)$$

One finds that the operator

$$U_a = \frac{1}{(4\pi)^{d/2} \Gamma(d/2) (d-1)} (-\square)^{\frac{d}{2}-1} \partial^b G_{ba} \quad (9.4.34)$$

satisfies

$$\langle \partial^a U_a(x) \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle = - \sum_{k=1}^n \omega_k \delta^{(d)}(x - x_k) \langle \mathcal{O}_1(\omega_1, x_1) \dots \mathcal{O}_n(\omega_n, x_n) \rangle. \quad (9.4.35)$$

Thus, $U_a(x)$ satisfies the current Ward identity corresponding to “energy” conservation. However, since ω is not a scalar charge, the current $U_a(x)$ is not a primary operator (though it has a well-defined scaling dimension $\Delta = d$). Acting on (9.4.35) with $-\frac{2}{d} \int d^d x \mathcal{I}_{ab}^{(d)}(y - x)$, one finds

$$G_{ab}(x) = -\frac{2}{d} \int d^d y \mathcal{I}_{ab}^{(d)}(x - y) \partial^c U_c(y). \quad (9.4.36)$$

Unlike the $U(1)$ current and the stress tensor, the leading soft graviton “current” is not related to the soft operator G_{ab} through a shadow transform. It may be possible to interpret (9.4.36) as some other (conformally) natural non-local transform of $U_c(x)$, but we do not pursue this here.¹⁴

¹⁴The shadow transform $(\Delta, s) \rightarrow (d - \Delta, s)$ is related to the \mathbb{Z}_2 symmetry of the quadratic and quartic Casimirs of the conformal group $c_2 = \Delta(d - \Delta) + s(2 - d - s)$, $c_4 = -s(2 - d - s)(\Delta - 1)(d - \Delta - 1)$. c_2 and c_4 are also invariant under another \mathbb{Z}_2 symmetry under which $(\Delta, s) \rightarrow (1 - s, 1 - \Delta)$. Equation (9.4.36) may be the integral representation of a shadow transform followed by the second \mathbb{Z}_2 transform which maps $(d + 1, 0) \rightarrow (-1, 0) \rightarrow (1, 2)$.

9.5 Conclusion

In this chapter we have taken steps to recast the $(d+2)$ -dimensional \mathcal{S} -matrix as a collection of celestial correlators, but many open questions remain. Our analysis relied on symmetry together with the universal behavior of the \mathcal{S} -matrix in certain kinematic regimes. It would be interesting to analyze the consequences of other universal properties of the \mathcal{S} -matrix for the celestial correlators. The analytic structure and unitarity of the \mathcal{S} -matrix should be encoded in properties of these correlation functions, although the mechanism may be subtle. It seems likely that the collinear factorization of the \mathcal{S} -matrix could be used to define some variant of the operator product expansion for local operators on the light cone. Although this chapter only addressed single soft insertions, double soft limits, appropriately defined, could be used to define OPEs between the conserved currents and stress tensors. We expect supergravity soft theorems to yield a variety of interesting operators, including a supercurrent. Finally, the interplay of momentum conservation with the CFT_d structure requires further clarification. We leave these questions to future work.

9.A Spacetime Picture

In this appendix we construct coordinates for Mink_{d+2} whose limiting metric on \mathcal{I}^+ has cross-sectional cuts given by \mathcal{M}_d . Consider the coordinate transformation from the flat Cartesian coordinates X^μ to the coordinates (u, r, x^a) given by

$$X^\mu(u, r, x^a) = rp^\mu(x^a) + uk^\mu(x^a) , \quad (9.A.1)$$

where $p^\mu(x^a)$ is given by (9.2.3) with ω set to one. We have

$$dp^\mu dp_\mu = \Omega^2(x) dx^a dx_a . \quad (9.A.2)$$

If the vector $k^\mu(x^a)$ is chosen to satisfy

$$\partial_a k^\mu = 0, \quad k^\mu \partial_a p_\mu = 0, \quad (9.A.3)$$

then one finds

$$dX^\mu dX_\mu = k^2 du^2 + 2(p \cdot k) dudr + r^2 dp^\mu dp_\mu. \quad (9.A.4)$$

Since neither p^μ nor k^μ scales with r , the limit $r \rightarrow \infty$ with u fixed yields a degenerate metric on \mathcal{I}^+ with cross-sectional metric

$$ds^2 = r^2 \Omega^2(x) \delta_{ab} dx^a dx^b = r^2 ds_{\mathcal{M}_d}^2. \quad (9.A.5)$$

For instance, the flat metric on \mathcal{M}_d corresponds to the choice

$$p^\mu(x) = \left(\frac{1+x^2}{2}, x^a, \frac{1-x^2}{2} \right), \quad k^\mu = \frac{1}{2} [1, 0^a, -1], \quad (9.A.6)$$

which yields the familiar coordinate transformation

$$X^\mu(u, r, x^a) = \left[\frac{u + r(1+x^2)}{2}, rx^a, \frac{r(1-x^2) - u}{2} \right] \quad (9.A.7)$$

and a metric of the form

$$ds^2 = -dudr + r^2 \delta_{ab} dx^a dx^b. \quad (9.A.8)$$

In order to achieve cross-sectional cuts of \mathcal{I}^+ which are metrically S^d , one chooses

$$p^\mu(x) = \frac{2}{1+x^2} \left(\frac{1+x^2}{2}, x^a, \frac{1-x^2}{2} \right), \quad k^\mu = [1, 0^a, 0]. \quad (9.A.9)$$

The coordinate transformation is then

$$X^\mu(u, r, x^a) = \left[u + r, \frac{2rx^a}{1+x^2}, r \frac{1-x^2}{1+x^2} \right] \quad (9.A.10)$$

and the spacetime metric is

$$ds^2 = -du^2 - 2dudr + \frac{4r^2}{(1+x^2)^2} \delta_{ab} dx^a dx^b . \quad (9.A.11)$$

In order to make the relationship between \mathcal{M}_d and \mathcal{I}^+ even more explicit, consider the flat coordinate system (9.A.7). A massless field in the plane wave basis takes the form

$$\Phi_{\mu_1 \dots \mu_s}(X) = \sum_{a_i} \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{1}{2\omega_q} \varepsilon_{\mu_1 \dots \mu_s}^{a_1 \dots a_s}(q) \left[\mathcal{O}_{a_1 \dots a_s}(q) e^{iq \cdot X} + \mathcal{O}_{a_1 \dots a_s}^\dagger(q) e^{-iq \cdot X} \right] . \quad (9.A.12)$$

To perform an asymptotic analysis near \mathcal{I}^+ , one considers the limit $r \rightarrow \infty$ with u fixed, so that $X \rightarrow rp^\mu(x^a)$. In this limit, the argument of the exponential

$$ir\omega_q q(y_a) \cdot p(x_a) = -\frac{i}{2} r\omega_q (x-y)^2 \quad (9.A.13)$$

is large so that the exponential is rapidly oscillating. At leading order in $\frac{1}{r}$, the only momenta that contribute to the integral are those for which the phase is stationary, i.e. for which $x = y$. Therefore in the large- r lightlike limit, one effectively trades the transverse coordinates on \mathcal{I}^+ for the momentum coordinates on \mathcal{M}_d .

Armed with this knowledge we can further elaborate on the results of section 9.4. The soft photon operator $S_a(x)$ is related to the boundary current $J_a(x)$ through a differential equation of the form

$$(-\square)^{\frac{d}{2}-1} S_a(x) = (4\pi)^{d/2} \Gamma(d/2) J_a(x) . \quad (9.A.14)$$

The physical picture is clear: charged particles passing through \mathcal{I}^+ act as a source $J_a(x)$ for the soft radiation $S_a(x)$ (see [36] for relevant expressions relating (9.A.14) to the soft charge for large $U(1)$ gauge transformations). Similar statements apply to the gravitational case [88]. Energetic particles passing through \mathcal{I}^+ act as an effective source (the boundary energy-momentum tensor T_{ab}) for soft graviton radiation $B_{ab}(x)$.

10

Area, Entanglement Entropy and Supertranslations at Null Infinity

10.1 Introduction

The Bekenstein-Hawking area-entropy law [162, 163]

$$S_{BH} = \frac{\text{Area}}{4\hbar G} \tag{10.1.1}$$

ascribes an entropy to a null surface proportional to its cross-sectional area in Planck units. This law has a number of fascinating generalizations [164–183], including the Bousso bound [184–189] which bounds the change in the area to the entropy flux through the null surface.

One of the most interesting null surfaces is future null infinity (\mathcal{I}^+), which is a future boundary

of asymptotically flat spacetimes. It is a universal observer horizon for all eternal observers which do not fall into black holes. It is natural to try to relate the change in the area of cross-sectional “cuts” Σ of \mathcal{I}^+ to the energy or entropy flux across \mathcal{I}^+ . An immediate obstacle is that both the areas and area changes of such cuts are infinite. The Bousso bound is obeyed but in a trivial manner.

In this chapter we define a finite renormalized area of cuts of \mathcal{I}^+ and conjecture a nontrivial bound relating it to the entropy radiated through \mathcal{I}^+ . A family of regulated null surfaces parametrized by r which approach \mathcal{I}^+ for $r \rightarrow \infty$ is introduced. For finite r these have finite area for any cut Σ . We then define a subtracted area by subtracting the area of the same cut in a fiducial vacuum geometry. The gravitational vacuum has an infinite degeneracy labeled by an arbitrary function C_0 on the sphere at \mathcal{I}^+ [30].¹ Under BMS supertranslations, also parameterized by an arbitrary function (denoted f) on the sphere, $C_0 \rightarrow C_0 + f$ and these vacua transform into one another. We show that the subtracted area, denoted A_0^Σ , is finite (and typically negative) in the $r \rightarrow \infty$ limit. However it retains “anomalous” dependence on the choice of a fiducial C_0 .

This renormalized area A_0^Σ is found to have several interesting properties. When C_0 coincides with the physical vacuum at the location of the cut, A_0^Σ is the negative of the so-called modular energy of the region $\mathcal{I}_>^+$ lying to the future of Σ , including “soft graviton” terms which are linear in the Bondi news. It tends to increase towards the far future, and asymptotically reaches zero from below when C_0 coincides with the asymptotic future vacuum. Moreover, under supertranslations it shifts by the supertranslation charge on Σ .

In quantum gravity, the outgoing quantum state is supported on \mathcal{I}^+ . The cut Σ divides \mathcal{I}^+ into two regions, and a quantum entanglement entropy S_0^{ent} of the portions of the outgoing quantum state on opposing sides of the cut is expected. In principle, unlike the entanglement across generic fluctuating interior surfaces, S_0^{ent} should be well defined because gravity is weakly coupled near the boundary. However, it is beset by both ultraviolet (UV) divergences from short wavelength entanglements near the cut and infrared (IR) divergences from soft gravitons. A choice of vacuum

¹Prescient early discussions of vacuum degeneracy are in [27, 28, 92].

is required for subtraction of UV divergences, so S_0^{ent} will also acquire an “anomalous” dependence on the fiducial C_0 . We will not try to give a precise definition of S_0^{ent} herein which would, among other things, require a decomposition of the soft graviton Hilbert space.² Nevertheless we will motivate a conjecture that a suitably defined S_0^{ent} obeys the bound

$$-\frac{A_0^\Sigma}{4\hbar G} \geq S_0^{\text{ent}}(\Sigma) \quad (10.1.2)$$

for any cut Σ of \mathcal{I}^+ . Typically, at late times both sides of this equation are positive and decreasing. This relation incorporates the BMS structure at \mathcal{I}^+ into the study of the relation between area and entanglement entropy.

Our results are plausibly relevant to, and were motivated by, the black hole information paradox. A unitary resolution of this paradox would amount, roughly speaking, to showing that late and early time Hawking emissions are correlated in such a way that, for a pure incoming state, the full quantum state on \mathcal{I}^+ is a pure state. However, a more precise BMS-invariant statement is needed. One would like to compute the entanglement entropy S_0^{ent} across any cut Σ . Naively, one expects that it approaches zero for all cuts in the far past and far future and has a maximum somewhere in the middle, possibly at the Page time [193, 194]. Given both the IR and UV subtractions needed to define S_0^{ent} , the resulting anomalies in supertranslation invariance and the discovery of soft hair [16, 45], it is not obvious to us what precisely the expectation following from unitarity should be. In particular, the requirement that S_0^{ent} vanish in the far future is not fully supertranslation invariant. We do not attempt to resolve these issues herein. Rather, we view the present effort as a first step in obtaining a precise statement of the black hole information paradox.

This chapter is organized as follows. Section 10.2 contains preliminaries and notation. In section 10.3 we define a renormalized area A_F^Σ in which we subtract the area associated to the asymptotic future vacuum and relate it to the “hard” modular energy of the region to the future of the cut. In section 10.4 we show that A_F^Σ varies under supertranslations into the hard part of

²A relevant discussion appears in [190–192].

the supertranslation charge. Section 10.5 introduces the more general renormalized A_0^Σ involving an arbitrary vacuum subtraction. Its variation under supertranslations is shown to involve the full modular energy including soft graviton contributions. In section 10.6 we motivate and conjecture a bound relating the renormalized area to the entanglement entropy which can be viewed as the second law of \mathcal{I}^+ .

Throughout this chapter we assume for simplicity that the geometry reverts to a vacuum in the far future and that all flux through \mathcal{I}^+ is gravitational. This highlights many of the salient features, but a treatment of more general cases would be of interest.

10.2 Preliminaries

In retarded Bondi coordinates, asymptotically flat metrics [1–3, 29] near \mathcal{I}^+ take the form

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ & + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^z C_{zz}dudz + D^{\bar{z}} C_{\bar{z}\bar{z}}dud\bar{z} + \dots \end{aligned} \quad (10.2.1)$$

Here, $\gamma_{z\bar{z}}$ is the round metric on the unit S^2 and D_z is the associated covariant derivative. Defining

$$\begin{aligned} N_{zz} &= \partial_u C_{zz} , \\ T_{uu} &= \frac{1}{2} N^{zz} N_{zz} , \\ U_z &= iD^z C_{zz} , \quad U = U_z dz + U_{\bar{z}} d\bar{z} , \\ V_z &= iD^z N_{zz} , \quad V = V_z dz + V_{\bar{z}} d\bar{z} , \\ \varepsilon &= i\gamma_{z\bar{z}} dz \wedge d\bar{z} , \end{aligned} \quad (10.2.2)$$

the leading order vacuum constraint equation reads

$$\partial_u m_B du \wedge \varepsilon = -\frac{1}{2} T_{uu} du \wedge \varepsilon - \frac{1}{4} du \wedge dV . \quad (10.2.3)$$

We could easily add a matter contribution to T_{uu} but we omit this for brevity. We assume that near the future boundary \mathcal{I}_+^+ of \mathcal{I}^+ the spacetime reverts to a vacuum so that

$$m_B|_{\mathcal{I}_+^+} = 0, \quad C_{zz}|_{\mathcal{I}_+^+} = -2D_z^2 C_F, \quad (10.2.4)$$

for some function $C_F(z, \bar{z})$. In the quantum theory we denote the corresponding vacuum state by $|C_F\rangle$. Given C_F and the Bondi news tensor N_{zz} , the mass aspect m_B is determined by integrating the constraint equation (10.2.3) backwards from \mathcal{I}_+^+ .

Asymptotically flat spacetimes admit an infinite-dimensional symmetry group, known as the Bondi-Metzner-Sachs (BMS) group [1–3]. The supertranslations are labeled by an arbitrary function $f(z, \bar{z})$ on the S^2 and are generated by the vector fields

$$\xi = f\partial_u - \frac{1}{r}(D^z f\partial_z + D^{\bar{z}} f\partial_{\bar{z}}) + \frac{1}{2}D^2 f\partial_r, \quad D^2 = 2D^z D_z. \quad (10.2.5)$$

Infinitesimal supertranslations act on the geometry as [12, 124]

$$\begin{aligned} \delta_f C_{zz} &= f N_{zz} - 2D_z^2 f, \\ \delta_f C_F &= f, \\ \delta_f U_z &= f V_z + iD^z f N_{zz} - iD^2 D_z f, \\ \delta_f m_B &= f\partial_u m_B + \frac{1}{4}D_z^2 f N^{zz} + \frac{1}{4}D_{\bar{z}}^2 f N^{\bar{z}\bar{z}} + \frac{i}{2}\partial_z f V^z - \frac{i}{2}\partial_{\bar{z}} f V^{\bar{z}}, \\ \delta_f T_{uu} &= f\partial_u T_{uu}. \end{aligned} \quad (10.2.6)$$

These transformations are generated by the supertranslation charges

$$Q[f] = \int_{\mathcal{I}_-^+} f(z, \bar{z}) m_B \varepsilon = \frac{1}{4} \int_{\mathcal{I}^+} f(z, \bar{z}) du \wedge dV + \frac{1}{2} \int_{\mathcal{I}^+} f(z, \bar{z}) T_{uu} du \wedge \varepsilon. \quad (10.2.7)$$

The first term, known as the soft charge, is linear in the gravitational field. It is written in terms of the zero-mode of the Bondi news, and creates soft gravitons when acting on physical states. The second term, or hard charge, is quadratic in the matter and gravitational fields and characterizes

the hard energy flux through \mathcal{I}^+ .

10.3 Renormalizing the Area

We wish to study the area of a cut Σ of \mathcal{I}^+ defined by

$$u = u_\Sigma(z, \bar{z}) \tag{10.3.1}$$

in the geometry (10.2.1), and also to study its variation under supertranslations of Σ

$$u_\Sigma \rightarrow u_\Sigma + f \tag{10.3.2}$$

with the geometry held fixed. Of course this area is infinite so we must introduce both a regulator and a subtraction. We regulate the area by the replacement of \mathcal{I}^+ with the past light cone of a point which approaches i^+ . For the flat Minkowski metric the null hypersurface

$$r = -\frac{1}{2}(u - u_0) \tag{10.3.3}$$

approaches \mathcal{I}^+ for $u_0 \rightarrow \infty$ with u held fixed. More generally we solve the ODE

$$\left(1 - \frac{2m_B}{r}\right) du + 2dr = 0, \tag{10.3.4}$$

which guarantees that the surface is null, and choose the integration constants at each (z, \bar{z}) so that the surface lies at large radius, approaching infinity, for finite u .³ The null condition (10.3.4) has $\frac{1}{r}$ corrections. For brevity such corrections are suppressed here and hereafter whenever they drop

³Such a surface will generically terminate at a cusp rather than a point, but this will not matter as the quantities considered below do not have contributions from the endpoint of the surface.

out of the large- r limit. Equation (10.3.4) can be rewritten as

$$\frac{dr^2}{du} = 2m_B - r . \quad (10.3.5)$$

The area of a cut Σ of \mathcal{I}^+ defined by $u = u_\Sigma(z, \bar{z})$ then follows from the metric induced from (10.2.1) and is given by

$$A(\Sigma, N_{zz}, C_F) = \int_\Sigma d^2z \sqrt{\det g} = \int_\Sigma \left(r^2 \varepsilon - \frac{1}{2} du_\Sigma \wedge U \right) . \quad (10.3.6)$$

Both the area (10.3.6) as well as its variation with respect to retarded time are divergent in the large- r limit of interest. A subtraction is necessary to obtain a finite result. We define a fiducial “ C_F -vacuum” in which the news N_{zz} vanishes and $C_{zz} = -2D_z^2 C_F$ on all of \mathcal{I}^+ . This flat geometry coincides with (10.2.1) at late times. A fiducial null hypersurface in this fiducial spacetime solving (10.3.5) (with $m_B = 0$) can then be found which coincides exactly with the solution of (10.3.4) in (10.2.1) at late times. A subtracted area, with a finite large- r limit, may then be obtained by subtracting the area of the fiducial hypersurface:

$$\begin{aligned} A_F^\Sigma &= A(\Sigma, N_{zz}, C_F) - A(\Sigma, 0, C_F) \\ &= \int_\Sigma \left[(r^2 - r_0^2) \varepsilon + \frac{1}{2} du_\Sigma \wedge \Delta U \right] . \end{aligned} \quad (10.3.7)$$

Here, $\Delta U = U_F - U_\Sigma$ is the change in U and r_0 is the radius in the fiducial vacuum. Using (10.3.5), we have

$$\frac{d}{du} (r^2 - r_0^2) = 2m_B . \quad (10.3.8)$$

Integrating this equation from \mathcal{I}_+^+ to u_Σ , one finds

$$r^2(u_\Sigma, z, \bar{z}) - r_0^2(u_\Sigma, z, \bar{z}) = - \int_{u_\Sigma}^\infty du \, 2m_B(u, z, \bar{z}) . \quad (10.3.9)$$

The finite renormalized area is then given by

$$A_F^\Sigma = - \int_{\mathcal{I}_>^+} du \wedge \left[2m_B \varepsilon - \frac{1}{2} du_\Sigma \wedge V \right] , \quad (10.3.10)$$

where $\mathcal{I}_>^+$ is the three-dimensional region of \mathcal{I}^+ lying to the future of the cut Σ . Using the constraint equation and the identity

$$\int_{\mathcal{I}_>^+} du \wedge du_\Sigma \wedge V = \int_{\mathcal{I}_>^+} (u - u_\Sigma) du \wedge dV = - \int_\Sigma u_\Sigma d\Delta U , \quad (10.3.11)$$

the renormalized area can be rewritten

$$A_F^\Sigma = - \int_{\mathcal{I}_>^+} (u - u_\Sigma) T_{uu} du \wedge \varepsilon = - \int_\Sigma d^2 z \gamma_{z\bar{z}} \int_{u_\Sigma}^\infty (u - u_\Sigma) T_{uu} du . \quad (10.3.12)$$

This expression could equivalently be derived through integration of the Raychaudhuri equation, and matches familiar expressions for the modular Hamiltonians of lightsheets [188, 195, 196]. We refer to this as the (negative of the) hard modular energy of the region $\mathcal{I}_>^+$. We note that A_F^Σ is typically negative and increases to zero in the far future due to the subtraction scheme.

10.4 Supertranslations

A_F^Σ is strictly invariant under coordinate transformations which both move the cut and transform the physical and subtraction geometries. In particular, A_F^Σ is invariant if we simultaneously shift the cut $u_\Sigma \rightarrow u_\Sigma + f$ and supertranslate the geometry by the inverse transformation. However, one can consider evaluating the subtracted area on a supertranslated cut, sending $u_\Sigma \rightarrow u_\Sigma + f$ while keeping the geometry fixed. Starting from either (10.3.10) or (10.3.12), one easily finds

$$\delta_f A_F^\Sigma = \int_\Sigma f \left[2m_B \varepsilon - \frac{1}{2} d\Delta U \right] = \int_{\mathcal{I}_>^+} f T_{uu} du \wedge \varepsilon . \quad (10.4.1)$$

The right hand side is the hard part of the supertranslation charge on $\mathcal{I}_>^+$. Alternately, (10.4.1) may be derived by infinitesimally supertranslating the geometry according to (10.2.6).

10.5 A General Subtraction

The subtraction used in (10.3.7) to obtain a finite area change has a teleological nature: we must know which vacuum the geometry settles into in the far future in order to define A_F^Σ . In this section we consider a more general, non-teleological subtraction of the area of Σ at $u = u_\Sigma$ in the null hypersurface defined by solving (10.3.5) in an arbitrary vacuum characterized by the arbitrary function C_0 with $C_{0zz} = -2D_z^2 C_0$. Unlike the case in (10.3.7), the subtracted geometry is not identical to the physical one at late times, and so the late-time contributions are not manifestly finite or well-defined. To characterize the resulting ambiguity we introduce a late-time cutoff by terminating both surfaces at a final cut Σ_F at $u = u_F(z, \bar{z})$, in the late-time vacuum region with $m_B = 0$.⁴ One finds

$$A_0^\Sigma = A_F^\Sigma + \frac{1}{2} \int_\Sigma d(u_\Sigma - u_F) \wedge (U_0 - U_F) , \quad (10.5.1)$$

where U_0 and U_F are constructed from C_0 and C_F according to (10.2.2). As may be easily verified, this expression is invariant if we supertranslate the physical geometry, the fiducial vacuum C_0 and both cuts at Σ and Σ_F . We now restrict consideration to the case $u_F = \text{constant}$, in which case this expression reduces to

$$A_0^\Sigma = - \int_{\mathcal{I}_>^+} du \wedge (u - u_\Sigma) (\varepsilon T_{uu} + \frac{1}{2} dV) + \frac{1}{2} \int_\Sigma du_\Sigma \wedge (U_0 - U) . \quad (10.5.2)$$

Fixing the geometry and varying $u_\Sigma \rightarrow u_\Sigma + f$, we find

$$\delta_f A_0^\Sigma = \int_\Sigma f \left[2m_B \varepsilon - \frac{1}{2} d(U_0 - U) \right] . \quad (10.5.3)$$

⁴It would be interesting to analyze the more generic case of the area change over a more general finite interval.

The right hand side is the full supertranslation charge in the special case $U = U_0$ on Σ .

10.6 An Area-Entropy Bound Conjecture

Given a cut Σ and a vacuum state $|C_0\rangle$ on all of \mathcal{I}^+ we may define a density matrix on the region $\mathcal{I}_{>}^+$ to the future of Σ by

$$\sigma_0 = \text{tr}_{<} |C_0\rangle\langle C_0| , \quad (10.6.1)$$

where the trace is over the region prior to Σ , and the dependence on the choice of cut is suppressed. Similarly, for an excited state $|\Psi\rangle$ we define the density matrix on $\mathcal{I}_{>}^+$

$$\rho = \text{tr}_{<} |\Psi\rangle\langle\Psi| . \quad (10.6.2)$$

We normalize so that $\text{tr}\rho = \text{tr}\sigma_0 = 1$. The modular hamiltonian which measures local Rindler energies relative to $|C_0\rangle$ is

$$-\ln \sigma_0 . \quad (10.6.3)$$

σ_0 has contributions from the entanglement of both hard and soft modes across the surface Σ . Hard mode entanglements contribute [195, 196]⁵

$$-\ln \sigma_0|_{\text{hard}} = \frac{1}{4\hbar G} \int_{\mathcal{I}_{>}^+} du \wedge (u - u_\Sigma) \varepsilon \hat{T}_{uu} + \text{constant} = -\frac{\hat{A}_F^\Sigma}{4\hbar G} + \text{constant} , \quad (10.6.4)$$

where here \hat{T}_{uu} and \hat{A} are both operators and the constants depend on the normal ordering prescription. It would be extremely interesting, but beyond the scope of this chapter, to regulate, define and compute the soft contributions to σ_0 . The precise form of σ_0 may well depend on the renormalization scheme. Here we simply conjecture, motivated by the structures encountered in

⁵Note that our normalization of the stress energy tensor as defined in (10.2.3) differs by a factor of $8\pi G$ from some other references.

the previous section, that these contributions can be defined in such a way that

$$-\ln \sigma_0 = -\frac{\hat{A}_0^\Sigma}{4\hbar G} + \text{constant} , \quad (10.6.5)$$

where the operator-valued area appearing here is

$$\hat{A}_0^\Sigma = - \int_{\mathcal{I}_>^+} du \wedge (u - u_\Sigma)(\varepsilon \hat{T}_{uu} + \frac{1}{2} d\hat{V}) + \frac{1}{2} \int_\Sigma du_\Sigma \wedge (U_0 - \hat{U}) . \quad (10.6.6)$$

We interpret the first term as the full modular Hamiltonian, including soft terms. The last is a soft term which vanishes when $U_0 = \hat{U}$ on the cut Σ .

The C_0 -vacuum subtracted modular energy of the state $|\Psi\rangle$ restricted to $\mathcal{I}_>^+$ is

$$K_0 = -\text{tr} \rho \ln \sigma_0 + \text{tr} \sigma_0 \ln \sigma_0 . \quad (10.6.7)$$

This expression vanishes for $\rho = \sigma_0$, as does A_0^Σ when the physical geometry is the C_0 vacuum. Hence, the constant is fixed so that

$$K_0 = -\frac{A_0^\Sigma}{4\hbar G} . \quad (10.6.8)$$

We further define the regulated entanglement entropy

$$S_0^{\text{ent}} = -\text{tr} \rho \ln \rho + \text{tr} \sigma_0 \ln \sigma_0 \quad (10.6.9)$$

and the relative entropy

$$S(\rho | \sigma_0) = \text{tr} \rho \ln \rho - \text{tr} \rho \ln \sigma_0 . \quad (10.6.10)$$

Evidently

$$S(\rho | \sigma_0) = K_0 - S_0^{\text{ent}} . \quad (10.6.11)$$

Positivity of relative entropy and the conjecture (10.6.5) then implies the bound

$$-\frac{A_0^\Sigma}{4\hbar G} \geq S_0^{\text{ent}} . \quad (10.6.12)$$

We note that the renormalized area A_0^Σ is typically negative while the entanglement entropy is typically positive. If the renormalized area and entanglement entropy both tend to zero when the cut Σ is taken to \mathcal{I}_+^+ , then it follows from (10.6.12) that the change (final minus initial) ΔA in the renormalized area and the change ΔS^{ent} in the entanglement entropy obey the “second law of \mathcal{I}^+ ”⁶

$$\frac{\Delta A}{4\hbar G} + \Delta S^{\text{ent}} \geq 0 . \quad (10.6.13)$$

In this inequality, ΔA is typically positive while ΔS is typically negative, reflecting the fact that the outgoing flux after the cut Σ is correlated with the flux prior to Σ if it is to restore quantum purity.

10.7 Future Directions

Our work leaves open several important questions meriting further investigation. It seems imperative to define, regulate, and compute the soft contributions to the modular Hamiltonian and the entanglement entropy at \mathcal{I}^+ . This question seems related to the careful treatment of soft quanta required at the horizons of black holes [16]. A more pressing question regards the apparent ambiguity in the subtraction scheme introduced in section 10.5. Our vacuum subtraction prescription is reminiscent of the subtraction procedure employed in the Euclidean approach to black hole thermodynamics in asymptotically flat spacetimes. There the boundary term in the on-shell action has a large-radius divergence which must be regulated with a vacuum subtraction. In that case the subtraction scheme is essentially fixed by requiring agreement with already known results calculable by other means. In the present context, there is no known answer to be reproduced. However, one might hope that a unique subtraction scheme could be singled out by other means, or that a covariant scheme exists, perhaps with a simple counterterm prescription. We leave this interesting question for future investigation.

⁶This highlights the differences with the situations typically considered in [184] involving area decreases and positive entropy fluxes.

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