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Smooth strings at large dimension

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Abstract

Surfaces embedded in d Euclidean dimensions with an extrinsic curvature term are investigated. To one loop order, the effective action for metric fluctuations has a Liouville form over both large and short distances. At large distances, the Liouville theory is the same as for the Nambu model, $\sim 26-d$. At short distances, the Liouville action is proportional to $26-2d$; this produces negative eigenvalues, and so instability, unless $d \leq 13$. At large d, this instability is overlooked to compute correlation functions by an expansion in $\sim 1/d$. There is a critical point when the renormalized string tension vanishes, with the only infrared singular correlations those of the Liouville theory over large distances, $\sim 26-d$. At the critical point, there is also a tachyon at non-zero momentum; tachyons probably do not occur if $d < 26$.

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I. Introduction and summary

It is natural to view theories of surfaces geometrically, but this still leaves open the question of what kind of geometry. The simplest assumption is that only intrinsic properties of the surface matter. Up to topological or non-renormalizable terms, the action must be proportional to the intrinsic area of the surface. This is customary in unified theories, where free parameters are undesirable.

If the surface represents a physical membrane, however, the limitation to intrinsic properties appears extreme. Neither a plane nor a cylinder have any intrinsic curvature, as a uniform rectangular grid can be laid down on each. Yet surely any real membrane has some stiffness, so if originally in the shape of a plane, resists being rolled up into a cylinder.

A term which responds to the bending of a surface is given by the square of the extrinsic curvature. Under the restrictions of general coordinate invariance and renormalizable couplings, an action formed from this and the area is unique in more than three dimensions. In three dimensions, the trace of the extrinsic curvature can also appear.

This theory has been applied to a wide variety of problems,[1-15] from determining the shape of a red blood cell,[1] to Polyakov's[8] suggestion that it approximates flux sheets in QCD.[9-15] The inclusion of the extrinsic curvature appears necessary for any realistic model of membranes. With only intrinsic terms, surfaces (with free boundaries) can crumple up over arbitrarily short distances, as long as their total area is preserved. The extrinsic curvature alone acts to give the surface rigidity, smoothing it out over short distances.

In conformal gauge, where the metric $g_{ab} = \rho\, \delta_{ab}$, the action for this model of smooth strings is

$$S = \int d^2z \, \left(\frac{1}{2\alpha} \frac{1}{\rho} \left(\partial^2 x\right)^2 + \mu\, \rho + \frac{i}{2\alpha} \lambda^{ab} \, \left(\partial_a x \cdot \partial_b x - \rho\, \delta_{ab}\right) \right) \, .$$

The surface is described by the d-dimensional vector x. The action includes three terms: the square of the extrinsic curvature, with a dimensionless coupling α; the Nambu term, with string tension μ; and lastly, a constraint to ensure the metric is that intrinsic to the surface. This constraint, which is unnecessary in Polyakov's approach to the Nambu model,[16-18] is needed for smooth strings, because they have no local conformal symmetry.

All surfaces contribute to the action through the metric field ρ. If by fiat ρ is set equal to one, what remains is a sum not over all surfaces, but only those that are intrinsically flat (as the intrinsic curvature $R \sim \partial^2 ln(\rho)$). Such a model of flat surfaces is far simpler than that of smooth strings: they both have a global $O(d)$ symmetry of the x's, but since the metric ρ does not appear in flat surfaces, they lack the local coordinate invariance of smooth strings. Even so, flat surfaces provide a useful analogy to smooth strings. In a different context, I previously proposed and studied this model of flat surfaces at large d;[19] these results help in studying smooth strings at large d.

I assume that the surfaces are embedded in Euclidean space-time. Due to the higher (time) derivatives in the extrinsic curvature term, in Minkowski space-time the energy functional is not positive semi-definite. This produces exponentially growing modes in time (see, *e.g.*, Braaten and Zachos[11]) and, probably, violations of unitarity.[19] *A priori*, there is nothing amiss with theories of Euclidean surfaces, either smooth or flat.

Helfrich,[2] Peliti and Leibler,[5] and Polyakov[8] found that, like flat surfaces,[20] smooth strings are asymptotically free in α. In the infrared, a critical point is expected as the renormalized string tension vanishes,[2-7] $\mu_{ren} \to 0$, but the critical behavior is as yet unknown.

Since x appears as an (iso-) vector in the action, in principle it is possible to study the theory over all distances by an expansion in large d. The large d expansion of flat surfaces is straightforward.[19] In momentum space, the x propagator is $\sim 1/(p^2)^2$, and it appears that there are power-like infrared divergences. Interactions always bring in two powers of momenta, though, so flat surfaces should be viewed as a minor variant of the usual non-linear sigma model, with a sigma field $\sim \partial_a x$. In perturbation theory, the two-point function of $\partial_a x$ is logarithmically divergent, $\sim 1/p^2$.

At infinite d, an $O(d)$ symmetric mass gap is generated dynamically, with the $\partial_a x$ two-point function $\sim 1/(p^2 + m^2)$, $m^2 \neq 0$. At large d,[19] and likely for any d in which flat surfaces are asymptotically free (which is $d > 2$[20]), correlation functions of $\partial_a x$, λ^{ab}, *etc.*, are all short-ranged.

The expansion of smooth strings at large d has been studied by David[9] and others.[10,12] At infinite d, the results are very similar to those of flat surfaces, sec. II. The renormalized string tension takes the place of a magnetic field in a spin system. The transverse propagator for $\partial_a x$ is $\sim 1/(p^2 + m^2)$, with $m^2 \sim \mu_{ren}$ at large μ_{ren}. As $\mu_{ren} \to 0$, a $m^2 \neq 0$ is dynamically generated.

Unlike flat surfaces, for smooth strings there are problems in going down from infinite to finite d. At finite d, fluctuations in ρ and λ^{ab} enter. Consider the one loop effective action which describes fluctuations in ρ. These effective actions of ρ are dominated by conformal anomalies over both large and short distances, sec. III.B. At large distances, these conformal anomalies generate the same Liouville action as found in the Nambu model, $\sim 26 - d$; this was suggested initially by Förster[6] and Polyakov.[8] At short distances, conformal anomalies generate a Liouville action $\sim 26 - 2d$.[15] I suggest that a non-renormalization theorem[21] applies to these conformal anomalies, so the Liouville actions derived to one loop order are exact.

These Liouville actions are negative if d is sufficiently large. That each Liouville action is negative for large d produces distinct pathologies over both large and short distances. There is no analogy to these Liouville actions in flat surfaces.

In ref. (15) I showed that at arbitrarily short distances, the one-loop effective action for smooth strings has negative eigenvalues unless $0 \leq d \leq 13$.[22] The condition that $d \leq 13$ follows from requiring the Liouville action at short distances be positive. This Liouville action is first seen at one-loop order, so the instability over short distances when $d > 13$ is manifestly of quantum origin. The condition that $d \geq 0$ is the same as for an $O(d)$ spin system, like flat surfaces; also, smooth strings are only asymptotically free at positive d.[8,20] Remarkably, the intersection of these two conditions does not include a stable large d limit for either sign of d. This is in contrast to most theories. Even the Liouville theory which arises in the Nambu model, $\sim 26 - d$, can be expanded consistently about $d = -\infty$.[18,23]

In sec. IV I ignore these negative eigenvalues at short distances to compute correlation functions at large but finite d.[24] This is the same thing as using perturbation theory to expand about an unstable point, the only difference being that I expand not in a coupling constant, but in $\sim 1/d$. Consequently, the correlation functions so obtained are, at best, only formally defined.

For calculational ease, I restrict myself to the critical point. At $\mu_{ren} = 0$, correlations of ρ and λ^{ab} exhibit a pole for a positive value of the momentum squared: a tachyon.

The infrared divergences of the critical point are computed by overlooking the tachyon at non-zero momentum. Doing so, I find that any two-point function involving $\partial_a x$ or λ^{ab} is infrared finite. The only infrared singular correlations are those of ρ with itself, and these are governed by the Liouville theory at large distances, $\sim 26 - d$.

The outline of the paper is the following. The solution at infinite d is developed in sec. II. The critical effective action at large d is described in sec. III.A; the one-loop effective action for fluctuations in ρ, in sec. III.B. While involved, the only subtlety involves the conformal anomalies. The conformal anomalies of smooth strings involve not only those of massless fields, which are familiar from Nambu strings,[16-18] but that of a massive field, which is less so.[15,25-27] The behavior of the critical correlation functions at large d is summarized in sec. IV.

There are three appendices. The massive conformal anomaly is checked in appendix A. Most of the details necessary to sec.'s III and IV are relegated to appendix B. In appendix C, an integral which typifies the short distance instability of smooth strings (for $d > 13$) is considered.

The expansion of smooth strings at large d has also been studied, independently, by David[9] and by David and Guitter.[10] David[9] initially concluded that smooth strings are unstable at large d over all distance scales. David and Guitter[10] then argued that smooth strings are stable at short distances, with the only instability occuring over large distances for small μ_{ren}.

In effect, the criterion that David and Guitter use to determine stability is that the two-point function of ρ be positive. This two-point function is proportional to a product of eigenvalues, and so is a necessary criterion for stability. It is not sufficient: over short distances when $d > 13$, this product of eigenvalues is positive only because there is a pair of negative eigenvalues. The example of appendix C illustrates this.

Up to this crucial difference, our results are similar. At the critical point, they find the phenomena described above. They also study $\mu_{ren} \neq 0$, and show that there are no tachyons for sufficiently large values of μ_{ren}. I stress that while the tachyons go away, the negative eigenvalues occur at arbitrarily short distances, and so are present for any $\mu_{ren} \geq 0$.

Although it is dangerous, I believe that the results for smooth strings at large d can be used as a qualitative guide to the physics over stable values of d, $0 \leq d \leq 13$.[22] To do so, I consider smooth strings as an amalgam of Liouville actions, which describe ρ, and flat surfaces, which describe x and λ^{ab}.

I assume that if for small d a Liouville action is positive, then the corresponding pathology found at large d, when that Liouville action is negative, disappears. The tachyons seen at large d for small μ_{ren} are related to the Liouville action over large distances, $\sim 26 - d$. Hence I expect that the tachyons are absent when $d < 26$. Similarly, the negative eigenvalues at short distances are related to the Liouville action $\sim 26 - 2d$. When $0 \leq d \leq 13$, then, I propose that smooth strings are stable, free of negative eigenvalues and of tachyons, over all distances, for all $\mu_{ren} \geq 0$.

Continuing, I suggest when $0 \leq d \leq 13$, that for $\mu_{ren} \neq 0$ correlation functions of $\partial_a x$, λ^{ab}, ρ, *etc.*, are short-ranged, damped by mass scales set by μ_{ren} and m^2. At the critical point, a $m^2 \neq 0$ is generated dynamically, so over large distances the two-point functions of $\partial_a x$ and λ^{ab} behave like those of flat surfaces, damped over scales $\sim 1/m$. Only correlations of ρ with itself are infrared singular at the critical point, in the universality class of the Liouville theory over large distances, $\sim 26 - d$.

The constraint field λ^{ab} is related to normals of the surface.[8] If the two-point functions of $\partial_a x$ and λ^{ab} are short-ranged, then while the extrinsic curvature smooths out surfaces over short distances, they remain crumpled at large scales.

A direct understanding of smooth strings over physical values of d might be gained from discrete forms of the model.[28] The numerical study of lattice variants of the Nambu model show that for a given d, discretizing the model in different ways alters the critical behavior.[29] Assuming that universality applies, only one type of discretization can be relevant to the critical point of smooth strings.

At the very least, smooth strings can be viewed as a regularization of the Liouville action $\sim 26 - d$. Their example demonstrates that it is sensible to regulate this Liouville action without maintaining local conformal symmetry.

A fundamental property of surfaces is their Hausdorff dimension. If the two-point function of $\partial_a x$ is $\sim 1/(p^2 + m^2)$, then that of x is $\sim 1/(m^2 p^2)$ as $p^2 \to 0$, and the Hausdorff dimension is infinite. For flat surfaces, this holds order by order in $\sim 1/d$ at large d,[19] and presumably for any $d > 2$. For smooth strings, the Hausdorff dimension is infinite at large d, sec. IV.B. Thus it seems probable that it remains so over $0 \leq d \leq 13$.

This assumes that the original action is real. If the original action is complex, as with Polyakov's θ-like term in four dimensions,[8] it may well be possible to obtain finite Hausdorff dimension. This seems to be the only way of reaching a phase in which the surfaces are smooth over large distances.

II. Smooth strings at infinite d

The action for smooth strings is

$$S_{smooth} = \int d^D z \; \sqrt{g} \left(\frac{1}{2\alpha} \left(\Box x \right)^2 + \mu \right) \; ; \tag{2.1}$$

$\Box$ is the covariant laplacian,

$$\Box = \frac{-1}{\sqrt{g}} \partial_a \left(\sqrt{g}\, g^{ab} \partial_b \right) . \tag{2.2}$$

The embedding of the surface in d Euclidean dimensions is described by the vector $x = x^\beta(z^a)$, where the space-time index β $(= 1 \ldots d)$ is generally suppressed. The coordinates of the world sheet are the z^a, $a, b \ldots = 1, 2$; $\partial_a \equiv \partial / \partial z^a$. I renormalize the theory with dimensional regularization by letting the number of dimensions of the surface be less than two, $D = 2 - 2\epsilon$, $0 \leq \epsilon \ll 1$.

The first term in eq. (2.1), which can be rewritten as the square of the second fundamental form for the surface,[8,11] is the extrinsic curvature term. The second is the usual Nambu action.

Eq. (2.1) is not complete. Unlike Polyakov's treatment of the Nambu model,[16] for smooth strings it is necessary to supplement the action by a term which fixes the metric g_{ab} to be that intrinsic to the surface. Following ref. (8), I introduce a constraint field λ^{ab} and add

$$S_{constraint} = \int d^D z \; \left(\frac{i}{2\alpha} \lambda^{ab} \left(\partial_a x \cdot \partial_b x - g_{ab} \right) \right) \; . \tag{2.3}$$

to the action. The factor of i is chosen so that integration over real λ^{ab} enforces the constraint. Like g_{ab}, λ^{ab} is symmetric in its indices.

The constraint field λ^{ab} is of physical significance. Using the equations of motion for g_{ab}, up to factors of $\sqrt{g}$ the constraint field $\lambda^{ab} \sim i\, T^{ab}$, where T^{ab} is the stress energy tensor for the original action of eq. (2.1).

Gauge fixing is done in conformal gauge, $g_{ab} = \rho\, \delta_{ab}$, which is allowed for $D \leq 2$. To account for the gauge degeneracy of the measure over g_{ab}, Faddeev-Popov ghosts for general coordinate invariance, S_{ghost}, must be included. Fortunately, the ghosts follow merely from the gauge symmetry and the measure over g_{ab} in the functional integral. Hence their contribution is identical to that for the ghosts in Polyakov's treatment of the Nambu string,[16-18]

$$S_{ghost} = \frac{13}{48\pi} \int d^2z \ \left(\partial_a \ln(\rho)\right)^2 \ . \tag{2.4}$$

Altogether, in conformal gauge the partition function for smooth strings is

$$Z_{smooth} = \int dx \ d\rho \ d\lambda^{ab} \ \ exp\left(-S_{smooth} - S_{constraint} - S_{ghost}\right) \ . \tag{2.5}$$

The coordinate x appears quadratically in the action of eq. (2.5), so it can be integrated out to produce an effective action,

$$S_{eff}(\rho, \lambda^{ab}) = \frac{d}{2} \ tr \ ln \ \Delta_x^{-1} + S_{ghost} + \int d^D z \left(\mu \, \rho^{1-\epsilon} - \frac{i}{2\alpha} \rho \, \lambda^{ab} \, \delta_{ab}\right) \ ; \tag{2.6}$$

This is to be compared to the effective action for flat surfaces, eq. (3.10) of ref. (19). Δ_x^{-1} is the inverse propagator for the x field,

$$\Delta_x^{-1} = \Box^2 - \frac{i}{\sqrt{g}} \partial_a \lambda^{ab} \partial_b \ , \tag{2.7a}$$

which in conformal gauge is

$$\Delta_x^{-1} = \left(\frac{1}{\rho^{1-\epsilon}} \partial_a \rho^{-\epsilon} \partial^a\right)^2 - \frac{i}{\rho^{1-\epsilon}} \partial_a \lambda^{ab} \partial_b \ . \tag{2.7b}$$

I determine the stationary point for a manifold with the topology of an infinite, flat disc. Given the topology, at infinite d the stationary point is the dominant configuration in the functional integral. For the action S_{smooth} to be $\sim d$ at large d, it is necessary to take αd and μ/d to be fixed numbers of ~ 1 at infinite d.

For an infinite disc, the stationary point is obvious: $\lambda^{ab} = -i \, \lambda \, \delta^{ab}$, with ρ and λ constants to be determined. For constant ρ, all that is needed is the free energy of the x fields. In dimensional regularization, only the free energy of the massive mode contributes,

$$tr \ ln \ \left(-\partial^2 + m^2\right) = A \, \frac{m^2}{4\pi} \left(\frac{1}{\epsilon} + \gamma + 1 + ln \left(\frac{4\pi\Lambda^2}{m^2}\right) + \ldots\right) \ , \tag{2.8}$$

up to terms $\sim \epsilon$. The area of the surface is A, Λ is a renormalization mass scale, and γ is Euler's constant. In eq. (2.7b), $m^2 = \lambda\, \rho^{1+\epsilon}$, and so

$$S_{eff}(\rho, -i\,\lambda\,\delta^{ab}) = A\,\rho \left(\mu - \frac{\lambda}{\alpha_{ren}} + \frac{d\,\lambda}{8\pi} \left(ln \left(\frac{\Lambda^2}{\lambda} \right) + 1 \right) \right) \; . \tag{2.9}$$

The renormalized coupling constant is α_{ren},

$$\frac{1}{\alpha_{ren}} \equiv \frac{Z_\alpha}{\alpha} = \frac{1}{\alpha} - \frac{d}{8\pi} \left(\frac{1}{\epsilon} + \gamma + ln\,(4\pi) \right) \; , \tag{2.10}$$

where Z_α is the renormalization consant for the coupling, $Z_\alpha = 1 - d\alpha/(8\pi\epsilon)$ up to finite terms. If Pauli-Villars is used instead of dimensional regularization, $1/\epsilon \sim ln(\Lambda^2_{uv}) + \ldots$, with Λ_{uv} the Pauli-Villars mass scale (appendix A). This value for Z_α agrees with Peliti and Leibler,[5] Polyakov,[8] and others,[6,7,9,10,12,15] and shows that the theory is asymptotically free. To one loop order, this Z_α is correct for any d.

Notice that there are no terms $\sim ln(\rho)$ in the action. In two dimensions, $m^2 = \lambda\rho$, so from eq. (2.8) it might be thought that such terms arise. Correctly taking $m^2 = \lambda\rho^{1+\epsilon}$, however, these terms cancel.

The absence of such terms is important. The stationary point is determined by requiring S_{eff} vanish with respect to variations in ρ and λ. Denoting these values by ρ_0 and λ_0, eq. (2.9) gives

$$\lambda_0 = -\frac{8\pi\mu}{d} = \Lambda^2\, exp \left(-\frac{8\pi}{\alpha_{ren} d} \right) \; . \tag{2.11}$$

For $\lambda_0 > 0$, the bare string tension μ must be negative.

There is a natural explanation for why the the constraint field is imaginary at the stationary point. As noted above, $\lambda^{ab} \sim iT^{ab}$, with T^{ab} the stress-energy tensor of eq. (2.1). Hence if $\langle \lambda^{ab} \rangle = -i\lambda_0 \delta^{ab}$, the vacuum expectation value of the stress-energy tensor is real, $\langle T^{ab} \rangle \sim \lambda_0 \delta^{ab}$.

For arbitrary ρ, S_{eff} in eq. (2.9) is strictly proportional to ρ. This has two consequences. First, at the stationary point the value of ρ_0 is not determined, so its value must be taken as a boundary condition. Secondly, the only way that S_{eff} can be stationary with respect to variations of ρ is if $S_{eff} = 0$ at the stationary point; thus the renormalized string tension vanishes.

That ρ_0 is not determined is merely a result of general coordinate invariance. An effective action can only be formed from invariant quantities such as a cosmological constant $\sim \sqrt{g}$, an Einstein term $\sim \sqrt{g}R$, etc. For constant fields, there is only a cosmological constant, $\sim \rho$.

To be definite, consider a global coordinate transformation, in which the surface is uniformly scaled by an amount κ,

$$x \to x \ , \ z^a \to \kappa \, z^a \ , \ g_{ab} \to \kappa^{-2} \, g_{ab} \ , \ \lambda^{ab} \to \kappa^{2\epsilon} \lambda^{ab} \ . \tag{2.12}$$

This scaling does not disturb the *ansatz* that the ρ and λ fields have constant values, and so is a symmetry of the effective action in eq. (2.9). Then ρ can only appear as $A\,\rho$, with no terms $\sim ln(\rho)$ in the effective action.

From eq. (2.7b), at infinite d $(D = 2)$, the x propagator is

$$\Delta_x \sim \frac{\rho_0^2}{-\partial^2(-\partial^2 + m^2)} \ , \tag{2.13}$$

where m is the mass for correlations of $\partial_a x$,

$$m^2 = \rho_0 \, \lambda_0 . \tag{2.14}$$

To avoid ending up with zero (renormalized) string tension, I integrate not over all x fields, as in eq. (2.6), but only with respect to a fixed, flat background:

$$x^1 = \varsigma\sqrt{Z}\, z^1 + x^1_{long} \ , \ x^2 = \varsigma\sqrt{Z}\, z^2 + x^2_{long} \ , \ x^{\beta'} = x^{\beta'}_{tr} \ , \beta' = 3 \ldots d \ . \tag{2.15}$$

Z is a wave-function renormalization constant, where at large d I can take $Z = Z_\alpha$.[8] ς is an arbitrary constant. For constant fields, the effective action for this configuration can easily be evaluated:[10,12]

$$S_{eff}(\rho, -i\,\lambda\,\delta^{ab}) = A\left(\mu\,\rho + \frac{\varsigma^2\lambda}{\alpha_{ren}} + \rho\,\lambda\left(-\frac{1}{\alpha_{ren}} + \frac{d}{8\pi}\left(ln\left(\frac{\Lambda^2}{\lambda}\right) + 1\right)\right)\right) \ . \tag{2.16}$$

This is stationary if

$$\frac{\varsigma^2}{\rho_0\alpha_{ren}} = \frac{\mu}{\lambda_0} + \frac{d}{8\pi} = \frac{1}{\alpha_{ren}} - \frac{d}{8\pi}\, ln\left(\frac{\Lambda^2}{\lambda_0}\right) \ . \tag{2.17}$$

Eqs. (2.16) and (2.17) are invariant under the scaling of eq. (2.12), as long as $\varsigma \to \kappa^{-1}\varsigma$. For fixed ς, however, the scale symmetry is no longer manifest. As a consequence, at the stationary point μ and α_{ren} are not related to each other, as in eq. (2.11), but can be varied independently. Similarly, the values of both ρ_0 and λ_0 are determined.

Consequently, integration over all x's inadvertantly imposes a scale symmetry; as such, it is reasonable that it produces zero renormalized string tension. For eq. (2.16), at the stationary point $S_{eff} = A\lambda_0/\alpha_{ren}$, so the renormalized string tension is $\mu_{ren} = \lambda_0/\alpha_{ren}$.

With dimensional regularization, both the bare and renormalized string tensions are finite quantities. Power-like divergences do arise with other regularizations, eq. (A.5a) of appendix A.

What is noteworthy is that there are no logarithmic divergences in the relation between μ and μ_{ren}. This happens because while the x and λ^{ab} fields require wave-function renormalization, ρ does not;[8] hence there is no anomalous dimension for μ.

Take $\varsigma = 1$. In weak coupling, $\alpha_{ren} \approx 0$, the solutions of eq. (2.17) are independent of the renormalization scale Λ^2,

$$\rho_0 \approx 1 \ , \ \lambda_0 \approx m^2 \ , \ \mu_{ren} \approx m^2/\alpha_{ren} \ \approx \mu \ . \tag{2.18}$$

Fluctuations are small when α_{ren} is, so naturally $\rho_0 \approx 1$ and $\mu_{ren} \approx \mu$. The result for m^2 can be understood by expanding the original action of eq. (2.1) to quadratic order in the transverse modes, x_{tr}:

$$S_{smooth} \approx \frac{1}{2\alpha}\int d^2z \ \left(\left(\partial^2 x_{tr}\right)^2 + \mu\,\alpha\left(\partial_a x_{tr}\right)^2\right) + \dots \ . \tag{2.19}$$

The correct relation for m^2 at small α_{ren}, $m^2 \approx \mu_{ren}\,\alpha_{ren}$, is obtained merely by replacing bare with renormalized quantities in eq. (2.19).

The relation $m^2 \approx \mu_{ren}\alpha_{ren}$ cannot be used in strong coupling, where it would imply that $m^2 \to 0$ as $\mu_{ren}/d \to 0$. As seen from the solution at the critical point, eq. (2.11), even when $\mu_{ren}/d = 0$, m^2 remains non-zero. In other words, while m^2 is generated perturbatively by μ_{ren} in weak coupling, at the critical point the theory dynamically generates a mass $m^2 \neq 0$.

The analogy of smooth strings to flat surfaces, used in the Introduction, is not precise. For flat surfaces, only the expansion in the $O(d)$ symmetric phase, as in eq. (2.6), correctly describes the vacuum.[19] Expanding with reference to a fixed background, as in eq. (2.13), spontaneously breaks the global $O(d)$ symmetry, and generates Goldstone bosons: for flat surfaces, $\mu = 0$ in eq. (2.19), so the two-point function of $\partial_a x_{tr}$ is $\sim 1/p^2$ in momentum space.

For smooth strings, expansion about a background field does not generate any massless modes. This occurs because the expansion of eq. (2.13) breaks not only the global $O(d)$ symmetry, but as well, local reparametrization invariance. Consequently, the mass m^2 that is generated for correlations of $\partial_a x$, as in eq. (2.19), is nothing more than the Anderson-Higgs effect.

III. Effective actions

In part A I compute the leading corrections in $\sim 1/d$ to the results at infinite d. As with flat surfaces,[19] for simplicity I work only at the $O(d)$ symmetric point; for smooth strings this is the critical point, $\mu_{ren} = 0$, eq. (2.6). In part B, I compute the one-loop effective action for fluctuations in ρ, at arbitrary d, in the limit of small and large distances. Most of the details are relegated to appendices, so what follows in this section is largely descriptive. In appendix A, terms dependent upon conformal anomalies are checked by using different regularizations. The detailed form of the inverse propagator is contained in appendix B.

A. The effective action at large d

At large d, with $\alpha \sim 1/d$ and $\mu \sim d$, the effective action of eq. (2.6) is $\sim d$. To compute effects of ~ 1, I expand S_{eff} in ρ and λ^{ab} about their stationary values:

$$S_{eff}(\rho, \lambda^{ab}) = S_{eff}(\rho_0, -i\,\lambda_0\,\delta^{ab}) +$$

$$\frac{1}{2}\iint \left(\rho_q\,\Delta^{-1}(\rho,\rho)\,\rho_q + 2\,\rho_q\,\Delta^{-1}(\rho,\lambda)\,\lambda_q + \lambda_q\,\Delta^{-1}(\lambda,\lambda)\,\lambda_q\right) + \ldots\,, \tag{3.1}$$

which defines the inverse propagator, Δ^{-1}; $\Delta^{-1}(\rho,\lambda) = \Delta^{-1}(\lambda,\rho)$. The terms linear in ρ_q and λ_q vanish by the stationary point condition. In eq. (3.1) and henceforth, the indices on λ_q^{ab} are often dropped to avoid notational clutter. Anticipating the results for Δ^{-1}, I define

$$\rho = \rho_0\left(1 + \frac{1}{m}\sqrt{\frac{8\pi}{d}}\,\rho_q\right)\;,\;\lambda^{ab} = \lambda_0\left(-i\,\delta^{ab} + \frac{1}{m}\sqrt{\frac{8\pi}{d}}\,\lambda_q^{ab}\right)\;, \tag{3.2a}$$

with m^2 as in eq. (2.14); remember that $m^2 \sim 1$ at large d. With this normalization of ρ_q and λ_q, $\Delta^{-1} \sim 1$ as well. Δ^{-1} depends on a single momentum p^a. Using m^2 to set the fundamental length scale, I exchange p^a for

$$P = \frac{p^2}{m^2}\;,\quad \hat{p}^a = \frac{p^a}{\sqrt{p^2}}\;. \tag{3.2b}$$

Originally, the ρ field is dimensionless, while λ has dimensions of (mass)^2; both ρ_q and λ_q have dimensions of mass. To estimate ultraviolet convergence by the use of regulator masses, it is crucial to use the correct dimensions of these fields, which are ρ_q/m and $m\,\lambda_q$, respectively.

I discuss the three elements of Δ^{-1} in turn.

$\underline{\Delta^{-1}(\lambda,\lambda)}$

To quadratic order in λ_q,

$$\iint \lambda_q\,\Delta^{-1}(\lambda,\lambda)\,\lambda_q =$$

$$+4\pi\; tr\left(\overleftarrow{\partial}_a\left(m\,\lambda_q^{ab}\right)\overrightarrow{\partial}_b\,\frac{1}{-\partial^2\left(-\partial^2+m^2\right)}\,\overleftarrow{\partial}_c\left(m\,\lambda_q^{cd}\right)\overrightarrow{\partial}_d\,\frac{1}{-\partial^2\left(-\partial^2+m^2\right)}\right)\;. \quad (3.3)$$

The theory of flat surfaces[15,19] is obtained from smooth surfaces by freezing out the metric degree of freedom, $\rho = 1$. Since to quadratic order in λ_q the fluctuations in ρ do not contribute to $\Delta^{-1}(\lambda,\lambda)$, at the critical point $\Delta^{-1}(\lambda,\lambda)$ is the same as for flat surfaces.

$\Delta^{-1}(\lambda,\lambda)$ is given in appendix B. The dependence on the world sheet indices a, $b\ldots$, eq. (B.3a), is defined by the functions K_1 — K_5, which span the space of two symmetric tensors, eq. (B.4). The momentum dependence is carried by the functions J_1 — J_4, eqs. (B.5a) — (B.5d).

The self energy in eq. (3.3) is completely ultraviolet finite. In terms of the physical field, $\sim m\,\lambda_q$, about zero (external) momentum the integral in eq. (3.3) depends on m as $\sim 1/m^2$. Thus if a regulator mass M replaces m, its effects vanish as $M \to \infty$ like $\sim 1/M^2$.

About zero momentum,

$$\Delta^{-1}(\lambda,\lambda) \sim \frac{1}{8}\left(K^1 + K^2\right)\;, \quad (3.4)$$

eqs. (B.3a) and (B.6a). The tensors K^1 and K^2 are combinations of delta-functions between a, $b\ldots$, eqs. (B.4a) and (B.4b). As for flat surfaces,[19] that $\Delta^{-1}(\lambda,\lambda) \neq 0$ at zero momentum represents the dynamical generation of mass $\sim m^2$ for correlations of λ^{ab}.

$\underline{\Delta^{-1}(\rho,\lambda)}$

There are two contributions to $\Delta^{-1}(\rho,\lambda)$. The first is at tree level, $\sim 1/(\rho_0\alpha_{ren})$. The second arises from the expansion of $tr\, ln\, \Delta_x^{-1}$:

$$\frac{d}{2}\, tr\, ln\, \Delta_x^{-1} \approx \ldots + i\, 4\pi\, \rho_0^{-\epsilon}\, tr\left(\frac{\rho_q}{m}\, \frac{1}{-\partial^2+m^2}\, \overleftarrow{\partial}_a \left(m\, \lambda_q^{ab}\right) \overrightarrow{\partial}_b\, \frac{1}{-\partial^2+m^2} \right) + \ldots \,. \tag{3.5}$$

An overall factor of $1/\rho$ in Δ_x^{-1} has been dropped. Also, ρ^ϵ is replaced by ρ_0^ϵ.

Because of the factor of i introduced with the constraint in eq. (2.3), $\Delta^{-1}(\rho,\lambda)$ is naturally an imaginary quantity. I pull out the explicit factor of i to define

$$\Delta^{-1}(\rho,\lambda^{ab}) \equiv -i\left(J_5\, \delta^{ab} + J_6\, \hat{p}^a \hat{p}^b\right) \,, \tag{3.6}$$

eq. (B.3b), with the functions J_5 and J_6 defined in eqs. (B.5e) and (B.5f).

About zero momentum,

$$\Delta^{-1}(\rho,\lambda) \approx -i\frac{P}{12}\left(\delta^{ab} + 2\hat{p}^a\hat{p}^b\right) + \ldots \,, \tag{3.7a}$$

eq. (B.6). The only divergent contribution to eq. (3.5) is in J_5 at zero momentum. That in all $J_5 = 0$ at $P = 0$ can be checked from eq. (2.6).

Because $\Delta^{-1}(\rho,\lambda)$ vanishes at zero momentum, over large distances correlations of λ^{ab} and ρ decouple. This decoupling is special to the critical point: when $\mu_{ren} \neq 0$, from eq. (2.16) it can be shown that

$$\Delta^{-1}(\rho,\lambda) \sim -i\frac{\mu_{ren}}{dm^2}\delta^{ab} + \ldots \,. \tag{3.7b}$$

Hence for $\mu_{ren} \neq 0$, over large distances correlations of ρ and λ^{ab} mix, developing common mass scales set by both μ_{ren} and m^2.

At large momentum,

$$\Delta^{-1}(\rho,\lambda) \approx -i\left(\frac{ln(P)}{2}\delta^{ab} + \hat{p}^a\hat{p}^b\right) \,, \tag{3.8a}$$

eq. (B.7). Again from eq. (2.16), for small α_{ren} $J_5 \approx 4\pi/(\alpha_{ren} d)$, so the logarithmic growth of J_5 implies

$$\alpha_{ren}(p^2) \approx \frac{8\pi}{d\, ln(p^2)} \,. \tag{3.8b}$$

This is precisely the behavior of α_{ren} expected from asymptotic freedom.[5-7,8-10,12-15]

$\Delta^{-1}(\rho,\rho)$

Setting $\lambda_q = 0$, the only terms of quadratic (or higher) order in ρ_q arise from the expansion of

$$\frac{d}{2}\, tr_{\epsilon\to 0}\, ln\, \Delta_x^{-1} \approx \frac{d}{2}\, tr_{\epsilon\to 0}\, ln\, (\Box) + \frac{d}{2}\, tr_{\epsilon\to 0}\, ln\, (\Box + \rho^{\epsilon}\lambda_0)\ . \tag{3.9}$$

(The differences between Δ_x^{-1} in eq. (2.7b) and (3.9) do not matter as $\epsilon \to 0$.) The ghosts are ~ 1 and negligible at large d.

Eq. (3.9) is the sum of the free energies for d massless and d massive scalars in a background gravitational field.

The massless free energy is the most familiar. Its dependence on the metric field is entirely through the conformal anomaly.[16-18,25-27] To quadratic order in ρ_q,

$$\frac{d}{2}\, tr_{\epsilon\to 0}\, ln\,(\Box) \approx \ldots + \frac{1}{2}\iint \rho_q\left(-\frac{P}{6}\right)\rho_q + \ldots\ . \tag{3.10}$$

The massive mode is more subtle. Expanding perturbatively in ρ_q directly in two dimensions,

$$\frac{d}{2}\, tr_{\epsilon=0}\, ln\,(\Box + \lambda_0) \approx \ldots - 2\pi\, m^4\, tr_{\epsilon=0}\left(\frac{\rho_q}{m}\,\frac{1}{-\partial^2+m^2}\,\frac{\rho_q}{m}\,\frac{1}{-\partial^2+m^2}\right) + \ldots\ , \tag{3.11a}$$

$$= \ldots + \frac{1}{2}\iint \rho_q\,(-2L_4(P))\,\rho_q + \ldots\ . \tag{3.11b}$$

The function $L_4(P)$ is defined in eq. (B.2). Up to differences in sign and normalization, exactly the same function appears in usual non-linear sigma model, eq. (C.10).

While eq. (C.10) is correct in the sigma model, eq. (3.11b) is not the complete free energy of a massive field to $\sim \rho_q^2$. Each arises from the expansion of $\sim tr\, ln\, \vartheta$: the operators ϑ are similar at $D = 2$, but differ when $D = 2 - 2\epsilon \neq 2$. Thus certain terms in the massive free energy are missed if ϵ is set to zero at the outset, rather than taking a non-zero $\epsilon \to 0$.

Another explanation for the difference can be found by using Pauli-Villars regularization, as in appendix A. The essential point is that the the mass dimension of the metric field and the constraint field differ. For a constraint field, of dimension $\sim$ $(\text{mass})^2$, the integral in eq. (3.11a) is like that of eq. (3.4): any effects of a regulator mass M vanish as $\sim 1/M^2$.

In contrast, the metric field is dimensionless, $\sim \rho_q/m$. In this instance, if the mass m is replaced by M, about zero (external) momentum the integral behaves as M^2; to order momentum squared, the integral is ~ 1, and is still sensitive to an arbitrarily large mass M. Ultraviolet convergence does not occur until quartic order in the momentum, $\sim P^2$, when the contributions of a field with mass M are $\sim 1/M^2$.

Consequently, terms ~ 1 and $\sim P$ must be added to eq. (3.11b). The term ~ 1 follows by noting that for constant ρ, the only thing missing from eq. (3.11) is the term $\sim \rho^\epsilon$ multiplying λ_0. Hence

$$\frac{d}{2}\left(tr_{\epsilon\to 0}\, ln\,(\Box + \rho^\epsilon\,\lambda_0) - tr_{\epsilon\to 0}\, ln\,(\Box + \lambda_0)\right) \approx \ldots + \frac{1}{2}\int\!\!\int \rho_q\,(+1)\,\rho_q + \ldots\,. \quad (3.12)$$

The term $\sim P$ can be read off from previous analysis, for it is known that for a field of arbitrary mass, the anomalous part of the trace of the stress-energy tensor is always the same.[25-27] By eq. (3.10) for the massless mode, the anomalous part of the free energy to $\sim \rho_q^2$ is $= -P/6$. Altogether, the total free energy for the massive mode is

$$\frac{d}{2}\, tr_{\epsilon\to 0}\, ln\,(\Box + \rho^\epsilon\,\lambda_0) \approx \ldots + \frac{1}{2}\int\!\!\int \rho_q\left(+1 - \frac{P}{6} - 2\,L_4(P)\right)\rho_q + \ldots\,. \quad (3.13)$$

Adding eqs. (3.10) and (3.13),

$$\Delta^{-1}(\rho,\rho) \equiv J_7(P) = +1 - \frac{P}{3} - 2\,L_4(P)\,, \quad (3.14)$$

eq. (B.5g). At small momentum,

$$\Delta^{-1}(\rho,\rho) \approx -\frac{P}{6} + \ldots\,, \quad (3.15)$$

eq. (B.6g). Eq. (3.15) shows that for the massive mode, all terms ~ 1 and $\sim$ P cancel identically about zero momentum: eq. (3.15) comes entirely from the massless mode, eq. (3.10). At large momentum, the massless and massive modes contribute equally to give

$$\Delta^{-1}(\rho,\rho) \approx -\frac{P}{3} + \dots \; , \tag{3.16}$$

eq. (B.7g).

When I first computed $\Delta^{-1}(\rho,\rho)$ in ref. 15, I missed the corrections to eq. (3.11b) that produce eq. (3.13). At large momentum, eq. (3.14) behaves as in eq. (B.7g), which agrees with eq. (7′) of the corrected version of ref. 15.

The results for Δ^{-1} can be compared to those of David[9] and of David and Guitter.[10] Up to differences in normalization, my expressions for $\Delta^{-1}(\rho,\rho)$ and $\Delta^{-1}(\rho,\lambda)$ agree with these works. For reasons that I do not understand, that for $\Delta^{-1}(\lambda,\lambda)$ does not.

It is worth discussing the regularized free energy of the massive mode in greater detail. The cancellation of the terms ~ 1 could have been anticipated in sec. II. The effective action of eq. (2.9) is valid for fields ρ and λ that have arbitrary, constant values; thus it can be used to read off n-point terms for ρ_q and λ_q at zero momentum. Since ρ only appears linearly in eqs. (2.9), at zero momentum any n-point term for ρ_q vanishes when $n \geq 2$. Eq. (3.15) verifies this for $n = 2$. From eq. (2.16), this remains true if $\mu_{ren} \neq 0$.

To understand the terms of order momentum squared, $\sim P$, instead of talking about the free energy I consider the trace of the stress energy tensor.[25–27] For a scalar field ϕ of mass m, in two dimensions the trace of the classical stress energy tensor is

$$T^a_a = -m^2\,\phi^2 \; . \tag{3.17}$$

The vacuum expectation value of eq. (3.17) is computed in a background gravitational field. For a massless field,

$$\langle T^a_a \rangle_{ren,m=0} = \frac{1}{24\pi}\, R \; . \tag{3.18}$$

Eq. (3.18) agrees with eq. (3.10) to leading order in $\rho - \rho_0$, as the intrinsic curvature $R = -(1/\rho)\partial^2 ln(\rho)$. By general coordinate invariance, the only term that can be formed from two derivatives of ρ is R, so eq. (3.18) is exact.

The complete expression for a massive field in an arbitrary gravitational field is not simple, but the limiting forms are. When the curvature R is much larger than m^2,

$$\langle T^a_{\ a}\rangle_{ren,\, m\neq 0} \approx \frac{1}{24\pi} R \; + O(m^2) \tag{3.19}$$

Eq. (3.19) agrees with (3.18), for if $R \gg m^2$, it doesn't matter if the field is strictly massless, or has a mass that is small on the scale set by R.

In the opposite limit, $R \ll m^2$, the terms $\sim R$ cancel identically:

$$\langle T^a_{\ a}\rangle_{ren,m\neq 0} \approx a\, m^2 + \frac{1}{120\pi}\, \frac{\Box R}{m^2} + \dots\,, \tag{3.20}$$

for some constant a. That there are no terms $\sim R$ in eq. (3.20) implies that the massive mode doesn't contribute to $\Delta^{-1}(\rho,\rho)$ until $\sim P^2$, eq. (3.15).

These results for $\langle T^a_{\ a}\rangle$ are most easily derived by Pauli-Villars regularization, appendix A. For a scalar field of mass m,

$$\langle \phi^2 \rangle \approx \frac{1}{24\pi}\, \frac{R}{m^2} + \dots\,, \tag{3.21}$$

in the limit of small R, $R \ll m^2$. To compute the renormalized value of $T^a_{\ a}$, it is necessary to add the contributions of all fields, physical and regulator. For the massless mode, there is no contribution from the physical field, and the entire result is from the regulators. Plugging eq. (3.21) into eq. (3.17) gives eq. (3.18), remembering that the sum of all regulator fields contribute as one physical field with the wrong spin-statistics connection, eqs. (A.12a) and (A.19).

For the massive mode in the limit of small R, the vanishing of terms $\sim R$, eq. (3.20), follows simply because each field contributes the same amount to $\langle T^a_{\ a}\rangle$, independent of its mass: the m^2 in eq. (3.17) cancels against $1/m^2$ in eq. (3.21). The physical field contributes $-R/(24\pi)$ to $\langle T^a_{\ a}\rangle$, but the sum of the regulator fields contributes $+R/(24\pi)$, eqs. (A.17) and (A.18).

For the massive mode at large R, as in the massless case the term $\sim R$ is due entirely to the regulators, eq. (3.19).

This behavior of the conformal anomaly at large and small R is implicit in the work of Bernard and Duncan,[26,30] and discussed at length by Bunch, Christensen, and Fulling.[27] Hopefully my discussion, while not novel, is at least illuminating.

B. The effective action for metric fluctuations

The results for $\Delta^{-1}(\rho,\rho)$ determine the effective action for fluctuations in ρ. For clarity, I use as the momentum variable the (original) momentum squared, p^2, instead of $P = p^2/m^2$, eq. (3.3).

The effective action for fluctuations in ρ is

$$\hat{S}(\rho) = S_{eff}(\rho, -i\,\lambda_0\,\delta^{ab}) - S_{eff}(\rho_0, -i\,\lambda_0\,\delta^{ab}) \; . \tag{3.22}$$

With ρ_0 and λ_0 the stationary point values, by definition $\hat{S}(\rho)$ starts out quadratic in ρ. In the limit of large and small distances, $\hat{S}(\rho)$ is dominated by conformal anomalies.

I consider $\hat{S}(\rho)$ to one-loop order at arbitrary d. For small d I expand about a background field in weak coupling, eqs. (2.15) — (2.18). The $d-2$ transverse modes, x_{tr}, and the 2 longitudinal modes, x_{long}, produce a total conformal anomaly of the x fields that is strictly $\sim d$.[22] From eq. (3.9), the conformal anomalies of d massless and d massive fields enter. At small d, the conformal anomaly of the ghosts must also be included, eq. (2.4).

From eqs. (3.18) and (3.20), over large distances only the d massless modes and the ghosts contribute,

$$\hat{S}(\rho) \approx \frac{26-d}{96\pi} \int d^2z \; \left(\partial_a\, ln\,(\rho)\right)^2 + \dots \, , \tag{3.23}$$

up to terms quartic in the momentum, $\sim \int \left(\partial^2 ln(\rho)\right)^2$.

Förster[6] and Polyakov[8] first suggested that this Liouville action, familiar from Polyakov's quantization of the Nambu string,[16] appears in smooth strings. Their argument is essentially dimensional analysis: in the original action, the extrinsic curvature term has more derivatives than the Nambu term. Over large distances, the term with the fewest derivatives dominates, so in this limit the Liouville action is that of Nambu strings.

To understand why the d massive modes don't contribute over large distances, remember that the Liouville action is non-local in an arbitrary gauge,[16]

$$\hat{S}(\rho) \approx \frac{26-d}{96\pi} \iint R\,\frac{1}{\Box}\,R \; . \tag{3.24}$$

In $\langle T^a_a \rangle$, the massive mode produces terms ~ 1 and $\sim \Box R$, but those $\sim R$ cancel, eq. (3.20); this cancellation is why the massive mode doesn't contribute to $\hat{S}(\rho)$ over large distances. For the effective action, the term ~ 1 in $\langle T^a_a \rangle$ becomes a cosmological constant; that $\sim \Box R$ contributes $\sim \int R^2$ to $\hat{S}(\rho)$. If the massive mode did contribute $\sim R$ in $\langle T^a_a \rangle$, this would represent a non-local term in $\hat{S}(\rho)$, eq. (3.24). But all contributions of a massive field to $\hat{S}(\rho)$ are infrared finite, constructed entirely from local expressions involving R, $\Box R$, *etc.*; there is no way a massive field could generate a non-local term.

From eqs. (3.18) and (3.19), at short distances

$$\hat{S}(\rho) \approx \frac{26-2d}{96\pi} \int d^2z \; (\partial_a \, ln\,(\rho))^2 - \hat{S}_1 + \dots \; ; \tag{3.25}$$

$\hat{S}_1$ represents terms of ~ 1, with eq. (3.25) valid up to corrections that are $\sim \int\int \rho\,(ln(p^2)/p^2)\,\rho$, *etc.*

Eq. (3.25) can be established directly. As in eq. (5.6) of Alvarez,[18] I expand the free energy of the d massive modes in powers of the mass:

$$\frac{d}{2}\, tr\, ln\left(\Box + m^2\right) \approx \frac{d}{2}\, tr\, ln\,(\Box) + \frac{d\,m^2}{2}\, tr\, \left(\frac{1}{\Box}\right) + \dots \, . \tag{3.26}$$

The first term is the massless free energy, and through its conformal anomaly contributes $\sim -d$ to the Liouville action; that $\sim m^2$ generates $\hat{S}_1$. While the other terms in eq. (3.26) are infrared divergent, they are all ultraviolet convergent, and only contribute to corrections to eq. (3.25).

Eq. (3.26) shows that in a strong gravitational field, the free energy of a massless and a massive particle are equal. Thus at short distances, the d massless modes and the d massive modes produce a Liouville action proportional to $\sim 26 - 2d$.

$\hat{S}_1$ can be computed, although eq. (3.26) isn't the way to do it. Power counting shows that the perturbative contribution to an n-point function for the ρ_q's (*i.e.*, with $\epsilon = 0$) vanishes at large momentum. For example, when $n = 2$, the contribution of eq. (3.11) is $\sim ln(p^2)/p^2$ at large p^2; when $n > 2$, the analogous term behaves as $\sim 1/p^2$. The only way that terms ~ 1 can arise at large momentum is as in eq. (3.12), through contributions in the massive free energy as $\epsilon \to 0$. For arbitrary, constant values of ρ,

$$\frac{d}{2}\left(tr_{\epsilon\to 0}\, ln\left(\Box+\rho^{\epsilon}\, m^2\right)-tr_{\epsilon\to 0}\, ln\left(\Box+m^2\right)\right) = A\,\frac{d\,m^2}{8\pi}\,\rho\, ln(\rho)\ . \tag{3.27}$$

From this, the value of this expression at $\rho=\rho_0$, and that part proportional to $\rho-\rho_0$, must be subtracted to give $\hat{S}_1$ in eq. (3.25),

$$\hat{S}_1 = A\,\frac{d\,m^2}{8\pi}\left(\rho\, ln\left(\frac{\rho}{\rho_0}\right)-(\rho-\rho_0)\right)\ . \tag{3.28}$$

The Liouville actions in eq. (3.23) and (3.25) hold to one loop order. I suggest that these leading forms are exact, to any order in the loop expansion, for any d, and any $\mu_{ren}\geq 0$. That is, following Alvarez[18] I propose that a non-renormalization theorem applies to the conformal anomalies of smooth strings.

By previous argument, this theorem holds in the (admittedly unstable) limit of large d. A formal argument can also be given at $d=0$. From eq. (2.6), at $d=0$ the matter fields drop out, and immediately $\hat{S}(\rho)=S_{ghost}$. This agrees with calculation of the β-function,[8,20] which vanishes to one-loop order when $d=0$, and presumably to all orders.

A non-renormalization theorem crucially affects the critical behavior of smooth strings. From sec. IV.A, the infrared singular correlations are those of the Liouville action over large distances, eq. (4.6). As the only thing which determines the universality class of the Liouville action is its coefficient $\sim 26-d$, if the critical point of smooth strings satisfies universality, then this coefficient cannot be renormalized.

IV. Correlation functions

To quadratic order in small fluctuations, stability is determined by the eigenvalues of Δ^{-1}. The functional integrals over small fluctuations are damped, and so well defined, if each eigenvalue has a positive real part over all momenta. (Remember that with the constraint of eq. (2.3), the functional integral is over real values of λ_q; of course ρ_q is real.)

Consider first flat surfaces,[15,19] which have no metric degree of freedom. Stability is determined the the eigenvalues of $\Delta^{-1}(\lambda, \lambda)$; as described in appendix B following eq. (B.9), the three eigenvalues of $\Delta^{-1}(\lambda, \lambda)$ were computed numerically. Each eigenvalue is positive over all momenta, so flat surfaces are completely stable at large d. This is standard for a sigma model.

Coupling a metric field to flat surfaces, to produce smooth strings, completely alters stability. Δ^{-1} is now a 4 by 4 matrix, with the diagonal component for the metric field dominated by Liouville terms at small and large momentum. From ref. (15), at large d there is a pair of negative eigenvalues over short distances. This is not surprising: as the Liouville action over short distances is negative, the action of a fluctuation about the stationary point can be made arbitrarily large and negative by moving in the direction of $\rho_q \neq 0$ with $\lambda_q = 0$. To understand why this leads to a pair of negative eigenvalues, see the example of appendix C, eqs. (C.1) — (C.5).

It is possible to use the results at large d, with $\mu_{ren} = 0$, to compute at small d. For small d, it is necessary to expand to one-loop order about a given background, eq. (2.15), in the region of weak coupling, where $\mu_{ren} \gg m^2$, eq. (2.18). As discussed in the previous section, in $\Delta^{-1}(\rho, \rho)$ the contribution of the ghosts must be included. At short distances, the results for $\Delta^{-1}(\rho, \lambda)$ and $\Delta^{-1}(\lambda, \lambda)$ can be carried over unchanged. This is because at large momentum, in each one channel dominates the others by a logarithm, eqs. (B.7b) and (B.7e). As in eq. (3.8), these logarithms reflect coupling constant renormalization. Since to one-loop order the β-function of smooth strings is strictly proportional to d,[8,20] this renormalization, computed at large d, is valid down to $d = 0$. At small d these results are only applicable over short distances to the order of leading logarithms; beyond this, details of the background, $\mu_{ren} \neq 0$, enter.

Using this, in ref. (15) I showed that there are negative eigenvalues unless $0 \leq d \leq 13$.[31] When $0 \leq d \leq 13$, it can be shown that at least at short distances, any fluctuation about the stationary point will give a smaller contribution to the functional integral than the stationary point. When $d > 13$, instability occurs because the Liouville action at short distances is negative.

I define the propagator Δ as the matrix inverse to Δ^{-1}. While the physical meaning of this Δ is questionable when $d > 13$, it does have the virtue of being mathematically well-defined; of course it is the right propagator over stable values of d, $0 \leq d \leq 13$. The correlations of ρ_q and λ_q found from this Δ are considered in sec. A. In sec. B, I show that the Hausdorff dimension of smooth strings is infinite at large d.

A. Correlations of ρ and λ^{ab}

In sec. III.B I showed that the effective action for fluctuations in ρ assume a Liouville form at small and large momentum. The appearance of these effective actions is deceptive, however, for while they do dominate $\hat{S}(\rho)$, usually they do not dominate correlation functions of ρ. If correlations of ρ could be computed from $\hat{S}(\rho)$ alone, the propagator for ρ would be $\Delta(\rho,\rho) \sim 1/P$ at large and small P. With one important exception, however, mixing between ρ and λ^{ab} removes the logarithmic singularities that arise from a Liouville action. (I revert to using $P = p^2/m^2$, eq. (3.2b), as the momentum variable.)

I start with the correlation functions at large momentum. As noted above, merely by including the ghosts it is possible to compute to leading logarithmic order at arbitrary d.

The two-point function of ρ, $\Delta(\rho,\rho)$, is given in eq. (B.16h). With the normalization of ρ_q and λ_q in eq. (3.2a), the ghosts show up in Δ^{-1} and Δ as terms $\sim 1/d$. Thus to find the Liouville action in $\Delta(\rho,\rho)$ at large P, it is enough to look for factors $\sim 1/d$. The only dependence on $\sim 1/d$ enters in the denominator of $\Delta(\rho,\rho)$, which is the function D_9, eq. (B.18a). At large P,

$$D_9 \approx \left(ln^3(P) - 3\,ln^2(P) + \left(\frac{13}{d}+3\right)\,ln(P)\right)\frac{1}{4P} + \dots\,, \qquad (4.1)$$

eq. (B.21a). While the ghosts are non-leading at large P, eq. (4.1) does show that the Liouville term in $\Delta^{-1}(\rho,\rho)$ is down by $\sim 1/ln^2(P)$ relative to the leading term.

This occurs because D_9 is proportional to the determinant of Δ^{-1}, eq. (B.10).

As $P \to \infty$, while $\Delta^{-1}(\rho,\rho) \sim P$, $\Delta^{-1}(\rho,\lambda) \sim ln(P)$, and $\Delta^{-1}(\lambda,\lambda) \sim ln(P)/P$, eq. (B.7). Thus in $det(\Delta^{-1})$, the product of the diagonal terms $\Delta^{-1}(\rho,\rho)$ and $\Delta^{-1}(\lambda,\lambda)$ is $\sim ln(P)$, which is smaller than the square of the off-diagonal term $\Delta^{-1}(\rho,\lambda)$, $\sim ln^2(P)$. When the detailed form of D_9 is computed, the contribution of $\Delta^{-1}(\rho,\rho)$ is actually suppressed by an extra $\sim 1/ln(P)$, $\sim 1/ln^2(P)$ altogether. See, also, the example of appendix C, eq. (C.4).

Using eqs. (B.16h) and (B.21g) to compute $\Delta(\rho,\rho)$ at large P,

$$\Delta(\rho,\rho) \approx \frac{+3}{ln^2(P)\, P} \,. \tag{4.2}$$

While at short distances $\Delta^{-1}(\rho,\rho)$ is negative when $d > 13$, in this limit $\Delta(\rho,\rho)$ is positive for any $d \geq 0$.

Consider the vacuum expectation value of ρ_q^2, $\langle\rho_q^2\rangle$. If the Liouville action could be used to compute $\langle\rho_q^2\rangle$, it would be $\sim \int dP/P \approx ln(\Lambda_{uv}^2)$, where Λ_{uv} is a cutoff in momentum space. Because of the factor of $1/ln^2(P)$ in eq. (4.2), however, $\langle\rho_q^2\rangle \sim 1/ln(\Lambda_{uv})$ at large P, and is completely ultraviolet finite.

At large momentum, the Liouville mode does not dominate D_9, and so $\Delta(\rho,\rho)$, because $\Delta^{-1}(\rho,\lambda) \sim ln(P)$. From eq. (3.8), this behavior of $\Delta^{-1}(\rho,\lambda)$ is a consequence of asymptotic freedom in the coupling α_{ren}. Thus it is possible to say that at short distances correlations of ρ are finite and well-behaved, and not logarithmically divergent as in a Liouville theory, because smooth strings are asymptotically free.

The two-point function of λ_q is involved, as there are five independent channels, eq. (B.11a). Most of the channels are like D_9, in that they are not dominated by the Liouville term at large momenta, eq. (B.21).

By itself, though, each channel is not of much significance. For instance, in computing any correlation that involves virtual λ_q's, all channels in $\Delta(\lambda,\lambda)$ contribute. To disentangle the channels, I form an isoscalar quantity from $\Delta(\lambda,\lambda)$ by computing the two-point function for the trace of λ_q^{ab}, $\lambda_a^a \equiv \lambda^a_{q,\,a}$. This is defined in eq. (B.19). From eq. (B.21), at large P

$$\Delta(\lambda_a^a, \lambda_b^b) \approx \left(\frac{13}{d} - 1 + \left(\frac{52}{d} - 1\right)\frac{1}{ln(P)}\right)\frac{4P}{3\, ln^2(P)} + \dots \; ; \tag{4.3}$$

The correlation function like that of eq. (4.3) can be computed in the $O(N)$

non-linear sigma model at large, positive N; from eq. (C.10), it is positive. In the model of flat surfaces at large d, some channels of $\Delta(\lambda,\lambda)$ are positive, and some are negative.[19] When $\Delta(\lambda^a_a,\lambda^b_b)$ is formed, however, it turns out to be positive over all distance scales (appendix B, the discussion following eq. (B.19)).

This indicates that positivity of $\Delta(\lambda^a_a,\lambda^b_b)$ can be used as a (necessary) criterion for stability in the theory of smooth surfaces. From eq. (4.3), at short distances $\Delta(\lambda^a_a,\lambda^b_b)$ is only positive if $d \leq 13$.

At first it seems odd that the instability at short distances, which arises from the Liouville mode in $\Delta^{-1}(\rho,\rho)$, should manifest itself not in $\Delta(\rho,\rho)$, which by eq. (4.2) is always positive, but in $\Delta(\lambda^a_a,\lambda^b_b)$, which is only positive if $d \leq 13$. The example studied in appendix C, however, shows that this has an elementary explanation, eqs. (C.7) and (C.8).

I next turn to studying Δ over the entire range of P at large d. I concentrate on the function D_9, as it appears in the denominator in most channels of Δ, eq. (B.16). As $P \to 0$,

$$D_9 \approx -\frac{P}{48}\,, \tag{4.4}$$

eq. (B.20a). From eq. (4.1), at large P D_9 is positive, so it must have at least one zero at $P \neq 0$. Numerical analysis, described in appendix B, shows that it has one (simple) zero, at a value of $P = P_t$,

$$P_t \approx 1.16943\ldots\,. \tag{4.5}$$

For $\mu_{ren} = 0$, D_9 is negative over $0 < P < P_t$, and positive over $P > P_t$.

As $1/D_9$ appears in the propagator, a simple zero in D_9 implies the spectrum has a tachyon at $P = P_t$. This assumes that the numerators in Δ are non-zero when $P = P_t$, which was checked: the tachyon occurs in all two-point functions of ρ and λ^{ab}, eq. (B.16).

To study the critical point at large d, I ignore the tachyon at non-zero momentum, and concentrate on the behavior of the correlation functions about zero momentum. From eqs. (B.16) and (B.20), the two-point function of ρ is infrared singular:

$$\Delta(\rho,\rho) \approx -\frac{6}{P} + \ldots\,, \tag{4.6}$$

as $P \to 0$. All other two-point functions are infrared finite at the critical point — each channel of $\Delta(\rho,\lambda)$ and $\Delta(\lambda,\lambda)$ is at least ~ 1 as $P \to 0$. For instance,

$$\Delta(\lambda^a_a, \lambda^b_b) \approx 4 + \dots \ , \tag{4.7}$$

so while $\Delta(\lambda^a_a, \lambda^b_b)$ is negative as short distances, eq. (4.3), it is positive about zero momentum.

Eq. (4.6) is nothing other than $1/\Delta^{-1}(\rho,\rho)$ at zero momentum, eq. (3.15). At the critical point, the ρ_q and λ_q fields decouple over large distances: from eq. (3.7a), $\Delta^{-1}(\rho,\lambda) \sim P$. The λ_q fields act like those of flat surfaces, with a non-zero mass gap, while correlations of the metric field are determined by the Liouville action at low momentum.

This result is surely general: about the critical point, the infrared singular correlations are those of the Liouville theory at large distances, eq. (3.23).

Given this, it appears likely that there are tachyons only if $d \geq 26$. When $d > 26$, the Liouville action $\sim 26 - d$, and thus $\Delta(\rho,\rho)$ as $P \to 0$, are negative; $\Delta(\rho,\rho)$ is positive at large momentum, eq. (4.2). From eq. (B.16h), $\Delta(\rho,\rho) = D_9'/D_9$. The function D_9' appears in flat surfaces as the denominator in most channels of $\Delta(\lambda,\lambda)$.[19] At large d, and probably for any $d > 2$,[20] D_9' is positive for any finite P, as it must be for flat surfaces to be stable and free of tachyons. Thus if $\Delta(\rho,\rho)$ changes sign, it can only do so through D_9; but a zero in the denominator means there's a tachyon. Conversely, when $d < 26$, $\Delta(\rho,\rho)$ is positive about $P = 0$; it is natural for it to remain positive and finite at $P \neq 0$, with no tachyon. $d = 26$ is a special case, as the Liouville action $\sim 26 - d$ vanishes. Of course, when $d = 26$ the usual Nambu string has a tachyon in a different correlation function.

The decoupling of ρ and λ^{ab} over large distances occurs only at the critical point, $\mu_{ren} = 0$. From eq. (3.7b), when $\mu_{ren} \neq 0$ these fields mix over large distances: D_9 is non-zero at zero momentum, as both fields develop mass gaps which depend on the values of μ_{ren} and m^2. Consequently, the logarithmically divergent correlations of ρ that arise with a Liouville action generally do not persist for smooth strings: never at short distances, and not over large distances away from the critical point. For smooth strings, the only place where divergent correlations arise is where expected — over large distances near the critical point.

I conclude this section by discussing the relationship between my results and those of David and Guitter.[10] These authors come to very different conclusions about the same problem. They determine stability not by the eigenvalues of Δ^{-1}, but by the condition that $\Delta(\rho,\rho) > 0$. If true, smooth strings would be stable at short distances for any $d > 0$.

While $\Delta(\rho,\rho)$ should be positive, by itself this does not guarantee stability. From eqs. (B.10) and (B.16h), $\Delta(\rho,\rho)$ has the same sign as the product of the eigenvalues of Δ^{-1}. But knowing that the product of eigenvalues is positive is not sufficient to ensure that each eigenvalue has a positive, real part. When $d > 13$, at short distances $\Delta(\rho,\rho)$ and the product of eigenvalues is positive because there is a pair of negative eigenvalues. Further, at short distances the instability appears not in $\Delta(\rho,\rho)$, but in $\Delta(\lambda^a_a, \lambda^b_b)$. Correlations of λ^{ab}, and in particular those of $\lambda^a{}_a$, were not computed in ref. (10). Appendix C discusses the short-distance instability in the context of a simple example.

Other than this, our results are in general agreement. At the critical point, they find that only correlations of ρ are critical, as in eq. (4.6), with a single tachyon at a non-zero value of P, eq. (4.5). Eq. (4.2) does not agree with ref. (10), presumably because our results for $\Delta^{-1}(\lambda,\lambda)$ differ.

David and Guitter also studied $\mu_{ren} \neq 0$. When $\mu_{ren} \neq 0$, D_9 is positive at $P = 0$; if μ_{ren} is sufficiently large, it stays positive over all P. Hence they find a certain value of μ_{ren} above which there are no tachyons.

B. Correlations of x

Using eq. (2.13), the two-point function of x at zero momentum is

$$\left\langle (x)^2 \right\rangle \approx \int_{1/A} \frac{dP}{P(P+1)} \approx ln(A)\ . \tag{4.8}$$

at infinite d. This shows that the mean square size of the surface grows logarithmically with the area, with infinite Hausdorff dimension. This result holds for all $\mu_{ren} \geq 0$, since m^2 is always non-zero, sec. II.

At finite d, the leading corrections to the transverse propagator Δ_x^{-1} are given by the self energy Σ_x,

$$\Delta_x^{-1} = \frac{m^4}{\rho\,\rho_0}\;\left(P\,(P+1) + \Sigma_x\right)\ . \tag{4.9}$$

To $\sim 1/d$,

$$\Sigma_x = \frac{8\pi}{d\,m^2}\;\left(\langle\rho_q^2\rangle\,P^2 - \langle V\,D_0\,V\rangle\right)\ , \tag{4.10}$$

where V is the vertex

$$V = \frac{1}{m^4}\left(\partial^2\,\rho_q\,\partial^2 + i\,m^2\;\overleftarrow{\partial}_a\,\lambda_q^{ab}\,\overrightarrow{\partial}_b\right)\ . \tag{4.11}$$

The external momentum p^a flows through the loop integral in the second term of eq. (4.10); Σ_x depends only on $P \sim p^2$.

In a leap of faith, at $\mu_{ren} = 0$ I ignore the tachyon at $P = P_t$, concentrating on the loop integral about zero momentum to see if the Hausdorff dimension changes. It doesn't, for about zero momentum the contribution of the two-point function of ρ_q to Σ_x is $\sim P^2\int dK$; this is infrared finite, and down relative to terms $\sim P$ from virtual λ_q's. This remains true for $\mu_{ren} \neq 0$, as a consequence of the mass gaps for ρ_q and λ_q.

The obvious limitation is that this calculation ignores the tachyon. For flat surfaces, however, it is possible to calculate (reliably) at large d. In this instance, the Hausdorff dimension remains infinite, as in eq. (4.8), order by order in $\sim 1/d$: see eq. (7.6) of ref. (19).

Acknowledgements

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Appendix A: Alternate regularizations

In this appendix I check some of the results in sections II and III by using heat-kernel and Pauli-Villars instead of dimensional regularization. For the results of sec. II, this is done to establish notation more than anything else. For the conformal anomaly of the massive mode in sec. III, however, the detailed way in which the final result emerges is very different with each regularization. Thus it is worth doing the calculation in other ways.

Heat-kernel regularization

For a massive scalar, the diffusion equation is

$$\left(\frac{\partial}{\partial t} + \Box + \lambda\right) G(z,0;t) = \frac{\delta^2(z)}{\sqrt{g}} , \qquad (A.1)$$

with t the diffusive "time", and $\Box$ the covariant Laplacian of eq. (2.2). For a constant metric field,

$$G(z,0;t) = \frac{\rho}{4\pi t} \; exp\left(-\rho \frac{|z|^2}{4t} - \lambda t\right) . \qquad (A.2)$$

The free energy of a massive scalar in a background ρ field is

$$tr\, ln\,(\Box + \lambda) = -\int_{\xi}^{\infty} \frac{dt}{t}\, tr\; e^{-t(\Box+\lambda)} , \qquad (A.3)$$

where ξ is the cutoff in proper time; it has dimensions of inverse mass squared, so it is related to an ultraviolet cutoff Λ_{uv} as $\xi \approx 1/\Lambda_{uv}^2$. Using properties of the incomplete gamma function,

$$tr\, ln\,(\Box + \lambda) = -\frac{\rho A}{4\pi}\left(\frac{1}{\xi} + \lambda\,(ln\,(\xi\,\lambda) + \gamma - 1)\right) + \ldots , \qquad (A.4)$$

dropping terms $\sim \xi$. Using eq. (A.4), the effective Lagrangian of eq. (2.9) can be recovered by taking

$$\mu \rightarrow \mu + \frac{d}{4\pi\xi} , \qquad (A.5a)$$

$$\frac{1}{\alpha_{ren}} = \frac{1}{\alpha} + \frac{d}{8\pi}\left(ln\left(\xi\Lambda^2\right) + \gamma\right) . \qquad (A.5b)$$

This is just a redefinition of the relation between the bare and renormalized parameters. Unlike dimensional regularization, there are now power-like divergences which renormalize the bare string tension; there are still no logarithmic divergences in the relation between μ and μ_{ren}. The divergent part of eq. (A.5b) agrees with eq. (2.10).

To obtain the conformal anomaly for the massive mode, I follow Alvarez.[18] Define $\rho = exp(-\sigma)$, and consider the variation of the free energy under an arbitrary variation with respect to σ:

$$\delta\, tr\, ln\,(\Box + \lambda) = \int_{\xi}^{\infty} dt\; tr\left(\delta\Box\; e^{-t\Box}\right) \; . \tag{A.6}$$

Since

$$\delta\Box = \delta\sigma\; \Box \; , \tag{A.7}$$

eq. (A.6) can be integrated by parts to yield

$$\delta\, tr\, ln\,(\Box + \lambda) = tr\left(\delta\sigma\; e^{-\xi(\Box+\lambda)}\right) - \lambda \int_{\xi}^{\infty} dt\; tr\left(\delta\sigma\; e^{-t(\Box+\lambda)}\right) \; . \tag{A.8}$$

The first term on the right hand side can be identified as the anomalous part of the free energy (eq. (3.57) of ref. (18)), while the second is the usual, perturbative contribution for a massive field.

I compute eq. (A.8) about zero momentum. To lowest order in σ,[18]

$$tr\left(e^{-t\Box}\right) \approx \frac{1}{4\pi t} + \frac{1}{24\pi}\partial^2\sigma + \dots \; , \tag{A.9}$$

where for convienience I expand about $\sigma_0 = 0$. Then

$$\delta\, tr\, ln\,(\Box + \lambda) \approx A\left(\frac{1}{4\pi\xi} + \dots\right)\delta\sigma + A\left(\frac{1}{24\pi} - \frac{1}{24\pi}\right)\partial^2\sigma\;\delta\sigma + \dots \; . \tag{A.10}$$

The constant term, proportional to $\sim \delta\sigma$, can be read off from eq. (A.4). What is of note is that the terms proportional to momentum squared, $\sim \partial^2\sigma\;\delta\sigma$, cancel. This cancellation is between the anomalous and perturbative terms in eq. (A.8), and persists to all orders in σ, $\sim \sigma^n\,\partial^2\sigma\,\delta\sigma$.

Pauli-Villars regularization

Pauli-Villars regularization can be used, since the addition of a mass term does not affect general coordinate invariance. The free energy of the massive mode is defined as:

$$tr_{PV}\, ln\,(\Box + \lambda) = \sum_{i=1}^{r} c_i\, tr\, ln\,(\Box + \lambda_i)\ . \tag{A.11}$$

There are r fields, each with weight c_i. The first field is physical, with $c_1 = 1$ and $\lambda_i = \lambda$. The other $r-1$ fields are regulator fields, whose $\lambda_i \sim \Lambda_{uv}^2$, where Λ_{uv} is a large mass scale that is ultimately taken to infinity. In two dimensions, eq. (A.11) is ultraviolet finite if two conditions are satisfied:

$$\sum_{i=1}^{r} c_i = 0\ , \tag{A.12a}$$

$$\sum_{i=1}^{r} c_i\, \lambda_i = 0\ . \tag{A.12b}$$

Because of the second condition, there must be at least two regulator fields; *e.g.*, $c_2 = -2$, $\lambda_2 \approx \Lambda_{uv}^2 + \ldots$, and $c_3 = 1$, $\lambda_3 \approx 2\,\Lambda_{uv}^2 + \ldots$. With eq. (A.12),

$$\begin{aligned} tr_{PV}\, ln\,(\Box + \lambda) &= -\frac{\rho A}{4\pi} \sum_{i=1}^{r} c_i\, \lambda_i\, ln\,(\lambda_i)\ , \\ &= -\frac{\rho A}{4\pi}\left(d_1 \Lambda_{uv}^2 + \lambda\, ln\left(\frac{\lambda}{d_2 \Lambda_{uv}^2}\right)\right)\ , \end{aligned} \tag{A.13}$$

with d_1 and d_2 constants that depend upon the choice of the c_i's. Eq. (A.13) is identical to eq. (A.4), up to inessential changes in the relations between bare and renormalized quantities.

To calculate the momentum dependence, I assume that the background metric $\rho_0 = 1$, and expand

$$\rho = 1 + \tilde{\rho}.$$

Since Pauli-Villars is just perturbation theory, in calculating the free energy of the massive scalar it is permissible to assume that

$$tr\, ln\left(-\frac{1}{\rho}\,\partial^2 + \lambda_i\right) = tr\, ln\left(-\partial^2 + \rho\,\lambda_i\right) \approx \ldots + \frac{1}{2}\int \tilde{\rho}\, \Delta_i^{-1}\, \tilde{\rho} + \ldots\ , \tag{A.14}$$

where

$$\Delta_i^{-1} = -\frac{\lambda_i}{2\pi} L_4 \left(\frac{p^2}{\lambda_i} \right) . \tag{A.15}$$

p^2 is the momentum squared; the function L_4 is given in eq. (B.2). From this, the Pauli-Villars regulated inverse propagator is

$$\Delta_{PV}^{-1} = \sum_{i=1}^{r} c_i \, \Delta_i^{-1} . \tag{A.16}$$

For the massive mode, this can be rewritten as

$$\Delta_{PV}^{-1} = -\frac{1}{4\pi} \sum_{i=1}^{r} c_i \, \lambda_i + \frac{p^2}{24\pi} \sum_{i=1}^{r} c_i + \frac{1}{4\pi} \sum_{i=1}^{r} c_i \, \lambda_i \left(1 - \frac{p^2}{6\lambda_i} - 2 \, L_4 \left(\frac{p^2}{\lambda_i} \right) \right) . \tag{A.17}$$

In eq. (A.17), the terms ~ 1 and $\sim p^2$ in the function L_4 have been pulled out explicitly, so about zero momentum, the last term begins at order $\sim (p^2)^2/\lambda_i$. From eq. (A.12), the terms ~ 1 and $\sim p^2$ cancel about zero momentum. For the last term, since order by order in p^2 they always depend on λ_i as $1/\lambda_i$, the contribution of the regulator fields can be dropped as $\Lambda_{uv} \to \infty$. Thus only $i = 1$ contributes,

$$\Delta_{PV}^{-1} = \frac{\lambda}{4\pi} \left(1 - \frac{p^2}{6\lambda} - 2 \, L_4 \left(\frac{p^2}{\lambda} \right) \right) , \tag{A.18}$$

eq. (3.13). From eq. (A.12), for the massive mode the cancellation of terms ~ 1 and $\sim p^2$ persists to all higher orders in $\tilde{\rho}$.

For the massless mode, $\lambda_1 = 0$, so $\Delta_1^{-1} = 0$, and the sum in eq. (A.17) runs only over the regulator fields, $i = 2$ to r. As $\sum_{i=2}^{r} c_i = -1$,

$$\Delta_{PV}^{-1} = -\frac{1}{24\pi} \, p^2 , \tag{A.19}$$

eq. (3.10). This explains why the conformal anomaly of a physical field has the "wrong" sign: it's all due to the regulator fields.

Appendix B: Δ^{-1} and Δ

In this appendix I give the detailed results for the inverse propagator Δ^{-1} and the propagator Δ used in sections III and IV.

The one loop diagrams which arise in the calculation of the inverse propagator Δ^{-1} all involve the exchange of two x-fields. The higher derivative x propagator, eq. (2.13), can be written as a difference of a massless and a massive mode,

$$\frac{1}{p^2(p^2+m^2)} = \frac{1}{m^2}\left(\frac{1}{p^2} - \frac{1}{p^2+m^2}\right), \tag{B.1}$$

so in Δ^{-1} there are functions which have discontinuities for Minkowski values of the momenta, $p^2 < 0$, corresponding to the exchange of two massless modes, one massless mode and one massive mode, and two massive modes.

In terms of the dimensionless variable $P = p^2/m^2$, these three functions are, respectively,

$$L_0 = ln(P)\,,\ L_1 = ln(P+1)\,,\ L_4 = \frac{1}{\sqrt{P(P+4)}}\, ln\left(\frac{\sqrt{P+4}+\sqrt{P}}{\sqrt{P+4}-\sqrt{P}}\right)\,. \tag{B.2}$$

These functions, like those which arise in Δ^{-1} and Δ, depend only upon P. For notational ease this dependence is suppressed, and should be taken for granted.

The inverse propagator is

$$\Delta^{-1}(\lambda,\lambda) = J_1\left(K^1+K^2\right) + J_2 K^3 + J_3 K^4 + J_4 K^5\,, \tag{B.3a}$$

$$\Delta^{-1}(\rho,\lambda^{ab}) = -i\left(J_5\,\delta^{ab} + J_6\,\hat{p}^a\hat{p}^b\right)\,, \tag{B.3b}$$

$$\Delta^{-1}(\rho,\rho) = J_7\,, \tag{B.3c}$$

with $\hat{p}^a$ the unit vector along p^a, eq. (3.2).

The K's are matrices which span the space of two symmetric tensors:

$$K^1_{ab,cd} = \delta^{ac}\,\delta^{bd} + \delta^{ad}\,\delta^{bc}\,, \tag{B.4a}$$

$$K^2_{ab,cd} = \delta^{ab}\,\delta^{cd}\,, \tag{B.4b}$$

$$K^3_{ab,cd} = \hat{p}^a\hat{p}^c\,\delta^{bd} + \hat{p}^a\hat{p}^d\,\delta^{bc} + \hat{p}^b\hat{p}^c\,\delta^{ad} + \hat{p}^b\hat{p}^d\,\delta^{ac}\,, \tag{B.4c}$$

$$K^4_{ab,cd} = \hat{p}^a\hat{p}^b\,\delta^{cd} + \hat{p}^c\hat{p}^d\,\delta^{ab}\ , \tag{B.4d}$$

$$K^5_{ab,cd} = \hat{p}^a\hat{p}^b\hat{p}^c\hat{p}^d\ ; \tag{B.4e}$$

the tensors

$$K^{4+}_{ab,cd} = \hat{p}^a\hat{p}^b\,\delta^{cd}\ , K^{4-}_{ab,cd} = \hat{p}^c\hat{p}^d\,\delta^{ab}\ , \tag{B.4f}$$

also arise. The multiplication table for the K's is given in eq. (4.20) of ref. (19).

The functions J_1 — J_4 can be read off from eq. (4.15) of ref. (19):

$$J_1 = \frac{1}{12P} + \frac{P}{24}\,L_0 - \frac{(P+1)^3}{12P^2}\,L_1 + \frac{(P+4)^2}{24}\,L_4\ , \tag{B.5a}$$

$$J_2 = -\frac{1}{3P} - \frac{P}{24}\,L_0 + \frac{(P^3+3P^2+6P+4)}{12P^2}\,L_1 - \frac{(P+4)^2}{24}\,L_4\ , \tag{B.5b}$$

$$J_3 = -\frac{1}{3P} + \frac{P}{12}\,L_0 - \frac{(P^3-3P-2)}{6P^2}\,L_1 + \frac{(P-2)(P+4)}{12}\,L_4\ , \tag{B.5c}$$

$$J_4 = 2\left(\frac{1}{P} - \frac{(P+1)}{P^2}\,L_1 + L_4\right)\ . \tag{B.5d}$$

There are two typographical errors in ref. (19): in eq. (4.15c), the coefficient of J_3 involves $P+4$, as in eq. (B.5c), while in eq. (4.13a), the coefficient of the last term should be -2 and not -1.

The others J's are:

$$J_5 = -1 + \frac{(P+4)}{2}\,L_4\,, \tag{B.5e}$$

$$J_6 = 1 - 2\,L_4\,, \tag{B.5f}$$

$$J_7 = 1 + \left(\frac{13}{d} - 1\right)\frac{P}{3} - 2\,L_4\ . \tag{B.5g}$$

The contribution of the ghosts, $\sim 1/d$, is included in eq. (B.5g). As discussed at the beginning of sec. IV, for arbitrary d this is only valid to the accuracy of leading logarithms at short distances. Thus I only keep terms $\sim 1/d$ at large P. For the purposes of discusssion in sec. IV, I often keep terms $\sim 1/d$ at large P that are not leading logarithms.

About zero momentum, $P \to 0$,

$$J_1 \approx \frac{1}{8} + \frac{(L_0 - 1)}{24}\,P + \dots\ , \tag{B.6a}$$

$$J_2 \approx -\frac{L_0}{24}P + \frac{3}{80}P^2 + \dots , \tag{B.6b}$$

$$J_3 \approx \frac{L_0}{12}P - \frac{P^2}{15} + \dots , \tag{B.6c}$$

$$J_4 \approx \frac{P}{6} - \frac{2}{15}P^2 + \dots , \tag{B.6d}$$

$$J_5 \approx \frac{P}{12} - \frac{P^2}{120} + \dots , \tag{B.6e}$$

$$J_6 \approx \frac{P}{6} - \frac{P^2}{30} + \dots , \tag{B.6f}$$

$$J_7 \approx -\frac{P}{6} - \frac{P^2}{30} + \dots . \tag{B.6g}$$

up to corrections $\sim P^3$. The only function which doesn't vanish at zero momentum is J_1.

About large momentum, $P \gg 1$,

$$J_1 \approx \frac{1}{4P} + \dots , \tag{B.7a}$$

$$J_2 \approx \left(\frac{L_0}{2} - 1\right)\frac{1}{2P} + \dots , \tag{B.7b}$$

$$J_3 \approx -\frac{1}{2P} + \dots , \tag{B.7c}$$

$$J_4 \approx \frac{2}{P} + \dots , \tag{B.7d}$$

$$J_5 \approx \frac{L_0}{2} - 1 + \dots , \tag{B.7e}$$

$$J_6 \approx 1 + \dots , \tag{B.7f}$$

$$J_7 \approx \left(\frac{13}{d} - 1\right)\frac{P}{3} + 1 + \dots , \tag{B.7g}$$

up to corrections $\sim 1/P$ times smaller. The greatest term at large momentum is J_7. Other than that, J_2 is larger than J_1, J_2, and J_4 by a factor of $ln(P)$; J_5 dominates J_6 in the same way. As illustrated by eq. (3.8), these $ln(P)$'s are due to asymptotic freedom.

The values of the J's for intermediate values of P were determined numerically. The only point of importance, which can be guessed from their limiting forms, is that J_1 and $J_1 + J_2$ are positive for all $\infty > P \geq 0$.

First I describe how to calculate the eigenvalues of Δ^{-1}. I take a basis given by:

$$\left(\rho,\ \lambda^{11},\ \lambda^{22},\ \left(\lambda^{12}+\lambda^{21}\right)/\sqrt{2}\right)\ . \tag{B.8}$$

The eigenvectors of Δ^{-1} depend on the direction of the vector $\hat{p}^a$, but the eigenvalues do not. Assuming $\hat{p}^a = \delta^{a1}$,

$$\Delta^{-1} = \begin{pmatrix} J_7 & -i(J_5+J_6) & -i\,J_5 & 0 \\ -i(J_5+J_6) & 3J_1+4J_2+2J_3+J_4 & J_1+J_3 & 0 \\ -i\,J_5 & J_1+J_3 & 3J_1 & 0 \\ 0 & 0 & 0 & 2(J_1+J_2) \end{pmatrix} . \tag{B.9}$$

In this form, one eigenvalue is immediately seen to be $2(J_1 + J_2)$.

The eigenvalues for the model of flat surfaces[15,19] can be determined from eq. (B.9). Flat surfaces only involve constraint field λ^{ab}, where the inverse propagator for λ^{ab} can be read off from eq. (B.9) by setting $J_5 = J_6 = J_7 = 0$. The three eigenvalues of $\Delta^{-1}(\lambda,\lambda)$ for flat surfaces are then $2(J_1 + J_2)$ and two others, which are found by solving a quadratic equation. By numerical means I showed that each eigenvalue is real and positive for all $0 \leq P < \infty$.

For smooth strings, the four eigenvalues of Δ^{-1} can be determined analytically at large momentum in any number of dimensions, ref. (15) and (31). The Liouville theory at high momentum, $\sim 13 - d$, dominates the eigenvalues for $d \neq 13$. At large momentum when $d > 13$, Δ^{-1} has two eigenvalues with a positive real part, and two with a negative real part.

The eigenvalues Δ^{-1} for smooth strings were determined numerically over all P at $\mu_{ren} = 0$. To understand them, it is simplest to consider the product of the eigenvalues:

$$det(\Delta^{-1}) = 2\,(J_1 + J_2)\,D_9\ , \tag{B.10}$$

where the function D_9 is defined in eq. (B.18a); remember that $J_1 + J_2$ is always positive. The behavior of the function D_9 is discussed in sec. IV; it is negative for $P < P_t$, and positive for $P > P_t$, eq. (4.5). When $P < P_t$, Δ^{-1} has three positive eigenvalues, and one negative one. At $P = P_t$, there is one zero eigenvalue, which corresponds to a simple zero in D_9. As P increases, it develops two eigenvalues with a negative real part.

For flat surfaces, $\Delta^{-1}(\lambda,\lambda)$, and its eigenvalues and eigenvectors, are all real. For smooth strings, because $\Delta^{-1}(\rho,\lambda)$ is imaginary. Δ^{-1} and its eigenvectors are complex. This complexity is innocuous: for example, eigenvalues with a non-zero imaginary part always come in complex conjugate pairs.

The propagator Δ is defined as the matrix inverse to Δ^{-1}. This is correct for physical values of d, $0 \le d \le 13$, and at least a unique prescription when $d > 13$. In terms of components,

$$\Delta(\lambda,\lambda) = \sum_{i=1}^{5} D_i\, K^i \ , \tag{B.11a}$$

$$\Delta(\rho,\lambda^{ab}) = i\left(D_6\, \delta^{ab} + D_7\, \hat{p}^a \hat{p}^b\right) \ , \tag{B.11b}$$

$$\Delta(\rho,\rho) = D_8 \ . \tag{B.11c}$$

From the equations

$$\Delta(\rho,\rho)\,\Delta^{-1}(\rho,\rho) + \Delta(\rho,\lambda)\,\Delta^{-1}(\lambda,\rho) = 1 \ , \tag{B.12a}$$

$$\Delta(\rho,\rho)\,\Delta^{-1}(\rho,\lambda) + \Delta(\rho,\lambda)\,\Delta^{-1}(\lambda,\lambda) = 0 \ , \tag{B.12b}$$

$$\Delta(\lambda,\rho)\,\Delta^{-1}(\rho,\rho) + \Delta(\lambda,\lambda)\,\Delta^{-1}(\lambda,\rho) = 0 \ , \tag{B.12c}$$

follow five relations:

$$(2J_5 + J_6)D_6 + (J_5 + J_6)D_7 + J_7 D_8 = 1 \ , \tag{B.13a}$$

$$(4J_1 + J_3)D_6 + (J_1 + J_3)D_7 - J_5 D_8 = 0 \ , \tag{B.13b}$$

$$(4J_2 + 2J_3 + J_4)D_6 + (2J_1 + 4J_2 + J_3 + J_4)D_7 - J_6 D_8 = 0 \ , \tag{B.13c}$$

$$2J_5 D_1 + (2J_5 + J_6)D_2 + (J_5 + J_6)D_4 - J_7 D_6 = 0 \ , \tag{B.13d}$$

$$2J_6 D_1 + 4(J_5 + J_6)D_3 + (2J_5 + J_6)D_4 + (J_5 + J_6)D_5 - J_7 D_7 = 0 \ . \tag{B.13e}$$

Eq. (B.13a) follows from eq. (B.12a); eqs. (B.12b) and (B.12c) each contain two channels, and produce eqs. (B.13b) — (B.13e). The equation

$$\Delta(\lambda,\rho)\,\Delta^{-1}(\rho,\lambda) + \Delta(\lambda,\lambda)\,\Delta^{-1}(\lambda,\lambda) = \frac{1}{2}K^1 \ , \tag{B.14}$$

produces six relations, for the channels K^1, K^2, K^3, K^{4+}, K^{4-}, and K^5. In order, these are:

$$J_1 D_1 = \frac{1}{4} \ , \tag{B.15a}$$

$$2J_1 D_1 + (4J_1 + J_3) D_2 + (J_1 + J_3) D_4 + J_5 D_6 = 0 \ , \tag{B.15b}$$

$$J_2 D_1 + (J_1 + J_2) D_3 = 0 \ , \tag{B.15c}$$

$$2J_3 D_1 + 4(J_1 + J_3) D_3 + (4J_1 + J_3) D_4 + (J_1 - J_3) D_5 + J_5 D_7 = 0 \ , \tag{B.15d}$$

$$2J_3 D_1 + (4J_2 + 2J_3 + J_4) D_2 + (2J_1 + 4J_2 + J_3 + J_4) D_4 + J_6 D_6 = 0 \ , \tag{B.15e}$$

$$\begin{aligned} 2J_4 D_1 &+ 4(2J_2 + J_3 + J_4) D_3 + (4J_2 - 2J_3 + J_4) D_4 \\ &+ (2J_1 + 4J_2 + J_3 + J_4) D_5 + J_6 D_7 = 0 \ . \end{aligned} \tag{B.15f}$$

The same relations follow if the transpose of eq. (B.15) is computed.

Eqs. (B.13) and (B.15) represent a set of eleven equations for eight unknowns. This overdetermined system has the unique solution:

$$D_1 = \frac{1}{4J_1} \ , \tag{B.16a}$$

$$D_2 = -\frac{D_{10}}{2J_1 D_9} \ , \tag{B.16b}$$

$$D_3 = -\frac{J_2}{4J_1(J_1 + J_2)} \ , \tag{B.16c}$$

$$D_4 = \frac{D_{11}}{2J_1 D_9} \ , \tag{B.16d}$$

$$D_5 = -\frac{D_{12}}{(J_1 + J_2) D_9} \ , \tag{B.16e}$$

$$D_6 = \frac{D_{13}}{D_9} \ , \tag{B.16f}$$

$$D_7 = -\frac{D_{14}}{D_9} \ , \tag{B.16g}$$

$$D_8 = \frac{D_9'}{D_9} \ , \tag{B.16h}$$

where I have introduced the functions

$$D_9' = 8J_1^2 + 12J_1 J_2 + 4J_1 J_3 + 3J_1 J_4 - J_3^2 \ , \tag{B.17a}$$

$$D'_{10} = 2J_1^2 + 4J_1J_2 + J_1J_4 - J_3^2 \ , \qquad (B.17b)$$

$$D'_{11} = 4J_1J_2 - 2J_1J_3 + J_1J_4 - J_3^2 \ , \qquad (B.17c)$$

$$D'_{12} = -2J_1J_4 + 4J_2^2 + 4J_2J_3 + J_2J_4 + J_3^2 \ , \qquad (B.17d)$$

and

$$D_9 = (4J_1 + 4J_2 + J_4)J_5^2 + 2(2J_1 - J_3)J_5J_6 + 3J_1J_6^2 + D'_9J_7 \ , \qquad (B.18a)$$

$$D_{10} = (2J_1 + 4J_2 + J_4)J_5^2 - 2J_3J_5J_6 + J_1J_6^2 + D'_{10}J_7 \ , \qquad (B.18b)$$

$$D_{11} = (4J_2 + J_4)J_5^2 - 2(J_1 + J_3)J_5J_6 + J_1J_6^2 + D'_{11}J_7 \ , \qquad (B.18c)$$

$$D_{12} = J_4J_5^2 - 2(2J_2 + J_3)J_5J_6 + 2(2J_1 - J_2)J_6^2 - D'_{12}J_7 \ , \qquad (B.18d)$$

$$D_{13} = (2J_1 + 4J_2 + J_3 + J_4)J_5 - (J_1 + J_3)J_6 \ , \qquad (B.18e)$$

$$D_{14} = (4J_2 + 2J_3 + J_4)J_5 - (4J_1 + J_3)J_6 \ . \qquad (B.18f)$$

This solution can be checked against the known results for flat surfaces. If by hand I set J_5 and J_6 to zero, the ρ and λ fields decouple, so that $\Delta(\lambda, \lambda)$ should be that for flat surfaces. With $J_5 = J_6 = 0$, eqs. (B.15) reduce to eqs. (4.21) of ref. (19). $\Delta(\lambda, \lambda)$ has the same form as in eqs. (B.16a) — (B.16e) (eqs. (4.22) of ref. (19)), if the functions $D_9 \rightarrow D_{12}$ are replaced by $D'_9 \rightarrow D'_{12}$, eqs. (B.17), respectively. The functions $D'_9 \rightarrow D'_{12}$ appeared before, in eqs. (4.22) of ref. (19), under a different name. In this limit, the other D's collapse to $D_6 = D_7 = 0$, $D_8 = 1/J_7$.

Besides the individual components of the propagator, the two point function for the trace of λ_q^{ab} is also of interest. This is given by

$$\Delta(\lambda_a^a, \lambda_b^b) \equiv \delta^{ab} \left(\sum_{i=1}^{5} D_i \, K^i_{ab,cd} \right) \delta^{cd}$$

$$= \frac{1}{J_1(J_1 + J_2)D_9} \left(J_1 \left(D_9 - D_{12} \right) + 2 \left(J_1 + J_2 \right) \left(-D_{10} + D_{11} \right) \right) \ , \qquad (B.19a)$$

$$= \frac{1}{D_9} \left(J_6^2 + \left(4 \left(J_1 + J_2 \right) + J_4 \right) J_7 \right) \ . \qquad (B.19b)$$

For flat surfaces, $\Delta(\lambda_a^a, \lambda_b^b)$ is given by replacing $D_9 \to D_{12}$ with $D_9' \to D_{12}'$ in eq. (B.19a). I checked numerically that for flat surfaces, $\Delta(\lambda_a^a, \lambda_b^b)$ is positive over all P.

In studying the behavior of the propagator, it is most convienient to study the functions of eq. (B.18). About zero momentum,

$$D_9 \approx -\frac{P}{48} - (15L_0 - 29)\frac{P^2}{1440} + \dots , \tag{B.20a}$$

$$D_{10} \approx -\frac{P}{192} + \frac{P^2}{240} + \dots , \tag{B.20b}$$

$$D_{11} \approx -\frac{P^2}{288} + \dots , \tag{B.20c}$$

$$D_{12} \approx O(P^3) , \tag{B.20d}$$

$$D_{13} \approx -(30L_0 - 23)\frac{P^2}{1440} + \dots , \tag{B.20e}$$

$$D_{14} \approx -\frac{P}{12} - (5L_0 - 7)\frac{P^2}{120} + \dots , \tag{B.20f}$$

$$D_9' \approx \frac{1}{8} - \frac{P}{48} + \dots , \tag{B.20g}$$

up to terms which are smaller by powers of P.

At large momentum, $P \gg 1$,

$$D_9 \approx \left(\frac{L_0^3}{4} - \frac{3}{4}L_0^2 + \left(\frac{13}{4d} + \frac{3}{4}\right)L_0\right)\frac{1}{P} + \dots , \tag{B.21a}$$

$$D_{10} \approx \left(\frac{L_0^3}{4} - \frac{7}{8}L_0^2 + \left(\frac{13}{12d} + \frac{11}{12}\right)L_0\right)\frac{1}{P} + \dots , \tag{B.21b}$$

$$D_{11} \approx \left(\frac{L_0^3}{4} - L_0^2 + \left(\frac{13}{12d} + \frac{7}{6}\right)L_0\right)\frac{1}{P} + \dots , \tag{B.21c}$$

$$D_{12} \approx \left(\left(-\frac{13}{d} + 1\right)\frac{L_0^2}{12} + \left(+\frac{13}{3d} - \frac{1}{12}\right)L_0\right)\frac{1}{P} + \dots , \tag{B.21d}$$

$$D_{13} \approx \left(\frac{L_0^2}{2} - L_0\right)\frac{1}{P} + \dots , \tag{B.21e}$$

$$D_{14} \approx \left(\frac{L_0^2}{2} - \frac{3}{2}L_0\right)\frac{1}{P} + \dots , \tag{B.21f}$$

up to terms $\sim 1/P$, and

$$D_9' \approx (3L_0 - 1)\,\frac{1}{4P^2} - \dots\,, \qquad (B.21g)$$

up to terms $\sim 1/P^3$.

Appendix C: An example of instability

In this appendix I give a simple example which illustrates the nature of the instability at short distances when $d > 13$. This is done in part to counter David and Guitter,[10] who assert there is no such instability; Kleinert[7,b] has followed their lead. The discussion does clarify some of the basic points, without the confusing complexity presented by the full problem.

Instead of treating all three components of λ^{ab}, I replace them with a single field, λ_t; the ρ_q field is represented by ρ_t. For the purposes of discussion, it is not necessary to consider the complete functional integral, but only the integral over fields with a given (large) momentum. This is modeled by

$$Z_t = \int_{-\infty}^{+\infty} d\rho_t \int_{-\infty}^{+\infty} d\lambda_t \; e^{-S_t} \, , \tag{C.1}$$

where the action S_t is

$$S_t = \frac{1}{2} (\rho_t \;\; \lambda_t) \begin{pmatrix} a & ib \\ ib & c \end{pmatrix} \begin{pmatrix} \rho_t \\ \lambda_t \end{pmatrix} . \tag{C.2}$$

I choose a, b, and c to behave like the corresponding components of Δ^{-1} at large P, eqs. (B.7):

$$a = \Delta_t^{-1}(\rho_t, \rho_t) = \left(\frac{13}{d} - 1\right) \frac{P}{3} + 1 \, , \; b = -i\, \Delta_t^{-1}(\rho_t, \lambda_t) = +\frac{ln(P)}{2} \, ,$$

$$c = \Delta_t^{-1}(\lambda_t, \lambda_t) = +\frac{ln(P)}{4P} \, . \tag{C.3}$$

Like eq. (2.5), the integration in eq. (C.1) is over real values of ρ_t and λ_t. Since the constraint of eq. (2.3) brings in an explicit factor of i, the off-diagonal components in Δ_t^{-1}, $\Delta_t^{-1}(\rho_t, \lambda_t)$, are imaginary.

The constants b and c are positive; a represents the Liouville mode at short distances, and is positive for $d \leq 13$, and negative when $d > 13$. If Δ_t^{-1} were purely real, then its determinant would be negative if a were. Because the off-diagonal elements of Δ_t^{-1} are imaginary, though, the determinant of Δ_t^{-1} is positive, regardless of the sign of a:

$$det\left(\Delta_t^{-1}\right) = ac + b^2 \approx b^2 \, , \tag{C.4}$$

since while $|a| \gg b \gg c$ at large P, $b^2 > |a|\, c$ by a factor of $ln(P)$. As discussed following eq. (4.3), this is similar to smooth strings, where the Liouville term does not dominate $det\,(\Delta^{-1})$ either.

At large P, the eigenvalues of Δ_t^{-1} are, approximately,

$$+a\ ,\ +\frac{b^2}{a}\ . \tag{C.5}$$

Up to corrections $\sim b^2/a$, the eigenvector with eigenvalue a is along ρ_t, while the eigenvector with eigenvalue b^2/a is along λ_t.

Looking just at the determinant of Δ_t^{-1}, one can formally define the partition function $Z_t \sim 1/\sqrt{b^2} = 1/b$, which is real. The question is whether the integrals which lead to this result are well-defined.

Everything is fine when $d \leq 13$. Both a, c, and the eigenvalues of Δ_t^{-1} are all positive, so no matter how the integral is done, in what order, at each step every integral is finite, and produce $Z_t \sim 1/b$.

The only contention concerns $d > 13$. Suppose that one first integrates over λ_t by completing the square. Since c is positive, this integral is well-defined, so

$$Z_t \sim \frac{1}{\sqrt{c}} \int_{-\infty}^{+\infty} d\rho_t\ exp\left(-\frac{1}{2}\,\rho_t\,\frac{1}{\Delta_t(\rho_t,\rho_t)}\rho_t\right)\ . \tag{C.6}$$

$\Delta_t(\rho_t,\rho_t)$ is the two-point function for ρ_t,

$$\Delta_t(\rho_t,\rho_t) = \frac{c}{ac+b^2} \approx \frac{c}{b^2}\ . \tag{C.7}$$

Whatever the sign of a, $\Delta_t(\rho_t,\rho_t)$ is positive, so the integral in eq. (C.6) is well-defined, and gives $Z_t \sim 1/b$. For smooth strings, $\Delta(\rho,\rho)$, eq. (4.2), behaves like $\Delta_t(\rho_t,\rho_t)$: it is positive, and independent of d, at large P.

For $d > 13$, however, the integrals are well-defined *only* if they are performed in this order. *Any* other order gives ill-defined integrals: since $a < 0$, it is not possible to integrate over λ_t first, while if one goes to the eigenvalues of Δ_t^{-1}, eq. (C.5), then as each eigenvalue is negative, Z_t is a product of two infinite integrals.

Consider what it would imply for the full functional integral if only one order of integration were allowed: to wit, integrating first over λ^{ab}, and only then over ρ.[10] To consider the effect of an infintesimal fluctuation in ρ at a point z on the world sheet, $\rho(z)$, it would be necessary to integrate over all fluctuations $\lambda^{ab}(z')$ — even if z' is arbitrarily distant from z.

I insist that if the functional integral makes sense, then it shouldn't matter in what order the integrals are performed. This is only true if $d \leq 13$.

One difference with ref. (10) is that they do not introduce an explicit factor of i when they exponentiate the constraint, eq. (2.3). They use $\tilde{\lambda}_t = i\lambda_t$, and integrate over purely imaginary values of $\tilde{\lambda}_t$. In eq. (C.2), a is unchanged, while $b \to -i\,\tilde{b}$ and $c \to -\tilde{c}$, for real $\tilde{b}$ and $\tilde{c}$. The entire discussion can be repeated in this instance. For example, suppose the integral is done by going to the eigenvectors of Δ_t^{-1}. Integrating along real ρ_t and imaginary $\tilde{\lambda}_t$, the integrals are finite only if the eigenvalue in the $\sim \rho_t$ direction, a, is positive, while that in the $\sim \tilde{\lambda}_t$ direction, $-\tilde{b}^2/a$, is negative. This again requires $a > 0$, or $d \leq 13$.

Moreover, this example indicates where the instability appears in correlation functions. Suppose that $d > 13$, so the Liouville action, $\Delta_t^{-1}(\rho_t, \rho_t) \sim a$, is negative. As an obvious consequence of inverting a two by two matrix, the negative sign of a shows up not in $\Delta_t(\rho_t, \rho_t)$, but in the other diagonal element,

$$\Delta_t(\lambda_t, \lambda_t) = \frac{a}{ac + b^2} \approx \frac{a}{b^2} \; . \tag{C.8}$$

which is the two-point function of λ_t. This is precisely what happens for $\Delta(\lambda^a_a, \lambda^b_b)$ in smooth strings, eq. (4.3): the instability for $d > 13$ shows up not in correlations of ρ, but in those of of λ^a_a.

For the sake of comparison, I note the analogous results for the usual non-linear sigma model. The action for an $O(N)$ iso-vector field σ is

$$S = \frac{1}{2\alpha} \int d^2z \, \left((\partial_a \sigma)^2 + i\,\lambda \left(\sigma^2 - 1 \right) \right) \; . \tag{C.9}$$

Using the equations of motion for σ and the constraint, $\lambda = +i\,(\partial_a \sigma)^2$.[32] This is similar to smooth strings, where $\lambda^{ab} \sim i\,T^{ab}$, with T^{ab} the stress-energy tensor of eq. (2.1).

Integrating out the σ field produces an effective action that depends only on the constraint field λ. Expanding about the stationary point $\lambda = -i\lambda_0 + \lambda_q$, $\lambda_0 > 0$, in momentum space the two-point function for λ_q is:

$$\Delta(\lambda_q, \lambda_q) = +\frac{4\pi}{N} \frac{\lambda_0^2}{L_4(P)} \,. \tag{C.10}$$

The function $L_4(P)$ is given in eq. (B.2), and is positive for all P ($P \equiv p^2/\lambda_0$).

In the sigma model, there is a stationary point for both $N = -\infty$ and $N = +\infty$, with the two-point function of λ_q having the same sign as N. For smooth strings, at short distances the two-point function of λ_a^a is negative at large d, eqs. (4.3) and (C.8). Consequently, the expansion of smooth strings about $d = +\infty$ is similar to the expansion of the $O(N)$ model around $N = -\infty$. For the $O(N)$ model, stability is recovered by going to $N = +\infty$; smooth strings are unstable for either sign of d.[15]

Obviously, these instabilities did not deter me from plowing ahead and computing at large d anyway, as David and Guitter[10] did. The only difference is that I insist that the computations be undertaken with apology, and treated with due circumspection. Even so, I believe that our results, in this unstable limit of large d, can be used to understand at least some properties of smooth strings over physical values of d.

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[20] From ref. (8), to one-loop order the β function for smooth strings is

$$\beta_{smooth} = -\frac{d}{4\pi}\,\alpha^2 + \dots \,.$$

Using Polyakov's methods, the β function of flat surfaces is found to be

$$\beta_{flat} = -\frac{(d-2)}{4\pi}\,\alpha^2 + \dots \,.$$

Thus smooth strings are asymptotically free for $d > 0$, while flat surfaces are only so when $d > 2$. To leading order, the anomalous dimension of x is the same for each model,

$$\gamma = \frac{(d-2)}{4\pi}\alpha + \dots .$$

Presumably, for flat surfaces all renormalization group functions vanish at $d = 2$: for surfaces embedded in two dimensions, there is no room in which to (extrinsically) bend an (intrinsically) flat surface. In contrast, smooth strings are non-trivial when $d = 2$, as $\beta_{smooth} \neq 0$.

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[22] Ignoring gauge invariance, a surface in d dimensions has 2 longitudinal degrees of freedom and $d-2$ transverse degrees of freedom. While the stable range of d for smooth strings is $0 \leq d \leq 13$, it must be noted that the number of transverse modes is only positive for $d > 2$; thus the physical range of d is $2 < d \leq 13$. This can be seen from the anomalous dimension, ref. (20), which is only positive when $d > 2$. Even so, it is interesting to consider smooth strings for $0 \leq d \leq 2$. When $d = 0$, for example, from eq. (2.6) the effective action for smooth strings reduces to that of Nambu strings, and only involves the ghosts, eq. (2.4). This shows that smooth strings are stable at $d = 0$, and suggests an expansion about $d = 0^+$, ref. (24). I also remark that the upper limit of stability in d, $d_u = 13$, is changed by adding extra fields on the world sheet, through the conformal anomalies of these new fields. World sheet scalars or fermions lower d_u, while gauge fields increase d_u. For the supersymmetric string with extrinsic curvature proposed by Curtright and van Nieuwenhuizen in ref. (13), perhaps $d_u = 5$.

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[24] Why do I expand about $d = +\infty$, and not $d = -\infty$, or $d = 0$? I exclude $d = -\infty$ because asymptotic freedom is lost at negative d, ref. (20). When $d = 0$, the effective action for smooth strings collapses to the Liouville action of the ghosts, ref. (22). Correlations of λ^{ab} first arise at $d \neq 0$, through the

expansion of $\sim d\, tr\, ln\, \Delta_z^{-1}$ in eq. (2.6). In contrast, while smooth strings are unstable at large d, all fields are non-zero at $d = +\infty$, sec. II.

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[30] The results of eqs. (3.19) and (3.20) can be deduced from eq. (2.11) of Bernard and Duncan, ref. (26).

[31] In ref. (15), I worked at $\mu_{ren} = 0$, instead of the perturbative regime. Since I worked only at short distances, however, the only change is that the m^2 of ref. (15) should be replaced by its perturbative value, $m^2 = \alpha_{ren}\mu_{ren}$, eq. (2.18). I also note that while the Liouville action at short distances vanishes at $d = 13$, the non-leading term ~ 1 in $\Delta^{-1}(\rho, \rho)$, eq. (B.7g), also derives from regularization of the massive mode, eq. (3.12). For this reason, $\Delta^{-1}(\rho,\rho)$ can be trusted to conclude that smooth strings are stable, over short distances, when $d = 13$.

[32] As for smooth strings, the vacuum expectation value of this composite operator is real: $\langle (\partial_a \sigma)^2 \rangle = -\lambda_0$. Consider the two-point function,

$$\langle (\partial_a\sigma)^2(z)\ (\partial_a\sigma)^2(0)\rangle = -\langle \lambda(z)\, \lambda(0)\rangle = +\lambda_0^2 - \langle \lambda_q(z)\, \lambda_q(0)\rangle\ .$$

By writing down the dominant diagrams at large N and summing them up, it is possible to show that the connected piece of this two point function does contribute with a negative sign; *i.e.*, that the two point function of λ_q, eq. (C.10), is positive.