Inflation – In the Early Universe and Today

Edmund J. Copeland

Department of Physics and Astronomy, University of Sussex, Brighton,
BN1 9QJ, U.K., e.j.copeland@susx.ac.uk

Abstract. In these lectures we will be giving a basic introduction to modern ideas in cosmology. Beginning with a review of the Standard Big Bang (SBB) scenario, we will introduce the observed cosmological parameters and indicate the features that the SBB can not explain, such as the initial conditions. This will lead to an introduction to the inflationary cosmology, which postulates a period of accelerated expansion during the Universe's earliest stages [1, 2, 3, 4, 5]. We will provide some examples of inflationary solutions and demonstrate how they can be used to make distinctive predictions which in principle can be tested with current observations. In particular it provides a possible model for the origin of structure in the Universe. The state of these observations will also be discussed with particular attention being given to the most recent experiments which have detected anisotropies in the cosmic microwave background radiation. We will discuss some of the most exciting developments that have recently emerged in cosmology, arising from string and M-theory models. A particular example of inflation arising out of branes will be given to emphasise the potential new features these solutions have. Finally we will discuss models of Quintessence, scalar field models used to explain the exciting results that the Universe is undergoing a period of acceleration today.

1 The Standard Big Bang Model

The standard hot big bang (SBB) theory is an extremely successful one, and has been around for over 60 years, since Gamow originally proposed it. Remarkably, for such a simple idea, it provides us with an understanding of many of the basic features of our Universe. All that you require in the cooking pot, are initial conditions of an expanding scale factor, gravity, plus the standard particle physics we are used to, to provide the matter in the Universe. It can then pass a number of crucial observational tests.

- The expansion of the Universe – $t_{\rm age} \sim 10 - 20$ Billion years.
- The existence and spectrum of the cosmic microwave background radiation (CBR) – Planck Back Body spectrum with $T \sim 2.73$K.
- The abundance of light elements in the Universe (nucleosynthesis).
- Gravitational collapse – responsible for the formation of structure in the Universe, although it relies on the presence of initial irregularities being present in the CBR consistent with that detected by the COBE satellite.

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However, as mentioned the hot big bang theory can successfully proceed only if the initial conditions are very carefully chosen, and even then it only really works at temperatures low enough, so that the underlying physics can be well understood. The very early Universe is out of bounds, yet there is a hope that accurate observations of the present state of the Universe may highlight the types of process occurring during these early stages, providing an insight on the nature of physical laws at energies which it would be inconceivable to explore by other means. Another unresolved issue is the cause of the apparent acceleration of the Universe today, as seen through the distribution of distant Type Ia Supernovae.

To begin with we will give a quick review of the big bang cosmology. Surprisingly, for a theory which is usually associated with solving highly non-linear Einstein equations, it is possible to obtain the key evolution equation (the Friedmann equation) simply from Newtonian cosmology. The hot big bang theory is based on the *cosmological principle*, which states that the Universe should look the same to all observers. That tells us that the Universe must be homogeneous and isotropic. With this in mind, imagine (you will need a healthy imagination throughout these lectures!) a uniform homogeneous 'dust' filled Universe of mass M, with a test particle at radius a. The acceleration experienced by the particle is (ignoring the mass outside of radius a),

$$\ddot{a} = -\frac{GM}{a^2},$$

where G is Newton's constant and $\dot{a} \equiv \frac{da}{dt}$. This can be integrated to give

$$\frac{\dot{a}^2}{2} - \frac{GM}{a} = -\frac{k}{2}, \tag{2}$$

where k is an integration constant. Equation (2) is simply the statement that energy is conserved. Now, for a uniform dust distribution we have

$$M = \frac{4\pi}{3}\rho a^3 = \text{constant},$$

where ρ is the energy density (i.e. mass per unit volume). Substituting for M in (2) we obtain the Friedmann equation,

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{4}$$

where H is the Hubble parameter. This derivation is perfectly adequate because of the assumption of homogeneity. Birkhoff's theorem allows us to consider a region of arbitrary small a, where we expect the Newtonian approximation to be a valid. Homogeneity allows us to then extend this to large a. The parameter $a(t)$ is an important one, it is the 'Scale factor' of the universe, so called because all length scales grow by the same factor $a(t)$ in a homogeneous Universe. It measures the physical size of the Universe. The constant k has a geometrical interpretation (although we need to return to the full General Relativity to see it), it measures the spatial curvature with k negative, zero or positive corresponding to open, flat and closed Universes respectively. The *cosmological principle* tells us which metric must be used to describe it. It is the Robertson–Walker metric, which is given by (3) in the contribution of J.L. Cervantes–Cota in this book.

In cosmology we often use comoving coordinates (r) which are rescaled to the physical coordinates (x) through $x = a(t)r$. Comoving coordinates are so useful, they allow for the expansion of the Universe to be removed from a problem. The crucial link that Einstein spotted was that the geometry of the Universe or its expansion is governed by the properties of material within it. This is specified by the energy density $\rho(t)$ and the pressure $p(t)$, usually related by an equation of state, $\rho = wp$ which gives p as a function of ρ. The key examples are

$$p = \frac{\rho}{3} \qquad \text{Radiation}\,, \tag{5}$$

$$p = 0 \qquad \text{Non-relativistic matter}\,, \tag{6}$$

$$p = -\rho \qquad \text{Vacuum Dominated}\,. \tag{7}$$

In general though there need not be a simple equation of state for example when there is a combination of radiation and non-relativistic matter. This matter satisfies energy momentum conservation, also known as the Fluid equation,

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{8}$$

In (8), $3H\rho$ is the reduction in density due to the increase in volume, and $3Hp$ is the reduction in energy caused by the thermodynamic work done by the pressure when this expansion occurs. Combining (4) and (8) we obtain the useful k-independent Acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p)\;. \tag{9}$$

1.1 Standard Big Bang Solutions

In most of what follows we will assume a flat Universe (although there will be exceptions which hopefully will be clear). When $k = 0$ (4) and (8) can be solved for the various equations of state to give the cosmological solutions

$$\text{Matter Domination} \qquad p = 0: \quad \rho \propto a^{-3} \qquad a(t) \propto t^{2/3} \tag{10}$$

$$\text{Radiation Domination} \qquad p = \rho/3: \rho \propto a^{-4} \qquad a(t) \propto t^{1/2} \tag{11}$$

$$\text{Vacuum Domination} \qquad p = -\rho: \quad \rho \propto \rho_0 \qquad a(t) \propto exp(Ht). \tag{12}$$

In both radiation and matter cases the density falls as t^{-2}, whereas in the vacuum (or *cosmological constant* case it remains constant (in general it will be constant or decrease less quickly than $1/t^2$). If there is both matter and radiation the Friedmann equation can be solved using conformal time $\tau = \int dt/a$, while, as we shall see, if there is matter and a non-zero curvature term the solution can be given either in parametric form using normal time t.

1.2 Characteristic Scales and Density Parameters

When the spatial geometry is flat, for a given H, (4) determines the critical density

$$\rho_{\mathrm{c}}(t) = \frac{3H^2}{8\pi G} \,. \tag{13}$$

Densities are usually measured as fractions of ρ_{c}:

$$\Omega(t) \equiv \frac{\rho}{\rho_{\mathrm{c}}} \,, \tag{14}$$

where Ω is known as the density parameter, and can be applied either to individual types of material or to the total density.

The present value of the Hubble parameter is not that well known, so it is parameterized as

$$H_0 = 100h \; \mathrm{km\,s^{-1}\,Mpc^{-1}} = \frac{h}{3000}\,\mathrm{Mpc}^{-1} \,, \tag{15}$$

where h is normally assumed to lie in the range $0.5 \leq h \leq 0.8$. The current most popular value for h is around $h \simeq .7$ based on a number of different observations. Note, the subscript '0' refers to the present day and reflects the value a parameter has today. Having defined the Hubble parameter and curvature scale, it follows that these can be used to define two length scales: The Hubble time (or length) $H_0^{-1} = 9.8 \times 10^9 h^{-1}$ years gives an approximation to the actual age of the Universe, providing the typical time scale of evolution for $a(t)$. Of course, the Hubble parameter is not constant, varying in general as t^{-1}. The second scale is the curvature scale $a|k|^{-1/2}$ and gives the distance up to which space can be taken as having a flat (Euclidean) geometry. The present critical density is

$$\rho_{\mathrm{c}}(t_0) = 1.88\,h^2 \times 10^{-29}\,\mathrm{g\,cm^{-3}}, \tag{16}$$

incredibly small bearing in mind what we are used to dealing with on earth.

Both the Hubble length and curvature length are physical scales; to obtain the corresponding comoving scale we must divide by $a(t)$. The ratio of these scales actually gives a measure of Ω; from the Friedmann equation we find

$$|\Omega - 1| = \frac{|k|}{H^2 a^2} \,. \tag{17}$$

A crucial property of the big bang Universe is that it possesses *horizons* which arise because light can only have traveled a finite distance since the start of the Universe t_*. To obtain the horizon, we simply use the fact that light travels on null geodesics ($ds^2 = 0$), see (3) in the contribution of J.L. Cervantes–Cota in this book, hence for fixed θ and ϕ, we obtain $dr = dt/a(t)$ which integrates to give the physical distance

$$d_{\mathrm{H}}(t) = a(t) \int_{t_*}^{t} \frac{dt}{a(t)} \, . \tag{18}$$

In a matter dominated Universe $d_{\mathrm{H}}(t) = 3t = 2H^{-1}$; see also Sect. 1 (horizon) in the contribution of Jorge L. Cervantes–Cota in this book.

1.3 Introducing the Cosmic Background Radiation

The redshift measures the expansion of the Universe via the stretching of light

$$1 + z = \frac{a(t_0)}{a(t_{\mathrm{emit}})} \, . \tag{19}$$

As a measure of time, the redshift refers to the time at which light would have to be emitted to have a present redshift z. As a measure of distance, it refers to the *present* distance to an object from which light is received with a redshift z. We can combine (4), (17) and (19) to solve for more general cosmologies involving the situation $k \neq 0$ (remember we are assuming that the cosmological constant vanishes here). Unfortunately it is generally too difficult to obtain explicit solutions in these cases for $a(t)$, rather we obtain $t(a)$. For example, in a matter dominated Universe we obtain

$$\begin{aligned} t_0 &= H_0^{-1} \frac{\Omega}{2(\Omega - 1)^{3/2}} \left[\cos^{-1}(2\Omega^{-1} - 1) - \frac{2}{\Omega}(\Omega - 1)^{1/2} \right] , \quad \Omega > 1 \\ &= H_0^{-1} \frac{\Omega}{2(1 - \Omega)^{3/2}} \left[\frac{2}{\Omega}(1 - \Omega)^{1/2} - \cosh^{-1}(2\Omega^{-1} - 1) \right] , \quad \Omega < 1 . \end{aligned} \tag{20}$$

Expanding about $\Omega = 1$ we obtain

$$t_0 \simeq \frac{2}{3} H_0^{-1} \left[1 - \frac{1}{5}(\Omega - 1) + \cdot \right] , \tag{21}$$

implying that for $\Omega < 1$, the Universe is older for a given value of h.

The Universe is full of *radiation*, a remnant of the big bang. The detection of this primordial soup in 1965 by Penzias and Wilson provided one of the major breakthroughs for cosmologists trying to understand the nature of the Universe. Its existence is a prediction of the model and was first proposed in the 1940's by George Gamow. The radiation was emitted at a red shift $z \sim 1100$ the epoch known as the surface of last scattering, corresponding to a time $t \sim 180,000(\Omega h^2)^{(-1/2)}$ years, the time when the photons decoupled from the the electrons as they found their way into their ground state. The temperature then was about 2500 K, and was the moment the Universe went from being opaque to being transparent. Gamow argued that as the Universe expanded and cooled, the photons would be stretched and would today have a temperature of order 10 K and be close to a perfect blackbody spectrum. In 1990, based on just 9 minutes of data, the **CO**smic microwave **B**ackground **E**xplorer satellite (COBE) detected the radiation and showed that it was almost perfectly isotropic, with a Planck blackbody spectrum of $T = 2.735 \pm 0.01$ K. This corresponds to a photon density in the Universe today of $n_\gamma = 422\text{cm}^{-3}$. The radiation is peaked at wavelength $\lambda = 2$ mm corresponding to a frequency $\nu = 150\text{GHz}$ (i.e. in the microwave region of the electromagnetic spectrum). Of course, this temperature is today much lower than it was in the early Universe. This is because as the Universe expands, it cools. We can determine the rate that it cools through the following simple argument. At high density, because of the high interaction rate, any matter rapidly approaches thermal equilibrium. For radiation, Planck showed us that a quanta of frequency ν had energy $E = h\nu = h/\lambda \propto a(t)^{-1}$ where h is Planck's constant, and the scaling with the scale factor simply represents the fact that all length scales are stretched by the expansion of the Universe. The corresponding energy density in radiation evolves as $\rho_\gamma(t) = E_\gamma/V \propto a(t)^{-4}$. From the world of statistical mechanics we have Stefan's law which tells us[1] $\rho_\gamma \propto T^4$, from which we see that the Universe cools as it expands according to the law

$$T \propto \frac{1}{a}\,. \tag{22}$$

In its earliest stages the Universe may have been arbitrarily hot and dense, so although matter dominated since nucleosynthesis, far enough in the past it will have been dominated by radiation.

1.4 The Mass of the Universe

How much mass is there in the Universe and can we determine the answer? This is a crucial issue in cosmology. Earlier, we defined the density parameter $\Omega(t) \equiv \rho/\rho_\text{c}$. This is the parameter which is important if we are to determine the future evolution of the Universe. Current bounds on its value place it between $.3 < \Omega < 1.2$, but what is it composed of? One of the most significant and successful predictions of the SBB is Nucleosynthesis, the formation of the lightest elements in the Universe. The spectrum of these elements can be predicted and has been compared to observation through numerous experiments. Basically as the Universe expands and cools, it reaches a critical temperature (around 1 MeV) when the reversible reaction of the neutron decaying into protons ceases and the neutron freezes out. The neutron's then

[1] Compare to (21) in the contribution of J.L. Cervantes–Cota in this book.

decay solely into protons and the lightest elements begin to form. This is not the place to discuss nucleosynthesis in detail, it involves analysing rate equations to determine the net fractions of light elements formed, but it is worth discussing the main results. The actual fraction of the light elements formed depends sensitively on the density of ordinary matter (the baryons) at the time of nucleosynthesis. This can vary from $10^{-30} - 10^{-31}$gm cc^{-1}. The light elements formed are Hydrogen (75% by mass), Helium (24% by mass), Deuterium (10^{-5} compared to hydrogen), Helium3 (10^{-5} compared to hydrogen), Lithium 7 (10^{-10} compared to hydrogen). These incredibly small numbers are crucial as slight variations in the baryon density lead to large changes in the abundance, changes that can be ruled out by observation. The key feature though as far as we are concerned are bounds on the total amount of **baryonic matter** in the Universe. The current Nucleosynthesis constraints give

$$\Omega_{\rm baryon} = (0.019 \pm .0012)h^{-2}. \tag{23}$$

So the maximum contribution to the total energy density arising from baryons is bounded by $\Omega_{\rm baryon} < 0.08$. However, we just saw earlier that the lower bound on Ω from all sources is $\Omega > .3$. This arises from a number of observations including analysing the dynamics of clusters of galaxies, from the gravitational lensing of distant quasars by rich clusters of galaxies and by determining the baryon abundance in the centres of clusters of galaxies. All of these long distance observations provide bounds of

$$\Omega_{\rm matter} = (0.3 \pm .05)h^{-(1/2)}. \tag{24}$$

The conclusion that emerges from comparing (23) and (24) are incredibly significant and one of the principle reasons why particle physicists should be interested in cosmology. Clearly, some of the matter in the Universe must be **non-baryonic** and it must be dark, we can not see it. This is clear simply from analysing the rotation curves of light emitted from neighbouring galaxies. These can be interpreted as placing constraints on the matter distribution in our own neighbourhood and points to the existence of large almost spherical dark halos around our visible galaxy.

The current boom in high precision cosmic microwave background experiments such as BOOMERANG, MAXIMA and DASI have enabled a new and exciting approach on the matter question. Since these lectures were given WMAP has come on the scene, and so we should really make use of their wonderful data (whilst bearing in mind the pioneering work of the other high precision experiments). Given a particular cosmological model, there is an associated distribution of peaks and troughs in the power spectrum associated with the anisotropies in the CBR, the position and height of which depend on the nature of the cosmological parameters. In particular the location in ℓ-space of the first 'doppler' peak depends on the quantity $\Omega_{\rm matter} + \Omega_\Lambda$, where Ω_Λ is the density parameter associated with a cosmological constant (and will be discussed later), through $\ell \sim 220/\sqrt{\Omega_{\rm matter} + \Omega_\Lambda}$. Based on parameter fits and combining their data with other astronomical data, the WMAP team find that the best fits are [6]: $\Omega_{\rm matter} + \Omega_\Lambda = 1.02 \pm .002$, with $\Omega_{\rm matter} h^2 = .135^{+.008}_{-.009}$ and $\Omega_{\rm baryons} h^2 = .0224 \pm .0009$.

This all opens up the intriguing question, what could be the source of this dark matter? There are a number of particle physics candidates including cold (non-relativistic at decoupling) particles such as Axions, neutralinos, primordial black holes and supermassive non-thermal relics. There are also Hot particles possible such as massive neutrinos. At first sight the amazing discovery by the Super-Kamiokande team of evidence for massive neutrinos could be thought to point in this direction, however, the proposed masses for the light neutrinos appear to be too light for them to play a significant cosmological role. Another fascinating aspect of the matter question is the fact that even with dark matter present there is still too little matter to cause the Universe to be flat as can be seen from (17). This amount of matter would lead to an open Universe but as we shall shortly discuss the Universe appears to be flat today. Where then is the remaining contribution required to yield $\Omega = 1$? We shall see that it appears to be coming from an unusual source, namely something is providing an energy contribution through an effective cosmological constant which is dominating the Universe today – a dark energy contribution!

1.5 The Timetable for the Universe

Up until the mid 1990's, any cosmology book would state with some authority that the present Universe is dominated by non-relativistic matter which scales as $\rho \propto a^{-3}$. Since we know radiation reduces more quickly with the expansion, this implies that at earlier times the Universe was radiation dominated. We can estimate this period by simply relating the two contributions to obtain $1/a_{eq} = 2.4. \times 10^4(\Omega h^2)$ where the two energy densities are equal when the scale factor is a_{eq}. During the radiation era, since $a \propto t^{1/2}$, temperature and time are related by

$$\frac{t}{1\,\text{sec}} \simeq \left(\frac{10^{10}\,\text{K}}{T}\right)^2 \simeq \left(\frac{\text{MeV}}{T}\right)^2 . \tag{25}$$

From this we quickly see that

$$\begin{aligned} &T \sim \text{MeV} \Longleftrightarrow t \sim \text{sec} \\ &T \sim \text{GeV} \Longleftrightarrow t \sim 10^{-6}\text{sec} \\ &T \sim 10^{15}\text{GeV} \Longleftrightarrow t \sim 10^{-35}\text{sec}. \end{aligned} \tag{26}$$

The highest energies accessible to terrestrial experiment, generated in particle accelerators, correspond to a temperature of about 10^{15} K, which was attained when the Universe was about 10^{-10} sec old. Earlier times rely on extrapolation of our known physics and possibly new mathematical insights (such as string theory may offer). Later times appear to be well understood, with a possible timetable being:

- 10^{-34} seconds: Grand unified phase transition, where strong force decouples from electroweak force.
- 10^{-10} seconds: Electroweak phase transition, where weak force decouples from electromagnetic force. Possible origin for the observed baryon asymmetry in the Universe.
- 10^{-4} seconds: Quarks condense to form protons and neutrons.
- 1 second: The Universe has cooled sufficiently that light nuclei are able to form – **nucleosynthesis**.
- 10^4 years: **Matter–radiation equality**. Subsequently the Universe is matter dominated.
- 10^5 years: Decoupling of radiation from matter leads to the formation of the microwave background. Similar time to recombination, when the up-to-now free electrons combine with the nuclei to form atoms.
- 10^{10} years: The present era where Beckham joins Real Madrid for a bargain of £20M – and the Universe is accelerating!.

We have seen that up to two years ago, this would have been the accepted folklore. However, it now appears that the Universe, rather than steadily decelerating is actually accelerating, going faster and faster. The evidence for this lies in the distribution of Type Ia Supernovae at very large scales, but the conclusion is dramatic. As we shall shortly discover, an accelerating universe requires an energy source which effectively has a negative pressure. There is something out there providing this constant source of energy density, and it is dominating over everything else present – there appears to be a cosmological constant present in the Universe today! We will return to this amazing issue later on.

2 Problems with the Big Bang

There are a number of issues that the SBB simply can not address and have to adopt as initial conditions. These provided the original motivation for the inflationary cosmology, and we now turn our attention to these issues.

2.1 The Flatness Problem

In the absence of a cosmological constant contribution, the Friedmann equation can be written in terms of the density parameter equation (17). During SBB evolution, a^2H^2 is decreasing, and so Ω moves away from one, for example

$$\text{Matter domination:} \quad |\Omega - 1| \propto t^{2/3} \tag{27}$$

$$\text{Radiation domination:} \quad |\Omega - 1| \propto t \tag{28}$$

where the solutions apply provided Ω is close to one. So $\Omega = 1$ is an *unstable* critical point. However, today Ω is certainly within an order of magnitude of one, so it must have been much closer in the past. Inserting the appropriate behaviours for the matter and radiation eras gives for example

$$\text{nucleosynthesis } (t \sim 1\,\text{sec}) : |\Omega - 1| < \mathcal{O}(10^{-16}) \tag{29}$$

That is, hardly any choices of the initial density lead to a Universe like our own. Typically, the Universe will either swiftly recollapse, or will rapidly expand and cool below 3K within its first second of existence.

2.2 The Horizon Problem

The observation by COBE that all cosmic microwave photons appear to be in thermal equilibrium at almost the same temperature is a puzzle? Why is it so isotropic? It is not difficult to see that in the SBB the Universe has not had enough time for different regions to reach a state of thermal equilibrium by today. The regions could not have interacted before the photons were emitted because of the finite horizon size,

$$\int_{t_*}^{t_{\text{dec}}} \frac{dt}{a(t)} \ll \int_{t_{\text{dec}}}^{t_0} \frac{dt}{a(t)} \,. \tag{30}$$

In other words, the distance light could travel before the microwave background was released is much smaller than the present horizon distance. In fact, any regions separated by more than about 2 degrees would be causally separated at decoupling in the hot big bang theory. In the big bang theory there is therefore no explanation of why the Universe appears so homogeneous.

The same argument that prevents the smoothing of the Universe also prevents the creation of irregularities. The COBE satellite has detected irregularities in the CMB on all large angular scales, too large to be accounted for as emerging in the period between the big bang and the time of decoupling, because the horizon size at decoupling subtends only a degree or so. Hence these perturbations must have been part of the initial conditions.

2.3 The Monopole Problem

Modern particle theories predict a variety of ‘unwanted relics’, which can not be present today as they would have dramatically altered the evolution of the Universe. These include magnetic monopoles, domain walls, gravitinos and moduli fields associated with the extra dimensions arising in superstring theories. They are all massive particles created in the very early Universe but are diluted less rapidly than radiation as the Universe expands. Hence they would rapidly come to dominate the dynamics, and lead to rapid closure of the Universe. We must eliminate them, while preserving the rest of the matter which we like.

2.4 The Cosmological Constant

When Einstein developed the theory of General Relativity, the consensus was that the Universe was static, after all there was no apparent movement of the galaxies in the night sky with respect to one another. However, the Friedmann equations were unstable to a static solution, not surprising since matter attracts. To resolve the problem, Einstein introduced a constant balancing term (Λ) which would allow for a static solution to exist. The modified Friedmann equation now became

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \tag{31}$$

with the new *cosmological constant* appearing as a constant contribution to the Hubble parameter. As soon as the Universe was discovered to be expanding, the need for this term went away. Unfortunately, the equations suggested otherwise, there was no apriori reason to set $\Lambda = 0$. This term has haunted cosmologists and particle physicists ever since and could well have just come back to visit us again today. What is the problem then?

From (31), as we are not dominated by the curvature term and since the present energy density is close to the critical value, we see that today,

$$|\Lambda| \leq H_0^2. \tag{32}$$

Thus the length scale $\ell_\Lambda \equiv |\Lambda|^{-1/2}$ associated with the cosmological constant must be larger than $H_0^{-1} = h_0^{-1} \times 10^{26}$ m, a macroscopic distance. In a classical regime this is fine, it is simply saying the cosmological constant length scale is larger than the Hubble length. Problems arise when we combine gravity and quantum mechanics. At the quantum level the natural scales which emerge at the Planck epoch are the Planck mass and Planck length given by (where we have reinserted Planck constant and the speed of light)

$$m_P = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18}\ \mathrm{GeV/c^2},$$
$$\ell_P = \frac{\hbar}{m_P c} = 8.1 \times 10^{-35}\ \mathrm{m}$$

The above constraint now reads :

$$\ell_\Lambda \equiv |\Lambda|^{-1/2} \geq \frac{1}{H_0} \sim 10^{60}\ \ell_P. \tag{33}$$

There are more than sixty orders of magnitude between the scale associated with the cosmological constant and the scale of quantum gravity. We could of course simply set $\Lambda = 0$, indeed this is generally what is done. Unfortunately when there is matter hanging around this is not such a good idea. The matter itself experiences quantum fluctuations (called zero-point fluctuations) and these can act like an effective cosmological constant ($\Lambda_{\mathrm{eff}} \sim \lambda^4/m_P^2$). In fact the natural value this constant should then have usually reflects the scale associated with these quantum fluctuations. So, λ would be typically of the order of 100 GeV in the case of the gauge symmetry breaking of the Standard Model or 1 TeV in the case of supersymmetry breaking. But the constraint (33) now reads:

$$\lambda \leq 10^{-30}\ m_P \sim 10^{-3}\ \text{eV}. \tag{34}$$

It is this very unnatural fine-tuning of parameters that is referred to as the *cosmological constant problem*, or more accurately the vacuum energy problem.

3 Enter Inflation

Inflation is defined to be any epoch where $\ddot{a} > 0$, an accelerated expansion. From (9) this corresponds to a negative pressure ($p < -\frac{\rho}{3}$) and from the definition $H = \frac{\dot{a}}{a}$, we see that it also corresponds to $\frac{d(H^{-1}/a)}{dt} < 0$, i.e. the Hubble length as measured in comoving coordinates, *decreases* during inflation. At any other time, the comoving Hubble length increases. This is the key property of inflation; although typically the expansion of the Universe is very rapid, the crucial characteristic scale of the Universe is actually becoming smaller, when measured relative to that expansion.

We have already seen an example of an inflationary solution, the vacuum dominated regime $p = -\rho$, has a solution is

$$a(t) \propto \exp{(Ht)}\ . \tag{35}$$

There are many many more! Of course, we know the SBB has many successes, and it is none inflating, so inflation can not last for ever, it must terminate and enter the SBB regime smoothly at some epoch.

3.1 The Flatness Problem

Inflation solves the flatness problem by rapidly forcing Ω towards unity rather than away from it. This is clear from the fact that the comoving Hubble length H^{-1}/a is decreasing. We require enough inflation to force Ω extremely close to unity to ensure that it will remain close to it today. Remember, as soon as we enter the SBB phase, $\Omega = 1$ is an unstable point. Including a possible cosmological constant contribution, modifies the Friedmann equation to

$$|\Omega + \Omega_\Lambda - 1| = \frac{|k|}{a^2 H^2}\ , \tag{36}$$

and so it is $\Omega + \Omega_\Lambda$ which is forced to one. In general, it is spatial flatness ($k \simeq 0$) that we are driven towards, not a critical matter density.

3.2 Relic Abundances

The rapid expansion of the inflationary stage rapidly dilutes the unwanted relic particles, because the energy density during inflation falls off more slowly than the relic particle density. Very quickly their density becomes negligible. Of course they do not disappear totally and will one day re-enter the horizon – the ultimate in sweeping something under the carpet.

We need to ensure that after inflation, the energy density of the Universe can be turned into conventional matter without recreating the unwanted relics. This *reheating* period must have a temperature that never gets hot enough to allow their thermal recreation. It will then allow for the particles we want to create and lead naturally into the SSB period, vital for the success of nucleosynthesis and the CMB.

3.3 The Horizon Problem and Homogeneity

Inflation rapidly increases the size of any region of the Universe, but it keeps its characteristic scale, the Hubble scale fixed. So, a small patch of the Universe, small enough for thermalisation before inflation, can expand to a patch much larger than the size of our presently observable Universe. This ensures that all the cosmic microwave radiation are in thermal equilibrium. Moreover, it also allows for irregularities to be generated in the CMB, irregularities which would then evolve to form structures. We can rephrase the horizon solution by saying that because of inflation, light can travel much further before decoupling than it can afterwards.

3.4 The Cosmological Constant

Unfortunately, a period of inflation says nothing about why the present value of the cosmological constant should be so small. In fact it should now be clear that inflation effectively relies on such a constant if only for a finite period of time.

4 Inflation out of Particle Physics

The most common framework in which inflation is obtained is based on the existence of scalar fields, in particular scalar field potentials. Needless to say, as far as particle physics is concerned they remain elusive – yet we really need them! They represent spin zero particles, transforming as a scalar (that is, it is unchanged) under coordinate transformations. In a homogeneous Universe, the scalar field is a function of time alone.

The traditional starting point for particle physics models is the action, which is an integral of the Lagrange density over space and time and from which the equations of motion can be obtained. A scalar field Lagrangian is like one for a particle, the difference between the kinetic energy and the potential energy of the field

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi). \tag{37}$$

The stress energy tensor is defined in terms of the Lagrangian

$$T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - L g_{\mu\nu}, \tag{38}$$

where $g_{\mu\nu}$ is the metric tensor. If ϕ represents an isotropic fluid then we can write down the pressure and energy density from the definition

$$T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p), \tag{39}$$

from which we obtain for a homogeneous field

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{40}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{41}$$

The potential energy $V(\phi)$ measures how much internal energy is associated with a particular field value. Normally, like all systems, scalar fields try to minimize this energy; however, a crucial ingredient which allows inflation is that scalar fields are not always very efficient at reaching this minimum energy state. In a given theory, there would be a specific form for the potential $V(\phi)$. However, we are not presently in a position where there is a well established fundamental theory that one can use, so, in the absence of such a theory, inflation workers tend to regard $V(\phi)$ as a function to be chosen arbitrarily, with different choices corresponding to different models of inflation. Some example potentials are

$$V(\phi) = \lambda\left(\phi^2 - M^2\right)^2 \qquad \text{Higgs potential} \tag{42}$$

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 \qquad \text{Massive scalar field} \tag{43}$$

$$V(\phi) = \lambda \phi^4 \qquad \text{Self-interacting scalar field} \tag{44}$$

4.1 Inflation Dynamics

The equations for an expanding Universe containing a homogeneous scalar field are easily obtained by substituting (40) and (41) into the Friedmann and fluid equations, giving

$$H^2 = \frac{8\pi G}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{45}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi)\,, \tag{46}$$

where prime indicates $d/d\phi$. Here we have ignored the curvature term k, since we know that by definition it will quickly become negligible once inflation starts. Since

$$\ddot{a} > 0 \Longleftrightarrow p < -\frac{\rho}{3} \Longleftrightarrow \dot{\phi}^2 < V(\phi) \tag{47}$$

we will have inflation whenever the potential energy dominates. This should be possible provided the potential is flat enough, as the scalar field would then be expected to roll slowly. The potential should also have a minimum or some other feature which would allow inflation to end.

To solve these equations we use the **slow-roll approximation** (SRA), which assumes that a term can be neglected in each of the equations of motion to leave the simpler set

$$H^2 \simeq \frac{8\pi G}{3} V \tag{48}$$

$$3H\dot{\phi} \simeq -V' \tag{49}$$

The **slow-roll parameters** fist introduced by Liddle and Lyth [14]

$$\epsilon(\phi) \equiv \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad ; \quad \eta(\phi) \equiv \frac{1}{8\pi G}\frac{V''}{V}\,, \tag{50}$$

measures the slope of the potential (ϵ), and the curvature (η), and the necessary conditions for the slow-roll approximation to hold are

$$\epsilon \ll 1 \quad ; \quad |\eta| \ll 1\,. \tag{51}$$

4.2 The Amount of Inflation

The amount of inflation is normally specified by the *the number of e-foldings* N, given by

$$N \equiv \ln\frac{a(t_{\rm end})}{a(t_{\rm initial})} = \int_{t_{\rm i}}^{t_{\rm e}} H\,dt\,, \tag{52}$$

$$\simeq -8\pi G\int_{\phi_{\rm i}}^{\phi_{\rm e}} \frac{V}{V'}\,d\phi\,, \tag{53}$$

where the final step uses the SRA. Notice that the amount of inflation between two scalar field values can be calculated without needing to solve the equations of motion, and also that it is unchanged if one multiplies $V(\phi)$ by a constant. We can estimate the amount of inflation required to solve the various cosmological problems. Consider the flatness problem. First we make a few plausible assumptions to ease the situation: inflation is of the exponential form ending at $t = 10^{-34}$ sec, with the Universe immediately entering a radiation era which persists until today some 3×10^{17} sec later. Imagine also that today $|\Omega - 1| \leq 0.01$, a reasonable constraint on the value of Ω. Now during the radiation era, from (17), $|\Omega-1| \propto t$, hence $|\Omega(10^{-34}\,\mathrm{sec})-1| \leq 3\times 10^{-54}$. During inflation H is constant, so $|\Omega - 1| \propto \frac{1}{a^2}$. From this it follows that in order to satisfy the constraint by the end of inflation, the scale factor has to grow during inflation by an amount

$$\frac{a_{\mathrm{tend}}}{a_{\mathrm{tbegin}}} \sim 10^{27} \sim exp(62), \tag{54}$$

corresponding to around 62 e-foldings. Although this looks large, inflation is typically so rapid that most inflation models give much more.

4.3 Some Examples of Inflation: Polynomial Chaotic Inflation

A particularly nice example of an inflaton potential is a simple polynomial potential first introduced by Linde (for a review see [7]). It could be a massive non-interacting field, $V(\phi) = m^2\phi^2/2$ where m is the mass of the scalar field, or it could be a massless self-interacting field, $V(\phi) = \lambda\phi^4$, where λ is the self coupling of the field. Consider the first case. The slow-roll equations are

$$3H\dot{\phi} + m^2\phi = 0 \quad ; \quad H^2 = \frac{4\pi G m^2\phi^2}{3}, \tag{55}$$

and the slow-roll parameters are

$$\epsilon = \eta = \frac{1}{4\pi G\phi^2}, \tag{56}$$

implying that inflation can proceed provided $|\phi| > 1/\sqrt{4\pi G}$, i.e. away from the minimum.

The solutions to the equations give

$$\phi(t) = \phi_{\mathrm{i}} - \frac{m}{\sqrt{12\pi G}}\, t, \tag{57}$$

$$a(t) = a_{\mathrm{i}} \exp\left[\sqrt{\frac{4\pi G}{3}}\, m\left(\phi_{\mathrm{i}} t - \frac{m}{\sqrt{48\pi G}} t^2\right)\right], \tag{58}$$

(where $\phi = \phi_{\mathrm{i}}$ and $a = a_{\mathrm{i}}$ at $t = 0$) and the total amount of inflation is

$$N_{\mathrm{tot}} = 2\pi G\, \phi_{\mathrm{i}}^2 - \frac{1}{2}. \tag{59}$$

An important thing to bear in mind is that we need to ensure that we are in a position where classical physics remains a valid approximation. This is simply the requirement $V \ll G^{-2}$, but it is still easy to get enough inflation provided m is small enough. In fact, m is required to be small from observational limits on the size of density perturbations produced.

As an exercise the reader may want to try and repeat the exercise for potential $V(\phi) = \lambda\phi^4$, assuming the field starts at $t = 0$ from rest rolling towards $\phi = 0$ from the positive side of the potential. Show that the SR equations give

$$\phi(t) = \phi_{\rm i}\, exp\left(-\sqrt{\frac{2\lambda}{3\pi G}}\, t\right) , \tag{60}$$

$$a(t) = a_{\rm i} \exp\left\{\phi_{\rm i}^2 \pi G \left[1 - exp\left(-\sqrt{\frac{4\lambda}{3\pi G}}\, t\right)\right]\right\} , \tag{61}$$

(where $\phi = \phi_{\rm i}$ and $a = a_{\rm i}$ at $t = 0$) and the total amount of inflation is

$$N_{\rm tot} = \pi G\, \phi_{\rm i}^2 - 1\,. \tag{62}$$

Since these lectures were given, the WMAP results are now beginning to place constraints on the viability of polynomial inflation models, and it appears that the $\lambda\phi^4$ marks the boundary between viable models (powers less than 4) and un-viable models (powers greater than 4)[8]. However, a word of caution. The 'real' inflaton potential is likely to be a bit more complicated than a simple single scalar field power law model, so lets not get too excited yet about ruling out large classes of potentials.

4.4 From Inflation to the SBB – Reheating

During inflation, all matter except the inflaton scalar field is redshifted to extremely low densities. **Reheating** is the process whereby the inflaton's energy density is converted back into conventional matter after inflation, re-entering the standard big bang theory.

As the slow-roll conditions break down, ϕ evolves from being overdamped to being underdamped, moving rapidly on the Hubble timescale and oscillating at the bottom of the potential, where it decays into conventional matter. This is an active and technically demanding area of research and there has recently been something of a revolution in the way we think reheating takes place. Traditional treatments (e.g. as given in Kolb & Turner[9]) added a phenomenological decay term; this was constrained to be very small with reheating being inefficient. In particular there was a long time delay (redshifting) between the end of inflation and the Universe returning to thermal equilibrium; hence a low reheat temperature compared to the energy density at the end of inflation.

In **preheating** [10], this picture is turned on its head. Kofman et al have shown that the decay can initially proceed through broad parametric resonance, with extremely efficient transfer of energy from the coherent oscillations of the inflaton field. The result is a very short reheating period, with most of the inflaton energy density at the end of inflation available for conversion into thermalized form. A higher reheat temperature is possible with some amazing possibilities, such as non-thermal phase transitions [11] and baryogenesis occurring at the electroweak scale[12, 13].

4.5 Inflation Models

There are a number of models on offer, some better motivated than others. **Chaotic inflation models** are the generic type found in a number of situations because they just require a single scalar field, rolling in a potential $V(\phi)$, which in some regions satisfies the slow-roll conditions, while also possessing a minimum with zero potential in which inflation is to end. the initial conditions place the field well up the potential, and could be due to large fluctuations at the Planck era. Examples include (see [14])

Polynomial chaotic inflation	$V(\phi) = \frac{1}{2}m^2\phi^2$
	$V(\phi) = \lambda\phi^4$
Power-law inflation	$V(\phi) = V_0 \exp(\sqrt{\frac{16\pi G}{p}}\,\phi)$
'Natural' inflation	$V(\phi) = V_0[1 + \cos\frac{\phi}{f}]$
Intermediate inflation	$V(\phi) \propto \phi^{-\beta}$

Hybrid inflation models are a very interesting class as they have more than one scalar field and appear to offer the possibility of occurring in particle physics contexts. An example is one with a potential

$$V(\phi,\psi) = \frac{\lambda}{4}\left(\psi^2 - M^2\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda'\phi^2\psi^2\,. \tag{63}$$

When ϕ^2 is large, the minimum of the potential is in the ψ-direction is at $\psi = 0$. The field rolls down this 'valley' until it reaches $\phi^2_{\rm inst} = \lambda M^2/\lambda'$, where $\psi = 0$ becomes unstable and the field rolls into one of the true minima at $\phi = 0$ and $\psi = \pm M$. Note for suitable choices of the potential, topological defects could form at the end of a period of inflation.

While in the 'valley', it is like a single field model with an effective potential for ϕ of the form

$$V_{\rm eff}(\phi) = \frac{\lambda}{4}M^4 + \frac{1}{2}m^2\phi^2\,. \tag{64}$$

The constant term would not normally be allowed as it would give a present-day cosmological constant. When it dominates, it allows both for the energy density during inflation to be much lower than normal while still giving suitably large density perturbations, and for ϕ to roll very slowly.

Models of inflation can also be found in scalar-tensor theories of gravity where the gravitational constant may vary. One interesting case arises from the low energy string action, where two scalar fields, the dilaton and moduli field lead to a period of inflation driven not by the potential energy of the fields (in fact the potential vanishes), rather by the kinetic energy of the fields. This interesting possibility is known as the **pre big bang model**, so called because this evolution occurs before the usual big bang singularity is met. These will be discussed in Sect. 5. There are also fascinating models which lead to Open universes, but likewise we do not have time to discuss them here.

4.6 Density Perturbations and Gravitational Waves

Perhaps the most important property of inflationary cosmology is that it produces spectra of both density perturbations and gravitational waves. The former would be responsible for the formation and clustering of galaxies, as well as creating anisotropies in the microwave background radiation. The gravitational waves do not affect the formation of galaxies, but may contribute extra microwave anisotropies on the large angular scales sampled by the COBE satellite.

The beauty of having models like inflation, or topological defects (which we do not discuss here) is that they are predictive. We can predict the form of the initial perturbation spectra, as opposed to simply assuming it as is often done in studies of large-scale structure. For example, the gravitational waves may be assumed not to be present, and the density perturbations to take on a simple form such as the scale-invariant Harrison–Zel'dovich spectrum, or a scale-free power-law spectrum. In his lectures Robert Brandenberger has gone into a great deal of detail describing the cosmological perturbations theory [15]. Here we will just be picking out the bits useful for inflation without deriving any of the formalism.

4.7 Perturbations Produced During Inflation

Inflation generates perturbations on large scales because the comoving Hubble length decreases during inflation, where as in the SBB the comoving Hubble length is always increasing, all scales are initially much larger than it, and hence unable to be affected by causal physics. Once they become smaller than the Hubble length, they remain so for all time. The fact that COBE sees perturbations on large scales at a time when they were much bigger than the Hubble length, means that in the standard picture no mechanism could have created them.

During inflation a given comoving scale could be well inside the Hubble length, and hence be affected by causal physics, thereby enabling it to generate homogeneity to solve the horizon problem and to superimpose small quantum perturbations. Before inflation ends, as the comoving Hubble length decreases, the given scale crosses outside the Hubble radius rendering causal physics ineffective. Any perturbations generated become imprinted, or, 'frozen in'. Long after inflation is over, as the comoving Hubble length increases the scales cross inside the Hubble radius again. Perturbations are created on a very wide range of scales, but the most readily observed ones range from about the size of the present Hubble radius (i.e. the size of the presently observable Universe) down to a few orders of magnitude less. Thus inflation allows perturbations to be generated causally. These quantum fluctuations are present simply from the Uncertainty principle, they have to be and can explain the initial inhomogeneities that later grow by gravitational collapse to the structures we see today.

The size of the irregularities depends on the energy scale at which inflation takes place. It is outside the scope of these lectures to describe in detail how this calculation is performed. We will just briefly outline the necessary steps and then quote the result, which we can go on to apply (see Liddle and Lyth [14] for details).

(a) Perturb the scalar field	$\phi = \phi(t) + \delta\phi(\mathbf{x}, t)$
(b) Expand in comoving wavenumbers	$\delta\phi = \sum(\delta\phi)_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}$
(c) Linearized equation for classical evolution	
(d) Quantize theory	
(e) Find solution with initial condition giving flat space quantum theory ($k \gg aH$)	
(f) Find asymptotic value for $k \ll aH$	$\langle\lvert\delta\phi_{\mathbf{k}}\rvert^2\rangle = H^2/2k^3$
(g) Relate field perturbation to metric or curvature perturbation	$\mathcal{R}_{\mathbf{k}} = H\,\delta\phi_{\mathbf{k}}/\dot{\phi}$

Unfortunately, analytic results are not known for general inflation models. The results given below are lowest-order in the SRA. There are results known to second-order in slow-roll for arbitrary inflaton potentials. Power-law inflation is the only standard model for which exact results are known.

The curvature perturbation $\mathcal{R}_{\mathbf{k}}$ is so important because it provides the vital bridge which allows us to link the primordial fluctuations in ϕ to the physically observed matter fluctuations present today. The reason is that it remains effectively constant for scales much greater than the co-moving Hubble length, hence the scales that 'freeze-in' as they leave the Hubble length during inflation, remain unaffected (apart from stretching due to the Universe expanding) until they re-enter the Hubble radius much later in the standard big bang era. Given the definition of the power spectra

$$\mathcal{P}_g(k) = \frac{k^3}{2\pi^2}\lvert\delta g_k\rvert^2 \,, \tag{65}$$

then the amplitude of the density perturbation $\delta^2_{\mathrm{H}}(k) = \frac{4}{25}\mathcal{P}_{\mathcal{R}}$ is given by

$$\begin{aligned} \delta^2_{\mathrm{H}}(k) &= \frac{4}{25}\frac{k^3}{2\pi^2}\left(\frac{H\delta\phi_{\mathbf{k}}}{\dot{\phi}}\right)^2 \\ &= \left|.\frac{4}{25}\left(\frac{H}{\dot{\phi}}\right)\left(\frac{H}{2\pi}\right)^2\right|_{k=aH} . \end{aligned} \tag{66}$$

Using the SRA this then gives

$$\delta_{\mathrm{H}}(k) = \sqrt{\frac{512\pi}{75}\frac{V^{3/2}G^{3/2}}{\lvert V'\rvert}}\Bigg|_{k=aH} . \tag{67}$$

A similar calculation gives the amplitude of the gravitational waves

$$A_{\mathrm{G}}(k) = \sqrt{\frac{32}{75}}\, V^{1/2} G \bigg|_{k=aH} , \tag{68}$$

where $A_{\mathrm{G}}^2(k) = \frac{1}{25}\mathcal{P}_g(k)$.

The right-hand sides of the above equations are to be evaluated at the time when the comoving wave number $k = aH$ during inflation, which for a given k corresponds to some particular value of ϕ. We see that the amplitude of perturbations depends on the properties of the inflaton potential at the time the scale crossed the Hubble radius during inflation. The relevant number of e-foldings from the end of inflation is approximately given by

$$N \simeq 62 - \ln \frac{k}{a_0 H_0} \tag{69}$$

Approximating this to to say that the scales of interest to us crossed outside the Hubble radius 60 e-foldings before the end of inflation then gives

$$N \simeq -8\pi G \int_{\phi}^{\phi_{\mathrm{end}}} \frac{V}{V'}\, d\phi\,, \tag{70}$$

which tells us the value of ϕ to be substituted into (67) and (68).

We can apply this formalism to the specific example of the $m^2\phi^2/2$ potential. Inflation ends when $\epsilon = 1$, so $\phi_{\mathrm{end}} \simeq 1/\sqrt{4\pi G}$. 60 e-foldings before this, gives $\phi_{60} \simeq \frac{3}{\sqrt{G}}$ from (59), which in turn upon substitution yields

$$\delta_{\mathrm{H}} \simeq 12\, m\sqrt{G} \quad ; \quad A_{\mathrm{G}} \simeq 1.4\, m\sqrt{G}.$$

The COBE result corresponds to $\delta_{\mathrm{H}} \simeq 2 \times 10^{-5}$ (provided $A_{\mathrm{G}} \ll \delta_{\mathrm{H}}$), hence $m\sqrt{G} \simeq 10^{-6}$ and we have an inflaton mass of $m = 10^{13}$ GeV. Such a small mass satisfies the condition $VG^2 < 1$, which implies that $\phi < 1/(mG) \simeq 10^6(1/\sqrt{G})$. Since $N_{\mathrm{tot}} \simeq 2\pi\phi^2 G$, we can get up to about 10^{13} e-foldings in principle, compared with the 70 or so actually required.

4.8 Observational Consequences

The current high precision CMB experiments like BOOMERANG, MAXIMA I and now WMAP are beginning to probe key features of the spectra, such as the scale-dependence and the relative size of the two spectra. Again the slow-roll parameters ϵ and η can be used to estimate these quantities for any given inflation potential. The standard approximation used to describe the spectra is the **power-law approximation**, where we take[2]

$$\delta_{\mathrm{H}}^2(k) \propto k^{n-1} \quad ; \quad A_{\mathrm{G}}^2(k) \propto k^{n_{\mathrm{G}}}\,, \tag{71}$$

[2] Cf. (53) in the contribution of J.L. Cervantes–Cota in this book.

where the spectral indices[3] n and $n_{\rm G}$ are given by

$$n-1=\frac{d\ln\delta_{\rm H}^2}{d\ln k} \quad ; \quad n_{\rm G}=\frac{d\ln A_{\rm G}^2}{d\ln k}\,. \tag{72}$$

The power-law approximation is usually valid because only a limited range of scales are observable, with the range 1 Mpc to 10^4 Mpc corresponding to $\Delta\ln k\simeq 9$.

The crucial equation we need is that relating ϕ values to when a scale k crosses the Hubble radius,

$$\frac{d\ln k}{d\phi}=8\pi G\,\frac{V}{V'}\,. \tag{73}$$

This comes from the noticing the right hand side of the amplitude equations are evaluated when $k=aH$, and during inflation $\dot{H}$ is very small compared to the rate of change of a. Hence we can take $d\ln\,k=Hdt$, from which it follows $k\simeq\exp N$. Then make use of (70). Direct differentiation then yields[14]

$$n=1-6\epsilon+2\eta\,, \tag{74}$$

$$n_{\rm G}=-2\epsilon\,, \tag{75}$$

where now ϵ and η are to be evaluated on the appropriate part of the potential.

A measure of the relevant importance of density perturbations and gravitational waves is seen in the microwave background which gives gives

$$R\equiv\frac{C_\ell^{\rm GW}}{C_\ell^{\rm DP}}\simeq 4\pi\epsilon\,. \tag{76}$$

where the C_ℓ are the contributions to the microwave multipoles. Briefly, the temperature difference between two regions of the sky separated by (θ,ϕ) is given in terms of spherical harmonics Y_m^ℓ as $\Delta T/T=\sum a_{\ell m}Y_m^\ell(\theta,\phi)$ where $C_\ell=\langle|a_{\ell m}|^2\rangle$; see (56) in the contribution of Jorge L. Cervantes–Cota in this book.

From n, n_G and R, it follows that if and only if $\epsilon\ll 1$ and $|\eta|\ll 1$ do we get $n\simeq 1$ and $R\simeq 0$ whereas gravitational waves can have a significant effect even if ϵ is quite a bit smaller than one.

Different models predict different things which implies that large-scale structure observations, and especially microwave background observations, can strongly discriminate between inflationary models. When they are made, most existing inflation models will be ruled out. As an example the recent WMAP data appears to be placing the $\lambda\phi^4$ model under some pressure [8]. Fortunately, inflation as an idea has one very useful and hopefully unique test, which will allow it to be verified or ruled out, independent of the particular

[3] The scalar spectral index, n, is sometimes referred as n_s.

Table 1. The spectral index and gravitational wave contribution for a range of inflation models – taken from Liddle - astro-ph/9910110.

MODEL	POTENTIAL	n	R
Polynomial	ϕ^2	0.97	0.1
chaotic inflation	ϕ^4	0.95	0.2
Power-law inflation	$\exp(-\lambda\phi)$	any $n < 1$	$2\pi(1-n)$
'Natural' inflation	$1+\cos(\phi/f)$	any $n < 1$	0
Hybrid inflation (standard)	$1+B\phi^2$	1	0
Hybrid inflation (extreme)	$1+B\phi^2$	$1 < n < 1.15$	~ 0

model being investigated. There exists a *consistency equation*

$$R = -2\pi n_{\rm G} \,, \tag{77}$$

independent of the choice of inflationary model (though it does rely on the slow-roll and power-law approximations). There are no other models that produce such a relation, unfortunately as we have already seen it may turn out that the gravitational wave contribution is so small that the consistency equation can never be verified!

4.9 The Cosmological Parameters

Cosmologists are aiming to fully understand and explain the origin and contents of our Universe, and this includes all the parameters that make it up. So far, we have discussed three primordial ones, δ_H , n and R which describe the initial perturbations laid down in the first 10^{-34} sec or so. Most of the perturbations except the largest ones just re-entering the horizon today, have been heavily processed by real astrophysics to give the non-linear features we observe. We can break the parameters up into cosmological and inflationary:

Inflationary parameters: $\delta_H \, , n \, , n_G \, , R \, , dn/d\ln k \cdot$

Cosmological parameters: $h \, , \Omega_{\rm baryon} \, , \Omega_{\rm CDM} \, , \Omega_{\rm HDM} \, , \Omega_{\rm Lambda} \, , k \, , g_* \, , \tau \cdot,$

where g_* is the number of massless species of particles and τ is the reionisation optical depth. As we mentioned earlier, through a combination of observations and parameter fitting techniques these parameters are already being constrained The recent WMAP data coupled with other astronomical data have led to the following published constraints: $n = 1.13 \pm 0.08$, $dn/d\ln k = -0.055^{+0.028}_{-0.029}$, $\Omega_{\rm matter} + \Omega_\Lambda = 1.02 \pm .002$, with $\Omega_{\rm matter} h^2 = .135^{+.008}_{-.009}$ and $\Omega_{\rm baryons} h^2 = .0224 \pm .0009$[6, 8]. Note there may be tentative evidence for a running of the spectral index, something that would be highly significant if it holds.

The COBE normalization allows the energy scale associated with inflation to be determined, since it is probing perturbations still in their primordial form, dependent only on the initial seed perturbations. Using the present Hubble scale, $\delta_{\rm H} \equiv \delta_{\rm H}(k = a_0 H_0)$, to be given by the COBE normalisation

$$\delta_{\rm H} \simeq 2 \times 10^{-5}\,, \tag{78}$$

then since

$$\delta_{\rm H}^2 = \frac{32}{75} G^2\, V\, \frac{1}{\epsilon}\,, \tag{79}$$

this implies

$$V^{1/4} \simeq 10^{-3}/\sqrt{G} \simeq 10^{16}\,\mathrm{GeV}\,, \tag{80}$$

at the time when observable scales crossed outside the horizon. A scale consistent with many GUT models.

5 String Cosmology

String theory, and its most recent incarnation, that of M-theory, has been accepted by many as the most likely candidate theory to unify the forces of nature as it includes General Relativity in a consistent quantum theory. If it is to play such a pivotal role in particle physics, it should also include in it all of cosmology. It should provide the initial conditions for the Universe, perhaps even explain away the singularity associated with the standard big bang. It should also provide a mechanism for explaining the observed density fluctuations, perhaps by providing the inflaton field or some other mechanism which would lead to inflation. Should the observations survive the test of time, string theory should be able to provide a mechanism to explain the current accelerated expansion of the Universe. In other words, even though it is strictly a theory which can unify gravity with the other forces in the very early Universe, for consistency, as a theory of everything it will have a great deal more to explain. In this article, we will introduce some of the developments that have occurred in string cosmology over the past decade or so, initially basing the discussion on an analyse of the low energy limit of string theory, and then later extending it to include branes arising in Heterotic M-theory.

5.1 Dilaton-Moduli Cosmology (Pre-Big Bang)

Strings live in 4+d spacetime dimensions, with the extra d dimensions being compactified. For homogeneous, four–dimensional cosmologies, where all fields are uniform on the surfaces of homogeneity, we can consider the compactification of the $(4+d)$–dimensional theory on an isotropic d–torus. The radius, or ‘breathing mode’ of the internal space, is then parameterized by a modulus field, β, and determines the volume of the internal dimensions. We can then assume that the $(4+d)$–dimensional metric is of the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j + e^{\sqrt{2/d}\beta}\delta_{ab}dX^a dX^b \tag{81}$$

where indices run from $(i,j) = (1,2,3)$ and $(a,b) = (4,\ldots,3+d)$ and δ_{ab} is the d–dimensional Kronecker delta. The modulus field β is normalized in such a way that it becomes minimally coupled to gravity in the Einstein frame.

The low energy action that is commonly used as a starting point for string cosmology is the four dimensional effective Neveu-Schwarz- Neveu-Schwarz (NS-NS) action given by:

$$S_* = \int d^4x\sqrt{|g|}e^{-\varphi}\left[R + (\nabla\varphi)^2 - \frac{1}{2}(\nabla\beta)^2 - \frac{1}{2}e^{2\varphi}(\nabla\sigma)^2\right], \tag{82}$$

where φ is the effective dilaton in four dimensions, and σ is the pseudo–scalar axion field which is dual to the fundamental NS–NS three–form field strength present in string theory, the duality being given by

$$H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\kappa}\, e^{\varphi}\, \nabla_\kappa\sigma. \tag{83}$$

The dimensionally reduced action (82) may be viewed as the prototype action for string cosmology because it contains many of the key features common to more general actions. Cosmological solutions to these actions have been extensively discussed in the literature – for a review see [16]. Some of them play a central role in the pre–big bang inflationary scenario, first proposed by Veneziano [17, 18]. An important point can be seen immediately in (82) where there is a non-trivial coupling of the dilaton to the axion field, a coupling which will play a key role later on when we are investigating the density perturbations arising in this scenario.

All homogeneous and isotropic external four–dimensional spacetimes can be described by the Friedmann-Robertson-Walker (FRW) metric. The general line element in the string frame can be written as

$$ds_4^2 = a^2(\eta)\left\{-d\eta^2 + d\Omega_\kappa^2\right\}, \tag{84}$$

where $a(\eta)$ is the scale factor of the universe, η is the conformal time and $d\Omega_\kappa^2$ is the line element on a 3-space with constant curvature κ:

$$d\Omega_\kappa^2 = d\psi^2 + \left(\frac{\sin\sqrt{\kappa}\psi}{\sqrt{\kappa}}\right)^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{85}$$

To be compatible with a homogeneous and isotropic metric, all fields, including the pseudo–scalar axion field, must be spatially homogeneous.

The models with vanishing form fields, but time-dependent dilaton and moduli fields, are known as *dilaton-moduli-vacuum* solutions. In the Einstein–frame, these solutions may be interpreted as FRW cosmologies for a stiff perfect fluid, where the speed of sound equals the speed of light. The dilaton and moduli fields behave collectively as a massless, minimally coupled scalar field, and the scale factor in the Einstein frame is given by

$$\tilde{a} = \tilde{a}_* \sqrt{\frac{\tau}{1+\kappa\tau^2}} \tag{86}$$

where $\tilde{a} \equiv e^{-\varphi/2}a$, $\tilde{a}_*$ is a constant and we have defined a new time variable:

$$\tau \equiv \begin{cases} \kappa^{-1/2}|\tan(\kappa^{1/2}\eta)| & \text{for } \kappa > 0 \\ |\eta| & \text{for } \kappa = 0 \\ |\kappa|^{-1/2}|\tanh(|\kappa|^{1/2}\eta)| & \text{for } \kappa < 0 \end{cases} . \tag{87}$$

The time coordinate τ diverges at both early and late times in models which have $\kappa \geq 0$, but $\tau \to |\kappa|^{-1/2}$ in negatively curved models. There is a curvature singularity at $\eta = 0$ with $\tilde{a} = 0$ and the model expands away from it for $\eta > 0$ or collapses towards it for $\eta < 0$. The expanding, closed models recollapse at $\eta = \pm\pi/2$ and there are no bouncing solutions in this frame.

The corresponding string frame scale factor, dilaton and modulus fields are given by the 'rolling radii' solutions [19]

$$a = a_* \sqrt{\frac{\tau^{1+\sqrt{3}\cos\xi_*}}{1+\kappa\tau^2}} , \tag{88}$$

$$e^{\varphi} = e^{\varphi_*}\tau^{\sqrt{3}\cos\xi_*} , \tag{89}$$

$$e^{\beta} = e^{\beta_*}\tau^{\sqrt{3}\sin\xi_*} \tag{90}$$

The integration constant ξ_* determines the rate of change of the effective dilaton relative to the volume of the internal dimensions. Figures 1 and 2 show the dilaton-vacuum solutions in flat FRW models when stable compactification has occurred, so that the volume of the internal space is fixed, with $\xi_* \bmod \pi = 0$.

The solutions just presented have a scale factor duality which when applied simultaneously with time reversal implies that the Hubble expansion parameter $H \equiv d(\ln a)/dt$ remains invariant, $H(-t) \to H(t)$, whilst its first derivative changes sign, $\dot{H}(-t) \to -\dot{H}(t)$. A decelerating, post–big bang solution – characterized by $\dot{a} > 0$, $\ddot{a} < 0$ and $\dot{H} < 0$ – is mapped onto a pre–big bang phase of inflationary expansion, since $\ddot{a}/a = \dot{H} + H^2 > 0$. The Hubble radius H^{-1} decreases with increasing time and the expansion is therefore super-inflationary. Thus, the pre-big bang cosmology ($\kappa = 0$ case in (88–90)) is one that has a period of super-inflation driven simply by the kinetic energy of the dilaton and moduli fields [17, 18]. This is related by duality to the usual FRW post–big bang phase. The two branches are separated by a curvature singularity, however, and it is not clear how the transition between the pre– and post–big bang phases might proceed. This will be the focus of attention in Sect. 5.

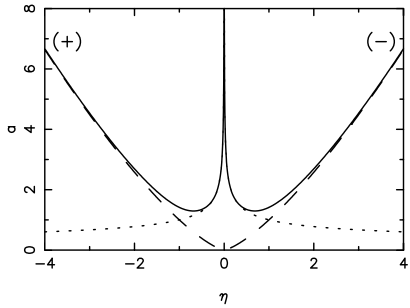

Fig. 1. String frame scale factor, a, as a function of conformal time, η, for flat $\kappa = 0$ FRW cosmology in dilaton-vacuum solution in (88) with $\xi_* = 0$ (dashed-line), $\xi_* = \pi$ (dotted line) and dilaton-axion solution in (93) with $r = \sqrt{3}$ (solid line). The (+) and (−) branches are defined in the text.

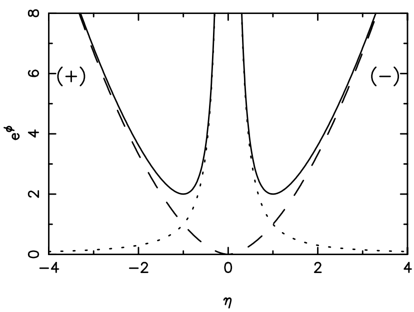

Fig. 2. Dilaton, e^{φ}, as a function of conformal time, η, for flat $\kappa = 0$ FRW cosmology in dilaton-vacuum solution in (89) with $\xi_* = 0$ (dashed-line), $\xi_* = \pi$ (dotted line) and dilaton-axion solution in (92) with $r = \sqrt{3}$ (solid line).

The solution for a flat ($\kappa = 0$) FRW universe corresponds to the well–known monotonic power-law, or 'rolling radii', solutions. For $\cos\xi_* < -1/\sqrt{3}$ there is accelerated expansion, i.e., inflation, in the string frame for $\eta < 0$ and $e^{\varphi} \to 0$ as $t \to -\infty$, corresponding to the weak coupling regime. The expansion is an example of 'pole–law' inflation [20, 21].

The solutions have semi-infinite proper lifetimes. Those starting from a singularity at $t = 0$ for $t \geq 0$ are denoted as the (–) branch in [22], while those which approach a singularity at $t = 0$ for $t \leq 0$ are referred to as the (+) branch (see Figs. 1–2). These (+/−) branches do *not* refer to the choice of sign for $\cos\xi_*$. On either the (+) or (−) branches of the dilaton-moduli-vacuum cosmologies we have a one-parameter family of solutions corresponding to the choice of ξ_*, which determines whether e^{φ} goes to zero or infinity as $t \to 0$. These solutions become singular as the conformally invariant time parameter $\eta \equiv \int dt/a(t) \to 0$ and there is no way of naively connecting the two branches based simply on these solutions [22].

In the Einstein frame, where the dilaton field is minimally coupled to gravity, the scale factor given in (86), becomes

$$\tilde{a} = \tilde{a}_* \, |\eta|^{1/2} \tag{91}$$

As $\eta \to 0$ on the $(+)$ branch, the universe is collapsing with $\tilde{a} \to 0$, and the comoving Hubble length $|d(\ln \tilde{a})/d\eta|^{-1} = 2|\eta|$ is decreasing with time. Thus, in both frames there is inflation taking place in the sense that a given comoving scale, which starts arbitrarily far within the Hubble radius in either conformal frame as $\eta \to -\infty$, inevitably becomes larger than the Hubble radius in that frame as $\eta \to 0$. The significance of this is that it means that perturbations can be produced in the dilaton, graviton and other matter fields on scales much larger than the present Hubble radius from quantum fluctuations in flat spacetime at earlier times – this is a vital property of any inflationary scenario.

For completeness, it is worth mentioning that these solutions can be extended to include a time-dependent axion field, $\sigma(t)$, by exploiting the $\mathrm{SL}(2,R)$ S-duality invariance of the four–dimensional, NS-NS action [19]. We now turn our attention to this fascinating case.

5.2 Dilaton-Moduli-Axion Cosmologies

The cosmologies containing a non–trivial axion field can be generated immediately due to the global $\mathrm{SL}(2,R)$ symmetry of the action (82). The resultant solutions are [19]:

$$e^{\varphi} = \frac{e^{\varphi_*}}{2}\left\{\left(\frac{\tau}{\tau_*}\right)^{-r} + \left(\frac{\tau}{\tau_*}\right)^{r}\right\}, \tag{92}$$

$$a^2 = \frac{a_*^2}{2(1+\kappa\tau^2)}\left\{\left(\frac{\tau}{\tau_*}\right)^{1-r} + \left(\frac{\tau}{\tau_*}\right)^{1+r}\right\}, \tag{93}$$

$$e^{\beta} = e^{\beta_*}\tau^s, \tag{94}$$

$$\sigma = \sigma_* \pm e^{-\varphi_*}\left\{\frac{(\tau/\tau_*)^{-r} - (\tau/\tau_*)^{r}}{(\tau/\tau_*)^{-r} + (\tau/\tau_*)^{r}}\right\}, \tag{95}$$

where the exponents are related via

$$r^2 + s^2 = 3 , \tag{96}$$

and without loss of generality we may take $r \geq 0$.

In all cases, the dynamics of the axion field places a *lower* bound on the value of the dilaton field, $\varphi \geq \varphi_*$. In so doing, the axion smoothly interpolates between two dilaton–moduli–vacuum solutions, where its dynamical influence asymptotically becomes negligible. The effects of time–dependent axion solutions for the scale-factor and dilaton are plotted in Figs. 1 and 2 for the flat FRW model when the modulus field is trivial ($s = 0$). When the internal space is static, it is seen that the string frame scale factors exhibit a bounce. However we still have a curvature singularity in the Einstein frame as $\tau \to 0$. The actual time-dependent axion solutions is shown in Fig. 3.

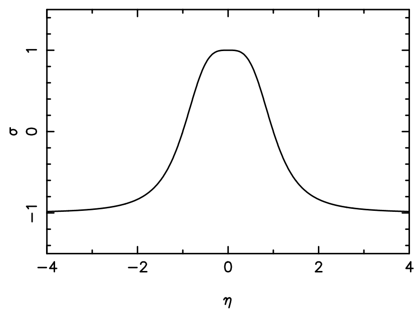

Fig. 3. Axion, σ, as a function of conformal time, η, for flat $\kappa = 0$ FRW cosmology in dilaton-axion solution in (95) with $r = \sqrt{3}$ (solid line).

The spatially flat solutions reduce to the power law, dilaton–moduli–vacuum solution given in (88–90) at early and late times. When $\eta \to \pm\infty$ the solution approaches the vacuum solution with $\sqrt{3}\cos\xi_* = +r$, while as $\eta \to 0$ the solution approaches the $\sqrt{3}\cos\xi_* = -r$ solution. Thus, the axion solution interpolates between two vacuum solutions related by an S-duality transformation $\varphi \to -\varphi$. When the internal space is static the scale factor in the string frame is of the form $a \propto t^{1/\sqrt{3}}$ as $\eta \to \pm\infty$, while as $\eta \to 0$ the solution becomes $a \propto t^{-1/\sqrt{3}}$. These two vacuum solutions are thus related by a scale factor duality that inverts the spatial volume of the universe. This asymptotic approach to dilaton–moduli–vacuum solutions at early and late times will lead to a particularly simple form for the semi-classical perturbation spectra that is independent of the intermediate evolution. However, there is a down side to these solutions from the standpoint of pre big bang cosmologies. As $\eta \to \pm\infty$ and as $\eta \to 0$ the solution approaches the strong coupling regime where $e^{\varphi} \to \infty$. Thus there is no weak coupling limit, the axion interpolates between two strong coupling vacuum solutions. We will shortly see how a similar affect arises when we include a moving brane in the dilaton-moduli picture, as it too mimics the behaviour of a non-minimally coupled axion field.

The overall dynamical effect of the axion field is negligible except near $\tau \approx \tau_*$, when it leads to a bounce in the dilaton field. Within the context of M–theory cosmology, the radius of the eleventh dimension is related to the dilaton by $r_{11} \propto e^{\varphi/3}$ when the modulus field is fixed. This bound on the dilaton may therefore be reinterpreted as a lower bound on the size of the eleventh dimension.

5.3 Fine Tuning Issues

The question over the viability of the initial conditions required in the pre Big Bang scenario has been a cause for many an argument both in print and in person. Since both $\dot{H}$ and $\dot{\varphi}$ are positive in the pre–big bang phase, the initial values for these parameters must be *very small.* This raises a number of important issues concerning fine–tuning in the pre–big bang scenario [23, 24, 25, 26, 27, 28, 29]. There needs to be enough inflation in a homogeneous patch in order to solve the horizon and flatness problems which means that the dilaton driven inflation must survive for a sufficiently long period of time. This is not as trivial as it may appear, however, since the period of inflation is limited by a number of factors.

The fundamental postulate of the scenario is that the initial data for inflation lies well within the perturbative regime of string theory, where the curvature and coupling are very small [18]. Inflation then proceeds for sufficiently homogeneous initial conditions [27, 28], where time derivatives are dominant with respect to spatial gradients, and the universe evolves into a high curvature and strongly–coupled regime. Thus, the pre–big bang initial state should correspond to a cold, empty and flat vacuum state. Initial the universe would have been huge relative to the quantum scale and hence should have been well described by classical solutions to the string effective action. This should be compared to the initial state which describes the standard hot big bang, namely a dense, hot, and highly curved region of spacetime. This is quite a contrast and a primary goal of pre–big bang cosmology must be to develop a mechanism for smoothly connecting these two regions, since we believe that the standard big bang model provides a very good representation of the current evolution of the universe.

Our present observable universe appears very nearly homogeneous on sufficiently large scales. In the standard, hot big bang model, it corresponded to a region at the Planck time that was 10^{30} times larger than the horizon size, $l_{\rm Pl}$. This may be viewed as an initial condition in the big bang model or as a final condition for inflation. It implies that the comoving Hubble radius, $1/(aH)$, must decrease during inflation by a factor of at least 10^{30} if the horizon problem is to be solved. For a power law expansion, this implies that

$$\left| \frac{\eta_f}{\eta_i} \right| \leq 10^{-30} \tag{97}$$

where subscripts i and f denote values at the onset and end of inflation, respectively. In the pre–big bang scenario, (89) implies that the dilaton grows as $e^{\varphi} \propto |\eta|^{-\sqrt{3}}$, and since at the start of the post–big bang epoch, the string coupling, $g_s = e^{\varphi/2}$, should be of order unity, the bound (97) implies that the initial value of the string coupling is strongly constrained, $g_{s,i} \leq 10^{-26}$. Turner and Weinberg interpret this constraint as a severe fine–tuning problem in the scenario, because inflation in the string frame can be delayed by the effects of spatial curvature [23]. It was shown by Clancy, Lidsey and Tavakol that the bounds are further tightened when spatial anisotropy is introduced, actually preventing pre–big bang inflation from occurring [24]. Moreover, as we have seen the dynamics of the NS–NS axion field also places a lower bound on the allowed range of values that the string coupling may take [19]. In the standard inflationary scenario, where the expansion is quasi–exponential, the Hubble radius is approximately constant and $a \propto (-\eta)^{-1}$. Thus, the homogeneous region grows by a factor of $|\eta_i/\eta_f|$ as inflation proceeds. During a pre–big bang epoch, however, $a \propto (-\eta)^{-1/1+\sqrt{3}}$ and the increase in the size of a homogeneous region is reduced by a factor of at least $10^{30\sqrt{3}/(1+\sqrt{3})} \approx 10^{19}$ relative to that of the standard inflation scenario. This implies that the initial size of the homogeneous region should exceed 10^{19} in string units if pre–big bang inflation is to be successful in solving the problems of the big bang model [17, 25]. The occurrence of such a large number was cited by Kaloper, Linde and Bousso as a serious problem of the pre–big bang scenario, because it implies that the universe must already have been large and smooth by the time inflation began [25].

On the other hand, Gasperini has emphasized that the initial homogeneous region of the pre–big bang universe is not larger than the horizon even though it is large relative to the string/Planck scale [30]. The question that then arises when discussing the naturalness, or otherwise, of the above initial conditions is what is the basic unit of length that should be employed. At present, this question has not been addressed in detail.

Veneziano and collaborators conjectured that pre–big bang inflation generically evolves out of an initial state that approaches the Milne universe in the semi–infinite past, $t \to -\infty$ [27, 28]. The Milne universe may be mapped onto the future (or past) light cone of the origin of Minkowski spacetime and therefore corresponds to a non–standard representation of the string perturbative vacuum. The proposal was that the Milne background represents an early time attractor, with a large measure in the space of initial data. If so, this would provide strong justification for the postulate that inflation begins in the weak coupling and curvature regimes and would render the pre-big bang assumptions regarding the initial states as 'natural'. However, Clancy *et al.* took a critical look at this conjecture and argued that the Milne universe is an unlikely past attractor for the pre–big bang scenario [31]. They suggested that plane wave backgrounds represent a more generic initial state for the universe [24]. Buonanno, Damour and Veneziano have subsequently proposed that the initial state of the pre–big bang universe should correspond to an ensemble of gravitational and dilatonic waves [29]. They refer to this as the state of 'asymptotic past triviality'. When viewed in the Einstein frame these waves undergo collapse when certain conditions are satisfied. In the string frame, these gravitationally unstable areas expand into homogeneous regions on large scales.

To conclude this section, it is clear that the question of initial conditions in the pre–big bang scenario is currently unresolved. We turn our attention now to another unresolved problem for the scenario – the Graceful Exit.

5.4 The Graceful Exit

We have seen how in the pre Big Bang scenario, the Universe expands from a weak coupling, low curvature regime in the infinite past, enters a period of inflation driven by the kinetic energy associated with the massless fields present, before approaching the strong coupling regime as the string scale is reached. There is then a branch change to a new class of solutions, corresponding to a post big bang decelerating Friedman-Robertson-Walker era. In such a scenario, the Universe appears to emerge because of the gravitational instability of the generic string vacua – a very appealing picture, the weak coupling, low curvature regime is a natural starting point to use the low energy string effective action. However, how is the branch change achieved without hitting the inevitable looking curvature singularity associated with the strong coupling regime? The simplest version of the evolution of the Universe in the pre-big bang scenario inevitably leads to a period characterised by an unbounded curvature. The current philosophy is to include higher-order corrections to the string effective action. These include both classical finite size effects of the strings (α' corrections arising in higher order derivatives), and quantum string loop corrections (g_s corrections). The list of authors who have worked in this area is too great to mention here, for a detailed list see [16, 32]. A series of key papers were written by Brustein and Madden, in which they demonstrated that it is possible to include such terms and successfully have an exit from one branch to the other [33, 34]. More recently this approach has been generalised by including combinations of classical and quantum corrections [35]. Brustein and Madden [33, 34] made use of the result that classical corrections can stabilize a high curvature string phase while the evolution is still in the weakly coupled regime[36]. The crucial new ingredient that they added was the inclusion of terms of the type that may result from quantum corrections to the string effective action and which induce violation of the null energy condition (NEC – The Null Energy Condition is satisfied if $\rho + p \geq 0$, where ρ and p represent the effective energy density and pressure of the additional sources). Such extra terms mean that evolution towards a decelerated FRW phase is possible. Of course this violation of the null energy condition can not continue indefinitely, and eventually it needs to be turned off in order to stabilise the dilaton at a fixed value, perhaps by capture in a potential minimum or by radiation production – another problem for string theory!

The analysis of [33] resulted in a set of necessary conditions on the evolution in terms of the Hubble parameters H_S in the string frame, H_E in the Einstein frame and the dilaton φ, where they are related by $H_E = e^{\varphi/2}(H_S - \frac{1}{2}\dot{\varphi})$. The conditions were:

– Initial conditions of a (+) branch and $H_S, \dot{\varphi} > 0$ require $H_E < 0$.
– A branch change from (+) to (−) has to occur while $H_E < 0$.
– A successful escape and exit completion requires NEC violation accompanied by a bounce in the Einstein frame after the branch change has occurred, ending up with $H_E > 0$.
– Further evolution is required to bring about a radiation dominated era in which the dilaton effectively decouples from the "matter" sources.

There is as yet no definitive calculation of the full loop expansion of string theory. This is of course a big problem if we want to try and include quantum effects in analysing the graceful exit issue. The best we can do, is to propose plausible terms that we hope are representative of the actual terms that will eventually make up the loop corrections. We believe that the string coupling g_S actually controls the importance of string-loop corrections, so as a first approximation to the loop corrections we multiplied each term of the classical correction by a suitable power of the string coupling [33, 34].

Not surprisingly the field equations need to be solved numerically, but this can be done and the solutions are very encouraging as they show there exists a large class of parameters for which successful graceful exits are obtained [35]. One such example is shown in Fig. 4.

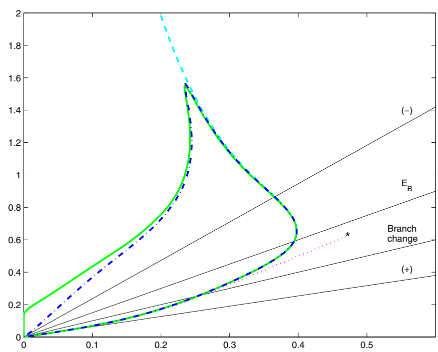

Fig. 4. Hubble expansion in the S-frame as a function of the dilaton for a successful exit. The y-axis corresponds to H, and the x-axis to $2\dot{\varphi}/3$. The initial conditions for the simulations have been set with respect to the lowest-order analytical solutions at $t_S = -1000$. For details see [35]

We should point out though, that although it is possible to have a successful exit, it is not so easy to ensure that the exit takes place in a weakly coupled regime, and typically we found that as the exit was approached $\varphi_{\text{final}} \sim 0.1 - -0.3$. Thus it is fair to say that although great progress has been made on the question of Graceful Exit in string cosmology, it remains a problem in search of the full solution. It is a fascinating problem, and not surprisingly alternative prescriptions which aim to address this issue have recently been proposed, involving colliding branes [37] and Cyclic universes [39]. We now turn our attention to the observational consequences of string cosmology, in particular the generation of the observed cosmic microwave background radiation.

5.5 Density Perturbations in String Cosmology

We have to consider inhomogeneous perturbations that may be generated due to vacuum fluctuations, and follow the formalism pioneered by Mukhanov and collaborators [40, 41]. During a period of accelerated expansion the comoving Hubble length, $|d(\ln a)/d\eta|^{-1}$, decreases and vacuum fluctuations which are assumed to start in the flat-spacetime vacuum state may be stretched up to exponentially large scales. The precise form of the spectrum depends on the expansion of the homogeneous background and the couplings between the fields. The comoving Hubble length, $|d(\ln \tilde{a})/d\eta|^{-1} = 2|\eta|$, does indeed decrease in the Einstein frame during the contracting phase when $\eta < 0$. Because the dilaton, moduli fields and graviton are minimally coupled to this metric, this ensures that small-scale vacuum fluctuations will eventually be stretched beyond the comoving Hubble scale during this epoch.

As we remarked earlier, the axion field is taken to be a constant in the classical pre-big bang solutions. However, even when the background axion field is set to a constant, there will inevitably be quantum fluctuations in this field. We will see that these fluctuations can not be neglected and, moreover, that they are vital if the pre-big bang scenario is to have any chance of generating the observed density perturbations.

In the Einstein frame, the first-order perturbed line element can be written as

$$d\widetilde{s}^2 = \widetilde{a}^2(\eta)\left\{-(1+2\widetilde{A})d\eta^2 + 2\widetilde{B}_{,i}d\eta dx^i + [\delta_{ij}+h_{ij}]\,dx^i dx^j\right\}, \tag{98}$$

where $\widetilde{A}$ and $\widetilde{B}$ are scalar perturbations and h_{ij} is a tensor perturbation.

5.6 Scalar Metric Perturbations

First of all we consider the evolution of linear metric perturbations about the four-dimensional spatially flat dilaton-moduli-vacuum solutions given in (88–90). Considering a single Fourier mode, with comoving wavenumber k, the perturbed Einstein equations yield the evolution equation

$$\widetilde{A}'' + 2\widetilde{h}\widetilde{A}' + k^2\widetilde{A} = 0\,, \tag{99}$$

plus the constraint

$$\widetilde{A} = -(\widetilde{B}' + 2\widetilde{h}\widetilde{B})\,, \tag{100}$$

where $\widetilde{h}$ is the Hubble parameter in the Einstein frame derived from (91), and $\widetilde{A}' \equiv \frac{d\widetilde{A}}{d\eta}$. In the spatially flat gauge we have the simplification that the evolution equation for the scalar metric perturbation, (99), is independent of the evolution of the different massless scalar fields (dilaton, axion and moduli), although they will still be related by the constraint

$$\widetilde{A} = \frac{\varphi'}{4\widetilde{h}}\,\delta\varphi + \frac{\beta'}{4\widetilde{h}}\,\delta\beta\,, \tag{101}$$

where $\delta\varphi$ and $\delta\beta$ are the perturbations in φ and β respectively. To first-order, the metric perturbation, $\widetilde{A}$, is determined solely by the dilaton and moduli field perturbations, although its evolution is dependent only upon the Einstein frame scale factor, $\widetilde{a}(\eta)$, given by (91), which in turn is determined solely by the stiff fluid equation of state for the homogeneous fields in the Einstein frame.

One of the most useful quantities we can calculate is the curvature perturbation on uniform energy density hypersurfaces (as $k\eta \to 0$). It is commonly denoted by ζ [42]and in the Einstein frame, we obtain

$$\zeta = \frac{\widetilde{A}}{3}\,, \tag{102}$$

in any dilaton–moduli–vacuum or dilaton–moduli–axion cosmology [43, 46].

The significance of ζ is that in an expanding universe it becomes constant on scales much larger than the Hubble scale ($|k\eta| \ll 1$) for purely adiabatic perturbations. In single-field inflation models this allows one to compute the density perturbation at late times, during the matter or radiation dominated eras, by equating ζ at "re-entry" ($k = \widetilde{a}\widetilde{H}$) with that at horizon crossing during inflation. To calculate ζ, hence the density perturbations induced in the pre-big bang scenario we can either use the vacuum fluctuations for the canonically normalised field at early times/small scales (as $k\eta \to -\infty$) or use the amplitude of the scalar field perturbation spectra to normalise the solution for $\widetilde{A}$. This yields, (after some work), the curvature perturbation spectrum on large scales/late times (as $k\eta \to 0$):

$$\mathcal{P}_\zeta = \frac{8}{\pi^2} l_{\rm Pl}^2 \widetilde{H}^2 (-k\eta)^3 [\ln(-k\eta)]^2\,, \tag{103}$$

where $l_{\rm Pl}$ is the Planck length in the Einstein frame and remains fixed throughout. The scalar metric perturbations become large on superhorizon scales ($|k\eta| < 1$) only near the Planck era, $\widetilde{H}^2 \sim l_{\rm Pl}^{-2}$.

The spectral index of the curvature perturbation spectrum is conventionally given as [44]

$$n \equiv 1 + \frac{d\ln\mathcal{P}_\zeta}{d\ln k} \tag{104}$$

where $n = 1$ corresponds to the classic Harrison-Zel'dovich spectrum for adiabatic density perturbations favoured by most models of structure formation in our universe. By contrast the pre–big bang era leads to a spectrum of curvature perturbations with $n = 4$. Such a steeply tilted spectrum of metric perturbations implies that there would be effectively no primordial metric perturbations on large (super-galactic) scales in our present universe if the post-Big bang era began close to the Planck scale. Fortunately, as we shall see later, the presence of the axion field could provide an alternative spectrum of perturbations more suitable as a source of large-scale structure. The pre-big bang scenario is not so straightforward as in the single field inflation case, because the full low-energy string effective action possesses many fields which can lead to non-adiabatic perturbations. This implies that density perturbations at late times may not be simply related to ζ alone, but may also be dependent upon fluctuations in other fields.

5.7 Tensor Metric Perturbations

The gravitational wave perturbations, h_{ij}, are both gauge and conformally invariant. They decouple from the scalar perturbations in the Einstein frame to give a simple evolution equation for each Fourier mode

$$h_k'' + 2\widetilde{h}\, h_k' + k^2 h_k = 0\,. \tag{105}$$

This is exactly the same as the equation of motion for the scalar perturbation given in (99) and has the same growing mode in the long wavelength ($|k\eta| \to 0$) limit given by (103). The spectrum depends solely on the dynamics of the scale factor in the Einstein frame given in (91), which remains the same regardless of the time-dependence of the different dilaton, moduli or axion fields. It leads to a spectrum of primordial gravitational waves steeply growing on short scales, with a spectral index $n_T = 3$ [18], in contrast to conventional inflation models which require $n_T < 0$ [44]. The graviton spectrum appears to be a robust and distinctive prediction of any pre-big bang type evolution based on the low-energy string effective action, although recently in the non-singular model of Sect. 5, we have demonstrated how passing through the string phase could lead to a slight shift in the tilt closer to $n_T \sim 2$ [45]

5.8 Dilaton–Moduli–Axion Perturbation Spectra

We will now consider inhomogeneous linear perturbations in the fields about a homogeneous background given by [46, 47]

$$\varphi = \varphi(\eta) + \delta\varphi(\mathbf{x},\eta), \quad \sigma = \sigma(\eta) + \delta\sigma(\mathbf{x},\eta), \quad \beta = \beta(\eta) + \delta\beta(\mathbf{x},\eta)\ . \tag{106}$$

The perturbations can be re-expressed as a Fourier series in terms of Fourier modes with comoving wavenumber k. Considering the production of dilaton, moduli and axion perturbations during a pre-big bang evolution where the background axion field is constant, $\sigma' = 0$, the evolution of the homogeneous background fields are given in (89–90). The dilaton and moduli fields both evolve as minimally coupled massless fields in the Einstein frame. In particular, the dilaton perturbations are decoupled from the axion perturbations and the equations of motion in the spatially flat gauge become

$$\delta\varphi'' + 2\widetilde{h}\delta\varphi' + k^2\delta\varphi = 0\,, \tag{107}$$

$$\delta\beta'' + 2\widetilde{h}\delta\beta' + k^2\delta\beta = 0\,, \tag{108}$$

$$\delta\sigma'' + 2\widetilde{h}\delta\sigma' + k^2\delta\sigma = -2\varphi'\delta\sigma'\,, \tag{109}$$

Note that these evolution equations for the scalar field perturbations defined in the spatially flat gauge are automatically decoupled from the metric perturbations, although as we have said they are still related to the scalar metric perturbation, $\widetilde{A}$ through (101).

On the (+) branch, i.e., when $\eta < 0$, we can normalise modes at early times, $\eta \to -\infty$, where all the modes are far inside the Hubble scale, $k \gg |\eta|^{-1}$, and can be assumed to be in the flat-spacetime vacuum. Whereas in conventional inflation where we have to assume that this result for a quantum field in a classical background holds at the Planck scale, in this case the normalisation is done in the zero-curvature limit in the infinite past. Just as in conventional inflation, this produces perturbations on scales far outside the horizon, $k \ll |\eta|^{-1}$, at late times, $\eta \to 0^-$.

Conversely, the solution for the (−) branch with $\eta > 0$ is dependent upon the initial state of modes far outside the horizon, $k \ll |\eta|^{-1}$, at early times where $\eta \to 0$. The role of a period of inflation, or of the pre-big bang (+) branch, is precisely to set up this initial state which otherwise appears as a mysterious initial condition in the conventional (non-inflationary) big bang model.

The power spectrum for perturbations is commonly denoted by

$$\mathcal{P}_{\delta x} \equiv \frac{k^3}{2\pi^2}|\delta x|^2\,, \tag{110}$$

and thus for modes far outside the horizon ($k\eta \to 0$) we have

$$\mathcal{P}_{\delta\varphi} = \frac{32}{\pi^2} l_{\rm Pl}^2 \widetilde{H}^2 (-k\eta)^3 [\ln(-k\eta)]^2\,, \tag{111}$$

$$\mathcal{P}_{\delta\beta} = \frac{32}{\pi^2} l_{\rm Pl}^2 \widetilde{H}^2 (-k\eta)^3 [\ln(-k\eta)]^2\,, \tag{112}$$

where $\widetilde{H} \equiv \widetilde{a}'/\widetilde{a}^2 = 1/(2\widetilde{a}\eta)$ is the Hubble rate in the Einstein frame. The amplitude of the perturbations grows towards small scales, but only becomes large for modes outside the horizon ($|k\eta| < 1$) when $\widetilde{H}^2 \sim l_{\rm Pl}^{-2}$, i.e., the Planck scale in the Einstein frame. The spectral tilt of the perturbation spectra is given by

$$n - 1 \equiv \Delta n_x = \frac{d\ln\mathcal{P}_{\delta x}}{d\ln k} \tag{113}$$

which from (111) and (112) gives $\Delta n_\varphi = \Delta n_\beta = 3$ (where we neglect the logarithmic dependence). This of course is the same steep blue spectra we obtained earlier for the metric perturbations, which of course is far from the observed near H-Z scale invariant spectrum. We have recently examined the case of the evolution of the field perturbations in the non-singular cosmologies of Sect. 5 and as with the metric-perturbation case, amongst a number of new features that emerge there is a slight shift produced in the spectral index [48].

While the dilaton and moduli fields evolve as massless minimally coupled scalar fields in the Einstein frame, the axion field's kinetic term still has a non-minimal coupling to the dilaton field. This is evident in the equation of motion, (109), for the axion field perturbations $\delta\sigma$. The non-minimal coupling of the axion to the dilaton leads to a significantly different evolution to that of the dilaton and moduli perturbations.

After some algebra, we find that the late time evolution in this case is logarithmic with respect to $-k\eta$, (for $\mu \neq 0$)

$$\mathcal{P}_{\delta\sigma} = 64\pi l_{\rm Pl}^2 C^2(\mu) \left(\frac{e^{-\varphi}\widetilde{H}}{2\pi} \right)^2 (-k\eta)^{3-2\mu} \,, \tag{114}$$

where $\mu \equiv |\sqrt{3}\cos\xi_*|$ and the numerical coefficient

$$C(\mu) \equiv \frac{2^\mu \Gamma(\mu)}{2^{3/2}\Gamma(3/2)} \,, \tag{115}$$

approaches unity for $\mu \to 3/2$.

The key result is that the spectral index can differ significantly from the steep blue spectra obtained for the dilaton and moduli fields that are minimally coupled in the Einstein frame. The spectral index for the axion perturbations is given by [46, 47]

$$\Delta n_\sigma = 3 - 2\sqrt{3}|\cos\xi_*| \tag{116}$$

and depends crucially upon the evolution of the dilaton, parameterised by the value of the integration constant ξ_*. The spectrum becomes scale-invariant as $\sqrt{3}|\cos\xi_*| \to 3/2$, which if we return to the higher-dimensional underlying theory corresponds to a fixed dilaton field in ten-dimensions. The lowest possible value of the spectral tilt Δn_σ is $3 - 2\sqrt{3} \simeq -0.46$ which is obtained when stable compactification has occurred and the moduli field β is fixed. The more rapidly the internal dimensions evolve, the steeper the resulting axion spectrum until for $\cos\xi_* = 0$ we have $\Delta n_\sigma = 3$ just like the dilaton and moduli spectra.

When the background axion field is constant these perturbations, unlike the dilaton or moduli perturbations, do not affect the scalar metric perturbations. Axion fluctuations correspond to isocurvature perturbations to first-order. However, if the axion field does affect the energy density of the universe at later times (for instance, by acquiring a mass) then the spectrum of density perturbations need not have a steeply tilted blue spectrum such as that exhibited by the dilaton or moduli perturbations. Rather, it could have a nearly scale-invariant spectrum as required for large-scale structure formation. Such an exciting possibility has received a great deal of attention recently, notably in [49, 50, 51, 52, 53], and could be a source for the 'curvaton' field recently introduced by Lyth and Wands as a way of converting isocurvature into adiabatic perturbations [54]. Time will tell if the axion has any role to play in cosmological density perturbations although already it is beginning to look as the curvaton route is an interesting one to follow in this context [55, 56].

5.9 Smoking Guns?

Are there any distinctive features that we should be looking out for which would act as an indicator that the early Universe underwent a period of kinetic driven inflation? We have already mentioned the possibility of observing the presence of axion fluctuations in the cosmic microwave background anisotropies. Some of the other smoking guns include:

- The spectrum of primordial gravitational waves steeply growing on short scales, with a spectral index $n_T = 3$, although of no interest on large scales, such a spectrum could be observed by the next generation of gravitational wave detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) if they are on the right scale [57, 58, 45]. The current frequency of these waves depends on the cosmological model, and in general we would require either an intermediate epoch of stringy inflation, or a low re-heating temperature at the start of the post-big bang era [59] to place the peak of the gravitational wave spectrum at the right scale. Nonetheless, the possible production of high amplitude gravitational waves on detector scales in the pre–big bang scenario is in marked contrast to conventional inflation models in which the Hubble parameter decreases during inflation.
- Because the scalar and tensor metric perturbations obey the same evolution equation, their amplitude is directly related. The amplitude of gravitational waves with a given wavelength is commonly described in terms of their energy density at the present epoch. For the simplest pre–big bang models this is given in terms of the amplitude of the scalar perturbations as
$$\Omega_{\rm gw} = \frac{2}{z_{\rm eq}} \mathcal{P}_\zeta \tag{117}$$
where $z_{\rm eq} = 24000 \Omega_o h^2$ is the red-shift of matter-radiation equality. The advanced LIGO configuration will be sensitive to $\Omega_{\rm gw} \approx 10^{-9}$ over a range of scales around 100Hz. However, the maximum amplitude of gravitational waves on these scales is constrained by limits on the amplitude of primordial scalar metric perturbations on the same scale [59]. In particular, if the fractional over-density when a scalar mode re-enters the horizon

during the radiation dominated era is greater than about 1/3, then that horizon volume is liable to collapse to form a black hole with a lifetime of the order the Hubble time and this would be evaporating today! If we find PBH's and gravitational waves together then this would indeed be an exciting result for string cosmology!

– Evidence of a primordial magnetic field could have an interpretation in terms of string cosmology. In string theory the dilaton is automatically coupled to the electromagnetic field strength, for example in the heterotic string effective action the photon field Lagrangian is of the form

$$\mathcal{L} = e^{-\varphi} F_{\mu\nu} F^{\mu\nu} \,, \tag{118}$$

where the field strength is derived from the vector potential, $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$.
Now in an isotropic FRW cosmology the magnetic field must vanish to zeroth-order, and thus the vector field perturbations are gauge-invariant and we can neglect the metric back-reaction to first-order. In the radiation gauge ($A^0 = 0$, $A^i_{|i} = 0$) then the field perturbations can be treated as vector perturbations on the spatial hypersurfaces. The field perturbation A_i turns out to have a clear unique dependence on the dilaton field. In fact the time dependence of the dilaton (rather than the scale factor) leads to particle production during the pre–big bang from an initial vacuum state [60, 61, 62]. Using the pre–big bang solutions given in (88)–(90), we find that the associated Power spectrum of the gauge fields have a minimum tilt for the spectral index for $\xi_* = 0$ when $\mu = (1+\sqrt{3})/2$ with a spectral tilt $\Delta n_{\rm em} = 4 - \sqrt{3} \approx 2.3$. This is still strongly tilted towards smaller scales, which currently is too steep to be observably acceptable.

6 Dilaton-Moduli Cosmology Including a Moving Five Brane

We turn our attention briefly to M-theory, and in particular to cosmological solutions of four-dimensional effective heterotic M-theory with a moving five-brane, evolving dilaton and T modulus [63]. It turns out that the five-brane generates a transition between two asymptotic rolling-radii solutions, in a manner analogous to the case of the NS-NS axion discussed in Sect. 3. Moreover, the five-brane motion generally drives the solutions towards strong coupling asymptotically. The analogous solutions to those presented in the pre-big-bang involves a negative-time branch solution which ends in a brane collision accompanied by a small-instanton transition. Such an exact solution should be of interest bearing in mind the recent excitement that has been generated over the Ekpyrotic Universe scenario, which involves solving for the collision of two branes [37, 38].

The four-dimensional low-energy effective theory we will be using is related to the underlying heterotic M-theory. Of particular importance for the interpretation of the results is the relation to heterotic M-theory in five dimensions, obtained from the 11-dimensional theory by compactification on a Calabi-Yau three-fold. This five-dimensional theory provides an explicit realisation of a brane-world. The compactification of 11 dimensional Horava-Witten theory, that is 11-dimensional supergravity on the orbifold $S^1/Z_2 \times M_{10}$, to five dimensions on a Calabi-Yau three fold, leads to the appearance of extra three-branes in the five-dimensional effective theory. Unlike the "boundary" three-branes which are stuck to the orbifold fix points, however, these three-branes are free to move in the orbifold direction, and this leads to a fascinating new cosmology.

Our starting point is the four dimensional action

$$S = -\frac{1}{2\kappa_P^2}\int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{4}(\nabla\varphi)^2 + \frac{3}{4}(\nabla\beta)^2 + \frac{q_5}{2}e^{(\beta-\varphi)}(\nabla z)^2\right] , \tag{119}$$

where φ is the effective dilaton in four dimensions, β is the size of the orbifold, z is the modulus representing the position of the five brane and satisfies $0 < z < 1$, and q_5 is the five brane charge. Due to the non-trivial kinetic term for z, solutions with exactly constant φ or β do not exist as soon as the five-brane moves. Therefore, the evolution of all three fields is linked and (except for setting $z =$ const) cannot be truncated consistently any further. Looking for cosmological solutions for simplicity, we assume the three-dimensional spatial space to be flat. Our Ansatz then reads

$$ds^2 = -e^{2\nu}d\tau^2 + e^{2\alpha}d\mathbf{x}^2 \tag{120}$$

$$\varphi = \varphi(\tau) \tag{121}$$

$$\alpha = \alpha(\tau) \tag{122}$$

$$\beta = \beta(\tau) \tag{123}$$

$$z = z(\tau) \tag{124}$$

The cosmological solutions are given by [63]

$$\alpha = \frac{1}{3}\ln\left|\frac{t-t_0}{T}\right| + \alpha_0 \tag{125}$$

$$\beta = p_{\beta,i}\ln\left|\frac{t-t_0}{T}\right| + (p_{\beta,f}-p_{\beta,i})\ln\left(\left|\frac{t-t_0}{T}\right|^{-\delta}+1\right)^{-\frac{1}{\delta}} + \beta_0 \tag{126}$$

$$\varphi = p_{\varphi,i}\ln\left|\frac{t-t_0}{T}\right| + (p_{\varphi,f}-p_{\varphi,i})\ln\left(\left|\frac{t-t_0}{T}\right|^{-\delta}+1\right)^{-\frac{1}{\delta}} + \varphi_0 \tag{127}$$

$$z = d\left(1+\left|\frac{T}{t-t_0}\right|^{-\delta}\right)^{-1} + z_0 \ . \tag{128}$$

where t is the proper time, the time-scales t_0 and T are arbitrary constants as are the constants d and z_0 which parameterise the motion of the five-brane. For $-\infty < t < t_0$ we are in the positive branch of the solutions and for $t_0 < t < \infty$ we are in the negative branch.

We see that both expansion powers for the scale factor α are given by 1/3, a fact which is expected in the Einstein frame. The initial and final expansion powers for β and φ are less trivial and are subject to the constraint

$$3p_{\beta,n}^2 + p_{\varphi,n}^2 = \frac{4}{3} \tag{129}$$

for $n = i, f$. These are mapped into one another by

$$\begin{pmatrix} p_{\beta,f} \\ p_{\varphi,f} \end{pmatrix} = P \begin{pmatrix} p_{\beta,i} \\ p_{\varphi,i} \end{pmatrix} , \qquad P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} . \tag{130}$$

This map is its own inverse, that is $P^2 = 1$, which is a simple consequence of time reversal symmetry. The power δ is explicitly given by

$$\delta = p_{\beta,i} - p_{\varphi,i} . \tag{131}$$

For $\delta < 0$ we are in the negative branch and for $\delta > 0$ we are in the positive time branch. Finally, we have

$$\varphi_0 - \beta_0 = \ln\left(\frac{2q_5 d^2}{3}\right) . \tag{132}$$

The solutions have the following interpretation: at early times, the system starts in the rolling radii solution characterised by the initial expansion powers p_i while the five-brane is practically at rest. When the time approaches $|t - t_0| \sim |T|$ the five-brane starts to move significantly which leads to an intermediate period with a more complicated evolution of the system. Then, after a finite comoving time, in the late asymptotic region, the five-brane comes to a rest and the scale factors evolve according to another rolling radii solution with final expansion powers p_f. Hence the five-brane generates a transition from one rolling radii solution into another one. While there are perfectly viable rolling radii solutions which become weakly coupled in at least one of the asymptotic regions, the presence of a moving five-brane always leads to strong coupling asymptotically, a phenomenon similar to what we observed in the dilaton-moduli-axion dynamics (see Fig. 2).

These general results can be illustrated by an explicit example. Focusing on the negative-time branch and considering the solutions with an approximately static orbifold at early time, Fig. 5 shows the evolution of β and φ, whereas Fig. 6 shows the evolution of the dynamical brane.

At early times, $|t - t_0| \gg |T|$, the evolution is basically of power-law type with powers p_i, because at early time the five-brane is effectively frozen at $z \simeq d + z_0$ and does not contribute a substantial amount of kinetic energy.

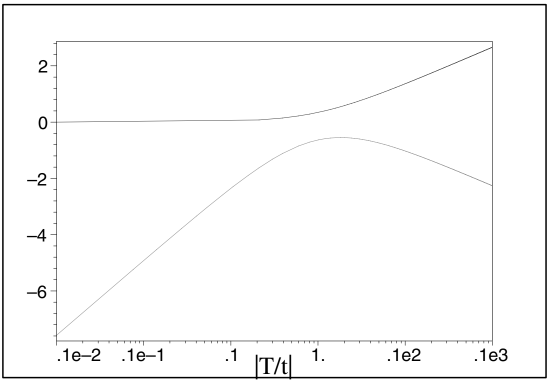

Fig. 5. Time-behaviour of β (upper curve) and φ (lower curve).

This changes dramatically once we approach the time $|t - t_0| \sim |T|$. In a transition period around this time, the brane moves from its original position by a total distance d and ends up at $z \simeq z_0$. At the same time, this changes the behaviour of the moduli β and φ until, at late time $|t| \ll |T|$, they correspond to another rolling radii solution with powers controlled by p_f. Concretely, the orbifold size described by β turns from being approximately constant at early time to expanding at late time, while the Calabi-Yau size controlled by φ undergoes a transition from expansion to contraction. We also find that as with the axion case discussed earlier, the solution runs into strong coupling in both asymptotic regions $t - t_0 \to -\infty$ and $t - t_0 \to 0$ which illustrates our general result.

In Fig. 6 we have shown a particular case which leads to brane collision. The five-brane is initially located at $d + z_0 \simeq 0.9$ and moves a total distance of $d = 1.5$ colliding with the boundary at $z = 0$ at the time $|t - t_0|/|T| \simeq 1$.

This represents an explicit example of a negative-time branch solution which ends in a small-instanton brane-collision. Solving for these systems has only just the begun, but already interesting features have emerged including a new mechanism for baryogenesis arising from the collision of two branes [64], and a more detailed understanding of the vacuum transitions associated with brane collisions [65].

7 Inflation Today – Quintessence

Now we will look at the general form Quintessence scenarios take. They are of course attempts to account for the observed accelerated expansion of the universe [66, 67], but are based on the evolution of as yet unobserved time dependent scalar fields. In particular they are not: a true cosmological constant; a time-dependent cosmological constant or solid dark energy such as arising from frustrated network of domain walls.

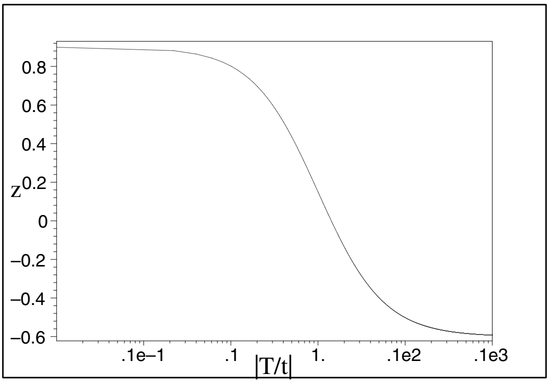

Fig. 6. Time-behaviour of the five-brane position modulus z for the example specified in the text. The boundaries are located at $z = 0, 1$ and the five-brane collides with the $z = 0$ boundary at $|t/T| \simeq 1$.

In Quintessence, the time dependent solutions arising out of scalar field potentials usually involves some form of tracking behaviour, where the energy density in the scalar field evolves so as to mimic that of the background fluid density for a period of time [68]. As we approach a redshift between $0.5 < z < 1$ the potential energy of the Quintessence field becomes the dominant contribution to the energy density and the Universe begins to accelerate [69, 70]. We will not go into details of the solutions in these lectures, rather we will discuss the general behaviour one expects from Quintessence scenarios. A nice review of the rich structure present in these models is presented in [71, 72], and Axel de la Macorra has given some detailed lectures here at the meeting [73].

Using a particular potential $V(\phi) = exp(0.3e^{0.3\phi})$ as an example, Fig. 7 shows the generic behaviour that is expected to be followed in Quintessence models.

Region 1 corresponds to the period where the initial potential energy in the scalar field is converted into kinetic energy as the field begins to roll down its potential. This scalar field kinetic energy soon comes to dominate the energy density of the scalar field as $\rho_\phi \propto a(t)^{-6}$ where $a(t)$ is the scale factor [region 2]. As the kinetic energy decreases rapidly, the system slows down again [region 3] leading to a constant field regime. This is then followed by the crucial period where the kinetic energy in the scalar field scales in proportion to its potential energy [region 4]. This is an attractor regime and as can be seen from Fig. 7 it corresponds to an extended period in which the energy density tracks that of the background energy density. These attractor properties are very useful because they make the reliance on initial conditions of the scalar field less important. Finally in region 5, we see the specific property of the scalar field potential coming into its own, as it determines when the scalar field potential energy density comes to dominate over the

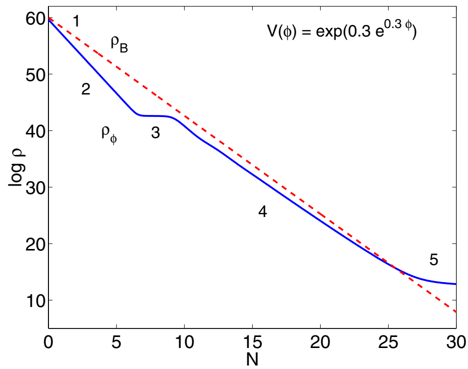

Fig. 7. Typical scaling properties of Quintessence potentials. ρ_B, ρ_ϕ are the energy densities in the background fluid and scalar field respectively[72].

background fluid energy density leading to the observed acceleration of the Universe. Fig. 7 gives a flavour for some of the fine tuning issues that arises in Quintessence. There are two obvious ones, the first is that the value of the energy density today must be very close to the critical density $10^{-3}eV^4$, the second is that domination had to occur very recently $z \leq 1$ in order to account for the fact that galaxy formation is not affected too much by the Quintessence field. There are also tight constraints on the energy density in the Quintessence field at the time of nucleosynthesis, as the field acts like an extra light degree of freedom and we already know that there are tight constraints on the number of families from nucleosynthesis. We will now go on to look at some individual models.

7.1 Specific Quintessence Models

The original Quintessence model [69, 70] has an inverse power law type of potential,

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}\,, \tag{133}$$

where α is thought of as a positive number (it could actually also be negative) and M is constant.

Most models of Quintessence are analysed through their effective equation of state,

$$w_\phi \rho_\phi,$$

where p_ϕ is the pressure in the field and ρ_ϕ is the energy density in the field. We know from Einstein's acceleration equation that for the Quintessence field to lead to acceleration of the Universe we require $\rho_\phi + 3p_\phi < 0$ or $w_\phi < -\frac{1}{3}$. Applying this to the inverse power case we find

$$w_\phi = \frac{\alpha w_B - 2}{2+\alpha},$$

where w_B is the background fluid equation of state. Where does the fine tuning arise in these models? Recall we need to match the energy density in the Quintessence field to the current critical energy density, which in terms of the Hubble parameter today H_0 and the Planck mass M_{pl} is given by

$$\rho_\phi < M_{pl}^2 H_0^2 \sim 10^{-47} GeV^4.$$

It turns out that during the tracking regime, $H^2 \sim \frac{V(\phi)}{\phi^2} \sim \frac{\rho_\phi}{M_{pl}^2}$, hence it follows that at the time the scalar field is dominating the energy density and leading to acceleration today, we must have $\phi_0 \sim M_{pl}$, the value of the scalar field today has to be of order the Planck scale. This is typical of virtually all Quintessence models. The real fine tuning now becomes clear, substituting for the value of ϕ_0 in to the bound on the energy density today ρ_ϕ^0, we see:

$$M = (\rho_\phi^0 M_{pl}^\alpha)^{\frac{1}{4+\alpha}}.$$

This then constrains the allowed combination of α, M. For example for $\alpha = 2$ the constraint implies $M = 1GeV$ etc... Within the class of parameters which satisfy the coincidence problem the inverse power law potentials suffer in that their predicted equation of state w_ϕ is only marginally compatible with the values emerging from observations. At the 1σ confidence level in the $\Omega_{\rm M} - w_\phi$ plane, the data prefer $w_\phi < -0.8$ with possibly a favoured cosmological constant $w_Q = -1$ whereas the values permitted by these tracker potentials (for $\alpha \geq 1$, have $w_Q > -0.8$. A general problem we will always have to tackle is finding such Quintessence models in particle physics. For an interesting attempt at this in the context of Supersymmetric QCD see the model proposed by Binetruy [74].

Multiple exponential potentials also offer interesting possibilities for a successful Quintessence scenario [75]. Such potentials are expected to arise as a result of compactifications in superstring models, hence are well motivated. Unfortunately we still have not obtained what one would call a 'natural' model for reasons we will discuss below. Nevertheless it remains a model with some potential for success in it as it delivers Quintessence scenarios for a wide range of initial conditions.

It has been known for some time that single exponential potentials lead to scaling solutions[68, 76, 77]. Consider the case of $V(\phi) = V_0 \exp(\alpha\kappa\phi)$, where $\kappa^2 \equiv 8\pi/M_{pl}^2$. The two late time attractor solutions depend on the values of α and the background's equation of state w_B:

(1) $\alpha^2 > 3(w_B+1)$: the scalar field mimics the evolution of the barotropic fluid with $w_\phi = w_B$, and the relation $\Omega_\phi = 3(w_B+1)/\alpha^2$ holds.

(2) $\alpha^2 < 3(w_B+1)$. The late time attractor is the scalar field dominated solution ($\Omega_\phi = 1$) with $w_Q = -1 + \alpha^2/3$.

By including two exponential terms it allows for the possibility of the system entering two scaling regimes which depend on the value of the slope of the two terms: one tracks radiation and matter, while the second one dominates at end. To be specific we can consider

$$V(\phi) = V_0 \left(e^{\alpha\kappa\phi} + e^{\beta\kappa\phi} \right) , \tag{134}$$

where for convenience we assume α to be positive (the case $\alpha < 0$ can always be obtained taking $\phi \to -\phi$). Figures 8 and 9 show the results of a typical run with such a potential leading to potential domination today and acceleration.

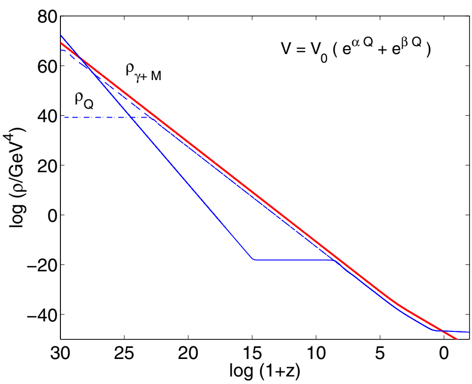

Fig. 8. Plot of the energy density, ρ_Q, for the 2EXP model with $\alpha = 20$, $\beta = 0.5$ and several initial conditions admitting an $\Omega_Q = 0.7$ flat universe today. The line labeled by $\rho_{\gamma+\mathrm{M}}$ is the evolution of radiation and matter.[72].

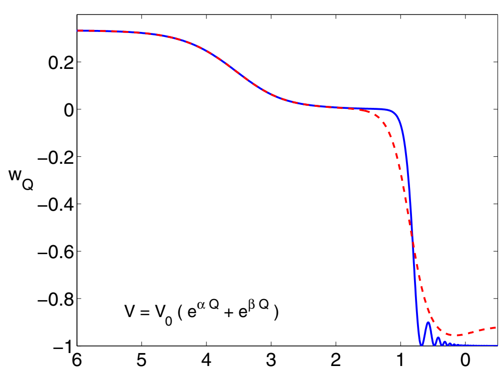

Fig. 9. The late time evolution of the equation of state, in the 2EXP model, for parameters $(\alpha, \beta) = (20{,}0.5)$ dashed line; $(20{,}-20)$ solid line for $\Omega_Q = 0.7$.[72].

It is clear from Fig. 9, where the evolution of the equation of state is shown and compared to the case with $\beta/\alpha > 0$, that the field mimics the radiation ($w_Q = 1/3$) and matter ($w_Q = 0$) evolution before settling in an accelerating ($w_Q < -1/3$) expansion. As a result of the scaling behaviour of attractor (1), it is clear that there exists a wide range of initial conditions that provide realistic results.

Where in this case is the fine tuning to be found then? Demanding the energy density in the field matches the critical density today, places the bound $V_0 \sim \rho_\phi^0 \sim 10^{-47} GeV^4 \sim (10^{-3} eV)^4$. This very low energy density converts into an extremely light scalar field, in particular its mass is given by

$$m \simeq \sqrt{\frac{V_0}{M_{pl}^2}} \sim 10^{-33} eV.$$

Such a tiny mass is very difficult to reconcile with fifth force experiments, unless there is a mechanism to prevent ϕ from having interactions with the other matter fields!

A model which can be related to the two exponential case has been suggested by Sahni and Wang [78]. The potential can be written as:

$$V(\phi) = V_0 \left[\cosh(\alpha\kappa\phi) - 1\right]^n . \tag{135}$$

It behaves as an exponential potential $V \to \exp(n\alpha\kappa\phi)$ for $|\alpha\kappa\phi| \gg 1$ and as a power law type of potential $V \to (\alpha\kappa\phi)^{2n}$ for $|\alpha\kappa\phi| \ll 1$. It follows that the evolution scales as radiation and matter when dominated by the exponential form and later enters into an oscillatory regime when the minimum is reached. In this regime the time average equation of state is

$$\langle w_\phi \rangle = \frac{n-1}{n+1} . \tag{136}$$

We see that for $n < 1/2$ then $w_\phi < -1/3$, implying late times accelerated expansion driven by the scalar field. The fine tuning in this case is similar to that of the two exponential potential discussed earlier.

Albrecht and Skiordis [79] have developed an interesting model which they have argued can be derived from String theory, in that they claim the parameters are all of order one in the underlying string theory. The potential has a local minimum which can be adjusted to have today's critical energy density value (this is where the fine tuning is to be found by the way). The actual potential is a combination of exponential and power-law terms:

$$V(\phi) = V_0 e^{-\alpha\kappa\phi} \left[A + (\kappa\phi - B)^2\right] . \tag{137}$$

In Fig. 10 we show the evolution of the equation of state. For early times the exponential term dominates the dynamics, with the energy density of ϕ scaling as radiation and matter. For suitable choices of the parameters the field gets trapped in the local minimum because the kinetic energy during

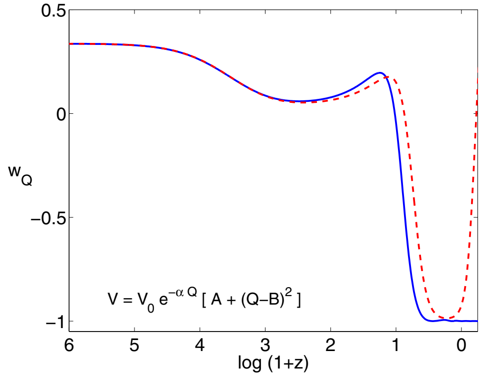

Fig. 10. The evolution of the equation of state of quintessence when the Albrecht-Skiordis potential has a local minimum (solid) and when it does not (dashed). In this case $\alpha = 10$, $V_0 = \kappa^{-4}$, $A = 0.9/\alpha^2$ and $B = 27.2$, for the former case and $\alpha = 6$, $V_0 = \kappa^{-4}$, $A = 1.1/\alpha^2$ and $B = 45.5$, for the latter.[72].

scaling is small. The field then enters a regime of damped oscillations leading to $w_\phi \to -1$ and an eternally expanding universe.

There are many other models which we could describe: coupled quintessence, extended quintessence, tracking oscillatory quintessence to name but three. They all have similar properties to those described above, but rather than concentrate on them we will turn our attention finally to the case of Quintessential Inflation, developed by Peebles and Vilenkin [80]. One of the major drawbacks often used to attack models of Quintessence is that it introduces yet another weakly interacting scalar field. Why can't we use one of those scalars already 'existing' in cosmology, to also act as the Quintessence field? This is precisely what Peebles and Vilenkin set about doing. They introduced a potential for the field ϕ which allowed it to play the role of the inflaton in the early Universe and later to play the role of the Quintessence field. To do this it was important that the potential did not have a minimum in which the inflaton field would completely decay at the end of the initial period of inflation. The potential they proposed was:

$$\begin{aligned} V(\phi) &= \lambda(\phi^4 + M^4) \quad \text{for } \phi < 0 \\ &= \frac{\lambda M^4}{(1 + (\frac{\phi}{M})^\alpha)} \quad \text{for } \phi \geq 0 \end{aligned}$$

For $\phi < 0$ we have ordinary chaotic inflation. When this ends the Universe is reheated via gravitational particle production. Much later on, for $\phi > 0$ the Universe once again begins to inflate but this time at the lower energy scale associated with Quintessence. Needless to say, Quintessential Inflation also requires a degree of fine tuning, in fact perhaps even more than before as there are no tracker solutions we can rely on for the initial conditions. The initial period of inflation must produce the observed density fluctuations, hence constrains $\lambda \sim 10^{-14}$. Demanding that $\Omega^0_\phi \sim 0.7$, we find we can constrain the parameter space of (α, M). For example, for $\alpha = 4$, we have $M = 10^5$ GeV. Time does not permit us to elaborate further on this aspect of Quintessence, but it is worth at least mentioning that there are some very nice resolutions of Quintessential Inflation in Brane world scenarios (for details see [81, 82, 83]. Neither have we time to go into the wealth of Quintessence models that have been proposed within the context of supergravity, apart from giving a brief flavour of the general idea. Brax and Martin [84] demonstrated that a supergravity model with Superpotential $W = \Lambda^{3+\alpha}\Phi^\alpha$ and Kahlar potential $K = \Phi\Phi^*$ (where Φ is the Chiral scalar field) leads to an associated scalar potential

$$V(\phi) = \frac{\Lambda^{6+\alpha}}{\phi^{2\alpha+2}} e^{\frac{\kappa^2}{2}\phi^2},$$

under the rather strict assumption that $< W >= 0$. A working example is the case $\alpha = 11$ which has an associated equation of state $w^0_\phi = -0.8$. There are more models that have been investigated [85, 86, 87]. A word of caution though about Quintessence in supergravity. Kolda and Lyth [88], have argued that all current supergravity inspired models suffer from the fact that loop corrections will always couple the Quintessence field to other sources of matter so as to lift the potential thereby breaking the flatness criteria required for Quintessence today.

7.2 Evidence for Quintessence?

If there is a scalar field responsible for the current acceleration of the Universe how might we see it? In this conference there have been many talks addressing this issue, so we will not go into great details here, other than remind the reader of some of the attempts that are under way and have been proposed recently. Ideally we would look for evidence of evolution in the equation of state, w_ϕ as a function of redshift. These include

- Precision CMB anisotropies – lots of models are currently compatible.
- Combined LSS, SN1a and CMB data tend to give $w_\phi < -0.8$, which is difficult to tell from a true cosmological constant.
- Look for more supernova of the type SN1a. The proposed satellite, SNAP will find over 2000 which may then enable us to start constraining the equation of state.
- Constraining the equation of state with Sunayaev-Zeldovich cluster surveys from which we can compute the number of clusters for a given set of cosmological parameters.
- Probing the Dark Energy with Quasar clustering in which redshift distortions constrain cosmological parameters.

– Reconstruct the equation of state from observations – this approach at least offers the hope of developing a method independent of potentials – an example is the Statefinder method developed by Sahni et al. [89].
– Look for evidence in the variation of the fine structure constant.

We finish off the lectures by discussing in a bit more detail one of the items just mentioned. Finding a suitable parameterisation of the equation of state an issue of importance for those interested in reconstructing w_ϕ from observation, such as those working on SNAP [90, 91]. Two approaches suggested to date involve a polynomial expansion either in terms of the red-shift, z (i.e. $w_\phi(z) = \sum_{i=0}^{N} w_i z^i$) [92] or in terms of the logarithm of the red-shift (i.e. $w_\phi(z) = \sum_{i=0}^{N} w_i \ln(1+z)^i$) [93]. A third approach has recently been developed by Corasaniti and Copeland [94]. It allows for tracker solutions in which there is a rapid evolution in the equation of state, something that the more conventional power-law behaviour can not accommodate. This has has some nice features in that it allows for a broad class of Quintessence models to be accurately reconstructed and it opens up the possibility finding evidence of quintessence in the CMB both through its contribution to the Integrated Sachs Wolfe Effect [95] and as a way of using the normalisation of the dark energy power spectrum on cluster scales, σ_8, to discriminate between dynamical models of dark energy (Quintessence models) and a conventional cosmological constant model[96].

8 Summary

In these lectures we have addressed a number of issues relating to inflationary cosmology, both in the early Universe and today. We have seen how inflation arises in both potential dominated cases and as a result of rolling radii solutions associated with the low energy string action. We have also seen how hard it is to relate inflation to realistic particle physics inspired models. This area is one of intense interest at the moment. In our attempt to bridge this gap, we have related these solutions to the exciting new solutions arising in M-theory cosmology, and showed how a moving five brane could act in a manner similar to the axion field in the pre Big Bang case. This is an exciting time for string and M-theory cosmology, the subject is developing at a very fast rate, and no doubt there will be new breakthroughs emerging over the next few years. Hopefully out of these we will be in a position to address a number of the issues we have raised in this article, as well as other key ones such as stabilising the dilaton and explaining the current observation of an accelerating Universe. We have investigated a number of Quintessence models and tried to argue why Quintessence offers a plausible explanation for the observational fact that the Universe is accelerating today. We have also tried to emphasise the issues that Quintessence as a model simply fails to answer naturally, requiring some form of fine tuning in order to do so. These include:

- Why is there a Λ type term dominating today?
- Why are the matter and Λ contributions comparable today – 'coincidence' problem?
- Why is Λ so small compared to typical particle physics scale?
- Is there any need for a quintessence field? Is it simply a cosmological constant?

There is little doubt that this very exciting field is being driven by observations, especially in the CMBR and LSS. They are constraining the cosmological parameters, even before Map or Planck arrives on the scene. Yet we do not know why the universe inflating today and through Quintessence we are hoping that particle physics provides an answer. The existence of scaling solutions and tracker behaviour may yet show up through time varying constants [97]. There is much going on in Brane inspired cosmology and it may provide important clues to the nature of dark energy. In general as we have seen, there are many models of Quintessence but they may yet prove too difficult to separate from a cosmological constant. We need to try though – it is too exciting a prospect not to!

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References

1. A. H. Guth, Phys. Rev. D**23**, 347 (1981).
2. A.D. Linde, Phys. Lett. B**108**, 389 (1982).
3. A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett.**48**, 1220 (1982).
4. S.W. Hawking, I.G. Moss and J.M. Stewart, Phys. Rev. D**26**, 2681 (1982).
5. A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
6. D. N. Spergel et al., Astrophys. J. Suppl. **148**, 175 (2003) [arXiv:astro-ph/0302209].
7. A. D. Linde: *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).
8. H. V. Peiris et al., Astrophys. J. Suppl. **148**, 213 (2003) [arXiv:astro-ph/0302225].
9. E.W. Kolb and M.S. Turner: *The Early Universe* (Addison-Wesley, Redwood City, 1994).
10. L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994) [arXiv:hep-th/9405187].
11. L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. **76**, 1011 (1996) [arXiv:hep-th/9510119].
12. E. J. Copeland, D. Lyth, A. Rajantie and M. Trodden, Phys. Rev. D **64**, 043506 (2001) [arXiv:hep-ph/0103231].

13. E. J. Copeland, S. Pascoli and A. Rajantie, Phys. Rev. D **65**, 103517 (2002) [arXiv:hep-ph/0202031].
14. A.R. Liddle and D.H. Lyth, Phys. Rep **231**, 1 (1993), [astro-ph/9303019]; A.R. Liddle and D.H. Lyth: *Cosmological inflation and large scale structure*, (Cambridge Univ. Press, 2000)
15. R. H. Brandenberger, Lectures on the Theory of Cosmological Perturbations, Lect. Notes Phys. 646, 127 (2004) [arXiv:hep-th/0306071].
16. J. H. Lidsey, D. Wands and E. J. Copeland, Phys. Rep. **337**, 343 (2000).
17. G. Veneziano, Phys. Lett. B**265**, 287 (1991).
18. M. Gasperini and G. Veneziano. Astropart. Phys. **1**, 317 (1993).
19. E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D**50**, 4868 (1994).
20. M. D. Pollock and D. Sahdev, Phys. Lett. B**222**, 12 (1989).
21. J. J. Levin and K. Freese, Phys. Rev. D**47**, 4282 (1993).
22. R. Brustein and G. Veneziano, Phys. Lett. B**329**, 429 (1994).
23. M. S. Turner and E. J. Weinberg, Phys. Rev. D**56**, 4604 (1997).
24. D. Clancy, J. E. Lidsey, and R. Tavakol, Phys. Rev. D **58**, 044017 (1998).
25. N. Kaloper, A. D. Linde, and R. Bousso, Phys. Rev. D**59**, 043508 (1999).
26. J. Maharana, E. Onofri, and G. Veneziano, J. High Energy Phys. **01**, 004 (1998).
27. G. Veneziano, Phys. Lett. B**406**, 297 (1997).
28. A. Buonanno, K. A. Meissner, C. Ungarelli, and G. Veneziano, Phys. Rev. D**57**, 2543 (1998).
29. A. Buonanno, T. Damour, and G. Veneziano, Nucl. Phys. B**543**, 275 (1999).
30. M. Gasperini, Phys. Rev. D**61** 087301 (2000).
31. D. Clancy, J. E. Lidsey, and R. Tavakol, Phys. Rev. D**59**, 063511 (1999).
32. M. Gasperini's web page, http://www.to.infn.it/ gasperin/
33. R. Brustein and R. Madden, Phys. Lett. B**410**, 110 (1997). R. Brustein and R. Madden, Phys. Lett. B**410**, 110 (1997).
34. R. Brustein and R. Madden, Phys. Rev. D**57**, 712 (1998). R. Brustein and R. Madden, Phys. Rev. D**57**, 712 (1998).
35. C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000).
36. M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. B**494**, 315 (1997).
37. J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D**64**, 123522 (2001) and hep-th/0108187
38. R. Kallosh, L. Kofman and A. Linde, Phys. Rev. **D64**, 123523 (2001).
39. P. J. Steinhardt and N. Turok, Phys. Rev. D**65** 126003 (2002) hep-th/0111030 and hep-th/0111098
40. V. F. Mukhanov, Sov. Phys. JETP **68**, 1297 (1988).
41. V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
42. J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D**28**, 679 (1983).
43. R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov, and G. Veneziano, Phys. Rev. D **51**, 6744 (1995).
44. A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).
45. C. Cartier, E. J. Copeland and M. Gasperini, Nucl. Phys. B**607**, 406 (2001).
46. E. J. Copeland, R. Easther, and D. Wands, Phys. Rev. D**56**, 874 (1997).
47. E. J. Copeland, J. E. Lidsey, and D. Wands, Nucl. Phys. B**506**, 407 (1997).
48. C. Cartier, J. Hwang and E. J. Copeland, Phys. Rev. D**64**, 103504 (2001).
49. R. Durrer, M. Gasperini, M. Sakellariadou, and G. Veneziano, Phys. Lett. B**436**, 66 (1998).

50. R. Durrer, M. Gasperini, M. Sakellariadou, and G. Veneziano, Phys. Rev. D**59**, 043511 (1999).
51. A. Melchiorri, F. Vernizzi, R. Durrer and G. Veneziano, Phys. Rev. Lett. **83**, 4464 (1999).
52. F. Vernizzi, A. Melchiorri and R. Durrer, Phys. Rev. D **63**, 063501 (2001).
53. K. Enqvist and M. S. Sloth, Nucl. Phys. B**626**, 395 (2002) [hep-ph/0109214].
54. D. Lyth and D. Wands, Phys. Lett. B**524**, 5 (2002).
55. V. Bozza, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **543**, 14 (2002).
56. V. Bozza, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. D**67**, 063514 (2003) [arXiv: hep-ph/0212112].
57. B. Allen and R. Brustein, Phys. Rev. D **55**, 3260 (1997).
58. M. Maggiore, Phys. Rept. **331**, 283 (2000).
59. E. J. Copeland, A. R. Liddle, J. E. Lidsey, and D. Wands, Phys. Rev. D**58**, 063508 (1998).
60. D. Lemoine and M. Lemoine, Phys. Rev. D**52**, 1955 (1995).
61. M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Rev. Lett. **75**, 3796 (1995).
62. M. Gasperini, M. Giovannini, and G. Veneziano, Phys. Rev. D**52**, 6651 (1995).
63. E.J. Copeland, J. Gray and A. Lukas, Phys. Rev. D**64**, 126003 (2001).
64. M. Bastero-Gill, E.J. Copeland, J. Gray, A. Lukas, M. Plumacher, Phys. Rev. **D66**, 066005 (2002)
65. N. Antunes, E.J. Copeland, M. Hindmarsh and A. Lukas, Phys. Rev. D**68**, 066005 (2003) [hep-th/0208219].
66. S. Perlmutter et al., Astrophys. J **517**, 565 (1999).
67. A. Riess, et al., Astrophys. J,**117**, 707 (1999).
68. C. Wetterich, Nucl. Phys. B**302**, 668 (1998).
69. B. Ratra, and P.J.E. Peebles, Phys. Rev. D**37**, 3406 (1988).
70. R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998). I. Zlatev, L. Wang, and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
71. C. Ng, N. Nelson, and F. Rosati, Phys. Rev. D **64** 083510 (2001).
72. N. Nelson. D. Phil thesis, University of Sussex (2001).
73. A. de la Macorra, Quintessence and Dark Energy, Lect. Notes Phys. 646, 225 (2004).
74. P. Binetruy, Phys. Rev. D**60**, 063502 (1999).
75. T. Barreiro, E.J. Copeland, and N. Nunes, Phys. Rev. D**61**, 127301 (2000).
76. E.J. Copeland, , A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).
77. P. Ferreira and M. Joyce, Phys. Rev. D**58**, 023503 (1998).
78. V. Sahni and L. Wang, Phys. Rev. D**62**, 103517 (2000).
79. A. Albrecht and C. Skordis, Phys. Rev. Lett. **84**, 2076 (1999).
80. P.J.E. Peebles and A. Vilenkin, Phys. Rev. D**59**, 063505 (1999).
81. E.J. Copeland, A. R. Liddle and J.E. Lidsey, Phys. Rev. D **64**, 023509 (2001)
82. G. Huey and J.E. Lidsey, Phys. Lett. B**514**, 217 (2001).
83. N. Nunes and E.J. Copeland, Phys. Rev. D**66**, 043524 (2002).
84. P. Brax and J. Martin, Phys. Lett. B**468**, 40 (1999).
85. K. Choi, Phys. Rev. D**62**, 043509 (2000).
86. A. Masiero, M. Pietroni and F. Rosati, Phys. Rev. D**61**, 023504 (2000).
87. E. J. Copeland, N. Nunes and F. Rosati, Phys. Rev. D**62**, 123503 (2000).
88. C. F. Kolda and D. H. Lyth, Phys. Lett. B**458**, 197 (1999).

89. V. Sahni et al., JETP Lett. **77**, 201 (2003) [astro-ph/0201498]; [astro-ph/0211084].
90. D. Huterer and M. Turner, Phys. Rev. D**60**, 081301 (1999).
91. D. Huterer and G. Starkman, Phys. Rev. Lett. **90**, 031301 (2003) [astro-ph/0207517].
92. J. Weller and A. Albrecht, Phys. Rev. Lett. **86**, 1939 (2001); Phys. Rev. D**65** 103512 (2002).
93. B. F. Gerke and G. Efstathiou, Mon. Not. Roy. Astron. Soc. **335**, 33 (2002).
94. P. S. Corasaniti and E. J. Copeland, Phys. Rev. D **67**, 063521 (2003) [arXiv:astro-ph/0205544].
95. P. S. Corasaniti, B. A. Bassett, C. Ungarelli and E. J. Copeland, Phys. Rev. Lett. **90**, 091303 (2003) [arXiv:astro-ph/0210209].
96. M. Kunz, P. S. Corasaniti, D. Parkinson and E. J. Copeland, arXiv:astro-ph/0307346.
97. E. J. Copeland, N. J. Nunes and M. Pospelov, Phys. Rev. D **69**, 023501 (2004), [arXiv:hep-ph/0307299].