Quantum Theory in Six Dimensions - Particles and Strings in Six Dimensions

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1 Prelude

Fourscore years after the passing of Gunnar Nordström, we are still struggling with many of the problems he considered in his works. In particular, the idea that extra space-time dimensions may explain the occurence of gauge symmetries in four-dimensional physics has reappeared in many guises. In my first talk, I described in rather non-technical terms a new way to do this, introduced by Witten in 1995 [1]. In the second talk, I discussed an ongoing project to construct the corresponding supersymmetric low-energy effective theory. In the beginning, we more or less went through superspace on foot, rather much in the spirit of a contemporary compatriot of Nordströms' [2], but at the time of the conference, we had found the first strand of the correct structures to use. The complete solution to this problem is described in our recent paper [3], which also contains some more references.

2 Introduction

Quantum phenomena are extremely diverse, so it is rather surprising that consistent *quantum theories* are so rare. Sofar, the most important classes have been the following:

- *Quantum mechanics* describes systems with a finite number of degrees of freedom.
- *Quantum field theory* describes everything except gravity, in at most four space-time dimensions.
- *String theory* also describes gravity, but lives in ten space-time dimensions.

Before 1995, a reasonable conjecture based on this experience, might have been that quantum theories in more than four space-time dimensions must derive from string theory and necessarily contain dynamical gravity. But in that year, Witten made the surprising discovery of the existence of

- *Quantum theories in six space-time dimensions* that do not include gravity.

For the last few years, the main goal of my research has been to learn more about these mysterious theories. A main problem is then that no proper definition of them is currently available directly in six dimensions. Instead we can take a

- *lower dimensional perspective*:
  Certain four-dimensional quantum field theories can be naturally viewed as arising upon compactification from six dimensions.

- *higher dimensional perspective*:
  Consider string theory on ten-dimensional space-time with a six-dimensional impurity (five-brane or codimension four singularity). Self-consistent theory at the locus of impurity may decouple from bulk degrees of freedom (e.g gravity, ...)

So we seem to have a choice between an a priori ambiguous (but possibly quite fruitful) characterization of the six-dimensional quantum theories as the origin of certain lower-dimensional theories, or a well-defined (but rather impractical) definition in terms of an isolated subsector of a higher dimensional theory. I thus see at least three good reasons to study six-dimensional quantum theories:

1. *Their intrinsic interest*:
   New classes of quantum theories appear very rarely.
   They are likely to involve interesting new concepts of mathematics and physics.
   They certainly deserve to be studied.

2. *Connection to quantum field theory*:
   Certain aspects of quantum field theories are rather mysterious in their conventional formulation, but are manifest when viewed from six dimensions.

3. *Connection to string theory*:
   We can learn more about string theory, while avoiding the conceptual subtleties of quantum gravity.

3 The lower dimensional perspective

Four dimensional quantum field theories are *Yang-Mills gauge theories* (non-linear generalizations of Maxwell's theory of electromagnetism). They are characterized by:

- A gauge group G and matter representation R.

- Some continuous parameters (coupling constant g, θ-angle, masses, Yukawa couplings, mixing matrices, ...)

They can have various amounts N of *supersymmetry*. A theory with $N = 4$ maximally extended SISU is uniquely characterized by its gauge group G and complexified coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

Maxwell's theory of electromagnetism is invariant under 'S-duality', interchanging electricity and magnetism. Montonen and Olive conjectured (1977) that this might generalize to Yang-Mills theory, taking $G \to \hat{G}$ (dual gauge group) and $\tau \to -1/\tau$. Together with $\tau \to \tau + 1$ these transformations generate a group isomorphic to $SL(2, \mathbf{Z})$.

For $G \simeq SU(2)$ spontaneously broken to $H \simeq U(1)$, S-duality predicts existence of a large spectrum of dyonic particles with various electric charges e and magnetic charges q. Osborn pointed out (1979) that this required an $N = 4$ theory. Sen gave striking evidence for S-duality (1994) by constructing the states with magnetic charge $q = 2$. Vafa and Witten tested S-duality (1994) for a topologically twisted Yang-Mills theory.

S-duality is a rather mysterious property of $N = 4$ theories, but it is naturally explained if these theories arise upon compactifying a six-dimensional theory containing strings on a small two-torus T^2:

- The coupling constant τ is interpreted as the geometric modulus ('shape') of the torus.
- Dyonic particles of electric charge e and magnetic charge q correspond to strings winding e times around a-cycle and q times around b-cycle of T^2.
- $SL(2, \mathbf{Z})$ is group of 'large' diffeomorphisms of T^2.

But the question is of course to what extent these properties uniquely define the six-dimensional quantum theories.

4 The higher dimensional perspective

Consider a four-dimensional space $\hat{Q}^4$ of the form

$$\hat{Q}^4 \simeq \mathbf{C}^2/\Gamma.$$

The group Γ belongs to the *ADE-classification* of discrete subgroups of $SU(2)$. Considered as subgroups of $SO(3) \simeq SU(2)/\mathbf{Z}_2$, these are:

- Cyclic groups, A_r, $r = 1, 2, \ldots$.
- Dihedral groups, D_r, $r = 4, 5, \ldots$.
- Tetrahedal group, E_r, $r = 6$.
- Octahedral group, E_r, $r = 7$.
- Icosahedral group, E_r, $r = 8$.

The space $\hat{Q}^4 \simeq \mathbf{C}^2/\Gamma$ is singular at the origin. Resolving the singularity results in a smooth space Q^4. The different resolutions are parametrized by some *moduli* Φ. The point $\Phi = 0$ of the moduli space corresponds to $Q^4 \simeq \hat{Q}^4$.

Q^4 contains r two-spheres S^2. The areas of these two-spheres are determined by the values of the moduli and go to zero at the origin of the moduli space.

The mutual intersection numbers of the two-spheres form the Dynkin matrix of a 'simply laced' Lie algebra:

- $su(r+1)$ for A_r, $r = 1, 2, \ldots$.
- $so(2r)$ for D_r, $r = 4, 5, \ldots$.
- The 'exceptional' Lie algebras E_6, E_7, E_8.

But no Lie group symmetry is apparent in Q^4!

We now consider type IIB string theory on a ten-dimensional space-time of the form

$$Y^{1,9} \simeq M^{1,5} \times Q^4,$$

where $M^{1,5}$ is six-dimensional Minkowski space. We are interested in the six-dimensional low-energy physics. There are then two distinct types of excitations to consider:

- Massless type IIB supergravity fields have zero modes supported on the two-spheres in Q^4. This leads to the appearance of *massless particles* in $M^{1,5}$.
- Tensile type IIB D3-branes may wrap around the two-spheres in Q^4. This leads to the appearance of *tensile strings* in $M^{1,5}$ with the tension given by the D3-brane tension times the area of the two-sphere. In particular, the tension vanishes at the origin of the moduli space.

5 The degrees of freedom

To get an idea of how these particles and strings are organized, we consider a Lie algebra of ADE-type in 'Cartan-Weyl' basis. There are two types of generators:

- The *Cartan generators* are a maximal set of linearly independent generators that commute with each other. We associate them with the massless particles.
- The *root generators* have non-zero eigenvalues with respect to commutation with the Cartan generators. We associate them with the tensile strings.

These assignements may seem artificial, since no Lie group symmetry is apparent in six dimensions, but become natural if we consider the compactification of the theory on a small circle S^1 down to five space-time dimensions. We then find that

- *Massless particles* in six dimensions give rise to *massless particles* in five dimensions (plus a Kaluza-Klein tower of massive particles). These are associated with the Cartan generators.
- *Tensile strings* in six dimensions wound around S^1 give rise to *massive particles* in five dimensions. These are associated with the root generators.

But this looks like the perturbative spectrum of five-dimensional Yang-Mills theory in the Coulomb phase, where the gauge group is spontaneously broken down to its maximal torus by the moduli. Furthermore:

- *Tensile strings* in six dimensions orthogonal to S^1 give rise to *tensile strings* in five dimensions.

But this looks like the non-perturbative magnetically charged solitons of five-dimensional Yang-Mills theory. So it appears that five-dimensional fundamental and solitonic degrees of freedom have a common origin in six dimensions.

One wound now like to understand what the mathematical structure underlying the six-dimensional theories are? This means that we need to find the six-dimensional origins of many well-known concepts in lower-dimensional field theory:

- Lie group symmetry
- Fibre bundles
- The Yang-Mills action
- Spontaneous symmetry breaking
- The Higgs mechanism
- Quantum consistency or renormalizability
- ...

To make progress on these problems, a natural (but probably very difficult) approach would be to work around the origin of the moduli space. But this means that we would need to understand an interacting, strongly coupled six-dimensional conformal theory describing tensionless strings. Furthermore, there is no small dimensionless parameter in which one could do perturbation theory. (Except possibly $1/N$, where $N = r + 1$ is the number of 'colors' for the A_r version of the theory.) Our approach is instead based on working around a generic point of the moduli space, where the theory has characteristic scale $\sqrt{\Phi}$ set by the values of the moduli. If we now consider some process of typical energy E (e.g. the energy of massless particle quanta in a scattering process), we could form a dimensionless parameter $\epsilon = E/\sqrt{\Phi}$, which could be used in a perturbative expansion.

We can now make a rough sketch of the Hilbert space of a six-dimensional quantum theory:

- We begin with a moduli space of inequivalent vacuum states $|\Phi\rangle$ parametrized by the expectation values of the moduli fields ϕ.
- A more general set of states are 'Ground states' $|\mathbf{n}_1, \mathbf{p}_1, \ldots, \mathbf{n}_k, \mathbf{p}_k; \Phi\rangle$ that describe some number k of straight, infinitely extended strings whose spatial directions are given by the vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ and whose spatial momenta are $\mathbf{p}_1, \ldots, \mathbf{p}_k$.
- The complete (perturbative) Hilbert space is now obtained as a Fock space by acting on a ground state with a sequence of creation operators $\alpha^\dagger$. There are two basic types of creation operators: $\alpha^\dagger_{\text{s.t.}}$ create particles propagating in space-time, and $\alpha^\dagger_{\text{w.s.}}$ create waves propagating on the string world-sheet.

An important point is now that when the theory is considered in uncompactified six-dimensional Minkowski space, the ground state data $(\Phi, n_1, p_1, \ldots, n_k, p_k)$ cannot change in process involving a finite amount of energy. This means that sectors with different such data define separate unitary quantum theories. The tensile strings need thus only be 'first quantized'. This is in contrast to the situation after compactification on a circle down to five space-time dimensions. Massive particles (originating from wound strings) can then be pair-created with a finite amount of energy. This means that only the complete theory is unitary, and all particles have to be 'second quantized'.

In the remaining part of these lectures, we will consider the simplest A_1 version of the six-dimensional quantum theories, which reduces to $SU(2)$ Yang-Mills theory upon compactification. This theory describes a single type of particle and a single type of string (together with the anti-string obtained by reversing the orientation of the string world-sheet). Our aim is to construct a supersymmetric model describing the interaction of second quantized particles and a first quantized string. This theory could then be used to calculate e.g. the amplitudes for scattering of particles and string waves.

6 $(2, 0)$ supersymmetry in six dimensions

The six-dimensional quantum theories are invariant under so called $(2, 0)$ supersymmetry. The notation refers to the fact that this algebra has twice as many fermionic generators as the smallest supersymmetry algebra in six dimensions (the $(1, 0)$ algebra), and that these generators are chiral spinors (of the same chirality, as opposed to the $(1, 1)$ algebra which has both chiral and anti-chiral fermionic generators).

The generators of the $(2, 0)$ algebra are

- The six-momentum P.
- The $SO(5, 1)$ Lorentz generators J.
- Generators R of an $SO(5)$ 'R-symmetry' group.
- A one-form central charge Z in the **5** representation of $SO(5)$.
- A self-dual three-form central charge W in the **10** representation of $SO(5)$)
- Chiral supercharges Q in the **4** of $SO(5)$.

The anti-commutator of two supercharges is of the form

$$\{Q, Q\} = P + Z + W.$$

We will have use for two particular representations of this algebra:

- The *tensor particle multiplet* is obtained by taking P lightlike and $Z = W = 0$. It describes massless particles transforming as self-dual tensor, spinors and scalars under the unbroken 'little' subgroup $SO(4) \subset SO(5, 1)$ of the

Lorentz group. (The $SO(5)$ R-symmetry group is unbroken.) Dimensional reduction of this representation gives rise to a massless vector multiplet in five dimensions.

- The *vector string multiplet* is obtained by taking P timelike, $Z = V \otimes \Phi$ and $W = 0$ for some Lorentz vector V such that $P \cdot V = 0$ and some R-symmetry vector Φ. It describes tensile strings (in V direction) in a background with moduli given by Φ. These strings transform as vector, spinors and scalars under the unbroken 'little' subgroup $SO(4) \subset SO(5,1)$ of the Lorentz group. (The R-symmetry is broken to $SO(4) \subset SO(5)$ by the moduli.) Dimensional reduction of this representation gives rise to a massive vector multiplet in five dimensions.

7 Free particles

The Fock space of the tensor particle multiplet can be obtained by quantizing a free field theory in six-dimensional Minkowski space M. The fields are:

- A two-form B in the singlet representation of $SO(5)$ with three-form field strength $H = H_+ + H_- = dB$. The anti self-dual part H_- is however *not* part of the tensor particle multiplet.
- Scalars Φ in the **5** representation of $SO(5)$.
- Chiral spinors Ψ in the **4** representation of $SO(5)$.

Their dynamics is governed by the action

$$S_{TM} = \int_M (H \wedge *H + d\Phi \wedge *d\Phi + \Psi \not{D} \Psi)\,,$$

where the bilinears in Φ and Ψ are $SO(5)$ invariants. The classical equations of motion that follow from this action are

$$\begin{array}{rcll} dH & = & 0 & \text{(Bianchi identity)} \\ d^*H & = & 0 & \\ \Box\Phi & = & 0 & \\ \not{D}\Psi & = & 0. & \end{array}$$

In addition to its manifest Lorentz and R-symmetries, the action is also invariant under $(2,0)$ supersymmetry. Infinitesimal transformations with parameters η that are anti-chiral spinors in the **4** representation of $SO(5)$ act as

$$\begin{array}{rcl} \delta B & = & \eta\Psi \\ \delta\Phi & = & \eta\Psi \\ \delta\Psi & = & \eta(H_+ + d\Phi). \end{array}$$

The transformation law for B implies that

$$\delta H_+ \;=\; \eta d\Psi$$

$$\delta H_- = 0 \quad \text{on shell.}$$

The fact that H_- is invariant on shell and that it does not appear in the right hand sides of the transformation laws of the other fields means that it is not really part of the tensor particle multiplet. We have to include it as a decoupled field, though, in order to construct an action. When we later construct interactions, we have to ensure that H_- remains decoupled from all other degrees of freedom.

8 Free strings

The physical fields on the world-sheet Σ of a string are the Goldstone modes corresponding to the symmetries broken by the string and transform linearly under the unbroken group. We thus get

- Four bosons $\hat{X}$ transforming as a vector under $SO(4) \subset SO(5,1)$ vector and as singlets under $SO(4) \subset SO(5)$.
- Left-moving fermions $\hat{\Theta}_L$ transforming as chiral spinor under both $SO(4)$ groups .
- Right-moving fermions $\hat{\Theta}_R$ transforming as anti-chiral spinor under both $SO(4)$ groups.

Quantization of the zero modes of $\hat{X}$ and their conjugate momenta gives a Heisenberg algebra which can be represented by string states with definite transverse momenta. Quantization of the zero-modes of Θ_L and $\hat{\Theta}_R$ gives a Clifford algebra which is represented on the states of the vector string multiplet described above. Finally, quantization of the non zero-modes of all these fields give rise to a Fock space of string waves.

It would be straightforward to write down a standard free Lagrangian for these massless world-sheet fields. However, in order to facilitate the task of coupling the string to a tensor multiplet, it is convenient to add further world-sheet fields to fill out complete representations of $SO(5,1) \times SO(5)$. We thus take the world-sheet fields to be

- Six bosons X transforming as a vector under $SO(5,1)$ and as singlets under $SO(5)$.
- Fermions Θ transforming as a chiral spinor under $SO(5,1)$ and as a spinor under $SO(5)$.

The $(2,0)$ supersymmetry transformations act on these according to

$$\begin{aligned} \delta X &= \eta\Theta \\ \delta\Theta &= \eta. \end{aligned}$$

To ensure that the added field components do not represent any physical degrees of freedom, we should construct the theory so that it is invariant under world-sheet reparametrizations and a further local fermionic 'κ-symmetry' by which these components can be eliminated.

9 Coupling it together

The couplings between the tensor multiplet and the string are uniquely determined by a few principles, that can be easily understood at least if we temporarily restrict attention to the bosonic degrees of freedom:

- As discussed above, the kinetic terms of the (bosonic) tensor multiplet fields B, Φ, and Ψ is
$$S_{TM} = \int_M (H \wedge *H + d\Phi \wedge *d\Phi) .$$
- The tension of the string is given by the local value of the moduli fields Φ. This leads us to construct a string kinetic term of Nambu-Goto type with the tension given by $\sqrt{\Phi \cdot \Phi}$, i.e.
$$S_{NG} = \int_\Sigma \sqrt{\Phi \cdot \Phi} \sqrt{- \det G}$$
where G denotes the induced metric on the string world-sheet Σ.
- The string couples electrically to the field B. We incorporate this by a standard electric coupling term
$$S_{el} = \int_\Sigma B = \int_D H.$$
Here we have used Stokes' theorem to rewrite the term as an integral over a 'Dirac membrane' world-volume D, the boundary of which is the string world-sheet Σ.
- The string also couples magnetically to the field B. We incorporate this by replacing the field strength $H = dB$ in the tensor multiplet kinetic term with
$$H^{\text{total}} = dB + \delta_D,$$
where δ_D is the Poincaré dual three-form of the Dirac-membrane world volume D. It follows that H^{total} obeys the Bianchi identity
$$dH^{\text{total}} = \delta_\Sigma,$$
where δ_Σ is the Poincaré dual of the string world-sheet Σ.

Given the string world-sheet Σ, the action is in fact independent of the choice of D as long as its boundary is given by Σ. Namely, a change of D changes the Poincaré dual δ_D by an exact form, so H^{total} is invariant if we simultaneously shift B in an appropriate way.

If we rewrite the complete action in terms of $H = dB$, we get a sum of four terms
$$S = S_{TM} + S_{NG} + S_{WZ} + S_{DD}.$$

Here S_{TM} is the original kinetic term for the tensor multiplet fields, and S_{NG} is as described above. The Wess-Zumino term is

$$S_{WZ} = \int_D (H + *H)$$

and the last term S_{DD} is

$$S_{DD} = \int_D *\delta_D.$$

It is now manifest that only the self-dual part of H couples to the string. An important point is that the action is completely unique: There are no coupling constants, neither dimensionful nor dimensionless that can be adjusted.

It remains to complete the action with the fermionic degrees of freedom Ψ in Minkowski space and Θ on the string world-sheet so that the resulting theory is supersymmetric. This is slightly complicated, and we will only give a brief description here: We have already described the supersymmetric tensor multiplet kinetic term S_{TM}. It is straightforward to supersymmetrize also the Nambu-Goto term S_{NG} and the Wess-Zumino term S_{WZ} by simply replacing the Minkowski space fields by superfields. In fact, Φ, Ψ, and H can be seen as component fields of a superfield defined over a superspace with bosonic coordinates x and fermionic coordinates θ. The string world-sheet degrees of freedom X and Θ can then be seen as describing the embedding of Σ in superspace. Furthermore, Σ is the boundary of a Dirac-membrane world-volume D embedded in superspace, and the action has to be supplemented by a fermionic version of S_{DD} to ensure that the choice of D is immaterial. Finally, the supersymmetry laws of the tensor multiplet fields have to be modified with terms supported on D in order for the complete action to be supersymmetric.

10 Classical Thomson scattering

A basic physical process is scattering of tensor multiplet quanta against a straight static string. If we neglect the fermionic degrees of freedom, this can be understood by a classical computation largely analogous to Thomson's calculation of light scattering by an electrically charged particle. An important difference is that Thomson's computation relied on the charge being small (i.e. it was a perturbative calculation in the charge), whereas the mass of the particle is usually taken to be finite. In our case, the charge of the string cannot be taken to be small. Instead, we have to assume that the tension of the string is large compared to the square of the energy of the incoming and outgoing tensor multiplet quanta.

- We start with incoming plane waves of some frequency ω at an angle Θ relative to string. There are three different polarizations of the B field and five different polarizations of the Φ field.
- We compute the induced string vibrations, which are suppressed by the factor $1/\sqrt{\Phi \cdot \Phi}$.

- We compute the reradiation of B and Φ field by the string, which is still suppressed by $1/|\Phi|$.

- Finally, we consider the outgoing radiation in the far-field region. It is still of frequency ω and at the same angle Θ relative to string, but rotated angle φ relative to incoming radiation. There is again a total of eight different polarizations.

- This gives us an 8×8 S-matrix which is a function of the frequency ω and the angles Θ and φ.

To include also the fermionic degrees of freedom, one has to do a quantum computation. Work on this is currently in progress.

References

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