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Kirchhoff's Integral Representation and a Cavity Wake Potential*

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Abstract

A method is proposed for the calculation of the short-range wake field potentials of an ultra-relativistic bunch passing near some irregularities in a beam pipe. The method is based on the space-time domain integration of Maxwell's equations using Kirchhoff's formulation. We demonstrate this method on two cases where we obtain the wake potentials for the energy loss of a bunch traversing an iris-collimator in a beam pipe and for a cavity. Likewise, formulas are derived for Green's functions that describe the transverse force action of wake fields. Simple formulas for the total energy loss of a bunch with a Gaussian charge density distribution are derived as well. The derived estimates are compared with computer results and predictions of other models.

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1 Introduction

The wake fields of a single bunch of charged particles traversing a resonant cavity or periodic accelerating structure is of considerable interest in high-energy particle accelerators, storage rings, light sources and Free Electron Lasers due to the strong effect these fields have on longitudinal and transverse particle dynamics. When a charged bunch is traveling close to the speed of light then the space charge forces are negligible compared

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to forces between thee charge self-field and an external environment. Naturally a correct knowledge of the wake field is also required for wake field accelerators.

The wake fields are usually described by the wake potentials. The wake potential gives us the integrated electromagnetic force response of the environment caused by the diffraction of the "self" field of the leading particle toward a particle behind the lead particle as a function of the distance between the particles. The optimal description of the wake potentials, in particular for computer simulations of beam dynamics, is when fields can be calculated by convoluting the bunch charge density distribution with the Green's wake function. A Green's function or a wake field function represents a wake potential of a point-like charge. Known analytic solution for the wake potential is an infinite sum of the resonant eigenfunctions. The series terms are analytically calculated only for a closed cylindrical cavity [1]. In other cases, computer methods for calculating and summing the series terms are used [2]. Unfortunately the summarized series converges rather slowly, in particular for short distances, therefore a high frequency approximation of the particle resonator model is added to computer summarized series methods [3].

The wake potential also can be evaluated based on the knowledge of the energy loss of a bunch of finite length and the causality condition for particles moving at the speed of light, which means that there is no wake field contribution from the particles which are behind the leading particle. Using an optic's approximation for a bunch self field scattering on a perturbing obstacle we can get obtain simple formulas for the wake potential [4]. Computer methods, which are widely used nowadays, are limited in calculating the wake fields of a point charge, because the computer time increases as the third power of the inverse bunch length.

There is another way to calculate wake potentials [5]. We can use the solution of the Kirchoff's Surface Integral Representation together with the boundary conditions. It may be difficult to get an exact solution over a broad range of distances, however in the region where distances compared to the structure dimensions it is possible to derive an accurate solution . Additionally, we get a set of simple formulas for wake fields of a Gaussian bunch traversing an iris and a cavity.

2 Kirchhoff's Surface Integral Representation

We shall use Maxwell equations in the form of integral relations between the field inside some volume and the field at the surface that covered this volume. This form is called the Kirchhoff's Surface-Integral Representation[6] or a space-time domain Kirchhoff - Kotler - Sobolcv integral equation [7]. The Kirchoff representation of the field inside the volume in terms of the values and its derivatives on the surface also works for the fields at the surface. That is why we can make it to be an integral equation if we include the boundary conditions. We know these conditions if the surface is a metallic surface. The integral equation can then be solved by iterating. However a brief study has revealed the fact that the number of iterations required to obtain a converged solution depends upon the time interval we are interested in. If this time interval multiplied by the speed of light is less than or comparable to the actual structure dimensions then the result of the first iteration is very close to the exact solution.

Consider a scalar field Ψ defined on and inside a closed surface S and satisfying the source-free Maxwell equations in that region. The field Ψ can be thought of a rectangular component of $\overrightarrow{E}$ or $\overrightarrow{H}$. Kirchhoff says that the value of Ψ inside S could be written in terms of the value of Ψ and its normal derivative on the surface as

$$\Psi(\overrightarrow{x},t) = \frac{1(2)}{4\pi}\oint_S\{\frac{1}{R}\overrightarrow{\nabla}'\Psi(\overrightarrow{x}',t')\cdot\overrightarrow{n} - -\frac{\overrightarrow{R}\cdot\overrightarrow{n}}{R^2}(\frac{1}{R}+\frac{\partial}{c\partial t'})\Psi(\overrightarrow{x}',t')\}dS' \qquad (1)$$

Where 1(2) means that value of the scalar field is doubled if it is on the surface. $\overrightarrow{R} = \overrightarrow{x} - \overrightarrow{x}'$ and $R = |\overrightarrow{R}|$, $t' = t - R/c$, $\overrightarrow{n}$ is the outwardly directed unit normal to the surface. From the given representation (1) a vector equivalent can be derived. For the magnetic field we have

$$\overrightarrow{H}(\overrightarrow{x},t) = \frac{1(2)}{4\pi}\oint_S([\frac{\overrightarrow{n}}{R}\times\epsilon_0\frac{d\overrightarrow{E}}{dt'}] - \frac{\overrightarrow{R}\cdot\overrightarrow{n}}{R^2}(\frac{1}{R}+\frac{\partial}{c\partial t'})(\overrightarrow{H}(\overrightarrow{x}',t')\}dS' \qquad (2)$$

3 A mathematical model

Superposition, Causality and the Conservation of energy are the basic principles of this model. The total electromagnetic field ($\overrightarrow{E}_{full}, \overrightarrow{H}_{full}$) in any (accelerating) structure can be considered to be the superposition of the bunch self-field ($\overrightarrow{E}_{bunch}, \overrightarrow{H}_{bunch}$) in a free space and the actual wake field ($\overrightarrow{E}_{wake}, \overrightarrow{H}_{wake}$)

$$\overrightarrow{E}_{full} = \overrightarrow{E}_{bunch} + \overrightarrow{E}_{wake}$$
$$\overrightarrow{H}_{full} = \overrightarrow{H}_{bunch} + \overrightarrow{H}_{wake}$$

We assume that we have a metallic surface and conductivity is infinite. The boundary conditions need to be fulfilled for the total field, so from this condition we have the relations

$$[\overrightarrow{n}\times\overrightarrow{E}_{wake}] = -[\overrightarrow{n}\times\overrightarrow{E}_{bunch}] \qquad (\overrightarrow{n}\cdot\overrightarrow{H}_{wake}) = -(\overrightarrow{n}\cdot\overrightarrow{H}_{bunch}) \qquad (3)$$

These boundary conditions (3) and the Kirchhoff's Surface Integral Representation (2) form an integral equation for the magnetic field H_{wake} at the boundary of a considered volume. We may try to solve this equation by iterations.

According to the energy conservation law the total energy loss U of a bunch is equal to the electromagnetic energy of the wake field left behind. We can calculate the electromagnetic energy in the volume by the time integration of the energy flux $P(t)$ coming through the surface boundary and directed inside the volume

$$U = \int P(t)dt, \qquad (4)$$

For a metallic surface this flux must be zero, however according to the boundary conditions (3) we can present it as a sum of incoming and reflected fluxes, which are equivalent and hence cancel each other at the boundary because they propagate in opposite directions. After we have found the magnetic field H_{wake} at the boundary using the integral equation then we can calculate the energy flux directed inside the volume using only the bunch field component $\overrightarrow{E}_{bunch}$.

$$P(t) = -\oint_S (\overrightarrow{n} \cdot [\overrightarrow{E}_{bunch} \times \overrightarrow{H}_{wake}]) dS$$

On the other hand the energy that a bunch has lost can be present in the form of an integral over the wake field potential $W(s)$ and the bunch charge distribution $q(s)$

$$U = \int W(s) q(s) ds, \tag{5}$$

Because of the energy conservation law we may assume that there must be a relation between the energy flux $P(t)$ and the wake potential $W(s)$

$$\int_{-\infty}^{\infty} \left[W(s)q(s) - P(t = \frac{s}{c}) \right] ds' = 0$$

To calculate the wake field potential we usually use a Green's function $q_{\|}(s)$, which is sometimes called a longitudinal wake function

$$W(s) = \int_0^{\infty} g_{\|}(s') q(s - s') ds' \tag{6}$$

4 Transverse wake function

Here we recall a Green's function $g_{\perp}(s)$, which determines the integrated transverse momentum kick $\Delta p_{\perp}(s)$ due to the action of the wake fields. Using this Green's function we cam calculate a momentum kick as a function of the longitudinal position s in a bunch

$$\Delta p_{\perp}(s) = \int_0^{\infty} g_{\perp}(s') q(s - s') ds'$$

Analogous to the Panofsky-Wenzel theorem for azimuthally symmetric structures, there is a relation between the "transverse" Green's function $g_{\perp}(s)$ and the "longitudinal" Green's function $g_{\|}(s)$:

$$g_{\perp}(s) = -\frac{1}{2} \frac{\partial}{\partial \Delta} \int_0^s g_{\|}(s') ds' \tag{7}$$

Δ is the displacement of the bunch trajectory against the coordinate axis. This statement can be easily derived based on a theorem in the frequency domain and using the Fourier transform.

5 A bunch field

If a bunch moves in a cylindrical tube in the z-direction at a speed close to that of light the bunch self-field has only transverse components

$$\overrightarrow{E}_{bunch} = I(z,t)\overrightarrow{R}(x,y)$$

$$\overrightarrow{H}_{bunch} = \epsilon_0 \overrightarrow{V} \times \overrightarrow{E}_{bunch}$$

$$I(z,t) = \frac{1}{2\pi\epsilon_0} q(s)\delta(s - ct + z)$$

$\delta(z)$ is the delta-function and q(s) is the charge density distribution for a bunch. Function $\overrightarrow{R}(x,y)$ can be expanded in an infinite sum of azimuthal modes. In a cylindrical tube of radius b it is

$$R_r(r,\varphi) = \frac{1}{r} + \sum_{m=1}^{\infty} \left(\frac{\Delta}{r}\right)^m \left[1 + \left(\frac{r}{b}\right)^{2m}\right] \frac{cos(m\varphi)}{r}$$

$$R_\varphi(r,\varphi) = \sum_{m=1}^{\infty} \left(\frac{\Delta}{r}\right)^m \left[1 - \left(\frac{r}{b}\right)^{2m}\right] \frac{sin(m\varphi)}{r}$$

m is the azimuthal number, Δ is, as it was defined before, the displacement of the bunch trajectory against the coordinate axis.

6 An iris in a cylindrical pipe

The geometry of an iris in a cylindrical pipe is a good example for us to demonstrate this method. Let b, a be consequently the radius of a pipe and the radius of the hole in an iris. The energy flux $P_m(t)$ of the m-th azimuthal field mode from both sides of an iris is

$$P_m(t) = 2\Delta^{2m} \frac{q(ct)c}{4\pi^2\epsilon_0} \int_a^b \frac{dr}{r^m} \int_a^b \frac{dr_0}{r_0^m} \int_0^{2\pi} \frac{\partial q(s')}{\partial s'} \frac{cos(m+1)\varphi}{R} d\varphi$$

where

$$s' = ct - R$$

$$R = \sqrt{r^2 - 2rr_0 + r_0^2}$$

To make the integration easier we assume that $s' << b-a$, so we may ignore the reflection from a corner. Next we change the variables in this integral and do an integration by parts to have the same presentation for the electromagnetic flux, as for the wake potential (5)

$$P(t = s/c) = q(s)W(s)$$

$$P(s/c)/q(s) = \frac{1}{\pi\epsilon_0} \left(\frac{\Delta}{a}\right)^{2m} \left\{q(s)K_m - \frac{1}{a\pi} \int_0^\infty q(s')F_m(\frac{s-s'}{a})ds\prime\right\} \quad (8)$$

function K_m and $F_m(x)$ are:

$$K_m = \ln(\frac{b}{a}) \quad if \quad m = 0; \qquad K_m = \frac{1}{2m} \quad if \quad m \geq 1$$

$$F_m(x) = \frac{1}{x}\int_0^{\alpha} \frac{\cos(m\varphi - \psi)}{\cos\psi\,(\cos\varphi + x\cos\psi)^m} d\varphi$$

$$\tan\psi = \frac{\sin\varphi}{\sqrt{x^2 - sin^2\varphi}}$$

$$\alpha = \arccos(1 - \frac{x^2}{2}), \qquad for \quad 2 \geq x \geq 0;$$

$$\alpha = \pi, \qquad for \quad x \geq 2$$

For the main mode ($m = 0$) this function has a more simpler presentation:

$$F_0(x) = \frac{2}{x}\arcsin(\frac{2}{x}), \qquad if \quad x < 2$$

$$F_0(x) = \frac{\pi}{x}, \qquad if \quad x \geq 2$$

We may suggest that we have found a formula for a wake potential (9), because it has the same presentation as (6) and we may assume that the Green's function for this case is

$$g_{\|}(s) = \frac{1}{\pi\epsilon_0}\left(\frac{\Delta}{a}\right)^{2m}\left\{K_m\delta(s) - \frac{1}{\pi a}F_m(\frac{s}{a})\right\} \tag{9}$$

The first part in this formula is a well known optical estimate of the wake potential.The second part gives an acceleration for the particles, which may follow a bunch. We check this result for the main mode ($m = 0$) using a computer code NOVO [8] and we found exactly same result, which is shown in Fig.1.

According to the derived formula the amplitude of the longitudinal wake field experienced by a central particle in a Gaussian bunch of the length σ is

$$W(s=0) = \frac{2Q}{(2\pi)^{3/2}\epsilon_0\sigma}\left\{\ln\frac{b}{a} - \frac{\sigma}{a\sqrt{2\pi}}\right\} \tag{10}$$

and the total energy loss of a bunch is

$$U = \frac{Q^2}{(2\pi)^{3/2}\epsilon_0\sigma}\left\{\ln\frac{b}{a} - \frac{\sigma}{a\sqrt{\pi}}\right\}$$

Green's function for the transverse kick is

$$g_{\perp}(s) = \frac{\Delta^{2m-1}}{2\pi\epsilon_0 a^{2m}}\left\{1 - \frac{2m}{\pi}\int_0^{s/a} F_m(s')ds'\right\}$$

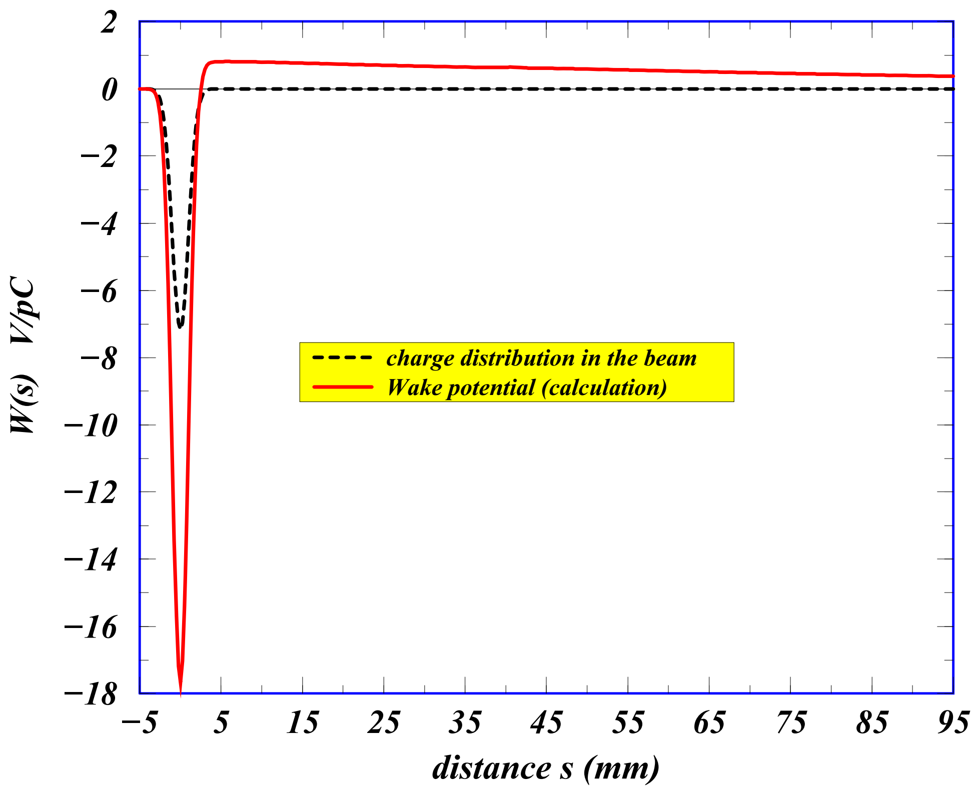

Figure 1: Wake potential of a 1 mm bunch passing an iris with a hole radius of 20 mm in a cylindrical pipe of radius of 70 mm

7 A single cavity

Let's consider a single cavity to be a pill box with incoming and outgoing pipes of infinite length. An additional geometric parameter to the pipe radius a and a cavity radius is the cavity gap length g. We assume that a bunch travels from the left to the right. The field energy fluxes are coming from two sides of the cavity. These fluxes are not summed but partially compensated. The field H_l, which is excited at the left side of a cavity, moves to the right side and then reflected back at the time when the other field energy flux begins at the right side. So the total flux is

$$P(t) = 2\int_a^b rdr \int_0^{2\pi} E_b(t)[H_r(t) - H_l(t + g/c)]d\phi$$

Inserting a bunch field into this formula gives us a formula for the wake function

$$g_{\parallel}(s) = \frac{\Delta^{2m}}{\pi^2\epsilon_0 a^{2m+1}} \left\{ \frac{s+g}{\sqrt{s(s+2g)}} F_m(\frac{\sqrt{s(s+2g)}}{a}) - F_m(\frac{s}{a}) \right\}$$

Comparison of the wake field potential, derived with this wake function and the result of a numerical calculation is presented at Fig. 2.

In a single cavity, for small distances s the wake function scales inversely with the square root of distance and become infinite at a vanishing distance (in ultra relativistic case)

$$w(s) = \frac{Z_0 c}{2\pi^2}\frac{1}{\sqrt{s_0 s}} \qquad\qquad s_0 = \frac{a^2}{2g}$$

This result is in very good agreement with the formula derived from the diffraction model [9] and [10]. However our result contains two interesting points, where a function F_0 has a second order singularity $F_0(x) = \pi$

$$s(s+2g) = (2a)^2 \qquad\qquad s = 2a$$

These points can be used for comparison of this formula and the numerical results, which are shown in Fig.3, 4 and 5. The same behavior of the wake field can be seen in other cavities, even of very complicated geometry. In Fig.6 the wake field in the 9-cell TESLA cavity is presented. The peak at the distance of the double aperture size is clearly seen. The equivalent gap is around 93 mm. Fig.7 shows the wake potential another very complicated cavity - a PEP-II cavity. We can see the same irregularly points.

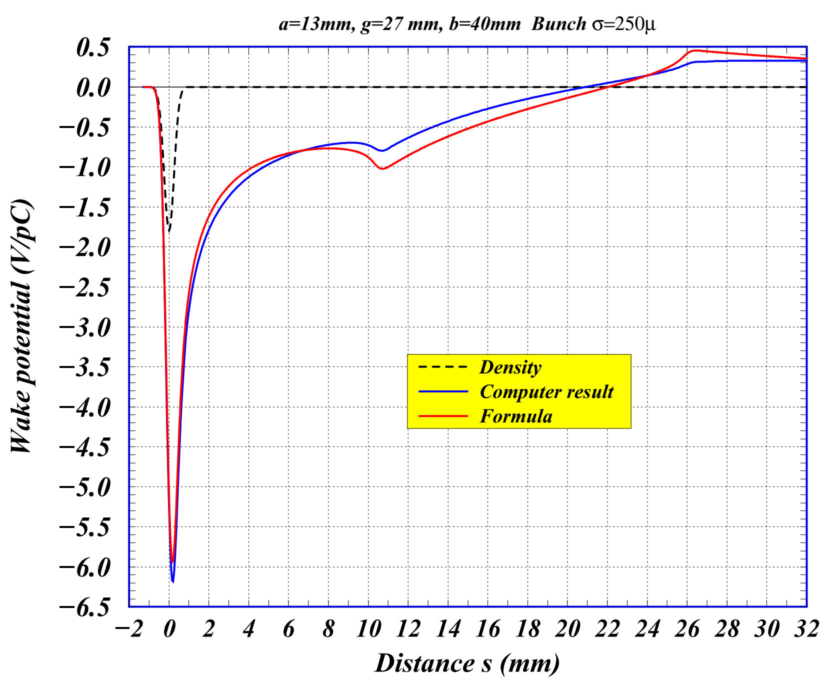

Figure 2: Comparison of the analytical and numerically computed wake potentials.

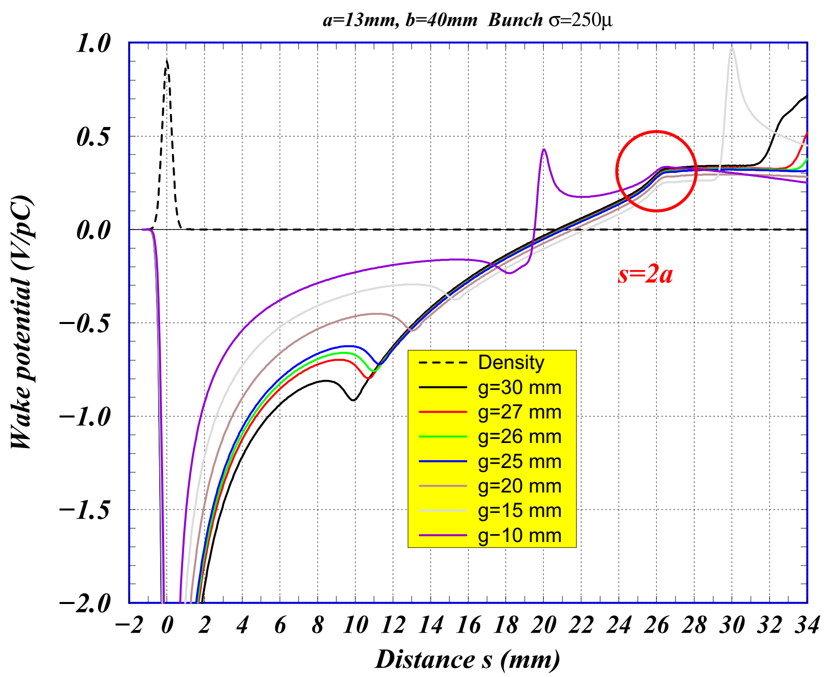

Figure 3: Wake potential of cavities of different gap size, but fixed radius of aperture.

Finally we derive a transverse wake function for a cavity

$$g_{\perp}(s) = \frac{m\Delta^{2m-1}}{2\pi\epsilon_0 a^{2m}} \left\{ 1 - \frac{2m}{\pi} \int_0^{s/a} F_m(s')ds' \right\}$$

8 Conclusion

Kirchhoff's Integral representation of Maxwell's equation is a useful instrument for wake field studies. We are able to easily derive formulas for simple geometries.

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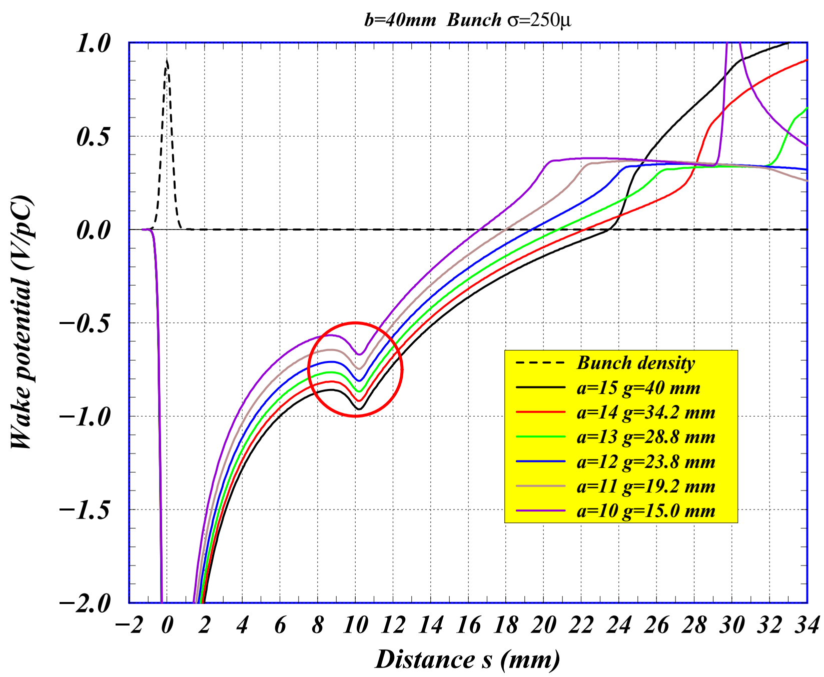

Figure 4: Wake potentials of cavities of the fixed time flight.

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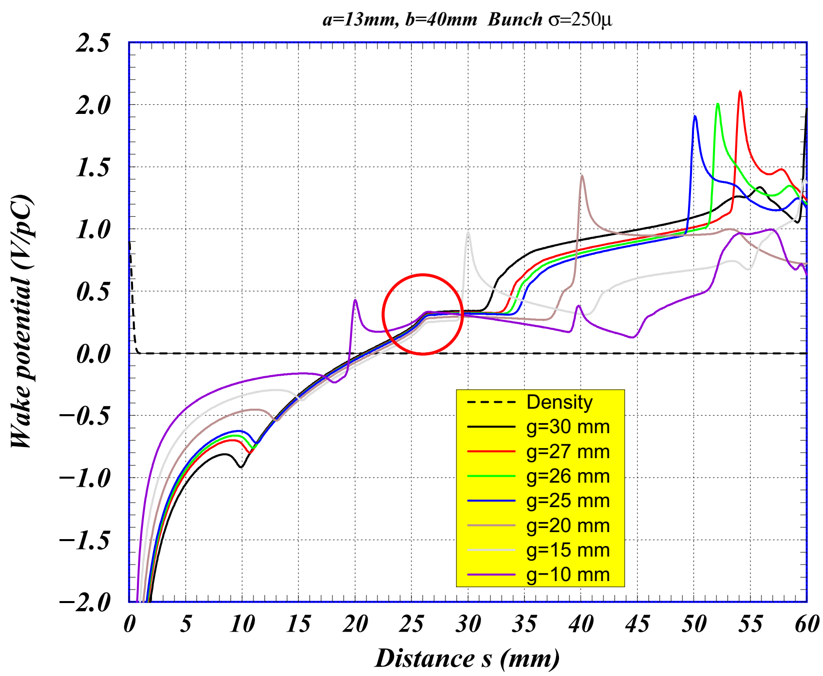

Figure 5: Peaks in wake potentials at the distance, equal to double size cavity gap.

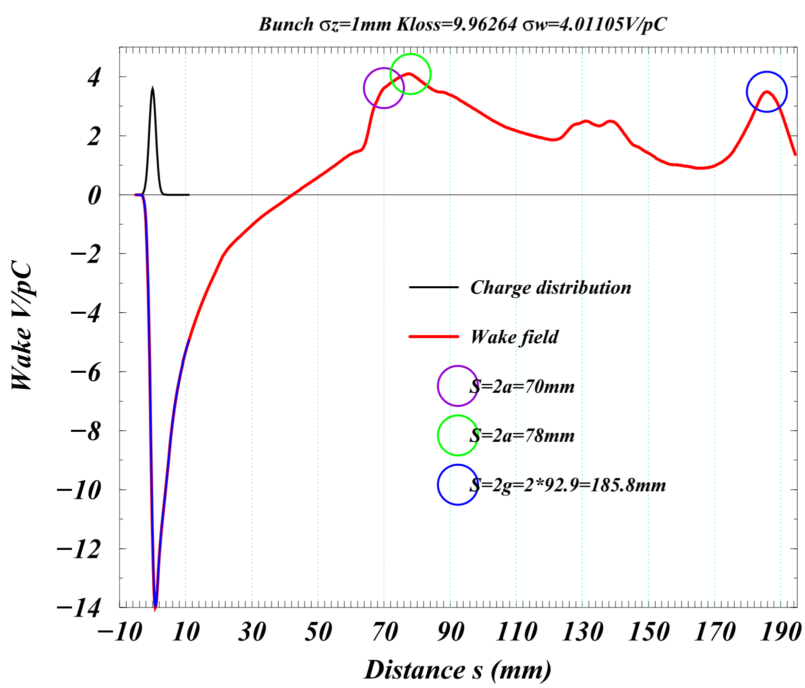

Figure 6: Wake potential of the TESLA cavity.

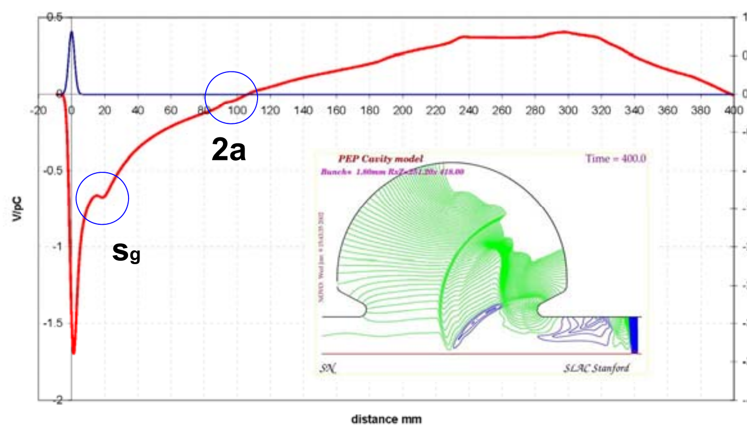

Figure 7: Wake potential of the PEP-II cavity for a 1.8 mm bunch.