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Stochastic fluctuations in multi-field inflation

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Abstract. We develop a stochastic framework for describing fluctuations in multi-field inflationary models. Unlike most previous such approaches, metric perturbations are taken into account and the basic dynamical quantities are invariant under changes of time slicing.

Inflation [1], an early period of accelerated expansion of the universe, is a concept which can explain many special features of the cosmos on large scales. Its most important merit is the provision of a natural mechanism for the creation of fluctuations in an otherwise homogeneous universe [2]. These fluctuations act as the primordial seeds which later lead to the formation of structures such as galaxies and clusters of galaxies. There are many models which implement inflation and practically all of them invoke one or more scalar fields to drive the accelerated expansion phase. Such a phase is capable of producing fluctuations in the energy density by amplifying the quantum fluctuations of any light scalar field present during inflation.

The study inflationary fluctuations has become a major activity in cosmology over the last two decades [3]. They are usually represented as linearised deviations from a homogeneous spacetime and are therefore described by non-interacting quantum fields. Linearity, along with the additional assumption that the initial state is the vacuum (as defined on scales much smaller then the Hubble radius), lead to the prediction that inflation creates Gaussian fluctuations. The smallness of the observed CMB anisotropy [4] certainly justifies the use of linear perturbation theory as a first approximation. However, some non-linearity will always be present in inflation due to the non-linear nature of gravity and the fact that the fields have a potential $V(\phi)$ which may be not be quadratic. Such non-linearities will lead to non-Gaussianity in the temperature anisotropies of the CMB. Given the increasing precision of CMB observations [5, 6], it is worth considering the possibility that CMB non-Gaussianity may be observable.

In order to address this question one must go beyond standard linear perturbation theory [7]. The extension of the latter to second order in perturbations is straightforward but technically tedious [8] and in such an approach quantisation can only be performed at linear order. A more proper quantum computation has been performed [9] but it refers to single field models and is constrained by the slow-roll assumption. In this paper we offer an alternative methodology to study non-Gaussianity, valid for generic multi-field inflation models without assuming slow-roll. The form of the equations is rather simple and contain non-linearity to all orders, at least at the classical level. Quantum fluctuations are modelled via stochastic noise terms which source the classical equations.

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The use of a stochastic picture to simulate quantum fluctuations in inflation is quite old [10]. In the simplest case of a scalar field evolving in deSitter space with Hubble rate H, one can split the inhomogeneous part of the field into long ($k < caH$) and short ($k > caH$) wavelength modes and derive for the slowly evolving long wavelength part

$$\dot{\phi} = -\frac{1}{3H}\frac{\partial V}{\partial \phi} + f(t, \mathbf{x})\,, \tag{1}$$

with f is a random source term satisfying

$$\langle f(t, \mathbf{x}) f(t', \mathbf{x}')\rangle = \frac{H^3}{4\pi^2}\delta(t - t')\frac{\sin(\tilde{c}aH|\mathbf{x} - \mathbf{x}'|)}{\tilde{c}aH|\mathbf{x} - \mathbf{x}'|}\,. \tag{2}$$

In the above, $\tilde{c}$ is an arbitrary number defining the separation between the short and the long wavelength regime. The quantum nature of the scalar field can adequately be described by a classical probability distribution on long wavelengths, hence the stochastic nature of f.

An equation of this sort provides a relatively simple way to study deviations from linearity since $H^2 \simeq \frac{8\pi}{m_{\rm pl}^2}V$ and $\frac{\partial V}{\partial \phi}$ will depend on ϕ in a non-linear way and introduce mode couplings. Indeed such approaches have been used in the past. Even though it has proved useful, equation (1) has various shortcomings. One is the appearance of $\tilde{c}$ when considering correlations at different spatial points. Since it is a totally arbitrary parameter, separating long and short wavelengths, it should not appear in the final results. Secondly, the time parameter t is supposed to be the cosmic time of homogeneous cosmology. However in the presence of perturbations there is no preferred time slicing. Thirdly, and most importantly, gravitational perturbations are totally ignored. The precision of forthcoming CMB measurements requires more fine tuned computational technology. We present a formalism which does not exhibit such shortcomings. More details and applications can be found in [11, 12, 13].

We start by considering a metric of the form

$$ds^2 = -N^2(t, \mathbf{x})dt^2 + a^2(t, \mathbf{x})h_{ij}(\mathbf{x})dx^i dx^j\,. \tag{3}$$

The matter content of the inflationary era is a set of scalar fields with the energy momentum tensor

$$T_{\mu\nu} = G_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - g_{\mu\nu}\left(\frac{1}{2}G_{AB}\partial^\lambda\phi^A\partial_\lambda\phi^B + V\right)\,. \tag{4}$$

We focus attention on lengths longer than the characteristic scale of the inflationary spacetime, namely the Hubble radius $(aH)^{-1}$. The main approximation is to drop from the Einstein equations terms that contain second order spatial gradients [14, 15]. After ignoring the traceless part of the extrinsic curvature $\bar{K}^i{}_j$ which is a decaying mode on such scales, we arrive at

$$\frac{dH}{dt} = -\frac{4\pi}{m_{\rm pl}^2}N\Pi_B\Pi^B\,, \tag{5}$$

$$\mathcal{D}_t\Pi^A = -3NH\Pi^A - NG^{AB}V_B\,, \tag{6}$$

$$H^2 = \frac{8\pi}{3m_{\rm pl}^2}\left(\frac{1}{2}\Pi_B\Pi^B + V\right)\,, \tag{7}$$

$$\partial_i H = -\frac{4\pi}{m_{\rm pl}^2}\Pi_B\partial_i\phi^B, \tag{8}$$

where $\Pi^B = \frac{\dot{\phi}^B}{N}$ and the symbol $\mathcal{D}$ appearing in (6) is a covariant derivative with respect to the field space metric G_{AB}. In particular, one can define

$$\mathcal{D}_t L^A = \partial_t L^A + \Gamma^A_{BC}\,\partial_t\phi^B L^C, \tag{9}$$

and similarly

$$\mathcal{D}_i L^A = \partial_i L^A + \Gamma^A_{BC}\,\partial_i\phi^B L^C\,, \tag{10}$$

with Γ^A_{BC} the symmetric connection formed from G_{AB}. The quantities $\partial_i\phi^B$ and $N\Pi^B$ transform as vectors in field space but ϕ^B does not. The latter is just a set of n scalar functions (w.r.t both field and spacetime transformations) parameterising the field manifold.

Cosmological perturbations are quantised in linear theory through the Sasaki-Mukhanov variables

$$\delta q^A = a\left(\delta\phi^A - \frac{\dot\phi^A}{H}\Psi\right), \tag{11}$$

where $\Psi = \delta a/a$ is the perturbation in the trace of the spatial metric. An appropriate non-linear generalisation of these variables is

$$\mathcal{Q}^A_i = a\left(\partial_i\phi^A - \frac{\Pi^A}{H}X_i\right), \tag{12}$$

with $X_i = \partial_i \ln a$. These combinations of spatial gradients have the same value for all choices of the lapse function N in the long wavelength limit. Their linear part is the gradient of (11). From (5) - (8) and using linear theory to describe short wavelengths, we derive the following equations for (12)

$$\mathcal{D}_t^2\mathcal{Q}^A_i - \left(\frac{\dot N}{N} - NH\right)\mathcal{D}_t\mathcal{Q}^A_i + \Omega^A{}_B\mathcal{Q}^B_i = \int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}\,k_i\,\xi^A(\mathbf{k})\,\mathrm{e}^{i\mathbf{kx}} + \text{c.c.}\,, \tag{13}$$

with c.c. the complex conjugate,

$$\begin{aligned}\Omega^A{}_B \;&=\; N^2V^A{}_B - \frac{m_{\rm pl}^2}{4\pi}(NH)^2R^A{}_{FCB}\omega^F\omega^C - (NH)^2\Big[(2-\tilde\epsilon)\,\delta^A{}_B \\ &+\; 2\,(3+\tilde\epsilon)\,\omega^A\omega_B + 2\,\sqrt{\tilde\epsilon}\left(\tilde\eta_B\omega^A + \tilde\eta^A\omega_B\right)\Big],\end{aligned} \tag{14}$$

$V^A{}_B = \mathcal{D}_B V^A$, $R^A{}_{FCB}$ is the curvature tensor of the field manifold and

$$\tilde\epsilon = \frac{4\pi}{m_{\rm pl}^2}\frac{\Pi^A\Pi_A}{H^2}\,, \tag{15}$$

$$\tilde\eta^A = \frac{1}{N}\frac{\mathcal{D}_t\Pi^A}{H\Pi}\,, \tag{16}$$

$$\omega^A = \sqrt{\tilde\epsilon}\frac{\Pi^A}{\Pi}\,. \tag{17}$$

The right hand side of (13) is a stochastic source term:

$$\xi^A(\mathbf{k}) = \delta q^A(\mathbf{k})\ddot{\mathcal{W}}(k) + \left[2D_t\delta q^A(\mathbf{k}) - \left(\frac{\dot N}{N} - NH\right)\delta q^A(\mathbf{k})\right]\dot{\mathcal{W}}(k) - \left(\frac{Nk}{a}\right)^2\mathcal{W}(k)\delta q^A(\mathbf{k})\,, \tag{18}$$

$$\delta q^A(\mathbf{k}) = \frac{1}{\sqrt2}\mathcal{Q}^A{}_B(k)\alpha^B(\mathbf{k})\,. \tag{19}$$

The matrix $\mathcal{Q}^A{}_B(k)$ obeys the linear equation of motion

$$\mathcal{D}_t^2\mathcal{Q}^A{}_B - \left(\frac{\dot N(t)}{N(t)} - N(t)H(t)\right)\mathcal{D}_t\mathcal{Q}^A{}_B + \left(\Omega^A{}_C(t) + \left(\frac{N(t)}{a(t)}k\right)^2\delta^A{}_C\right)\mathcal{Q}^C{}_B = 0\,. \tag{20}$$

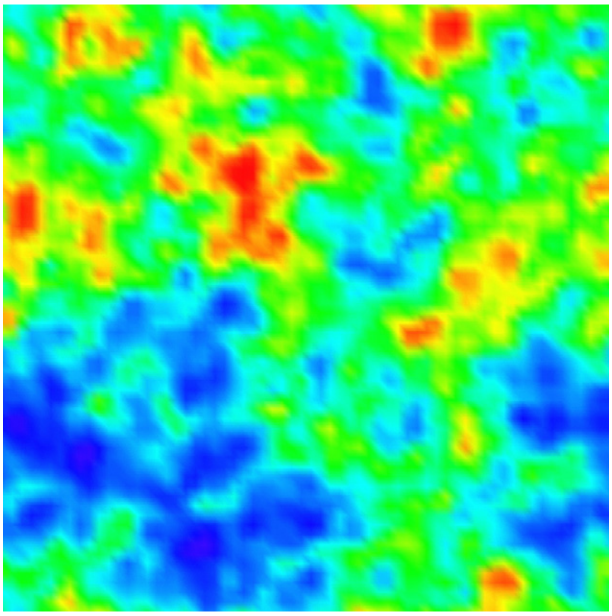

Figure 1. A slice through a small 64^3 simulation of equation (13) for a single field model. Different colours represent fluctuations in the expansion of each spatial point near the end of inflation. The statistics of the fluctuations show small deviations (skewness) from Gaussianity.

The quantities α^A are complex stochastic constants satisfying

$$\langle \alpha^A(\mathbf{k})\,\alpha^{B*}(\mathbf{k}')\rangle = \delta^{AB}\delta(\mathbf{k}+\mathbf{k}')\,, \tag{21}$$

$$\langle \alpha^A(\mathbf{k})\,\alpha^{B}(\mathbf{k}')\rangle = 0\,, \tag{22}$$

where $\langle ...\rangle$ denotes an ensemble average and $\mathcal{W}(k)$ is an appropriate window function that filters out short wavelength modes. So far no choice of time slicing has been made. It turns out that a useful choice of time for calculating $\xi(k)$ is

$$t = \ln aH\,, \tag{23}$$

which gives $NH = (1-\tilde{\epsilon})^{-1}$.

Equation (13) describes the non-linear evolution on long wavelengths, sourced by random linear short-wavelength fluctuations. The coefficients appearing on the left hand side depend on $\mathcal{Q}_i^A$. For the gauge choice (23) the following constraints hold:

$$X_i = \left(\frac{4\pi}{m_{\rm pl}^2}\right)^{\frac{1}{2}} \frac{1}{a(1-\tilde{\epsilon})}\omega_A \mathcal{Q}_i^A = -\partial_i(\ln H)\,, \tag{24}$$

$$\partial_i\phi^A = \frac{1}{a}\left[\delta^A{}_B + \frac{\omega^A\omega_B}{(1-\tilde{\epsilon})}\right]\mathcal{Q}_i^B\,, \tag{25}$$

and

$$\begin{aligned}\partial_i\Pi^A = {} & \frac{H}{a}(1-\tilde{\epsilon})\mathcal{D}_t\mathcal{Q}_i^A - \frac{H}{a}\left[\delta^A{}_C + \omega^A\omega_C + \left(\frac{m_{\rm pl}^2}{4\pi}\right)^{\frac{1}{2}}\Gamma^A_{BC}\omega^B\right]\mathcal{Q}_i^C \\ & + \frac{H}{a}\left[\sqrt{\tilde{\epsilon}}\tilde{\eta}^A - \left(\frac{m_{\rm pl}^2}{4\pi}\right)^{\frac{1}{2}}\Gamma^A_{BC}\omega^B\omega^C\right]\frac{\omega_D}{(1-\tilde{\epsilon})}\mathcal{Q}_i^D\,. \end{aligned} \tag{26}$$

Equation (13) along with the constraints (24) - (26) provide a stochastic framework for studying the generation of primordial non-Gaussianity from inflation. Unlike most previous approaches, metric perturbations are incorporated and no slow-roll assumption is made. The choice of time slicing is addressed and the basic variables used, the $\mathcal{Q}_i^A$, are invariant under changes of the latter. These features allow for a wide class of inflationary models to be accurately studied using this formalism. This seems to the present authors likely to be a more fruitful and elegant approach to incorporating further nonlinearity than the technically much more complicated alternative of applying higher order perturbation theory to the original Einstein equations [8]. Analytic solutions to the equations can be derived in some simple cases under the slow-roll approximation [13]. However, the stochastic system can be simulated numerically for any model without the need for such approximations. A large code has been developed to perform these nonlinear stochastic simulations for multifield inflation. Figure 1 illustrates preliminary results for a simple single-field inflation model, which confirms analytic expectations of a distinct non-Gaussian signature (see the three-point correlation function discussed in [13]), though one which is well below current observational limits. A number of multifield models are already believed to produce much larger observable non-Gaussian signatures, and using these numerical methods calculations are underway to produce example realizations of cosmological perturbations from these models.

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