Super-extended noncommutative Landau problem and conformal symmetry

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ABSTRACT: A supersymmetric spin-1/2 particle in the noncommutative plane, subject to an arbitrary magnetic field, is considered, with particular attention paid to the homogeneous case. The system has three different phases, depending on the magnetic field. Due to supersymmetry, the boundary critical phase which separates the sub- and super-critical cases can be viewed as a reduction to the zero-energy eigensubspace. In the sub-critical phase the system is described by the superextension of exotic Newton-Hooke symmetry, combined with the conformal $so(2,1) \sim su(1,1)$ symmetry; the latter is changed into $so(3) \sim su(2)$ in the super-critical phase. In the critical phase the spin degrees of freedom are frozen and supersymmetry disappears.

Contents

1 Introduction

The Landau problem in the noncommutative plane belongs to a family of non-relativistic systems with so called "exotic" symmetries, meaning that the symmetry algebras have two central charges. This is possible only in $d = 2 + 1$ dimensions; for $d \neq 2 + 1$, the mass is the unique central charge [1].[1] Examples of doubly centrally extended symmetries are provided by exotic Galilei [3–9], and exotic Newton-Hooke (ENH) [10–12] symmetries. In both cases, the second central charge corresponds to the non-commutativity of the boost generators. The non-commutative Landau problem (NCLP) carries, in particular, an exotic Newton-Hooke symmetry [13], which becomes exotic Galilean symmetry in the free limit.

The NCLP was investigated in the context of the quantum Hall effect [7], and the physics of anyons [14]. In the spinless case it possesses three different phases [14],[2] namely a sub- ($\beta < 1$) and a super- ($\beta > 1$) critical ones, separated by a critical phase when $\beta = 1$. Here $\beta = \theta B$ the product of the (homogeneous) magnetic field $B = const$ and the non-commutativity parameter θ is a dimensionless parameter. The critical phase is characterized by a loss of degrees of freedom. In the generic case of inhomogeneous magnetic field $B(x)$, all three phases can simultaneously be present in total phase space.

Non-relativistic conformal symmetry [17–19] is attracting much current attention, particularly in the context of the AdS/CFT correspondence [20, 21]. The usual ($\theta = 0$) Landau problem has non-relativistic conformal symmetry. This raises the following question: *What happens with this symmetry in the noncommutative case?* A priori, the answer is

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[1]In the relativistic case all central extensions are trivial (central charges can be absorbed by redefining the generators) [2].

[2]Sub- and super-critical phases were discussed before in ref. [15]. The framework used there, however, is only consistent if the magnetic field is homogeneous, see [16]. Neither conformal nor supersymmetric extensions are considered in that paper.

not obvious. The reason is that conformal symmetry involves scale invariance, while the noncommutativity parameter, θ, introduces an independent length scale, $[\theta] = \ell^2$, in addition to those associated with the mass parameter and the magnetic field. Scale invariance, nevertheless, could be expected in the sub-critical phase, whose properties are, in many aspects, similar to those of the usual Landau problem with $\theta = 0$. But it is not clear what happens with this symmetry in the super-critical phase, and for a spin-1/2 particle in particular.

The commutative Landau problem for a non-relativistic electron with spin $\frac{1}{2}$ has an $N = 2$ supersymmetry [22], see also [23], and the conformal $so(2, 1)$ symmetry is extended into a superconformal $osp(2|2)$ symmetry, with the angular momentum as central charge [18, 24]. However, in the NCLP, the angular momentum behaves in an essentially different way in the sub- and super- critical phases: it takes values of both signs for $\beta < 1$, but values of only one sign (depending on the sign of magnetic field) when $\beta > 1$ [14].

The present paper is devoted to the investigation of the symmetries of the super-extended NCLP for a spin-1/2 particle of gyromagnetic ratio $g = 2$.[3]

The paper is organized as follows. In section 2 we start with an arbitrary magnetic field and in section 3 we restrict our considerations to a homogeneous one. We discuss the three phases of the system and the differences between them. Section 4 is devoted to the investigation of the super-symmetries of the system, related to exotic Newton-Hooke symmetry and its extension by adding dilatations and expansions. In section 5 we switch to an alternative basis of quadratic generators that allow us to reveal the essential difference between the sub- and super- critical phases from the viewpoint of conformal symmetry. Section 6 summarizes the results.

2 $N = 2$ supersymmetry: arbitrary magnetic field

The first order Lagrangian of a spinless "exotic" particle in an arbitrary planar magnetic field $B = B(x)$ [7, 13],

$$L = P_i \dot{x}_i - \frac{1}{2m} P_i^2 - \frac{\theta}{2} \epsilon_{ij} \dot{P}_i P_j - \frac{B}{2} \epsilon_{ij} \dot{x}_i x_j, \tag{2.1}$$

corresponds, at the quantum level, to the commutation relations,

$$[x_i, x_j] = i \frac{\theta}{1 - \beta} \epsilon_{ij}, \qquad [x_i, P_j] = i \frac{1}{1 - \beta} \delta_{ij}, \qquad [P_i, P_j] = i \frac{B}{1 - \beta} \epsilon_{ij}, \tag{2.2}$$

where ϵ_{ij} is the antisymmetric tensor with $\epsilon_{12} = 1$.[4] The parameter θ is related to the noncommutativity of the coordinates and has the dimension of a squared length ℓ^2. The dimension of the magnetic field, $B(x)$, is ℓ^{-2}. $\beta = \beta(x) = \theta B(x)$ is then dimensionless.

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[3]For previous works on non-commutative supersymmetry the reader is referred to [25–31], where the superextension of exotic Galilei and Schrödinger symmetries, and some aspects of superextended NCLP were discussed. Questions related to the super-extension of conformal and exotic Newton-Hooke symmetries for the NCLP were not considered there.

[4]We have chosen units $\hbar = 1 = c$ and put the electric charge equal to one.

Spin degrees of freedom are introduced by supplementing (2.2) with

$$\{S_i, S_j\} = \frac{1}{2}\delta_{ij}, \qquad [S_i, x_j] = 0, \qquad [S_i, P_j] = 0. \tag{2.3}$$

The coordinates x_i and momenta P_i are bosonic, while the spin-1/2 operators S_i are fermionic variables. The relations (2.2)–(2.3) specify a consistent quantum structure (Jacobi identities hold for an arbitrary inhomogeneous magnetic field [16]).

In the critical case $\beta = 1$ the bosonic differential two-form associated with (2.2) becomes degenerate, and, in the spinless case, it requires special consideration [7, 14]. In the superextended NCLP we consider below, the critical phase $\beta = 1$ will be obtained by reducing the system to the (infinitely degenerate) zero energy subspace.

The fermionic operators

$$Q_1 = \sqrt{\frac{2}{m}}\, P_i S_i, \qquad Q_2 = \sqrt{\frac{2}{m}}\, \epsilon_{ij} P_i S_j \tag{2.4}$$

generate an $sl(1|1)$ superalgebra [32],

$$\{Q_a, Q_b\} = 2H\delta_{ab}, \qquad [Q_a, H] = 0, \tag{2.5}$$

where

$$H = \frac{1}{2m}P_i^2 - \omega S_3, \qquad \omega = \frac{B}{m^*}, \qquad m^* = m(1 - \beta), \tag{2.6}$$

and S_3 is defined by

$$S_3 = -i\epsilon_{ij}S_i S_j. \tag{2.7}$$

Choosing H as the Hamiltonian, we get a system that generalizes the usual $N = 2$ supersymmetry of a spin-1/2 particle with gyromagnetic ratio $g = 2$ in arbitrary magnetic field [32] to the non-commutative case. The operator S_3, like the supercharges Q_a, is an integral of the motion, which acts as a $\mathbb{Z}_2$-grading operator Γ for the $N = 2$ supersymmetry, $\Gamma = 2S_3$, $\Gamma^2 = 1$. One can choose, in particular, a representation where S_3 is proportional to the diagonal Pauli matrix, $S_3 = \frac{1}{2}\sigma_3$.

3 Three phases of the superextended NCLP

From now on we consider a homogeneous field $B = const \neq 0$. It is convenient to define a linear combination of the bosonic operators x_i and P_i,

$$\mathcal{P}_i = P_i - B\epsilon_{ij}x_j. \tag{3.1}$$

For nonzero magnetic field, the set formed by the $\mathcal{P}_i$ and P_i is an alternative to the initial set of bosonic variables. The advantage is that the $\mathcal{P}_i$ commute with the P_j. From the form of the Hamiltonian (2.6) we infer that $\mathcal{P}_i$ is an integral of the motion. Since $[x_i, \mathcal{P}_j] = i\delta_{ij}$ and

$$[\mathcal{P}_i, \mathcal{P}_j] = -iB\epsilon_{ij}, \tag{3.2}$$

the $\mathcal{P}_i$, $i = 1, 2$, are identified as the non-commuting generators of space translations. Another bosonic operator,

$$J = \frac{1}{2B} \left(\mathcal{P}_i^2 - (1 - \beta) P_i^2 \right) + S_3 = \frac{B}{2} \left(x_i + \frac{1}{B} \epsilon_{ij} P_j \right)^2 - \frac{1 - \beta}{2B} P_i^2 + S_3, \qquad (3.3)$$

is identified as the angular momentum, since it generates the rotations, $[J, R_i] = i\epsilon_{ij} R_j$ for $R_i = x_i$, P_i, S_i.

Putting $\varepsilon_z = \text{sgn}(z)$, we define bosonic and fermionic creation and annihilation operators

$$a^+ = (a^-)^\dagger = \sqrt{\frac{|1 - \beta|}{2 |B|}} \left(P_1 - i\varepsilon_B \varepsilon_{(1-\beta)} P_2 \right),$$

$$b^+ = (b^-)^\dagger = \frac{1}{\sqrt{2 |B|}} \left(\mathcal{P}_1 + i\varepsilon_B \mathcal{P}_2 \right), \qquad (3.4)$$

$$f^+ = (f^-)^\dagger = S_1 - i\varepsilon_B \varepsilon_{(1-\beta)} S_2. \qquad (3.5)$$

Their nonzero (anti)-commutation relations are

$$\left[a^-, a^+ \right] = 1, \qquad \left[b^-, b^+ \right] = 1, \qquad \left\{ f^-, f^+ \right\} = 1. \qquad (3.6)$$

The definition of the bosonic creation and annihilation operators depends, in view of (2.2) and (3.2), on the signs of the magnetic field and of the quantity $1 - \beta$ that defines the phase of the system. The dependence is included into the definition of the fermionic operators. This allows us to present the Hamiltonian in both the sub- ($\beta < 1$) and super- ($\beta > 1$) critical phases in a universal form,

$$H = |\omega| \left(N_a + N_f \right). \qquad (3.7)$$

Here we introduced the bosonic, $N_a = a^+ a^-$ and $N_b = b^+ b^-$, and fermionic, $N_f = f^+ f^-$, number operators with eigenvalues $n_a, n_b = 0, 1, \ldots,$ and $n_f = 0, 1$, respectively. The angular momentum reads,

$$J = \varepsilon_B \left(N_b + \frac{1}{2} \right) - \varepsilon_B \varepsilon_{(1-\beta)} \left(N_a + N_f \right). \qquad (3.8)$$

According to (3.7), in both non-critical phases the system has a typical $N = 2$ supersymmetric spectrum with zero ground state energy corresponding to $n_a = n_f = 0$, and supersymmetric energy doublets with quantum numbers $n_a > 1$, $n_f = 0$, and $n_a - 1$, $n_f = 1$, respectively. Each energy level has an additional infinite degeneracy ($n_b = 0, 1, \ldots$), associated with the translation invariance generated by the $\mathcal{P}_i$.

On the other hand, eq. (3.8) reveals the essential difference between two non-critical phases. In the subcritical phase, the angular momentum takes half-integer values of any sign, while in super-critical phase it only takes half-integer values of one sign (the sign of the magnetic field).

It is worth noting that the difference between the two non-critical phases reveals itself also in another aspect. Proceeding from the quantum structure (2.2), one can construct

vector variables q_i and p_i with canonical commutation relations $[q_i, q_j] = [p_i, p_j] = 0$, $[q_i, p_j] = i\delta_{ij}$. Up to a unitary transformation, they can be presented in a simple form in terms of the mutually commuting operators $\mathcal{P}_i$ and P_i,

$$q_i = \frac{1}{B}\epsilon_{ij}\left(\mathcal{P}_j - \sqrt{1-\beta}\,P_j\right), \qquad p_i = \frac{1}{2}\left(\mathcal{P}_i + \sqrt{1-\beta}\,P_i\right). \tag{3.9}$$

In the limit $B \to 0$, q_i becomes the canonical coordinate for a free particle on the non-commutative plane, $q_i = x_i + \frac{\theta}{2}\epsilon_{ij}P_j$, and $p_i = P_i$, see refs. [7–9]. Eq. (3.9) provides us with canonical coordinates and momenta both in the sub- and supercritical phases. However, in the super-critical case, unlike in the sub-critical phase, the operators (3.9) are *non-hermitian.*

Consistently with eq. (2.6), in the limit $\beta \to 1$ the frequency tends to infinity, $|\omega| \to \infty$. The critical phase $\beta = 1$ can be obtained by reduction of the system to the lowest energy level $E = 0$, where $n_a = n_f = 0$. In this phase the system is described by the oscillator variables $b^\pm$ (non-commuting translation generators $\mathcal{P}_i$), and $H = 0$. Taking into account eqs. (2.7) and (3.5), we find that $S_3 = \frac{1}{2}\varepsilon_B\varepsilon_{(1-\beta)}$, i.e., the spin projection is fixed. Curiously, its value depends on the phase from which the reduction is realized. In the critical phase the spin degrees of freedom, like those associated with the bosonic oscillator variables $a^\pm$, are frozen, and supersymmetry disappears.

The Virasoro algebra can be realized in terms of the remaining bosonic integrals $b^\pm$ [33]. Restricting ourselves to integrals of degree not higher than 2 in the operators $b^\pm$, provides us with the symmetry algebra of the planar Euclidean group, spanned by the angular momentum, $J = \varepsilon_B(N_b + \frac{1}{2})$, and by the non-commuting translation generators $\mathcal{P}_i$, see eqs. (3.2) and (3.6).

In what follows we suppose $\beta \neq 1$.

4 Symmetries

Here we identify the other symmetries of the system in addition to those described in the previous section. For this purpose, we consider the Hamiltonian equations of motion,

$$\dot{x}_i = \frac{1}{m^*}P_i, \qquad \dot{P}_i = \omega\epsilon_{ij}P_j, \qquad \dot{S}_i = \omega\epsilon_{ij}S_j. \tag{4.1}$$

In the sub- and super- critical phases the evolution is of the same form, but (assuming a given sign for the field B) the sign of the effective mass, m^*, (and of the frequency, ω,) is opposite for $\beta < 1$ and $\beta > 1$. Remarkably, the same effect can be produced by a time reflection, $t \to -t$. The integration of the equations of motion gives,

$$x_i(t) = \frac{1}{B}\left(\epsilon_{ij}\mathcal{P}_j - \Delta_{jk}^{-1}(t)P_k(0)\right), \quad P_i(t) = \Delta_{ij}^{-1}(t)P_j(0), \quad S_i(t) = \Delta_{ij}^{-1}(t)S_j(0). \tag{4.2}$$

Here $\Delta_{ij}(t) = \cos\omega t\,\delta_{ij} - \sin\omega t\,\epsilon_{ij}$ is a rotation matrix, $\Delta_{ij}^{-1}(t) = \Delta_{ji}(t) = \Delta_{ij}(-t)$ is its inverse, and $\frac{1}{B}\epsilon_{ij}\mathcal{P}_j = \frac{1}{B}\epsilon_{ij}\mathcal{P}_j(0) = x_i(0) + \frac{1}{B}\epsilon_{ij}P_j(0)$.

Now, we identify the boost generators as the integrals which, when acting on $x_i(0)$ and $\dot{x}_i(0)$, produce the necessary form of the infinitesimal transformations, $[x_i(0), \mathcal{K}_j] = 0$,

$[\dot{x}_i(0), \mathcal{K}_j] = -i\delta_{ij}$. This gives $\mathcal{K}_i = m^* (x_i(0) + \theta\epsilon_{ij}P_j(0))$. Using the solution of the equations of motion, the generators can be rewritten in terms of the variables x_i and P_i,

$$\mathcal{K}_i = m^* \left(x_i + \frac{1}{B}\epsilon_{ij}\rho_{jk}P_k \right), \qquad \rho_{jk} \equiv (\delta_{jk} - (1 - \beta)\Delta_{jk}(t)). \tag{4.3}$$

The boost generators (4.3) are dynamical integrals of motion in the sense that they explicitly depend on time, $\frac{d}{dt}\mathcal{K}_j = \frac{\partial}{\partial t}\mathcal{K}_j + \frac{1}{i}[\mathcal{K}_j, H] = 0$.

For $\theta = 0$, (4.3) reproduces correctly the boost generators of the usual Landau problem [34]. In the free case $B = 0$, (4.3) reduces to the boost generators of the free particle in the non-commutative plane.

The commutators between $\mathcal{P}_i$ and $\mathcal{K}_i$ are given by

$$[\mathcal{K}_i, \mathcal{K}_j] = -i\theta m^{*2}\epsilon_{ij}, \qquad [\mathcal{K}_i, \mathcal{P}_j] = im^*\delta_{ij}, \qquad [\mathcal{P}_i, \mathcal{P}_j] = -im^*\omega\epsilon_{ij}, \tag{4.4}$$

and their commutation relations with H are

$$[\mathcal{P}_i, H] = 0, \qquad [\mathcal{K}_i, H] = i(\mathcal{P}_i + \omega\epsilon_{ij}\mathcal{K}_j). \tag{4.5}$$

The bosonic integrals H, J, $\mathcal{P}_i$ and $\mathcal{K}_i$ generate the exotic Newton-Hooke symmetry algebra, in which ω is a parameter, while $C = m^*$ and $\tilde{C} = \theta m^{*2}$ are central charges [13]. The commutators with the supercharges Q_a show that they are translation-, but not boost-invariant,

$$[\mathcal{P}_i, Q_a] = 0, \qquad [\mathcal{K}_i, Q_a] = i(\delta_{a1}\Sigma_i + \delta_{a2}\epsilon_{ij}\Sigma_j). \tag{4.6}$$

Here we have identified a new, fermionic vector generator $\Sigma_i = (1 - \beta)\sqrt{2m}S_i(0)$. This is again a dynamical integral,

$$\Sigma_i = (1 - \beta)\sqrt{2m}\Delta_{ij}(t)S_j. \tag{4.7}$$

The anticommutation relations

$$\{\Sigma_i, \Sigma_j\} = \mathcal{C}\delta_{ij}, \qquad \mathcal{C} = C - \omega\tilde{C} = m(1 - \beta)^2 > 0, \tag{4.8}$$

imply that Σ_i is the square root of a suitable positive definite linear combination of the central charges.

The commutation relations of Σ_i with $\mathcal{K}_i$, $\mathcal{P}_i$, J and H are

$$[\Sigma_i, \mathcal{K}_j] = 0, \qquad [\Sigma_i, \mathcal{P}_j] = 0, \qquad [J, \Sigma_i] = i\epsilon_{ij}\Sigma_j, \qquad [\Sigma_i, H] = i\omega\epsilon_{ij}\Sigma_j. \tag{4.9}$$

As it follows from (4.6), the Σ_i inherit the explicit time dependence of the $\mathcal{K}_i$. The anticommutators with the supercharges Q_a are

$$\{\Sigma_i, Q_a\} = (\delta_{a1}\delta_{ij} - \delta_{a2}\epsilon_{ij})(\mathcal{P}_j + \omega\epsilon_{jk}\mathcal{K}_k). \tag{4.10}$$

The integrals H, J, $\mathcal{P}_i$, $\mathcal{K}_i$, Q_a and Σ_i generate a closed Lie-type superalgebra centrally extended by C and $\tilde{C}$, in which the frequency, ω, plays the role of a parameter.

Let us now inquire about conformal symmetry. To identify its generators, we first consider the direct analogs of the dilatation and special conformal symmetry (expansion) generators of a free particle [20],

$$D = \frac{1}{4m^*} \left(\mathcal{K}_i \mathcal{P}_i + \mathcal{P}_i \mathcal{K}_i \right), \qquad K = \frac{1}{2m^*} \mathcal{K}_i^2. \tag{4.11}$$

The commutation relations,

$$[K, H] = 2iD, \qquad [D, H] = \frac{i}{2} \Big((2 - \beta) H + \omega \left(J + (1 - \beta) S_3 - \omega K \right) \Big), \tag{4.12}$$

$$[K, D] = \frac{i}{2} \Big((2 - \beta) K - m^* \theta \left(J - (1 - \beta) S_3 + m^* \theta H \right) \Big), \tag{4.13}$$

generalize the "exotic" relations found before for a free spinless "Moyal" field [19].

In contrast with the usual spinless and free case $\theta = B = 0$, the commutators do not close to an $so(2,1)$ algebra. But, since J and S_3 commute with H, D and K, one could conclude that we have a kind of central extension of $so(2,1)$. As we shall see below, this is only true in the sub-critical phase, while in the super-critical phase the noncompact $so(2,1)$ algebra is changed into the compact $so(3)$. Since ω plays a role of a parameter in Newton-Hooke symmetry, in the commutative case $\theta = 0$ the relations (4.12), (4.13) correspond to an $so(2,1)$ algebra, centrally extended by $J + S_3$. In non-commutative case $\theta \neq 0$, however, we do not have a Lie-algebraic structure, due to the dependence of the coefficients in (4.12) and (4.13) on the central charges of exotic Newton-Hooke symmetry. In the next section we will consider an alternative choice of the generators that linearizes a superalgebraic structure.

To identify the complete superalgebraic structure, we will also need the commutators of D and K with the other generators of the super-extended exotic Newton-Hooke symmetry. The commutators with $\mathcal{P}_i$, $\mathcal{K}_i$ and Σ_i are

$$[K, \mathcal{P}_i] = i\mathcal{K}_i, \qquad\qquad [K, \mathcal{K}_i] = im^* \theta \epsilon_{ij} \mathcal{K}_j, \qquad\qquad [K, \Sigma_i] = 0, \quad (4.14)$$

$$[D, \mathcal{P}_i] = \frac{i}{2} \left(\mathcal{P}_i + \omega \epsilon_{ij} \mathcal{K}_j \right), \quad [D, \mathcal{K}_i] = \frac{i}{2} m^* \theta \epsilon_{ij} \left(\mathcal{P}_j + \omega \epsilon_{jk} \mathcal{K}_k \right), \quad [D, \Sigma_i] = 0, \quad (4.15)$$

where, again, the nonlinearity is manifest. The commutators with the supercharges Q_a,

$$[K, Q_a] = iZ_a, \qquad [D, Q_a] = \frac{i}{2} \Big((1 - \beta) Q_a + \omega \epsilon_{ab} Z_b \Big), \tag{4.16}$$

generate a new set of the scalar supercharges Z_a,

$$Z_1 = \frac{1}{m^*} \mathcal{K}_i \Sigma_i, \qquad Z_2 = \frac{1}{m^*} \epsilon_{ij} \mathcal{K}_i \Sigma_j, \tag{4.17}$$

with anticommutators

$$\{Z_a, Z_b\} = 2(1 - \beta) \delta_{ab} \left(K + m^* \theta S_3 \right). \tag{4.18}$$

The (anti)-commutation relations of Z_a with other symmetry generators,

$$[Z_a, H] = iQ_a,$$
$$[Z_a, K] = im^*\theta\epsilon_{ab}Z_b,$$
$$[Z_a, D] = \frac{i}{2}(1-\beta)\left(Z_a + m^*\theta\epsilon_{ab}Q_b\right), \tag{4.19}$$
$$[Z_a, \mathcal{P}_i] = i\left(\delta_{a1}\delta_{ij} + \delta_{a2}\epsilon_{ij}\right)\Sigma_j,$$
$$[Z_a, \mathcal{K}_i] = im^*\theta\left(\delta_{a1}\epsilon_{ij} - \delta_{a2}\delta_{ij}\right)\Sigma_j, \tag{4.20}$$
$$\{Z_a, \Sigma_i\} = (1-\beta)\left(\delta_{a1}\delta_{ij} - \delta_{a2}\epsilon_{ij}\right)\mathcal{K}_j,$$
$$\{Z_a, Q_b\} = 2D\,\delta_{ab} - \left(J + (1-\beta)S_3 - \omega K + m^*\theta H\right)\epsilon_{ab}, \tag{4.21}$$

show that a closed super-algebraic structure is obtained, and no new independent integrals are generated. In the commutative case $\theta = 0$ this super-algebraic structure reduces to the Schrödinger superalgebra studied in [18].

5   Alternative basis

In this section we show that, changing the basis of the conformal symmetry generators, the nonlinearity due to the presence of central charges in the coefficients in the (anti)commutation relations can be removed, and we get a certain Lie-superalgebraic extension of the conformal symmetry. The linearization procedure can be extended to include also the generators of translations and boosts, and the vector supercharge Σ_i. We consider

$$\mathcal{J}^0 = \frac{1}{\omega}H + \frac{1}{2}(J + S_3), \qquad \mathcal{J}^1 = \frac{\varepsilon_{1-\beta}}{2\sqrt{|1-\beta|}}\left(\frac{2-\beta}{\omega}H + J + (1-\beta)S_3 - \omega K\right),$$
$$\mathcal{J}^2 = \frac{1}{\sqrt{|1-\beta|}}\,D, \tag{5.1}$$

instead of H, D, K. Note that $\mathcal{J}^1$ and $\mathcal{J}^2$ depend nontrivially on the noncommutative parameter θ via β. All three integrals (5.1) depend only on the bosonic variables $a^\pm$, $b^\pm$ but not on the fermionic operators $f^\pm$,

$$\mathcal{J}^0 = \frac{1}{2}\varepsilon_B(N_b + \varepsilon_{1-\beta}N_a) + \frac{1}{4}\varepsilon_B(1 + \varepsilon_{(1-\beta)}), \tag{5.2}$$

$$\mathcal{J}^1_{\text{sub}} = \frac{1}{2}\varepsilon_B\left(a^+(0)b^+ + a^-(0)b^-\right), \qquad \mathcal{J}^2_{\text{sub}} = \frac{-i}{2}\left(a^+(0)b^+ - a^-(0)b^-\right), \tag{5.3}$$

$$\mathcal{J}^1_{\text{sup}} = \frac{1}{2}\varepsilon_B\left(a^+(0)b^- + a^-(0)b^+\right), \qquad \mathcal{J}^2_{\text{sup}} = \frac{i}{2}\left(a^+(0)b^- - a^-(0)b^+\right), \tag{5.4}$$

where the subscripts *sub* and *sup* refer to the sub- and super- critical phases, and $a^\pm(0) = a^\pm e^{\mp i|\omega|t}$. The commutation relations of $\mathcal{J}^\mu$, $\mu = 0, 1, 2$, are given by

$$\left[\mathcal{J}^1, \mathcal{J}^2\right] = -i\varepsilon_{1-\beta}\mathcal{J}^0, \qquad \left[\mathcal{J}^2, \mathcal{J}^0\right] = i\mathcal{J}^1, \qquad \left[\mathcal{J}^0, \mathcal{J}^1\right] = i\mathcal{J}^2. \tag{5.5}$$

In the sub-critical case this is the $so(2,1) \sim su(1,1)$ algebra, but in the supercritical case the operators (5.1) generate the $so(3) \sim su(2)$ algebra.

Together with the new basis of bosonic generators, we define the following linear combinations of the scalar supercharges Q_a and Z_a,

$$\mathcal{Q}_a^+ = \frac{1}{2|\omega|}\left((1 + \sqrt{|1-\beta|})Q_a + \frac{\omega}{\sqrt{|1-\beta|}}\epsilon_{ab}Z_b\right),$$

$$\mathcal{Q}_a^- = \frac{1}{2|\omega|}\epsilon_{ab}\left((1 - \sqrt{|1-\beta|})Q_b - \frac{\omega}{\sqrt{|1-\beta|}}\epsilon_{bc}Z_c\right). \qquad (5.6)$$

Then, in the sub-critical phase, we get the (anti-)commutation relations

$$\left\{\mathcal{Q}_a^+, \mathcal{Q}_b^+\right\} = 2\varepsilon_B\delta_{ab}\left(\mathcal{J}^0 + \mathcal{J}^1\right), \qquad \left\{\mathcal{Q}_a^-, \mathcal{Q}_b^-\right\} = 2\varepsilon_B\delta_{ab}\left(\mathcal{J}^0 - \mathcal{J}^1\right), \quad (5.7)$$

$$\left\{\mathcal{Q}_a^+, \mathcal{Q}_b^-\right\} = 2\varepsilon_B\left(\delta_{ab}\mathcal{J}^2 + \epsilon_{ab}\frac{1}{2}\left(J + S_3\right)\right), \qquad (5.8)$$

$$\left[\mathcal{J}^1, \mathcal{Q}_a^\pm\right] = -\frac{i}{2}\mathcal{Q}_a^\mp, \qquad \left[\mathcal{J}^2, \mathcal{Q}_a^\pm\right] = \pm\frac{i}{2}\mathcal{Q}_a^\pm. \qquad (5.9)$$

In the super-critical case we instead find

$$\left\{\mathcal{Q}_a^+, \mathcal{Q}_b^+\right\} = 2\varepsilon_B\delta_{ab}\left(\frac{1}{2}\left(J + S_3\right) - \mathcal{J}^1\right), \quad \left\{\mathcal{Q}_a^-, \mathcal{Q}_b^-\right\} = 2\varepsilon_B\delta_{ab}\left(\frac{1}{2}\left(J + S_3\right) + \mathcal{J}^1\right),$$

$$(5.10)$$

$$\left\{\mathcal{Q}_a^+, \mathcal{Q}_b^-\right\} = 2\varepsilon_B\left(\epsilon_{ab}\mathcal{J}^0 - \delta_{ab}\mathcal{J}^2\right), \qquad (5.11)$$

$$\left[\mathcal{J}^1, \mathcal{Q}_a^\pm\right] = \mp\frac{i}{2}\epsilon_{ab}\mathcal{Q}_b^\pm, \qquad \left[\mathcal{J}^2, \mathcal{Q}_a^\pm\right] = \frac{i}{2}\epsilon_{ab}\mathcal{Q}_b^\mp. \qquad (5.12)$$

In both phases, we also have,

$$\left[\mathcal{J}^0, \mathcal{Q}_a^\pm\right] = \pm\frac{i}{2}\mathcal{Q}_a^\mp, \qquad \left[S_3, \mathcal{Q}_a^\pm\right] = i\epsilon_{ab}\mathcal{Q}_b^\pm. \qquad (5.13)$$

Let us emphasise that all these relations are linear, as advertised.

Note that the commutators of the supercharges with $\mathcal{J}^0$ are the same for the sub- and super- critical phases, but those with $\mathcal{J}^1$ and $\mathcal{J}^2$ are different. The relations (5.7), (5.8) and (5.10), (5.11) are transformed mutually under the change $\mathcal{J}^0 \leftrightarrow \frac{1}{2}(J+S_3)$, $\mathcal{J}^1 \leftrightarrow -\mathcal{J}^1$ and $\mathcal{J}^2 \leftrightarrow -\mathcal{J}^2$.

The relations (5.5) and (5.7)–(5.13) show that the system is described by the centrally extended supersconformal $osp(2|2)$ symmetry in the sub-critical phase, and by the analogous Lie-superalgebraic extension of the compact $so(3)$ symmetry in the super-critical phase. In both cases the angular momentum J plays the role of central charge in these superalgebras, while $\mathcal{R} = 2S_3$ is the generator of R-symmetry.

Let us take

$$\mathcal{E}_i^+ = \sqrt{\frac{|1-\beta|}{2|\omega|}}\,\mathcal{P}_i, \qquad \mathcal{E}_i^- = \frac{1}{\sqrt{2|\omega|}}\left(\mathcal{P}_i + \omega\epsilon_{ij}\mathcal{K}_j\right) \qquad (5.14)$$

instead of the translation and boost generators. The nontrivial (anti-)commutators of the integrals $\mathcal{E}_i^\pm$ and Σ_i between themselves are given by (4.8) and

$$\left[\mathcal{E}_i^+, \mathcal{E}_j^+\right] = -\frac{i}{2}\varepsilon_B\mathcal{C}\epsilon_{ij}, \qquad \left[\mathcal{E}_i^-, \mathcal{E}_j^-\right] = \frac{i}{2}\varepsilon_B\varepsilon_{(1-\beta)}\mathcal{C}\epsilon_{ij}. \qquad (5.15)$$

Their (anti)commutation relations with the rest of generators, $\mathcal{J}^\mu$, J, S_3 and $\mathcal{Q}_\alpha^\pm$, are

$$[\mathcal{J}^0, \mathcal{E}_i^\pm] = \pm\frac{i}{2}\epsilon_{ij}\mathcal{E}_j^\pm, \qquad\qquad\qquad [J, \mathcal{E}_i^\pm] = i\epsilon_{ij}\mathcal{E}_j^\pm,$$

$$[S_3, \mathcal{E}_i^\pm] = 0, \tag{5.16}$$

$$[\mathcal{E}_i^+, \mathcal{J}^\mu] = -\frac{i}{2}\varepsilon_{(1-\beta)}\left(\delta^{\mu 1}\epsilon_{ij} + \delta^{\mu 2}\delta_{ij}\right)\mathcal{E}_j^-, \quad [\mathcal{E}_i^-, \mathcal{J}^\mu] = \frac{i}{2}\left(\delta^{\mu 1}\epsilon_{ij} - \delta^{\mu 2}\delta_{ij}\right)\mathcal{E}_j^+,$$

$$\mu = 1, 2, \tag{5.17}$$

$$[S_3, \Sigma_i] = i\epsilon_{ij}\Sigma_j, \qquad\qquad\qquad [J, \Sigma_i] = i\epsilon_{ij}\Sigma_j,$$

$$[\mathcal{J}^\mu, \Sigma_i] = 0, \tag{5.18}$$

$$[\mathcal{E}_i^+, \mathcal{Q}_a^\pm] = \frac{i}{2}\varepsilon_B\left(-\delta_{a1}\epsilon_{ij} \pm \delta_{a2}\delta_{ij}\right)\Sigma_j, \qquad [\mathcal{E}_i^-, \mathcal{Q}_a^\pm] = \frac{i}{2}\varepsilon_B\varepsilon_{(1-\beta)}\left(\pm\delta_{a1}\epsilon_{ij} \mp \delta_{a2}\delta_{ij}\right)\Sigma_j, \tag{5.19}$$

$$\{\Sigma_i, \mathcal{Q}_a^+\} = (\delta_{a1}\delta_{ij} - \delta_{a2}\epsilon_{ij})(\mathcal{E}_j^+ + \mathcal{E}_j^-), \qquad \{\Sigma_i, \mathcal{Q}_a^-\} = (\delta_{a1}\epsilon_{ij} + \delta_{a2}\delta_{ij})(\mathcal{E}_j^+ - \mathcal{E}_j^-). \tag{5.20}$$

The indices of the bosonic generators in (5.14) correspond to the signs in their commutators with $\mathcal{J}^0$ in (5.16).

According to (4.8) and (5.15), even, $\mathcal{E}_i^\pm$, and odd, Σ_i, the integrals generate the superextended two-dimensional Heisenberg algebra with $\mathcal{C}$ as central charge, which, in the commutative case $\theta = 0$, becomes the mass, m. This is the unique central charge of the resulting complete symmetry Lie superalgebra given by eqs. (4.8), (5.5), (5.7)–(5.13), (5.15)–(5.20). For $\theta = 0$ it provides us with an alternative form of the Schrödinger superalgebra [18].

Let us emphasize that the linearization of the unified super-extended exotic Newton-Hooke and conformal symmetries is achieved by inclusion of the dependence on θ in the base changing coefficients. The noncommutativity parameter itself is a function of the exotic Newton-Hooke symmetry central charges, $\theta = \tilde{C}/C^2$.

Note that while the scalar supercharges $\mathcal{Q}_a^\pm$ generate via anticommutation relations the even scalar integrals $\mathcal{J}^\mu$ and $J + S_3$, their anticommutation relations with Σ_i reproduce the even vector generators $\mathcal{E}_i^\pm$. The generator $\mathcal{R} = 2S_3$ of the R-symmetry is related to the vector supercharge Σ_i via one of the Casimir operators of the superalgebra, $i\epsilon_{ij}\Sigma_i\Sigma_j + \mathcal{R}\mathcal{C}$, that takes here zero value.

6 Conclusion and outlook

Let us summarize our results.

We observed that the energy levels in all three phases are infinitely degenerate, due to magnetic translation invariance. In the sub- and super- critical phases nonzero energy levels reveal also an additional double degeneration, associated with $N = 2$ supersymmetry. Due to supersymmetry, the critical boundary phase can be obtained by a simple reduction of the system to the zero energy eigenspace. In this phase one bosonic and the spin degrees of freedom are frozen, and SUSY disappears. The symmetry associated with the integrals

of degree not higher than two in the residual bosonic oscillator variables corresponds to the Euclidean group of motions in two dimensions, generated by non-commuting magnetic translations and by rotations. The angular momentum generator takes here values of one sign that coincides with that of the magnetic field.

The two non-critical phases have essentially different properties. Angular momentum takes half-integer values of both signs in the sub-critical phase, but it takes half-integer values of one sign only (correlated to the sign of the magnetic field) in the super-critical phase. In both of these phases canonical *vector* coordinates and momenta can be constructed from the initial non-commuting coordinates and momenta. In the sub-critical phase such operators are hermitian, but in the super-critical phase they are not hermitian. In the sub-critical phase the bosonic part of the super-conformal symmetry is described by the non-compact $so(2,1) \sim su(1,1)$ algebra. In the super-critical phase it is changed into the compact $so(3) \sim su(2)$ algebra.

When we try to unify the two-fold central extension of the superextended Newton-Hooke symmetry with super-conformal symmetry, the structure coefficients of the symmetry superalgebra transform into certain functions depending on central charges. Linear, Lie-superalgebraic structure can be achieved via the change of the basis with coefficients depending on the noncommutativity parameter θ. The resulting Lie superalgebraic structure has only one central charge.

In ref. [12], it was shown that the exotic Newton-Hooke symmetry with associated coordinate non-commutativity can be obtained from relativistic AdS$_3$ via a certain Wigner-Inönü contraction. It would be interesting to extend the analysis of conformal and supersymmetries to the context of AdS/CFT correspondence [21], and to noncommutative fields [35].

Acknowledgments

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