CROSS-GRADIENT FOCUSING IN LINEAR ACCELERATORS

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(Presented by P. LAPOSTOLLE)

1. INTRODUCTION

It is more of less generally admitted that, in a linear accelerator, it is impossible to get the total stability (that is for phase and transverse motions simultaneously) under the sole effect of the electric field, **whatever the configuration of this field would be,** and that the remedy to this impossibility is to be found only in the use of foils or grids, or else in external focusing. This opinion rests upon an argument due to E. M. McMillan [1] and another from E. L. Ginzton, W. W. Hansen and W. R. Kennedy [2]; in a way or another both reasonings lead to the conclusion that there would be an incompatibility between the harmonic character of the electric potential (expressed by the Laplace equation or Gauss theorem) and the stability requirements.

However, several authors made attempts to overcome this incompatibility. In particular Ya. B. Fainberg has shown [3] that, keeping the cylindrical symmetry of the field, it is possible to imagine for it a periodic structure in order to provide both axial and radial stability, the focusing effect being in this case due to an alternating phase gradient. One can think too of an alternative method of obtaining strong focusing, by abandoning the cylindrical symmetry for the drift tubes: one may, for example, supply them with wodges in the gap region, in order to manage a dissymetric structure of the electris field; this is the idea which will be resumed here. Since the various attempts and the general incompatibility theorem do not agree, it is first necessary to clear up the matter from the theoretical point of view.

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In this respect the main result may be summarized as follows: assuming that the only condition imposed on the electric field of a linear accelerator is to be periodic along the axis (thus disregarding the adiabatic variation of energy), there exists a family of theoretical fields depending on two arbitrary (periodic) functions and two arbitrary (in limited ranges) constants, which simultaneously obey the equation of Laplace and provide the total stability. The incompability between these two requirements occurs only if one limits oneself to very particular sorts of fields (one of which happens to be the conventional linac field).

The general proof of this statement being beyond the frame of this paper, will be given in a later publication; we will only examine here a restricted class of fields, called hereafter «cross-gradient» fields, which seem to be practically workable, and about which we will give direct demonstrations. However, it is not useless to point out briefly the fundamental basis of the compatibility between the Laplace equation and the stability for general fields.

Let us define the following notations to be used in the sequel:

— the axis of the accelerator is taken as the x-axis, y horizontal and z vertical axes;

— ξ, η, ζ are the components of the deviation which goes from the rectilinear trajectory to a neighbouring one **at constant time;**

— v is the velocity of the particle along the x-axis, $\beta = \frac{V}{C}$, and τ is the proper time;

— E_x, E_y, E_z are the components of the electric field, functions of x, y, z, t;

— χ is the ratio of the charge to the rest mass: $\chi = q/m_0$.

On the other hand, we make the following assumptions:

— the field possesses a symmetry plane, for example the xy plane;
— the magnetic field of the electric wave is negligible; as a matter of fact this hypothesis is made only for the sake of simplicity, and one can show that introducing the magnetic field in the calculations brings no qualitative, but only quantitative, changes in the conclusions.

Under these assumptions, one shows that the linear approximation of the equations of oscillations can be written, in the relativistic form

$$\frac{d}{d\tau}\left(\frac{1}{1-\beta^2}\frac{d\xi}{d\tau}\right)=\frac{x}{\sqrt{1-\beta^2}}\frac{\partial E_x}{\partial x}\xi, \quad (1a)$$

$$\frac{d^2\eta}{d\tau^2}=\frac{\chi}{\sqrt{1-\beta^2}}\frac{\partial E_y}{\partial y}\eta, \quad (1b)$$

$$\frac{d^2\zeta}{d\tau^2}=\frac{\chi}{\sqrt{1-\beta^2}}\frac{\partial E_z}{\partial z}\zeta \quad (1c)$$

the gradients $\frac{\partial E_x}{\partial x}$, . . . being evaluated of course along the x-axis.

The equation (1a), which rules the longitudinal motion, can be easily transformed in terms of phase : if ξ is the spatial displacement from the rectilinear trajectory at constant time, the phase-shift $\partial\varphi=-\frac{\omega}{v}\xi$ corresponds to it, and one obtains then for $\partial\varphi$ the equation $\frac{d}{d\tau}\left(\frac{v^2}{1-\beta^2}\frac{d\partial\varphi}{d\tau}\right)+\frac{\chi v}{\sqrt{1-\beta^2}}\frac{\partial E_x}{\partial t}\partial\varphi=0$, of the usual form. But we keep equation (1a) just as it is, since it is more convenient for the subsequent calculations. Furthermore, the coefficients of the second members of (1) obey the Laplace equation

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 \quad (2)$$

(since the magnetic field is neglected).

Let us now examine the case where the field is periodic along x-axis, β being practically a constant within the period. Suppose then, as McMillan assumes, that the stability condition for any of the oscillations $\xi\ \eta\ \zeta$ is that its temporary average coefficient $\overline{\frac{\partial E_x}{\partial x}}$ . . . , is negative; then, owing to the equation (2), it is impossible that the three mean values should be negative at once, and therefore we are faced again with the incompatibility between the Laplace equation and the stability.

But, as it is well known after the numerous studies made about the alternating gradient, the condition $\overline{\frac{\partial E_x}{\partial x}}<0$ is neither necessary nor sufficient to get the stability of the oscillation ξ; and one knows [4] that the characteristic exponent ν_x of this oscillation is given by an expansion which, disregarding the constant $\frac{\chi}{\sqrt{1-\beta^2}}$ begins by

$$\nu_x^2=-\overline{\frac{\partial E_x}{\partial x}}+\left[\int\left(\frac{\partial E_x}{\partial x}-\overline{\frac{\partial E_x}{\partial x}}\right)d\tau\right]^2+\ldots,$$

and similar expressions for ν_y^2 and ν_z^2; thus it is clear that, when the remainder of the series is negligible with regard to the two first terms, a positive value of $\overline{\frac{\partial E_x}{\partial x}}$ can be balanced by the quadratic term, and the equation of Laplace is no more in contradiction with ν_x^2, ν_y^2 and ν_z^2 being simultaneously positive. It can be shown that the phenomenon of the presence of an ever-positive term which can balance the negative value of $\overline{\frac{\partial E_x}{\partial x}}$ . . . , remains valid even when one takes into account the full expressions of ν_x^2, ν_y^2 and ν_z^2.

Thus one sees that the above mentioned incompatibility exists only when one assigns *a priori* to the field a form which prevents from taking advantage of the quadratic term; this will be the case when $\frac{\partial E_x}{\partial x}$ . . . are constants, or nearly so; it will also occur, as we will see below in § 2, when the focal length of the gap-lens repeats itself identically at each period.

Finally, let us examine the argument based on the Gauss theorem. It can be summarized as follows : consider a travelling wave, the electric field of which has the form $E(xyzt)=e(xyz)\cos\omega\left(t-\frac{x}{v}\right)$; in the particle moving frame, the field appears to be static and therefore, as an harmonic potentiel has no extremum (Gauss theorem), there is no stable equilibrium in this frame, thus no stability for the trajectories of the laboratory frame. But, to be valid, this reasoning supposes that $e(xyz)$ is a constant along any parallel to the x-axis; if one introduces a modulation on $e(xyz)$, the field is no more static in the particle frame, and the proof fails; the same will be true for a stationary wave.

Thus it is seen that the theoretical possibilities of alternating focusing are exactly the same in electric fields as in magnetic ones. Unfortunately the practical possibilities are

considerably lower, simply because the electric forces, and the subsequent focusing power are independent of the velocities, while magnetic forces increase with them; thus, an electric focusing device can only be efficient for low energies.

2. TRANSFER MATRICES AND FOCUSING FOR A TWO-LENSES CELL

Consider the cell constituted by two different units OI and IS, each, of them including a zero-field section OG or IR, and an accelerating gap GI or RS which in addition acts

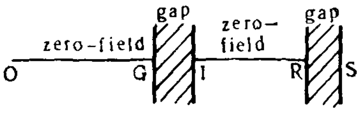

Fig. 1.

as a lens. In order to shorten and simplify the explanation, we suppose that the accelerating gaps can be assimilated to thin lenses for any of the motions $\xi\eta\zeta$; but it will be shown briefly at the end of this paragraph that a thick lens is, in the present case, nearly equivalent to a thin one, which gives quite a good approximation.

We suppose moreover that the transfer matrices of each gap, as related to the proper time τ, have, for any of the oscillation $\xi\eta\zeta$, determinants equal to one (this is rigourously exact for η and ζ but involves a slight approximation for the ξ motion), and we write

$$\begin{vmatrix} 1 & 0 \\ P & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 \\ P' & 1 \end{vmatrix}$$

the transfer matrices corresponding to the gaps GI and RS, the symbols P and P' being indexed by x, y or z according to the oscillation which is considered.

Let us now call θ and θ' the transit (proper) times through the two units. It is easily shown that the $\cos\mu$ of the total transfer matrix of the lens is given by

$$2\cos\mu = 2 + \theta\theta' PP' + (\theta + \theta')(P + P'). \quad (3)$$

In the P, P' plane the values $\cos\mu = \pm 1$ define two hyperbolas H ($\cos\mu = 1$) and h ($\cos\mu = -1$); both these hyperbolas lie in the first and third quadrants of coordinates and have the behaviour shown in Fig. 2; the point Ω is on H and the common centre Γ of H and h has $-\frac{\theta+\theta'}{\theta\theta'}$, $-\frac{\theta+\theta'}{\theta\theta'}$ as its coordinates. One verifies easily that, in order to have stability for the oscillation characterized by P, P', the representative point P, P' must lie inside the region bounded by the two hyperbolas (hatched region in Fig. 2).

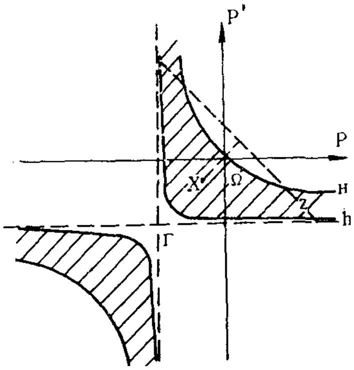

Fig. 2.

Consider now condition (2). For the oscillations η and ζ a classical calculation about thin lenses shows, from equations (1b) and (1c), that one has

$$P_y = \chi \int\limits_{GI} \frac{1}{\sqrt{1-\beta^2}} \frac{\partial E_y}{\partial y} d\tau = \chi \int\limits_{GI} \frac{\partial E_y}{\partial y} dt;$$

$$P'_y = \chi \int\limits_{RS} \frac{\partial E_y}{\partial y} dt, \quad (4)$$

and similar expressions for P_z and P'_z. As for the coefficients P_x and P'_x we see that, if we neglect in equation (1a) the variation of β inside the gap GI or RS, we obtain

$$P_x = (1-\beta_I^2) \int\limits_{GI} \frac{\partial E_x}{\partial x} dt, \quad P'_x = (1-\beta_S^2) \int\limits_{RS} \frac{\partial E_x}{\partial x} dt. \quad (5)$$

Let us also neglect the difference between β_I and β_S; if inserted in equations (4) and (5), the Laplace equation (2) leads to the conditions

$$\frac{P_x}{1-\beta^2} + P_y + P_z = 0, \quad \frac{P'_x}{1-\beta^2} + P'_y + P'_z = 0, \quad (6)$$

the geometric signification of which is obvious: let XYZ be the representative points of the three oscillations — that is the points of respective coordinates (P_x, P'_x), (P_y, P'_y),

(P_z, P'_z) — the origin Ω of the P, P' plane is the centre of gravity of the points XYZ of weights $\frac{1}{1-\beta^2}$, 1,1 respectively.

The possibilities of compatibility appear then immediately.

a) If the two units OI and IS of Fig. 1 are identical (apart from a slight difference due to the variation of energy, that we have already neglected), one has $P_x = P'_x$, $P_y = P'_y$, $P_z = P'_z$; the three points XYZ lie on the first bisectrix (for circular drift-tubes Y and Z are even confounded); it is then impossible, due to the condition (6) of the centre of gravity, that the three points lie together inside the stability region of Fig. 2: one finds the incompatibility of the conventional linac.

b) But suppose we have a «cross-gradient» device that we define as follows (see § 4 for a practical realization) : the field has two symmetry planes xOy and xOz; secondly, in each gap there is a symmetry of the field with respect to the two bisectrices of the yz plane passing by the middle point of the gap; and thirdly the fields of two successive gaps derive one from another by a rotation of $\pi/2$ around the x-axis. Neglecting always the variation of energy between the two gaps of one cell, we have then $P'_x = P_x$, $P'_y = P_z$, $P'_z = P_y$, this is the arrangement seen in Fig. 2 which shows that, if it is possible to separate conveniently Y and Z one from another, the compatibility is obtained. Many other devices are of course possible, such as for example the alternating phase gradient device.

A more elaborate computation can be made in the case where the lenses GI and RS cannot be any longer considered as thin lenses. One can show that the approximation of the thin lens is good apart from a quantity of the third order in the length of the gaps, and we can keep it without great error.

3. CALCULATION OF THE FOCAL COEFFICIENTS P, P', FOR THE CROSS-GRADIENT DEVICE

We have now to make sure whether it is possible to realize a sufficiently great dissymetry between the two representative points Y and Z of the Fig. 2 to produce stability.

Let $e(x)$, $f(x)$, $g(x)$ be the peak values of the field gradients $\frac{\partial E_x}{\partial x}$, $\frac{\partial E_y}{\partial y}$, $\frac{\partial E_z}{\partial z}$ in any point of the gap. For any type of accelerator fed by a stationary wave, we can write :

$$\frac{\partial E_x}{\partial x} = e(x)\cos(\varphi_0 + \omega t),$$

$$\frac{\partial E_y}{\partial y} = f(x)\cos(\varphi_0 + \omega t),$$

$$\frac{\partial E_z}{\partial z} = g(x)\cos(\varphi_0 + \omega t),$$

t being considered in these equations as a function of x defined by the relation $\frac{dx}{dt} = v$. We introduce now the following notations :

— the middle point M of the gap is taken as the origin of abscissas x, and the time at which the particle passes in M as the origin for t; thus φ_0 is the RF phase when the particle passes in M;

— we call $-l$ the abscissa where the field practically arises, and l the abscissa where it falls practically to zero (these two lengths play rather a mathematical part in the following but, as will be seen later on, they are practically of small importance in the expression of the focusing strength); φ_1 and φ_2 will be the phase in $-l$ and $+l$;

— L is the length of the unit (length of zero-field $+2l$);

— λ is the «wave length» corresponding to the particle velocity, that is the product of v by the RF period (in real time) $2\pi/\omega$.

After the equations (4) and (5) we have to calculate quantities such as $\int e(x)\cos(\varphi_0 + \omega t)\,dt$, and so on. In this calculation two difficulties arise, the first one due to the variation of the phase of the cosine together with that of $f(x)$, and the second to the variation of the velocity v, and one has to be careful about these variations in order to avoid setting too rough approximations, as the focal factors P are small. The first difficulty will be overcome by expanding the integrals by partial integration and one can show that, if limiting the developments to the two first terms, the error on P is about 1% for a phase variation $\varphi_2 - \varphi_1 = 50°$, which is quite convenient. On the other hand one may neglect the variation of the velocity, excepted in the first cells of the accelerator; and the error made is about 5% for a variation of velocity of 20%.

Let us write

$$F(x) = \int_0^x f(x)\,dx, \qquad F_1(x) = \int_0^x F(x)\,dx,$$

and similar functions $G(x)$ and $G_1(x)$ for the gradient $g(x)$. With the above mentioned method of approximation we obtain easily

$$P_y = \frac{\chi}{v}[F(l)\cos\varphi_2 - F(-l)\cos\varphi_1] + \frac{\chi\omega}{v^2}[F_1(l)\sin\varphi_2 - F_1(-l)\sin\varphi_1], \quad (7)$$

and an homologous expression for P_z by changing F in G and F_1 in G_1. P_x will be obtained from P_y and P_z by using equation (6).

These formulas are valid for any form of field. Let us now apply them to a cross-gradient

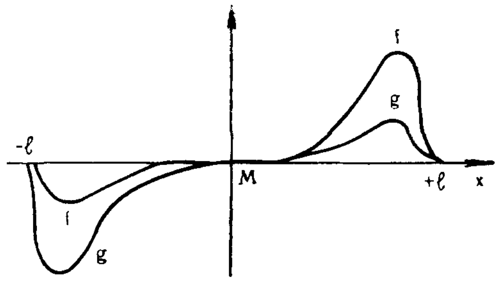

Fig. 3.

field, as defined previously. For such a field, and owing to the symmetries of the device, the functions $f(x)$ and $g(x)$ behave as shown of Fig. 3, one has then

$$f(x) = g(-x),$$

and consequently

$$F(-l) = G(l) \qquad G(-l) = F(l).$$
$$F_1(-l) = -G_1(l) \qquad G_1(-l) = -F_1(l).$$

Furthermore, equation (2) implies $e(x) + f(x) + g(x) = 0$; let F_0 define the peak (with respect to time) value of E_x in the middle point M of the gap, V_0 the peak voltage. By integration of the relation $e + f + g = 0$ between 0 and l one gets easily

$$F(l) + G(l) = E_0, \quad F_1(l) + G_1(l) = F_0 l - \frac{V_0}{2}.$$

The quantities

$$R = F(l) - G(l); \qquad R_1 = 2[F_1(l) - G_1(l)]$$

characterize the dissymetry between the two gradients $\frac{\partial E_y}{\partial y}$ and $\frac{\partial E_z}{\partial z}$. One may notice that one can also write:

$$R = \int_{-l}^{+l} f(x)\,dx = -\int_{-l}^{+l} g(x)\,dx. \quad (8)$$

Let us now come back to equation (7). One can nearly replace $\cos\varphi_2 - \cos\varphi_1$ by $-\sin\varphi_0 \times (\varphi_2 - \varphi_1)$, $\cos\varphi_2 + \cos\varphi_1$ by $2\cos\varphi_0$, and do the same with the sines, moreover with the previously defined notations, one has $\varphi_2 - \varphi_1 = 4\pi l/\lambda$. With these arrangements the equation (7) turns into

$$P_y = -\pi\frac{\chi}{v}\frac{V_0}{\lambda}\sin\varphi_0 + \frac{\chi}{v}\left(R + \frac{2\pi^2 l}{\lambda^2}R_1\right)\cos\varphi_0. \quad (9)$$

P_z in deduced from P_y by changing R in $-R$, R_1 in $-R_1$. From (6) results then

$$\frac{P_x}{1-\beta^2} = 2\pi\frac{\chi}{v}\frac{V_0}{\lambda}\sin\varphi_0. \quad (10)$$

We have thus obtained the focal factors P_x, P_y, P_z for a unit of the cell as functions of the dissymetry of the two transverse gradients, and with an accuracy of about 5%.

Let us now consider the next unit. Owing to the rotation of $\pi/2$ of the field lines in the cross-gradient device, $\frac{\partial E_y}{\partial y}$ and $\frac{\partial E_z}{\partial z}$ exchange themselves, R and R_1 change sign, and one has really $P'_y = P_z$, $P'_z = P_y$ and $P'_x = P_x$ if one neglects, as before, the slight variation of v between the two gaps. Then we are lead to our conclusions concerning stability.

a) From the diagram of Fig. 2, the phase stability required $P_z = P'_x < 0$, therefore $\sin\varphi_0 < 0$ from equation (10): we have obviosly the same condition as in the usual accelerator, since the two gaps of the cell have the same longitudinal gradient.

b) As for the transverse stability and using always the thin lens approximation, we have to refer to equation (3) and insert in its right member the values of P_y and $P'_y = P_z$ in order to get $\cos\mu_y$; obviously $\cos\mu_y = \cos\mu_z$. We can take now $\theta' \neq \theta = \frac{L}{v}\sqrt{1-\beta^2}$, we introduce in addition the following quantities

$$\Delta W = qV_0, \quad W_i = m_0\frac{v^2}{2}, \quad g = \frac{V}{E_0}.$$

ΔW is the maximum possible energy gain per gap (energy gain for $\cos\varphi_0 = 1$); W_i is a fictitious energy which reduces itself to the real kinetic energy W_c at the non-relativistic approximation, and is less than this kinetic energy in the relativistic case, g is a **fictitious length** of the gap, different from the real gap between the electrodes and from the length $2l$ of the field region; obviously $g < 2l$. With these notations, equations (3) and (9) yield

$$4 \sin^2 \frac{\mu}{2} = \frac{2\pi L}{\lambda} \frac{\Delta W}{W_i} \sqrt{1-\beta^2} \sin \varphi_0 \times \times \left(1 - \frac{\pi}{8} \frac{L}{\lambda} \frac{\Delta W}{W_i} \sin \varphi_0 \right) + + \frac{L^2}{g^2} \frac{\Delta W^2}{4W_i^2} (1-\beta^2) \left(1 + \frac{2\pi^2 l}{\lambda^2} \frac{R_1}{R}\right)^2 \times \times \left(\frac{R}{E_0}\right)^2 \cos^2 \varphi_0. \tag{11}$$

Because of the phase stability, the first line of the right member must be negative; then, as previously said, **the possibility of obtaining nevertheless the transverse stability is given by the second line, which is essentially positive, but requires sufficiently high values of the dissymmetry factor $\frac{R}{E_0}$ in order to overcome the first term.**

Let us apply this formula to an accelerator of the Sloan-Lawrence type. We have then $\lambda = 2L$. In a first evaluation we may neglect the quantities $\frac{\pi}{8} \frac{L}{\lambda} \frac{\Delta W}{W_i} \sin \varphi_0$ and $\frac{2\pi^2 l}{\lambda^2} \frac{R_1}{R}$, which are small as compared with one (the first factor is unfavourable, but of a few percent except in the very first sections of the accelerator; the second is of the same order of magnitude, but always positive, that is to say favourable. Equation (11) gives then

$$2 \sin \frac{\mu}{2} = K \sqrt{\frac{\Delta W}{W_i}} \tag{12}$$

with

$$K^2 = \pi \sqrt{1-\beta^2} \sin \varphi_0 + + \frac{L^2}{g^2} \frac{\Delta W}{4W_i} (1-\beta^2) \left(\frac{R}{E_0}\right)^2 \cos^2 \varphi_0. \tag{13}$$

The condition of stability is that μ should be real, that is $0 < K^2 \frac{\Delta W}{W_i} < 4$. As it will appear clearly from the orders of magnitude which will be discussed later on, the right inequality is practically always satisfied. Then the left inequality remains, which is

$$\frac{L^2}{g^2} \frac{\Delta W}{W_i} \left(\frac{R}{E_0}\right)^2 > \frac{4\pi}{\sqrt{1-\beta^2}} \frac{|\sin \varphi_0|}{\cos^2 \varphi_0}. \tag{14}$$

One can obtain another form of this condition by bringing into sight the number of charges n, the atomic mass A in proton units, and the RF frequency ν; this yields to

$$\frac{n}{A} \frac{38 E_0^2}{\nu^2 V_0} \left(\frac{R}{E_0}\right)^2 > \frac{1}{\sqrt{1-\beta^2}} \frac{|\sin \varphi_0|}{\cos^2 \varphi_0}, \tag{15}$$

E_0 being measured in kV/cm, V_0 in kV, and ν in MHz.

The numerical inferences of these formulas will be discussed in the next paragraph. Let us finish this one by the calculation of the acceptances. This does not need going into great detail, the principle of it being the same as in any alternating gradient accelerator. Let us consider, for any of the three oscillations $\xi\eta\zeta$, one of the ellipses invariant by the transfer matrix of the whole cell, and let a be its area. One knows that the maximum possible value of the square amplitude of the oscillation is $\frac{a}{\pi \sin \mu} f$, f being the maximum of the form factor. It can be shown easily, under the assumptions and the approximations already made, that the maximum form factor is $\theta\,(2 + \theta P_{\max})$, $P_{\max}$ being the larger of both focal factors P, P'.

For the transverse oscillations P is given by (9) (with $R > 0$), and we get neglecting the factor R_1

$$\theta P = \frac{L}{2g} \left(\frac{R}{E_0} \cos \varphi_0 - \frac{\pi}{2} \frac{g}{L} \sin \varphi_0 \right) \frac{\Delta W}{W_i}.$$

Then, together with equations (12) and (13) we have, for the maximum square amplitude, the value

$$\frac{A}{\pi} \frac{\theta}{K} \left[2 \sqrt{\frac{W_i}{\Delta W}} + + \frac{L}{2g} \left(\frac{R}{E_0} \cos \varphi_0 - \frac{\pi}{2} \frac{g}{L} \sin \varphi_0 \right) \sqrt{\frac{\Delta W}{W_i}} \right]. \tag{16}$$

Let us now restrict ourselves to the non-relativistic approximation. $\frac{R}{E_0}$ is a constant along the accelerator, and the same is true for K, since L^2 is proportional to W_i. Then it is clear that the above value is ruled practically by the first term of its bracket, and that the amplitude is an increasing function of energy. The transversal acceptance will be given by the section where the ratio of amplitude to aperture is the most unfavourable. By equating the expression (16) to the square of the drift tube half aperture a^2, we will obtain the value of a, in the variables η, $\frac{d\eta}{dt}$ or ζ, $\frac{d\zeta}{dt}$. In the usual phase space η, $\frac{d\eta}{dx}$, one will have then for the **transverse acceptance**

$$A = \frac{\pi a^2}{L_{in}} K \left[2 \sqrt{\frac{W_i}{\Delta W}} + \left(\frac{L}{2g} \frac{R}{E_0} \cos \varphi_0 - \frac{\pi}{4} \sin \varphi_0 \right) \sqrt{\frac{\Delta W}{W_i}} \right]^{-1} \quad (17)$$

L_{in} being the length of the initial unit, W_i and L being taken along the accelerator where the acceptance is really limited.

The same calculation can be applied to the longitudinal acceptance. One will find that the amplitude of the longitudinal oscillation ξ increases as $W^{1/4}$; but, as we saw in § 1, equations (1), the phase shift $\partial\varphi$ from the synchronous trajectory is proportional to $\frac{\xi}{v}$. Thus, at the non-relativistic approximation, **the phase oscillation is damped as $W^{-1/4}$ or $v^{-1/4}$.**

4. APPLICATION OF THE CROSS-GRADIENT FOCUSING

The idea of using a dissymetric accelerating field in linear accelerators is not new; already in 1956, during the Symposium on High Energy Accelerators at CERN, Professor Vladimirsky has suggested to produce accelerating field with quadrupolar distribution by putting horns on the drift tubes, at each side of the gaps. Such horns, fitting into each other on opposite sides, could have produced a strong focusing effect; but the practical design was not very attractive.

More recently, and especially since the development of sector-focused cyclotrons, use is made in these machines of axial slits to define and focus the beam along the first orbits in the central region, where magnetic axial focusing is very weak. Cross gradient focusing originated from a mixture of these two ideas: a particularly attractive design to produce Vladimirskiy's quadrupolar distribution is obtained with drift tubes having rectangular instead of circular holes.

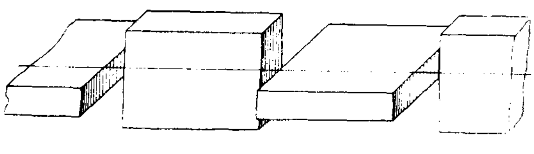

Fig. 4.

Fig. 4 shows a schematic representation, where drift tubes appear in the form of match boxes, whose orientation is changed by 90° from one tube to the next. Such a device has of course a dissymmetric character of the type described in the previous paragraphs. Ideally it can be imagined that, with such a configuration the fields at the entrance of an accelerating gap are independent of one of the transverse co-ordinates and at the exit independent of the other one.

The focusing effect at the output of a drift tube and the defocusing at the input of the following one achieve AG focusing. In this way also the maximum dissymetry $\left(\frac{R}{E_0} = 1\right)$ is obtained. We shall see later on that the actual situation is not so far from this ideal case. So that its range of application still rests where grid focusing has presently to be used. In fact it may even be argued that grid focusing acts by a similar process to the one described here, but in a less efficient way, because alternating gradient is not achieved on the orbits in a very satisfactory way.

5. FIELD MEASUREMENTS IN A CROSS-GRADIENT DEVICE

We shall now describe some measurements which have been made in order to obtain some numerical values of the focusing factors which can be obtained with practical accelerating structures. For this purpose, we have studied in an electrolytic tank the electric fields obtained with electrodes of rectangular cross-section; other studies need to be made about the best shape to be used for the lips of the drift tubes in order to minimize breakdown possibility; but it is not the intention of this report to consider that point and we shall restrict ourselves to the feasibility of dissymmetry achievement.

Parameters to be measured. The stability condition is given for example by (15)

$$\frac{n}{A} \frac{38}{V^2} \frac{E_0^2}{V_0} \left(\frac{R}{E_0}\right)^2 > \frac{1}{\sqrt{1-\beta^2}} \frac{|\sin \varphi_0|}{\cos^2 \varphi_0},$$

in which the dissymmetry factor $\frac{R}{E_0}$ is depending on the shape of the electrodes. We have

$$E_0 = F(l) + G(l)$$
$$R = F(l) - G(l),$$

with

$$F(l) = \int_0^l f(x)\,dx \quad \text{and} \quad G(l) = \int_0^{+l} g(x)\,dx,$$

and

$$\frac{\partial E_y}{\partial y} = f(x) \cos(\varphi_0 + \omega t)$$

$$\frac{\partial E_z}{\partial z} = g(x) \cos(\varphi_0 + \omega t).$$

According to the symmetries of the system $f(x) = -g(-x)$ and one only has to study $f(x)$.

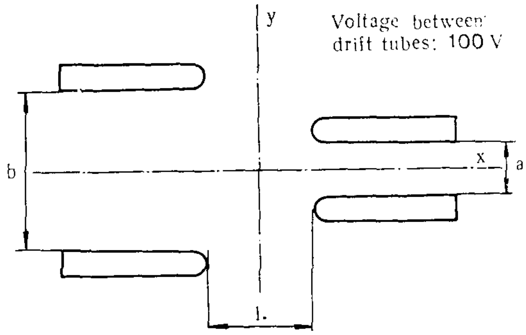

Fig. 5.

One calculates $\frac{\partial E_y}{\partial y}$ and $\frac{\partial E_z}{\partial z}$ from the measurements of E_y and of E_z made close to the axis where

$$E_y = \frac{\partial E_y}{\partial y} \cdot y + \dots$$

$$E_z = \frac{\partial E_z}{\partial z} \cdot z + \dots$$

Eventually one can check that: $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.$

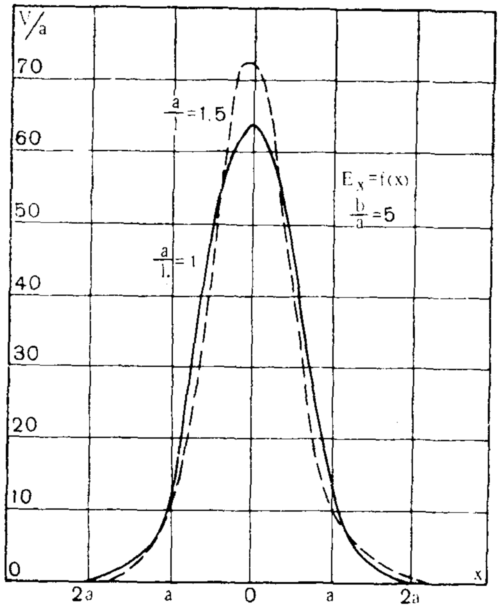

Fig. 6.

With a planimeter one computes the integrals $F(l)$ and $G(l)$ from which are deduced R and E_0.

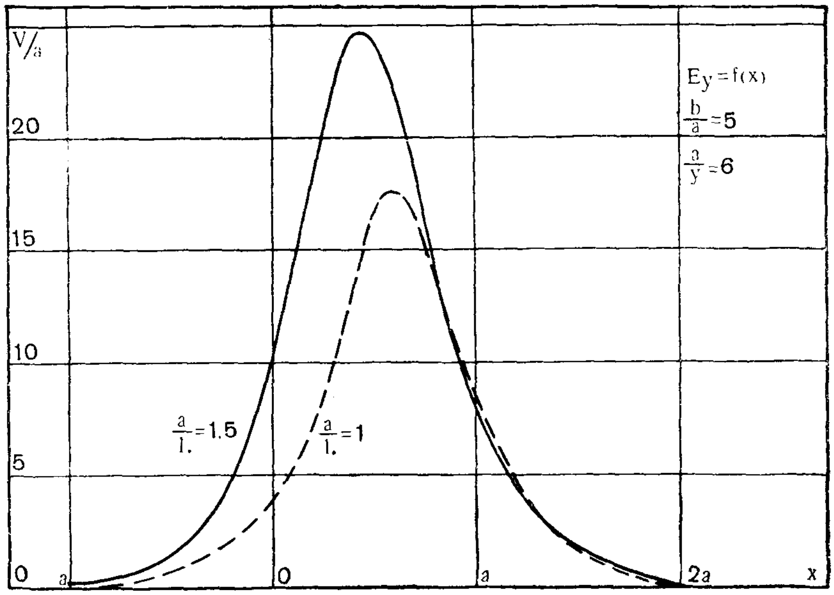

Fig. 7.

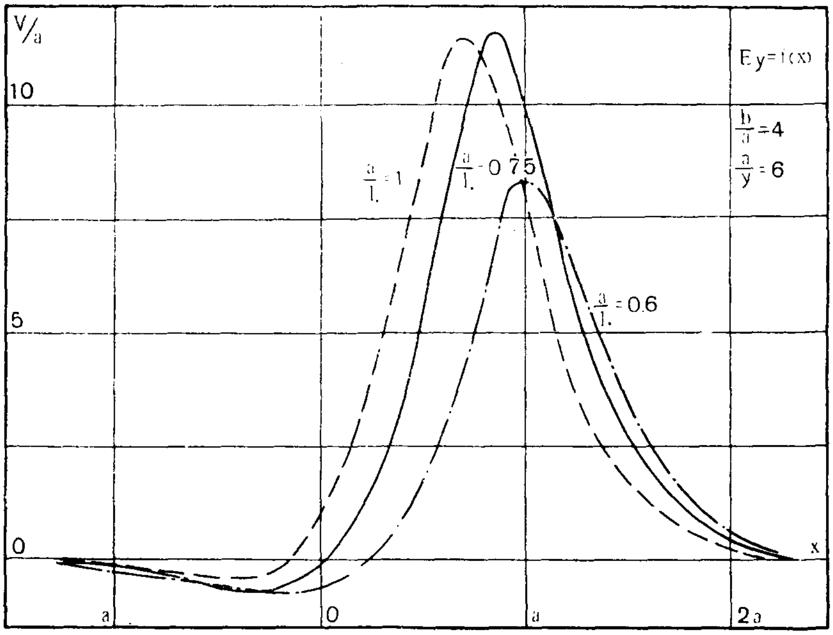
V/a
10
5
0
Ey=f(x)
b/a = 4
a/y = 6
a/L = 1
a/L = 0.75
a/L = 0.6
a
0
a
2a
x

Fig. 8.

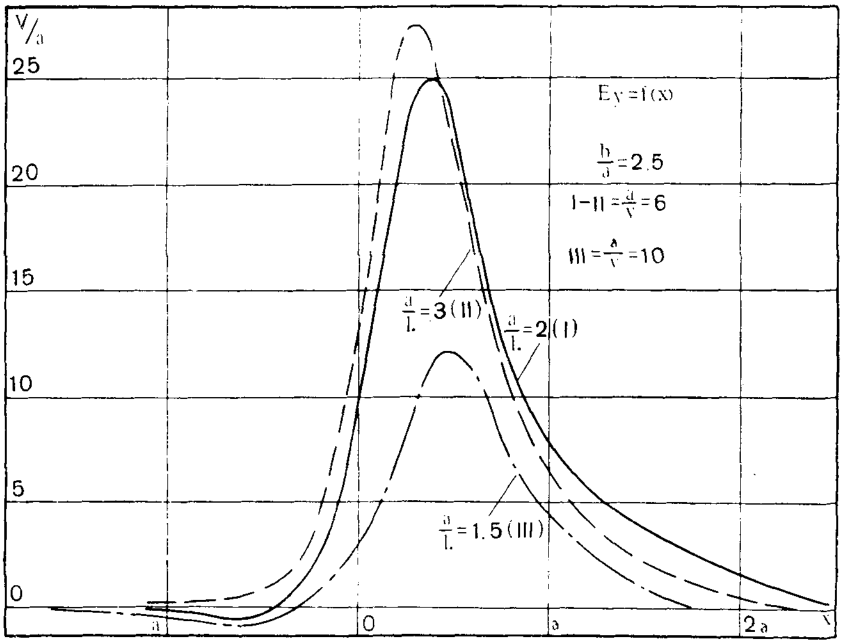
V/a
25
20
15
10
5
0
Ey=f(x)
b/a = 2.5
I-II = a/y = 6
III = a/y = 10
a/L = 3 (II)
a/L = 2 (I)
a/L = 1.5 (III)
a
0
a
2a
x

Fig. 9.

Measurements made in the electrolytic tank. A model of one half section of the electrodes is put into the tank, the water level being one of the symmetry planes of the system. It is not difficult to measure E_x with a bridge circuit. The measurement of E_y has proved to be more delicate because this field is very weak near the axis, especially in the negative x region. In order to ensure a greater measurement accuracy (two-probe measuring system, stability and constancy of water level, temperature, amplifier gains) one must take serious precautions.

Results. The measurements have been made with the electrodes shown in Fig. 5. Measured

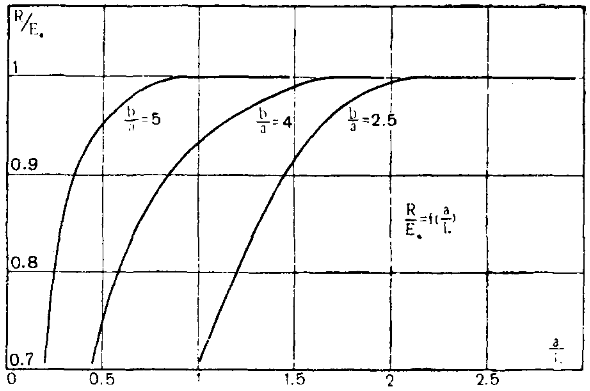

Fig. 10

values of E_x and E_y are shown on Figs. 6 to 10 and the equipment used appears on photographs 1 and 2. Some numerical results of integration are given in Table 1. One can see that one has obtained dissymmetry factors $\frac{R}{E_0}$ close to 1, that is to say the maximum focusing effect.

6. CONCLUSIONS

The method of using rectangular holes instead of circular ones can be applied to any kind of linear accelerator, for electrons, protons or heavy ions. However, it should not replace present focusing techniques, wherever they prove to be satisfactory. In particular for proton accelerators magnetic quadrupole focusing is more efficient and far more flexible than cross gradient focusing could be. However cross gradient could supplement in some cases quadrupole focusing. But for very low velocities in heavy ion linacs magnetic quadrupoles cannot be inserted in the drift tubes. One has to make use of grid focusing, which is not very good. In this case, the cross gradient device would be an attractive alternative. Numerical applications of eqs. (14) and (17) show that large acceptances in phase and transverse directions reqiure small gaps, low accelerating voltages and low frequencies, condition which lead to increase the length of an accelerator; this is not a serious drawback for low energies or for the beginning of an accelerator but it would be if one intended to reach high energies with this scheme.

b/a	a/l_0	a/y	R/E_0	$2l/l_0$	g/l_0
2.5	1.5	10	0.88	5	1.5
"	2	6	0.985	7.1	1.56
"	3	6	1	10.5	2.6
4	0.6	6	0.78	2.5	1.37
"	0.75	6	0.88	2.9	1.3
"	1	6	0.926	3.85	1.7
5	1	6	1	3.5	1.12
"	1.5	6	1	5.25	1.15

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DISCUSSION

I. M. Kapchinskiĭ

By what value is the shunt-impedance of the accelerating system decreased in relation to the appearance of focusing components in the RF field?

P. Lapostolle

Yes, it is certain that energy must be introduced to provide transverse focusing. In practice acceleration rate has to be reduced and the length of accelerator increased.

N. C. Christophilos

Did you investigate the effect of the alternating field on the phase stability?

P. Lapostolle

It is certain that the cross gradient can effect phase motion. One might also mix it with alternate phase focusing in order to obtain better stability. But we did not investigate this possibility in detail. We only considered constant phase acceleration.