REGGE POLES AND STRIP APPROXIMATION

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1. INTRODUCTION

My aim in these lectures is to outline an approximation scheme for calculating scattering amplitudes in dispersion theory by the use of which one would hope to overcome the difficulties associated with previous approximation schemes. At the moment a fully consistent set of equations has not been written down, but it is hoped that the materials for a solution of the problem are at hand. We shall see, probably in the next lecture, that the concept of "Regge poles" will play an important part in the analysis. In fact, it was in this connection that they were originally introduced into elementary particle physics.

First let me outline why we were unable to get consistent equations by the previous approach, used for instance by Chew and me in the pion-pion problem [1]. (This approach has also been treated in a paper by CINI and FUBINI [2]. One started with the double-dispersion representation [3],

$$A(s,t) = \frac{1}{\pi}\int ds'dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi}\int du'dt' \frac{A_{23}(u',t')}{(u'-u)(t'-t)}$$

$$+ \frac{1}{\pi}\int ds'du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)} . \qquad (1)$$

For the purpose of this lecture we have taken the case of neutral scalar particles with equal mass, when the variables s, t and u are related by the equation

$$s + t + u = 4\mu^2.$$

We have not written the subtraction terms explicitly, but it is understood that such terms may and in fact will be present. The essence of the old approach was to assume that the scattering amplitude at low energies was dominated by the nearest singularities. Accordingly, one neglected terms where "one" was large. In the first approximation, one neglected contributions which began at the inelastic threshold.

In pion-pion scattering, this amounted to neglecting the double-spectral functions completely. The reason is that it is impossible to draw a diagram where the $\pi\pi$ scattering, both in the s and t- channels, took place through a two-pion intermediate state. The processes with the lowest intermediate states were in fact as in Fig. 1.

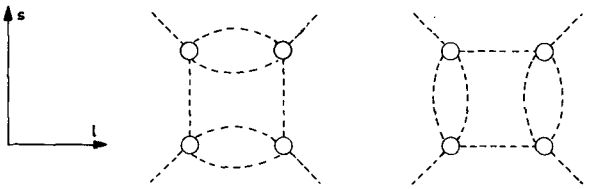

Fig. 1

Graphs for pion-pion scattering with the lowest intermediate state.

The first diagram goes through a two-pion state in the s-channel but through a four-pion state in the t-channel, the second through a four-pion state in the s-channel but through a two-pion state in the t-channel. Accordingly, the double-spectral function starts at a high threshold in at least one of the variables s and t, and must consequently be neglected.

We should emphasize that the neglect of the double-spectral function is purely due to the absence of a three-pion vertex. If there were a three-pion vertex, the following process (Fig. 2) would contribute, and the double-

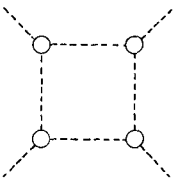

Fig. 2

Pion-pion scattering through a three-pion vertex

spectral function could not be neglected as all double-spectral functions are neglected, the representation would consist purely of subtraction terms, and would appear as follows

$$A(s,t) = \frac{1}{\pi}\int \frac{f_S(s')\,ds'}{s'-s} + \frac{1}{\pi} P_1\left(1+\frac{2t}{s-4\mu^2}\right)\int \frac{f_P(s')\,ds'}{(s'-s)} + \text{crossed terms} + \lambda \cdot \quad (2)$$

We have taken the case of two subtractions in the t-variable, so that there will be a constant term and a term linear in t. We have re-grouped them into a constant term and a term involving the factor $P_1\ [1+2t/(s-4\mu^2)]$ $= 1 + 2t/(s-4\mu^2)$ as these terms correspond to S- and P-waves. Thus, if P-waves are important, as they are in practice in π-π scattering, one would expect to have to perform at least two subtractions in t. We have been a little careless in writing (2), as P-waves cannot occur in neutral (pseudo) scalar pion-pion scattering but, since we are really interested in charged pions we shall ignore this.

When Chew and I attempted to solve the problem, we arrived at singular integral equations which did not have a unique solution. The difficulty was

due to the fact that the diagram (Fig. 3) represents the exchange of a P-wave pion pair.

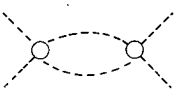

Fig. 3

Exchange of a P-wave pion pair.

Now the exchange of a P-wave system corresponds to a very singular potential. If one were solving the problem by any other method, the singularity would be rounded off by the fact that a composite system such as a pion pair has an extension in space. Dispersion theory operates in terms of the S-matrix, however, and concepts such as spatial extension do not enter directly into the theory. In fact, in the simplest approximation as Chew and I treated it, the exchange of a composite system is treated on exactly the same footing as the exchange of an elementary particle, and leads to singular equations.

The difficulty actually arises from the factor $1 + 2t/(s-4\mu^2)$ in the second term of (2), which approaches infinity with infinite t. Now, the function A (s, t) in addition to representing direct pion-pion scattering, also represents crossed pion-pion scattering, and now being the energy and s the momentum transfer. The amplitude for crossed pion-pion scattering then approaches infinity with the energy, and such a behaviour can lead one into conflict with the unitarity condition.

2. CALCULATION OF THE DOUBLE SPECTRAL FUNCTION

In the face of these difficulties, a much more ambitious approximation scheme was suggested independently by CHEW and FRAUTSCHI, McCAULEY, TER-MARTIROSYAN and WILSON [3]. The proposal was essentially to calculate all that one can with neglect of multi-particle states in the unitarity condition. Our next problem will therefore be to investigate how the double-spectral function may be calculated.

Let us suppose for the moment that we know the double-spectral function A_{23}. In practice, we do not know it in advance, of course, but must calculate it by means of an iteration procedure. We may then re-write (1) as

$$A(s,t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{V_3(s,t)}{(t'-t)} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} du' \frac{V_2(s,u')}{(u'-u)}$$

$$+ \frac{1}{\pi}\int ds'dt' \frac{A_{13}(s,t')}{(s'-s)(t'-t)} + \frac{1}{\pi}\int ds'du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)} .$$

$$+ \text{subtraction terms involving } s + \lambda. \qquad (3)$$

In this equation, the first two integrals come from two sources:

(i) Substraction terms in t and u;
(ii) The second term of (1).

It is shown in [1] how the latter term can be written in the form of the first two terms of (3). If we have subtractions, the integrals in (3) will really have a more complicated form. This is also explained in [1]. Such complications are inessential and we shall ignore them here. We shall explicitly exclude more than one subtraction in each variable, however. For the moment we are interested in the calculation of A_{13} and A_{12} from unitarity when V_3 and V_2 are known. Later we shall have something to say on the iteration procedure for calculating V_3 and V_2. We shall formulate all our equations in terms of neutral pion-pion scattering. Generalization to problems with spin, isotopic spin and unequal mass can be made, and they do not change the essential features of the calculation.

From (3), we can write a dispersion relation in the momentum transfer (for fixed s):

$$A(s,t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{A_3(s,t')}{(t'-t)} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} du' \frac{A_2(s,u')}{(u'-u)} \tag{4}$$

where

$$A_3(s,t) = V_3(s,t) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{A_{13}(s',t)}{(s'-s)} \,. \tag{5a}$$

$$A_2(s,u) = V_2(s,u) + \frac{1}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{A_{12}(s,u)}{(s'-s)} \,. \tag{5b}$$

We now insert (4) into the unitarity relation, so as to obtain A, in terms of the spectral functions A_3, A_2 instead of in terms of A. The unitarity relation is

$$A_1(s,t) = \frac{q}{32\pi^2 W} \int_{-1}^{1} dz_1 \int_0^{2\pi} d\varphi \quad A^* \{s, t(z_1)\} \; A\{s, t(z z_1 - \sqrt{(1-z^2)(1-z_1^2)}\cos\varphi\} \tag{6}$$

The notation should be fairly evident. The symbol z denotes the cosine of the ongle of scattering from the initial to the intermediate state, and z_1 the cosine of the angle from the initial to the final state. The symbol φ is the azimuthal angle between the initial and the intermediate state, measured from the plane of the initial and the final state, and the integral is thus over all intermediate states. The cosine angle between the intermediate and the final states will thus be

$$z\, z_1 - \sqrt{(1 - z^2)\,(1 - z_1^2)}\,\cos\varphi\,.$$

The variable z will be related to the momentum transger t by the formula

$$z = 1 + t/2q^2 \tag{7a}$$

while the expressions $t\,(z_1)$ and $t\,(zz_1 - \sqrt{(1-z^2)\,(1-z_1^2)}\cos\varphi)$ in (6) indicate that t is to be expressed in terms of the cosine of the angle of scattering by the formula

$$t\,(z_1) = 2q^2(z_1 - 1) \tag{7b}$$

$$t\,(zz_1 - \sqrt{(1-z^2)\,(1-z_1^2)}\cos\varphi) = 2q^2(z\,z_1 - \sqrt{(1-z^2)\,(1-z_1^2)}\cos\varphi - 1) \tag{7c}$$

In all these formulae, t is the centre-of-mass momentum given by

$$q^2 = \tfrac{1}{4}\,(s - 4\mu^2) \tag{8a}$$

and W the centre-of-mass energy $s^{\frac{1}{2}}$.

At this point we shall first simplify the calculations by assuming that the second term of (4) is absent. We can then insert (4) into (6) and, on expressing $t\,(z_1)$ and $t\,(zz_1 - \sqrt{(1-z^2)\,(1-z_1^2)}\cos\varphi)$ by (7c), we arrive at the equation

$$A_1(s,t) = \frac{q}{32\pi^2 W}\int_{-1}^{1} dz_1 \int_0^{2\pi} d\varphi\, \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt'\, \frac{A_3(s,t')}{t' - 2q^2(z_1 - 1)}$$

$$\times \int_{4\mu^2}^{\infty} dt''\, \frac{A_3(s,t'')}{t'' - 2q^2\,(zz_1 - \sqrt{(1 - z^2)\,(1-z_1^2)}\cos\varphi - 1)}\,. \tag{9}$$

The integrations over z and φ on the one hand, and t′ and t″ on the other hand, can now be interchanged. As the variables z and φ only occur in the denominators, the integrations over these can be performed. After expressing z in terms of t by (7a), the result is

$$A_1(s,t) = \frac{1}{32\pi^3\, qW}\int d\sigma' d\sigma''\, A_3(s,t')\,A_3(s,t'')\, \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}}$$

$$\times \log \frac{\alpha\,(q^2;\, t,t',t'') + \{K(q^2,\, t,t',t'')\}^{\frac{1}{2}}}{\alpha\,(q^2;\, t,t',t'') - \{K(q^2,\, t,t',t'')\}^{\frac{1}{2}}} \tag{10}$$

where

$$K(q^2; t, t', t'') = t^2 + t'^2 + t''^2 - 2(tt' + tt'' + t't'') - t\,t'\,t''/q^2 \quad (11a)$$

$$\alpha(q^2, t, t', t'') = t - t' - t'' - t'\,t''/2q^2 \quad (11b)$$

Eq.(10) gives A_1 in terms of A_3. The equation as it stands is not particularly useful; however, it can easily be rewritten as an equation for A_{13} in terms of A_3. To see this, we observe that

$$\frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}} \log \frac{\alpha(q^2;t,t',t'') + \{K(q^2;t,t',t'')\}^{\frac{1}{2}}}{\alpha(q^2;t,t',t'') - \{K(q^2;t,t',t'')\}^{\frac{1}{2}}}$$

$$= 2\int \frac{dt'''}{t''' - t} \frac{1}{\{K(q^2;t,t',t''')\}^{\frac{1}{2}}} \quad (12a)$$

where the integral is taken over the region

$$(t''')^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t'')^{\frac{1}{2}}$$

$$K > 0. \quad (12b)$$

Eq.(12) may be derived by observing that the logarithm is analytic except for a cut along the real axis when the inequalities (12b) are satisfied, the discontinuity across the cut being $2\pi K^{-\frac{1}{2}}$. On substituting (12) into (10), we arrive at the equation

$$A_1(s,t) = \frac{1}{\pi}\int \frac{dt'''}{t''' - t} \frac{1}{16\pi^2 q W} \int dt'\,dt'' \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}}$$

$$\times A_3^*(s,t')\,A_3(s,t'') \quad (13a)$$

where the integrals of t' and t'' are taken over the region for which

$$(t''')^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t'')^{\frac{1}{2}}$$

$$K > 0. \quad (13b)$$

We can now compare Eq.(13a) with the dispersion relation for $A_1(s,t)$

$$A_1(s,t) = \frac{1}{\pi}\int dt' \frac{A_{13}(s,t')}{(t' - t)}. \quad (14)$$

(The integral over $A_{23}(s,u)$ does not contribute when the second term of (4) is neglected). From Eq. (14) we observe at once that we can identify the integral over t' and t'' in Eq. (13a) with A_{13}:

$$A_{13}(s,t) = \frac{1}{16\pi^2 qW} \int dt'dt'' \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}} A_3(s,t')A_3(s,t''), \quad (15)$$

for $$(t)^{\frac{1}{2}} > (t')^{\frac{1}{2}} + (t'')^{\frac{1}{2}}, \quad K > 0$$

otherwise

$$A_{13}(s,t) = 0,$$

Eq. (14) is the unitarity equation for the double-spectral function which we require [3].

We now have two equations, Eqs. (5a) and (15), between A_3 and A_{13}. One is linear, the other quadratic. Because of the limitations on the range of integration in Eq. (15) it turns out that one can obtain A_{13} and A_3 without solving an integral equation. To see how this can be done, we observe that $A_{13}(s,t)$ will be zero if $t < 4\mu^2$. As the integration in Eq. (15) is taken only over the region $t^{\frac{1}{2}} > t'^{\frac{1}{2}} + t''^{\frac{1}{2}}$, it follows that

$$A_{13}(s,t) = 0 \quad , \qquad t < 16\mu^2. \quad (16a)$$

Thus, from Eq. (5a)

$$A_3(s,t) = V_3(s,t), \qquad t < 16\mu^2. \quad (16b)$$

And $A_3(s,t)$ will thus be known for this range of t.

Next, we observe from the inequality $t^{\frac{1}{2}} > t'^{\frac{1}{2}} + t''^{\frac{1}{2}}$ that, if $t < 36\mu^2$, t' and t'' will both be less than $16\mu^2$ (since both are greater than $4\mu^2$). However, $A_3(s,t)$ is known for $t < 16\mu^2$, so that $A_{13}(s,t)$ can be calculated for $t < 36\mu^2$. Using the dispersion relation (4), $A_3(s,t)$ can then be calculated for $t < 36\mu^2$.

The process of using successively Eqs. (15) and (4) may now be continued indefinitely. In the next stage, for instance, $A_{13}(s,t)$ can be calculated from Eq. (15) for $t < \sigma 4\mu^2$ if $A_3(s,t)$ is known for $t < 36\mu^2$. $A_3(s,t)$ can then be calculated for $t < 64\mu^2$ from Eq. (4). We can thus construct the double-spectral function for successively larger ranges of t, and can reach any given value of t in a finite number of steps. The elastic unitarity and analyticity properties thus provide us with a means of calculating the double spectral function.

We may remark that a similar equation can be derived for calculating the double spectral function for a superposition of Yakuwa potentials, as was shown by BLANKENBECLER, GOLDBERGER, KHURI and TREIMAN [4]. There are two differences:

(i) The factor $1/W$ in Eq. 15 is absent and the numerical factors are different;

(ii) The function $V(s,t)$ is known in advance. It depends on t alone and is given by

$$U(t) = \frac{1}{\pi} \int dt' \frac{N(t')}{(t'-t)} \tag{17}$$

where $U(t)$ is the potential. A superposition of Yukawa potentials can always be written in the form (17). In the potential case the determination of the double-spectral function provides a complete solution of the problem, in the relativistic case it does not as $V_3(s,t)$ is not known in advance and must be determined from the solution. Because of the similarities between the equations for the potential and relativistic theories, CHEW and FRAUTSCHI [5] have dubbed the function $A_3(s,t)$ a "generalized potential". We should emphasize however, that the innocent-looking factor $1/W$ in the relativistic case is sufficient to destroy the equivalence between the present method and any Schroedinger-like equation, and one is forced to solve the problem using the procedure just outlined.

We have thus far simplified the problem by omitting the second term in (2). When such a term is present a similar procedure can be used, but the equations corresponding to (15) are slightly more complicated. They are:

$$A_{13}(s,t) = \frac{1}{16\pi^2 qW}\left[\int_R dt'dt'' \frac{1}{\{K(q^2;t,t',t'')\}^{\frac{1}{2}}} A_3^*(s,t') A_3(s,t'')\right.$$

$$\left. + \int_R du'du'' \frac{1}{\{K(q^2,t,u',u'')\}^{\frac{1}{2}}} A_2^*(s,u') A_2^*(s,u'')\right], \tag{18a}$$

$$A_{12}(s,u) = \frac{1}{16\pi^2 qW} \int_R dt'du' \frac{1}{\{K(q^2;u,t',u'')\}^{\frac{1}{2}}} \left[A_3^*(s,t') A_2(s,u'')\right.$$

$$\left. + A_2^*(s,u') A_3(s,t')\right]. \tag{18b}$$

The sub-script R indicates that the integral is to be taken only over the region for which the last three arguments of K satisfy the inequality in Eq. (15).

We shall now outline very briefly the iteration procedure suggested by the authors named at the beginning of the lecture for calculating V_3 and V_2. We shall not go into details, both because the scheme will probably be discussed in other lectures, and also because, as it stands at the moment, it does not appear to be free of divergences and will probably have to be modified. The iteration scheme is based on the crossing relation, which takes the form (for neutral pion-pion scattering)

$$A_{13}(s,t) = A_{13}(t,s) = A_{12}(s,t) = A_{12}(t,s) = A_{23}(s,t) = A_{23}(t,s). \quad (19)$$

Now the function A_{13} calculated according to the iteration procedure is certainly not symmetric in its arguments. We shall therefore define

$$A_{13}(s,t) = A_{13\,el}(s,t) + A_{13\,in}(s,t) \quad (20a)$$

where

$$A_{13\,in}(s,t) = A_{13\,el}(t,s). \quad (20b)$$

The procedure outlined above gives $A_{13\,el}(s,t)$, and corresponds to diagrams such as Fig. 4 (a). To maintain the crossing relation one must then include

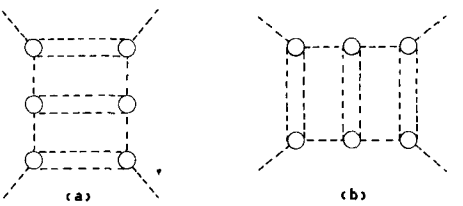

Fig. 4

Elastic and inelastic contributions to pion-pion scattering

diagrams corresponding to Fig. 4(b) as well, and they will correspond to the spectral function A_{13} in inelastic processes as they take into account to a certain extent, in fact, as will probably be discussed in Frautschi's lectures, they are taken into account in the "physical approximation". For the moment, however, we simply remark that we must include them in order to maintain the crossing relation.
Thus, in the iteration scheme, we would define (from Eq. (20b))

$$A^{(n)}_{13\,in}(s,t) = A^{(n-1)}_{13\,el}(t,s), \quad (21a)$$

$$A^{(n)}_{12\,in}(s,u) = A^{(n-1)}_{12\,el}(s,u), \quad (21b)$$

and, from (19)

$$A^{(n)}_{12}(t,u) = A^{(n-1)}_{13el}(u,t) + A^{(n-1)}_{12el}(t,s). \quad (21c)$$

The quantities $V_3(s,t)$ and $V_2(s,u)$ are obtained by inserting A_{13}, A_{12} and A_{23} in the dispersion relations for A_3 and A_2

$$V_3^{(n)}(s,t) = \int ds' \frac{A_{13}^{(n)}{}_{in}(s',t)}{(s'-s)} + \int du' \frac{A_{23}^{(n)}{}_{in}(t,u')}{(u'-u)}$$

+ subtraction terms, (22a)

$$V_2^{(n)}(s,u) = \int ds' \frac{A_{12}^{(n)}{}_{in}(s',u)}{(s'-s)} + \int dt' \frac{A_{23}^{(n)}(t',u)}{(t'-t)}$$

+ subtraction terms. (22b)

One obtains the subtraction terms (if there are any) by solving the S-wave equations by the N/D method; the connection between the subtraction terms and the S-waves is outlined in [1]. The method given above can then be used to calculate $A_{13}^{(n)}{}_{el}(s,t)$ and $A_{12}^{(n)}{}_{el}(s,t)$, and we thus have an iteration procedure which provides a solution of the problem on the assumption that it converges, of course.

3. THE SCATTERING AMPLITUDE IN THE CASE OF SUBTRACTIONS.

In the previous section we discussed the construction of the double spectral function A_{13} and the single spectral function A_3 from unitarity. (Of course, in the general case, we also calculate A_{12} and A_2). We pointed out that, in the absence of subtractions, one could now find the scattering amplitude simply by using the dispersion relation in the momentum transfer:

$$A(s,t) = \frac{1}{\pi} \int dt' \frac{A_3(s,t')}{(t'-t)}. \tag{23}$$

However, this is only true if there are no subtractions in the t dispersion relation. Now we want to discuss the question: what happens if there are subtractions? Can we still get the scattering amplitude by this procedure, knowing A_{13} and A_3?

Just from ordinary, naive, common-or-garden dispersion theory, it does not look as though we can. There are no subtractions if $A_3 \to 0$ as $t \to \infty$. However suppose we only have the weaker condition $A_3/t \to 0$ as $t \to \infty$. We then need one subtraction, and have to write the equation as

$$A(s,t) = A(s,t_0) + \frac{t-t_0}{\pi} \int dt' \frac{A_3(s,t')}{(t'-t_0)(t'-t)}. \tag{24}$$

Now you see that we don't know the scattering amplitude if we know A_3 - we also have to know $A(s,t_0)$. This doesn't depend on the momentum transfer or the angle so it just corresponds to the S-wave. Therefore, if we have one subtraction, it means that the S-waves are apparently not determined by the double spectral function, but have to be calculated separately - for instance by the N/D method.

Similarly, if A behaves like t at infinity, so that only $A/t^2 \to 0$, we need two subtractions and we have to calculate both the S and P-waves separately - they are not determined by the double spectral function.

Now to answer the question: how does A actually go to zero in the problem of interest? We know what it does in perturbation theory. For potential scattering in perturbation theory, $A(s,t) \to 0$ as $t \to \infty$, and in the relativistic case it goes to a constant. If the potential is sufficiently small, we can use perturbation theory, and everything is then determined by the spectral function A_3. However, suppose we increase the strength of the potential, till we get an S-wave bound state. Then A will have a term $1/(s - s_B)$, which is constant as $t \to \infty$. Similarly, if we increase the potential up to the point where we get a bound P-state, then we will have a term

$$(1 + t/2a^2)/(s - s_B). \tag{25}$$

The numerator is just the first Legendre polynomial of the scattering angle. And you see that when the potential reaches this strength, $A \to \infty$, as $t \to \infty$, and we need two subtractions. So, as one increases the strength of the potential, the asymptotic behaviour gets worse and worse and, without any further information than we have already put in, one has to perform more and more subtractions.

Now, this, although it is not obviously wrong, does sound a bit paradoxical, because it would be funny if the double spectral function determined everything until the potential reached a certain strength, and then suddenly at this strength of the potential we lost information and couldn't get the S-waves from the double spectral function, and then when the potential reached another strength we lost still more information and couldn't get the P-waves from the spectral functions, and so on. I think that most people would consider this a rather implausible, although not necessarily ridiculous, situation. Now this itself is not so serious, but we see that, in the relativistic case, once $A \to \infty$ at infinite t, then in the crossed reaction $A \to \infty$ at infinite energy, and we get those troubles I was speaking about yesterday. So for the relativistic case, it is vital to analyse this asymptotic behaviour in more detail to see whether we can get rid of this trouble. As a matter of fact, that is really why I am going to the trouble of doing all this complicated procedure, instead of using the simpler procedure that we used in earlier calculations.

The solution to this problem of the asymptotic behaviour was solved completely by REGGE [6] in the potential theory - this is where he comes into the picture - and he showed that one can, in fact, get rid of this paradox, and that it is possible to get the whole scattering amplitude from the spectral functions A_3 and A_{13}, even in the case where we have subtractions.

As I have just said, the problem was originally solved by the potential theory, but it can also be solved - and we get the same solution - for the relativistic case that I have just been discussing. What I have to say now will be adequate both for the potential case and the relativistic case. Now let me be careful what we are interested in. We are interested in the construction of the double spectral function and scattering amplitude from the successive procedure which I outlined yesterday which makes use of the

elastic unitarity approximation. The question whether the exact scattering amplitude in field theory has the properties that I am going to describe, is very much more complicated, and we can only apply conjectures at the moment. You are certainly going to hear a lot more about it in other lectures at this Seminar. But for the moment we are interested in the question of whether the functions constructed according to our approximation scheme have certain asymptotic properties, because we want to use these asymptotic properties in solving these equations, and for that we don't need to apply conjectures - everything has now been proved.

The essence of the Regge analysis is to look at everything in the complex ℓ - plane, ℓ being the angular momentum. So this is a new analytic continuation.

The only physical values of ℓ are the positive integers: at $\ell = 0$, we have S-waves, at $\ell = 1$ we have P-waves, at $\ell = 2$ D-waves, and so on. Now what

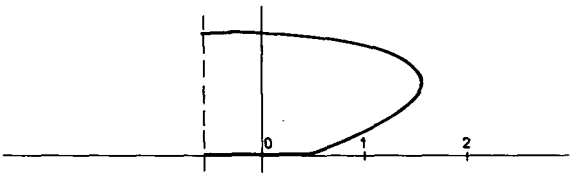

Fig. 5

The complex ℓ-plane

Regge showed was that this function $A(s,\ell)$, which is equal to the S-wave at $\ell = 0$, the P-wave at $\ell = 1$, the D-wave at $\ell = 2$, can be continued analytically both to non-integral and complex ℓ to the right of the line Re $\ell = -\frac{1}{2}$. Actually, in potential theory, one can get to the left of this line, but in field theory it seems a bit difficult to do so, so I think I will keep my analysis to what happens to the right of this line Re$\ell = -\frac{1}{2}$. Our scattering amplitude is meromorphic in this region. There may be poles in the upper half plane, and these are in fact the Regge poles (Fig. 5).

We now want to use these properties to try and get the asymptotic behaviour of our scattering amplitude as a function of t, the momentum transfer. So let us use the ordinary partial wave expansion

$$A(s,t) = \epsilon \, (2\ell + 1) \, A_\ell(s) \, P_\ell(z), \quad z = 1 + t/2q^2. \tag{26}$$

What Regge did was to replace this sum by an integral over a contour like that in Fig. 4, of the following expression

$$A = \int_C d\ell \, \frac{(2\ell + 1) \, A(s,1) \, P_\ell(z)}{\sin(\pi)}, \tag{27}$$

Here ℓ is not restricted to an integer any more. The contour must not enclose any of the poles of $A(s,\ell)$, so that the only singularities of the integrand are given by $\sin(\pi\ell) = 0$, which of course are just the positive integers.

So evaluating the integral by the residue theorem, we get just the partial wave expansion. We put $P_\ell(-z)$ instead of $P_\ell(z)$ in order to cancel the alternation of sign of the residues of $\sin(\pi\ell)$ between the even and odd integers. ($P_\ell(z) = (-1)^\ell P_\ell(z)$ for integer ℓ).

The next thing one does is to deform the contour of integration until it goes along the line $\mathrm{Re}\,\ell = -\frac{1}{2}$. However, in doing so we have to cross these poles of $A(s,\ell)$, so we must add

$$\Sigma\ (2\alpha+1)\,\beta\ P\alpha\ (z)/\mathrm{Sin}\ \pi\alpha. \tag{28}$$

Now let me call the positions of these poles α_1, α_2 and so on, and the residues at each of these poles we will call β. α and β will depend on the energy, so the equation becomes

$$A(s,z) = \int_{\mathrm{Re}\,\ell = -\frac{1}{2}} d\ell\ \frac{(z\ell+1)\,A_\ell(s)\,P_\ell(-z)}{\mathrm{Sin}\,(\pi\ell)} + \sum_n \frac{(2\alpha_n(s)+1)\,\beta_n(s)\,P_{\alpha_n(s)}(z)}{\mathrm{Sin}\,(\pi\alpha_n(s))}. \tag{29}$$

We can use this formula to find at once the asymptotic behaviour of A as a function of z. The reason is that we know that

$$P_\alpha(z) \sim z^\alpha, \text{ as } z \to \infty.$$

Now in the first (integral) term, the real part of ℓ is $-\frac{1}{2}$, so this part goes down like $|z|^{-\frac{1}{2}}$ at infinite z. So we will forget about that, since it goes down very nicely. Anything that goes to zero, we are not interested in. The pole terms, however, behave like z^α as $z \to \infty$. In particular, the pole that dominates is the one that has the largest real part. So the asymptotic behaviour will be $\sim z^{\alpha_1}$ where α_1 is the pole furthest to the right.

This gives the results in principle, but in order really to be able to see what is going on, we have to know how the function α depends on s, so let me take the ℓ-plane again. What normally * happens is that for sufficiently large, negative values of the energy, the poles all lie to the left of the line $\mathrm{Re}\,\ell = -\frac{1}{2}$, so that we just do not see them., As the energy increases, the poles move to the right along the real axis. Now it may happen that, at some energy $s_0 < 4\mu^2$, before the threshold, one of the poles passes through $\ell = 0$. At that point the scattering amplitude becomes infinite, it therefore has a pole as a function of s, at $s = s_0$. At $\ell = 0$, $P = 1$, so the residue at the pole does not involve the angle at all, and what we have is a bound S-state. As we continue to increase the energy, it may happen that, for a very strong potential, the pole actually passes through $\ell = 1$ before we reach the threshold $s = 4\mu^2$. Again we have a bound state, because $\sin(\pi\ell)$ becomes zero. However, the residue is now proportional to $P_1(z)$, so we have a bound P-state. In general we have bound states at those values of s, for which $\alpha(s)$ = a positive integer.

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* In relativistic theory, when one solves the N/D equations, they will sometimes give poles with rather unphysical properties ("ghosts"), which may be due to the failure of the approximation theory, I will ignore these.

Suppose that we have now got to the threshold $s = 4\mu^2$. Then what happens as we continue to increase s is that the poles move out into the complex plane. They go a certain distance further to the right, but eventually come back again, and when s is sufficiently large the poles disappear to the left of the line $\mathrm{Re}\,\ell = -\frac{1}{2}$.

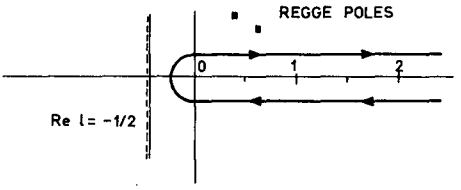

Fig. 6

Trajectory of a Regge pole in the complex ℓ-plane

Now for positive kinetic energy, seeing that the poles are complex, they never pass through a positive integer, so we don't get bound states. However, it may happen, as in the case I have drawn, that a pole passes near a positive integer. Then $\sin(\pi\ell)$ will be very small, and the scattering amplitude, although it does not get infinite, gets very large. So when the pole passes near $\ell = 1$ we get not a P-wave bound state but a P-wave resonance.

Well one can draw this in a different way if one likes: suppose one just plots $\mathrm{Re}\,\alpha$ against s (Fig. 7). The line at the bottom is $\mathrm{Re}\,\alpha = -\frac{1}{2}$. We don't know what is going on below this. The vertical line is the threshold $s = 4\mu^2$.

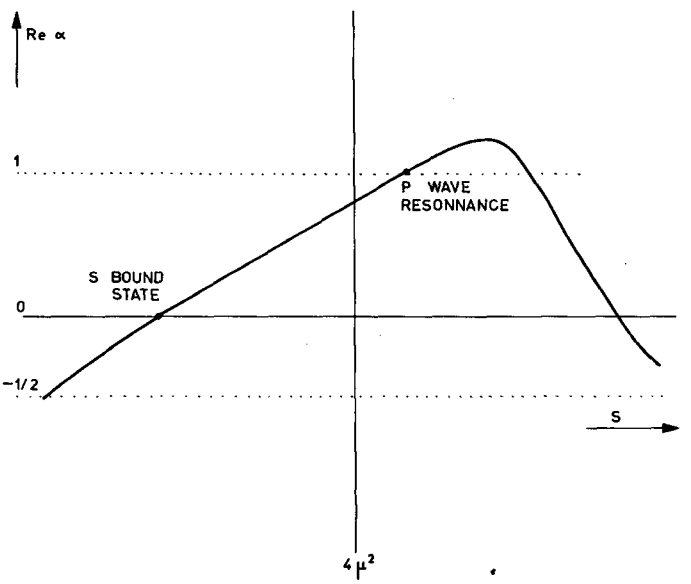

Fig. 7

A Regge trajectory: Re α as function of s

The points where the curve passes through integers are bound states to the left of this line, and resonances to the right of it. These curves are sometimes called Regge trajectories. For potential scattering, we have an α for each value of the radial quantum number. All that the Regge trajectory then does is to interpolate between the known bound states, like a Bohr angular momentum plot turned sideways.

Let us turn back to this question of how we can find the scattering amplitude from the spectral functions if we need subtractions, without introducing any optional quantities. We can do this, given the fact that the scattering amplitude satisfies the Regge formula. The reason is the following: we know the analytic properties of $P_\alpha(-z)$ in the z plane. It is analytic in z, except for a cut along the real axis from z = 1 onwards. The discontinuity across this cut is $P_\alpha(z) \sin(\pi\alpha)$. In particular for α an integer, $P_n(z)$ is analytic all the way, because this discontinuity is then equal to zero. If therefore the scattering amplitude satisfies the Regge formula, the spectral function A_3, which is the discontinuity as a function of z (or as a function of t, which is the same thing), will satisfy the formula

$$A_3(s,t) = A_{3B}(s,t) + \sum_n (2\alpha_n(s) + 1)\beta_n(s)\, P_{\alpha_n(s)}(z). \qquad (30)$$

The first term, which I will call the background term, comes from taking the discontinuity of the integral. We can get the discontinuity of the pole terms from the discontinuity of $P_{\alpha_n}(z)$, because this is the only place in them where the momentum transfer is involved.

Now, remember our problem is to calculate A, given A_3. We cannot do this by putting A_3 into the dispersion relation for A, because we have subtractions. However, the "background" term goes down like $t^{-\frac{1}{2}}$. So, for the background term, we can find A from A_3 by using the dispersion relation without subtractions. Now, if we know A_3 numerically, then we can separate it into Regge pole terms and the "background" term by equation (30), and find the α's and β's of the Regge pole terms. This is practical numerically [7]. And therefore all one needs to do is to put the α's and β's into Eq. on (29) and one has obtained the whole scattering amplitude from the spectral function without introducing any arbitrary subtractions.

All right then, so this is how we get over this apparent paradox of not being able to calculate the scattering amplitude from the spectral functions. We see that, once we know the Regge formula, we can calculate the scattering amplitude from the spectral functions, even when we have lots of subtractions, and therefore we do not lose information when the asymptotic behaviour gets worse and worse, as we increase the strength of the potential or the strength of the coupling. Now let us go on to the second point: can this get us over our difficulties of bad asymptotic behaviour, which gave us singular integral equations in the previous scheme? Now you remember that in the old method we found that if we only had large S-waves then things go like a constant at large t, which is all right, we do not mind that sort of behaviour. However, if we have large P-waves as well, then things go proportional to t at large t, and similarly, if we have large S, P and D-waves, then things go proportional to t^2. Now what we have in the Regge formula is slightly different. According to the Regge formula, the asymptotic behaviour as $t \to \infty$ depends on s. At those points where we have an S-wave

bound state, $\alpha = 0$, and the asymptotic behaviour is a constant just as before. Similarly, the asymptotic behaviour is still proportional to t at a P-wave bound state. However, even if we have a P-wave resonance or bound state, the asymptotic behaviour is no longer proportional to t everywhere. Now you know that the trouble resulted from the fact that, if the direct reaction has a bad asymptotic behaviour as a function of t, which does not matter. then the crossed reaction would have a bad asymptotic behaviour as a function of the energy, which does matter. However, the interchange of s and t only takes us into the physical region of the crossed reaction if s is negative, because in the crossed reaction t is the energy, s the momentum transfer, and for a physical reaction the momentum transfer is always negative. The energy is always positive. So we only expect to get into trouble if we have a bad asymptotic behaviour as $t \to \infty$, where s is negative. Therefore, in a Regge curve like Fig. 6, so long as we keep $\alpha < 1$ when s is negative, we would not expect to get into trouble, even though $\alpha > 1$ when s is positive, so that we could get P-wave resonances, and resonances of any higher angular momentum. So it is this dependence of α on s which can probably get us over the difficulty that the old procedure led us into. And it is, as a matter of fact, the equivalent of the spreading out of the wave function of a composite system in space that one gets if one uses any method other than dispersion relations.

I may say that the only way α can depend on s is if the spectral function oscillates. (In this case you observe that the spectral function does oscillate, because we have $t^{\alpha}(s)$, with α complex for positive s, and a number to a complex power is an oscillatory function). This follows from the ordinary dispersion relation in the energy:

$$A(s,t) = \frac{1}{\pi} \int ds' \frac{A_1(s',t)}{(s'-s)}. \tag{31}$$

Now suppose for a certain value of s', A_1 had bad asymptotic behaviour as a function of t, and suppose there were no oscillations, so that there could be no cancellation in sign. Then, if one performs the integral, the expression on the left will have the same bad asymptotic behaviour, whatever the value of s. So, if our spectral function does not oscillate, we cannot have an asymptotic behaviour as a function of t which depends on s. However, if A_1 does oscillate, this bad asymptotic behaviour may cancel out in the dispersion relation, and we can have the asymptotic behaviour depending on s, which is what does happen in the Regge formula.

Now you may think that this construction of the spectral function by the method of successive approximations, and subsequently isolating these Regge pole terms to get α's and β's, is something rather complicated However, it has actually been carried out in model calculations by BURKE and TATE [7] well, not for the complete relativistic case, but for the relativistic case where they assumed $V_{2,3}$ known, and they also did it for the potential case. The calculation is almost the same, and doing it for the potential case, they find that the results agree with the direct calculations of the Regge trajectories from the Schroedinger equation, which was made both by them and by LOVELACE at Imperial College [8]. So it looks as if this method is feasible. However, at the moment one does not have a

consistent set of integral equations for the problem [9]. The difficulty comes from the fact that if we have an input with Regge asymptotic behaviour in s, and use the unitarity equation in the s channel, we obtain an output with still worse asymptotic behaviour. The situation has not been clarified, that is all I can say at the moment.

In order to conclude this lecture, let me now go on to the conjectures, which have not been made till now, that the exact scattering amplitude also has a Regge asymptotic behaviour, and see what experimental consequences that will lead to.

We shall now use crossing symmetry, so that s and t are interchanged, and we shall assume that $A(s,t)$ behaves like $\beta(t)\, s^{\alpha(t)}$ as $s \to \infty$. And we now conjecture that this is true of the exact scattering amplitude, not only the scattering amplitude which is constructed from the strip approximation. Let us use this formula to analyse what happens in the diffraction peak region, where s is large and t is small and negative. This was first done independently, I think, by CHEW, by FRAUTSCHI, GELL-MANN and ZACHARIASEN, and by LOVELACE [10]. First of all, we shall take it as an experimental fact that the cross-section is constant for large values of the energy.

It follows from the optical theorem that $A(s,t)$ will go like s, as $s \to \infty$, at $t = 0$, which is the forward direction. Such a result does not follow from the Regge analysis, one has to put this in. We thus observe that $\alpha = 1$ at $t = 0$, and we can rewrite the Regge asymptotic behaviour as

$$A(s,t) \approx i\,\beta(t)\, s\, e^{-\eta(t)} \log s \qquad (32)$$

η is a decreasing function of t, so that as we go into the physical region (negative t), the scattering amplitude falls off, which is what one expects it to do. However, it does not fall off in the same way as one would expect in the optical model, for two reasons.

Firstly, it follows from the double dispersion relation that η must be analytic near $t = 0$, so we can put $\eta \approx \gamma t$, and we see that the scattering amplitude will go down exponentially as a function of t. In the optical model, if one assumes that the diffracting object has a Yukawa shape, which is a reasonable thing to do, the diffraction peak would go down much more slowly than exponentially, it would go down like an inverse power. Experimentally one definitely finds an exponential type of behaviour, rather than anything like an inverse power behaviour.

The second thing is that the width of the diffraction peak depends on the energy, because you could say that the width of the diffraction peak is essentially that value of t where $A(s,t)$ reaches some given value, and you see that the bigger the value of log s, the less distance you will have to go in t in order to reach any particular value. The width of the diffraction peak therefore shrinks logarithmically as the energy s is increased. Now, if that was the only thing, one would not be so surprised, because a shrinking diffraction peak corresponds to an increase in the size of the diffracting object. The bigger the diffracting object, the smaller the diffraction peak. So one could say that the peripheral regions of the nucleon were just getting more and more effective as the energy was being increased. However, the thing that is very surprising from any classical sort of analysis is that the total cross-section is remaining constant at the same time, so not only is the

nucleon getting bigger and bigger, but the inner part is getting more and more transparent at the same time, in order to keep the total cross-section constant. Such a feature is in conflict with any sort of physical visualization by means of an optical model, so if the Regge pole model is right then the optical model is wrong. And the experimentalists - I am not quite sure just under how much brain-washing - say they see a narrowing of the diffraction peak [11]. That would therefore mean that the optical model visualization is bad, and that there is an essential truth in this method of visualization. It does not prove that the conjecture of applying the Regge pole formula to the exact scattering amplitude is necessarily right. For instance, if the AMATI-FUBINI-STANGHELLINI analysis [12] is correct, and there are Regge cuts, in addition to Regge poles, then I think one could still fit the results to present experiments. So I would say that the experiments show that a Regge formula, or something of a similar sort which is rather more complicated, is correct.

In doing the unitarity condition, obviously, different quantum numbers do not get mixed up, so we get different uncrossed Regge trajectories associated with different quantum numbers. In terms of the crossed process, where the Regge asymptotic behaviour is in s, these correspond to different quantum numbers being exchanged. Pure diffraction scattering can't exchange any quantum number, so the Regge trajectory which produces the diffraction scattering, and goes through the value 1 at $t = 0$, must have the quantum numbers of the vacuum. Gell-Mann has called the object that gets exchanged a Pomeranchon, because if this Regge trajectory dominates, the Pomeranchuk theorem is valid.

Now, of course, for other kinds of elastic scattering, there will be other trajectories [13] coming below this one, and GELL-MANN, FRAUTSCHI and ZACHARIASEN [10] have proposed experiments to look at these lower trajectories, I do not think I need go into them, because we are sure to hear a lot more about that in further lectures.

But let me end by referring very briefly to a further conjecture made by CHEW and FRAUTSCHI [10] that fits in here. Remember, when we were speaking about the potential theory, I said that, in writing all these dispersion relations and double dispersion relations down, we might be prepared to include the S-wave subtraction explicitly. We do not want to include higher subtractions explicitly, because that would give much trouble with the unitarity equation. Thus, in addition to these double dispersion terms, we could have terms like

$$\int \frac{ds' f(s')}{(s'-s)} \qquad (33)$$

which do not depend on t. If one were doing the calculation in such a case, one would get all the angular momentum states other than the S-wave by integrating over the double spectral function, but for the S-wave one would conclude by performing an N/D calculation to find the function f. We thus observe that the asymptotic behaviour of the scattering amplitude consists of a Regge term $t^{\alpha(s)}$, coming from the double spectral function, plus another term which is asymptotically just a constant. In the Regge term the asymptotic behaviour does depend on s, in this other term it does not. The point is that, when we do have a subtraction, there is extra information which we

can introduce. For instance, we might put into the calculation a pole $a/(s - s_P)$, with two constants. The constants a and s_P have to be known beforehand, they are not given to us by the theory. And if we put in the pole like this, then in order to have consistency at least in the approximation schemes that have been tried up till now we would have to do a subtraction in t. We would have to calculate the S-waves by the N/D technique, they would not be given from the double spectral function, and the asymptotic behaviour would be given by the sum of two terms, one which does depend on s and one which does not.

Now the question is often raised whether there is really a distinction between elementary particles and non-elementary particles, and I do not think it is one that one can really answer definitely. There will probably always be conflicting views until we have a complete theory, and I rather think that, if we do ever get a complete theory, it is not going to make any distinction between elementary particles and non-elementary particles. But, if we do not have a complete theory at the moment, there may be some particles that one cannot get from the calculations - that one has to put in at the beginning - which provisionally one would call elementary particles, and some other particles which one can calculate, which one would not call elementary particles. And generally one would expect to have to introduce the masses and coupling constants for elementary particles, but to be able to get the masses and coupling constants for non-elementary particles in principle from the calculations.

When we put in the subtraction term from the beginning, we actually have to put in the mass s_P - the position of the pole - and the coupling constant a. So one may therefore take the viewpoint that terms like this, where the asymptotic behaviour does not depend on the energy, correspond to elementary particles, whereas terms where the asymptotic behaviour does depend on the energy correspond to bound states. If we subtract P-waves or higher angular momentum waves, it is going to give us the old trouble again, and therefore, from this way of looking at it, we can only have S-wave elementary particles, not P-wave and higher angular momenta. In other words, we get the same results we get from the renormalization theory studied by perturbation methods. And this again fits in with what I said about this s-dependence of the asymptotic behaviour corresponding to the spreading out of particles in space, because the elementary particle, which one naively supposes at least to have something in the middle which is not spread out in space, does not have this s-dependence of α in the asymptotic behaviour. The proposal made by Chew and Frautschi is, in fact, that, even at the present moment, there are no particles that one has to put in and call elementary, but that all particles correspond to the points where these Regge trajectories pass through the positive integers, and all resonances to the points when they pass near the positive integers. Fairly detailed graphs have been drawn up with these Regge trajectories for all the different quantum numbers, and the various resonances have been put in. I think at the moment the number of resonances is not yet so much greater than the number of quantum states that one would ascribe much significance to this fit, and I do not think that the authors claim that one should. But at least it is interesting to see how, if we assume that it is correct, the various particles could be fitted into the Regge scheme.

For those of you whose family keep on asking you what sort of work you are doing, there is an article by Chew, Gell-Mann and Rosenfeld which is going to appear in the Scientific American, so you will be able to get them to read that. There, all these Regge trajectories are drawn in a nice coloured diagram.

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