ALGEBRAIC ASPECTS OF WIGHTMAN QUANTUM FIELD THEORY

H. J. Borchers

Institut für Theoretische Physik Göttingen

Since the existence of quantum mechanics there are two schemes for the description of physical systems each with its own advantage.

Method one uses bounded operators and has the advantage of using mathematically nice objects so that a great technical machinery is at hand. Fundamental physical objects, however, do not belong to this class. By rewriting a physical theory in terms of bounded operators some detailed information is usually getting lost and one ends up with a scheme of quite general nature. This setting is extremely helpful for answering questions of general nature, but, I believe, that it is useless for constructing models.

The second method uses unbounded operators and has the advantage that we can formulate the physical principles better than in the first scheme. This leads to the fact that the objects of interest are usually explicitly defined. The disadvantage lies purely on the mathematical side since one runs into all kinds of problems and pathologies associated with unbounded operators.

In relativistic field theory the first scheme is known as Araki [1] - Haag-Kastler [2] theory of local rings. The second scheme exists in two versions, the LSZ formalism [3] and the Wightman field theory [4] . Since 1962 it is known [5] that a Wightman field theory is nothing else than representation of a well defined algebra, the so called test-function algebra. So the "only" problem which remains is the study of the representation theory of this algebra. In my lecture I will try to give a report on our knowledge of this algebra.

Let $\mathcal{S}(\mathbb{R}^4)$ be the Schwartz space of strongly decreasing C^∞ functions then $\underline{\mathcal{S}}$ will denote the tensor algebra with identity generated by $\mathcal{S}(\mathbb{R}^4)$, i.e. $\underline{\mathcal{S}}$ consists of terminating sequences of functions $\{f_o, f_1, \dots f_n \dots\}$ with $f_o \in \mathbb{C}$ and $f_n \in \mathcal{S}(\mathbb{R}^{4n})$. $\underline{\mathcal{S}}$ is a topological $*$ -algebra furnished with the direct sum topology. For details see e.g. W. Wyss [6] , G. Lassner and A. Uhlmann [7], H. J. Borchers [8]. A Wightman field theory can be identified with a cyclic representation of this algebra given by a state W which is invariant and which anihilates the twosided locality-ideal and the spectral-left-ideal.

The topology on $\underline{\mathcal{S}}$ is a locally convex vector-space topology so that addition and multiplication by scalars are continuous operations. The product, however, is only separately continuous.

When we talk about a representation we will assume that we have a dense subspace $\mathcal{D}$ in a Hilbert-space $\mathcal{H}$ and in general unbounded operators $\pi(f)$; $f \in \underline{\mathcal{S}}$, which are defined on $\mathcal{D}$ having the property $\pi(f)\,\mathcal{D} \subset \mathcal{D}$ and $(\psi, \pi(f)\varphi) = (\pi(f^*)\psi, \varphi)$. This seems to be the most general notation of $*$- representations. One can define more restricted representations as it is done for instance by R. Powers [9], [10] which leads to results which are closer to results similar to those of rings of bounded operators, see also Y. Itagaki [11]. But since we want to stay with Wightman's axioms we have to stick to our definition of representations.

One further requirement is the assumption that the representation is weakly continuous. This means the mapping

$$f \longrightarrow (\psi, \pi(f)\,\varphi) \qquad ; \qquad \psi, \varphi \in \mathcal{D}$$

shall be continuous. If the $*$- algebra is barrelled, as it is the case for $\underline{\mathcal{S}}$ and the representation is weakly continuous then it is also strongly continuous i.e.

$$f \longrightarrow \| \pi(f)\,\varphi \| \qquad ; \qquad \varphi \in \mathcal{D}$$

is continuous [12] thm 4.1 or [13] thm 3.7. A more detailed discussion of the whole problematic of continuity of representations can be found by G. Lassner [13]. Before discussing the representations of $\underline{\mathcal{S}}$ more thoroughly let us first look at the algebra itself.

Let us first look at the purely algebraic properties. One knows:

a) $\underline{\mathcal{S}}$ contains an identity

b) is free of divisors of zero

c) has a trivial center

d) only multiples of the identity are invertible

e) the identity is the only idempotent element

f) has a trivial radical.

Furthermore A. Uhlmann showed [14]

g) In $\underline{\mathcal{S}}$ exists an euclidian algorithm.

This allows to define prime elements in $\underline{\mathcal{S}}$ and to disentangle the structure of left ideals which are generated by a finite number of elements.

If we denote by $\underline{\mathcal{S}}_h$ the real subspace of hermitian elements, then one defines the positive cone by

$$\underline{\mathcal{S}}^+ = \left\{ \sum_i f_i^* f_i \qquad ; \text{ the sum is convergent} \right\}$$

Since $\underline{\mathcal{S}}$ contains the identity follows that $\underline{\mathcal{S}}^+ - \underline{\mathcal{S}}^+ = \underline{\mathcal{S}}_h$. Furthermore $\underline{\mathcal{S}}^+$ is a proper cone this means $\underline{\mathcal{S}}^+ \cap - \underline{\mathcal{S}}^+ = \{0\}$. W. Wyss [6] observed that $\underline{\mathcal{S}}^+$ is a strict b-cone that is for every bounded set $\mathcal{B} \subset \underline{\mathcal{S}}_h$ exists a bounded set $\mathcal{B}' \subset \underline{\mathcal{S}}^+$ such that $B \subset B' - B'$. H. J. Borchers [15] proved that $\underline{\mathcal{S}}^+$ is a closed cone. J. Yngvason [16] showed that $\underline{\mathcal{S}}^+$ is a cone with base. F. Brauer [17] and H. J. Borchers [18] showed that the positive cone $\underline{\mathcal{S}}^+$ is generated by its extremal rays, more precisely every element $f^* f$ can be written in the form

$$f^* f = \sum_i g_i^* g_i$$

where $g_i^* g_i$ lies on an extremal ray and the sum converges in $\underline{\mathcal{S}}$. Finally one has to mention that the positive cone $\underline{\mathcal{S}}^+$ has no topological (and algebraical) interior point. This fact has the consequence that almost all extension problems of states are unsolvable.

Extremely little is known about the structure of closed left ideals. In particular one would like to have an algebraic characterization of those left ideals which are the intersection of left kernels of states. This property is automatic for closed left ideals in C^*-algebras, but not for $\underline{\mathcal{S}}$ since one can construct examples of left-ideals which do not have this property. But the two interesting ideals have the property that they are intersections of left-kernels. This can be derived using the explicit structure of these ideals.

Next we are turning to the dual-space $\underline{\mathcal{S}}'$. We will denote by $\underline{\mathcal{S}}'^+$ the set of $\omega \in \underline{\mathcal{S}}'$ with the property $\omega(f^* f) \geq 0$ for all $f \in \underline{\mathcal{S}}$ and by $\underline{\mathcal{S}}_h'$ the set of real functionals.

Using the fact that $\underline{\mathcal{S}}^+$ is a proper cone it follows directly that $\underline{\mathcal{S}}'^+ - \underline{\mathcal{S}}'^+$ is a dense subspace of $\underline{\mathcal{S}}_h'$. But J. Yngvason [17] gave an example of a continuous linear functional which cannot be decomposed into positive ones. In the same paper he derived a necessary and sufficient condition that a functional can be decomposed into positive functionals. We know already that the product is only separately continuous. Let $\mathcal{T}$ be the original topology and $\mathcal{N}$ be the final topology such that the product map

$$\underline{\mathcal{S}}(\mathcal{T}) \times \underline{\mathcal{S}}(\mathcal{T}) \longrightarrow \mathcal{S}(\mathcal{N})$$

is simultaneous continuous, then a functional is decomposable iff it is continuous in the $\mathcal{N}$ topology. This result is in agreement with the earlier result that for any state ω the function $f \rightarrow \omega(f^* f)^{1/2}$ is continuous.

That there exist linear functionals on $\underline{\mathcal{S}}$ which are not linear combinations of states follows from the fact that $\underline{\mathcal{S}}$ admits representations by unbounded operators. Namely if π is such a representation and $\psi \in \mathcal{D}$ but φ is arbitrary in $\mathcal{H}$ then the matrixelement

$$f \rightarrow (\varphi, \pi(f)\psi)$$

is continuous on $\underline{\mathcal{S}}$ but if it is not continuous in ψ then one cannot make a decomposition into states. One further result of Yngvason [17] is that every continuous linear functional is of this form showing that there exist no strange continuous functionals on $\underline{\mathcal{S}}$.

Since $\underline{\mathcal{S}}$ is an algebra with identity follows that the set of states (normalized positive functionals) form a base of the cone $\underline{\mathcal{S}}'^{+}$. One also can show that this cone contains no topological interior point. More interesting is the fact that the cone $\underline{\mathcal{S}}'^{+}$ is generated by its extremal rays. This has been proved by Borchers and Yngvason [12] and I think I should make some remarks about the results of this paper.

If ω is a state and π_ω the representation given by the G.N.S. construction and if ω can be decomposed this means

$$\omega = \lambda\,\omega_1 + (1-\lambda)\,\omega_2 \quad ; \quad 0 < \lambda < 1$$

then exists a bounded positive operator C on $\mathcal{H}_\omega$ such that $\omega_1(f^* \cdot g) = (\pi_\omega(f)\Omega, C\,\pi_\omega(g)\Omega)$. Since the operators $\pi_\omega(f)$ are generally unbounded follows that the expression $\pi_\omega(f)\,C\,\Omega$ must not necessarily be defined. With other words the operator C belongs only to the weak commutant of π_ω . As in the case of bounded operators the weak commutant is invariant under taking the adjoint of an operator and it is also closed in the weak operator topology. But in contrast to the bounded case it is not an algebra in general. This has the consequence that one has to alter the decomposition theory in such a way that it is also applicable to rings of unbounded operators. The main clue to this program is following observation: If π_1 and π_2 are two cyclic representations and if π_1 and π_2 are strange then the vector $\sqrt{\lambda}\,\Omega_1 + \sqrt{1-\lambda}\,\Omega_2$ need not necessarily be a cyclic vector for $\pi_1 \oplus \pi_2$, with other words the cyclic subspace generated by the vector $\sqrt{\lambda}\,\Omega_1 + \sqrt{1-\lambda}\,\Omega_2$ need not be dense in $\mathcal{H}_1 \oplus \mathcal{H}_2$. Therefore if one wants to decompose the restriction of $\pi_1 \oplus \pi_2$ to the cyclic subspace one first has to recover the full Hilbert-space $\mathcal{H}_1 \oplus \mathcal{H}_2$. That this can be done is essential part of the paper [12]. The outcome is the following:

Theorem: Let π be a representation of $\underline{\mathcal{S}}$ defined on a dense nuclear domain $\mathcal{D}$ of a separable Hilbert space $\mathcal{H}$. Then exists a standard measure space (Λ, μ) a separable Hilbert space

$$\mathcal{H}' = \int_\Lambda^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda)$$

such that there exist dense domains $\mathcal{D}_\lambda \subset \mathcal{H}_\lambda$, representations π_λ defined on $\mathcal{D}_\lambda$ with :

a) if $\lambda \longrightarrow \varphi_\lambda \in \mathcal{D}_\lambda$ is μ-measurable then also
$\lambda \longrightarrow \pi_\lambda(f)\,\varphi_\lambda$ is μ-measurable
and fulfills some continuity property.

b) $\mathcal{H}$ is a closed subspace of $\mathcal{H}'$ such that the projection of $\mathcal{D}$ into $\mathcal{H}_\lambda$ is $\mathcal{D}_\lambda$.

c) The representations π_λ have a trivial weak commutant.
(all these properties hold μ a.e.)

That one can also decompose unbounded representations is probably only true when $\mathcal{D}$ is a nuclear space which is the case for cyclic representations of nuclear algebras. Since this is the case for $\underline{\mathcal{S}}$ we can decompose every state on $\underline{\mathcal{S}}$ into extremal ones :

$$\omega = \int_\Lambda \omega_\lambda \, d\mu(\lambda)$$

where ω_λ are μ - a.e. extremal states that is the representation π_{ω_λ} all have a trivial weak commutant.

If the state ω is a Wightman functional then all the states ω_λ in the above decomposition are also Wightman functionals. This shows that the set of Wightman states form a face in the set of all states. If in a cyclic Wightman theory with cyclic vector Ω this Ω is also the only invariant vector in the representation Hilbert space then this representation has a trivial weak commutant, which means that the corresponding Wightman state is extremal. As one can show by examples the converse conclusion is not correct this means there exists extremal Wightman functionals such that the corresponding representation has an infinite dimensional subspace as invariant subspace. This effect does not occur in the theory of local rings, and is due to pathologies in the theory of commuting unbounded operators.

Symanzik [19] introduced euclidean fields as some kind of "analytic continuation" of Wightman fields. It is natural to apply the above decomposition theory also to such fields. This has been done by Borchers and Yngvason [20]. The question here is not the decomposition of a state on an abelian algebra into extremal states but rather the question when can a state of an abelian algebra be decomposed into characters. This is not always the case, but necessary and sufficient for this is that the state must be strictly positive, that is, if $P(x_1 \dots x_n) \geq 0$ is a positive polynomial then $\omega\,(P(f_1, \dots f_n))$ has to be non negative for all $f_i \in \underline{\mathcal{S}}_h$. If the Schwinger functional is strict positive then it can be represented by a cylinder-measure over $\mathcal{S}'(\mathbb{R}^4)$.

If one starts from a Wightman state then we can analytically continuate to the Schwinger points. This, however, defines a continuous linear functional only on some subspace of the symmetric part of $\underline{\mathcal{S}}$. As Osterwalder and Schrader [22] remarked this can be extended to a continuous linear functional to the whole symmetric part of $\underline{\mathcal{S}}$. Since the positive cone in $\underline{\mathcal{S}}$ has now interior points it seems to me a hopeless problem to show (except in concrete examples) that there exists an extension as positive functional. Since one does not know explicitly the structure of all positive polynomials,it is even much more hopeless to the extension as a strict positive functional. The beauty of strict positive functionals is that they can be represented by cylinder measures. In order to get a wider class one can study functionals which are representable by not necessarily positive cylinder measures, i.e. functionals which can be decomposed into strict positive ones. This program is under investigation in Göttingen and we could already say that such functionals can be characterized by pure continuity property. The question whether every analytic continuation of a Wightman functional has this continuity property is unsettled.

One group of problems which is still completely in the dark is the extension of positive functionals given on subspaces or subalgebras to a positive functional on all of $\underline{\mathcal{S}}$. For Wightman functionals W. Wyss [22] could give some conditions. They are still in such a form that it is practically impossible to use them. But nevertheless Lassner and Hofmann [23] could use these conditions and give an abstract proof that there exist Wightman fields different from the generalized free fields.

It is known that for Wightman fields only unbounded representations of are of interest. But, nevertheless, it is known from [8] and [15] that $\underline{\mathcal{S}}$ admits a large number of representations by bounded operators. This means we have on $\underline{\mathcal{S}}$ a great family of continuous norms and semi-norms which are also C^*-norms. M. Dubois-Violette [24] is trying to explore this fact. His present investigations are in the direction of the infinite dimensional moment problem. But I think that here is a new technique which might add to the understanding of the algebra $\underline{\mathcal{S}}$ and which has to be explored in the future.

As final subject I will talk on combination of states. This is a method of combining two states to a new one in a nonlinear fashion. These combinations are mainly possible because $\underline{\mathcal{S}}$ is a graded algebra. The first subject is the s-product which is well known in statistical mechanics to obtain the Meyer expansion from correlation function. In field theory it is used for obtaining the truncated functional ω^t from ω . The relation is

$$\omega = \exp \omega^t |_s$$

where we have to take the powers in the s-product. If now ω_1 and ω_2 are two states on $\underline{\mathcal{S}}$ then

$$\omega = \omega_1 \, s \, \omega_2$$

defines a new state on $\underline{\mathcal{S}}$. This operation amounts in the language of operators to the following: Let $A(x)$ and $B(x)$ be the field obtained from ω_1 and ω_2 and $C(x)$ the field obtained from ω then we have:

$$C(x) = \text{cyclic part of } \big(A(x) \quad 1_2 + 1_1 \quad B(x) \big).$$

(For the proofs see [8]). Of interest is now the inverse process namely the decomposition of states into s-products. The reason for this is an observation made by

Symanzik, namely, if $A(x)$ is a field which has complete asymptotic fields and if W is the Wightman functional belonging to $A(x)$ then the field $B(x)$ constructed from the functional $W s W$ has in general no longer complete asymptotic fields. This makes the interest in the s-product evident. Some results in this direction have been obtained by Hegerfeldt [25], but, I need some notations before I can describe them. A state ω on $\underline{\mathcal{S}}$ is called a prime state if it cannot be written as the s-product of two other states. On the other hand a state is called infinitely divisible if it is the n-th s-power of another state for every n = 1, 2, ... Hegerfeldt's result is now the following: Every state ω on $\underline{\mathcal{S}}$ can be decomposed into the s-product of at most a countable number of prime states and a rest where the rest is either trivial or infinitely divisible.

His proof is a pure existence proof. What would be of great interest, any characterisation of prime states, is still missing. But nevertheless one can prove the existence of prime states since one can characterize the infinitely divisible states and show that the Wick polynomials do not belong to this class. The characterization of this class is by means of the truncated functional ω^t. The result is: A state ω is infinitely divisible if and only if ω^t is positive on the ideal of element with vanishing zeroth component. The generalized free fields belong to this class.

There is still another combination of states which works only for Wightman functionals. But since nothing has been done with it I will refrain from explaining it. There is also another group of results which I will not bring here. These are results obtained in connection with constructive field theory. Most of them start from additional assumptions suggested by models and not from a general axiomatic.

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Discussion

Question by R. Arens:

On the algebra $\mathcal{S}$, is the map $x \to xx^*$ perhaps continuous?

Answer:

No, since the algebra contains an identity, continuity of the map $x \to xx^*$ and joint continuity of the product are the same.

Question by M. Winnink:

Can the decompositions you have been discussing be related to a simplicial structure of the set of states of interest?

Answer:

For the decomposition treated here the set of states do not form a simplex even not for the set of Wightman states. I expect that no face of the set of states, which might appear in physics, will form a simplex. The reason for this is that the moment problem has a unique solution only under very restrictive conditions. And I cannot imagine that such condition will appear naturally.