ASYMPTOTIC COMPLETENESS IN THREE-PARTICLE QUANTUM MECHANICAL SCATTERING

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The purpose of this note is to report on some recent results in three-particle non-relativistic spinless quantum mechanical potential scattering [1,2,3]. The principal result of these investigations can be formulated in the following theorem: Let

$$H = H_o + \Sigma V = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} + \sum_{i<j} V_{ij}(\underset{\sim}{x}_{ij})$$

acting in $L^2(\mathbb{R}^6)$ be the Hamiltonian for the three-particle system with center of mass motion removed, where (i) $V_{ij}(\underset{\sim}{x}) \varepsilon L^{3/2+\delta}(\mathbb{R}^3) \cap L^{3/2-\delta}(\mathbb{R}^3)$ for some $\delta > 0$; (ii) the two body Hamiltonians

$$h_{ij} = \frac{k_{ij}^2}{2m_{ij}} + V_{ij}(\underset{\sim}{x}_{ij}),$$

acting in $L^2(\mathbb{R}^3)$, have only strictly negative energy eigenstates $\{\phi_{ij}^n\}_n$; and (iii) the operator-valued functions $W_{ij}(x \pm i0)$ $= |V_{ij}|^{1/2}\, \hat{r}_{ij}(x \pm i0)\, |V_{ij}|^{1/2}, x \varepsilon \mathbb{R}$, acting in $L^2(\mathbb{R}^3)$ are bounded. Here,

$$\hat{r}_{ij}(z) = r_{ij}(z) - \sum_n \frac{|\phi_{ij}^n\rangle\langle\phi_{ij}^n|}{\lambda_{ij}^n - z},$$

$r_{ij}(z)$ is the resolvent for h_{ij}, and λ_{ij}^n is the eigenvalue corresponding to ϕ_{ij}^n. Then the wave operators exist, and are complete in the sense that the direct sum of the ranges of the wave operators is the absolutely continuous subspace of the Hilbert space with respect to H. The scattering operator is therefore unitary. (Condition (iii) is assured if in addition to conditions (i) and (ii) being satisfied, V_{ij} is of the form $V_{ij}(\underset{\sim}{x}) = (1+|\underset{\sim}{x}|)^{-1-\varepsilon}(v_{ij}^p(\underset{\sim}{x}) + v_{ij}^\infty(\underset{\sim}{x}))$ with $\varepsilon > 0$, $v_{ij}^p \varepsilon L^p(\mathbb{R}^3)$, $p > 3/2$, and $v_{ij}^\infty \varepsilon L^\infty(\mathbb{R}^3)$, and the operators $W_{ij}(0)$ exist [4,5].)

These results generalize the work of Faddeev on the three body problem because more singular local behavior and less restrictive long range behavior of the potentials are both accommodated. A similar theorem holds for space dimensions n>3 although no analogous theorem is known for one or two space dimensions. An additional result of Ginibre and Moulin establishes that the negative spectrum of H contains no singular continuous part [1]. The positive singular spectrum is contained in a closed set of measure zero, but the existence of positive singular continuous spectrum with these interactions remains an open question.

The proof of the theorem is carried out primarily in configuration space using time independent techniques. Weighted Hilbert spaces [1] or L^p spaces [2] replace the Banach spaces of Hölder continuous functions of momenta employed by Faddeev [6]. Well established methods (e.g. criteria for compactness of operators in L^p, Kato's theory of smooth operators [7] and Agmon's theory using weighted Hilbert spaces [4]) become available with a consequent simplification of the analysis.

In outline, the proof begins by considering Faddeev's equations, which are modified appropriately to exploit the fact that the operator valued functions $|V_{ij}|^{1/2}(H_o-z)^{-1}|V_{ik}|^{1/2}$: $L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$, $(ij)\neq(ik)$ are compact and norm Holder continuous in z for $z = x\pm i0$, $x\varepsilon\ \mathbb{R}$. This follows easily from the estimate [8],

$$||V_{ij}|^{1/2}e^{-itH_o}|V_{ik}|^{1/2}|| < c(q)t^{-3/q}||V_{ij}{}^{1/2}||_{L^q(\mathbb{R}^3)}\,||V_{ik}{}^{1/2}||_{L^q(\mathbb{R}^3)}.$$

The components of the solution to these equations, which are operator valued analytic functions of z for Im $z \neq 0$, may be continued to the real axis where they exist as bounded operators in appropriate weighted Hilbert spaces or L^p spaces for $z = x\pm i0$, $x\varepsilon\ \mathbb{R}$ and x away from a closed set of measure zero. The components are then used to construct the full resolvent R(z) and give a meaning to it for $z = x\pm i0$ away from the singular set.

In reference [2], the wave operators are defined in terms of spectral integrals, i.e. Riemann Stieltjes integrals, with operator-valued integrands and projection valued measures. For example, the wave operators relating to the free channel are given by

$$\Omega^o_\pm = 1 - \operatorname*{s-lim}_{\varepsilon\to 0} \int R(\mu\pm i\varepsilon)\sum_{i<j} V_{ij}\,dE_o(\mu)$$

with $E_o(\mu)$ the spectral family corresponding to H_o, cf. [9]. The control over R(z) for z near the real axis obtained through the modified Faddeev equations is sufficient to imply the existence of these expressions along with the completeness of the wave operators.

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