FIELD THEORY IN A STRONG MAGNETIC FIELD AND QUANTUM HALL EFFECT

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ABSTRACT

We develop a field theory of 2d electrons in a strong magnetic field on a representation in which electrons are described by multi-component local fields defined on lattice sites. The modified Ward-Takahashi identity, and the exact low energy theorem for σ_{xy} are derived. The absence of higher order QED correction to the quantized σ_{xy} in the localized state region is shown and a small correction in the presence of extended states at Fermi energy is computed.

Electrons in condensed matter physics is described by non-relativistic Schrödinger field in which spin degree of freedom is decoupled. Propagator is a single-component complex function and is topologically rather trivial compared to the Dirac propagator which is expressed by a matrix. Axial anomaly and related phenomena of relativistic quantum field theory are connected with the multi-component character of the Dirac field and would not exist in single-component fermion theory. Anomaly-related phenomena would not be there in many-electron system where the Schrödinger field expresses the electron.

Magnetic field, however, changes one particle properties drastically. Electrons in the strong magnetic field can be described by many-component field even in the low energy region because the electron's energy is split into Landau levels. With a suitable representation,the electron field has many components. It is seen clearly that the many electron system in the magnetic field [1] has anomaly-related phenomena in that representation. We develop a theory [2] based on such representation in two spatial dimensions and prove an exact low energy theorem for the Hall conductance. We show further that the electrons are localized if their energies are different from the bare Landau level's energies, and the Hall conductance is given by the exactly quantized value $e^2/h \times N$ in the localized state regions of a system with impurities. The absence of higher order correction and of finite size effect in the localized state regions are shown and a small correction due to extended states at Fermi energy is computed.

An action

$$\begin{aligned} S = \int dt d\vec{x} \Big[&\varphi^\dagger(x) \Big[i\hbar \frac{\partial}{\partial t} + eA_0 \\ &- \frac{(\vec{p} + e\vec{A}^{ext} + e\vec{A})^2}{2m} \Big] \varphi(x) \\ &- \varphi(x)^\dagger \varphi(x) V(x)^{im} + L_{int} \Big] \\ \partial_1 A_2^{ext} - \partial_2 A_1^{ext} &= B \end{aligned} \tag{1}$$

describes a system with magnetic field B and impurity $V_{(x)}^{im}$. Relative coordinates and center coordinates which defined by

$$\begin{aligned} \xi &= \frac{1}{eB}(Py + eAy), \qquad X = x - \xi \\ \eta &= \frac{-1}{eB}(Px + eAx), \qquad Y = y - \eta \end{aligned} \tag{2}$$

and satisfy

$$\begin{aligned} [\xi, \eta] &= -[X, Y] = -i\hbar(eB)^{-1} \\ [\xi, X] &= [\xi, Y] = [\eta, X] = [\eta, Y] = 0 \end{aligned} \tag{3}$$

are used in the description of the electrons in a magnetic field. One body Hamiltonian is proportional to the sum of squares of relative coordinates and has no center coordinates. We use direct products of harmonic oscillator eigenfunctions of relative coordinates and coherent states of center coordinates,

$$
\begin{aligned}
&U_l(\xi,\eta)|R_1,R_2\rangle \\
&\frac{e^2B^2}{2m}(\xi^2+\eta^2)U_l = E_lU_l \\
&(X+iY)|R_1,R_2\rangle = (R_1+iR_2\rangle|R_1,R_2\rangle
\end{aligned} \tag{4}
$$

as a complete set of functions.

The completeness leads the center coordinates lying defined on lattice sites of spacing a [3],

$$
\begin{aligned}
(R_1,R_2) &= a(m,n) \\
a &= \sqrt{\frac{2\pi\hbar}{eB}}
\end{aligned} \tag{5}
$$

We can construct dual bases as satisfying

$$
\begin{aligned}
\langle \widetilde{R_1,R_2}|R_1',R_2'\rangle &= \delta_{R_1,R_1'}\delta_{R_2,R_2'} - \frac{1}{\Sigma_{R_1,R_2}} \\
\sum_{R_i}\langle \widetilde{R_1,R_2}|R_1',R_2'\rangle &= \sum_{R_i'}\langle \widetilde{R_1,R_2}|R_1',R_2'\rangle = 0
\end{aligned} \tag{6}
$$

The coherent states have minimum uncertainty $\Delta X^2\Delta Y^2$ that is allowed from the commutation relations and are most localized in both directions. Harmonic oscillator eigenfunctions $U_l(\xi,\mu)$ are also well localized and hence the product are well localized in both directions around their centers. This property is important when we apply a multi-pole expansion later.

The field expansions

$$
\begin{aligned}
\varphi &= \sum_{e,\vec{R}} a_l(\vec{R},t)U_l(\xi,\varphi)\otimes|R_1,R_2\rangle \\
\varphi^\dagger &= \sum_{e,\vec{R}} b_l(\vec{R},t)U_l(\xi,\varphi)\otimes\langle\widetilde{R_1,R_2}|
\end{aligned} \tag{7}
$$

are substituted to the action. The operators $\{a_l(\vec{R}), b_l(\vec{R})\}$ satisfy the equal time commutation relation and describe the system. They have Landau level index l in addition to the coordinates defined on the lattice sites. The propagator becomes a matrix and has a non-trivial topological property.

The electromagnetic current is conserved and equal time commutation relation between the charge density and the field is modified from that of local field. It is possible to apply a multi-pole expansion,however. Using the commutation relations, we are able to derive a relation between the vertex part and the propagator,Ward-Takahashi identity. The Ward-Takahashi identity is different from the ordinary form but is transformed to the ordinary form by a suitable transformation. The Hall conductance is given by the first derivative of the current correlation function and is expressed as

$$
\begin{aligned}
\sigma_{xy} &= \frac{e^2}{3!}\epsilon^{\mu\nu\rho}\frac{\partial}{\partial p_\rho}\pi_{\mu\nu}(p)\bigg|_{p=0} \\
&= \frac{e^2}{2\pi}\frac{1}{24\pi^2}\int dq\,\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left[\frac{\partial\tilde{S}^{-1}}{\partial q_\mu}\tilde{S}\frac{\partial\tilde{S}^{-1}}{\partial q_\nu}\tilde{S}\frac{\partial\tilde{S}^{-1}}{\partial q_\rho}\tilde{S}\right]
\end{aligned} \tag{8}
$$

when the Ward-Takahashi identity is substituted. The last formula shows that σ_{xy} is a topological invariant of the mapping defined by the propagator $\tilde{S}(p)$ and is not modified by interactions and disorders[4]. In fact a theorem that σ_{xy} is computed from the one loop diagrams and is not modified by higher order corrections has been proved [5] in the system without infra-red divergence. σ_{xy} agrees exactly to $e^2/h \times N$ from both theorem in the absence of infra-red divergence.

Localization and plateau

The short range impurity potential term in the action is expressed by

$$
\int dt \sum_{l_i,R_i} g b_{l_1}(\vec{R}_1)a_{l_2}(\vec{R}_2)V_{l_1l_2}(\vec{R}_1,\vec{R}_2;\vec{x}_1) \tag{9}
$$

with a transformation function $V_{l_1l_2}(\vec{R}1,\vec{R}2;\vec{x}_1)$. The $V_{l_1l_2}(\vec{R}_1,\vec{R}_2;\vec{x}_1)$ decreases fast as $|\vec{R}_1-\vec{x}_1|$ or $|\vec{R}_2-\vec{x}_1|$ increasees, so the eigenvalue equation

$$[E_{l_1}\delta_{l_1 l_2}\delta(\vec{R}_1,\vec{R}_2)+gV_{l_1 l_2}(\vec{R}_1,\vec{R}_2;\vec{x}_1)]U_{l_2}^{(\alpha)}(\vec{R}_2) = E^{(\alpha)}U_{l_1}^{(\alpha)}(\vec{R}_1) \tag{10}$$

has localized eigenfunctions if $E^{(\alpha)}$ is different from E_{ei}. Dilute short range impurities are treated similarly. The results are equivalent. Wave functions are localized if the energy E_α is different from the Landau level's energy E_l. If all the states near the Fermi energy are localized, there is no infra-red divergence. The above theorem can be applied,and σ_{xy} is given by one of the exactly quantized values. σ_{xy} stays constant in the localized state regions.

Higher order QED corrections.

The formula that σ_{xy} is expressed by a propagator is correct in four dimensional space time as well if the current is integrated in one direction. The re-scaling of the propagator, which is necessary in the renormalization program, does not modify it. Hence the fermion field renormalization does not modify σ_{xy}. The change renormalization modifies the absolute value of the electric current and of the vector potential. The effect is equivalent to the both values due to the Ward-Takahashi identity and the final value of σ_{xy} is given by $e^2/h \times N$ with the electron charge in the vacuum. Hence there is no higher order QED corrections.

Extended states at fermi energy.

If there are few extended states near the Fermi energy, σ_{xy} deviates from the quantized value and σ_{xx} does not vanish. Both of them are computed in the presence of dilute short range impurities. We have

$$\Delta\sigma_{xy} = \frac{c^2}{2m^2}\sum_\gamma \frac{1}{\langle |a_0(k_F,\gamma)|^2\rangle}\sigma_{xx}\Delta k_F \tag{11}$$

using parameters of the theory. $\Delta\sigma_{xy}$ is proportional to σ_{xx}.

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