DISS. ETH NO. 19597

Three Results about the Vacuum Einstein Equations

A dissertation submitted to
ETH ZURICH

for the degree of
Doctor of Sciences

presented by

MICHAEL REITERER
Master of Science ETH
Born February 7, 1983
Citizen of Italy

accepted on the recommendation of

Prof. Dr. Jürg Fröhlich, examiner
Prof. Dr. Gian Michele Graf, co-examiner
Prof. Dr. Eugene Trubowitz, co-examiner

2011

Abstract

This thesis comprises three mathematically rigorous papers about the vacuum Einstein equations. Their respective titles and abstracts are:

A formalism for analyzing vacuum spacetimes: The Einstein vacuum equations in the formulation developed by Newman, Penrose and Friedrich are expressed in terms of a Lie superbracket. Differential identities are derived from the super Jacobi identity. This perspective clarifies the covariance properties of the equations. The equations are intended as a tool for the analytic study of vacuum spacetimes.

Strongly Focused Gravitational Waves: Christodoulou proved that trapped spheres can form in evolution from a generic initial state, through the focusing of gravitational waves. His work is the motivation for the present paper, in which we consider the same physical problem, using very different mathematical methods. Our approach is based on a controlled "far field expansion". By a systematic use of scaling symmetries, we regularize Christodoulou's singular "short pulse method", rigorously track vacuum solutions by the far field expansion and exhibit trapped spheres that first appear deep inside the far field region. Our presentation is self-contained. In the final section, we present a detailed outline of the construction of another, more subtle, expansion that allows us to continue the solutions beyond the far field region to within any fixed "finite distance" from the (expected) singularity. From a methodological perspective, the underlying aim of this paper is the development of a general method for constructing solutions to the vacuum Einstein equations by controlled expansions.

The BKL Conjectures for Spatially Homogeneous Spacetimes: We rigorously construct and control a generic class of spatially homogeneous (Bianchi VIII and Bianchi IX) vacuum spacetimes that exhibit the oscillatory BKL phenomenology. We investigate the causal structure of these spacetimes and show that there is a "particle horizon".

These are three collaborations with Dr. Eugene Trubowitz.

Zusammenfassung

Diese Dissertation umfasst drei mathematisch rigorose Arbeiten über die Einsteinschen Vakuumfeldgleichungen, nämlich:

Ein Formalismus zur Untersuchung von Vakuumraumzeiten: Für die Newman-Penrose-Friedrich Formulierung der Einsteinschen Vakuumfeldgleichungen wird eine Lie-Super-Klammer eingeführt. Wichtige differentielle Identitäten dieser Formulierung folgen aus der Super-Jacobi-Identität. Durch diesen Zugang werden die Kovarianzeigenschaften der Gleichungen deutlich gemacht.

Stark fokussierte Gravitationswellen: Christodoulou hat gezeigt, dass das Fokussieren von Gravitationswellen zur Entstehung von "trapped spheres" führen kann. In der vorliegenden Arbeit untersuchen wir dasselbe physikalische Phänomen, verwenden dazu aber andere mathematische Methoden, insbesondere rigorose Fernfeld-Entwicklungen. Mittels geeigneter Skalierungen regularisieren wir Christodoulous "short pulse method" und zeigen, dass "trapped spheres" bereits tief in der Fernfeldzone entstehen können. Schliesslich skizzieren wir eine zweite Entwicklung, die es erlaubt, die Lösungen auch jenseits der Fernfeldzone zu kontrollieren. Ziel dieser Arbeit ist es auch, eine allgemeine Methode für die rigorose Konstruktion von Vakuumraumzeiten durch Entwicklungen zu erarbeiten.

Die BKL-Vermutungen für räumlich homogene Raumzeiten: Wir konstruieren eine generische Familie von räumlich homogenen (Bianchi VIII und Bianchi IX) Vakuumraumzeiten, die BKL-artige Oszillationen aufweisen. Wir untersuchen die kausale Struktur dieser Raumzeiten und zeigen, dass sie "Teilchenhorizonte" haben.

Diese drei Arbeiten entstanden in Zusammenarbeit mit Dr. Eugene Trubowitz.

Acknowledgments

I would like to express my gratitude to Dr. Jürg Fröhlich, for his advice, encouragement and support. I thank Dr. Gian Michele Graf for his interest in this thesis and many thought-provoking questions.

I am especially indebted to my collaborator Dr. Eugene Trubowitz. His broad interest in science and his unique adventurous character are a great source of inspiration. I enjoyed our many conversations from which I learned so much, and I look forward to future collaboration.

I wish to thank Dr. Joel Feldman and Dr. Horst Knörrer for going through the bigger part of this thesis, and more generally for their help and support. I also thank Dr. Thomas Spencer, for his ongoing interest and advice at all levels.

Last but not least, I enjoyed working with fellow members of assistant group 5 at the Mathematics Department of ETH Zurich.

General Introduction

This thesis comprises three papers:

[RT1] A formalism for analyzing vacuum spacetimes (p. 15-36)
[RT2] Strongly Focused Gravitational Waves (p. 37-132)
[RT3] The BKL Conjectures for Spatially Homogeneous Spacetimes (p. 133-181)

These are three collaborations with Dr. Eugene Trubowitz.

This thesis is a mathematically rigorous piece of work in theoretical physics. The common topic of [RT1], [RT2], [RT3] are the vacuum Einstein equations. The main mathematical methods used are: differential geometry and the language of vector bundles [RT1]; quasilinear symmetric hyperbolic systems and expansions [RT2]; ordinary differential equations [RT3].

The three papers are included in their entirety, each with a dedicated technical introduction, and with a separate list of references. The purpose of the present general introduction is to put these works in a common, wider context, to explain their mutual relationships, and foremost to emphasize the original scientific contributions, by a comparison with existing literature. Original scientific contributions are highlighted visually and thus easy to spot.

Fix a *real, constant, symmetric* matrix (g^{ab}) with signature $(-,+,+,+)$. Let $\mathrm{O}(1,3)$ be the 6-dimensional group of real matrices $({\Lambda_a}^b)$ with $g^{ab}{\Lambda_a}^m{\Lambda_b}^n = g^{mn}$, the *homogeneous Lorentz group*. Here and in the rest of this introduction, small Latin indices always run over the ordered set $\{1,2,3,4\}$.

A *spacetime* (all the discussion here is local) is a pair $(M,[\,E\,])$, where M is a four dimensional manifold, $E=(E_1,E_2,E_3,E_4)$ is a frame of vector fields on M, and two frames belong to the same equivalence class $[\,E\,]$ iff the associated field $g^{ab}\,E_a\otimes E_b$ is the same. Equivalently, $[\,E\,]=\{\Lambda E \bigm| \Lambda : M\to \mathrm{O}(1,3)\}$, where $(\Lambda E)_a = {\Lambda_a}^b E_b$.

The spacetime *metric* g is given by $g(E_a,E_b)=g_{ab}$ and depends only on $[\,E\,]$. Here, (g_{ab}) is the matrix inverse of (g^{ab}). The *inverse metric* is the field $g^{ab}\,E_a\otimes E_b$.

A spacetime is *flat* iff there is a representative frame E such that the vector field commutators $[E_a,E_b]$, $a,b=1,2,3,4$, all vanish identically. Equivalently, given *any* representative frame E, the spacetime is flat iff there is a $\Lambda : M\to \mathrm{O}(1,3)$ for which the commutators $[(\Lambda E)_a, (\Lambda E)_b]$ all vanish identically. The *integrability condition* for this first order differential system[1] for Λ is the vanishing of

$$R_{mna}{}^b = E_m(\Gamma_{na}{}^b) - E_n(\Gamma_{ma}{}^b) + \Gamma_{na}{}^\ell \Gamma_{m\ell}{}^b - \Gamma_{ma}{}^\ell \Gamma_{n\ell}{}^b + (\Gamma_{nm}{}^\ell - \Gamma_{mn}{}^\ell)\Gamma_{\ell a}{}^b$$

with $(\Gamma_{na}{}^b)$ determined by $[E_m, E_n] = (\Gamma_{mn}{}^\ell - \Gamma_{nm}{}^\ell)E_\ell$ and $\Gamma_{a\ell}{}^n g^{\ell m} + \Gamma_{a\ell}{}^m g^{\ell n} = 0$. Equivalently, $\Gamma_{am}{}^\ell = \frac{1}{2} g^{\ell n}(c_{mna} - c_{amn} + c_{anm})$ where $[E_m, E_n] = c_{mna} g^{ab} E_b$.

By construction, the field $(R_{mna}{}^b)$ represents the equivalent concepts: obstruction to integrability of the flatness condition; measure of deviation from flatness; curvature. It is called *Riemann curvature*.

The Riemann curvatures associated to two equivalent frames E and ΛE are related in a *pointwise* manner, $R(\Lambda E)_{mna}{}^b \Lambda_b{}^d = \Lambda_m{}^k \Lambda_n{}^\ell \Lambda_a{}^c R(E)_{k\ell c}{}^d$. Thus, at each point, the Riemann curvature can be analyzed by the group representation theory of $\mathrm{O}(1,3)$. It decomposes invariantly into a sum of three pieces, that transform according to three irreducible real representations of $\mathrm{O}(1,3)$, denoted $(0,0)$ and $(1,1)$ and $(2,0) \oplus (0,2)$, with dimensions 1 and 9 and 10, respectively. In the context of four-dimensional geometry, the three pieces in this invariant decomposition have the following names:

$$\text{(Riemann curvature)} = \begin{pmatrix} \text{Trace of Ricci curvature} \\ \text{= Scalar curvature} \end{pmatrix} \oplus \text{(Traceless Part of Ricci curvature)} \oplus \text{(Weyl curvature)}$$

Einstein proposed to interpret spacetimes for which the Riemann curvature is purely of type $(2,0) \oplus (0,2)$, that is purely a Weyl curvature, as physical *vacuum spacetimes*. By the above discussion, this is a representative-independent condition for a spacetime. Equivalently,

$$\textit{Vacuum Einstein Equations:} \quad R_{nab}{}^n = 0 \quad \text{for all } a, b = 1, 2, 3, 4.$$

A fundamental property of the vacuum Einstein equations is *causality*, that is, *finite speed of propagation*. To discuss causality properly, local gauge-transformations have to be taken into account. Causality of the vacuum Einstein equations may be exhibited by a process known as *hyperbolic reduction* (essentially, complete gauge fixing, and deriving hyperbolic partial differential equations). Such a reduction yields L^2-type estimates, usually referred to as *energy estimates*, a basic analytic tool to gain rigorous control over the solutions.

The traditional hyperbolic reduction of the vacuum Einstein equations uses the *harmonic gauge*[2]. It was used by Choquet-Bruhat [CB] to obtain a rigorous local existence and uniqueness theorem.

An alternative hyperbolic reduction uses the *Newman-Penrose-Friedrich* orthonormal frame formalism. Newman and Penrose [NP] introduced the basic unknown fields of this formalism (frame, connection, Weyl curvature) and the corresponding vacuum

[1] Modulo the $\mathrm{O}(1,3)$-condition $g^{ab}\Lambda_a{}^m \Lambda_b{}^n = g^{mn}$, the system $[(\Lambda E)_a, (\Lambda E)_b] = 0$ is equivalent to the linear system $E_n(\Lambda_a{}^b) + \Lambda_a{}^\ell \Gamma_{n\ell}{}^b = 0$ for Λ.

[2] Coordinates $(x^\mu)_{\mu=0,1,2,3}$ satisfy the harmonic gauge with respect to a metric $g = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ iff $\partial_\mu(g^{\mu\nu}\sqrt{-g}\,) = 0$ for all ν. Here, g is the determinant of the matrix $(g_{\mu\nu})_{\mu,\nu=0,1,2,3}$. The harmonic gauge bears some similarity to the Lorentz gauge of electromagnetism.

Einstein equations . Their equations are not independent, but satisfy general differential identities, that were derived by Friedrich [Fr].

In [RT1], the vacuum Einstein equations as formulated by Newman and Penrose [NP] are expressed in terms of a Lie superbracket. The general differential identities of Friedrich [Fr] are derived from the associated super Jacobi identity. Special care is taken to exhibit the covariance properties of the equations.

The first hyperbolic reduction in the Newman-Penrose-Friedrich formalism was discovered by Friedrich [Fr]. He showed, by choosing a particular gauge, that the vacuum equations contain a symmetric hyperbolic subsystem[3] that determines the evolution of all unknown fields (frame, connection, Weyl curvature). To show that the remaining equations, called constraints, are also fulfilled, he used the general differential identities. The importance of [Fr] is the insight that a hyperbolic reduction in the Newman-Penrose-Friedrich formalism is at all possible, using symmetric hyperbolic systems. The particular gauge introduced in [Fr] is secondary, and is not used in this thesis.

The discussion below makes frequent reference to characteristic coordinates. A function u on a spacetime with nonvanishing differential, $\mathrm{d}u \neq 0$, is *characteristic* iff it solves the *eikonal equation*, $g^{ab}E_a(u)E_b(u) = 0$. The eikonal equation depends only on the equivalence class $[\,E\,]$. It is a Hamilton-Jacobi equation, and the associated Hamiltonian equation of motion is the geodesic equation. Explicitly, the *gradient* vector field $g^{ab}E_a(u)\,E_b$ is tangent to the level sets of u, and its integral curves are (affinely parametrized) null geodesics. Hence, the level sets of u are ruled by null geodesics.

Gauges in which *one* of the four coordinates are characteristic appear, for example, in [BBM] and in [Fr].

A gauge in which *two* of the four coordinates are characteristic, sometimes called *double-null-gauge*, is a natural choice for some problems in general relativity[4]. See [KN] and [Chr]. Many calculations and estimates are conveniently done in this gauge. However, in the absence of a hyperbolic reduction directly in the double-null gauge, rigorous works using this gauge had to carry out parts of the argument (local existence) in a different gauge, say the harmonic gauge. This technical detour can be avoided:

In [RT2], a hyperbolic reduction for the Newman-Penrose-Friedrich formalism is given directly in the double-null-gauge, using symmetric hyperbolic systems.

The geometric notion of a *closed trapped surface* was introduced by Penrose [Pen]. The definition of this notion assumes that the spacetime is *time-oriented*: at each point, a choice is made which half of the light cone is *future*-directed, and this choice is made in a continuous way. A closed 2-dimensional surface in a time-oriented spacetime is *trapped* iff, every tangent space to the surface is spacelike, and the traces of both future-directed null second fundamental forms[5] are negative everywhere on the surface. The original definition of Penrose [Pen] is more direct: “[...] a closed, spacelike, two-surface

---

[3] For a general discussion of quasilinear symmetric hyperbolic systems, see [Tay].

[4] A basic example are the coordinates $t + r + 2m\log(r/(2m) - 1)$ and $t - r - 2m\log(r/(2m) - 1)$ on Schwarzschild spacetime, when $r > 2m > 0$. Here, (t, r) are two of the standard Schwarzschild coordinates.

[5] A spacelike 2-dimensional surface can locally always be written as the intersection of the zero-level sets of two functions u and $\underline{u}$, both solutions to the eikonal equation, $g^{ab}E_a(u)E_b(u) = g^{ab}E_a(\underline{u})E_b(\underline{u}) = 0$, with $\mathrm{d}u$ and $\mathrm{d}\underline{u}$ pointwise linearly independent. The two *null second fundamental forms* describe the *extrinsic* geometry of the spacelike 2-dimensional surface with respect to these two zero-level sets.

[...] with the property that the two systems of null geodesics which meet [the surface] orthogonally converge locally in future directions at [the surface]." A closed trapped surface diffeomorphic to the two-sphere S^2, will be referred to as a *trapped sphere*.

The prototypical trapped spheres in a vacuum spacetime are the $\mathrm{SO}(3)$ orbits inside the horizon of a Schwarzschild spacetime. Closed trapped surfaces appear in the formulation of Penrose's incompleteness theorem [Pen].

Christodoulou [Chr] has proved that trapped spheres can form *in evolution* through the focusing of incoming gravitational waves. To be sure, the spacetimes constructed in [Chr] are solutions to the vacuum Einstein equations, and the theorems in [Chr] apply to a generic class of initial data (in particular, there are no assumptions of symmetry).

> *[RT2] contains a new and logically independent proof of the main results of [Chr], including the formation of trapped spheres. [RT2] is based on exactly the same physical mechanism/the same geometrical setup that was exploited in [Chr]. However, the proof of [RT2] uses only a combination of traditional and well-known tools, in particular symmetric hyperbolic systems and formal expansions, thereby achieving a technical simplification over [Chr].*

The focusing of gravitational waves in [Chr] is implemented by a dedicated geometric optics argument, called *short pulse method* in [Chr]. It is instructive to first consider an idealized limiting case: an infinitely short (or instantaneous) pulse. This discussion uses spherically symmetric non-vacuum spacetimes. The bearing of this discussion on [Chr] and [RT2], which deal with vacuum spacetimes without assumptions of symmetry, is explained afterward.

On the manifold $M = \{(\underline{u}, r, \theta, \phi) \in \mathbb{R} \times (0,\infty) \times (0,\pi) \times (0,2\pi)\}$ set

$$\begin{aligned} E_1 &= \partial_{\underline{u}} + \tfrac{1}{2}\big(1 - 2r^{-1}m(\underline{u})\big)\,\partial_r & \qquad E_3 &= r^{-1}\partial_\theta \\ E_2 &= -\partial_r & E_4 &= (r\sin\theta)^{-1}\partial_\phi \end{aligned}$$

and use

$$(g^{ab}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The function $m = m(\underline{u})$ is assumed to be given, with $\mathrm{d}m/\mathrm{d}\underline{u} \geq 0$. Observe that $E = (E_1, E_2, E_3, E_4)$ is a frame. By direct calculation:

$$R_{nab}{}^{n} = 2k_a k_b \qquad \text{with} \qquad k_a = \delta_{a1}\, r^{-1}\sqrt{\mathrm{d}m/\mathrm{d}\underline{u}}$$

The coordinate $\underline{u}$ is a characteristic coordinate, $g^{ab}E_a(\underline{u})E_b(\underline{u}) = 0$. The spacetime $(M, [\,E\,])$ is spherically symmetric, and $R_{nab}{}^{n} = 2k_a k_b$ are the Einstein Equations *with a null fluid matter field*, because the wave vector field (k_a) is null, $g^{ab}k_a k_b = 0$. This spacetime was introduced by Vaidya [Vai]. From now on,

$$m(\underline{u}) = \begin{cases} 0 & \text{if } \underline{u} \leq 0 \\ m_0 & \text{if } \underline{u} > 0 \end{cases} \qquad\qquad (m_0 > 0 \text{ constant})$$

The corresponding spacetime will be referred to as the *distributional Vaidya spacetime*. Then, $2k_a k_b = 2\delta_{a1}\delta_{b1} r^{-2} m_0\, \delta(\underline{u})$ has singular support on the incoming characteristic hypersurface $\underline{u} = 0$. The spacetime $(M, [E])$ is vacuum when $\underline{u} < 0$ (Minkowski spacetime) and $\underline{u} > 0$ (Schwarzschild spacetime in Eddington-Finkelstein coordinates with mass m_0).

The level sets of $(\underline{u}, r)$ as a map $M \to \mathbb{R} \times (0, \infty)$ are spheres with area $4\pi r^2$, and hence are trapped iff[6] $E_1(r) < 0$ and $E_2(r) < 0$, that is, $\underline{u} > 0$ and $r < 2m_0$. Therefore, in the distributional Vaidya spacetime, trapped spheres form *in evolution*. Recall that this is a *non-vacuum* spacetime.

The null fluid term $2k_a k_b$ in the distributional Vaidya spacetime may be interpreted as a *massless* radiation, possibly *gravitational* radiation. Therefore, heuristically speaking, one may attempt to high-frequency-modify the distributional Vaidya spacetime near $\underline{u} = 0$ to obtain a *vacuum* spacetime. Necessarily, one has to abandon spherical symmetry (Birkhoff theorem).

The distributional Vaidya spacetime is not discussed in [Chr], but could serve as a natural motivation for [Chr], or at least some aspects of it. In fact, the spacetimes constructed in [Chr] do also contain a complete Minkowskian past cone, the boundary of which 'carries' a wave. This wave is non-spherical, and purely gravitational, in the sense that it is a solution to the *vacuum* Einstein equations. The technical implementation of this picture is the short pulse method of [Chr]. In [Chr], the Minkowskian region and the pulse region are considered.

> *[RT2] in addition controls the transition from the pulse region to the (approximate) Schwarzschild region. This result is established under certain natural assumptions. In particular, the 'incoming energy per unit solid angle' in the finite-duration pulse is spherically symmetric. See Section 9 of [RT2].*

The distributional Vaidya spacetime gives a simple, intuitive picture for some aspects of the focusing problem, but not all aspects. In particular, it gives no direct information about the short pulse region itself.

In [Chr], the short pulse method is presented as a self-consistent way of introducing a small parameter $\delta > 0$ into the problem. More precisely, it is a self-consistent scheme of bounds for all the unknown quantities in terms of δ. Many quantities in [Chr] have bounds of the form $\mathcal{O}(\delta^{-\alpha})$, with $\alpha > 0$. The limit (of the bound) as $\delta \downarrow 0$ does not exist, it is *singular*. Whether this is necessarily so, or whether it is possible to define in a mathematically meaningful way *the actual limit*, is not discussed in [Chr]. Can the $\delta \downarrow 0$ limit be *regularized*?
The next statement is formulated in the coordinate system $(\xi^1, \xi^2, \underline{u}, u)$ that is used in [RT2]. The first two are 'angular coordinates', the last two characteristic coordinates. The small parameter $\mathfrak{A}$ in [RT2] is equivalent to δ of [Chr] through $\delta = \mathfrak{A}^4$.

> *[RT2] uses (dependent and independent) variables for which the limit $\mathfrak{A} \downarrow 0$ exists / is regular, and for which the equations* remain *symmetric hyperbolic even at $\mathfrak{A} = 0$. The inverse metric $g^{ab}\, E_a \otimes E_b$ degenerates from signature $(-,+,+,+)$ to $(-,0,0,+)$, which causes all partial derivatives with respect to ξ^1, ξ^2 to drop out. For each value of the now passive parameters (ξ^1, ξ^2), one obtains $1+1$ dimensional symmetric hyperbolic systems with respect to just $\underline{u}$, u. The solutions to these systems break down, along a curve in the $(\underline{u}, u)$-plane that is explicitly calculated in [RT2] in terms of the data at past null infinity.*

[6] This uses the implicit assumption that $E_1 + E_2$ determines the future direction.

The $\mathfrak{A} = 0$ solutions have a degenerate frame (rank 2 rather than rank 4). Therefore, they are not spacetimes, and their breakdown *does not* describe the breakdown of vacuum spacetimes. However, in Subsection 9.5 of [RT2], an informal but careful argument is given to the effect that the vacuum spacetimes of [RT2] can be extended, using nothing more than the methods of [RT2], to within any 'finite distance' of the $\mathfrak{A} = 0$ breakdown. No statement is made about later 'times'. Nevertheless, the informal argument, and the heuristic picture that comes with it, together identify a direction for *promising future research*. Subsection 9.5 stands apart from the rest of [RT2], and will not be discussed further in this introduction.

A few more details about [RT2] will now be given. To keep things reasonably short, the following compromises are made. The vacuum spacetimes of [RT2] are discussed in terms of just the 16 components of an orthonormal frame $E = (E_1, E_2, E_3, E_4)$. The components of the connection and Weyl curvature are ignored, even though in the Newman-Penrose-Friedrich formalism they are on an operationally equal footing with the frame, and are treated as such in [RT2]. A single stereographic-type coordinate patch $(\xi^1, \xi^2) \in \mathbb{R}^2$ is mentioned for the two-sphere. It is implicit that two such patches are used to cover the two-sphere, and that everything is compatible on the overlap. It is assumed that the vacuum spacetimes have been shown to exist beforehand, and they are just *described* here. The discussion is incomplete and a little informal.

The discussion uses the conventions of the *High Amplitude Picture* with $a = \mathfrak{A}$ in Section 9 of [RT2], because it is the simplest to explain. (All the technical parts of [RT2] are done using another picture, the *Regularized Picture*, because it is much better suited for making calculations. The two pictures are equivalent, and are related by scaling symmetries of the Newman-Penrose-Friedrich formalism, see Section 9 [RT2].)

Set $M = \big\{(x^\mu)_{\mu=1,2,3,4} = (\xi^1, \xi^2, \underline{u}, u) \in \mathbb{R}^2 \times (0,1) \times (-\infty, u_0)\big\}$ with $u_0 < 0$. The components of the frame vector fields $E_a = E_a{}^\mu \partial_\mu$ are

$$(E_a{}^\mu) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 1 \\ 0 & 0 & * & 0 \end{pmatrix} \qquad (g^{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Entries with asterisk $*$ can be nonzero. Some of them are *necessarily* nonzero, because E is a frame. The entry $E_4{}^3 > 0$. The seemingly wrong signature of (g^{ab}) is explained by the fact that the frame vector fields are *complex* (this is convenient!). The complex conjugate of (E_1, E_2, E_3, E_4) is (E_2, E_1, E_3, E_4). Therefore, $g^{ab} E_a \otimes E_b$ is real and has signature $(-, +, +, +)$. The asterisk pattern of $(E_a{}^\mu)$ is discussed in more detail a few paragraphs down from here.

Let $M', M'' \subset M$ be the subsets given by $0 < \underline{u} < \frac{1}{2}$ and $\frac{1}{2} < \underline{u} < 1$, respectively. Then $(M', [\,E\,])$ is flat/Minkowskian, $(M'', [\,E\,])$ carries the gravitational wave.

More precisely, the subset M', on which

$$(E_a{}^\mu) = \begin{pmatrix} \rho^{-1}\mathbf{e} & +i\rho^{-1}\mathbf{e} & 0 & 0 \\ \rho^{-1}\mathbf{e} & -i\rho^{-1}\mathbf{e} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} \rho &= \underline{u} - u \\ \mathbf{e} &= \tfrac{1}{2}\big(1 + (\xi^1)^2 + (\xi^2)^2\big) \end{aligned}$$

is isometric to a subset of Minkowski spacetime: $\underline{u} = 2^{-1/2}(t + r)$, $u = 2^{-1/2}(t - r)$, with standard Minkowskian time t and radius r, and standard stereographic coordinates

(ξ^1, ξ^2). Therefore, the past light cone $t + r \leq 0$ in Minkowski spacetime can be smoothly attached to $(M, [E])$.

There is a map[7] $\mathbf{DATA}_{\mathrm{H}} : \mathbb{R}^2 \times (0,1) \to \mathbb{C}$ with

$$\lim_{u \to -\infty} u^2 (E_1{}^1 - \rho^{-1}\mathbf{e}) = \lim_{u \to -\infty} i u^2 (E_1{}^2 - i\rho^{-1}\mathbf{e}) = \mathbf{e} \textstyle\int_0^{\underline{u}} \mathrm{d}\underline{u}' \, \mathbf{DATA}_{\mathrm{H}}(\xi^1, \xi^2, \underline{u}')$$

The limits are taken at constant ξ^1, ξ^2, $\underline{u}$. Here, $u \to -\infty$ is interpreted as *past null infinity*, and $\mathbf{DATA}_{\mathrm{H}}$ is interpreted as initial data at past null infinity. More informally, $\mathbf{DATA}_{\mathrm{H}}$ describes the incoming radiation. Necessarily, $\mathbf{DATA}_{\mathrm{H}} = 0$ when $\underline{u} < \frac{1}{2}$. Given the Minkowskian data on M', the map $\mathbf{DATA}_{\mathrm{H}}$ uniquely determines $(M, [E])$. To make rigorous sense of the last statement, technical assumptions about the decay as $u \to -\infty$ of various unknowns are made in [RT2]. These assumptions are not discussed here.

A minimal requirement for the uniqueness statement in the last paragraph is the *complete fixing* of the gauge degrees of freedom. Here, the asterisk pattern of $(E_a{}^\mu)$ comes in. The three zeros in the third (resp. fourth) column imply that $\underline{u}$ (resp. u) solves the eikonal equation and that its gradient[8] is $-E_4{}^3 E_3$ (resp. $-E_4$). The two zeros in the lower-left corner imply that ξ^1, ξ^2 are transported along E_4. Thus, *two eikonal equations and two transport equations*, together with the Minkowskian data on M' and the additional assumption $\lim_{u \to -\infty} E_4{}^3 = 1$, fix the coordinates. An additional condition is needed to also fix the frame, because there still is the local $U(1)$ gauge degree of freedom $(E_1, E_2) \to (e^{+i\theta} E_1, e^{-i\theta} E_2)$, but this is not discussed here.

If E is rescaled by a constant positive factor, it will still satisfy the vacuum Einstein equations. To satisfy the gauge conditions, u and $\underline{u}$ have to be rescaled by the inverse of the same factor. Therefore, the initial assumption that $\underline{u}$ has range $(0,1)$ is a choice of scale. That is, the coordinate-width of the wave is fixed to ~ 1.

Let $\|\mathbf{DATA}_{\mathrm{H}}\|$ (the *amplitude*) be equal to a suitable C^m norm of $\mathbf{DATA}_{\mathrm{H}}$, that also takes into account the two patches for the two-sphere. The value of m is technical. The results of [RT2] apply with $m = 10$.

There is no smallness condition on the amplitude $\|\mathbf{DATA}_{\mathrm{H}}\|$ in [RT2], in fact the analysis is tailored to large amplitude. The theorems of [RT2] control / assert existence of the vacuum spacetime for $u \in (-\infty, u_0)$, with $u_0 < 0$ becoming more negative as the amplitude grows. More precisely:

[RT2] yields $|u_0| \sim \|\mathbf{DATA}_{\mathrm{H}}\|$ *as* $\|\mathbf{DATA}_{\mathrm{H}}\| \to \infty$.

Whether the statement $|u_0| \sim \|\mathbf{DATA}_{\mathrm{H}}\|^\kappa$ as $\|\mathbf{DATA}_{\mathrm{H}}\| \to \infty$ can be proved for some $\kappa \in (0,1)$ is not known, but $\kappa = 1$ may well be optimal. By comparison, [Chr] only directly implies the weaker statement with[9] $\kappa = 2$.

---

[7] The subscript in $\mathbf{DATA}_{\mathrm{H}}$ is for *High Amplitude Picture*. See Section 9 of [RT2].

[8] The gradient of $f : M \to \mathbb{R}$ is the vector field $g^{ab} E_a(f) E_b$.

[9] To make this conclusion, the vacuum spacetimes of [Chr] have to be expressed in the same gauge, by a straightforward global rescaling. There are two remarks. First, in [Chr] the range of u is actually a finite interval, but the results are uniform in the left (i.e. more negative) endpoint of the interval. Second, the amplitude $\|\mathbf{DATA}_{\mathrm{H}}\|$ can be taken to be a C^7 norm in [Chr], as opposed to C^{10} in [RT2]. Thus, from this particular point of view, [Chr] is stronger than [RT2]. However, in view of the fact that the focusing of gravitational waves is an infrared problem, this remark is technical.

The vacuum Einstein equations are a system of nonlinear partial differential equations. There are two simplifications that are commonly used to facilitate / allow a rigorous quantitative treatment. One is *perturbation theory* (a small parameter) and this is used in [RT2]. The other is *symmetry* (dimensional reduction) and this is used in [RT3].

In [RT3], the spacetimes are spatially homogeneous, Bianchi type VIII and IX. The vacuum Einstein equations become a nonlinear system of six ordinary differential equations and one propagating algebraic constraint, namely the following equations with $n = (n_1, n_2, n_3) = (1, 1, 1)$:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_1 &= -(n_1\beta_1)^2 + (n_2\beta_2)^2 + (n_3\beta_3)^2 - 2n_2n_3\beta_2\beta_3 \\
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_2 &= -(n_2\beta_2)^2 + (n_3\beta_3)^2 + (n_1\beta_1)^2 - 2n_3n_1\beta_3\beta_1 \\
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_3 &= -(n_3\beta_3)^2 + (n_1\beta_1)^2 + (n_2\beta_2)^2 - 2n_1n_2\beta_1\beta_2 \\
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_1 &= \beta_1\alpha_1 \\
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_2 &= \beta_2\alpha_2 \\
\tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_3 &= \beta_3\alpha_3 \\
0 &= \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 - (n_1\beta_1)^2 - (n_2\beta_2)^2 - (n_3\beta_3)^2 \\
&\qquad\qquad + 2n_2n_3\beta_2\beta_3 + 2n_3n_1\beta_3\beta_1 + 2n_1n_2\beta_1\beta_2
\end{aligned}$$

(Other values $n = (n_1, n_2, n_3)$ are used later.) The condition $(\alpha_1 + \alpha_2 + \alpha_3)|_{\tau=0} < 0$ breaks the $\tau \to -\tau$ symmetry and implies that the solutions $\alpha(\tau), \beta(\tau) \in \mathbb{R}^3$ exist for all $\tau \geq 0$, with $\alpha_1 + \alpha_2 + \alpha_3 < 0$. However, the half-infinite interval $\tau \geq 0$ corresponds to a *finite* physical duration of the associated spatially homogeneous vacuum spacetime. In [RT3], $\beta_1, \beta_2, \beta_3 \neq 0$.

The pioneering calculations and heuristic picture of Belinskii, Khalatnikov, Lifshitz [BKL] and Misner [Mis] suggest that a generic class of solutions are oscillatory as $\tau \to +\infty$ and that the dynamics of one degree of freedom is closely related to the discrete dynamics of the Gauss map $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, a non-invertible map from $(0, 1) \setminus \mathbb{Q}$ to itself. Every element of $(0, 1) \setminus \mathbb{Q}$ admits a unique infinite continued fraction expansion

$$\langle k_1, k_2, k_3, \ldots \rangle \;=\; \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \cdots}}}$$

where $(k_n)_{n \geq 1}$ are strictly positive integers. The Gauss map is the left-shift,

$$G\big(\langle k_1, k_2, k_3, \ldots \rangle\big) = \langle k_2, k_3, k_4, \ldots \rangle$$

Rigorous results about spatially homogeneous spacetimes have been obtained by Rendall [Ren] and Ringström [Ri1], [Ri2]. See also Heinzle and Uggla [HU1]. See [HU2] for a detailed discussion.

The first rigorous proofs that there exist spatially homogeneous vacuum spacetimes whose asymptotic behavior is related, in a precise sense, to iterates of the Gauss map, have been obtained by Béguin [Be] and by Liebscher, Härterich, Webster and Georgi [LHWG]. These theorems apply to a dense subset of $(0, 1) \setminus \mathbb{Q}$. A basic restriction of both these works is that the sequence $(k_n)_{n \geq 1}$ has to be bounded, a condition fulfilled only by a Lebesgue measure zero subset of $(0, 1) \setminus \mathbb{Q}$.

The results of [RT3] apply to any sequence $(k_n)_{n\geq 1}$ that grows at most polynomially. The corresponding subset of $(0,1)\setminus\mathbb{Q}$ has full Lebesgue measure one.

The structure of each solution constructed in [RT3] can be informally described by a sequence $(\tau_j)_{j\geq 0}$, with $0=\tau_0<\tau_1<\ldots<\tau_{j-1}<\tau_j<\ldots$ and $\lim_{j\to\infty}\tau_j=\infty$, and by a sequence $(\mathbf{a}_j)_{j\geq 1}$ with $\mathbf{a}_j\in\{1,2,3\}$ and $\mathbf{a}_j\neq\mathbf{a}_{j+1}$ for all $j\geq 1$. These two sequences specify a *semi-global approximation scheme*: for each $j\geq 1$ the vacuum Einstein equations $n=(n_1,n_2,n_3)=(1,1,1)$ on $[\tau_{j-1},\tau_j]$ are approximated by $n=(1,0,0)$ if $\mathbf{a}_j=1$, by $n=(0,1,0)$ if $\mathbf{a}_j=2$, by $n=(0,0,1)$ if $\mathbf{a}_j=3$. (The three approximate systems are explicitly solvable.) This scheme is good enough to construct semi-global solutions. To make rigorous sense of this, it is shown in [RT3] that the accumulated error stays finite as one is coming in from $j\to+\infty$, or $\tau\to+\infty$ that is.

To explain the role of $(k_n)_{n\geq 1}$, suppose the sequence $(\mathbf{a}_j)_{j\geq 1}$ contains the segment

$$(\ldots,2,1,2,1^*,3,1,3,1^*,2^*,3,2^*,1^*,3,1,3,1,\ldots)$$

The element $\mathbf{a}_j$ has been marked by an asterisk iff $\mathbf{a}_{j-1}\neq\mathbf{a}_{j+1}$. The leftmost element and the rightmost element are not marked, because it was assumed that the next element to the left of the segment is 1, and the next to the right is 3. The elements of the sequence $(k_n)_{n\geq 1}$ measure the distance between neighboring asterisks. In the present example, $(k_n)_{n\geq 1}$ contains the segment $(\ldots,4,1,2,1,\ldots)$.

In [Mis], billiard game jargon is introduced to informally describe the dynamics: the billiard ball is in free motion near τ_{j-1}, a short but finite-duration billiard bounce occurs somewhere in $[\tau_{j-1},\tau_j]$, the billiard ball is again in free motion near τ_j, and so forth. There are three walls, labeled 1, 2, 3, respectively. The bounce in $[\tau_{j-1},\tau_j]$ is off the wall labeled $\mathbf{a}_j$.

In [RT3], a dimensionless parameter $\mathbf{h}_j>0$ is defined. Essentially, $\mathbf{h}_j$ is the duration of the billiard bounce in $[\tau_{j-1},\tau_j]$, divided by $|\tau_j-\tau_{j-1}|$. Proving rigorously the validity of the semi-global approximation scheme goes hand in hand with decay estimates for $\mathbf{h}_j$, as $j\to+\infty$.

It is shown in [RT3] that, under appropriate smallness conditions,

$$\mathbf{h}_j=\mathcal{O}\Big(\big(\tfrac{1}{2}(1+\sqrt{5})\big)^{-2(\mathbf{D}^{-1}j)^{1/(\gamma+1)}}\Big)\qquad \textit{as}\qquad j\to+\infty$$

Here, $\mathbf{D}\geq 1$ and $\gamma\geq 0$ are constants such that $k_n\leq\mathbf{D}\,n^\gamma$ for all $n\geq 1$.

A basic question regarding the causal structure is whether, for every pair of points p,p' in the spatially homogeneous spacetime, there is a point q that lies in the causal future of both p and p'. (Here, *future* corresponds to *increasing* τ.) If the answer to this question is *negative*, the spacetime is said to have a *particle horizon*.

All spatially homogeneous vacuum spacetimes constructed in [RT3] have particle horizons.

The heuristic work of Belinskii, Khalatnikov, Lifshitz [BKL] concerns very general (inhomogeneous) spacetime singularities. It heavily relies on intuition about the homogeneous case. The existence of particle horizons in the homogeneous case, established in [RT3] under certain smallness assumptions, seems to be a *necessary* condition for the homogeneous case to have any bearing on the inhomogeneous case.

The solutions constructed in [RT3] are generic in the sense that 'they depend on the right number of free parameters' (for a precise statement, see [RT3]). It would be desirable to have a stronger genericity statement.

References

[RT1] Reiterer, M. and Trubowitz, E., pages 15-36 of this thesis.
[RT2] Reiterer, M. and Trubowitz, E., pages 37-132 of this thesis.
[RT3] Reiterer, M. and Trubowitz, E., pages 133-181 of this thesis.

[BBM] Bondi, H., van der Burg, M.G.J. and Metzner, A.W.K., Proc. Roy. Soc. Lond. A **269**, (1962) 21-52
[Be] Béguin, F., Class. Quantum Grav. **27**, (2010) 185005
[BKL] Belinskii, V.A., Khalatnikov, I.M., and Lifshitz, E.M., Adv. Phys. **19** (1970) 525-573
[CB] Choquet-Bruhat, Y., Acta Math. **88**, (1952) 141-225
[Chr] Christodoulou, D., *The Formation of Black Holes in General Relativity* (EMS, 2009)
[Fr] H. Friedrich, Proc. Roy. Soc. Lond. A **375**, (1981) 169-184.
[HU1] Heinzle, J.M. and Uggla, C., Class. Quantum Grav. **26**, (2009) 075015
[HU2] Heinzle, J.M. and Uggla, C., Class. Quantum Grav. **26**, (2009) 075016
[KN] Klainerman, S. and Nicoló, F., *The Evolution Problem in General Relativity* (Birkhäuser, 2003)
[LHWG] Liebscher, S., Härterich, J., Webster, K. and Georgi, M., arXiv 1004.1989 (2010)
[Mis] Misner, Phys. Rev. Lett. **22**, (1969) 1071-1074
[NP] Newman, E. and Penrose, R., J. Math. Phys. **3**, (1962) 566-578
[Pen] Penrose, R., Phys. Rev. Lett. **14**, (1965) 57-59
[Ren] Rendall, A., Class. Quantum Grav. **14**, (1997) 2341-2356
[Ri1] Ringström, H., Class. Quantum Grav. **17**, (2000) 713
[Ri2] Ringström, H., Annales Henri Poincaré **2**, (2001) 405
[Tay] Taylor, M. E., *Partial Differential Equations III* (Springer, 1997)
[Vai] Vaidya, P.C., Proc. Indian Acad, Sci. A **33**, (1951) 264; reprinted Gen. Rel. and Grav. **31**, (1999) 121

A formalism for analyzing vacuum spacetimes

Michael Reiterer, Eugene Trubowitz

Department of Mathematics, ETH Zurich, Switzerland

Abstract: The Einstein vacuum equations in the formulation developed by Newman, Penrose [NP] and Friedrich [Fr] are expressed in terms of a Lie superbracket. Differential identities are derived from the super Jacobi identity. This perspective clarifies the covariance properties of the equations. The equations are intended as a tool for the analytic study of vacuum spacetimes.

1. Introduction

In this paper, we discuss a formalism that is suited to the analysis of solutions to the Einstein vacuum equations. In this formalism, the vacuum equations

- become a quasilinear, first order system of partial differential equations, that
- are quadratically nonlinear, and
- through gauge-fixing, can be brought into symmetric hyperbolic form.

Newman and Penrose [NP] introduced the basic unknown fields of this formalism (frame, connection, Weyl curvature) and the corresponding Einstein vacuum equations. Their equations are not independent, but satisfy general differential identities, that were derived by Friedrich [Fr].

Friedrich [Fr] showed, by choosing an appropriate gauge, that the vacuum equations contain a symmetric hyperbolic subsystem that determines the evolution of all unknown fields. To show that the remaining equations, called *constraints*, are also fulfilled, he used the general differential identities.

In this paper, the vacuum equations as formulated by Newman and Penrose are expressed in terms of a Lie superbracket, see (5.1) and (5.2). The general differential identities, see (5.6b), are derived from the associated super Jacobi identity. We take special care to exhibit the covariance properties of the equations.

We used a forerunner of the present formalism to analyze strongly focused gravitational waves, see Appendix B of [RT]. The point of the refined presentation of this paper is the derivation of the equations in Section 8 from an invariant point of view. They are intended to be used as a tool in the analysis of other problems in classical general relativity.

Important remark: We expect that there is a close relationship between the notion of a Cartan connection, see [Sh], and the formalism of this paper, which is not made here.

This relationship ought to be clarified. However, we have not pursued this relationship, since the equations of Section 8 can be derived without it.

2. A Lie Superalgebra Identity

We recall the definition of a real Lie superalgebra:

Definition 2.1. *A (non-associative) $\mathbb{Z}_2$-graded real algebra $\big(L = L_0 \oplus L_1, [\![\,\cdot\,,\,\cdot\,]\!]\big)$, with even parts L_0 and odd parts L_1, satisfying for all $x_1 \in L_{k_1}$, $x_2 \in L_{k_2}$, $x_3 \in L_{k_3}$*

(a) $[\![x_1, x_2]\!] \in L_\ell$ *with* $\ell = k_1 + k_2 \pmod 2$
(b) $[\![x_1, x_2]\!] = (-1)^{1+k_1k_2}[\![x_2, x_1]\!]$
(c) $(-1)^{k_1k_3}[\![x_1, [\![x_2, x_3]\!]]\!] + (-1)^{k_2k_1}[\![x_2, [\![x_3, x_1]\!]]\!] + (-1)^{k_3k_2}[\![x_3, [\![x_1, x_2]\!]]\!] = 0$

is called a real Lie superalgebra. *In this context, $[\![\,\cdot\,,\,\cdot\,]\!]$ is the Lie superbracket, (b) is super skew-symmetry, and (c) is the super Jacobi identity.*

Let $\big(L = L_0 \oplus L_1, [\![\,\cdot\,,\,\cdot\,]\!]\big)$ be a Lie superalgebra as above. Set $A_0 = L_0 \times L_1$ and $A_1 = L_1 \times L_0$, that is $A_\ell = L_\ell \times L_{\ell+1}$ for all $\ell \in \mathbb{Z}_2$.

Definition 2.2. *For $\ell \in \mathbb{Z}_2$, let*

$$\mathcal{D}^{(\ell)} : A_1 \times A_\ell \to A_{\ell+1} \qquad (x, y) \mapsto \mathcal{D}^{(\ell)}_x y$$

where

$$\mathcal{D}^{(\ell)}_x y = z = (z_1, z_2) \in A_{\ell+1} \qquad \text{with} \qquad \begin{cases} z_1 = y_2 - \epsilon_\ell [\![x_1, y_1]\!] \\ z_2 = \epsilon_\ell [\![x_1, y_2]\!] + \epsilon_\ell [\![y_1, x_2]\!] \end{cases} \tag{2.1}$$

for all $x = (x_1, x_2) \in A_1$ and all $y = (y_1, y_2) \in A_\ell$. Here, $\epsilon_0 = 1$ and $\epsilon_1 = \frac{1}{2}$.

Equation (2.1) is consistent, because $x_1 \in L_1$, $x_2 \in L_0$, $y_1 \in L_\ell$, $y_2 \in L_{\ell+1}$ imply $z_1 \in L_{\ell+1}$, $z_2 \in L_\ell$, as required.

Convention 2.1. From now on, we will drop the superscripts (0), (1) on the operator $\mathcal{D}$, with the understanding that "the arguments determine the superscript".

Proposition 2.1. $\mathcal{D}_x\mathcal{D}_x x = 0$ *for all* $x \in A_1$

Proof. Let $y = \mathcal{D}_x x$ and $z = \mathcal{D}_x y$. We have to show that $z = 0$. We have

$$y_1 = x_2 - \tfrac{1}{2}[\![x_1, x_1]\!] \tag{2.2a}$$
$$y_2 = [\![x_1, x_2]\!] \tag{2.2b}$$

and therefore

$$z_1 = y_2 - [\![x_1, y_1]\!] = \tfrac{1}{2}[\![x_1, [\![x_1, x_1]\!]]\!] \tag{2.2c}$$
$$z_2 = [\![x_1, y_2]\!] + [\![y_1, x_2]\!] = [\![x_1, [\![x_1, x_2]\!]]\!] - \tfrac{1}{2}[\![[\![x_1, x_1]\!], x_2]\!] + [\![x_2, x_2]\!] \tag{2.2d}$$

Recalling that $x_1 \in L_1$ and $x_2 \in L_0$, the super skew symmetry (b) and the super Jacobi identity (c) in Definition 2.1 imply $[\![x_1, [\![x_1, x_1]\!]]\!] = 0$, $[\![x_2, x_2]\!] = 0$ and $[\![[\![x_1, x_1]\!], x_2]\!] = 2[\![x_1, [\![x_1, x_2]\!]]\!]$. For example,

$$\begin{aligned} 0 &= [\![x_1, [\![x_1, x_2]\!]]\!] + [\![x_2, [\![x_1, x_1]\!]]\!] - [\![x_1, [\![x_2, x_1]\!]]\!] \\ &= [\![x_1, [\![x_1, x_2]\!]]\!] - [\![[\![x_1, x_1]\!], x_2]\!] + [\![x_1, [\![x_1, x_2]\!]]\!] \end{aligned}$$

Therefore, $z = 0$. $\square$

Remark 2.1. In Section 5 the abstract equation $\mathcal{D}_x x = 0$ for the unknown ”field” $x \in A_1$ will be interpreted as ”Einstein vacuum equations”. There are too many equations. The system is apparently overdetermined. The remedy is the identity of Proposition 2.1, that holds for all $x \in A_1$.

3. Diamonds

Convention 3.1. In this paper, all manifolds are real, smooth and finite dimensional. For any fiber bundle $\pi : E \to B$, the fiber over $p \in B$ is denoted by $E_p = \pi^{-1}(\{p\})$. For any section $X \in \Gamma(E)$ the map $X : B \to E$ is given by $p \mapsto X_p \in E_p$. For any vector bundle $\pi : E \to B$ we denote by E^*, $\mathrm{Sym}^2 E$, $\mathbb{S}\, E$, the dual bundle, the subbundle of symmetric elements of $E \otimes E$, and the sphere bundle associated with E. That is, for $p \in B$, we have $(\mathbb{S}\, E)_p = (E_p \setminus \{0\})/\mathbb{R}_+$. Finally, $\mathrm{End}(E) = E^* \otimes E$ is the endomorphism bundle associated with E.

Convention 3.2. For a bundle $\pi : E \to B$ we denote by $\mathcal{T}(E)$ the algebraic direct sum of all tensor products of E and E^*.

For the **rest of this paper**, fix

- a 4-dimensional manifold M,
- a real vector bundle $\pi_V : V \to M$ with 4 dimensional fibers,
- a section $\mathfrak{H} \in \Gamma(\mathbb{S}\,\mathrm{Sym}^2 V^*)$ with signature $(-,+,+,+)$.

In other words, $\mathfrak{H}$ defines a *conformal* Lorentzian inner product on each fiber of V.

Definition 3.1. *For every integer $k \geq 0$, let $\mathcal{P}^k$ be the set of all maps $\Diamond$,*

$$\Diamond :\ \Gamma(\mathcal{T}(V)) \to \Gamma\big(\wedge^k V^* \otimes \mathcal{T}(V)\big) \tag{3.1}$$

so that for all $u, v \in \Gamma(\mathcal{T}(V))$, all representatives $\mathfrak{h} \in \Gamma(\mathrm{Sym}^2 V^*)$ of the conformal Lorentzian inner product $\mathfrak{H} \in \Gamma(\mathbb{S}\,\mathrm{Sym}^2 V^*)$, and all $Y \in \Gamma(V^{\otimes k})$, we require, with Convention 3.3 below:

(a) $\Diamond$ is linear over $\mathbb{R}$,
(b) $\Diamond_Y$ maps $C^\infty(M) \to C^\infty(M)$ and $\Gamma(V) \to \Gamma(V)$ and $\Gamma(V^*) \to \Gamma(V^*)$,
(c) $\Diamond_Y(u \otimes v) = (\Diamond_Y u) \otimes v + u \otimes (\Diamond_Y v)$,
(d) $\Diamond I = 0$ if $I \in \Gamma(\mathrm{End}(V))$ is the identity on the fibers of V,
(e) $\Diamond \mathfrak{h} = \mu \otimes \mathfrak{h}$ for some $\mu \in \Gamma(\wedge^k V^*)$.

The vertical subspace $\mathcal{P}^k_\perp \subset \mathcal{P}^k$ is the set of all $\Diamond \in \mathcal{P}^k$ such that $\Diamond f = 0$ for all $f \in C^\infty(M)$.

Convention 3.3. For each $Y \in \Gamma(V^{\otimes k})$ and $u \in \Gamma(\mathcal{T}(V))$ set

$$\Diamond_Y u = i_Y\big(\Diamond u\big) \ \in\ \Gamma(\mathcal{T}(V))$$

Here i_Y is interior multiplication by Y acting on the first k factors of $\Diamond u$.

Remark 3.1. Observe that $\Diamond_Y$ acts on the ring $C^\infty(M)$ as a derivation, by (c).

Remark 3.2. Every element of $\mathcal{P}^k$ can be written as a finite sum of ”pure” elements $\theta \otimes \Diamond$, where $\theta \in \Gamma(\wedge^k V^*)$ and $\Diamond \in \mathcal{P}^0$. The Leibniz rule (c) for $\theta \otimes \Diamond$ reads

$$\big(\theta \otimes \Diamond\big)(u \otimes v) = \theta \otimes (\Diamond u) \otimes v + \theta \otimes u \otimes (\Diamond v)$$

Remark 3.3. Let $\mathcal{I}$ be an index set, $|\mathcal{I}| = 4$. Let $F_{(a)}$, $a \in \mathcal{I}$, be local sections of V that are a frame for fibers of V. Let $\lambda^{(a)}, a \in \mathcal{I}$, be the dual frame. For every $\lozenge \in \mathcal{P}^k$ and $Y \in \Gamma(V^{\otimes k})$, property (d) in Definition 3.1 implies

$$0 = \lozenge_Y I = \lozenge_Y \big(\lambda^{(a)} \otimes F_{(a)}\big) = \big(\lozenge_Y \lambda^{(a)}\big) \otimes F_{(a)} + \lambda^{(a)} \otimes \big(\lozenge_Y F_{(a)}\big)$$

Consequently, $(\lozenge_Y \lambda^{(a)})(F_{(b)}) = -\lambda^{(a)}\big(\lozenge_Y F_{(b)}\big)$ for all $a, b \in \mathcal{I}$. Now, the Leibniz rule (c) implies

$$(\lozenge_Y \xi)(Z) = \lozenge_Y(\xi(Z)) - \xi(\lozenge_Y Z) \tag{3.2}$$

for all $\xi \in \Gamma(V^*)$ and $Z \in \Gamma(V)$.

Definition 3.2. *Let $m, k_1, \ldots, k_\ell \geq 0$ be integers, and $k = k_1 + \ldots + k_\ell$. The multi $(k_1, \ldots, k_\ell)$ wedge product operator shifted by m is the linear map*

$$\begin{aligned}\wedge^{(m)}_{k_1,\ldots,k_\ell} :\ \Gamma\Big((\wedge^m V^*) \otimes (\wedge^{k_1} V^*) \otimes \cdots \otimes (\wedge^{k_\ell} V^*) \otimes \mathcal{T}(V)\Big)& \\ \to \Gamma\Big((\wedge^m V^*) \otimes (\wedge^k V^*) \otimes \mathcal{T}(V)\Big)&\end{aligned}$$

determined by $\xi \otimes \nu_1 \otimes \cdots \otimes \nu_\ell \otimes u \mapsto \xi \otimes \big(\nu_1 \wedge \cdots \wedge \nu_\ell\big) \otimes u$. Set $\wedge_{k_1,\ldots,k_\ell} = \wedge^{(0)}_{k_1,\ldots,k_\ell}$.

Remark 3.4. We have

$$\begin{aligned}\wedge_{k_1,k_2+k_3} \wedge^{(k_1)}_{k_2,k_3} &= \wedge_{k_1,k_2,k_3} \\ \lozenge \wedge_{k_2,k_3} &= \wedge^{(k_1)}_{k_2,k_3} \lozenge\end{aligned}$$

for any $\lozenge \in \mathcal{P}^{k_1}$.

Proposition 3.1. *For all $\lozenge \in \mathcal{P}^k$, $\not\lozenge \in \mathcal{P}^\ell$, set*

$$[\![\lozenge, \not\lozenge]\!] = \wedge_{k,\ell} \lozenge \not\lozenge - (-1)^{k\ell} \wedge_{\ell,k} \not\lozenge \lozenge \tag{3.3}$$

Then $[\![\lozenge, \not\lozenge]\!] \in \mathcal{P}^{k+\ell}$ and moreover, $\big(\mathcal{P}_0 \oplus \mathcal{P}_1, [\![\,\cdot\,,\,\cdot\,]\!]\big)$ is a Lie superalgebra, with $\mathcal{P}_0 = \bigoplus_{k \geq 0 \text{ even}} \mathcal{P}^k$ and $\mathcal{P}_1 = \bigoplus_{k \geq 0 \text{ odd}} \mathcal{P}^k$.

Proof. To see that $[\![\lozenge, \not\lozenge]\!] \in \mathcal{P}^{k+\ell}$, consider first the special case when $k = \ell = 0$. In this case $[\![\lozenge, \not\lozenge]\!] = \lozenge\not\lozenge - \not\lozenge\lozenge$. Properties (a), (b), (d) in Definition 3.1 hold. The Leibniz rule (c) holds:

$$\begin{aligned}[\![\lozenge, \not\lozenge]\!](u \otimes v) &= \lozenge\not\lozenge(u \otimes v) - \not\lozenge\lozenge(u \otimes v) \\ &= \lozenge\big((\not\lozenge u) \otimes v\big) + \lozenge\big(u \otimes (\not\lozenge v)\big) - \not\lozenge\big((\lozenge u) \otimes v\big) - \not\lozenge\big(u \otimes (\lozenge v)\big) \\ &= (\lozenge\not\lozenge u) \otimes v + (\not\lozenge u) \otimes (\lozenge v) + (\lozenge u) \otimes (\not\lozenge v) + u \otimes (\lozenge\not\lozenge v) \\ &\quad - (\not\lozenge\lozenge u) \otimes v - (\lozenge u) \otimes (\not\lozenge v) - (\not\lozenge u) \otimes (\lozenge v) - u \otimes (\not\lozenge\lozenge v) \\ &= \big([\![\lozenge, \not\lozenge]\!] u\big) \otimes v + u \otimes \big([\![\lozenge, \not\lozenge]\!] v\big)\end{aligned}$$

For property (e), note that there are $\mu, \not\mu \in \mathbb{C}^\infty(M)$ such that $\lozenge\mathfrak{h} = \mu\mathfrak{h}$ and $\not\lozenge\mathfrak{h} = \not\mu\mathfrak{h}$.

$$[\![\lozenge, \not\lozenge]\!]\mathfrak{h} = \lozenge(\not\mu\mathfrak{h}) - \not\lozenge(\mu\mathfrak{h}) = (\lozenge\not\mu)\mathfrak{h} + \not\mu\mu\mathfrak{h} - (\not\lozenge\mu)\mathfrak{h} - \mu\not\mu\mathfrak{h} = \big(\lozenge\not\mu - \not\lozenge\mu\big)\mathfrak{h}$$

Therefore, (e) holds. For general k, ℓ, (a), (b) and (d) still hold. For the Leibniz rule (c), observe that both sides of (3.3) are bilinear over $\mathbb{R}$ in $\lozenge$ and $\not\lozenge$. It therefore suffices to consider the case when $\Diamond = \theta \otimes \Diamond_0$ and $\not\Diamond = \not\theta \otimes \not\Diamond_0$, where $\Diamond_0, \not\Diamond_0 \in \mathcal{P}^0$ and $\theta \in \Gamma(\wedge^k V^*)$ and $\not\theta \in \Gamma(\wedge^\ell V^*)$. In this case,

$$\begin{aligned}
[\![\Diamond, \not\Diamond]\!] &= \wedge_{k,\ell}\, \theta \otimes \Diamond_0\big(\not\theta \otimes \not\Diamond_0\big) - (-1)^{k\ell} \wedge_{\ell,k} \not\theta \otimes \not\Diamond_0\big(\theta \otimes \Diamond_0\big) \\
&= (\theta \wedge \not\theta) \otimes [\![\Diamond_0, \not\Diamond_0]\!] + \big(\theta \wedge (\Diamond_0 \not\theta)\big) \otimes \not\Diamond_0 - \big((\not\Diamond_0 \theta) \wedge \not\theta\big) \otimes \Diamond_0
\end{aligned} \tag{3.4}$$

Each term separately satisfies the Leibniz rule (the first one by the special case $k = \ell = 0$), and (c) holds. Property (e) also follows from (3.4).
To see that $[\![\,\cdot\,,\,\cdot\,]\!] : \mathcal{P}^k \times \mathcal{P}^\ell \to \mathcal{P}^{k+\ell}$ is a Lie superbracket, observe that

$$\begin{aligned}
[\![\not\Diamond, \Diamond]\!] &= \wedge_{\ell,k} \not\Diamond \Diamond - (-1)^{k\ell} \wedge_{k,\ell} \Diamond \not\Diamond \\
&= (-1)^{1+k\ell}\big(\wedge_{k,\ell} \Diamond \not\Diamond - (-1)^{k\ell} \wedge_{\ell,k} \not\Diamond \Diamond\big) \\
&= (-1)^{1+k\ell} [\![\Diamond, \not\Diamond]\!]
\end{aligned}$$

Let $\Diamond_1 \in \mathcal{P}^{k_1}$, $\Diamond_2 \in \mathcal{P}^{k_2}$, $\Diamond_3 \in \mathcal{P}^{k_3}$. Then

$$\begin{aligned}
[\![\Diamond_1, [\![\Diamond_2, \Diamond_3]\!]]\!] = {} & \wedge_{k_1,k_2+k_3} \Diamond_1 \wedge_{k_2,k_3} \Diamond_2 \Diamond_3 - (-1)^{k_2 k_3} \wedge_{k_1,k_2+k_3} \Diamond_1 \wedge_{k_3,k_2} \Diamond_3 \Diamond_2 \\
& - (-1)^{k_1(k_2+k_3)} \wedge_{k_2+k_3,k_1} \wedge_{k_2,k_3} \Diamond_2 \Diamond_3 \Diamond_1 \\
& + (-1)^{k_1(k_2+k_3)+k_2k_3} \wedge_{k_2+k_3,k_1} \wedge_{k_3,k_2} \Diamond_3 \Diamond_2 \Diamond_1
\end{aligned}$$

By Remark 3.4,

$$\begin{aligned}
&(-1)^{k_1 k_3} [\![\Diamond_1, [\![\Diamond_2, \Diamond_3]\!]]\!] \\
&= (-1)^{k_1 k_3} \wedge_{k_1,k_2,k_3} \Diamond_1 \Diamond_2 \Diamond_3 - (-1)^{k_1 k_2} \wedge_{k_2,k_3,k_1} \Diamond_2 \Diamond_3 \Diamond_1 \\
&\quad - (-1)^{k_3(k_1+k_2)} \wedge_{k_1,k_3,k_2} \Diamond_1 \Diamond_3 \Diamond_2 + (-1)^{k_2(k_1+k_3)} \wedge_{k_3,k_2,k_1} \Diamond_3 \Diamond_2 \Diamond_1
\end{aligned}$$

Adding,

$$(-1)^{k_1 k_3} [\![\Diamond_1, [\![\Diamond_2, \Diamond_3]\!]]\!] + (-1)^{k_2 k_1} [\![\Diamond_2, [\![\Diamond_3, \Diamond_1]\!]]\!] + (-1)^{k_3 k_2} [\![\Diamond_3, [\![\Diamond_1, \Diamond_2]\!]]\!] = 0$$

□

Convention 3.4. The symbol $\mathcal{J}$ denotes a finite index set. The set $\mathcal{J}$ and its length $|\mathcal{J}|$ *may change from occurrence to occurrence*. Boldface small Latin indices $\mathbf{a}, \mathbf{b}, \ldots$ take values in $\mathcal{J}$. Boldface Capital Latin indices are multiindices, that is, elements of $\mathcal{J}^k$ for some $k \geq 0$. The length of a multiindex $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ will be denoted $|\mathbf{A}| = k$. We write $X_\mathbf{A} = X_{\mathbf{a}_1} \otimes \cdots \otimes X_{\mathbf{a}_k}$, for various types of objects X.

Definition 3.3. *Let $\mathcal{J}$ be an index set and let $\mathbf{A}$, $\mathbf{B}_1, \ldots, \mathbf{B}_\ell$ be $\mathcal{J}$-multiindices such that $|\mathbf{A}| = |\mathbf{B}_1| + \ldots + |\mathbf{B}_\ell| = k$. Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ and let $\mathbf{B}_1 || \cdots || \mathbf{B}_\ell = (\mathbf{b}_1, \ldots, \mathbf{b}_k)$ be the concatenation of $\mathbf{B}_1$ through $\mathbf{B}_\ell$. Set*

$$\mathbf{A_A}^{\mathbf{B}_1 \cdots \mathbf{B}_\ell} = \tfrac{1}{|\mathbf{B}_1|! \cdots |\mathbf{B}_\ell|!} \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \delta_{\mathbf{a}_{\pi(1)}}{}^{\mathbf{b}_1} \cdots \delta_{\mathbf{a}_{\pi(k)}}{}^{\mathbf{b}_k} \tag{3.5}$$

The index set $\mathcal{J}$ is implicit in (3.5) *and will be specified every time it is used.*

Remark 3.5. $\mathbf{A_A}^{\mathbf{BC}} \mathbf{A_B}^{\mathbf{DE}} = \mathbf{A_A}^{\mathbf{DEC}}$ where $|\mathbf{A}| = |\mathbf{B}| + |\mathbf{C}| = |\mathbf{D}| + |\mathbf{E}| + |\mathbf{C}|$.

Remark 3.6. Let $\Diamond \in \mathcal{P}^k$, $Y \in \Gamma(V^{\otimes k})$ and $z \in \Gamma(\mathcal{T}(V))$. Then

$$[z \otimes, \Diamond_Y] = -(\Diamond_Y z) \otimes \tag{3.6}$$

as operators acting on $\Gamma(\mathcal{T}(V))$, and $[\,\cdot\,,\,\cdot\,]$ is the commutator of operators.

Remark 3.7. Equation (3.3) is equivalent to

$$\begin{aligned} &i_{Y_{\mathbf{A}}} [\![\Diamond, \not\Diamond]\!] \\ &= \mathbf{A_A}^{\mathbf{BC}} \Big(i_{Y_{\mathbf{B}} \otimes Y_{\mathbf{C}}} \Diamond \not\Diamond - i_{Y_{\mathbf{C}} \otimes Y_{\mathbf{B}}} \not\Diamond \Diamond \Big) && \text{(3.7a)} \\ &= \mathbf{A_A}^{\mathbf{BC}} \big(\Diamond_{Y_{\mathbf{B}}} \not\Diamond_{Y_{\mathbf{C}}} - \not\Diamond_{Y_{\mathbf{C}}} \Diamond_{Y_{\mathbf{B}}} \big) - \mathbf{A_A}^{\mathbf{BcE}} \not\Diamond_{(\Diamond_{Y_{\mathbf{B}}} Y_{\mathbf{c}}) \otimes Y_{\mathbf{E}}} + \mathbf{A_A}^{\mathbf{bDC}} \Diamond_{(\not\Diamond_{Y_{\mathbf{C}}} Y_{\mathbf{b}}) \otimes Y_{\mathbf{D}}} && \text{(3.7b)} \end{aligned}$$

Here $Y_1, \ldots, Y_{k+\ell}$ are any sections of V. Moreover, $\mathcal{J} = \{1, \ldots, k+\ell\}$ and $\mathbf{A} = (1, \ldots, k+\ell)$, see Convention 3.4. The $\mathcal{J}$-multiindices have length $|\mathbf{A}| = k+\ell$, $|\mathbf{B}| = k$, $|\mathbf{C}| = \ell$. Also, i is interior multiplication as in Convention 3.3.
To check (3.7b), use (3.6) with $Y = Y_{\mathbf{B}}$ and $z = Y_{\mathbf{C}}$ and apply it to $\not\Diamond u$. Then,

$$Y_{\mathbf{C}} \otimes \big(\Diamond_{Y_{\mathbf{B}}} \not\Diamond u \big) - \Diamond_{Y_{\mathbf{B}}} \big(Y_{\mathbf{C}} \otimes \not\Diamond u \big) = -\big(\Diamond_{Y_{\mathbf{B}}} Y_{\mathbf{C}} \big) \otimes \not\Diamond u$$

Both sides are sections of $\Gamma(V^{\otimes \ell} \otimes \wedge^\ell V^* \otimes \mathcal{T}(V))$. Contracting the first ℓ with the second ℓ factors, we obtain (since diamonds commute with contractions)

$$i_{Y_{\mathbf{C}}} \big(\Diamond_{Y_{\mathbf{B}}} \not\Diamond u \big) - \Diamond_{Y_{\mathbf{B}}} \big(i_{Y_{\mathbf{C}}} \not\Diamond u \big) = -i_{\Diamond_{Y_{\mathbf{B}}} Y_{\mathbf{C}}} \big(\not\Diamond u \big)$$

This is equivalent to (since $i_{Y_{\mathbf{C}}} i_{Y_{\mathbf{B}}} = i_{Y_{\mathbf{B}} \otimes Y_{\mathbf{C}}}$)

$$i_{Y_{\mathbf{B}} \otimes Y_{\mathbf{C}}} \big(\Diamond \not\Diamond u \big) = \Diamond_{Y_{\mathbf{B}}} \big(\not\Diamond_{Y_{\mathbf{C}}} u \big) - \big(\not\Diamond_{\Diamond_{Y_{\mathbf{B}}} Y_{\mathbf{C}}} \big)\, u \tag{3.8}$$

With Remark 3.7, we obtain the following corollary of Proposition 3.1.

Corollary 3.1. *For all $\Diamond \in \mathcal{P}^1$ and $Y_1, Y_2 \in \Gamma(V)$,*

$$\begin{aligned} \tfrac{1}{2} [\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2} &= \big(i_{Y_1 \otimes Y_2} - i_{Y_2 \otimes Y_1} \big) \Diamond \Diamond \\ &= \Diamond_{Y_1} \Diamond_{Y_2} - \Diamond_{Y_2} \Diamond_{Y_1} - \Diamond_{\Diamond_{Y_1} Y_2 - \Diamond_{Y_2} Y_1} \end{aligned}$$

Definition 3.4. $\mathfrak{g}(V, \mathfrak{H})$ *is the subbundle of* $\mathrm{End}(V)$ *whose fiber at* $p \in M$ *is all* $A \in \mathrm{End}(V)_p$ *for which there is a* $\lambda \in \mathbb{R}$ *so that*

$$\mathfrak{h}_p(AY_1, Y_2) + \mathfrak{h}_p(Y_1, AY_2) = \lambda\, \mathfrak{h}_p(Y_1, Y_2) \tag{3.9}$$

for all $Y_1, Y_2 \in V_p$. *Here* $\mathfrak{h}_p \in (\mathrm{Sym}^2 V^*)_p$ *is a representative for* $\mathfrak{H}_p$. *For each* $k \geq 0$, *set*

$$\mathcal{R}^k = \Gamma\big(\wedge^k V^* \otimes \mathfrak{g}(V, \mathfrak{H}) \big)$$

Remark 3.8. The definition of the vector bundle $\mathfrak{g}(V, \mathfrak{H})$ does not depend on the choice of a representative $\mathfrak{h}$. The fibers of $\mathfrak{g}(V, \mathfrak{H})$ have dimension 7. Each fiber is a Lie algebra isomorphic to the Lie algebra of the group $\mathbb{R}_+ \times O(1,3)$, the direct product of the multiplicative group of positive real numbers with the Lorentz group.

Proposition 3.2. *For all* $\Diamond \in \mathcal{P}_{\perp}^{k}$ *and* $Y \in \Gamma(V^{\otimes k})$ *and* $Z \in \Gamma(V)$ *set*

$$\beta(\Diamond)_Y Z = \Diamond_Y Z \ \in \ \Gamma(V)$$

Then $\beta(\Diamond)_Y \in \Gamma(\mathfrak{g}(V,\mathfrak{H})) \subset \Gamma(\mathrm{End}(V))$ *and* $\beta(\Diamond) \in \mathcal{R}^k$. *The map*

$$\begin{aligned} \beta : \ \mathcal{P}_{\perp}^{k} &\to \mathcal{R}^k \\ \Diamond &\mapsto \beta(\Diamond) \end{aligned}$$

is a bijection.

Proof. First, $\beta(\Diamond) \in \Gamma(\wedge^k V^* \otimes \mathrm{End}(V))$ because $\beta(\Diamond)_Y Z$ is linear over $C^\infty(M)$ in both Y *and* Z, by the assumption that $\Diamond \in \mathcal{P}_{\perp}^{k}$. We have to show that $\beta(\Diamond) \in \mathcal{R}^k$. Let $\mathfrak{h}$ be a representative of $\mathfrak{H}$. Then

$$\begin{aligned} 0 &= \Diamond_Y(\mathfrak{h}(Z_1, Z_2)) \\ &= (\Diamond_Y \mathfrak{h})(Z_1, Z_2) + \mathfrak{h}(\Diamond_Y Z_1, Z_2) + \mathfrak{h}(Z_1, \Diamond_Y Z_2) \\ &= \mu(Y)\,\mathfrak{h}(Z_1, Z_2) + \mathfrak{h}(\beta(\Diamond)_Y Z_1, Z_2) + \mathfrak{h}(Z_1, \beta(\Diamond)_Y Z_2) \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(V)$, and μ as in (e) of Definition 3.1. Hence, $\beta(\Diamond) \in \mathcal{R}^k$. Also,

- β is injective. In fact, $\beta(\Diamond) = 0$ implies that $\Diamond$ annihilates functions, sections of V and, by equation (3.2), sections of V^*. By (a), (c) in Definition 3.1, we have $\Diamond = 0$.
- β is surjective. Given $\Upsilon \in \mathcal{R}^k$, set

$$\Diamond_Y f = 0 \qquad \Diamond_Y Z = \Upsilon_Y Z \qquad (\Diamond_Y \xi)(Z) = -\xi(\Upsilon_Y Z)$$

  for all $f \in C^\infty(M)$, $Z \in \Gamma(V)$, $\xi \in \Gamma(V^*)$ and all $Y \in \Gamma(V^{\otimes k})$. Together with (a),(c) in Definition 3.1, they uniquely determine $\Diamond_Y u$ for all $u \in \Gamma(\mathcal{T}(V))$, and (b), (d), (e) in Definition 3.1 are automatic. $\Diamond \in \mathcal{P}_{\perp}^{k}$ satisfies $\beta(\Diamond) = \Upsilon$.

  □

4. From Diamonds of degree one to Lorentzian Geometry

In this section, we characterize the elements of $\mathcal{P}^1$ that correspond to Lorentzian geometries. Conversely, we show that every Lorentzian manifold (locally) arises from an element of $\mathcal{P}^1$. The Einstein vacuum equations are reinterpreted as conditions on elements of $\mathcal{P}^1$, to motivate their reformulation in Section 5.
This section is outside the overall technical development of this paper. Its purpose is to connect the present formalism with traditional approaches.

Proposition 4.1. *For all* $\Diamond \in \mathcal{P}^1$ *there is a unique vector bundle homomorphism*

$$\mathcal{E}^{\Diamond} : \ V \to TM \qquad \text{or, equivalently,} \qquad \mathcal{E}^{\Diamond} \in \Gamma(V^* \otimes TM) \tag{4.1}$$

such that $(\mathcal{E}^{\Diamond}(Y))(f) = \Diamond_Y(f)$ *for all* $Y \in \Gamma(V)$ *and* $f \in C^\infty(M)$.

Proof. The operator $\Diamond_Y$ acts as a derivation on $C^\infty(M)$ and is linear over $C^\infty(M)$ in Y, by Definition 3.1. □

Definition 4.1. *A* $\Diamond \in \mathcal{P}^1$ *is called* non-degenerate *if and only if* $\mathcal{E}^\Diamond$ *is a vector bundle isomorphism. The canonical extension of* $\mathcal{E}^\Diamond$ *from* V *to* $\mathcal{T}(V)$ *is also denoted by*

$$\mathcal{E}^\Diamond : \quad \mathcal{T}(V) \to \mathcal{T}(TM)$$

The extension is a vector bundle isomorphism determined by

- $\mathcal{E}^\Diamond(f) = f$ *for all* $f \in C^\infty(M)$
- $\mathcal{E}^\Diamond(u \otimes v) = \mathcal{E}^\Diamond(u) \otimes \mathcal{E}^\Diamond(v)$ *for all* $u, v \in \Gamma(\mathcal{T}(V))$
- $\mathcal{E}^\Diamond(I_V) = I_{TM}$ *where* $I_V \in \Gamma(\mathrm{End}(V))$, $I_{TM} \in \Gamma(\mathrm{End}(TM))$ *are the identities*

Proposition 4.2. *Let* $\Diamond \in \mathcal{P}^1$ *be non-degenerate. Let* $\mathcal{E} = \mathcal{E}^\Diamond$ *and set*

$$\nabla^\Diamond : \Gamma(\mathcal{T}(TM)) \to \Gamma(T^*M \otimes \mathcal{T}(TM)) \qquad \nabla^\Diamond_X u = \mathcal{E}\Big(\Diamond_{\mathcal{E}^{-1}(X)} \mathcal{E}^{-1}(u)\Big)$$

for all $X \in \Gamma(TM)$ *and* $u \in \Gamma(\mathcal{T}(TM))$*. Then* $\nabla^\Diamond$ *is a connection on the tensor bundle* $\mathcal{T}(TM)$ *such that for all* $X \in \Gamma(TM)$*,*

- $\nabla^\Diamond$ *is linear over* $\mathbb{R}$
- $\nabla^\Diamond_X f = X(f)$ *for all* $f \in C^\infty(M)$
- $\nabla^\Diamond_X$ *maps* $C^\infty(M) \to C^\infty(M)$, $\Gamma(TM) \to \Gamma(TM)$ *and* $\Gamma(T^*M) \to \Gamma(T^*M)$
- $\nabla^\Diamond_X(u \otimes v) = (\nabla^\Diamond_X u) \otimes v + u \otimes (\nabla^\Diamond_X v)$ *for all* $u, v \in \Gamma(\mathcal{T}(TM))$
- $\nabla^\Diamond I = 0$ *where* $I \in \Gamma(\mathrm{End}(TM))$ *is the identity.*

Proof. By direct verification. □

Lemma 4.1. *Let* $\Diamond \in \mathcal{P}^1$ *be non-degenerate. Let* $\nabla = \nabla^\Diamond$, $\mathcal{E} = \mathcal{E}^\Diamond$*. For all* $X_i \in \Gamma(TM)$, $i = 1, 2$, *and* $v \in \Gamma(\mathcal{T}(TM))$ *and corresponding* $Y_i = \mathcal{E}^{-1}(X_i) \in \Gamma(V)$, $i = 1, 2$, *and* $z = \mathcal{E}^{-1}(v) \in \Gamma(\mathcal{T}(V))$*:*

(a) $\big(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{\nabla_{X_1}X_2 - \nabla_{X_2}X_1}\big)v = \frac{1}{2}\,\mathcal{E}\big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2} z\big)$
(b) $[\![\Diamond, \Diamond]\!] \in \mathcal{P}^2_\perp$ *if and only if* ∇ *is torsion-free*

Let $\mathfrak{h}$ *be a representative for* $\mathfrak{H}$ *and let* $\Diamond\mathfrak{h} = \mu \otimes \mathfrak{h}$ *as in (e) of Definition 3.1. Let* $\nu = \mathcal{E}(\mu) \in \Gamma(T^*M)$*. For all* X_i *and* Y_i *as above,* $i = 1, 2$*, and all* $f \in C^\infty(M)$*:*

(c) $\nabla_{X_1}\big(\mathcal{E}(e^f \mathfrak{h})\big) = e^f(\mathrm{d}f + \nu)(X_1)\,\mathcal{E}(\mathfrak{h})$
(d) $\mathrm{d}\nu(X_1, X_2)\,\mathfrak{h} - \nu\Big(\nabla_{X_1}X_2 - \nabla_{X_2}X_1 - [X_1, X_2]\Big)\mathfrak{h} = \frac{1}{2}[\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\mathfrak{h}$

Proof. We verify (a) through (d):

(a) The left hand side is equal to $\mathcal{E}\big(\big(\Diamond_{Y_1}\Diamond_{Y_2} - \Diamond_{Y_2}\Diamond_{Y_1} - \Diamond_{\Diamond_{Y_1}Y_2 - \Diamond_{Y_2}Y_1}\big)z\big)$, by the definition of $\nabla = \nabla^\Diamond$, see Proposition 4.2. Now use Corollary 3.1.
(b) Let $v \in C^\infty(M)$ in (a). Then $z = v$. We obtain $-\nabla_{T(X_1, X_2)}v = \frac{1}{2}[\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2} v$. The torsion T of ∇ vanishes if and only if $[\![\Diamond, \Diamond]\!] \in \mathcal{P}^2_\perp$.
(c)

$$\nabla_{X_1}\big(\mathcal{E}(e^f \mathfrak{h})\big) = e^f\,\mathrm{d}f(X_1)\,\mathcal{E}(\mathfrak{h}) + e^f\,\nabla_{X_1}\big(\mathcal{E}(\mathfrak{h})\big)$$
$$\nabla_{X_1}\big(\mathcal{E}(\mathfrak{h})\big) = \mathcal{E}\big(\Diamond_{\mathcal{E}^{-1}(X_1)}\mathfrak{h}\big) = \mathcal{E}\big(\mu(\mathcal{E}^{-1}(X_1))\,\mathfrak{h}\big) = \nu(X_1)\,\mathcal{E}(\mathfrak{h})$$

(d) Let $z = \mathfrak{h}$ in (a). Then $v = \mathcal{E}(\mathfrak{h})$. Rewrite the result using (c) with $f = 0$.

This concludes the proof. □

Convention 4.1. Let $\pi : E \to B$ be a vector bundle. For every $S \in \Gamma(\mathrm{End}(E))$ we denote by $\mathrm{tr}(S) \in C^\infty(B)$ its trace as a linear map.

Proposition 4.3. *Let M be simply connected. Let $\Diamond \in \mathcal{P}^1$ and suppose*

(a) $\Diamond$ is non-degenerate
(b) $\frac{1}{2}[\![\Diamond, \Diamond]\!] \in \mathcal{P}^2_\perp$
(c) $\mathrm{tr}(\varUpsilon_{Y_1 \otimes Y_2}) = 0$ for all $Y_1, Y_2 \in \Gamma(V)$, where

$$\varUpsilon = \beta(\tfrac{1}{2}[\![\Diamond, \Diamond]\!]) \in \mathcal{R}^2$$

Fix any representative $\mathfrak{h}'$ for $\mathfrak{H}$, and let $\Diamond \mathfrak{h}' = \mu \otimes \mathfrak{h}'$ as in (e) of Definition 3.1.
Part 1: *The 1-form $\nu = \mathcal{E}^\Diamond(\mu) \in \Gamma(T^*M)$ is exact, $\nu = -\mathrm{d}f$ with $f \in C^\infty(M)$.*
Part 2: *Let $\mathfrak{h}$ be a representative of $\mathfrak{H}$. Then $\nabla^\Diamond$ is the Levi-Civita connection for the Lorentzian metric $\mathcal{E}^\Diamond(\mathfrak{h}) \in \Gamma(\mathrm{Sym}^2 T^*M)$ if and only if $\mathfrak{h} = e^{f+C}\mathfrak{h}'$ for some $C \in \mathbb{R}$.*
Part 3: *The associated Riemann curvature $R^\Diamond$ is given by*

$$R^\Diamond(X_1, X_2)X_3 = \mathcal{E}^\Diamond\Big(\varUpsilon_{Y_1 \otimes Y_2} Y_3\Big) \quad \in \quad \Gamma(TM) \tag{4.2}$$

for all $X_i \in \Gamma(TM)$ and $Y_i = (\mathcal{E}^\Diamond)^{-1}(X_i) \in \Gamma(V)$, $i = 1, 2, 3$.

Remark 4.1. Part 2 of Proposition 4.3 implies that: *There is a representative $\mathfrak{h}$ of $\mathfrak{H}$, unique up to an overall constant multiplicative factor, such that $\nabla^\Diamond$ is the Levi-Civita connection for $\mathcal{E}^\Diamond(\mathfrak{h})$.* In particular, the assignment $\Diamond \mapsto \mathcal{E}^\Diamond(\mathfrak{h})$ is canonical (independent of the choice of $\mathfrak{h}'$), modulo an overall constant multiplicative factor.

Proof. We use Lemma 4.1 with the understanding that the representative for $\mathfrak{H}$ in Lemma 4.1 is $\mathfrak{h}'$. Then ν in Lemma 4.1 coincides with ν in Proposition 4.3.
Part 1: (b) implies that $\nabla^\Diamond$ is torsion-free by Lemma 4.1.(b). Then $\mathrm{d}\nu(X_1, X_2)\mathfrak{h}' = \frac{1}{2}[\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\mathfrak{h}'$ by Lemma 4.1.(d). Contracting with $(\mathfrak{h}')^{-1}$ gives $4\,\mathrm{d}\nu(X_1, X_2) = \frac{1}{2}\, i_{(\mathfrak{h}')^{-1}}\big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\mathfrak{h}'\big)$ where i denotes interior multiplication. That is, both factors of $\big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\mathfrak{h}'\big) \in \Gamma(\mathrm{Sym}^2 V^*)$ are contracted with $(\mathfrak{h}')^{-1} \in \Gamma(\mathrm{Sym}^2 V)$. Let $\mathcal{I}$, $F_{(a)}$ and $\lambda^{(a)}$, $a \in \mathcal{I}$, be as in Remark 3.3. Let h_{ab} be the components of $\mathfrak{h}'$, that is, $\mathfrak{h}' = h_{ab}\lambda^{(a)} \otimes \lambda^{(b)}$ and $\mathfrak{h}' = h^{ab}F_{(a)} \otimes F_{(b)}$, where (h^{ab}) is the inverse of (h_{ab}). By direct calculation,

$$\begin{aligned} \tfrac{1}{2}\, i_{(\mathfrak{h}')^{-1}}\big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\mathfrak{h}'\big) = \big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\,\lambda^{(a)}\big)(F_{(a)}) = -\lambda^{(a)}\big([\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2}\,F_{(a)}\big) \\ = -2\,\mathrm{tr}\big(\varUpsilon_{Y_1 \otimes Y_2}\big) \end{aligned}$$

For the last equality, bear in mind that $\frac{1}{2}[\![\Diamond, \Diamond]\!]$ and $\varUpsilon$ coincide in their actions on sections of V. It follows from the last identity that $\mathrm{d}\nu(X_1, X_2) = -\frac{1}{2}\,\mathrm{tr}\big(\varUpsilon_{Y_1 \otimes Y_2}\big)$, which vanishes for all $X_1, X_2 \in \Gamma(TM)$ by (c). Therefore, $\mathrm{d}\nu = 0$. Since M is simply connected, there is, by the Poincare Lemma, an $f \in C^\infty(M)$ with $\mathrm{d}f = -\nu$.
Part 2: $\nabla^\Diamond$ is torsion-free. $\nabla^\Diamond$ is compatible with the Lorentzian metric $\mathcal{E}(e^F\mathfrak{h}')$ if and only if $F = f + C$ for some $C \in \mathbb{R}$, see Lemma 4.1.(c).
Part 3: Use Lemma 4.1.(a) with $v = X_3$ and recall that $\nabla^\Diamond$ is torsion-free. $\square$

Remark 4.2. To connect the Lie superalgebra identity $[\![\Diamond, [\![\Diamond, \Diamond]\!]]\!] = 0$ with the classical algebraic and differential Bianchi identities for $R^\Diamond$, we derive an identity. First of all, suppose that $\Diamond \in \mathcal{P}^1$ and $\not\Diamond \in \mathcal{P}^2_\perp$. Then $\varUpsilon = \beta(\not\Diamond) \in \mathcal{R}^2$ is defined. Let $\mathcal{J} = \{1, 2, 3\}$.

For all $Y_i \in \Gamma(V)$, $i \in \mathcal{J}$, and $Z \in \Gamma(V)$, we have for any $\mathcal{J}$-multiindex $\mathbf{C}$ with $|\mathbf{C}| = 2$,

$$i_{Y_\mathbf{b} \otimes Y_\mathbf{C} \otimes Z}\big(\Diamond \varUpsilon\big) = i_{Y_\mathbf{b} \otimes Y_\mathbf{C}}\big(\Diamond \not\Diamond Z\big) - i_{Y_\mathbf{C} \otimes Y_\mathbf{b}}\big(\not\Diamond \Diamond Z\big) - \Diamond_{\varUpsilon_{Y_\mathbf{C}} Y_\mathbf{b}} Z$$

Multiply by $\mathbf{A_A}^{\mathbf{bC}}$, where $\mathbf{A} = (1, 2, 3)$, sum and obtain, by equation (3.7a),

$$\mathbf{A_A}^{\mathbf{bC}} i_{Y_\mathbf{b} \otimes Y_\mathbf{C} \otimes Z}\big(\Diamond \varUpsilon\big) = [\![\Diamond, \not\Diamond]\!]_{Y_\mathbf{A}} Z - \mathbf{A_A}^{\mathbf{bC}} \Diamond_{\varUpsilon_{Y_\mathbf{C}} Y_\mathbf{b}} Z$$

In the special case when $[\![\Diamond, \Diamond]\!] \in \mathcal{P}^2_\perp$ and when $\not\Diamond = \frac{1}{2}[\![\Diamond, \Diamond]\!]$, the first term on right hand side vanishes by the Lie superalgebra identity $[\![\Diamond, [\![\Diamond, \Diamond]\!]]\!] = 0$. The left hand side is linear over $C^\infty(M)$ in Z, and so must be the right hand side. If $\Diamond$ is non-degenerate, the last observation implies the "algebraic Bianchi identity"

$$\mathbf{A_A}^{\mathbf{bC}} \varUpsilon_{Y_\mathbf{C}} Y_\mathbf{b} = 0$$

where $\varUpsilon = \beta\big(\frac{1}{2}[\![\Diamond, \Diamond]\!]\big)$. Consequently, we also have the "differential Bianchi identity"

$$\mathbf{A_A}^{\mathbf{bC}} i_{Y_\mathbf{b} \otimes Y_\mathbf{C} \otimes Z}\big(\Diamond \varUpsilon\big) = 0$$

Finally, if $\Diamond$ satisfies all the assumptions of Proposition 4.3, then we obtain the traditional Bianchi identities for the associated Riemann curvature $R^\Diamond$.

Proposition 4.4. *Let M be simply connected, and assume we are given*

(a) a vector bundle isomorphism $\not\mathcal{E} : V \to TM$
(b) a representative $\not\mathfrak{h}$ for $\mathfrak{H}$

Let $\mathfrak{h}' = \not\mathfrak{h}$ in Proposition 4.3. Then, there is a unique $\Diamond \in \mathcal{P}^1$ which satisfies the assumptions of Proposition 4.3 such that $\mathcal{E}^\Diamond = \not\mathcal{E}$ and such that $\mu = 0$ in Proposition 4.3.

Remark 4.3. Observe that (a) and (b) induce the Lorentzian metric $\not\mathcal{E}(\not\mathfrak{h})$ on M. Conversely, every Lorentzian metric arises locally from such a construction.

Proof. We use Lemma 4.1 with the understanding that the representative for $\mathfrak{H}$ in Lemma 4.1 is $\not\mathfrak{h}$. Then ν in Lemma 4.1 coincides with ν in Proposition 4.3.
We first prove existence. The canonical extension of $\not\mathcal{E}$ from V to $\mathcal{T}(V)$ is also denoted by $\not\mathcal{E} : \mathcal{T}(V) \to \mathcal{T}(TM)$ (just as in Definition 4.1). Let $\not\nabla$ be the Levi-Civita connection associated with $\not\mathcal{E}(\not\mathfrak{h}) \in \Gamma(\mathrm{Sym}^2 T^*M)$, a metric with signature $(-,+,+,+)$. For all $Y \in \Gamma(V)$ and $u \in \Gamma(\mathcal{T}(V))$, set $\Diamond_Y u = \not\mathcal{E}^{-1}(\not\nabla_{\not\mathcal{E}(Y)} \not\mathcal{E}(u)) \in \Gamma(\mathcal{T}(V))$. By direct inspection, $\Diamond \in \mathcal{P}^1$ (see Definition 3.1). Then $\not\mathcal{E} = \mathcal{E}^\Diamond$ and $\not\nabla = \nabla^\Diamond$. In particular, $\Diamond$ is non-degenerate. Lemma 4.1.(b) implies that $[\![\Diamond, \Diamond]\!] \in \mathcal{P}^2_\perp$, because $\nabla^\Diamond = \not\nabla$ is torsion-free. Lemma 4.1.(c) implies $\nu = 0$ because $\nabla^\Diamond = \not\nabla$ is compatible with the metric $\not\mathcal{E}(\not\mathfrak{h})$. Now Lemma 4.1.(d) implies $\frac{1}{2}[\![\Diamond, \Diamond]\!]_{Y_1 \otimes Y_2} \not\mathfrak{h} = 0$ for all Y_1, Y_2. This implies $\mathrm{tr}(\varUpsilon_{Y_1 \otimes Y_2}) = 0$, where $\varUpsilon = \beta(\frac{1}{2}[\![\Diamond, \Diamond]\!])$. This concludes the existence proof. To prove uniqueness, assume there are two such $\Diamond \in \mathcal{P}^1$. Then their $\mathcal{E}^\Diamond = \not\mathcal{E}$ coincide, and their $\nabla^\Diamond$ coincide, because they are the Levi-Civita connection for the same metric $\mathcal{E}^\Diamond(\not\mathfrak{h})$ by Proposition 4.3. Then the two $\Diamond$'s must be the same. $\square$

Proposition 4.5. *Let M be simply connected. Let $\Diamond \in \mathcal{P}^1$ be non-degenerate. The following are equivalent:*

(a) $\Diamond$ *satisfies the assumptions of Proposition 4.3, and the associated Lorentzian manifold is Ricci-flat*
(b) $\frac{1}{2}[\![\Diamond,\Diamond]\!] \in \mathcal{P}^2_{vac}$
(c) there is an $\not\Diamond \in \mathcal{P}^2_{vac}$ *such that* $\not\Diamond = \frac{1}{2}[\![\Diamond,\Diamond]\!]$ *and* $[\![\Diamond,\not\Diamond]\!] = 0$

See Definition 5.1 below for $\mathcal{P}^2_{vac}$.

Proof. (c) implies (b), and conversely, (b) implies (c) by setting $\not\Diamond = \frac{1}{2}[\![\Diamond,\Diamond]\!]$ and using the super Jacobi identity to conclude that $[\![\Diamond,\not\Diamond]\!] = \frac{1}{2}[\![\Diamond,[\![\Diamond,\Diamond]\!]]\!] = 0$. The equivalence of (a) and (b) follows by comparing for each row of the following table the corresponding condition/assumption in Proposition 4.3 and Definition 5.1:

Proposition 4.3	*Definition 5.1 with* $\not\Diamond = \frac{1}{2}[\![\Diamond,\Diamond]\!]$, $k=2$
(b)	the assumption $\not\Diamond \in \mathcal{P}^2_{\perp}$
alg. Bianchi identity for $R^{\Diamond}$ and (4.2)	(a)
(c)	(b)
Ricci flatness and (4.2)	(c.2)

This concludes the proof. $\square$

5. Reformulation of the Einstein vacuum equations

In the next definition, the index set $\mathcal{J} = \{1,\dots,k+1\}$ and $\mathbf{A} = (1,\dots,k+1)$, and $\mathbf{B}$ is a $\mathcal{J}$-multiindex of length $|\mathbf{B}| = k$.

Definition 5.1. *The "vacuum subspace"* $\mathcal{P}^k_{vac} \subset \mathcal{P}^k_{\perp}$, $k = 2,3,4$, *is the set of all* $\Diamond \in \mathcal{P}^k_{\perp}$ *such that the associated* $\Upsilon = \beta(\Diamond) \in \mathcal{R}^k$ *satisfies for all* $Y_i \in \Gamma(V)$, $i \in \mathcal{J}$,

(a) $\mathbf{A_A}^{\mathbf{Bc}}\Upsilon_{Y_\mathbf{B}} Y_\mathbf{c} = 0$
(b) $\mathrm{tr}(\Upsilon_{Y_\mathbf{B}}) = 0$ *for* $\mathbf{B} = (1,\dots,k)$
(c.2) for $k=2$*:* $\mathbf{C}(\Upsilon) = 0$ *where* $\mathbf{C}$ *is the contraction operator for the index pair* $(2,4)$
(c.3) for $k=3$*:* $\mathbf{C}(\Upsilon \otimes \mathfrak{h}^{-1}) = 0$ *where* $\mathbf{C}$ *contracts* $(1,5),(3,6)$ *and* $(4,7)$

In (c.2) we regard Υ *as a section of* $(V^*)^{\otimes 3} \otimes V \supset \wedge^2 V^* \otimes \mathfrak{g}(V,\mathfrak{H})$.
In (c.3) we regard $\Upsilon \otimes \mathfrak{h}^{-1}$ *as a section of* $(V^*)^{\otimes 4} \otimes V^{\otimes 3} \supset \wedge^3 V^* \otimes \mathfrak{g}(V,\mathfrak{H}) \otimes \mathrm{Sym}^2 V$.
Here, $\mathfrak{h}$ *is any representative of* $\mathfrak{H}$*. All contractions are natural pairings of* V *with* V^*.

See Definition 6.1 and Proposition 6.1 for a discussion of $\mathcal{P}^k_{\mathrm{vac}}$ in index notation.

We now adopt verbatim, from Section 2, the definitions of $\mathcal{D}$, A_0 and A_1, with the understanding that $L_0 = \mathcal{P}_0$ and $L_1 = \mathcal{P}_1$, see Proposition 3.1. In particular, for all $\blacklozenge = (\Diamond,\not\Diamond) \in \mathcal{P}^1 \times \mathcal{P}^2 \subset A_1$ and $\blacklozenge' = (\Diamond',\not\Diamond') \in \mathcal{P}^2 \times \mathcal{P}^3 \subset A_0$, we have

$$\mathcal{D}_{\blacklozenge}\blacklozenge = \Big(\not\Diamond - \tfrac{1}{2}[\![\Diamond,\Diamond]\!],\ [\![\Diamond,\not\Diamond]\!]\Big) \qquad \in \mathcal{P}^2 \times \mathcal{P}^3 \subset A_0 \tag{5.1a}$$

$$\mathcal{D}_{\blacklozenge}\blacklozenge' = \Big(\not\Diamond' - [\![\Diamond,\Diamond']\!],\ [\![\Diamond,\not\Diamond']\!] + [\![\Diamond',\not\Diamond]\!]\Big) \qquad \in \mathcal{P}^3 \times \mathcal{P}^4 \subset A_1 \tag{5.1b}$$

The Einstein vacuum equations are now reformulated as:

$$\text{Find } \blacklozenge \in \mathcal{P}^1 \times \mathcal{P}^2_{\mathrm{vac}} \text{ such that } \mathcal{D}_{\blacklozenge}\blacklozenge = 0. \tag{5.2}$$

Remark 5.1. Proposition 4.5 justifies the expression "*reformulation* of the Einstein vacuum equations". Notice that, in contrast to Proposition 4.5, we do not require ω to be non-degenerate. Degenerate solutions may not be physically interesting in themselves. However, they can be used as a mathematical tool, to construct nearby non-degenerate solutions.

We now derive algebraic and differential identities.

Lemma 5.1. *For all* $(k,\ell) \in \{(1,2),(2,2),(1,3)\}$*, all* $\Diamond \in \mathcal{P}^k$ *and all* $\not\Diamond \in \mathcal{P}^\ell_{vac}$*, we have* $[\![\Diamond, \not\Diamond]\!] \in \mathcal{P}^{k+\ell}_{vac}$*.*

Proof. In this proof, the index set $\mathcal{J} = \{1,\dots,k+\ell\}$ and $\mathbf{A} = (1,\dots,k+\ell)$. First show that $[\![\Diamond, \not\Diamond]\!] \in \mathcal{P}^{k+\ell}_\perp$. Equations (3.7b), (3.8) and Definition 5.1.(a) for $\not\Diamond \in \mathcal{P}^\ell_{\mathrm{vac}}$ imply

$$[\![\Diamond, \not\Diamond]\!]_{Y_\mathbf{A}} u = \mathbf{A}_\mathbf{A}{}^{\mathbf{BC}} \Big(i_{Y_\mathbf{B} \otimes Y_\mathbf{C}} \big(\Diamond \not\Diamond u\big) - \not\Diamond_{Y_\mathbf{C}} \big(\Diamond_{Y_\mathbf{B}} u\big) \Big) \tag{5.3}$$

We have used that $\varUpsilon^{\not\Diamond} = \beta(\not\Diamond) \in \mathcal{R}^\ell$ satisfies $\varUpsilon^{\not\Diamond}_Y Z = \not\Diamond_Y Z$ for all $Y \in \varGamma(V^{\otimes \ell})$ and $Z \in \varGamma(V)$. The assumption $\not\Diamond \in \mathcal{P}^\ell_\perp$ implies that $\not\Diamond f = 0$ for all $f \in C^\infty(M)$, and consequently by equation (5.3), $[\![\Diamond, \not\Diamond]\!] f = 0$ for all $f \in C^\infty(M)$. Therefore, $[\![\Diamond, \not\Diamond]\!] \in \mathcal{P}^{k+\ell}_\perp$.

We can now define $\varUpsilon^{\not\Diamond} = \beta(\not\Diamond) \in \mathcal{R}^\ell$ and $\varUpsilon^{[\![\Diamond,\not\Diamond]\!]} = \beta([\![\Diamond, \not\Diamond]\!]) \in \mathcal{R}^{k+\ell}$. Equation (5.3) with $u = Z \in \varGamma(V)$ is equivalent to

$$\varUpsilon^{[\![\Diamond,\not\Diamond]\!]}_{Y_\mathbf{A}} Z = \mathbf{A}_\mathbf{A}{}^{\mathbf{BC}} i_{Y_\mathbf{B} \otimes Y_\mathbf{C} \otimes Z} \big(\Diamond \varUpsilon^{\not\Diamond}\big) \tag{5.4}$$

Here $\Diamond \varUpsilon^{\not\Diamond}$ is a section of $(V^*)^{\otimes(k+\ell+1)} \otimes V \supset \wedge^k V^* \otimes \wedge^\ell V^* \otimes \mathfrak{g}(V, \mathfrak{H})$. We now check that $[\![\Diamond, \not\Diamond]\!] \in \mathcal{P}^{k+\ell}_{\mathrm{vac}}$, by showing (a), (b) in Definition 5.1 for $[\![\Diamond, \not\Diamond]\!]$. When $(k,\ell) = (1,2)$ we also have to check (c.3).

- The totally antisymmetric part of the right hand side of equation (5.4) with respect to $Y_1, \dots, Y_{k+\ell}, Z$ vanishes by (a) for $\not\Diamond \in \mathcal{P}^\ell_{\mathrm{vac}}$. Therefore, (a) holds for $[\![\Diamond, \not\Diamond]\!]$.
- $\Diamond_{Y_\mathbf{A}}$ commutes with natural contractions (pairings of V with V^*). Therefore, (b) for $\not\Diamond \in \mathcal{P}^\ell_{\mathrm{vac}}$ and equation (5.4) imply (b) for $[\![\Diamond, \not\Diamond]\!]$.

This concludes the proof when $(k,\ell) \in \{(2,2),(1,3)\}$. From here, $(k,\ell) = (1,2)$.

- We must show (c.3). We must show that $\mathbf{C}(\varUpsilon^{[\![\Diamond,\not\Diamond]\!]} \otimes \mathfrak{h}^{-1}) = 0$, where $\mathbf{C}$ contracts the index-pairs $(1,5),(3,6),(4,7)$ (see the explanation at the end of Definition 5.1). By writing out the sum on the right hand side of (5.4) (there are $|P(1,2)| = 3$ terms), we see that it suffices to show that the contractions

  $$(3,5),(2,6),(4,7) \quad \text{or} \quad (2,5),(1,6),(4,7) \quad \text{or} \quad (1,5),(3,6),(4,7) \tag{5.5}$$

  of $(\Diamond \varUpsilon^{\not\Diamond}) \otimes \mathfrak{h}^{-1} \in \varGamma((V^*)^{\otimes 4} \otimes V^{\otimes 3})$ all vanish. Recall that there is a $\mu \in \varGamma(V^*)$ such that $\Diamond \mathfrak{h} = \mu \otimes \mathfrak{h}$. Consequently, $\Diamond(\mathfrak{h}^{-1}) = -\mu \otimes (\mathfrak{h}^{-1})$. By the Leibniz rule,

  $$\big(\Diamond \varUpsilon^{\not\Diamond}\big) \otimes (\mathfrak{h}^{-1}) = \Big(\Diamond + \mu \otimes\Big)\big(\varUpsilon^{\not\Diamond} \otimes \mathfrak{h}^{-1}\big)$$

  The contractions listed in (5.5) indeed vanish, because $\not\Diamond \in \mathcal{P}^2_{\mathrm{vac}}$. In the first set of pairings, the contraction $(3,5)$ suffices. In the second, $(2,5)$ suffices. In the third, $(3,6)$ and $(4,7)$ together suffice, by (b) and (c.2) for $\not\Diamond \in \mathcal{P}^2_{\mathrm{vac}}$.

This concludes the proof. $\square$

Proposition 5.1. *For all* $\blacklozenge \in \mathcal{P}^1 \times \mathcal{P}^2_{vac}$ *and all* $\blacklozenge' \in \mathcal{P}^2 \times \mathcal{P}^3_{vac}$,

$$\mathcal{D}_{\blacklozenge}\blacklozenge \in \mathcal{P}^2 \times \mathcal{P}^3_{vac} \tag{5.6a}$$

$$\mathcal{D}_{\blacklozenge}\mathcal{D}_{\blacklozenge}\blacklozenge = 0 \tag{5.6b}$$

$$\mathcal{D}_{\blacklozenge}\blacklozenge' \in \mathcal{P}^3 \times \mathcal{P}^4_{vac} \tag{5.6c}$$

Proof. Equations (5.6a) and (5.6c) follow from Lemma 5.1, equation (5.6b) follows from Proposition 2.1. $\square$

Remark 5.2. Equations (5.6a) and (5.6b) are, respectively, algebraic and differential identities for the left hand side of the equation $\mathcal{D}_{\blacklozenge}\blacklozenge = 0$.

6. Components and Multiindices

In this section, the previous constructions are made concrete by introducing local coordinates and components. For this purpose, fix

- an index set $\mathcal{I}$ with $|\mathcal{I}| = 4$
- a constant symmetric matrix $(g_{ab})_{a,b\in\mathcal{I}}$ with signature $(-,+,+,+)$
- an open set $U \subset M$
- a coordinate diffeomorphism $\rho : U \to \mathcal{U} \subset \mathbb{R}^4,\ p \mapsto (\rho^\mu(p))_{\mu=1,2,3,4}$
- a representative $\mathfrak{h}$ of $\mathfrak{H}$ over U
- sections $F_{(a)}$ of V over U, $a \in \mathcal{I}$, such that $\mathfrak{h}(F_{(a)}, F_{(b)}) = g_{ab}$

Convention 6.1. We denote by $(g^{ab})_{a,b\in\mathcal{I}}$ the inverse of $(g_{ab})_{a,b\in\mathcal{I}}$.

Convention 6.2. $(\lambda^{(a)})_{a\in\mathcal{I}}$ are the sections of V^* over U dual to $(F_{(a)})_{a\in\mathcal{I}}$.

Convention 6.3. Standard Cartesian coordinates on $\mathcal{U} \subset \mathbb{R}^4$ are denoted $(x^\mu)_{\mu=1,2,3,4}$.

Convention 6.4. Small Latin indices take values in the index set $\mathcal{I}$. Capital Latin indices are multiindices, that is, elements of $\mathcal{I}^k$ for some $k \geq 0$. For example, $A = (a_1 \dots a_k)$ where $a_1, \dots, a_k \in \mathcal{I}$. The length of a multiindex will be denoted by $|A| = k$. Moreover, $\mathbf{A}_A{}^{BC}$ is introduced just as in Definition 3.3, with the understanding that ordinary Latin indices refer to the index set $\mathcal{J} = \mathcal{I}$.

Convention 6.5. For any multiindex $A = (a_1 \dots a_k)$, write $F_{(A)} = F_{(a_1)} \otimes \cdots \otimes F_{(a_k)}$.

Definition 6.1. *S^k is the real vector space of all $(\sigma, \tau) = (\sigma_A{}^\mu, \tau_{Am}{}^n)$, where A is an $\mathcal{I}$-multiindex of length $|A| = k$ and $m, n \in \mathcal{I}$ and $\mu = 1, 2, 3, 4$, such that*

(a) σ, τ are totally antisymmetric in their first k lower indices,
(b) $\tau_{Am}{}^\ell g_{\ell n} + \tau_{An}{}^\ell g_{\ell m} = \frac{1}{2}\tau_{A\ell}{}^\ell g_{mn}$ where $|A| = k$

The "vertical subspace" $S^k_\perp$ is the set of all $(\sigma, \tau) \in S^k$ such that

(c) $\sigma = 0$

The "vacuum subspace" S^k_{vac}, $2 \leq k \leq 4$, is the set of all $(\sigma, \tau) \in S^k_\perp$ such that

(d) $\mathbf{A}_A{}^B \tau_B{}^n = 0$ where $|A| = |B| = k + 1$

(e) $\tau_{Am}{}^{\ell} g_{\ell n} + \tau_{An}{}^{\ell} g_{\ell m} = 0$ *where* $|A| = k$
(f.2) for $k = 2$*:* $\tau_{anm}{}^{n} = 0$
(f.3) for $k = 3$*:* $g^{bm} \tau_{abnm}{}^{n} = 0$

Remark 6.1. Property (e) in Definition 6.1 implies $\tau_{An}{}^{n} = 0$.

Remark 6.2. We have $\dim_{\mathbb{R}} S^k = 11\binom{4}{k}$ and $\dim_{\mathbb{R}} S^k_{\perp} = 7\binom{4}{k}$ and

$$\dim_{\mathbb{R}} S^2_{\text{vac}} = 10 \qquad \dim_{\mathbb{R}} S^3_{\text{vac}} = 16 \qquad \dim_{\mathbb{R}} S^4_{\text{vac}} = 6$$

Let $\mathcal{P}^k(U)$ be defined just as in Definition 3.1, with U instead of M. Similarly for $\mathcal{P}^k_{\perp}(U)$ and $\mathcal{P}^k_{\text{vac}}(U)$.

Proposition 6.1. Part 1*: Let* $\Diamond \in \mathcal{P}^k(U)$*. Set*

$$(\sigma^{\Diamond})_A{}^{\mu} \circ \rho = \Diamond_{F_{(A)}} \rho^{\mu} \tag{6.1a}$$

$$\big((\tau^{\Diamond})_{Am}{}^{n} \circ \rho\big)\, F_{(n)} = \Diamond_{F_{(A)}} F_{(m)} \tag{6.1b}$$

Then $(\sigma^{\Diamond}, \tau^{\Diamond}) \in C^{\infty}(\mathcal{U}, S^k)$*.*
Part 2*: For all* $(\sigma^{\Diamond}, \tau^{\Diamond}) \in C^{\infty}(\mathcal{U}, S^k)$ *there is a unique* $\Diamond \in \mathcal{P}^k(U)$ *so that* (6.1) *hold.*
Part 3*:* $(\sigma^{\Diamond}, \tau^{\Diamond}) \in C^{\infty}(\mathcal{U}, S^k_{\perp})$ *if and only if* $\Diamond \in \mathcal{P}^k_{\perp}(U)$*.*
Part 4*:* $(\sigma^{\Diamond}, \tau^{\Diamond}) \in C^{\infty}(\mathcal{U}, S^k_{\mathit{vac}})$ *if and only if* $\Diamond \in \mathcal{P}^k_{\mathit{vac}}(U)$*.*

Remark 6.3. For all $\Diamond \in \mathcal{P}^k(U)$, equations (6.1) imply that for all $f \in C^{\infty}(U)$:

$$\Diamond_{F_{(A)}} f = \Big((\sigma^{\Diamond})_A{}^{\mu} \tfrac{\partial}{\partial x^{\mu}}\big(f \circ \rho^{-1}\big)\Big) \circ \rho \tag{6.2a}$$

$$\Diamond_{F_{(A)}} \lambda^{(m)} = -\big((\tau^{\Diamond})_{An}{}^{m} \circ \rho\big)\, \lambda^{(n)} \tag{6.2b}$$

Proof (Proposition 6.1). Recall that $\mathfrak{h} = g_{ab} \lambda^{(a)} \otimes \lambda^{(b)}$ is a representative for $\mathfrak{H}$ over U. Part 1: Use $\Diamond \mathfrak{h} = \mu \otimes \mathfrak{h}$, where $\mu \in \Gamma(\wedge^k V^*|_U)$, substitute $\mathfrak{h} = g_{ab} \lambda^{(a)} \otimes \lambda^{(b)}$ and use the Leibniz rule to show that $((\tau^{\Diamond})_{Am}{}^{\ell} \circ \rho) g_{\ell n} + ((\tau^{\Diamond})_{An}{}^{\ell} \circ \rho) g_{m\ell} = -\mu(F_{(A)})\, g_{mn}$. Multiply with g^{mn}, sum and obtain $-\frac{1}{2}((\tau^{\Diamond})_{A\ell}{}^{\ell} \circ \rho) = \mu(F_{(A)})$. This implies (b) in Definition 6.1. Part 2: Equations (6.1b), (6.2a) and (6.2b) together with (a), (c) in Definition 3.1 determine $\Diamond$ uniquely. Properties (b), (d), (e) in Definition 3.1 are then automatic. This proves existence. $\Diamond$ is unique, because for every $\Diamond \in \mathcal{P}^k(U)$, the equations (6.1) imply (6.2a), (6.2b). Part 3: $\Diamond \in \mathcal{P}^k_{\perp}(U)$ iff $\Diamond f = 0$ for all $f \in C^{\infty}(U)$ iff $\sigma^{\Diamond} = 0$, by equation (6.2a). Part 4 follows from Definition 5.1. $\square$

Proposition 6.2. *Let* $\Diamond \in \mathcal{P}^k(U)$*,* $\rlap{/}\Diamond \in \mathcal{P}^{\ell}(U)$*. The superbracket* $[\![\Diamond, \rlap{/}\Diamond]\!] \in \mathcal{P}^{k+\ell}(U)$ *has the components*

$$\begin{aligned}(\sigma^{[\![\Diamond,\rlap{/}\Diamond]\!]})_A{}^{\mu} &= \mathbf{A}_A{}^{BC}\Big((\sigma^{\Diamond})_B{}^{\nu} \tfrac{\partial}{\partial x^{\nu}}(\sigma^{\rlap{/}\Diamond})_C{}^{\mu} - (\sigma^{\rlap{/}\Diamond})_C{}^{\nu} \tfrac{\partial}{\partial x^{\nu}}(\sigma^{\Diamond})_B{}^{\mu}\Big) \\ &\quad - \mathbf{A}_A{}^{BcE}(\tau^{\Diamond})_{Bc}{}^{\ell}(\sigma^{\rlap{/}\Diamond})_{\ell E}{}^{\mu} + \mathbf{A}_A{}^{bDC}(\tau^{\rlap{/}\Diamond})_{Cb}{}^{\ell}(\sigma^{\Diamond})_{\ell D}{}^{\mu}\end{aligned}$$

and

$$\begin{aligned}(\tau^{[\![\Diamond,\rlap{/}\Diamond]\!]})_{Am}{}^{n} &= \mathbf{A}_A{}^{BC}\Big((\sigma^{\Diamond})_B{}^{\mu} \tfrac{\partial}{\partial x^{\mu}}(\tau^{\rlap{/}\Diamond})_{Cm}{}^{n} - (\sigma^{\rlap{/}\Diamond})_C{}^{\mu} \tfrac{\partial}{\partial x^{\mu}}(\tau^{\Diamond})_{Bm}{}^{n}\Big) \\ &\quad + \mathbf{A}_A{}^{BC}\Big((\tau^{\rlap{/}\Diamond})_{Cm}{}^{\ell}(\tau^{\Diamond})_{B\ell}{}^{n} - (\tau^{\Diamond})_{Bm}{}^{\ell}(\tau^{\rlap{/}\Diamond})_{C\ell}{}^{n}\Big) \\ &\quad - \mathbf{A}_A{}^{BcE}(\tau^{\Diamond})_{Bc}{}^{\ell}(\tau^{\rlap{/}\Diamond})_{\ell Em}{}^{n} + \mathbf{A}_A{}^{bDC}(\tau^{\rlap{/}\Diamond})_{Cb}{}^{\ell}(\tau^{\Diamond})_{\ell Dm}{}^{n}\end{aligned}$$

The multiindices have length

$$|A| = k + \ell \qquad |B| = k \qquad |C| = \ell \qquad |D| = k - 1 \qquad |E| = \ell - 1$$

Proof. By direct calculation, using (3.7b) and Proposition 6.1. Equation (3.7b) with $Y_i = F_{(a_i)}$ and $A = (a_1 \dots a_{k+\ell})$ implies

$$\begin{aligned}
[\![\Diamond, \not\Diamond]\!]_{F_{(A)}} u &= \mathbf{A}_A{}^{BC}\Big(\Diamond_{F_{(B)}}\big(\not\Diamond_{F_{(C)}} u\big) - \not\Diamond_{F_{(C)}}\big(\Diamond_{F_{(B)}} u\big)\Big) \\
&\quad - \mathbf{A}_A{}^{BcE}\not\Diamond_{(\Diamond_{F_{(B)}} F_{(c)}), F_{(E)}} u + \mathbf{A}_A{}^{bDC}\Diamond_{(\not\Diamond_{F_{(C)}} F_{(b)}), F_{(D)}} u \\
&= \mathbf{A}_A{}^{BC}\Big(\Diamond_{F_{(B)}}\big(\not\Diamond_{F_{(C)}} u\big) - \not\Diamond_{F_{(C)}}\big(\Diamond_{F_{(B)}} u\big)\Big) \\
&\quad - \mathbf{A}_A{}^{BcE}\big((\tau^{\Diamond})_{Bc}{}^{\ell} \circ \rho\big)\not\Diamond_{F_{(\ell)}, F_{(E)}} u \\
&\quad + \mathbf{A}_A{}^{bDC}\big((\tau^{\not\Diamond})_{Cb}{}^{\ell} \circ \rho\big)\Diamond_{F_{(\ell)}, F_{(D)}} u
\end{aligned}$$

To calculate $\sigma^{[\![\Diamond, \not\Diamond]\!]}$, set $u = \rho^{\mu}$ and use (6.1) and (6.2) repeatedly. To calculate $\tau^{[\![\Diamond, \not\Diamond]\!]}$, set $u = F_{(m)}$. $\square$

Propositions 6.1 and 6.2 enable us to write down all the equations of Section 5 explicitly. See Section 8.

7. Covariance

For this section, fix

- $M, V, \mathfrak{H}$ just as at the beginning of Section 3
- another such triple $\widetilde{M}, \widetilde{V}, \widetilde{\mathfrak{H}}$
- open subsets $U \subset M$ and $\widetilde{U} \subset \widetilde{M}$
- a diffeomorphism $\psi : \widetilde{U} \to U$
- a vector bundle isomorphism $\phi : \widetilde{W} = \widetilde{V}|_{\widetilde{U}} \to W = V|_U$ so that $\pi_W \circ \phi = \psi \circ \pi_{\widetilde{W}}$

We require that

- for each representative $\widetilde{\mathfrak{h}}$ of $\widetilde{\mathfrak{H}}$ over $\widetilde{U}$, $\phi(\widetilde{\mathfrak{h}}) \in \Gamma(\mathrm{Sym}^2 W^*)$ is a representative for $\mathfrak{H}$ over U.

Convention 7.1. As always, there is a canonical extension of ϕ to a vector bundle isomorphism $\mathcal{T}(\widetilde{W}) \to \mathcal{T}(W)$, which we also denote as ϕ. For every section $u \in \mathcal{T}(\widetilde{W})$ we denote by $\phi(u) = \phi \circ u \circ \psi^{-1}$ the corresponding section of $\mathcal{T}(W)$.

Let $\mathcal{P}^k(U)$ and $\mathcal{P}^k(\widetilde{U})$ be defined just as in Definition 3.1.

Proposition 7.1. *For all $\Diamond \in \mathcal{P}^k(U)$ and all $\widetilde{Y} \in \Gamma(\widetilde{W}^{\otimes k})$ and $\widetilde{u} \in \mathcal{T}(\widetilde{W})$, set*

$$\widetilde{\Diamond}_{\widetilde{Y}}\widetilde{u} = \phi^{-1}\big(\Diamond_Y u\big) \tag{7.1}$$

where $Y = \phi(\widetilde{Y}) \in \Gamma(W^{\otimes k})$ and $u = \phi(\widetilde{u}) \in \Gamma(\mathcal{T}(W))$. Then $\widetilde{\Diamond} \in \mathcal{P}^k(\widetilde{U})$. The map $\mathcal{P}^k(U) \to \mathcal{P}^k(\widetilde{U})$, $\Diamond \mapsto \widetilde{\Diamond} = \phi^{-1}(\Diamond)$

- *is a bijection that maps $\mathcal{P}^k_{\perp}(U) \to \mathcal{P}^k_{\perp}(\widetilde{U})$ and $\mathcal{P}^k_{vac}(U) \to \mathcal{P}^k_{vac}(\widetilde{U})$,*

- $[\![\phi^{-1}(\Diamond_1), \phi^{-1}(\Diamond_2)]\!] = \phi^{-1}\big([\![\Diamond_1, \Diamond_2]\!]\big)$ *for all* $\Diamond_1, \Diamond_2 \in \mathcal{P}^k(\widetilde{U})$.

Proof. By construction. $\square$

We will now spell out the transformation law $\Diamond \mapsto \widetilde{\Diamond}$ (see Proposition 7.1) in components. For this purpose, we fix additional objects, as at the beginning of Section 6:

- $\mathcal{I}$ and (g_{ab})
- $\rho : U \to \mathcal{U} \subset \mathbb{R}^4$ and $\mathfrak{h}$ and $(F_{(a)})$
- $\widetilde{\rho} : \widetilde{U} \to \widetilde{\mathcal{U}} \subset \mathbb{R}^4$ and $\widetilde{\mathfrak{h}}$ and $(\widetilde{F}_{(a)})$

Define

- $\chi : \widetilde{\mathcal{U}} \to \mathcal{U}$ by the following commuting diagram:

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\phi} & W \\ {\scriptstyle \pi_{\widetilde{W}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_W} \\ \widetilde{U} & \xrightarrow{\psi} & U \\ {\scriptstyle \widetilde{\rho}}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \widetilde{\mathcal{U}} & \xrightarrow{\chi} & \mathcal{U} \end{array} \tag{7.2}$$

- $\varOmega : \widetilde{\mathcal{U}} \to (0, \infty)$ by

$$\widetilde{\mathfrak{h}} = \phi^{-1}(\mathfrak{h})\ (\varOmega \circ \widetilde{\rho})^{-2} \tag{7.3}$$

- a matrix valued map $(\varLambda^a{}_b)_{a,b \in \mathcal{I}}$ on $\widetilde{\mathcal{U}}$ by

$$\widetilde{F}_{(a)} = \phi^{-1}\big(F_{(b)}\big)\ \big(\varLambda^b{}_a \circ \widetilde{\rho}\big) \tag{7.4}$$

- the components $J_\nu{}^\mu \in C^\infty(\widetilde{\mathcal{U}})$ of the inverse of the Jacobian of χ by

$$J_\nu{}^\mu = \Big(\tfrac{\partial}{\partial x^\nu}(\chi^{-1})^\mu\Big) \circ \chi \qquad \text{or, equivalently,} \qquad \big(\tfrac{\partial}{\partial \widetilde{x}^\nu} \chi^\alpha\big) J_\alpha{}^\mu = \delta_\nu{}^\mu \tag{7.5}$$

Convention 7.2. Standard Cartesian coordinates on $\mathcal{U} \subset \mathbb{R}^4$ and $\widetilde{\mathcal{U}} \subset \mathbb{R}^4$ are denoted $(x^\mu)_{\mu=1,2,3,4}$ and $(\widetilde{x}^\mu)_{\mu=1,2,3,4}$ respectively.

Remark 7.1. Equations (7.3), (7.4) and $\mathfrak{h}(F_{(a)}, F_{(b)}) = g_{ab}$, $\widetilde{\mathfrak{h}}(\widetilde{F}_{(a)}, \widetilde{F}_{(b)}) = g_{ab}$ imply

$$g_{ab} = g_{k\ell}\ \big(\tfrac{1}{\varOmega}\varLambda^k{}_a\big)\ \big(\tfrac{1}{\varOmega}\varLambda^\ell{}_b\big) \tag{7.6}$$

on $\widetilde{\mathcal{U}}$. In other words, $(\frac{1}{\varOmega}\varLambda^a{}_b)$ is a Lorentz transformation matrix.

Proposition 7.2. *Let* $\Diamond \in \mathcal{P}^k(U)$ *and* $\widetilde{\Diamond} = \phi^{-1}(\Diamond) \in \mathcal{P}^k(\widetilde{U})$. *Let* (σ, τ) *and* $(\widetilde{\sigma}, \widetilde{\tau})$ *be the components of* $\Diamond$ *and* $\widetilde{\Diamond}$*, respectively, as in Proposition 6.1. (These components are functions on* $\mathcal{U}$ *and* $\widetilde{\mathcal{U}}$*.) We have on* $\widetilde{\mathcal{U}}$

$$\widetilde{\sigma}_A{}^\mu = \big(\sigma_B{}^\nu \circ \chi\big)\ \varLambda^B{}_A\ J_\nu{}^\mu \tag{7.7a}$$

$$\widetilde{\tau}_{Am}{}^n = \tfrac{1}{\varOmega^2}(\tau_{Bk}{}^\ell \circ \chi)\varLambda^B{}_A \varLambda^k{}_m \varLambda_\ell{}^n + \tfrac{1}{\varOmega^2}(\sigma_B{}^\nu \circ \chi)\varLambda^B{}_A J_\nu{}^\mu\big(\tfrac{\partial}{\partial \widetilde{x}^\mu}\varLambda^\ell{}_m\big)\varLambda_\ell{}^n \tag{7.7b}$$

Here $A = (a_1 \dots a_k)$, $B = (b_1 \dots b_k)$, $\varLambda^B{}_A = \varLambda^{b_1}{}_{a_1} \cdots \varLambda^{b_k}{}_{a_k}$ *and* $\varLambda_\ell{}^n = g_{\ell a}\varLambda^a{}_b g^{bn}$.

Remark 7.2. Set $\varphi = \chi^{-1}$ and $K_\nu{}^\mu = \frac{\partial}{\partial x^\nu}\varphi^\mu$ and $\Theta = \Omega \circ \varphi$ and $\Delta^a{}_b = \Lambda^a{}_b \circ \varphi$. Then (7.7) is equivalent to

$$\widetilde{\sigma}_A{}^\mu \circ \varphi = \sigma_B{}^\nu \, \Delta^B{}_A \, K_\nu{}^\mu \tag{7.8a}$$
$$\widetilde{\tau}_{Am}{}^n \circ \varphi = \tfrac{1}{\Theta^2} \, \tau_{Bk}{}^\ell \Delta^B{}_A \Delta^k{}_m \Delta_\ell{}^n + \tfrac{1}{\Theta^2} \, \sigma_B{}^\nu \Delta^B{}_A \big(\tfrac{\partial}{\partial x^\nu} \Delta^\ell{}_m\big) \Delta_\ell{}^n \tag{7.8b}$$

Proof (Proposition 7.2). Calculate

$$\begin{aligned}
\widetilde{\sigma}_A{}^\mu \circ \widetilde{\rho} &= \widetilde{\Diamond}_{\widetilde{F}_{(A)}} \widetilde{\rho}^\mu \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \, \widetilde{\Diamond}_{\phi^{-1}(F_{(B)})} \Big((\chi^{-1})^\mu \circ \rho \circ \psi \Big) \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \, \Big(\Diamond_{F_{(B)}} \big((\chi^{-1})^\mu \circ \rho \big) \Big) \circ \psi \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \, \Big(\sigma_B{}^\nu \tfrac{\partial}{\partial x^\nu} (\chi^{-1})^\mu \Big) \circ \rho \circ \psi
\end{aligned}$$

Compose with $\widetilde{\rho}^{-1}$ from the right, and obtain equation (7.7a). To show (7.7b), use

$$\big(\widetilde{\tau}_{Am}{}^n \circ \widetilde{\rho}\big) \widetilde{F}_{(n)} = \widetilde{\Diamond}_{\widetilde{F}_{(A)}} \widetilde{F}_{(m)}$$

(see equation (6.1b)) and calculate

$$\begin{aligned}
&\big(\widetilde{\tau}_{Am}{}^n \circ \widetilde{\rho}\big) \, \phi^{-1}(F_{(\ell)}) \, \big(\Lambda^\ell{}_n \circ \widetilde{\rho}\big) \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \, \widetilde{\Diamond}_{\phi^{-1}(F_{(B)})} \Big(\phi^{-1}(F_{(\ell)}) \, \big(\Lambda^\ell{}_m \circ \widetilde{\rho}\big) \Big) \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \, \Big\{ \big(\Lambda^k{}_m \circ \widetilde{\rho}\big) \, \widetilde{\Diamond}_{\phi^{-1}(F_{(B)})} \phi^{-1}(F_{(k)}) \, + \phi^{-1}(F_{(\ell)}) \, \widetilde{\Diamond}_{\phi^{-1}(F_{(B)})} \big(\Lambda^\ell{}_m \circ \widetilde{\rho}\big) \Big\} \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \Big\{ \big(\Lambda^k{}_m \circ \widetilde{\rho}\big) \, \phi^{-1}\big(\Diamond_{F_{(B)}} F_{(k)}\big) \\
&\qquad\qquad + \Big(\Diamond_{F_{(B)}} \big(\Lambda^\ell{}_m \circ \chi^{-1} \circ \rho\big) \Big) \circ \psi \Big\} \phi^{-1}(F_{(\ell)}) \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \Big\{ \big(\Lambda^k{}_m \circ \widetilde{\rho}\big) \big(\tau_{Bk}{}^\ell \circ \rho \circ \psi\big) \\
&\qquad\qquad + \Big(\sigma_B{}^\nu \tfrac{\partial}{\partial x^\nu} \big(\Lambda^\ell{}_m \circ \chi^{-1}\big) \Big) \circ \rho \circ \psi \Big\} \, \phi^{-1}(F_{(\ell)}) \\
&= \big(\Lambda^B{}_A \circ \widetilde{\rho}\big) \Big\{ \big(\Lambda^k{}_m \circ \widetilde{\rho}\big) \big(\tau_{Bk}{}^\ell \circ \rho \circ \psi\big) \\
&\qquad\qquad + \Big(\big(\tfrac{\partial}{\partial \widetilde{x}^\mu} \Lambda^\ell{}_m\big) \circ \widetilde{\rho} \Big) \, \Big(\sigma_B{}^\nu \tfrac{\partial}{\partial x^\nu} \big(\chi^{-1}\big)^\mu \Big) \circ \rho \circ \psi \Big\} \, \phi^{-1}(F_{(\ell)})
\end{aligned}$$

From both sides, factor out $\phi^{-1}(F_{(\ell)})$, compose with $\widetilde{\rho}^{-1}$ from the right, and obtain (7.7b). $\square$

8. Instruction manual

The purpose of this section is to state, in a self-contained and ready-to-use manner, definitions and propositions that express the reformulated Einstein vacuum equations (5.2), in explicit coordinate/index notation on an open subset of $\mathbb{R}^4$.

For this section, fix

- a simply connected open subset $\mathcal{U} \subset \mathbb{R}^4$
- an index set $\mathcal{I}$ with $|\mathcal{I}| = 4$
- a constant symmetric matrix $(g_{ab})_{a,b\in\mathcal{I}}$ with signature $(-,+,+,+)$

The **statements** of all definitions and propositions in this section are completely self-contained and make no reference to previous sections. The proofs, on the other hand, rely on the previous sections. We consider Definition 3.3, Conventions 6.1, 6.3, 6.4 and Definition 6.1 as being part of this section.

In the next proposition, A, B, C, D are $\mathcal{I}$-multiindices with length

$$|A| = |B| = 2 \qquad |C| = 3 \qquad |D| = 4$$

Proposition 8.1. Part 1: *For all* $\blacklozenge = ((E,\Gamma),(0,W)) \in C^\infty(\mathcal{U}, S^1 \times S^2_{\mathit{vac}})$ *set*

$$T_A{}^\mu = -\mathbf{A}_A{}^{bc}\Big(E_b{}^\nu \tfrac{\partial}{\partial x^\nu} E_c{}^\mu - \Gamma_{bc}{}^\ell E_\ell{}^\mu\Big) \tag{8.1a}$$

$$U_{Am}{}^n = W_{Am}{}^n - \mathbf{A}_A{}^{bc}\Big(E_b{}^\mu \tfrac{\partial}{\partial x^\mu}\Gamma_{cm}{}^n + \Gamma_{cm}{}^\ell\Gamma_{b\ell}{}^n - \Gamma_{bc}{}^\ell\Gamma_{\ell m}{}^n\Big) \tag{8.1b}$$

$$V_{Cm}{}^n = \mathbf{A}_C{}^{bA}\Big(E_b{}^\mu \tfrac{\partial}{\partial x^\mu} W_{Am}{}^n + \Gamma_{b\ell}{}^n W_{Am}{}^\ell - \Gamma_{bm}{}^\ell W_{A\ell}{}^n - 2\Gamma_A{}^\ell W_{\ell bm}{}^n\Big) \tag{8.1c}$$

Then $\blacklozenge' = ((T,U),(0,V))$ *is in* $C^\infty(\mathcal{U}, S^2 \times S^3_{\mathit{vac}})$*. In other words, there is a map*

$$\begin{aligned} C^\infty(\mathcal{U}, S^1 \times S^2_{\mathit{vac}}) &\to C^\infty(\mathcal{U}, S^2 \times S^3_{\mathit{vac}}) \\ \blacklozenge &\mapsto \blacklozenge' \end{aligned} \tag{8.2}$$

which we again write as $\mathcal{D}_\blacklozenge \blacklozenge = \blacklozenge'$*.*
Part 2: *For all*

$$\begin{aligned} \blacklozenge &= ((E,\Gamma),(0,W)) \in C^\infty(\mathcal{U}, S^1 \times S^2_{\mathit{vac}}) \\ \blacklozenge' &= ((T,U),(0,V)) \in C^\infty(\mathcal{U}, S^2 \times S^3_{\mathit{vac}}) \end{aligned}$$

not necessarily $\blacklozenge' = \mathcal{D}_\blacklozenge\blacklozenge$*, set*

$$\mathfrak{T}_C{}^\mu = \mathbf{A}_C{}^{bA}\Big(-E_b{}^\nu \tfrac{\partial}{\partial x^\nu} T_A{}^\mu + T_A{}^\nu \tfrac{\partial}{\partial x^\nu} E_b{}^\mu + 2\Gamma_A{}^\ell T_{\ell b}{}^\mu - U_{Ab}{}^\ell E_\ell{}^\mu\Big) \tag{8.3a}$$

$$\begin{aligned} \mathfrak{U}_{Cm}{}^n = V_{Cm}{}^n - \mathbf{A}_C{}^{bA}\Big(&E_b{}^\mu \tfrac{\partial}{\partial x^\mu} U_{Am}{}^n - T_A{}^\mu \tfrac{\partial}{\partial x^\mu}\Gamma_{bm}{}^n + U_{Am}{}^\ell \Gamma_{b\ell}{}^n \\ &- \Gamma_{bm}{}^\ell U_{A\ell}{}^n - 2\Gamma_A{}^\ell U_{\ell bm}{}^n + U_{Ab}{}^\ell \Gamma_{\ell m}{}^n\Big) \end{aligned} \tag{8.3b}$$

$$\begin{aligned} \mathfrak{V}_{Dm}{}^n = \;&\mathbf{A}_D{}^{bC}\Big(E_b{}^\mu \tfrac{\partial}{\partial x^\mu} V_{Cm}{}^n + V_{Cm}{}^\ell \Gamma_{b\ell}{}^n - \Gamma_{bm}{}^\ell V_{C\ell}{}^n + 3\, U_C{}^\ell W_{\ell bm}{}^n\Big) \\ &+ \mathbf{A}_D{}^{AB}\Big(T_A{}^\mu \tfrac{\partial}{\partial x^\mu} W_{Bm}{}^n + W_{Bm}{}^\ell U_{A\ell}{}^n - U_{Am}{}^\ell W_{B\ell}{}^n - 2\,\Gamma_A{}^\ell V_{\ell Bm}{}^n\Big) \end{aligned} \tag{8.3c}$$

Then $\blacklozenge'' = ((\mathfrak{T},\mathfrak{U}),(0,\mathfrak{V}))$ *is in* $C^\infty(\mathcal{U}, S^3 \times S^4_{\mathit{vac}})$*. In other words, there is a map*

$$\begin{aligned} C^\infty(\mathcal{U}, S^1 \times S^2_{\mathit{vac}}) \times C^\infty(\mathcal{U}, S^2 \times S^3_{\mathit{vac}}) &\to C^\infty(\mathcal{U}, S^3 \times S^4_{\mathit{vac}}) \\ (\blacklozenge, \blacklozenge') &\mapsto \blacklozenge'' \end{aligned} \tag{8.4}$$

which we again write as $\mathcal{D}_\blacklozenge\blacklozenge' = \blacklozenge''$*.*
Part 3: *For all* $\blacklozenge \in C^\infty(\mathcal{U}, S^1 \times S^2_{\mathit{vac}})$*,*

$$\mathcal{D}_\blacklozenge \mathcal{D}_\blacklozenge \blacklozenge = 0 \tag{8.5}$$

Proof. Warning, in this proof we consciously abuse notation, the symbols U and V are both given two meanings.
Let K be the 4-dimensional real vector space spanned by elements $(k_{(a)})_{a\in\mathcal{I}}$. We use the previous sections, with the understanding that M, V, $\mathfrak{H}$ at the beginning of Section 3 and U, ρ, $\mathfrak{h}$, $F_{(a)}$ at the beginning of Section 6 are:

- $M = \mathcal{U} \subset \mathbb{R}^4$ with trivial bundle $V = \mathcal{U} \times K$
- $\rho : U = \mathcal{U} \to \mathcal{U}$ the identity transformation
- $F_{(a)} : \mathcal{U} \ni x \mapsto (x, k_{(a)}) \in \mathcal{U} \times K$ constant sections
- $\mathfrak{H}$ is defined by declaring $\mathfrak{h}$ to be a representative, where $\mathfrak{h}(F_{(a)}, F_{(b)}) = g_{ab}$.

Part 1: We identify $\blacklozenge = ((E,\Gamma),(0,W)) \in C^\infty(\mathcal{U}, S^1 \times S^2_{\text{vac}})$ with the corresponding $\blacklozenge \in \mathcal{P}^1 \times \mathcal{P}^2_{\text{vac}}$, in the sense of Proposition 6.1. Let $\blacklozenge' = \mathcal{D}_\blacklozenge \blacklozenge \in \mathcal{P}^2 \times \mathcal{P}^3$ be given by equation (5.1a). By Proposition 5.1, $\blacklozenge' \in \mathcal{P}^2 \times \mathcal{P}^3_{\text{vac}}$. Identify $\blacklozenge'$ with the corresponding $\blacklozenge' = ((T,U),(0,V)) \in C^\infty(\mathcal{U}, S^2 \times S^3_{\text{vac}})$, in the sense of Proposition 6.1. It follows from equation (5.1a) and Proposition 6.2 that T, U, V are given by equations (8.1). For the last term in (8.1c), recall that $2\mathbf{A}_C{}^{bA}\Gamma_A{}^\ell = \mathbf{A}_C{}^{bmn}\Gamma_{mn}{}^\ell$, see Definition 3.3.
Part 2: Analogous to Part 1, using equation (5.1b).
Part 3: This is now a corollary of Proposition 5.1. $\square$

The Einstein vacuum equations are reformulated as:

$$\text{Find } \blacklozenge \in C^\infty(\mathcal{U}, S^1 \times S^2_{\text{vac}}) \text{ such that } \mathcal{D}_\blacklozenge \blacklozenge = 0. \tag{8.6}$$

Remark 8.1. By the proof of Proposition 8.1, the coordinate construction of $\mathcal{D}$ and the abstract construction of $\mathcal{D}$ coincide. Therefore, (5.2) and (8.6) are equivalent.

Proposition 8.2. *Suppose $\blacklozenge = ((E,\Gamma),(0,W)) \in C^\infty(\mathcal{U}, S^1 \times S^2_{vac})$ satisfies*

$$\mathcal{D}_\blacklozenge \blacklozenge = 0$$

and $(E_a{}^\mu)$ is invertible as a matrix at each point of $\mathcal{U}$, so that the four vector fields $E_a = E_a{}^\mu \frac{\partial}{\partial x^\mu}$, $a \in \mathcal{I}$, are a frame for each fiber of $T\mathcal{U}$.
Part 1: *$\nu \in \Gamma(T^*\mathcal{U})$ given by $\nu(E_a) = -\frac{1}{2}\Gamma_{an}{}^n$ is exact, $\nu = -\mathrm{d}f$ with $f \in C^\infty(\mathcal{U})$.*
Part 2: *The Lorentzian metric g on $\mathcal{U}$ given by $g(E_a, E_b) = e^f g_{ab}$ has Levi-Civita connection $\nabla_{E_a} E_m = \Gamma_{am}{}^n E_n$.*
Part 3: *The associated Riemann curvature is given by $R(E_a, E_b)E_m = W_{abm}{}^n E_n$. In particular, the Ricci-curvature vanishes.*

Proof. We adopt the conventions in the proof of Proposition 8.1, up to and including the four bullets. We identify $\blacklozenge = ((E,\Gamma),(0,W))$ with the corresponding $\blacklozenge = (\lozenge, \not\lozenge) \in \mathcal{P}^1 \times \mathcal{P}^2_{\text{vac}}$, in the sense of Proposition 6.1. Recall Proposition 4.1, Definition 4.1 and (6.2a). Observe that

- $\lozenge \in \mathcal{P}^1$ is non-degenerate, because $(E_a{}^\mu)$ is invertible, and $\mathcal{E}^\lozenge(F_{(a)}) = E_a$. In fact, for every $q \in C^\infty(\mathcal{U})$ we have $E_a(q) = E_a{}^\mu \frac{\partial}{\partial x^\mu} q = \lozenge_{F_{(a)}} q$.
- (a) in Proposition 4.5 holds, because $M = \mathcal{U}$ is simply connected, $\lozenge \in \mathcal{P}^1$ is non-degenerate, (c) in Proposition 4.5 holds by $\mathcal{D}_\blacklozenge \blacklozenge = 0$, and (c) implies (a).

Let $\mathfrak{h}'$ in Proposition 4.3 be given by $\mathfrak{h}'(F_{(a)}, F_{(b)}) = g_{ab}$, and let $\Diamond\mathfrak{h}' = \mu \otimes \mathfrak{h}'$. Then the 1-form $\nu \in \Gamma(T^*\mathcal{U})$ in Proposition 4.3 is given by $\nu(E_a) = \mu(F_{(a)}) = -\frac{1}{2}\Gamma_{an}{}^{n}$. For the last equality, use $\Diamond_{F_{(a)}}\mathfrak{h}' = \mu(F_{(a)})\mathfrak{h}'$ and equations (6.1) and (6.2).
We apply Parts 1, 2, 3 of Proposition 4.3. $\nu = -\mathrm{d}f$ with $f \in C^\infty(\mathcal{U})$. $\nabla^\Diamond$ is the Levi-Civita connection of $g = \mathcal{E}^\Diamond(e^f\mathfrak{h}') = e^f\mathcal{E}^\Diamond(\mathfrak{h}')$, and $g(E_a, E_b) = e^f g_{ab}$. The connection $\nabla^\Diamond_{E_a}E_m = \mathcal{E}^\Diamond(\Diamond_{F_{(a)}}F_{(m)}) = \mathcal{E}^\Diamond(\Gamma_{am}{}^{n}F_{(n)}) = \Gamma_{am}{}^{n}E_n$. The Riemann curvature is $R(E_a, E_b)E_m = \frac{1}{2}\mathcal{E}^\Diamond(\llbracket\Diamond,\Diamond\rrbracket_{F_{(a)}\otimes F_{(b)}}F_{(m)}) = \mathcal{E}^\Diamond(\not\Diamond_{F_{(a)}\otimes F_{(b)}}F_{(m)}) = \mathcal{E}^\Diamond(W_{abm}{}^{n}F_{(n)}) = W_{abm}{}^{n}E_n$. The Ricci-curvature vanishes by $W_{anm}{}^{n} = 0$. $\square$

Proposition 8.3. *Suppose $(E_a)_{a\in\mathcal{I}}$ is a frame for each fiber of $T\mathcal{U}$ and the Lorentzian metric g given by $g(E_a, E_b) = g_{ab}$ is Ricci-flat. Then g arises from a solution to*

$$\mathcal{D}_{\blacklozenge}\blacklozenge = 0$$

as in Proposition 8.2.

Proof. We adopt the conventions in the proof of Proposition 8.1, up to and including the four bullets. Define a vector bundle isomorphism $\not{\mathcal{E}} : V \to T\mathcal{U}$ by $\not{\mathcal{E}}(F_{(a)}) = E_a$. Let $\not{\mathfrak{h}}$ be given by $\not{\mathfrak{h}}(F_{(a)}, F_{(b)}) = g_{ab}$. It is a representative of $\mathfrak{H}$. Then $\not{\mathcal{E}}(\not{\mathfrak{h}})(E_a, E_b) = \not{\mathfrak{h}}(F_{(a)}, F_{(b)}) = g_{ab} = g(E_a, E_b)$, that is, $g = \not{\mathcal{E}}(\not{\mathfrak{h}})$. Let $\Diamond \in \mathcal{P}^1$ be as in Proposition 4.4. Then $\Diamond$ satisfies the assumptions of Proposition 4.3, $\mathcal{E}^\Diamond = \not{\mathcal{E}}$, $\Diamond\not{\mathfrak{h}} = 0$, and the Lorentzian metric associated with $\Diamond$ (see Remark 4.1) is $\not{\mathcal{E}}(\not{\mathfrak{h}}) = g$, which by assumption is Ricci-flat. By (a) $\Longrightarrow$ (c) in Proposition 4.5 (recall that $\mathcal{U}$ is simply connected) there is a $\not\Diamond \in \mathcal{P}^2_{\text{vac}}$ so that $\blacklozenge = (\Diamond, \not\Diamond)$ satisfies $\mathcal{D}_{\blacklozenge}\blacklozenge = 0$. Identify $\blacklozenge \in \mathcal{P}^1 \times \mathcal{P}^2_{\text{vac}}$ with the corresponding $\blacklozenge = ((\sigma^\Diamond, \tau^\Diamond), (0, \tau^{\not\Diamond})) \in C^\infty(\mathcal{U}, S^1 \times S^2_{\text{vac}})$, in the sense of Proposition 6.1. Then $(\sigma^\Diamond)_a{}^{\mu}\frac{\partial}{\partial x^\mu} = \mathcal{E}^\Diamond(F_{(a)}) = \not{\mathcal{E}}(F_{(a)}) = E_a$, that is, $(\sigma^\Diamond)_a{}^{\mu} = E_a{}^{\mu}$. Moreover, $\nu = 0$ in Proposition 8.2 and we can choose $f = 0$. Then the g's in Proposition 8.2 and 8.3 coincide. $\square$

Proposition 8.4. *Let $\widetilde{\mathcal{U}} \subset \mathbb{R}^4$ be open. We use Convention 7.2. Let (χ, Λ) be a pair,*

- *$\chi : \widetilde{\mathcal{U}} \to \mathcal{U}$ a diffeomorphism*
- *$(\Lambda^a{}_b)_{a,b\in\mathcal{I}} = \Omega\,(L^a{}_b)_{a,b\in\mathcal{I}}$ where $\Omega : \widetilde{\mathcal{U}} \to (0,\infty)$ and $(L^a{}_b)_{a,b\in\mathcal{I}}$ is a matrix valued map on $\widetilde{\mathcal{U}}$ such that $g_{ab} = g_{k\ell}L^k{}_a L^\ell{}_b$.*

and let

- *$J_\nu{}^{\mu}$ be given by* (7.5)

To each $\Diamond = (\sigma, \tau) \in C^\infty(\mathcal{U}, S^k)$ we associate $\widetilde{\Diamond} = (\widetilde{\sigma}, \widetilde{\tau}) \in C^\infty(\widetilde{\mathcal{U}}, S^k)$ by equations (7.7)*, or, equivalently,* (7.8)*. To each $\blacklozenge = (\Diamond, \not\Diamond) \in C^\infty(\mathcal{U}, S^k \times S^{k+1})$ we associate $\widetilde{\blacklozenge} = (\widetilde{\Diamond}, \widetilde{\not\Diamond}) \in C^\infty(\widetilde{\mathcal{U}}, S^k \times S^{k+1})$. Then:*
Part 1: *For all $\Diamond \in C^\infty(\mathcal{U}, S^k)$, $\blacklozenge \in C^\infty(\mathcal{U}, S^1 \times S^2_{vac})$ and $\blacklozenge' \in C^\infty(\mathcal{U}, S^2 \times S^3_{vac})$:*

(a) $\Diamond \in C^\infty(\mathcal{U}, S^k_\perp)$ if and only if $\widetilde{\Diamond} \in C^\infty(\widetilde{\mathcal{U}}, S^k_\perp)$
(b) $\Diamond \in C^\infty(\mathcal{U}, S^k_{vac})$ if and only if $\widetilde{\Diamond} \in C^\infty(\widetilde{\mathcal{U}}, S^k_{vac})$
(c) $\widetilde{\blacklozenge} \in C^\infty(\widetilde{\mathcal{U}}, S^1 \times S^2_{vac})$ and $\mathcal{D}_{\widetilde{\blacklozenge}}\widetilde{\blacklozenge} = \widetilde{\mathcal{D}_{\blacklozenge}\blacklozenge}$
(d) $\widetilde{\blacklozenge}' \in C^\infty(\widetilde{\mathcal{U}}, S^2 \times S^3_{vac})$ and $\mathcal{D}_{\widetilde{\blacklozenge}}\widetilde{\blacklozenge}' = \widetilde{\mathcal{D}_{\blacklozenge}\blacklozenge'}$

Especially, $\widetilde{\blacklozenge}$ *is a solution to* (8.6) *on* $\widetilde{\mathcal{U}}$ *if and only if* $\blacklozenge$ *is a solution to* (8.6) *on* $\mathcal{U}$.

Part 2: *The composition of* (χ, Λ) *and* $(\widetilde{\chi}, \widetilde{\Lambda})$, *where* $\widetilde{\chi} : \widetilde{\widetilde{\mathcal{U}}} \to \widetilde{\mathcal{U}}$ *and* $\widetilde{\Lambda}$ *is defined on* $\widetilde{\mathcal{U}}$, *is given by* $(\chi \circ \widetilde{\chi}, (\Lambda \circ \widetilde{\chi})\widetilde{\Lambda})$. *The inverse to* (χ, Λ) *is* $(\chi^{-1}, \Lambda^{-1} \circ \chi^{-1})$.

Proof. We adopt the conventions in the proof of Proposition 8.1, up to and including the four bullets. We make the same conventions for all quantities with tildes. We use Section 7, with the understanding that the diffeomorphism $\psi : \widetilde{U} = \widetilde{\mathcal{U}} \to U = \mathcal{U}$ is given by $\psi = \chi$, and the vector bundle isomorphism $\phi : \widetilde{\mathcal{U}} \times \widetilde{K} \to \mathcal{U} \times K$ maps $(\widetilde{x}, \widetilde{k}_{(a)})$ to $(\chi(\widetilde{x}), k_{(b)} \Lambda^b{}_a(\widetilde{x}))$. With these definitions, χ, Ω, $\Lambda^a{}_b$, $J_\nu{}^\mu$ as defined in Section 7 coincide with χ, Ω, $\Lambda^a{}_b$, $J_\nu{}^\mu$ in Proposition 8.4. In other words, the diagram (7.2) commutes, and equations (7.3), (7.4), (7.5) hold.

- We verify equation (7.4): For every $\widetilde{x} \in \widetilde{\mathcal{U}}$,

$$\big(\phi \circ \widetilde{F}_{(a)}\big)_{\widetilde{x}} = \phi(\widetilde{x}, \widetilde{k}_{(a)}) = (\chi(\widetilde{x}), k_{(b)} \Lambda^b{}_a(\widetilde{x})) = (F_{(b)} \circ \chi)_{\widetilde{x}}\, \Lambda^b{}_a(\widetilde{x})$$

  That is, $\phi \circ \widetilde{F}_{(a)} = (F_{(b)} \circ \chi)\, \Lambda^b{}_a$. Compose with ϕ^{-1} from the left to obtain (7.4).
- We verify equation (7.3):

$$\begin{aligned}\phi^{-1}\big(\mathfrak{h}\big)(\widetilde{F}_{(a)}, \widetilde{F}_{(b)}\big) = \mathfrak{h}\big(\phi(\widetilde{F}_{(a)}), \phi(\widetilde{F}_{(b)})\big) \circ \chi = \big(\mathfrak{h}(F_{(k)}, F_{(\ell)}) \circ \chi\big)\ \Lambda^k{}_a\, \Lambda^\ell{}_b \\ = g_{k\ell} \Lambda^k{}_a \Lambda^\ell{}_b = \Omega^2 g_{ab} = \Omega^2 \widetilde{\mathfrak{h}}\big(\widetilde{F}_{(a)}, \widetilde{F}_{(b)}\big)\end{aligned}$$

We identify abstract diamonds and their components, in the sense of Proposition 6.1. With this understanding, the maps

- $C^\infty(\mathcal{U}, S^k) \to C^\infty(\widetilde{\mathcal{U}}, S^k)$, $\lozenge \mapsto \widetilde{\lozenge}$ in Proposition 8.4
- $\mathcal{P}^k(U) \to \mathcal{P}^k(\widetilde{U})$, $\lozenge \mapsto \phi^{-1}(\lozenge)$ in Proposition 7.1

coincide, by Proposition 7.2. Part 1: Now (a), (b) follow from Proposition 6.1 and Proposition 7.1. The first statements in (c) and (d) follow from (a) and (b). The second statements in (c) and (d) follow from Remark 8.1, equations (5.1) and the fact that the map $\lozenge \mapsto \widetilde{\lozenge}$ commutes with the Lie superbracket, see Proposition 7.1. Part 2: Let ψ, ϕ and $\widetilde{\psi}$, $\widetilde{\phi}$ be the diffeomorphism and vector bundle isomorphism corresponding to the pairs (χ, Λ) and $(\widetilde{\chi}, \widetilde{\Lambda})$. Then the pair $(\chi \circ \widetilde{\chi}, (\Lambda \circ \widetilde{\chi})\widetilde{\Lambda})$ corresponds to $\psi \circ \widetilde{\psi}$, $\phi \circ \widetilde{\phi}$. Now, Part 2 follows from Proposition 7.1. □

We conclude this section with a few remarks:

Remark 8.2. The (coordinate) first order differential operators $\mathcal{D}$ in Part 1 and Part 2 of Proposition 8.1 are classically defined when $\blacklozenge$ and $\blacklozenge'$ are of class C^1. Especially, the left hand side of the Einstein vacuum equation $\mathcal{D}_{\blacklozenge}\blacklozenge = 0$ is well defined for any $\blacklozenge$ of class C^1. By continuity, the differential identity (8.5) holds for every $\blacklozenge$ of class C^2.

Remark 8.3. It is essential to observe that there is a canonical subformalism of the formalism of this paper, which informally speaking is obtained by putting all the $\mu \in \Gamma(V^*)$ and $\nu \in \Gamma(T^*M)$ to zero. More precisely, at the beginning of Section 3, we choose a section $\mathfrak{h}_0 = \Gamma(\mathrm{Sym}^2 V^*)$ with signature $(-, +, +, +)$ instead of a conformal section $\mathfrak{H}$. From this point on, every representative of $\mathfrak{H}$ is replaced by $\mathfrak{h}_0$. Then, in (e) of Definition 3.1 we also require that $\mu = 0$. Definition 3.4 for $\mathfrak{g}(V, \mathfrak{H})$ is replaced by a Definition of $\mathfrak{g}(V, \mathfrak{h}_0)$ by putting $\lambda = 0$ in (3.9), giving a vector bundle with fibers of dimension 6, and the definition of $\mathcal{R}^k$ is changed accordingly. In this subformalism, condition (c) in Proposition 4.3 is vacuous. Definition 6.1.(b) is replaced by $\tau_{Am}{}^{\ell} g_{\ell n} + \tau_{An}{}^{\ell} g_{\ell m} = 0$. (The new condition differs from the old condition by $\tau_{A\ell}{}^{\ell} = 0$.) Remark 6.2 is replaced by $\dim_{\mathbb{R}} S^k = 10\binom{4}{k}$ and $\dim_{\mathbb{R}} S^k_{\perp} = 6\binom{4}{k}$, while $\dim_{\mathbb{R}} S^k_{\text{vac}}$ is unchanged. Now $\Omega \equiv 1$ in (7.3). We emphasize the consequences that the subformalism has for Proposition 8.1. In Part 1, we have the new condition $\Gamma_{am}{}^{m} = 0$, and the new conclusion $U_{Am}{}^{m} = 0$. In Part 2, we have the new conditions $\Gamma_{am}{}^{m} = 0$ and $U_{Am}{}^{m} = 0$, and the new conclusion $\mathfrak{U}_{Cm}{}^{m} = 0$. Finally, Propositions 8.2 and 8.3 as well as all the other propositions hold for the subformalism, with the understanding that in Proposition 8.4, $\Omega \equiv 1$.

Remark 8.4. The discussion of Appendix B to [RT] is a precursor to the formalism of this paper, more precisely to the subformalism elaborated on in Remark 8.3. To compare the two developments, one must be aware that:

- The ordering of the indices may differ.
- Combinatorial factors may differ.
- In contrast to [RT], indices are neither raised nor lowered in this paper.

References

[Fr] Friedrich, H., Proc. R. Soc. Lond. A **375**, (1981) 169-184
[NP] Newman, E. and Penrose, R., J. Math. Phys. **3**, (1962) 566-578
[Sh] Sharpe, R.W. , *Differential Geometry*, (Springer, 1997)
[RT] Reiterer, M. and Trubowitz, E., arxiv.org/abs/0906.3812

Strongly Focused Gravitational Waves

Michael Reiterer, Eugene Trubowitz

Department of Mathematics, ETH Zurich, Switzerland

Abstract: Christodoulou [Chr] proved that trapped spheres can form in evolution from a generic initial state, through the focusing of gravitational waves. His work is the motivation for the present paper, in which we consider the same physical problem, using very different mathematical methods. Our approach is based on a controlled "far field expansion". By a systematic use of scaling symmetries, we regularize Christodoulou's singular "short pulse method", rigorously track vacuum solutions by the far field expansion and exhibit trapped spheres that first appear deep inside the far field region. Our presentation is self-contained. In the final section, we present a detailled outline of the construction of another, more subtle, expansion that allows us to continue the solutions beyond the far field region to within any fixed "finite distance" from the (expected) singularity. From a methodological perspective, the underlying aim of this paper is the development of a general method for constructing solutions to the vacuum Einstein equations by controlled expansions.

1. Introduction

Formal and controlled (perturbation) expansions are common tools in mathematics and physics. In general relativity, see, for example, [AnRe], [BBM], [Cha]. This paper is a first step in the development of a hybrid method, combining formal expansions and simple tools from the theory of hyperbolic partial differential equations (such as energy estimates), to construct generic classical solutions to the vacuum Einstein equations, for a wide variety of well posed problems with natural small parameters. The purpose of this paper is to illustrate the methodology, by carrying it out in all detail for a concrete, physically interesting situation.

Christodoulou [Chr] showed that strongly focused gravitational waves, coming in from past null infinity, generate trapped spheres. One of the most important innovations of [Chr] is the introduction of a small parameter δ and a picture in which δ represents the duration of a spherical pulse traveling along a null hypersurface. The amplitude of the pulse is scaled so that, roughly speaking, the total incoming energy per unit advanced time is proportional to $\frac{1}{\delta}$. Christodoulou refers to this picture as his "short pulse method". This physical setup triggered our interest in this problem.

To illustrate our approach, we shall construct strongly focused gravitational wave solutions to the vacuum Einstein equations by means of controlled expansions that exhibit trapped spheres.

In Christodoulou's picture, the limit $\delta \downarrow 0$ is singular, and "the initial data are no longer confined to a suitable neighborhood of trivial data". In this paper we adopt from the outset a different (regularized) picture, in which there is a small parameter $\mathfrak{A} \neq 0$, a regular limit $\mathfrak{A} \to 0$, and the initial data is contained in a small ball around zero. These apparently contradictory pictures can be reconciled, in fact they are equivalent, see Section 9, in particular Remark 9.3. The relationship is given by setting $\delta = \mathfrak{A}^4$ and using the one-parameter group of exact (anisotropic) scaling transformations, indexed by $\mathfrak{A} \neq 0$, that is introduced in Section 3.

The construction of spacetimes in [Chr] and [ChrKl] adopts a strictly geometric formalism and emphasizes the classical geometric point of view, in which the unknown field is a Lorentzian metric on a manifold, subject to the vacuum Einstein field equations **Ricci** $= 0$. In this paper we use the formalism of Friedrich [Fr] and Newman-Penrose [NP], in which the unknown field is a triple (E, Γ, W) of (a priori unrelated) fields, where

- $E = ({E_a}^\mu)$ are the components of a frame $E_a = {E_a}^\mu \frac{\partial}{\partial x^\mu}$ that is declared, by fiat, to be an orthonormal Lorentz frame,
- $\Gamma = (\Gamma_{abc})$ are the components of a connection ∇ with respect to the frame E,
- $W = (W_{abcd})$ are the components of a Weyl field with respect to the frame E.

In this formalism, one associates to (E, Γ, W) three additional fields (T, U, V) that are quadratic expressions in E, Γ, W and their first derivatives, see Appendix B. Derivatives of E only appear in T, those of Γ only in U, those of W only in V. Here:

- T is the torsion of ∇,
- U is the difference between the curvature of ∇ and the Weyl field W,
- $V = (V_{abijk})$ where $V_{abijk} = \nabla_i W_{abjk} + \nabla_j W_{abki} + \nabla_k W_{abij}$.

The vacuum Einstein field equations become $(T, U, V) = 0$. In fact, to every solution (E, Γ, W) of these equations one can canonically associate a solution of **Ricci** $= 0$, and conversely every solution to **Ricci** $= 0$ arises locally in this manner. We emphasize that the equations $(T, U, V) = 0$

- are a quasilinear, first order system of partial differential equations, that
- are quadratically nonlinear, and
- through gauge-fixing, can be brought into symmetric hyperbolic form, see [Fr].

In sharp contrast to the geometric approach tailored to **Ricci** $= 0$ of [Chr] and [ChrKl], we find it advantageous to ignore the geometric content of $(T, U, V) = 0$ altogether. In this paper, geometry appears only in the formulation of the problem and the interpretation of the final results.

We have tried to present our construction in a transparent form with as many details as possible, so that it is accessible to the general reader without any specific background in general relativity or hyperbolic partial differential equations. For example, we include the derivation of the single (L^2) Sobolev inequality that is used. In fact, the discussion, up to the formation of trapped spheres, is entirely self contained, apart from the reference, in the proof of Proposition 7.3, to [Tay] for a simple local existence theorem for quasilinear symmetric hyperbolic systems defined on the product of a time interval with a torus. For these reasons, this paper is longer than it might be. We have, however, omitted lengthy, but straight forward, direct verifications (typical of general relativity) of many equations and algebraic identities. We have included an index of notation (Appendix A).

This paper naturally divides into two parts, one algebraic and the other analytic. The algebraic part culminates in the three Propositions 5.2, 5.3, 5.4 that we refer to as the relevant/irrelevant form of the equations. This form exhibits the essential constituents that have to be treated carefully (relevant terms), and sweeps everything else into "generic terms" about which only general structural properties need to be known (irrelevant terms). The relevant part dictates the analysis that follows. Everything is organized around it. For example, the energy estimates, Proposition 7.4 and 7.7, are designed to accommodate the most delicate terms in the relevant part of the equations. One payoff of the lengthy algebraic preliminaries is that the analysis can be carried out, for formal solutions in Section 6 and for classical solutions in Sections 7 and 8, with elementary tools.

The rest of this introduction is an overview of the contents of this paper. The algebraic part comprises Appendices B, C, D and Sections 2, 3, 4, 5. If the reader is not concerned about the derivation of the equations, he / she can read the self-contained Sections 2, 3, 4, 5. Alternatively, the natural order is B, C, D, 2, 3, 4, 5. We now discuss the algebraic appendices and sections of this paper:

- Appendix B is a self-contained review of the general formalism of Friedrich [Fr] and Newman-Penrose [NP] of the vacuum Einstein equations.
- Appendix C introduces a gauge adapted to the focusing problem, in the language of Lorentzian geometry. This gauge requires two coordinates u and $\underline{u}$ to be solutions to the eikonal equation (also called null or characteristic coordinates). The other two coordinates are denoted ξ^1, ξ^2 that may be interpreted as "angular coordinates". Also, there are four frame vector fields: two future-directed null vector fields that are tangent to the level sets of u and $\underline{u}$, respectively, and two spacelike vector fields that span the tangent space to the intersections of the level sets of u and $\underline{u}$. It is shown that this gauge is locally realizable. That is, every point on every Lorentzian manifold has a neighborhood on which such coordinates and frame can be introduced. More colloquially, no spacetimes are left out.
- Appendix D reinterprets and abstracts the gauge of Appendix C as a set of (pointwise) affine linear algebraic constraints on the unknowns (E, Γ, W). We introduce a field $\Phi = (e, \gamma, w)$, with 31 real components, that takes values in the affine gauge subspace at each point. This Φ is the basic unknown field that appears throughout the paper. The equations $(T, U, V) = 0$ split into two parts. The first part is a quasilinear symmetric hyperbolic system for Φ, referred to as **(SHS)** . The second part (constraint equations) is written $\Phi^\sharp = 0$, where $\Phi^\sharp$ is the associated "constraint field", with 32 real components. It is an important fact that for every solution Φ of **(SHS)** , the associated $\Phi^\sharp$ is a solution to a linear homogeneous symmetric hyperbolic system referred to as $\widehat{\textbf{(SHS)}}$. Therefore, the basic strategy carried out in Sections 6 and 8 is to first construct a solution Φ to **(SHS)** for which $\Phi^\sharp$ vanishes initially, and then use $\widehat{\textbf{(SHS)}}$ to show that $\Phi^\sharp$ vanishes everywhere.
  It follows from the equivalence of $(T, U, V) = 0$ and **Ricci** $= 0$ and from the local realizability of the gauge, that to every solution Φ of **(SHS)** and $\Phi^\sharp = 0$ one can canonically associate a solution to **Ricci** $= 0$, and conversely every solution to **Ricci** $= 0$ arises locally in this way.
- In Section 2, which can be read independently of Appendices B, C, D, we write out **(SHS)** , the constraint field $\Phi^\sharp$ and $\widehat{\textbf{(SHS)}}$ explicitly. Observe the simple structure of the principal part of **(SHS)** , equation (2.5). The reader may be put off by the multi-page

equations of Section 2. However, he or she should not be discouraged, because later, in Propositions 5.2, 5.3, 5.4, we will derive the more transparent relevant / irrelevant forms of these equations, mentioned above.

- Section 3 introduces a number of exact symmetry transformations of the equations of Section 2. In particular, the global anisotropic scaling $\mathfrak{A}$ (Definition 3.5) plays a central role for everything we do.
- In Section 4 we define a two-parameter family of fields $\mathcal{M}_{a,\mathfrak{A}}$ on the open subset $(\xi, \underline{u}, u) \in \textbf{Strip}_\infty = \mathbb{R}^2 \times (0,\infty) \times (-\infty, 0) \subset \mathbb{R}^4$ that are solutions, in the role of $\varPhi$, to **(SHS)** and the constraint equations. $\mathcal{M}_{a,\mathfrak{A}}$ is obtained from $\mathcal{M}_{1,1}$ by applying the scaling symmetries of Section 3. For all parameter values $a, \mathfrak{A} \neq 0$, the field $\mathcal{M}_{a,\mathfrak{A}}$ corresponds to Minkowski space. The family $\mathcal{M}_{a,\mathfrak{A}}$ will be the starting point for our expansion (see below).
- In Section 5 we make a change of variables and write $\varPhi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\varPsi$, where M is a diagonal matrix with strictly positive integral entries and $\varPsi$ is the new unknown field. We make a similar change of variables $\varPhi^\sharp = u^{-M^\sharp}\varPsi^\sharp$ for the constraint field. Then we rewrite **(SHS)** and $\widehat{\textbf{(SHS)}}$ in terms of $\varPsi$ and $\varPsi^\sharp$. This section completes the algebraic part of the paper with the relevant / irrelevant form of the equations for $\varPsi$ and $\varPsi^\sharp$. Notice that only terms that are either principal in the number of derivatives or leading order in powers of $\frac{1}{u}$ are written out explicitly. The rest is summarized in symbolic form, that keeps track of only certain overall structural properties. Therefore, equations (5.7), (5.8), (5.9) deserve the name relevant / irrelevant form. It may be surprising that the there are only two nonlinear terms in the relevant part on the right hand side of equation (5.7). The one appearing in the last line of (5.7c) is actually irrelevant, see Remark 5.3. The one in the sixth line of (5.7a) generates trapped spheres.

We now discuss the analytic part of the paper.

- In Section 6 the unique formal solution $[\varPhi] = [\mathcal{M}_{a,\mathfrak{A}}] + u^{-M}[\varPsi]$ on $\textbf{Strip}_\infty \subset \mathbb{R}^4$ to a formal characteristic initial value problem for **(SHS)** and $[\varPhi^\sharp] = 0$ is constructed. The asymptotic characteristic initial value problem is motivated by [Chr]. Data is prescribed on
  - the characteristic hypersurface $\underline{u} = 0$,
  - the asymptotic characteristic hypersurface $u \to -\infty$ (past null infinity).

  The data along $\underline{u} = 0$ is $[\varPsi] = 0$ or, equivalently, $[\varPhi] = [\mathcal{M}_{a,\mathfrak{A}}]$. The data along $u \to -\infty$ is generic and consists of two real valued functions depending on $(\xi, \underline{u})$. In this paper, we will consistently use the notation

$$[f] \ = \ \textstyle\sum_{k=0}^\infty \left(\frac{1}{u}\right)^k f(k)(\xi, \underline{u})$$

  for a formal power series in $\frac{1}{u}$ with coefficient functions $f(k)$, defined for $(\xi, \underline{u}, u)$ in the open $\textbf{Strip}_\infty \subset \mathbb{R}^4$ introduced above. The expansion parameter $\frac{1}{u}$ is, morally, the distance to past null infinity. For this reason, we refer to expansions of this kind as far field expansions. Formally $\lim_{u\to-\infty} u^M[\varPhi - \mathcal{M}_{a,\mathfrak{A}}] = \varPsi(0)$ is the asymptotic initial data at past null infinity, constructed out of the two free functions via the constraint equations. All the coefficient functions $\varPsi(k)$, $k = 0, 1, 2, \ldots$ can be written down explicitly. Trapped spheres already appear in the lowest order term. The formal solution and the relevant / irrelevant form of the equations are used in the subsequent sections to construct a unique classical solution such that the formal expansion is an asymptotic expansion to this classical solution. At the conclusion of Section 6, all the notation, definitions and concepts required for the statement of the main theorem of this paper (Theorem 8.1) have been introduced. It can now be read on its own.

- In Section 7 we collect all the analytical tools that are required for the proof of Theorem 8.1. More precisely, standard results about symmetric hyperbolic systems are adapted to the very specific applications we have in mind. In particular, finite speed of propagation, a local existence theorem with a breakdown criterion, and energy estimates. The long list of hypotheses for the energy estimates (see Subsection 7.4) are dictated by the relevant / irrelevant form of the equations. We then prove a refined (localized) version of the energy estimate (Proposition 7.7) that exploits finite speed of propagation. This is done by using the sets $\mathcal{O}(\xi_0, b, t)$ that are introduced in Subsection 7.5 to capture the causal structure of the solutions. It is this refined energy estimate that is applied in Section 8. Observe that Proposition 7.7 is actually an energy estimate for a *linear*, inhomogeneous symmetric hyperbolic system, which in Section 8 is used in a self-consistent manner to obtain an estimate for the *nonlinear* problem.
- In Section 8 we write $\Psi = \Psi_K + \text{(Error)}$. Here Ψ_K is the truncation of the formal power series solution at order $K+1$. Proposition 8.3 states in particular that the quasilinear symmetric hyperbolic system for (Error) satisfies the hypotheses for the refined energy estimate. More generally, Proposition 8.3 provides a list of sufficient conditions under which the abstract propositions of Section 7 can be applied to the various symmetric hyperbolic systems that are required for the proof of Theorem 8.1. Its setup, formulation and proof are lengthy, even somewhat tedious, but elementary. The section concludes with the formulation and proof of Theorem 8.1, which states the existence and uniqueness of a classical solution to the characteristic initial value problem for **(SHS)** and $\Psi^\sharp = 0$, under appropriate smallness conditions. The solution is constructed on

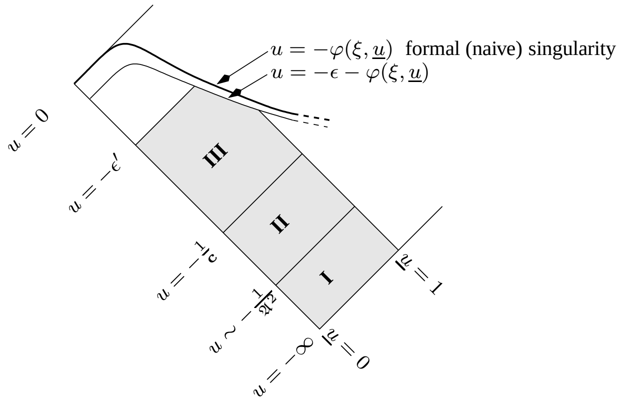
$$\textbf{Strip}(1, \mathbf{c}) = \mathbb{R}^2 \times (0,1) \times (-\infty, -\mathbf{c}^{-1}) \ \subset \ \textbf{Strip}_\infty$$

where $\mathbf{c} > 0$ is a constant. Roughly speaking, Theorem 8.1 says that

$$\psi \;=\; \textstyle\sum_{k=0}^{K} \big(\frac{1}{u}\big)^k \,\psi(k) \,+\, \mathcal{O}\big(\frac{1}{|u|^{K+1}}\big) \qquad (u \to -\infty)$$

We show that $\mathbf{c}$ can be chosen independent of $\mathfrak{A} \neq 0$. Therefore, this theorem is compatible with the limit $\mathfrak{A} \to 0$.
- In Section 9 we extract a number of corollaries of Theorem 8.1.
  - Proposition 9.1 states that the far field formal power series is an asymptotic expansion for the classical solution of Theorem 8.1 in **I** $\cup$ **II**, see the figure below. In this sense, the solution is quasi-explicit. Using this result, we can systematically exploit Theorem 8.1 by transferring properties of the formal solution to corresponding properties of the classical solution.
  - The high point of [Chr] is the demonstration that strongly focused gravitational waves generate trapped spheres. In Proposition 9.2 we recover this result by showing that trapped spheres form at the end of region **I**, that is, at $u \sim \mathfrak{A}^{-2}$, using the fact that they appear in the lowest order term of the formal expansion.
  - Proposition 9.5 states that for special initial conditions, the solutions of Theorem 8.1 become arbitrarily close to the Schwarzschild solution on the upper edge of

region I ($\underline{u} = 1$). This would be used in conjunction with a controlled perturbation expansion around the Schwarzschild / Kerr family to construct the global exterior of a black hole. Our proof of this fact uses four terms of the expansion.



– In Subsection 9.5 we sketch a method for continuing the solutions out of the "far field region" I $\cup$ II using a more powerful expansion in $\mathfrak{A}$. The expansion in $\mathfrak{A}$ around the regular limit $\mathfrak{A} \to 0$ is both more fundamental and more subtle. It is an expansion around a two parameter family of decoupled, fully nonlinear two dimensional systems. Higher order terms in $\mathfrak{A}$ can be constructed and controlled. The expansion breaks down at $u = -\varphi(\xi, \underline{u})$, where φ can be explicitly expressed in terms of the initial data and does not depend on $\mathfrak{A}$. See the figure. Even though all the details are not carried out, the methods of this paper together with the expansion in $\mathfrak{A}$ suffice to extend Theorem 8.1 from I $\cup$ II to the larger domain I $\cup$ II $\cup$ III, where $\epsilon, \epsilon' > 0$ are arbitrary constants and $|\mathfrak{A}|$ is sufficiently small depending on ϵ, ϵ'.
– In [Chr] solutions are constructed on I. In this paper, we rigorously construct solutions on I $\cup$ II, and outline their extension to I $\cup$ II $\cup$ III. Furthermore, in Subsection 9.6 we obtain a class of solutions distinct from that of [Chr], corresponding to more general initial data at past null infinity. This class arises from the limit $\mathfrak{A} \to 0$ with $a > 0$ fixed.

We appreciate the great effort that Demetrios Christodoulou invested over many years to nurture the mathematical study of general relativity at ETH Zurich.

We thank Lydia Bieri, Joel Feldman, Horst Knörrer and Martin Lohmann for encouragement and helpful conversations.

2. A Reformulation of the Vacuum Einstein Equations

Our method for constructing solutions to the vacuum Einstein equations has two parts. The first is algebraic, the second analytic. Here, we present the purely algebraic part. It is a reformulation of the vacuum Einstein equations that is carefully tailored to the constructive analytic tools used in the second, purely analytic part.

This section compresses the intuition and logic of the three leisurely Appendices B, C and D into elementary, but very lengthy, totally unmotivated and, to the contemporary eye, unsightly, definitions and statements.

Let

$$\left(x^1\,,\, x^2\,,\, x^3\,,\, x^4 \right) \;=\; \left(\xi^1\,, \xi^2\,,\, \underline{u}\,,\, u \right)$$

be a coordinate system on the open subset $\mathcal{U}$ of $\mathbb{R}^4$, and

$$\Phi(x) \;=\; \big(\Phi_1(x)\,,\, \Phi_2(x)\,,\, \Phi_3(x) \big) \;=\; \big(e(x)\,,\, \gamma(x)\,,\, w(x) \big)$$

any sufficiently differentiable field on $\mathcal{U}$ taking values in

$$\mathcal{R} = \big\{ (e, \gamma, w) \in \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 \;\big|\; e_3,\, e_4,\, e_5,\, \gamma_2,\, \gamma_6 \in \mathbb{R} \,\big\}, \tag{2.1}$$

a real vector space of dimension 31. Throughout this paper, $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$.

Remark 2.1. Later on, the complex coordinate $\xi = \xi^1 + i\xi^2$ will play the role of an "angular" coordinate. See, for instance, Remark 4.1.

Definition 2.1. *To any sufficiently differentiable field $\Phi : \mathcal{U} \to \mathcal{R}$, satisfying the conditions*

$$\begin{array}{rcl} (\star) & : & e_3 > 0 \\ (\star\,\star) & : & \Im\big(e_1\,\overline{e}_2\big) \neq 0 \end{array}$$

at every point of $\mathcal{U}$, we associate three fields $\mathbf{F}_a{}^{\mu}$, $\boldsymbol{\Gamma}_{ajk}$, $\mathbf{W}_{abjk}$ and a complex frame $F_a = \mathbf{F}_a{}^{\mu} \frac{\partial}{\partial x^{\mu}}$ on $\mathcal{U}$. Here and below, small Latin and small Greek indices run from one to four. The fields are uniquely determined by:

- $\boldsymbol{\Gamma}_{ajk} = -\boldsymbol{\Gamma}_{akj}$ *and* $\mathbf{W}_{abjk} = -\mathbf{W}_{abkj}$ *and* $\mathbf{W}_{abjk} = -\mathbf{W}_{bajk}$.
-

$$\big(\mathbf{F}_a{}^{\mu} \big) \;=\; \begin{pmatrix} e_1 & e_2 & 0 & 0 \\ \overline{e}_1 & \overline{e}_2 & 0 & 0 \\ e_4 & e_5 & 0 & 1 \\ 0 & 0 & e_3 & 0 \end{pmatrix}$$

$$\big(\boldsymbol{\Gamma}_{a(jk)} \big) \;=\; \begin{pmatrix} \gamma_3 + \overline{\gamma}_4 & \overline{\gamma}_7 & \gamma_6 & \gamma_1 & \gamma_2 & \gamma_3 - \overline{\gamma}_4 \\ -\gamma_4 - \overline{\gamma}_3 & \gamma_6 & \gamma_7 & \gamma_2 & \overline{\gamma}_1 & -\gamma_4 + \overline{\gamma}_3 \\ \gamma_8 - \overline{\gamma}_8 & 0 & 0 & -\gamma_3 + \overline{\gamma}_4 & \gamma_4 - \overline{\gamma}_3 & \gamma_8 + \overline{\gamma}_8 \\ 0 & \overline{\gamma}_5 & \gamma_5 & 0 & 0 & 0 \end{pmatrix}$$

$$\big(\mathbf{W}_{(ab)(jk)} \big) \;=\; \begin{pmatrix} w_3 + \overline{w}_3 & \overline{w}_4 & -w_4 & w_2 & -\overline{w}_2 & w_3 - \overline{w}_3 \\ \overline{w}_4 & \overline{w}_5 & 0 & 0 & -\overline{w}_3 & -\overline{w}_4 \\ -w_4 & 0 & w_5 & -w_3 & 0 & -w_4 \\ w_2 & 0 & -w_3 & w_1 & 0 & w_2 \\ -\overline{w}_2 & -\overline{w}_3 & 0 & 0 & \overline{w}_1 & \overline{w}_2 \\ w_3 - \overline{w}_3 & -\overline{w}_4 & -w_4 & w_2 & \overline{w}_2 & w_3 + \overline{w}_3 \end{pmatrix}$$

The matrix indices (ab), (jk) run over the ordered sequence

$$(12) \quad (31) \quad (32) \quad (41) \quad (42) \quad (34)$$

The complex frame is written as:

- $\big(F_1, F_2, F_3, F_4\big) = \big(D, \overline{D}, N, L\big)$ *or, equivalently,*

$$D \;=\; e_1\,\tfrac{\partial}{\partial \xi^1} + e_2\,\tfrac{\partial}{\partial \xi^2} \quad , \quad N = e_4\,\tfrac{\partial}{\partial \xi^1} + e_5\,\tfrac{\partial}{\partial \xi^2} + \tfrac{\partial}{\partial u} \quad , \quad L = e_3\,\tfrac{\partial}{\partial \underline{u}} \tag{2.2}$$

  The vector fields N and L are always real.

Proposition 2.1. *The field $\mathbf{W}_{abjk}$ has the symmetries*

$$\begin{aligned} \mathbf{W}_{abjk} &= -\mathbf{W}_{bajk} & \qquad \mathbf{W}_{ajk\ell} + \mathbf{W}_{a\ell jk} + \mathbf{W}_{ak\ell j} &= 0 \\ \mathbf{W}_{abjk} &= \mathbf{W}_{jkab} & \mathbf{g}^{aj}\,\mathbf{W}_{abjk} &= 0 \end{aligned}$$

where the matrix $\mathbf{g}_{ab}$ and its inverse $\mathbf{g}^{ab}$ are given by

$$\big(\,\mathbf{g}_{ab}\,\big) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \big(\,\mathbf{g}^{ab}\,\big) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.3}$$

That is, $\mathbf{W}_{abjk}$ has the algebraic properties of a Weyl field.

Proof. By direct inspection. □

Remark 2.2. Later on it will be important to drop $(\star\star)$. That is, to allow the frame to collapse.

The next definition singles out an important class of fields $\varPhi(x)$ and relates them to Ricci-flat Lorentzian manifolds, that is, vacuum spacetimes.

Definition 2.2. *A field $\varPhi : \mathcal{U} \to \mathcal{R}$ is a* **vacuum field** *when:*

- *Conditions $(\star)$ and $(\star\star)$ are all satisfied at every point of $\mathcal{U}$.*
- *The Levi-Civita connection for the complex linear metric $\mathbf{g}$ on $\mathcal{U}$ is given by*

$$\mathbf{g}\big(\nabla_{F_a} F_j,\, F_k\big) = \boldsymbol{\varGamma}_{ajk}$$

  where $\mathbf{g}(F_a, F_b) \overset{def}{=} \mathbf{g}_{ab}$ (see, (2.3)*) and*

$$\mathbf{g}\big(\nabla_{F_a} F_j,\, F_k\big) \;\overset{def}{=}\; \tfrac{1}{2}\Big(-\mathbf{g}\big(F_a, [F_j, F_k]\big) + \mathbf{g}\big(F_k, [F_a, F_j]\big) + \mathbf{g}\big(F_j, [F_k, F_a]\big)\Big)$$

- *The Riemann tensor for the Levi-Civita connection is given by*

$$\mathbf{R}_{abjk} \;\overset{def}{=}\; \mathbf{g}\Big(\big[\nabla_{F_j}, \nabla_{F_k}\big] F_b - \nabla_{[F_j, F_k]} F_b,\, F_a\Big) \;=\; \mathbf{W}_{abjk}$$

  Consequently, the Ricci curvature vanishes, since $\mathbf{W}_{abjk}$ is traceless.
- *The coordinate functions $\underline{u}$ and u are both solutions to the eikonal equation. More precisely, $e_3 N$ and L are null geodesic vector fields that are minus the gradients of $\underline{u}$ and u.*

Remark 2.3. For any $\mathcal{R}$-valued field $\varPhi$ satisfying the conditions $(\star)$ and $(\star\star)$ the metric given by $\mathbf{g}(F_a, F_b) = \mathbf{g}_{ab}$ is real in the sense that $\mathbf{g}(X, Y)$ is real whenever X and Y are real vector fields. Over the reals, it has signature $(-,+,+,+)$. Bear in mind that $\mathbf{g}_{ab}$ are the components with respect to the *complex* frame F_a. If $\varPhi$ is a vacuum field, then $(\mathcal{U}, \mathbf{g})$ is a Ricci-flat Lorentzian manifold, that is, a solution to the vacuum Einstein equations.

Remark 2.4. It is natural to ask whether all Ricci-flat Lorentzian manifolds arise, at least locally, in this way. The answer to this question is given in Appendices C and D.

Remark 2.5. Assume $\Phi(x)$ is a vacuum field. Declare $L+N$ to be future directed. Let $S_{\underline{u},u}$ be the intersection of the level sets of $\underline{u}$ and u, which by Definition 2.2 are null hypersurfaces. The traces of the future-directed second fundamental forms of $S_{\underline{u},u}$ relative to the level sets of u and $\underline{u}$ are given by $\mathbf{g}(\nabla_D L, \overline{D}) + \mathbf{g}(\nabla_{\overline{D}} L, D) = 2\gamma_2$ and $\mathbf{g}(\nabla_D N, \overline{D}) + \mathbf{g}(\nabla_{\overline{D}} N, D) = 2\gamma_6$. By (2.1), they are real, as they should be. By definition, $S_{\underline{u},u}$ is a *trapped surface* when γ_2 and γ_6 are strictly negative everywhere on $S_{\underline{u},u}$. Equivalently, $S_{\underline{u},u}$ is trapped if an infinitesimal shift of $S_{\underline{u},u}$ along either L or N (both future-directed null vector fields, orthogonal to $S_{\underline{u},u}$) induces a pointwise decrease of the area element.
The basic examples of (closed) trapped surfaces in a vacuum spacetime are the spherical $\mathrm{SO}(3)$ orbits inside the horizon of a Schwarzschild spacetime. Closed trapped surfaces appear in the the formulation of Penrose's incompleteness theorem, see [Pen].

We need a criterion for a field to be a vacuum field. To this end, we make two more definitions.

Definition 2.3. *Suppose, condition $(\star)$ is satisfied at every point of the domain $\mathcal{U}$. Let $\Phi(x) = \big(e(x),\, \gamma(x),\, w(x)\big) : \mathcal{U} \to \mathcal{R}$ be a sufficiently differentiable field, and let the weights $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$ be strictly positive functions on $\mathcal{U}$. The* **Quasilinear Symmetric Hyperbolic System (SHS)** *for the field $\Phi(x)$ is*

$$\mathbf{A}(\Phi)\,\Phi \;=\; \mathbf{f}(\Phi) \tag{2.4}$$

Here, $\mathbf{A}(\Phi) = \mathbf{A}_1(\Phi) \oplus \mathbf{A}_2(\Phi) \oplus \mathbf{A}_3(\Phi)$ is the first order, matrix differential operator, with coefficients that are affine linear functions (over $\mathbb{R}$) of Φ, given by

$$\mathbf{A}_1(\Phi) \;=\; \mathrm{diag}\big(L,\, L,\, N,\, L,\, L\big) \tag{2.5a}$$

$$\mathbf{A}_2(\Phi) \;=\; \mathrm{diag}\big(L,\, L,\, L,\, L,\, N,\, N,\, N,\, L\big) \tag{2.5b}$$

$$\mathbf{A}_3(\Phi) \;=\; \begin{pmatrix} \lambda_1 N & \lambda_1 D & 0 & 0 & 0 \\ \lambda_1 \overline{D} & \lambda_1 L + \lambda_2 N & \lambda_2 D & 0 & 0 \\ 0 & \lambda_2 \overline{D} & \lambda_2 L + \lambda_3 N & \lambda_3 D & 0 \\ 0 & 0 & \lambda_3 \overline{D} & \lambda_3 L + \lambda_4 N & \lambda_4 D \\ 0 & 0 & 0 & \lambda_4 \overline{D} & \lambda_4 L \end{pmatrix} \tag{2.5c}$$

Observe that the "angular" operators D, $\overline{D}$ only appear in $\mathbf{A}_3(\Phi)$. Also,

$$\mathbf{f}(\Phi) \;=\; \mathbf{f}_1(\Phi) \oplus \mathbf{f}_2(\Phi) \oplus \mathbf{f}_3(\Phi) \;=\; \begin{pmatrix} \mathbf{f}_{11} \\ \vdots \\ \mathbf{f}_{51} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{f}_{12} \\ \vdots \\ \mathbf{f}_{82} \end{pmatrix} \oplus \begin{pmatrix} \mathbf{f}_{13} \\ \vdots \\ \mathbf{f}_{53} \end{pmatrix}$$

is the quadratically nonlinear vector valued function given by

$$\mathbf{f}_{j1} = \begin{cases} -\, e_1\gamma_2 - \overline{e}_1\gamma_1 & \\ -\, e_2\gamma_2 - \overline{e}_2\gamma_1 & \\ e_3\; 2\,\Re\,\gamma_8 & \star \\ 2\,\Re\,\big(-\, e_1\gamma_4 + e_1\gamma_5 + e_1\overline{\gamma}_3\big) & \star \\ 2\,\Re\,\big(-\, e_2\gamma_4 + e_2\gamma_5 + e_2\overline{\gamma}_3\big) & \star \end{cases}$$

$$\mathbf{f}_{j2} = \begin{cases} -\,2\gamma_1\gamma_2 - w_1 & \\ -\,|\gamma_1|^2 - \gamma_2\gamma_2 & \star \\ +\,\gamma_1\gamma_4 - \gamma_1\gamma_5 - \gamma_2\gamma_3 - w_2 & \\ -\,\gamma_2\gamma_4 + \gamma_2\gamma_5 + \gamma_3\overline{\gamma}_1 & \\ -\,\gamma_3\gamma_7 + \gamma_4\gamma_6 - \gamma_5\gamma_6 - \gamma_5\gamma_8 + \gamma_5\overline{\gamma}_8 - \gamma_6\overline{\gamma}_3 + \gamma_7\overline{\gamma}_4 - \gamma_7\overline{\gamma}_5 + w_4 & \\ -\,\gamma_6\gamma_6 - \gamma_6\ 2\,\Re\,\gamma_8 - |\gamma_7|^2 & \star \\ -\,2\gamma_6\gamma_7 - 3\gamma_7\gamma_8 + \gamma_7\overline{\gamma}_8 - w_5 & \\ -\,2\gamma_3\gamma_4 + 2\gamma_3\gamma_5 + \gamma_3\overline{\gamma}_3 + \gamma_4\overline{\gamma}_4 - \gamma_4\overline{\gamma}_5 - \gamma_5\overline{\gamma}_4 + w_3 & \end{cases}$$

$$\mathbf{f}_{j3} = \begin{cases} -\,\lambda_1\big(3\gamma_1 w_3 - 6\gamma_3 w_2 + \gamma_6 w_1 - 4\gamma_8 w_1 + 4\overline{\gamma}_4 w_2\big) \\ -\,\lambda_1\big(4\gamma_2 w_2 + 4\gamma_4 w_1 + \gamma_5 w_1\big) \\ \qquad\qquad -\,\lambda_2\big(2\gamma_1 w_4 - 3\gamma_3 w_3 + 2\gamma_6 w_2 - 2\gamma_8 w_2 + 3\overline{\gamma}_4 w_3\big) \\ -\,\lambda_2\big(3\gamma_2 w_3 + 2\gamma_4 w_2 + 2\gamma_5 w_2 + \gamma_7 w_1\big) - \lambda_3\big(\gamma_1 w_5 + 3\gamma_6 w_3 + 2\overline{\gamma}_4 w_4\big) \\ -\,\lambda_3\big(2\gamma_2 w_4 + 3\gamma_5 w_3 + 2\gamma_7 w_2\big) - \lambda_4\big(3\gamma_3 w_5 + 4\gamma_6 w_4 + 2\gamma_8 w_4 + \overline{\gamma}_4 w_5\big) \\ -\,\lambda_4\big(\gamma_2 w_5 - 2\gamma_4 w_4 + 4\gamma_5 w_4 + 3\gamma_7 w_3\big) \end{cases}$$

Definition 2.4. *Let $\Phi(x) = \big(e(x), \gamma(x), w(x)\big) : \mathcal{U} \to \mathcal{R}$ be a sufficiently differentiable field, and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be strictly positive weight functions on $\mathcal{U}$. The associated* **constraint field**

$$\Phi^\sharp(x) \;=\; \big(\Phi^\sharp_1(x),\, \Phi^\sharp_2(x),\, \Phi^\sharp_3(x)\big) \;=\; \big(t(x),\, u(x),\, v(x)\big) \;=\; \begin{pmatrix} t_1 \\ \vdots \\ t_5 \end{pmatrix} \oplus \begin{pmatrix} u_1 \\ \vdots \\ u_9 \end{pmatrix} \oplus \begin{pmatrix} v_1 \\ \vdots \\ v_3 \end{pmatrix}$$

on $\mathcal{U}$ taking values in

$$\widehat{\mathcal{R}} = \big\{(t,u,v) \in \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3 \;\big|\; t_1,\, t_2 \in \mathbb{R}\,\big\} \tag{2.6}$$

is given by

$$t_j = \begin{cases} -\,2\,\Im\,\big(D(\overline{e}_1) + \overline{e}_1\gamma_3 + \overline{e}_1\overline{\gamma}_4\big) \\ -\,2\,\Im\,\big(D(\overline{e}_2) + \overline{e}_2\gamma_3 + \overline{e}_2\overline{\gamma}_4\big) \\ D(e_3) - e_3\gamma_3 + e_3\overline{\gamma}_4 + e_3\overline{\gamma}_5 \\ \overline{D}(e_4) - N(\overline{e}_1) - e_1\gamma_7 - \overline{e}_1\gamma_6 + \overline{e}_1\overline{\gamma}_8 - \overline{e}_1\gamma_8 \\ \overline{D}(e_5) - N(\overline{e}_2) - e_2\gamma_7 - \overline{e}_2\gamma_6 + \overline{e}_2\overline{\gamma}_8 - \overline{e}_2\gamma_8 \end{cases} \tag{2.7a}$$

$$u_j = \begin{cases} D(\gamma_2) - \overline{D}(\gamma_1) + w_2 - 3\gamma_1\gamma_4 - \gamma_1\overline{\gamma}_3 - \gamma_2\gamma_3 + \gamma_2\overline{\gamma}_4 \\ D(\gamma_4) + \overline{D}(\gamma_3) - w_3 + 2\gamma_3\gamma_4 + \gamma_1\gamma_7 - \gamma_2\gamma_6 + \gamma_3\overline{\gamma}_3 + \gamma_4\overline{\gamma}_4 \\ \overline{D}(\gamma_3) - \overline{D}(\overline{\gamma}_4) + N(\gamma_2) \\ \qquad - w_3 + 2\gamma_3\gamma_4 - 2\gamma_4\overline{\gamma}_4 + \gamma_1\gamma_7 + \gamma_2\gamma_6 - \gamma_2\gamma_8 - \gamma_2\overline{\gamma}_8 \\ D(\overline{\gamma}_4) - D(\gamma_3) - N(\gamma_1) \\ \qquad + 3\gamma_1\gamma_8 + 2\gamma_3\gamma_3 - 2\gamma_3\overline{\gamma}_4 - \gamma_1\gamma_6 - \gamma_1\overline{\gamma}_8 - \gamma_2\overline{\gamma}_7 \\ \overline{D}(\gamma_5) - L(\gamma_7) - \gamma_2\gamma_7 - \gamma_4\gamma_5 + \gamma_5\gamma_5 - \gamma_5\overline{\gamma}_3 - \gamma_6\overline{\gamma}_1 \\ L(\gamma_6) - D(\gamma_5) - w_3 + \gamma_1\gamma_7 + \gamma_2\gamma_6 - \gamma_3\gamma_5 - \gamma_5\overline{\gamma}_4 - \gamma_5\overline{\gamma}_5 \\ D(\gamma_8) - N(\gamma_3) - 2\gamma_3\gamma_6 + \gamma_3\gamma_8 - \gamma_3\overline{\gamma}_8 + \gamma_4\overline{\gamma}_7 + \gamma_6\overline{\gamma}_4 \\ N(\gamma_4) + \overline{D}(\gamma_8) + w_4 - 2\gamma_3\gamma_7 + \gamma_4\gamma_6 + \gamma_4\gamma_8 - \gamma_4\overline{\gamma}_8 + \gamma_7\overline{\gamma}_4 \\ D(\gamma_7) - \overline{D}(\gamma_6) - w_4 + 3\gamma_3\gamma_7 + \gamma_4\gamma_6 - \gamma_6\overline{\gamma}_3 + \gamma_7\overline{\gamma}_4 \end{cases} \tag{2.7b}$$

$$v_j = \begin{cases} \lambda_1\big(\,\overline{D}(w_1) + L(w_2) + 4\gamma_2 w_2 + 4\gamma_4 w_1 + \gamma_5 w_1\big) \\ \lambda_2\big(\,\overline{D}(w_2) + L(w_3) + 3\gamma_2 w_3 + 2\gamma_4 w_2 + 2\gamma_5 w_2 + \gamma_7 w_1\big) \\ \lambda_3\big(\,\overline{D}(w_3) + L(w_4) + 2\gamma_2 w_4 + 3\gamma_5 w_3 + 2\gamma_7 w_2\big) \end{cases} \tag{2.7c}$$

Proposition 2.2. *Suppose, conditions $(\star)$ and $(\star\star)$ are all satisfied at every point of $\mathcal{U}$. Then, the field $\varPhi(x)$ is a vacuum field if and only if there exist strictly positive weight functions λ_1, λ_2, λ_3, λ_4 on $\mathcal{U}$ such that $\varPhi(x)$ is a solution to the quasilinear symmetric hyperbolic system* **(SHS)** *and the constraint field $\varPhi^\sharp(x) = 0$ everywhere on $\mathcal{U}$.*

Proof. Follows from the Appendices B, C, D. See, Propositions B.1, D.1, D.2, D.3. □

Remark 2.6. Proposition 2.2, together with Definition 2.2, is a reformulation of the vacuum Einstein equations. When does the solution $\varPhi$ to a well posed problem for **(SHS)** also satisfy $\varPhi^\sharp = 0$?

Proposition 2.3. *Suppose, that condition $(\star)$ is satisfied and there are strictly positive weight functions λ_1, λ_2, λ_3, λ_4 on $\mathcal{U}$, such that $\varPhi(x)$ is a C^2 solution to* **(SHS)** *. Then, the constraint field $\varPhi^\sharp$ is a classical solution to the "dual"* **Homogeneous, Linear (over $\mathbb{R}$) Symmetric Hyperbolic System $\widehat{\text{(SHS)}}$** *:*

$$\widehat{\mathbf{A}}(\varPhi)\,\varPhi^\sharp \;=\; \widehat{\mathbf{f}}(\varPhi, \partial_x\varPhi)\,\varPhi^\sharp$$

In particular, if the data for any well posed problem for the system $\widehat{\text{(SHS)}}$ vanishes, then the constraint field $\varPhi^\sharp(x)$ vanishes everywhere. Here, $\widehat{\mathbf{A}}(\varPhi) = \widehat{\mathbf{A}}_1 \oplus \widehat{\mathbf{A}}_2 \oplus \widehat{\mathbf{A}}_3$ is the first order, matrix differential operator

$$\widehat{\mathbf{A}}_1 \;=\; \operatorname{diag}\big(L,\, L,\, N,\, L,\, L\big) \tag{2.8a}$$

$$\widehat{\mathbf{A}}_2 \;=\; \operatorname{diag}\big(L,\, L,\, L,\, L,\, N,\, N,\, L,\, L,\, N\big) \tag{2.8b}$$

$$\widehat{\mathbf{A}}_3 \;=\; \begin{pmatrix} \frac{1}{\lambda_1}N + \frac{1}{\lambda_2}L & \frac{1}{\lambda_2}D & 0 \\ \frac{1}{\lambda_2}\overline{D} & \frac{1}{\lambda_2}N + \frac{1}{\lambda_3}L & \frac{1}{\lambda_3}D \\ 0 & \frac{1}{\lambda_3}\overline{D} & \frac{1}{\lambda_3}N + \frac{1}{\lambda_4}L \end{pmatrix} \tag{2.8c}$$

and $\widehat{\mathbf{f}}(\Phi, \partial_x \Phi)$ *is a linear (over* $\mathbb{R}$*) transformation acting on* $\Phi^\sharp = (t, u, v)$*:*

$$\widehat{\mathbf{f}}(\Phi, \partial_x \Phi)\, \Phi^\sharp \;=\; \big(\widehat{\mathbf{f}}_1 \oplus \widehat{\mathbf{f}}_2 \oplus \widehat{\mathbf{f}}_3\big)\Phi^\sharp \;=\; \begin{pmatrix} \widehat{\mathbf{f}}_{11} \\ \vdots \\ \widehat{\mathbf{f}}_{51} \end{pmatrix} \Phi^\sharp \oplus \begin{pmatrix} \widehat{\mathbf{f}}_{12} \\ \vdots \\ \widehat{\mathbf{f}}_{92} \end{pmatrix} \Phi^\sharp \oplus \begin{pmatrix} \widehat{\mathbf{f}}_{13} \\ \vdots \\ \widehat{\mathbf{f}}_{33} \end{pmatrix} \Phi^\sharp$$

where Φ *has been suppressed on the right hand side, and*

$$\widehat{\mathbf{f}}_{j1}\, \Phi^\sharp = \begin{cases} -\,2\gamma_2 t_1 + 2\,\Im\left(t_3 \frac{\partial \overline{e}_1}{\partial \underline{u}}\right) + 2\,\Im\left(\overline{e}_1 u_1\right) \\ -\,2\gamma_2 t_2 + 2\,\Im\left(t_3 \frac{\partial \overline{e}_2}{\partial \underline{u}}\right) + 2\,\Im\left(\overline{e}_2 u_1\right) \\ (-\gamma_6 + 2\gamma_8) t_3 - \overline{\gamma}_7 \overline{t}_3 - \overline{t}_4 \frac{\partial e_3}{\partial \xi^1} - \overline{t}_5 \frac{\partial e_3}{\partial \xi^2} + e_3 u_7 + e_3 \overline{u}_8 \\ i(\overline{\gamma}_3 - \gamma_4 + \gamma_5) t_1 - \gamma_2 t_4 - \overline{\gamma}_1 \overline{t}_4 - \overline{t}_3 \frac{\partial e_4}{\partial \underline{u}} + \overline{e}_1 u_3 - e_1 \overline{u}_4 + e_1 u_5 - \overline{e}_1 \overline{u}_6 \\ i(\overline{\gamma}_3 - \gamma_4 + \gamma_5) t_2 - \gamma_2 t_5 - \overline{\gamma}_1 \overline{t}_5 - \overline{t}_3 \frac{\partial e_5}{\partial \underline{u}} + \overline{e}_2 u_3 - e_2 \overline{u}_4 + e_2 u_5 - \overline{e}_2 \overline{u}_6 \end{cases}$$

$$\widehat{\mathbf{f}}_{j2}\, \Phi^\sharp =$$

$$\begin{cases} -\,t_3 \frac{\partial \gamma_2}{\partial \underline{u}} + \overline{t}_3 \frac{\partial \gamma_1}{\partial \underline{u}} - 3\gamma_2 u_1 + \gamma_1 \overline{u}_1 + \frac{1}{\lambda_1} v_1 \\ -\,t_3 \frac{\partial \gamma_4}{\partial \underline{u}} - \overline{t}_3 \frac{\partial \gamma_3}{\partial \underline{u}} - \gamma_4 u_1 + \gamma_5 u_1 - \gamma_3 \overline{u}_1 - 2\gamma_2 u_2 - \gamma_1 u_5 - \gamma_2 u_6 - \frac{1}{\lambda_2} v_2 \\ -\,\overline{t}_3 \frac{\partial \gamma_3}{\partial \underline{u}} + \overline{t}_3 \frac{\partial \overline{\gamma}_4}{\partial \underline{u}} + \overline{\gamma}_3 u_1 - \gamma_4 u_1 + \gamma_5 u_1 \\ \qquad\qquad -\,2\gamma_2 u_3 + \overline{\gamma}_1 u_4 + \gamma_1 \overline{u}_4 - \gamma_1 u_5 + \gamma_2 \overline{u}_6 - \frac{1}{\lambda_2} v_2 \\ +\,t_3 \frac{\partial \gamma_3}{\partial \underline{u}} - t_3 \frac{\partial \overline{\gamma}_4}{\partial \underline{u}} + (\gamma_3 - \overline{\gamma}_4 + \overline{\gamma}_5) u_1 + \gamma_1 u_3 + \gamma_1 \overline{u}_3 - 2\gamma_2 u_4 + \gamma_2 \overline{u}_5 - \gamma_1 u_6 \\ -\,t_4 \frac{\partial \gamma_5}{\partial \xi^1} - t_5 \frac{\partial \gamma_5}{\partial \xi^2} - \gamma_7 u_3 + \gamma_6 \overline{u}_4 - 2\gamma_6 u_5 - 2\gamma_8 u_5 + 2\overline{\gamma}_8 u_5 \\ \qquad\qquad +\,\gamma_7 u_6 + \gamma_7 \overline{u}_6 + \gamma_5 \overline{u}_7 - \gamma_5 u_8 - \gamma_4 u_9 + \gamma_5 u_9 + \overline{\gamma}_3 u_9 \\ +\,\overline{t}_4 \frac{\partial \gamma_5}{\partial \xi^1} + \overline{t}_5 \frac{\partial \gamma_5}{\partial \xi^2} + \gamma_6 \overline{u}_3 - \gamma_7 u_4 + \overline{\gamma}_7 u_5 + \gamma_7 \overline{u}_5 \\ \qquad\qquad -\,2\gamma_6 u_6 + \gamma_5 u_7 - \gamma_5 \overline{u}_8 + \gamma_3 u_9 - \overline{\gamma}_4 u_9 + \overline{\gamma}_5 u_9 + \frac{1}{\lambda_3} v_2 \\ -\,t_3 \frac{\partial \gamma_8}{\partial \underline{u}} - \gamma_3 u_2 + \overline{\gamma}_4 u_2 - \overline{\gamma}_5 u_2 + \gamma_3 \overline{u}_3 + \gamma_4 u_4 - \gamma_5 u_4 \\ \qquad\qquad -\,\gamma_4 \overline{u}_5 - 2\gamma_3 u_6 + \overline{\gamma}_4 u_6 - \gamma_2 u_7 - \gamma_1 u_8 - \frac{1}{\lambda_2} v_1 \\ -\,\overline{t}_3 \frac{\partial \gamma_8}{\partial \underline{u}} + \overline{\gamma}_3 u_2 - \gamma_4 u_2 + \gamma_5 u_2 - \gamma_4 u_3 + \gamma_5 u_3 - \gamma_3 \overline{u}_4 \\ \qquad\qquad +\,2\gamma_3 u_5 - \overline{\gamma}_4 u_5 + \gamma_4 \overline{u}_6 - \overline{\gamma}_1 u_7 - \gamma_2 u_8 + \frac{1}{\lambda_3} v_3 \\ +\,t_4 \frac{\partial \gamma_6}{\partial \xi^1} - \overline{t}_4 \frac{\partial \gamma_7}{\partial \xi^1} + t_5 \frac{\partial \gamma_6}{\partial \xi^2} - \overline{t}_5 \frac{\partial \gamma_7}{\partial \xi^2} - 3\gamma_7 u_7 + \gamma_6 \overline{u}_7 \\ \qquad\qquad +\,\gamma_6 u_8 + \gamma_7 \overline{u}_8 - 3\gamma_6 u_9 - 2\gamma_8 u_9 + \gamma_7 \overline{u}_9 + \frac{1}{\lambda_4} v_3 \end{cases}$$

$$\begin{aligned} \widehat{\mathbf{f}}_{13}\, \Phi^\sharp = & -\, i t_1 \frac{\partial w_2}{\partial \xi^1} - i t_2 \frac{\partial w_2}{\partial \xi^2} + t_3 \frac{\partial w_3}{\partial \underline{u}} - t_4 \frac{\partial w_1}{\partial \xi^1} - t_5 \frac{\partial w_1}{\partial \xi^2} \\ & +\, 3 w_3 u_1 + 2 w_2 u_2 + 4 w_2 u_3 - 2 w_2 u_6 + 4 w_1 u_8 + w_1 u_9 \\ & +\, \Big[-\, \tfrac{4}{\lambda_2} \gamma_2 - \tfrac{2}{\lambda_1} \gamma_6 + \tfrac{3}{\lambda_1} \gamma_8 + \tfrac{1}{\lambda_1} \overline{\gamma}_8 + (\tfrac{1}{\lambda_1})^2 N(\lambda_1) + (\tfrac{1}{\lambda_2})^2 L(\lambda_2) \Big] v_1 \\ & +\, \Big[(\tfrac{1}{\lambda_2})^2 D(\lambda_2) + \tfrac{4}{\lambda_2} \gamma_3 - \tfrac{4}{\lambda_2} \overline{\gamma}_4 - \tfrac{1}{\lambda_2} \overline{\gamma}_5 \Big] v_2 - \tfrac{2}{\lambda_3} \gamma_1 v_3 \end{aligned}$$

$$
\begin{aligned}
\widehat{\mathbf{f}}_{23}\,\Phi^\sharp = & - it_1 \tfrac{\partial w_3}{\partial \xi^1} - it_2 \tfrac{\partial w_3}{\partial \xi^2} + t_3 \tfrac{\partial w_4}{\partial \underline{u}} - t_4 \tfrac{\partial w_2}{\partial \xi^1} - t_5 \tfrac{\partial w_2}{\partial \xi^2} \\
& + 2w_4 u_1 + 3w_3 u_3 - 3w_3 u_6 + 2w_2 u_8 + 2w_2 u_9 \\
& + \big[- \tfrac{2}{\lambda_2}\gamma_4 - \tfrac{2}{\lambda_2}\gamma_5 + (\tfrac{1}{\lambda_2})^2 \overline{D}\,(\lambda_2) \big] v_1 \\
& + \big[- \tfrac{3}{\lambda_3}\gamma_2 - \tfrac{3}{\lambda_2}\gamma_6 + \tfrac{1}{\lambda_2}\gamma_8 + \tfrac{1}{\lambda_2}\overline{\gamma}_8 + (\tfrac{1}{\lambda_2})^2 N\,(\lambda_2) + (\tfrac{1}{\lambda_3})^2 L\,(\lambda_3) \big] v_2 \\
& + \big[\tfrac{1}{\lambda_3}\gamma_3 - \tfrac{3}{\lambda_3}\overline{\gamma}_4 - \tfrac{1}{\lambda_3}\overline{\gamma}_5 + (\tfrac{1}{\lambda_3})^2 D\,(\lambda_3) \big] v_3
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\mathbf{f}}_{33}\,\Phi^\sharp = & - it_1 \tfrac{\partial w_4}{\partial \xi^1} - it_2 \tfrac{\partial w_4}{\partial \xi^2} + t_3 \tfrac{\partial w_5}{\partial \underline{u}} - t_4 \tfrac{\partial w_3}{\partial \xi^1} - t_5 \tfrac{\partial w_3}{\partial \xi^2} \\
& + w_5 u_1 - 2w_4 u_2 + 2w_4 u_3 - 4w_4 u_6 + 3w_3 u_9 \\
& - \tfrac{2}{\lambda_2}\gamma_7 v_1 + \big[- \tfrac{3}{\lambda_3}\gamma_5 + (\tfrac{1}{\lambda_3})^2 \overline{D}\,(\lambda_3) \big] v_2 \\
& + \big[- \tfrac{2}{\lambda_4}\gamma_2 - \tfrac{4}{\lambda_3}\gamma_6 - \tfrac{1}{\lambda_3}\gamma_8 + \tfrac{1}{\lambda_3}\overline{\gamma}_8 + (\tfrac{1}{\lambda_3})^2 N\,(\lambda_3) + (\tfrac{1}{\lambda_4})^2 L\,(\lambda_4) \big] v_3
\end{aligned}
$$

Proof. Follows from the Appendices B, C, D. See, Proposition D.4. □

Remark 2.7. Write $\mathbf{A}(\Phi) = \mathbf{A}^\mu \frac{\partial}{\partial x^\mu} = ({\mathbf{A}_1}^\mu \oplus {\mathbf{A}_2}^\mu \oplus {\mathbf{A}_3}^\mu)\frac{\partial}{\partial x^\mu}$. Explicitly,

$$
\begin{aligned}
\mathbf{A}^1 &= e_4\,\mathrm{diag}(0,0,1,0,0) \oplus e_4\,\mathrm{diag}(0,0,0,0,1,1,1,0) \oplus \begin{pmatrix} \lambda_1 e_4 & \lambda_1 e_1 & 0 & 0 & 0 \\ \lambda_1 \overline{e}_1 & \lambda_2 e_4 & \lambda_2 e_1 & 0 & 0 \\ 0 & \lambda_2 \overline{e}_1 & \lambda_3 e_4 & \lambda_3 e_1 & 0 \\ 0 & 0 & \lambda_3 \overline{e}_1 & \lambda_4 e_4 & \lambda_4 e_1 \\ 0 & 0 & 0 & \lambda_4 \overline{e}_1 & 0 \end{pmatrix} \\
\mathbf{A}^3 &= e_3\,\mathrm{diag}(1,1,0,1,1) \oplus e_3\,\mathrm{diag}(1,1,1,1,0,0,0,1) \oplus e_3\,\mathrm{diag}(0,\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\
\mathbf{A}^4 &= \quad \mathrm{diag}(0,0,1,0,0) \oplus \quad \mathrm{diag}(0,0,0,0,1,1,1,0) \oplus \quad \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,0)
\end{aligned}
$$

The matrices $\mathbf{A}^\mu$ are Hermitian matrices whose entries are linear (over $\mathbb{R}$) functions of $\Phi_1 = e$. The matrix $\mathbf{A}^3 + \mathbf{A}^4$ is strictly positive definite by the requirement $e_3 > 0$ of condition ($\star$). We see that, **(SHS)** is truly a quasilinear, symmetric hyperbolic system. An entirely similar discussion applies to $\widehat{\textbf{(SHS)}}$. See [John], [Tay], for linear and quasilinear symmetric hyperbolic systems (in the sense of Friedrichs).

Remark 2.8. By definition, the fields Φ and $\Phi^\sharp$ take values in $\mathcal{R} \cong \mathbb{R}^{31}$ and $\widehat{\mathcal{R}} \cong \mathbb{R}^{32}$, respectively. The left and right hand sides of **(SHS)** are in $\mathcal{R}$, because $\mathbf{f}_{31}$, $\mathbf{f}_{41}$, $\mathbf{f}_{51}$, $\mathbf{f}_{22}$, $\mathbf{f}_{62}$, marked by $\star$, are real. The left and right hand sides of $\widehat{\textbf{(SHS)}}$ are in $\widehat{\mathcal{R}}$. In other words, **(SHS)** is equivalent to a *real* quasilinear symmetric hyperbolic system for an $\mathbb{R}^{31}$ valued field, and $\widehat{\textbf{(SHS)}}$ is equivalent to a *real* linear homogeneous symmetric hyperbolic system for an $\mathbb{R}^{32}$ valued field.

Remark 2.9. Let $\mathfrak{P}$ be the parity transformation $(\mathfrak{P} \cdot \Phi)(x) = (-1)^A\,\Phi(x)$ with

$$
A = \mathrm{diag}(1,1,0,0,0) \oplus \mathrm{diag}(0,0,1,1,1,0,0,0) \oplus \mathrm{diag}(0,1,0,1,0).
$$

The field $\mathfrak{P} \cdot \Phi$ solves **(SHS)** if and only if Φ solves **(SHS)** . The constraint $(\mathfrak{P} \cdot \Phi)^\sharp = 0$ if and only if $\Phi^\sharp = 0$. Clearly, $\mathfrak{P} \circ \mathfrak{P} =$ Identity, and Φ splits naturally into 11 $\mathfrak{P}$-even and 7 $\mathfrak{P}$-odd components. If the $\mathfrak{P}$-odd components of Φ vanish at $x \in \mathcal{U}$, that is $(\mathfrak{P} \cdot \Phi)(x) = \Phi(x)$, then **(SHS)** implies that $L(e_4) = L(e_5) = 0$ at x.

Proposition 2.4. *Suppose* $(\star)$. *Set all the* $\mathfrak{P}$*-odd components,* e_1, e_2, γ_3, γ_4, γ_5, w_2, w_4, *and the two* $\mathfrak{P}$*-even components* e_4, e_5 *of the field* Φ *equal to zero, and introduce the field* $\widetilde{\Phi} = e_3 \oplus (\gamma_1, \gamma_2, \gamma_6, \gamma_7, \gamma_8) \oplus (w_1, w_3, w_5)$. *In this case, the frame collapses to* $D = 0$, $N = \frac{\partial}{\partial u}$ *and* $L = e_3 \frac{\partial}{\partial \underline{u}}$ *and the system* **(SHS)** *reduces to*

$$\begin{aligned} \widetilde{\mathbf{A}}\big(\widetilde{\Phi}\big)\widetilde{\Phi} &= \widetilde{\mathbf{f}}\big(\widetilde{\Phi}\big) \qquad \textbf{(subSHS)} \\ \widetilde{\mathbf{A}}\big(\widetilde{\Phi}\big) &= N \oplus \operatorname{diag}(L, L, N, N, L) \oplus \operatorname{diag}(\lambda_1 N,\ \lambda_2 L + \lambda_3 N,\ \lambda_4 L) \\ \widetilde{\mathbf{f}}\big(\widetilde{\Phi}\big) &= \mathbf{f}_{31} \oplus (\mathbf{f}_{12}, \mathbf{f}_{22}, \mathbf{f}_{62}, \mathbf{f}_{72}, \mathbf{f}_{82}) \oplus (\mathbf{f}_{13}, \mathbf{f}_{33}, \mathbf{f}_{53}) \qquad \textit{(see, Definition 2.3)} \end{aligned}$$

It is, separately for each ξ, *a quasilinear symmetric hyperbolic system for* $\widetilde{\Phi}$ *in the* $(\underline{u}, u)$ *plane. The components* t, u_1, u_7, u_8, u_9, v_1, v_3 *of* $\Phi^\sharp$ *vanish.* $\widehat{\textbf{(SHS)}}$ *reduces to a linear (over* $\mathbb{R}$*), homogeneous symmetric hyperbolic system for* $\widetilde{\Phi}^\sharp = (u_2, u_3, u_4, u_5, u_6) \oplus v_2$.

Corollary 2.1. *Suppose,* Φ *is a solution to any well posed problem for* **(SHS)** *, such that all its* $\mathfrak{P}$*-odd components and* e_4, e_5 *vanish initially. Then, they vanish everywhere.*

3. Symmetries

A **field transformation** S with respect to the open subsets $\mathcal{U}, \mathcal{U}'$ of $\mathbb{R}^4$ consists of

- a diffeomorphism from $\mathcal{U}$ to $\mathcal{U}'$,
- a map from fields $\Phi = (e, \gamma, w) : \mathcal{U} \to \mathcal{R}$ to fields $\Phi' = (e', \gamma', w') : \mathcal{U}' \to \mathcal{R}$,
- a map from strictly positive weight functions $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ on $\mathcal{U}$ to strictly positive weight functions $\Lambda' = (\lambda_1', \lambda_2', \lambda_3', \lambda_4')$ on $\mathcal{U}'$.

Let x, x' be Cartesian coordinates on $\mathcal{U}$ and $\mathcal{U}'$. We write

$$x' = S \cdot x \qquad \Phi'(x') = (S \cdot \Phi)(x') \qquad \Lambda'(x') = (S \cdot \Lambda)(x')$$

In this section, S will always be linear over real valued functions in its action on Φ and Λ. That is,

$$\big(S \cdot (f\Phi)\big)(x') = f(S^{-1} \cdot x')\, (S \cdot \Phi)(x') \quad , \quad \big(S \cdot (f\Lambda)\big)(x') = f(S^{-1} \cdot x')\, (S \cdot \Lambda)(x')$$

for all $f \in C(\mathcal{U}, \mathbb{R})$. Therefore, S acts pointwise. For this reason, it suffices to make a local analysis. For the rest of this section, we make the assumption that $x' = S \cdot x$ is a local diffeomorphism on $\mathbb{R}^4$. With this understanding, it is unnecessary to specify the domains $\mathcal{U}$ and $\mathcal{U}'$.

Definition 3.1. *A field transformation* S *is a* **field symmetry** *if:*

- $(\star)$ *and* $(\star\star)$ *are preserved (see, Definition 2.1).*
- Φ *satisfies* **(SHS)** *on* $\mathcal{U}$ *if and only if* $S \cdot \Phi$ *satisfies* **(SHS)** *on* $\mathcal{U}'$.
- $\Phi^\sharp$ *vanishes on* $\mathcal{U}$ *if and only* $(S \cdot \Phi)^\sharp$ *vanishes on* $\mathcal{U}'$.

It is implicit in the last two statements that the weights Λ *appear on* $\mathcal{U}$ *and the weights* $S \cdot \Lambda$ *appear on* $\mathcal{U}'$. *For a field symmetry* S, *it follows that* Φ *is a vacuum field on* $\mathcal{U}$ *if and only if* $S \cdot \Phi$ *is a vacuum field on* $\mathcal{U}'$.

As in Section 2, let $x = (x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$. We now define a number of field transformations.

Definition 3.2. Angular coordinate transformation $\mathfrak{C}$. *Let $\mathfrak{C}^1$, $\mathfrak{C}^2$ be real functions.*

$$
\begin{aligned}
x' &= \mathfrak{C}\cdot x = \big(\mathfrak{C}^1(x^1,x^2),\ \mathfrak{C}^2(x^1,x^2),\ x^3,\ x^4\big) \\
e'_A(x') &= \sum_{B=1}^{2} \tfrac{\partial \mathfrak{C}^A}{\partial x^B}(x)\ e_B(x)\,\big|_{x=\mathfrak{C}^{-1}\cdot x'} && A=1,2 \\
e'_{A+3}(x') &= \sum_{B=1}^{2} \tfrac{\partial \mathfrak{C}^A}{\partial x^B}(x)\ e_{B+3}(x)\,\big|_{x=\mathfrak{C}^{-1}\cdot x'} && A=1,2 \\
(e'_3,\gamma',w',\Lambda')(x') &= (e_3,\gamma,w,\Lambda)(\mathfrak{C}^{-1}\cdot x')
\end{aligned}
$$

We will also use the notation $\mathfrak{C}(\xi)=\mathfrak{C}^1(\xi)+i\mathfrak{C}^2(\xi)$, where $\xi=\xi^1+i\xi^2$.

Definition 3.3. $U(1)$ transformation $\mathfrak{Z}$. *Let $\zeta=\zeta(x^1,x^2)\in U(1)$.*

$$x' = \mathfrak{Z}\cdot x = x$$

$$\Phi'(x') = (\mathfrak{Z}\cdot\Phi)(x') = \zeta^A\Phi(x) \quad + \left. \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \oplus \begin{pmatrix}0\\0\\ \frac{1}{2}\,D(\zeta)\\ \frac{1}{2}\,\overline{D(\zeta)}\\0\\0\\0\\ \frac{1}{2}\,\zeta^{-1}N(\zeta)\end{pmatrix}(x) \oplus \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \right|_{x=\mathfrak{Z}^{-1}\cdot x'}$$

$$
\begin{aligned}
A &= \mathrm{diag}\big(1,1,0,0,0,\ \ 2,0,1,-1,-1,0,-2,0,\ \ 2,1,0,-1,-2\big) \\
\Lambda'(x') &= (\mathfrak{Z}\cdot\Lambda)(x') = \Lambda(x)\,\big|_{x=\mathfrak{Z}^{-1}\cdot x'}
\end{aligned}
$$

Here $D(\zeta)=(e_1\frac{\partial}{\partial x^1}+e_2\frac{\partial}{\partial x^2})(\zeta)$ and $N(\zeta)=(e_4\frac{\partial}{\partial x^1}+e_5\frac{\partial}{\partial x^2})(\zeta)$.

Definition 3.4. Global Isotropic Scaling $\mathfrak{J}$. *Let $\mathfrak{J}>0$ be a constant.*

$$
\begin{aligned}
x' &= \mathfrak{J}\cdot x = (x^1,\ x^2,\ \mathfrak{J}x^3,\ \mathfrak{J}x^4) \\
\Phi'(x') &= (\mathfrak{J}\cdot\Phi)(x') \ = \ \mathfrak{J}^A\Phi(x)\,\big|_{x=\mathfrak{J}^{-1}\cdot x'} \\
A &= \ (-1)\ \mathrm{diag}\big(1,1,0,1,1,\ \ 1,1,1,1,1,1,1,1,\ \ 2,2,2,2,2\big) \\
\Lambda'(x') &= (\mathfrak{J}\cdot\Lambda)(x') = \Lambda(x)\,\big|_{x=\mathfrak{J}^{-1}\cdot x'}
\end{aligned}
$$

Definition 3.5. Global Anisotropic Scaling $\mathfrak{A}$. *Let $\mathfrak{A}\neq 0$.*

$$
\begin{aligned}
x' &= \mathfrak{A}\cdot x = \big(\tfrac{1}{\mathfrak{A}}x^1,\ \tfrac{1}{\mathfrak{A}}x^2,\ x^3,\ \mathfrak{A}^2x^4\big) \\
\Phi'(x') &= (\mathfrak{A}\cdot\Phi)(x') \ = \ \mathfrak{A}^A\Phi(x)\big|_{x=\mathfrak{A}^{-1}\cdot x'} \\
A &= \ (-1)\ \mathrm{diag}\big(2,2,0,3,3,\ \ 0,0,1,1,1,2,2,2,\ \ 0,1,2,3,4\big) \\
\Lambda'(x') &= (\mathfrak{A}\cdot\Lambda)(x') = \mathrm{diag}(1,\mathfrak{A}^2,\mathfrak{A}^4,\mathfrak{A}^6)\ \Lambda(x)\,\big|_{x=\mathfrak{A}^{-1}\cdot x'}
\end{aligned}
$$

The transformation $\mathfrak{A}$ plays a central role in this paper. For a sample calculation, see the proof of Proposition 4.1.

Proposition 3.1. $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ *are field symmetries.*

Proposition 3.1 is proven in Appendix E. We will now define an additional transformation. It is a composition of two field symmetries, and therefore itself a field symmetry.

Definition 3.6. Pole-Flip transformation $\mathbf{Flip}_\alpha$. *Let $\alpha \neq 0$ be a constant. The Pole-Flip transformation $\mathbf{Flip}_\alpha$ is the composition of a $U(1)$ transformation and an angular coordinate transformation. Precisely,*

$$\mathbf{Flip}_\alpha \;=\; \mathfrak{Z} \circ \mathfrak{C} \qquad \textit{where} \qquad \zeta(\xi) = -\frac{\xi}{\bar{\xi}} \qquad \mathfrak{C}(\xi) = \frac{\alpha^2}{\xi}$$

and $\xi = \xi^1 + i\xi^2$ and $\mathfrak{C}(\xi) = \mathfrak{C}^1(\xi) + i\mathfrak{C}^2(\xi)$.

Remark 3.1. With this choice of ζ and $\mathfrak{C}$, we have $\mathfrak{C}^{-1} \circ \mathfrak{Z} \circ \mathfrak{C} = \mathfrak{Z}^{-1}$ and $\mathfrak{C} \circ \mathfrak{C} =$ Identity. Therefore, $\mathbf{Flip}_\alpha \circ \mathbf{Flip}_\alpha = \mathrm{Identity}$. The field symmetry $\mathbf{Flip}_\alpha$ will be used to match constructions between two "angular" coordinate patches. For Minkowski, ξ will be stereographic coordinates (scaled by α) based on the north and south poles.

4. The Doubly Scaled Minkowski Field $\mathcal{M}_{a,\mathfrak{A}}$

Fix the coordinates $(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$ as in the preceding sections. For all pairs $\mu, \lambda > 0$, set

$$\begin{aligned} \mathbf{Strip}(\mu, \lambda) &= \mathbb{R}^2 \times (0, \mu) \times (-\infty, -\lambda^{-1}) \\ \mathbf{Strip}_\infty = \mathbf{Strip}(\infty, \infty) &= \mathbb{R}^2 \times (0, \infty) \times (-\infty, 0) \end{aligned} \tag{4.1}$$

Definition 4.1. *For all $a, \mathfrak{A} \neq 0$, let $\mathcal{M}_{a,\mathfrak{A}} : \mathbf{Strip}_\infty \to \mathcal{R}$ (see (2.1)) be the field*

$$\mathcal{M}_{a,\mathfrak{A}} = \begin{pmatrix} \rho_{a,\mathfrak{A}}^{-1}\, \mathbf{e}_{a,\mathfrak{A}} \\ i\, \rho_{a,\mathfrak{A}}^{-1}\, \mathbf{e}_{a,\mathfrak{A}} \\ 1 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \rho_{a,\mathfrak{A}}^{-1}\, \mathfrak{A}^2 \\ \rho_{a,\mathfrak{A}}^{-1}\, \boldsymbol{\lambda}_{a,\mathfrak{A}} \\ \rho_{a,\mathfrak{A}}^{-1}\, \overline{\boldsymbol{\lambda}}_{a,\mathfrak{A}} \\ 0 \\ -\rho_{a,\mathfrak{A}}^{-1} \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

Here $\xi = \xi^1 + i\xi^2$ and

$$\rho_{a,\mathfrak{A}}(u, \underline{u}) = \mathfrak{A}^2 \underline{u} - u \qquad \mathbf{e}_{a,\mathfrak{A}}(\xi) = \tfrac{a}{2}\big(1 + \tfrac{\mathfrak{A}^2}{a^2}|\xi|^2\big) \qquad \boldsymbol{\lambda}_{a,\mathfrak{A}}(\xi) = -\tfrac{\mathfrak{A}^2}{2a}\,\xi \tag{4.3}$$

We will often consciously suppress the subscripts a, $\mathfrak{A}$ on the functions ρ, $\mathbf{e}$ and $\boldsymbol{\lambda}$. Set

$$S(u, \underline{u}) = \tfrac{u^2}{\rho} + u. \tag{4.4}$$

The decomposition $\frac{1}{\rho} = -\frac{1}{u} + \frac{S}{u^2}$ will be used over and over again.

Proposition 4.1. *For all $a, \mathfrak{A} \neq 0$:*

(a) $\mathcal{M}_{a,\mathfrak{A}} = (\mathfrak{C} \circ \mathfrak{A}) \cdot \mathcal{M}_{1,1}$ *on* $\mathbf{Strip}_\infty$*, where* $\mathfrak{C}(\xi) = a\,\xi$.
(b) $\mathcal{M}_{a,\mathfrak{A}} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{M}_{a,\mathfrak{A}}$ *on* $\mathbf{Strip}_\infty \cap \{\xi \neq 0\}$.
(c) $\mathcal{M}_{a,\mathfrak{A}} = \mathfrak{J} \cdot \mathcal{M}_{a,\mathfrak{A}}$ *on* $\mathbf{Strip}_\infty$*, for all* $\mathfrak{J} > 0$.
(d) $\mathcal{M}_{a,\mathfrak{A}}$ *is a* vacuum field *on* $\mathbf{Strip}_\infty$ *(see, Definition 2.2).*

(e) The Lorentzian manifold associated to $\mathcal{M}_{a,\mathfrak{A}}$ is isometric to the open subset of Minkowski space given by (4.5) *below. For this reason, we refer to $\mathcal{M}_{a,\mathfrak{A}}$ as the* **doubly scaled Minkowski field**.

Here, $\mathfrak{C}$, $\mathfrak{A}$, $\mathfrak{J}$ and **Flip**$_\alpha$ *are field transformations defined in Section 3.*

Proof. For (a), set $C_a = \operatorname{diag}(a,a,1,a,a) \oplus \mathbb{1}_8 \oplus \mathbb{1}_5$ and let A be the matrix in Definition 3.5. Then,

$$\begin{aligned}
\big((\mathfrak{C}\circ\mathfrak{A})\cdot\mathcal{M}_{1,1}\big)(x'') &= \big(\mathfrak{C}\cdot(\mathfrak{A}\cdot\mathcal{M}_{1,1})\big)(x'') \\
&= C_a\,(\mathfrak{A}\cdot\mathcal{M}_{1,1})(x')\,\big|_{x'=\mathfrak{C}^{-1}\cdot x''} \\
&= C_a\,\mathfrak{A}^A\,\mathcal{M}_{1,1}(x)\big|_{x=\mathfrak{A}^{-1}\cdot x'=\mathfrak{A}^{-1}\cdot(\mathfrak{C}^{-1}\cdot x'')} \\
&= C_a\,\mathfrak{A}^A\,\mathcal{M}_{1,1}\big(\tfrac{\mathfrak{A}}{a}\xi'', \underline{u}'', \tfrac{1}{\mathfrak{A}^2}u''\big) \\
&= \mathcal{M}_{a,\mathfrak{A}}(x'')
\end{aligned}$$

Similarly for (b) and (c). Part (d) is also verified by direct calculation. It suffices to check (d) for $\mathcal{M}_{1,1}$, because the general case follows from (a) and Proposition 3.1. Recall that the definition of a vacuum field is independent of the choice of weight functions $\lambda_1,\dots,\lambda_4$. For (e), see Remark 4.1 below. □

Remark 4.1. The Riemann curvature tensor of the Lorentzian manifold associated to the vacuum field $\mathcal{M}_{a,\mathfrak{A}} = (e,\gamma,w)$ on **Strip**$_\infty$ vanishes, because $w=0$ (see, Definition 2.2). It is isometric to the open subset of Minkowski space given by

$$\Big\{(X^0,\mathbf{X})\in\mathbb{R}\times\mathbb{R}^3 \;\Big|\; |X^0|<|\mathbf{X}|,\ \mathbf{X}\notin\{0\}\times\{0\}\times[0,\infty)\Big\} \tag{4.5}$$

where $(X^0,\mathbf{X})$ are the standard Minkowski coordinates, and

$$X^0 = \frac{1}{\sqrt{2}\,|\mathfrak{A}|}(\mathfrak{A}^2\underline{u}+u) \qquad \begin{pmatrix}X^1\\X^2\\X^3\end{pmatrix} = \frac{1}{\sqrt{2}\,|\mathfrak{A}|}\,\frac{\mathfrak{A}^2\underline{u}-u}{1+\frac{\mathfrak{A}^2}{a^2}|\xi|^2}\begin{pmatrix}\frac{2\mathfrak{A}}{a}\xi^1\\ \frac{2\mathfrak{A}}{a}\xi^2\\ -1+\frac{\mathfrak{A}^2}{a^2}|\xi|^2\end{pmatrix}$$

The level sets of $u = 2^{-\frac12}|\mathfrak{A}|\,\big(X^0-|\mathbf{X}|\big)<0$ and $\underline{u} = 2^{-\frac12}|\frac{1}{\mathfrak{A}}|\big(X^0+|\mathbf{X}|\big)>0$ are null hypersurfaces. They intersect in a standard sphere of radius $|\mathbf{X}| = 2^{-\frac12}|\frac{1}{\mathfrak{A}}|\rho$, with the north pole removed, on which $\frac{\mathfrak{A}}{a}\,\xi$ is the standard stereographic coordinate system. The southern hemisphere corresponds to $|\xi|<|\frac{a}{\mathfrak{A}}|$.

Remark 4.2. The limit $\mathcal{M}_{0,0} = \lim_{\mathfrak{A}\downarrow0}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ exists on **Strip**$_\infty$. By taking the limit of (d) in Proposition 4.1, it is a solution to **(SHS)** with $\mathcal{M}^\sharp_{0,0} = 0$. Observe that the associated frame is degenerate, because $D=0$ here. (See, Proposition 2.4.)

Remark 4.3. For each $a\neq0$, the limit $\mathcal{M}_{a,0} = \lim_{\mathfrak{A}\downarrow0}\mathcal{M}_{a,\mathfrak{A}}$ exists on **Strip**$_\infty$. By taking the limit of (d) in Proposition 4.1, it is a vacuum field. The Lorentzian manifold associated to $\mathcal{M}_{a,0}$ is isometric to the open subset of Minkowski space given by $\{|X^0|<|\mathbf{X}|,\ X^0+X^3<0\}$:

$$\tfrac{1}{\sqrt2}(X^0+X^3) = u \qquad \tfrac{1}{\sqrt2}(X^0-X^3) = \underline{u}+u\,\tfrac{1}{a^2}\,|\xi|^2 \qquad \begin{pmatrix}X^1\\X^2\end{pmatrix} = -\sqrt2\,u\begin{pmatrix}\frac1a\xi^1\\ \frac1a\xi^2\end{pmatrix}$$

Observe that $-(X^0)^2+|\mathbf{X}|^2 = -2u\underline{u}$. The field $\mathcal{M}_{a,0}$ is independent of ξ and $\underline{u}$, that is, translation invariant in these directions. The level sets of u and $\underline{u}$ are null hypersurfaces. They intersect in standard Euclidean planes.

5. The Far (Weak) Field Ansatz

Fix the coordinate system $(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$ on the infinitely wide strip $\mathbf{Strip}_\infty \subset \mathbb{R}^4$, see (4.1), and make the far ("past null infinity") field Ansatz

$$\Phi(x) \;=\; \mathcal{M}_{a,\mathfrak{A}}(x) \;+\; u^{-M}\Psi(x) \tag{5.1}$$

on $\mathcal{U}$. See, (4.2) for the definition of $\mathcal{M}_{a,\mathfrak{A}}(x)$. Here,

$$\begin{aligned} M \;&=\; \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(1,2,2,2,2,2,2,3) \oplus \mathrm{diag}(1,2,3,4,4) \\ \Psi(x) \;&=\; \big(\Psi_1(x),\, \Psi_2(x),\, \Psi_3(x)\big) \;=\; \big(f(x),\, \omega(x),\, z(x)\big) \;\in\; \mathcal{R} \end{aligned}$$

Our basic Ansatz (5.1), Minkowski plus asymptotically small corrections (assuming, $\Psi = \mathcal{O}(1)$ as $u \to -\infty$), is completely naive. The only subtlety, lies in the choice of the diagonal matrix M that prescribes the far field asymptotics of the system, and is ultimately a statement about the physics of propagating gravitational waves. Anyone who has made formal or rigorous perturbative calculations or constructions in classical or quantum physics knows from experience that one must "play with the expansion" until, "one sees what is going on". We have followed this traditional route to the matrix M. However, the only real justification is that it works.

In this section we bring the equations of Section 2 into a relevant / irrelevant form that exhibits the essential constituents that have to be treated carefully, and sweeps everything else into "generic terms" that we don't need to know much about.

Proposition 5.1. *In this proposition, ignore Definition 2.4, and regard $\Phi(x)$ and $\Phi^\sharp(x)$ as independent, sufficiently differentiable fields on $\mathbf{Strip}_\infty$ with values in $\mathcal{R}$ and $\widehat{\mathcal{R}}$, respectively. Set*

$$\begin{aligned} M &= \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(1,2,2,2,2,2,2,3) \oplus \mathrm{diag}(1,2,3,4,4) && (5.2\text{a}) \\ E &= \mathrm{diag}(4,4,4,6,6) \oplus \mathrm{diag}(2,4,4,4,4,4,4,6) \oplus \mathrm{diag}(0,0,0,0,0) && (5.2\text{b}) \\ M^\sharp &= \mathrm{diag}(2,2,2,3,3) \oplus \mathrm{diag}(2,2,2,2,2,2,3,3,3) \oplus \mathrm{diag}(0,-1,-2) && (5.2\text{c}) \\ E^\sharp &= \mathrm{diag}(4,4,4,6,6) \oplus \mathrm{diag}(4,4,4,4,4,4,6,6,6) \oplus \mathrm{diag}(2,2,2) && (5.2\text{d}) \end{aligned}$$

and

$$\begin{aligned} \Phi(x) &= \mathcal{M}_{a,\mathfrak{A}}(x) + u^{-M}\,\Psi(x) && \Psi(x) \in \mathcal{R} && (5.3\text{a}) \\ \Phi^\sharp(x) &= u^{-M^\sharp}\,\Psi^\sharp(x) && \Psi^\sharp(x) \in \widehat{\mathcal{R}} && (5.3\text{b}) \\ \lambda_j(x) &= u^{2j} && j = 1,2,3,4 \quad \textit{(see, Definition 2.3)} && (5.3\text{c}) \end{aligned}$$

The systems (see, Section 2) $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$ and $\widehat{\mathbf{A}}(\Phi)\,\Phi^\sharp = \widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\,\Phi^\sharp$ for Φ and $\Phi^\sharp$ are equivalent to the following systems for Ψ and $\Psi^\sharp$:

$$\begin{aligned} \mathbf{A}_{a,\mathfrak{A}}(x,\Psi)\,\Psi \;&=\; \mathbf{f}_{a,\mathfrak{A}}(x,\Psi) && (5.4\text{a}) \\ \widehat{\mathbf{A}}_{a,\mathfrak{A}}(x,\Psi)\,\Psi^\sharp \;&=\; \widehat{\mathbf{f}}_{a,\mathfrak{A}}(x,\Psi,\partial_x\Psi)\,\Psi^\sharp && (5.4\text{b}) \end{aligned}$$

where $\mathbf{A}_{a,\mathfrak{A}}(x,\Psi) = \mathbf{A}^\mu_{a,\mathfrak{A}}(x,\Psi)\frac{\partial}{\partial x^\mu}$ *and* $\widehat{\mathbf{A}}_{a,\mathfrak{A}}(x,\Psi) = \widehat{\mathbf{A}}^\mu_{a,\mathfrak{A}}(x,\Psi)\frac{\partial}{\partial x^\mu}$ *and*

$$\mathbf{A}^\mu_{a,\mathfrak{A}}(x,\Psi) = u^E\big(u^{-M}\mathbf{A}^\mu(\Phi)u^{-M}\big) \tag{5.5a}$$

$$\mathbf{f}_{a,\mathfrak{A}}(x,\Psi) = u^{E-M}\Big(- \mathbf{A}^\mu(\Phi)\big(\tfrac{\partial}{\partial x^\mu}u^{-M}\big)\,\Psi + \mathbf{f}(\Phi) - \mathbf{A}^\mu(\Phi)\tfrac{\partial}{\partial x^\mu}\mathcal{M}_{a,\mathfrak{A}}\Big) \tag{5.5b}$$

$$\widehat{\mathbf{A}}^\mu_{a,\mathfrak{A}}(x,\Psi) = u^{E^\sharp}\big(u^{-M^\sharp}\,\widehat{\mathbf{A}}^\mu(\Phi)\,u^{-M^\sharp}\big) \tag{5.5c}$$

$$\widehat{\mathbf{f}}_{a,\mathfrak{A}}(x,\Psi,\partial_x\Psi) = u^{E^\sharp-M^\sharp}\Big(- \widehat{\mathbf{A}}^\mu(\Phi)\,\big(\tfrac{\partial}{\partial x^\mu}u^{-M^\sharp}\big) + \widehat{\mathbf{f}}(\Phi,\partial_x\Phi)\,u^{-M^\sharp}\Big) \tag{5.5d}$$

In (5.5), Φ, $\partial_x\Phi$ *have to be expressed in terms of* Ψ, $\partial_x\Psi$ *using* (5.3a)*. We will sometimes drop the* a, $\mathfrak{A}$ *and write* $\mathbf{A}(x,\Psi)$, $\mathbf{f}(x,\Psi)$, $\widehat{\mathbf{A}}(x,\Psi)$, $\widehat{\mathbf{f}}(x,\Psi,\partial_x\Psi)$. *They are notationally distinguished from* $\mathbf{A}(\Phi)$, $\mathbf{f}(\Phi)$, $\widehat{\mathbf{A}}(\Phi)$, $\widehat{\mathbf{f}}(\Phi,\partial_x\Phi)$ *by the number of arguments.*

Remark 5.1. The matrices $\mathbf{A}^\mu(x,\Psi)$, $\widehat{\mathbf{A}}^\mu(x,\Psi)$ are Hermitian, so that (5.4a) and (5.4b) are also symmetric hyperbolic. They are affine linear (over $\mathbb{R}$) functions of the field Ψ. The linear (over $\mathbb{R}$) transformation $\widehat{\mathbf{f}}(x,\Psi,\partial_x\Psi)$ depends affine linearly (over $\mathbb{R}$) on $\Psi \oplus \partial_x\Psi$. On the other hand, $\mathbf{f}(x,\Psi)$ is a quadratic polynomial in the components of Ψ, $\overline{\Psi}$ without constant term. There is no constant term, because $\mathcal{M}_{a,\mathfrak{A}}$ is a vacuum field. By direct inspection, neither derivatives of $\mathbf{e}_{a,\mathfrak{A}}$ nor derivatives of $\boldsymbol{\lambda}_{a,\mathfrak{A}}$ appear in the term $\mathbf{A}^\mu(\Phi)\frac{\partial}{\partial x^\mu}\mathcal{M}_{a,\mathfrak{A}}$. See, (2.5) and (4.2).

Definition 5.1. *Let* S *be defined as in equation* (4.4)*.*

- $\mathcal{P}$ *is a generic symbol for a quadratic polynomial in the components of the fields* Ψ *and* $\overline{\Psi}$ *without constant term, whose coefficients are (complex) polynomials in* $\frac{1}{u}$, $\mathfrak{A}$, S, $\mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda}_{a,\mathfrak{A}}$, $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$.
- $\mathcal{P}^\sharp$ *is a generic symbol for a polynomial in the components of the fields* Ψ *and* $\overline{\Psi}$ *and all their first order coordinate derivatives, whose coefficients are (complex) polynomials in* $\frac{1}{u}$, $\mathfrak{A}$, S, $\mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda}_{a,\mathfrak{A}}$, $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$, *and all their first order coordinate derivatives.*

We use the same symbols $\mathcal{P}$ *and* $\mathcal{P}^\sharp$ *for a vector or matrix all of whose entries are polynomials of this kind.*

Remark 5.2. The vector fields D, N and L corresponding to $\Phi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi$ are:

$$\begin{aligned} D &= -\tfrac{2}{u}\,\mathbf{e}_{a,\mathfrak{A}}\,\tfrac{\partial}{\partial\overline{\xi}} + \tfrac{2}{u^2}\,\mathbf{e}_{a,\mathfrak{A}}\,S\,\tfrac{\partial}{\partial\overline{\xi}} + \tfrac{1}{u^2}\big(f_1\tfrac{\partial}{\partial\xi^1} + f_2\tfrac{\partial}{\partial\xi^2}\big) \\ N &= \tfrac{\partial}{\partial u} + \tfrac{1}{u^3}\big(f_4\tfrac{\partial}{\partial\xi^1} + f_5\tfrac{\partial}{\partial\xi^2}\big) \\ L &= \tfrac{\partial}{\partial\underline{u}} + \tfrac{1}{u^2}\,f_3\,\tfrac{\partial}{\partial\underline{u}} \end{aligned} \tag{5.6}$$

Here, $\frac{\partial}{\partial\overline{\xi}} = \frac{1}{2}(\frac{\partial}{\partial\xi^1} + i\frac{\partial}{\partial\xi^2})$.

Below, $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$.

Proposition 5.2. *Let* $\Psi = (f, \omega, z)$*. The system* (5.4a) *takes the relevant / irrelevant form:*

$$L \begin{pmatrix} f_1 \\ f_2 \\ f_4 \\ f_5 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_8 \end{pmatrix} = \begin{pmatrix} \mathbf{e}\,\omega_1 \\ -i\,\mathbf{e}\,\omega_1 \\ 2\,\mathbf{e}\,\Re(\omega_4 - \overline{\omega}_3 - \omega_5) \\ -2\,\mathbf{e}\,\Im(\omega_4 - \overline{\omega}_3 - \omega_5) \\ -z_1 \\ -|\omega_1|^2 \\ -z_2 - \overline{\boldsymbol{\lambda}}\,\omega_1 \\ -\boldsymbol{\lambda}\,\overline{\omega}_1 \\ z_3 + 2i\,\Im\left(\boldsymbol{\lambda}\,(\omega_4 - \omega_5 - \overline{\omega}_3)\right) \end{pmatrix} + \frac{1}{u}\mathcal{P} \tag{5.7a}$$

$$N \begin{pmatrix} f_3 \\ \omega_5 \\ \omega_6 \\ \omega_7 \end{pmatrix} = \frac{1}{u} \begin{pmatrix} 2\,(f_3 + \Re\,\omega_8) \\ \omega_4 + \omega_5 - \overline{\omega}_3 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{u^2}\mathcal{P} \tag{5.7b}$$

$$\begin{pmatrix} N & \frac{1}{u}D & 0 & 0 & 0 \\ \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L & \frac{1}{u}D & 0 & 0 \\ 0 & \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L & \frac{1}{u}D & 0 \\ 0 & 0 & \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L & D \\ 0 & 0 & 0 & \overline{D} & L \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \frac{1}{u} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4\boldsymbol{\lambda} z_5 \\ \mathfrak{A}^2 z_5 - 2\overline{\boldsymbol{\lambda}} z_4 - 3\omega_7 z_3 \end{pmatrix} + \frac{1}{u^2}\mathcal{P} \tag{5.7c}$$

Proof. By direct (machine) calculation. □

Remark 5.3. We comment on the relevant / irrelevant equations (5.7):

- The terms containing the generic symbol $\mathcal{P}$ and the term $\frac{1}{u}\left(\mathfrak{A}^2 z_5 - 2\overline{\boldsymbol{\lambda}} z_4 - 3\omega_7 z_3\right)$ on the right hand side of (5.7c) are referred to as irrelevant, the rest are referred to as relevant. Observe that the relevant terms are all either principal in the number of derivatives or leading order in powers of $\frac{1}{u}$. In particular, in the last line of (5.7c), the term $L(z_5)$ is of order zero in powers of $\frac{1}{u}$, so that all the terms on the right hand side are irrelevant.
- If we just keep track of the terms that are leading order in powers of $\frac{1}{u}$ and ignore the number of derivatives they contain, then the differential operators D, N, L reduce to $-\frac{2}{u}\,\mathbf{e}\,\frac{\partial}{\partial\overline{\xi}}$ and $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial \underline{u}}$, see (5.6).
- The relevant terms on the right hand sides of (5.7) are linear in the components of the unknown $\Psi = (f, \omega, z)$ and their complex conjugates, with *exactly one* exception, namely $-|\omega_1|^2$ in the sixth line of (5.7a). This term generates the trapped spheres. Observe that this is the only equation in which ω_2 appears in the relevant part.
- The linear terms in the relevant part of the right hand sides of (5.7) are of two kinds. Either there is an explicit factor of $\mathbf{e}$, $\boldsymbol{\lambda}$, $\overline{\boldsymbol{\lambda}}$ or there is a numerical factor (besides powers of $\frac{1}{u}$). In the first case, we can make the factor small by requiring $|\mathfrak{A}| \le |a|$ and making $|a|$ small. In the second case, we arrange the terms into a linear over $\mathbb{R}$ matrix applied to Ψ. We exploit the structure of this matrix, it motivates some of the hypotheses of the energy estimate, see **(E9)**, **(E11a)**, **(E11b)** in Subsection 7.4.

Proposition 5.3. *Let* $\Psi^\sharp = (s,p,y)$*, see* (5.3b)*, and recall Definition 2.4. Then,*

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \mathcal{P}^\sharp \qquad \begin{pmatrix} s_4 \\ s_5 \end{pmatrix} = -u\,N\,\begin{pmatrix} \overline{f}_1 \\ \overline{f}_2 \end{pmatrix} + \mathcal{P}^\sharp \tag{5.8a}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathcal{P}^\sharp \qquad \begin{pmatrix} p_4 \\ p_7 \\ p_8 \end{pmatrix} = -u\,N\,\begin{pmatrix} \omega_1 \\ \omega_3 \\ -\omega_4 \end{pmatrix} + \mathcal{P}^\sharp \tag{5.8b}$$

$$\begin{pmatrix} p_5 \\ p_6 \\ p_9 \end{pmatrix} = \mathcal{P}^\sharp = \begin{pmatrix} -L(\omega_7) - \overline{\omega}_1 \\ L(\omega_6) \\ u\,D(\omega_7) - u\,\overline{D}(\omega_6) + \omega_4 - \overline{\omega}_3 - 4\boldsymbol{\lambda}\,\omega_7 \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp \tag{5.8c}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \mathcal{P}^\sharp = \begin{pmatrix} u\,\overline{D} - 4\overline{\boldsymbol{\lambda}} & L & 0 & 0 \\ \omega_7 & u\,\overline{D} - 2\overline{\boldsymbol{\lambda}} & L & 0 \\ 0 & 2\,\omega_7 & u\,\overline{D} & L \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp \tag{5.8d}$$

Proof. By direct (machine) calculation. □

Remark 5.4. Every generic symbol $\mathcal{P}^\sharp$ that appears in (5.8), has no constant term as a polynomial in the components of Ψ, $\overline{\Psi}$ and their first coordinate derivatives. There is no constant term, because $\mathcal{M}_{a,\mathfrak{A}}$ is a vacuum field.

Proposition 5.4. *Let* $\Psi^\sharp = (s,p,y)$*, see* (5.3b)*. The dual system* (5.4b) *takes the relevant / irrelevant form:*

$$L\begin{pmatrix} s_1 \\ s_2 \\ s_4 \\ s_5 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{e}\,(\overline{p}_4 - p_5) + \mathbf{e}\,(\overline{p}_6 - p_3) \\ i\,\mathbf{e}\,(\overline{p}_4 - p_5) - i\,\mathbf{e}\,(\overline{p}_6 - p_3) \\ y_1 \\ 0 \\ 0 \\ 0 \\ \boldsymbol{\lambda}(p_6 - \overline{p}_3) - \overline{\boldsymbol{\lambda}}(p_4 - \overline{p}_5) \\ -\overline{\boldsymbol{\lambda}}(\overline{p}_6 - p_3) + \boldsymbol{\lambda}(\overline{p}_4 - p_5) \end{pmatrix} + \frac{1}{u}\mathcal{P}^\sharp\Psi^\sharp \tag{5.9a}$$

$$N\begin{pmatrix} s_3 \\ p_5 \\ p_6 \\ p_9 \end{pmatrix} = \frac{1}{u}\begin{pmatrix} s_3 + p_7 + \overline{p}_8 \\ \overline{p}_4 \\ \overline{p}_3 \\ \overline{p}_7 + p_8 \end{pmatrix} + \frac{1}{u^2}\mathcal{P}^\sharp\Psi^\sharp \tag{5.9b}$$

$$\begin{pmatrix} N + \frac{1}{u^2}L & \frac{1}{u}D & 0 \\ \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L & \frac{1}{u}D \\ 0 & \frac{1}{u}\overline{D} & N + \frac{1}{u^2}L \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{u^2}\mathcal{P}^\sharp\Psi^\sharp \tag{5.9c}$$

Above, the symbols $\mathcal{P}^\sharp$ *are linear over* $\mathbb{R}$ *generic transformations, in the sense of Definition 5.1.*

Proof. By direct (machine) calculation. □

Remark 5.5. The overall factors u^E and $u^{E^\sharp}$ appear in (5.4a) and (5.4b), so that these systems are line by line (up to a permutation of the lines) equivalent to their relevant / irrelevant counterparts in Propositions 5.2 and 5.4.

6. Formal Solutions

In this section we consider formal power series

$$[\Psi](x) \;=\; \sum_{k=0}^{\infty} (\tfrac{1}{u})^k \, \Psi(k)(\xi, \underline{u}) \tag{6.1}$$

on $\mathbf{Strip}_\infty \subset \mathbb{R}^4$, see (4.1), where for each $k \geq 0$, the coefficient function $\Psi(k) = \Psi(k)(\xi, \underline{u})$ is a smooth field on $\mathbb{R}^2 \times (0, \infty)$ taking values in $\mathcal{R}$. By Proposition 5.3, the associated formal constraint field $[\Psi^\sharp]$ is itself a formal power series in $\frac{1}{u}$, that is, $[\Psi^\sharp](x) = \sum_{k=0}^{\infty} (\frac{1}{u})^k \, \Psi^\sharp(k)(\xi, \underline{u})$.

Remark 6.1. It also follows from Proposition 5.3 that, for each $k \geq 0$, the coefficient function $\Psi^\sharp(k)$ depends only on $\Psi(\ell)$, $0 \leq \ell \leq k$.

The characteristic initial problem in Proposition 6.1 is motivated by [Chr].

Proposition 6.1. *For all $a, \mathfrak{A} \neq 0$, $\underline{u}_0 > 0$, all smooth $\mathbf{DATA}(\xi, \underline{u}) : \mathbb{R}^2 \times (0, \infty) \to \mathbb{C}$ that vanish when $\underline{u} < \underline{u}_0$, there is a unique formal power series $[\Psi]$ on $\mathbf{Strip}_\infty$, which satisfies* (5.4a) *and $[\Psi^\sharp] = 0$ and (the formal characteristic initial conditions)*

$$[\Psi] = 0 \;\; \textit{when } \underline{u} < \underline{u}_0 \qquad\qquad \omega_1(0) = \mathbf{DATA} \tag{6.2}$$

Moreover, for all $k \geq 0$, the value of $\Psi(k)$ at $(\xi, \underline{u}) \in \mathbb{R}^2 \times (0, \infty)$ depends only on the restriction of $\mathbf{DATA}(\xi, \underline{u})$ and its derivatives of all orders to the half-open line segment $\{\xi\} \times (0, \underline{u}]$ (formal finite speed of propagation). Explicitly, $\Psi(0)$ is given by:

$$\begin{aligned}
\omega_1(0) &= \mathbf{DATA} & z_5(0) &= 0 \\
\omega_7(0) &= -\partial_{\underline{u}}^{-1} \overline{\omega_1(0)} & \omega_2(0) &= -\partial_{\underline{u}}^{-1} |\omega_1(0)|^2 \\
z_1(0) &= -\tfrac{\partial}{\partial \underline{u}} \omega_1(0) & \omega_4(0) &= -\boldsymbol{\lambda}\, \partial_{\underline{u}}^{-1} \overline{\omega_1(0)} \\
z_2(0) &= 2\big(\mathbf{e} \tfrac{\partial}{\partial \xi} + 2\overline{\boldsymbol{\lambda}}\big)\, \partial_{\underline{u}}^{-1} z_1(0) & \omega_6(0) &= 0 \\
z_3(0) &= 2\big(\mathbf{e} \tfrac{\partial}{\partial \xi} + \overline{\boldsymbol{\lambda}}\big)\, \partial_{\underline{u}}^{-1} z_2(0) - \partial_{\underline{u}}^{-1} \big(\omega_7(0) z_1(0)\big) & f_1(0) &= \,\mathbf{e}\, \partial_{\underline{u}}^{-1} \omega_1(0) \\
z_4(0) &= 2\, \mathbf{e} \tfrac{\partial}{\partial \xi}\, \partial_{\underline{u}}^{-1} z_3(0) - 2 \partial_{\underline{u}}^{-1} \big(\omega_7(0) z_2(0)\big) & f_2(0) &= -i\, \mathbf{e}\, \partial_{\underline{u}}^{-1} \omega_1(0) \\
\omega_3(0) &= -\partial_{\underline{u}}^{-1} z_2(0) - \overline{\boldsymbol{\lambda}}\, \partial_{\underline{u}}^{-1} \omega_1(0) & f_3(0) &= -\Re\, \omega_8(0) \\
\omega_5(0) &= -\partial_{\underline{u}}^{-1} \overline{z_2(0)} & f_4(0) &= -4\, \mathbf{e}\, \partial_{\underline{u}}^{-1} \Re\, \omega_5(0) \\
\omega_8(0) &= \partial_{\underline{u}}^{-1} z_3(0) - 4i\, \partial_{\underline{u}}^{-1} \Im\big(\boldsymbol{\lambda}\, \omega_5(0)\big) & f_5(0) &= 4\, \mathbf{e}\, \partial_{\underline{u}}^{-1} \Im\, \omega_5(0)
\end{aligned} \tag{6.3}$$

where $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$ are defined in (4.3) *and $\frac{\partial}{\partial \xi} = \frac{1}{2}\big(\frac{\partial}{\partial \xi^1} - i \frac{\partial}{\partial \xi^2}\big)$, and*

$$\big(\partial_{\underline{u}}^{-1} g\big)(\underline{u}) = \int_0^{\underline{u}} \mathrm{d}\underline{u}'\, g\big(\underline{u}'\big). \tag{6.4}$$

We now prepare for the proof of Proposition 6.1, which appears on page 24.

Definition 6.1. *Let* $a, \mathfrak{A} \neq 0$. *Let*

$$[\,\mathcal{M}_{a,\mathfrak{A}}\,](x) \;=\; \sum_{k=0}^{\infty} (\tfrac{1}{u})^k\, \mathcal{M}_{a,\mathfrak{A}}(k)(\xi, \underline{u})$$

be the formal expansion in $\frac{1}{u}$ *for the Minkowski vacuum field* $\mathcal{M}_{a,\mathfrak{A}}$, *see* (4.2), *in which*

$$[\,\tfrac{1}{\rho}\,] \;=\; -\tfrac{1}{u} + \tfrac{1}{u^2}\,[\,S\,] \qquad\qquad [\,S\,] \;=\; -\sum_{k=0}^{\infty} (\tfrac{1}{u})^k\, \mathfrak{A}^{2(k+1)}\, \underline{u}^{k+1}. \tag{6.5}$$

Definition 6.2. *Regard the components of* $\Psi(k)$ *and* $\overline{\Psi(k)}$, $k \geq 0$, *and their formal first coordinate derivatives, as an infinite family of independent abstract variables.*
Set $\mathcal{P}_0 = 0$. *The generic symbol* $\mathcal{P}_k$, $k \geq 1$, *is an arbitrary polynomial in the components of* $\Psi(\ell)$ *and* $\overline{\Psi(\ell)}$, $0 \leq \ell \leq k-1$, *and all their first coordinate derivatives* ($\frac{\partial}{\partial x^\mu}\Psi(\ell)$ *and* $\frac{\partial}{\partial x^\mu}\overline{\Psi(\ell)}$, $\mu = 1,2,3$)*, whose coefficients are (complex) polynomials in* $\mathfrak{A}$, $\underline{u}$, $\mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda}_{a,\mathfrak{A}}$, $\overline{\boldsymbol{\lambda}_{a,\mathfrak{A}}}$, *and all their first coordinate derivatives. It is further required that the polynomial* $\mathcal{P}_k$ *have no constant term, that is,* $\mathcal{P}_k$ *vanishes when* $\Psi(\ell)$ *and* $\frac{\partial}{\partial x^\mu}\Psi(\ell)$ *vanish for all* $0 \leq \ell \leq k-1$ *and* $\mu = 1,2,3$. *We use the same symbol* $\mathcal{P}_k$ *for a vector or matrix all whose entries are polynomials of this kind.*

Proposition 6.2. *Substitute* $[\,\mathcal{M}_{a,\mathfrak{A}}\,]$ *(see, Definition 6.1) for* $\mathcal{M}_{a,\mathfrak{A}}$ *and* $[\,\Psi\,]$ *(see,* (6.1)*) for* Ψ *in* (5.4a). *Then* $[\,\Psi\,]$ *is a formal power series solution to* (5.4a) *if and only if its coefficients* $\Psi(k)$, $k \geq 0$, *satisfy a system of the form*

$$\begin{aligned}
z_1(k) &= \mathcal{P}_k && k > 0 && \text{(6.6a)}\\
z_2(k) &= \mathcal{P}_k && k > 0 && \text{(6.6b)}\\
z_3(k) &= \mathcal{P}_k && k > 0 && \text{(6.6c)}\\
\tfrac{\partial}{\partial \underline{u}}\, z_5(k) &= \mathcal{P}_k && k \geq 0 && \text{(6.6d)}\\
(1-\delta_{k0}) z_4(k) &= -\tfrac{2}{k-\delta_{k0}}\left(\mathbf{e}\,\tfrac{\partial}{\partial \overline{\xi}} + 2\boldsymbol{\lambda}\right) z_5(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6e)}\\
\tfrac{\partial}{\partial \underline{u}}\, \omega_1(k) &= -z_1(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6f)}\\
\tfrac{\partial}{\partial \underline{u}}\, \omega_2(k) &= -(2-\delta_{k0})\, \Re\big(\omega_1(0)\,\overline{\omega_1(k)}\,\big) + \mathcal{P}_k && k \geq 0 && \text{(6.6g)}\\
\tfrac{\partial}{\partial \underline{u}}\, \omega_3(k) &= -z_2(k) - \overline{\boldsymbol{\lambda}}\, \omega_1(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6h)}\\
\tfrac{\partial}{\partial \underline{u}}\, \omega_4(k) &= -\boldsymbol{\lambda}\, \overline{\omega_1(k)} + \mathcal{P}_k && k \geq 0 && \text{(6.6i)}\\
\omega_5(k) &= -\tfrac{1}{k+1}\big(\omega_4(k) - \overline{\omega_3(k)}\,\big) + \mathcal{P}_k && k \geq 0 && \text{(6.6j)}\\
\omega_6(k) &= \mathcal{P}_k && k > 0 && \text{(6.6k)}\\
\omega_7(k) &= \mathcal{P}_k && k > 0 && \text{(6.6l)}\\
\tfrac{\partial}{\partial \underline{u}}\, \omega_8(k) &= z_3(k) + 2i\Im\left(\boldsymbol{\lambda}\big(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\big)\right) + \mathcal{P}_k && k \geq 0 && \text{(6.6m)}\\
\tfrac{\partial}{\partial \underline{u}}\, f_1(k) &= \quad\;\; \mathbf{e}\, \omega_1(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6n)}\\
\tfrac{\partial}{\partial \underline{u}}\, f_2(k) &= -i\,\mathbf{e}\, \omega_1(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6o)}\\
f_3(k) &= -\tfrac{2}{k+2}\, \Re\, \omega_8(k) + \mathcal{P}_k && k \geq 0 && \text{(6.6p)}\\
\tfrac{\partial}{\partial \underline{u}}\, f_4(k) &= \quad 2\,\mathbf{e}\, \Re\left(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\right) + \mathcal{P}_k && k \geq 0 && \text{(6.6q)}\\
\tfrac{\partial}{\partial \underline{u}}\, f_5(k) &= -2\,\mathbf{e}\, \Im\left(\omega_4(k) - \overline{\omega_3(k)} - \omega_5(k)\right) + \mathcal{P}_k && k \geq 0 && \text{(6.6r)}
\end{aligned}$$

Here, $\mathbf{e} = \mathbf{e}_{a,\mathfrak{A}}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{a,\mathfrak{A}}$. *In* (6.6g), (6.6k), (6.6p), (6.6q), (6.6r), *the generic symbol* $\mathcal{P}_k$ *is real valued when* $\Psi(\ell)(\xi, \underline{u}) \in \mathcal{R}$ *for all* $0 \leq \ell \leq k$.

Proof. Substitute the formal series (6.1) into the relevant / irrelevant form of system (5.4a) given in Proposition 5.2. Collect all coefficients of common powers of $\frac{1}{u}$. □

Lemma 6.1 just below is simpler than Proposition 6.1, because it *assumes* equations (6.3) and it makes no statement about the formal constraint field $[\,\Psi^\sharp\,]$.

Lemma 6.1. *For all* $a, \mathfrak{A} \neq 0$, $\underline{u}_0 > 0$, *all smooth* **DATA**$(\xi, \underline{u}) : \mathbb{R}^2 \times (0,\infty) \to \mathbb{C}$ *that vanish when* $\underline{u} < \underline{u}_0$, *there is a unique formal power series* $[\,\Psi\,]$ *on* **Strip**$_\infty$, *which satisfies* (5.4a) *and*

$$[\,\Psi\,] = 0 \;\; \textit{when } \underline{u} < \underline{u}_0 \qquad\qquad \Psi(0) \textit{ is given by } (6.3) \tag{6.7}$$

Proof. $\Psi(0)$, as given by (6.3), satisfies the $k = 0$ equations in (6.6). The coefficient functions $\Psi(k)$, $k \geq 1$, are constructed by induction. For each step k, equations (6.6a) to (6.6r) are solved exactly in this order to obtain $\Psi(k)$. The right hand side is explicitly known by induction and the "upper triangular" structure of (6.6a) to (6.6r). Whenever $\frac{\partial}{\partial \underline{u}}$ appears on the left hand side, it is inverted using $\partial_{\underline{u}}^{-1}$, because the constant of integration is zero by the first condition in (6.7). By induction, one also verifies that $\Psi(k)$, $k \geq 0$, vanishes when $\underline{u} < \underline{u}_0$, so that the first condition in (6.7) is satisfied at all orders. It is essential at precisely this point that the generic polynomial $\mathcal{P}_k$ in Definition 6.2 has no constant term. Finally, by Proposition 6.2, there exists a formal power series solutions satisfying the hypothesis of the lemma. The construction given here is forced at every step, and therefore generates a unique formal power series. □

Proof (of Proposition 6.1). We first prove existence. It suffices to show that the formal power series $[\,\Psi\,]$ produced by Lemma 6.1 satisfies $[\,\Psi^\sharp\,] = 0$. (The formal finite speed of propagation statement in Proposition 6.1 follows from an examination of the construction of $[\,\Psi\,]$ in the proof of Lemma 6.1.) Note that

- $[\,\Psi^\sharp\,]$ is a formal power series solution to the linear homogeneous system (5.4b).
- $[\,\Psi^\sharp\,] = 0$ when $\underline{u} < \underline{u}_0$.
- $\Psi^\sharp(0) = 0$ on $\mathbb{R}^2 \times (0,\infty)$.

The first bullet follows from Proposition 2.3, because $[\,\Psi\,]$ is a formal power series solution to (5.4a). The second bullet follows from the first condition in (6.7), which implies $[\,\Phi\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,]$ when $\underline{u} < \underline{u}_0$, and $[\,\Phi^\sharp\,] = [\,\mathcal{M}^\sharp_{a,\mathfrak{A}}\,] = 0$. For the third bullet, note that $p_5(0)$, $y_1(0)$, $y_2(0)$, $y_3(0)$, $p_6(0)$ all vanish on $\mathbb{R}^2 \times (0,\infty)$ by the second condition in (6.7). By the first two bullets and by equation (5.9a), we conclude, step by step, that $s_1(0)$, $s_2(0)$, $p_1(0)$, $p_2(0)$, $p_3(0)$, $p_4(0)$, $s_4(0)$, $s_5(0)$, $p_7(0)$, $p_8(0)$ also vanish. The first equation in (5.9b) gives $s_3(0) = 0$. It remains to show that $p_9(0) = 0$ on $\mathbb{R}^2 \times (0,\infty)$. By (5.8c),

$$p_9(0) = -2\big(\mathbf{e}\,\tfrac{\partial}{\partial \overline{\xi}} + 2\,\boldsymbol{\lambda}\big)\,\omega_7(0) + 2\,\mathbf{e}\,\tfrac{\partial}{\partial \xi}\omega_6(0) + \omega_4(0) - \overline{\omega_3(0)}.$$

The second condition in (6.7) implies $(\frac{\partial}{\partial \underline{u}})^2 p_9(0) = 0$. By the second bullet, $p_9(0) \equiv 0$.

The three bullets imply, by induction on $k \geq 1$, that $\Psi^\sharp(k) = 0$ on $\mathbb{R}^2 \times (0,\infty)$. In fact, at each step k, one verifies, in the given order, that $y_1(k)$, $y_2(k)$, $y_3(k)$ all vanish by (5.9c), $p_1(k)$, $p_2(k)$, $p_3(k)$, $p_4(k)$ all vanish by (5.9a), $p_5(k)$, $p_6(k)$ both vanish by (5.9b), $p_7(k)$, $p_8(k)$ both vanish by (5.9a), $p_9(k)$, $s_3(k)$ both vanish by (5.9b), and $s_1(k)$, $s_2(k)$, $s_4(k)$, $s_5(k)$ all vanish by (5.9a). This concludes the existence proof.

Uniqueness in Lemma 6.1 implies uniqueness in Proposition 6.1, because we now show that (5.4a) and $[\Psi^\sharp] = 0$ and (6.2) together imply (6.3), which is the second condition in (6.7). Condition (6.2) and the $k = 0$ equations in (6.6) imply (6.3), apart from the formulas for $\omega_7(0)$, $z_2(0)$, $z_3(0)$, $z_4(0)$, $\omega_6(0)$. The remaining five formulas follow from the vanishing of $p_5(0)$, $y_1(0)$, $y_2(0)$, $y_3(0)$ and $p_6(0)$, see (5.8c) and (5.8d). Here, $\Psi^\sharp(0) = (s(0), p(0), y(0))$. $\square$

Proposition 6.3. *For all $k, R \geq 0$, all $0 < |\mathfrak{A}| \leq |a| \leq 1$, and all* **DATA**,

$$\|\Psi(k)\|_{C^R(\mathcal{Q})} \;\leq\; p_{k,R}\big(\|\textbf{DATA}\|_{C^{R+2k+3}(\mathcal{Q})}\big) \qquad\qquad \mathcal{Q} = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0,2)$$

where $[\Psi]$ is the corresponding formal solution in Proposition 6.1, and $p_{k,R} : \mathbb{R} \to \mathbb{R}$, is an infinite family, indexed by $k, R \geq 0$, of universal polynomials without constant term. Here, $D_r(0)$ is the open disk of radius $r > 0$ in the (ξ^1, ξ^2)-plane. Here $\|\Psi(k)\|_{C^R(\mathcal{Q})} = \sup_{|\alpha| \leq R} \|\partial^\alpha \Psi(k)\|_{C^0(\mathcal{Q})}$, where $\alpha \in \mathbb{N}_0^3$.

Remark 6.2. The uniformity of the estimate in a, $\mathfrak{A}$, when $0 < |\mathfrak{A}| \leq |a| \leq 1$, will be exploited later. In particular, it is compatible with taking the limit $a = \mathfrak{A} \downarrow 0$.

Proof. Observe that:

- $\|\mathbf{e}_{a,\mathfrak{A}}\|_{C^R(\mathcal{Q})} \leq \frac{17}{2}$ and $\|\boldsymbol{\lambda}_{a,\mathfrak{A}}\|_{C^R(\mathcal{Q})} \leq \frac{17}{2}$ for all $R \geq 0$.
- $\|\partial_{\underline{u}}^{-1} g\|_{C^R(\mathcal{Q})} \leq 2\,\|g\|_{C^R(\mathcal{Q})}$ for all $R \geq 0$ and all functions $g(\xi, \underline{u})$ on $\mathcal{Q}$.

The existence of polynomials $p_{0,R}$, $R \geq 0$, follow by direct inspection of (6.3). The existence of polynomials $p_{k,R}$, $R \geq 0$, is shown by induction over $k \geq 0$. At each step $k \geq 1$, we use (6.6). By the inductive hypothesis and Definition 6.2 there is a polynomial $p'_{k,R}$ (depending only on k and R) so that each generic term $\mathcal{P}_k$ on the right hand sides of (6.6) satisfies $\|\mathcal{P}_k\|_{C^R(\mathcal{Q})} \leq p'_{k,R}\big(\|\textbf{DATA}\|_{C^{R+2k+2}(\mathcal{Q})}\big)$. We can assume that $p'_{k,R}$ has no constant term, because $\mathcal{P}_k$ does not have one (see, Definition 6.2). Now, the existence of $p_{k,R}$, $R \geq 0$ follows directly from estimating the non generic terms on the right hand sides of (6.6a) to (6.6r), exploiting the upper triangular structure. Only in one equation, (6.6e), a coordinate derivative appears. $\square$

Remark 6.3. Fix **DATA** and let $[\Psi_{a,\mathfrak{A}}]$ be the formal power series solution in Proposition 6.1. The indices have been added to make the dependence on $a, \mathfrak{A} \neq 0$ explicit. One can show, by induction, that $\Psi_{\mathfrak{A},\mathfrak{A}}(k)(\xi, \underline{u})$, $k \geq 0$, are polynomials in $\mathfrak{A}$. Just follow the construction of $[\Psi_{\mathfrak{A},\mathfrak{A}}]$ given in the proof of Lemma 6.1, and use the observation that $\mathbf{e}_{\mathfrak{A},\mathfrak{A}}$ and $\boldsymbol{\lambda}_{\mathfrak{A},\mathfrak{A}}$ are polynomials in $\mathfrak{A}$.

Let $\mathfrak{P}$ be the parity field symmetry, see Remark 2.9. Then $\mathfrak{P} \cdot [\Psi_{\mathfrak{A},\mathfrak{A}}] = [\Psi_{-\mathfrak{A},-\mathfrak{A}}]$. This is a direct consequence of $\mathfrak{P} \cdot \mathcal{M}_{\mathfrak{A},\mathfrak{A}} = \mathcal{M}_{-\mathfrak{A},-\mathfrak{A}}$, the uniqueness statement in Proposition 6.1 and the fact that $\omega_1(0)$ is $\mathfrak{P}$-even. Therefore, the $\mathfrak{P}$-even ($\mathfrak{P}$-odd) components of $\Psi_{\mathfrak{A},\mathfrak{A}}(k)$, $k \geq 0$, are even (odd) polynomials in $\mathfrak{A}$.

Let $[\Psi_{0,0}] = \lim_{\mathfrak{A}\downarrow 0} [\Psi_{\mathfrak{A},\mathfrak{A}}]$ (the limit is taken coefficient by coefficient). We have $[\Phi_{0,0}] = [\mathcal{M}_{0,0}] + u^{-M}[\Psi_{0,0}]$. The $\mathfrak{P}$-odd components all vanish. By inspection, the e_4, e_5 components also vanish (see, Remark 2.9). Therefore, $[\Phi_{0,0}]$ satisfies the hypothesis of Proposition 2.4, in the sense of formal power series. The field $[\widetilde{\Phi}_{0,0}]$ is a formal solution to **(subSHS)** , and $[\widetilde{\Phi}^\sharp_{0,0}] = 0$.

We now match constructions between two stereographic charts.

Proposition 6.4. *Choose* $a, \mathfrak{A} \neq 0$*. Pick* $\mathbf{DATA}^{\sigma}(\xi, \underline{u})$ *as in Proposition 6.1, for* $\sigma = -, +$*, and let* $[\,\Psi^{\sigma}\,]$ *be the associated solution in Proposition 6.1. The following statements are equivalent:*

- $\frac{|\xi|^2}{\xi^2}\mathbf{DATA}^{\sigma}\big(\frac{a}{\mathfrak{A}}\xi,\, \underline{u}\big) \;=\; \frac{\xi^2}{|\xi|^2}\,\mathbf{DATA}^{-\sigma}\big(\frac{a}{\mathfrak{A}}\,\frac{1}{\xi},\, \underline{u}\big)$ *when* $\xi \neq 0$*.*
- $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot [\,\Phi^{\sigma}\,] = [\,\Phi^{-\sigma}\,]$ *when* $\xi \neq 0$*. Here,* $[\,\Phi^{\sigma}\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,] + u^{-M}[\,\Psi^{\sigma}\,]$*.*
- $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot [\,\Psi^{\sigma}\,] = [\,\Psi^{-\sigma}\,]$ *when* $\xi \neq 0$*.*

Here, $-\sigma = +$ *when* $\sigma = -$*, and conversely,* $-\sigma = -$ *when* $\sigma = +$*.*

Proof. The equivalence of the last two bullets follows from Proposition 4.1, (b), and the fact that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ commutes with multiplication by u^{-M}. Each of the last two bullets implies the first. Just look at how $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ acts on the component ω_1. The first bullet implies the last two, because $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is a field symmetry, and by uniqueness in Proposition 6.1 (more precisely, by formal finite speed of propagation). □

It is convenient (see Subsection 8.2) to make the

Definition 6.3. *For all* $(\xi, \underline{u}) \in \mathbb{R}^2 \times (0, \infty)$ *with* $\xi \neq 0$*, set*

$$\big(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathbf{DATA}\big)(\xi, \underline{u}) \;=\; \frac{\xi^2}{\bar{\xi}^2}\,\mathbf{DATA}\big(\frac{a^2}{\mathfrak{A}^2}\,\frac{1}{\xi},\, \underline{u}\big)$$

Remark 6.4. Proposition 6.1, the main result of this section, is analogous to Theorem 8.1, the main theorem of this paper. Proposition 6.1 concerns formal vacuum fields $[\,\Phi\,] = [\,\mathcal{M}_{a,\mathfrak{A}}\,] + u^{-M}[\,\Psi\,]$, Theorem 8.1 concerns classical vacuum fields $\Phi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi$. They are solutions to an asymptotic characteristic initial value problem that is motivated by [Chr]. Informally:

$$\lim_{u \to -\infty} \Psi(\xi, \underline{u}, u) \;=\; \Psi(0)(\xi, \underline{u}) \tag{6.8a}$$

$$\Psi(\xi, \underline{u}, u) \;=\; 0 \qquad \text{when } \underline{u} < \underline{u}_0 \tag{6.8b}$$

with the understanding that $\Psi(0)$ is given in terms of $\mathbf{DATA}(\xi, \underline{u})$ by equations (6.3). Equation (6.8b) stipulates that Φ coincides with the Minkowski vacuum field $\mathcal{M}_{a,\mathfrak{A}}$ when $\underline{u} < \underline{u}_0$. On the other hand, (6.8a) is an asymptotic initial condition at “past null infinity” $u \to -\infty$. *At this point, all the notation, definitions and concepts required for Theorem 8.1 have been introduced. In can now be read on its own.*

7. Energy Estimates

In this section, we prove an abstract local existence theorem for a general class of quasilinear symmetric hyperbolic systems, with a concrete breakdown criterion. Then, we develop appropriate energy estimates. These tools are applied in Section 8.

Convention 7.1. In this section,

$$\begin{aligned} x &= (x^1,\, x^2,\, x^3,\, x^4) = (\xi^1,\, \xi^2,\, \underline{u},\, u) \\ q &= (q^0,\, q^1,\, q^2,\, q^3) = (t,\, \xi^1,\, \xi^2,\, \underline{u}) \\ t &= u + \underline{u} \\ \mathbf{q} &= (q^1,\, q^2,\, q^3) = (\xi^1,\, \xi^2,\, \underline{u}) \end{aligned}$$

Let $D_r(\xi) \subset \mathbb{R}^2$ be the open disk of radius $r > 0$ around ξ and, generally, $B_r(p) \subset \mathbb{R}^N$ be the open ball of radius $r > 0$ around the point p.

For any parameter vector $a = (a_1, \ldots, a_k) \in (\mathbb{R}_+)^k$, the notation $X \lesssim_a Y$ signifies that $X \leq C\,Y$ for a constant $C = C(a) > 0$ that depends only on a. Dropping the subscript a, the notation $X \lesssim Y$ means $Y \leq C\,Y$ for a universal constant $C > 0$.

7.1. Sobolev inequality.

Lemma 7.1. *Let* $\partial_{\mathbf{q}} = (\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3})$. *If* $b \in (0, 2]$, *then for all* C^2*-functions* $f(\mathbf{q}) = f(q^1,\, q^2,\, q^3)$ *on the cylinder* $\mathbf{CYL} = D_{1/4}(0) \times (0, b) \subset \mathbb{R}^3$ *which vanish for* $q^3 < 1/4$,

$$\sup_{\mathbf{q} \in \mathbf{CYL}} |f(\mathbf{q})| \;\lesssim\; \Big(\sum_{|\alpha|=2} \|\partial_{\mathbf{q}}^{\alpha} f\|^2_{L^2(\mathbf{CYL})} \Big)^{1/2}, \qquad \alpha \in \mathbb{N}_0^3.$$

Proof. Let $B = D_{1/4}(0) \times \{0\}$ be the base of $\mathbf{CYL}$ and S^2 the unit sphere in $\mathbb{R}^3$. For each $\mathbf{q} \in \mathbf{CYL}$, let $\mathit{\Gamma}_{\mathbf{q}} \subset S^2$ be the set of all quotients $\boldsymbol{\zeta} = \frac{\mathbf{p}-\mathbf{q}}{|\mathbf{p}-\mathbf{q}|}$ where $\mathbf{p} \in B$. Set $l(\boldsymbol{\zeta}) = |\mathbf{p} - \mathbf{q}|$. We have

$$|f(\mathbf{q})| \;\leq\; \frac{1}{|\mathit{\Gamma}_{\mathbf{q}}|_{S^2}} \int_{\mathit{\Gamma}_{\mathbf{q}} \subset S^2} \mathrm{d}A_{S^2}(\boldsymbol{\zeta})\, F(\boldsymbol{\zeta})$$

where

$$F(\boldsymbol{\zeta}) \;=\; \int_0^{l(\boldsymbol{\zeta})} \mathrm{d}r\; r\, \Big| \Big\langle \boldsymbol{\zeta},\ \mathrm{H}(f)\big(\mathbf{q} + r\boldsymbol{\zeta}\big)\, \boldsymbol{\zeta} \Big\rangle \Big|$$

since, Taylor's theorem and the support properties of f imply $|f(\mathbf{q})| \leq F(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \mathit{\Gamma}_{\mathbf{q}}$. Here, $H(f)$ is the Hessian of f. Let $C_{\mathbf{q}} \subset \mathbb{R}^3$ be the convex hull of $B \cup \{\mathbf{q}\}$. By the Schwarz inequality,

$$|f(\mathbf{q})| \;\leq\; \frac{1}{|\mathit{\Gamma}_{\mathbf{q}}|_{S^2}} \Big(\int_{\mathit{\Gamma}_{\mathbf{q}} \subset S^2} \mathrm{d}A_{S^2}(\boldsymbol{\zeta}) \int_0^{l(\boldsymbol{\zeta})} 1\, \mathrm{d}r \Big)^{1/2} \Big(\int_{C_{\mathbf{q}}} \mathrm{d}^3\mathbf{y}\, |H(f)(\mathbf{y})|^2 \Big)^{1/2}$$

where, r^2 has disappeared into the measure $\mathrm{d}^3\mathbf{y}$ and $|M| = (\mathrm{tr}\, M^T M)^{1/2}$ is the Euclidean matrix norm. Also, observe that $l(\boldsymbol{\zeta}) \leq 3$ and $|\mathit{\Gamma}_{\mathbf{q}}|_{S^2}$ is bounded below by a universal constant, for instance $\pi/100$. By construction, $C_{\mathbf{q}} \subset \mathbf{CYL}$, and the proof is finished. $\square$

Lemma 7.2. *Let* $b \in [1, 2]$. *Then for all* C^2*-functions* $f(\mathbf{q}) = f(q^1, q^2, q^3)$ *on the cylinder* $\mathbf{CYL} = D_{1/4}(0) \times (0, b) \subset \mathbb{R}^3$,

$$\sup_{\mathbf{q} \in \mathbf{CYL}} |f(\mathbf{q})| \;\lesssim\; \Big(\sum_{|\alpha|\leq 2} \|\partial_{\mathbf{q}}^{\alpha} f\|^2_{L^2(\mathbf{CYL})} \Big)^{1/2}, \qquad \alpha \in \mathbb{N}_0^3. \tag{7.1}$$

Proof. By reflection symmetry, it suffices to show (7.1) when $\mathbf{q} \in \mathbf{CYL}$ satisfies $q^3 \geq \frac{b}{2} \geq \frac{1}{2}$. Fix a smooth transition function $\psi = \psi(q^3) : \mathbb{R} \to [0, 1]$ equal to 0 on $(-\infty, \frac{1}{4}]$ and equal to 1 on $[\frac{1}{2}, \infty)$. Then,

$$\begin{aligned} |f(\mathbf{q})|^2 = |(\psi f)(\mathbf{q})|^2 \quad &\overset{\text{Lemma 7.1}}{\lesssim} \quad \sum_{|\alpha|=2} \|\partial_{\mathbf{q}}^{\alpha}(\psi f)\|^2_{L^2(\mathbf{CYL})} \\ &\lesssim \;\; \|\psi\|^2_{C^2([0,1])} \sum_{|\alpha|\leq 2} \|\partial_{\mathbf{q}}^{\alpha} f\|^2_{L^2(\mathbf{CYL})} \qquad \square \end{aligned}$$

7.2. Finite speed of propagation for a general class of symmetric hyperbolic systems. We show finite speed of propagation in the context of the following hypotheses:

(FS0) $\mathcal{U} \subset \mathbb{R}^4$ is open and $\mathcal{A} = \mathcal{U} \times B_r(0)$ where $B_r(0) \subset \mathbb{R}^P$, $0 < r \leq +\infty$, $P \in \mathbb{N}$.
(FS1) $\mathbf{M}^\mu(q, \Theta)$, $\mu = 0, 1, 2, 3$, is a symmetric $P \times P$ matrix on $\mathcal{A}$. Moreover, $h(q, \Theta)$ is an $\mathbb{R}^P$ valued function on $\mathcal{A}$. Both $\mathbf{M}^\mu$ and h are C^1 on $\mathcal{A}$ and extend, with their derivatives, continuously to $\overline{\mathcal{A}}$, and $\mathbf{M}^0 > 0$ on $\overline{\mathcal{A}}$.
(FS2) Θ_1, Θ_2 are C^1-functions on $\mathcal{U}$ with values in $B_r(0) \subset \mathbb{R}^P$, that are solutions to the symmetric hyperbolic system

$$\mathbf{M}(q, \Theta)\Theta = h(q, \Theta), \qquad \mathbf{M} = \mathbf{M}^\mu \tfrac{\partial}{\partial q^\mu}. \tag{7.2}$$

Both Θ_1, Θ_2 extend, with their derivatives, continuously to $\overline{\mathcal{U}}$.
(FS3) $\mathcal{U}$ is either the set **Cone** or the intersection **Cone** $\cap$ ($\mathbb{R} \times$ **Half**).

Above, all the quantities are real. **Cone** is any set of the form

$$\big\{(t, \mathbf{q}) \in \mathbb{R}^4 : \ |\mathbf{q} - \mathbf{q}_0|_{\mathbb{R}^3} < v|t_1 - t|, \ t \in (t_0, t_1)\big\} \tag{7.3}$$

where $v > 0$, $\mathbf{q}_0 \in \mathbb{R}^3$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ are arbitrary, and **Half** is any open half-space in $\mathbb{R}^3$. We refer to v as the velocity of the set **Cone**.

Lemma 7.3. *Suppose* **(FS0)**, **(FS1)**, **(FS2)**, *then the difference $\Upsilon = \Theta_2 - \Theta_1$ satisfies the linear homogeneous symmetric hyperbolic system*

$$\mathbf{M}(q)\Upsilon = H(q)\Upsilon \tag{7.4}$$

where $\mathbf{M}(q) = \mathbf{M}(q, \Theta_1(q))$ and $H(q)$ is a square matrix. Moreover, $\mathbf{M}(q)$, its first derivatives and $H(q)$ are continuous on $\mathcal{U}$ and extend continuously to $\overline{\mathcal{U}}$.

Proof. Adding and subtracting,

$$\mathbf{M}(q, \Theta_1)\Upsilon = -\big(\mathbf{M}(q, \Theta_2) - \mathbf{M}(q, \Theta_1)\big)\Theta_2 + h(q, \Theta_2) - h(q, \Theta_1).$$

Set $\Theta_s = (1-s)\,\Theta_1 + s\,\Theta_2$, and

$$H_{ij}(q) = -\sum_k \Big(\int_0^1 \mathrm{d}s\, \frac{\partial (\mathbf{M}^\mu)_{ik}}{\partial \Theta^j}(q, \Theta_s(q)) \Big) \frac{\partial \Theta_2^k}{\partial q^\mu}(q) + \int_0^1 \mathrm{d}s\, \frac{\partial h_i}{\partial \Theta^j}(q,\, \Theta_s(q)).$$

The proposition follows from the fundamental theorem of calculus. □

Suppose **(FS3)**. In this case, let

$$\begin{aligned} S &= (\partial \mathcal{U}) \cap \big((t_0, t_1) \times \mathbb{R}^3\big) \\ B &= (\partial \mathcal{U}) \cap \big(\{t_0\} \times \mathbb{R}^3\big) \end{aligned}$$

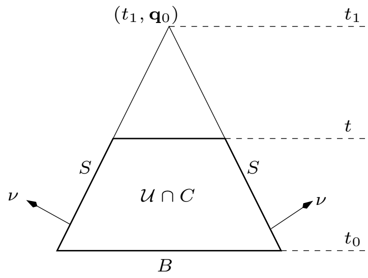
be the "lateral boundary" and "base" of $\mathcal{U}$. Note that S is a piecewise smooth hypersurface in $\mathbb{R}^4$. Let $\theta = \theta_\mu\, \mathrm{d}q^\mu$ be a smooth 1-form on the smooth components of S, such that $\theta(X) > 0$ for every vector X pointing *out* of $\mathcal{U}$.

Proposition 7.1. *Given* **(FS0)**, **(FS1)**, **(FS2)** *and* **(FS3)**, *the difference $\Upsilon = \Theta_2 - \Theta_1$ vanishes identically on $\overline{\mathcal{U}}$ when*

$$\Upsilon|_B \;=\; 0, \tag{7.5a}$$

$$\theta_\mu\, \mathbf{M}^\mu(q, \Theta_1(q)) \;\geq\; 0, \quad \textit{along the smooth components of } S. \tag{7.5b}$$

Proof. We abbreviate $\mathbf{M} = \mathbf{M}(q, \Theta_1(q))$. Define the C^1 "energy current" vector field $j^\mu = \Upsilon^T \mathbf{M}^\mu \Upsilon$ on $\mathcal{U}$. By hypothesis, j^μ and its derivatives extend continuously to $\overline{\mathcal{U}}$. For $t \in [t_0, t_1]$, let $E(t)$ be the integral of the component j^0 over $\mathcal{U} \cap (\{t\} \times \mathbb{R}^3)$. The Euclidean divergence theorem gives



$$E(t) = \int_{\mathcal{U}\cap C} \partial_\mu j^\mu - \int_{S\cap C} \langle j, \nu \rangle$$

where $C = (t_0, t) \times \mathbb{R}^3$, because, by (7.5a), $E(t_0) = 0$. For $\mathcal{U} = $ **Cone**, see the nearby figure. By (7.5b), the outward flux $\int_{S\cap C} \langle j, \nu \rangle$ of j through $S \cap C$ is positive. By construction, $\partial_\mu j^\mu = \Upsilon^T K \Upsilon$, where $K = \partial_\mu \mathbf{M}^\mu + H^T + H$, since $\mathbf{M}^\mu$ is symmetric, and Υ is a solution to (7.4). By Proposition 7.3, K is continuous on $\overline{\mathcal{U}}$ and therefore bounded. Also, there is a constant $k > 0$ such that $|\Upsilon^T K \Upsilon| \le k j^0$, because $\mathbf{M}^0$ is strictly uniformly positive definite on the compact set $\overline{\mathcal{U}}$. It follows that

$$E(t) \le \int_{\mathcal{U}\cap C} |\Upsilon^T K \Upsilon| \le k \int_{t_0}^{t} \mathrm{d}s\, E(s),$$

for all $t \in [t_0, t_1]$. Consequently, $E(t) = 0$ on the interval $[t_0, t_1]$. In other words, $\Upsilon = 0$. □

Proposition 7.2. *Let $\mathcal{U} = (t_0, t_1) \times \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^3$ is open, and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$. Suppose* **(FS0)**, **(FS1)**, **(FS2)** *and let $\Upsilon = \Theta_2 - \Theta_1$. Further suppose*

(i) $\Upsilon|_{\{t_0\}\times\mathcal{X}} = 0$.

(ii) $\mathbf{M}^0 \ge a$ *and* $-b \le \mathbf{M}^i \le b,\ i = 1, 2, 3,$ *on* $\overline{\mathcal{A}}$*, for constants* $a, b > 0$.

Part 1. *Set $v^* = \sqrt{3}\,(b/a) > 0$. Then Υ vanishes at $(t, \mathbf{q}) \in \mathcal{U}$ if*

$$\mathrm{dist}_{\mathbb{R}^3}\big(\mathbb{R}^3 \setminus \mathcal{X}, \mathbf{q}\big) > v^* |t_1 - t_0|. \tag{7.6}$$

Part 2. *If, in addition, $\mathbf{M}^3 \ge 0$ on $\overline{\mathcal{A}}$, then Υ vanishes at $(t, \mathbf{q}) \in \mathcal{U}$ if*

$$\mathrm{dist}_{\mathbb{R}^3}\Big(\overline{\mathbf{Half_q}} \cap \big(\mathbb{R}^3 \setminus \mathcal{X}\big), \mathbf{q}\Big) > v^* |t_1 - t_0| \tag{7.7}$$

where $\mathbf{Half_q} = \{\mathbf{y} \in \mathbb{R}^3 \mid y^3 < q^3\}$.

Proof (of Part 1). Let $\mathbf{q} \in \mathcal{X}$ satisfy (7.6). Observe that, for the set **Cone** $\subset \mathcal{U}$ with velocity v^*, base at time t_0 and vertex at $(t_1, \mathbf{q})$, we have

$$\theta_\mu\, \mathbf{M}^\mu|_{S\times B_r(0)} \ge 0 \tag{7.8}$$

by the choice of v^* and (ii). Here, the lateral boundary S and the 1-form θ are just as in the discussion above Proposition 7.1. We can now apply Proposition 7.1 to **Cone**, and conclude $\Upsilon|_{\overline{\mathbf{Cone}}} = 0$. □

Proof (of Part 2). Suppose that $\mathbf{M}^3 \ge 0$ on $\overline{\mathcal{A}}$. Let $\mathbf{q} \in \mathcal{X}$ satisfy (7.7). The boundary of the set **Cone** $\cap$ $\mathbf{Half_q} \subset \mathcal{U}$, where **Cone** has velocity v^*, base at time t_0 and vertex at $(t_1, \mathbf{q})$, has two smooth components. On the "round" one, the inequality (7.8) holds again by the choice of v^*, and on the "flat" one by $\theta_\mu\, \mathbf{M}^\mu = \mathbf{M}^3 \ge 0$ for θ proportional to $(0, 0, 1, 0)$. We have $\Upsilon|_{\overline{\mathbf{Cone} \cap \mathbf{Half_q}}} = 0$, by Proposition 7.1. □

7.3. Existence / breakdown theorem.
Assumptions for the Existence / breakdown theorem: All the quantities here are real.

(EB0) Let $\mathcal{U} = (-\infty, T) \times \mathbb{R}^3$, where $T \in \mathbb{R}$ and let $\mathcal{A} = \mathcal{U} \times B_2(0) \subset \mathbb{R}^4 \times \mathbb{R}^P$.
(EB1) $\mathbf{M}^\mu(q, \Theta)$ is a symmetric $P \times P$ matrix on $\mathcal{A}$. Particularly, $\mathbf{M}^0 \geq \frac{1}{2}$.
(EB2) $h(q, \Theta)$ is an $\mathbb{R}^P$ valued function on $\mathcal{A}$.
(EB3) Both h and $\mathbf{M}^\mu$ are smooth on $\mathcal{A}$ and their derivatives of all orders extend continuously to $\overline{\mathcal{A}}$.
(EB4) $\mathcal{K} \subset \mathcal{Q} \subset \mathbb{R}^3$ with $\mathcal{K}$ compact, $\mathcal{Q}$ open, such that on $(-\infty, T) \times (\mathbb{R}^3 \setminus \mathcal{K}) \times B_2(0)$, the matrix $\mathbf{M}^\mu$ is constant and denoted by $\not{\mathbf{M}}^\mu$ and $h(q, \Theta) = \not{H}(t)\, \Theta$, where $\not{H}(t)$ is a matrix depending only on $t = q^0$. It is assumed, $\not{\mathbf{M}}^0 \geq \frac{1}{2}$. We can naturally extend $\mathbf{M}^\mu$ and h, by $\not{\mathbf{M}}^\mu$ and $\not{H}(t)\Theta$, to $(-\infty, T) \times (\mathbb{R}^3 \setminus \mathcal{K}) \times \mathbb{R}^P$.

We now formulate and prove an existence theorem for the quasilinear symmetric hyperbolic system

$$\mathbf{M}(q, \Theta)\, \Theta = h(q, \Theta) \qquad \mathbf{M} = \mathbf{M}^\mu \tfrac{\partial}{\partial q^\mu}. \tag{7.9}$$

Proposition 7.3. *Suppose* **(EB0)**, **(EB1)**, **(EB2)**, **(EB3)**, **(EB4)**.

Part 1. *For each $t_0 < T$, there is a $t_1 \in (t_0, T]$ and a smooth solution $\Theta : [t_0, t_1) \times \mathbb{R}^3 \to \mathbb{R}^P$ of* (7.9) *with trivial initial data, $\Theta(t_0, \cdot\,) = 0$, such that* $\operatorname{supp} \Theta \subset [t_0, t_1) \times B_r(0)$ *for some finite $r > 0$, and*

$$\Theta\big([t_0, t_1) \times \mathcal{Q}\big) \subset B_1(0) \subset \mathbb{R}^P \tag{7.10}$$

and such that $t_1 \neq T$ implies either one or both of:

$($**Break**$)_1$*:* $\overline{\Theta([t_0, t_1) \times \mathcal{Q})} \not\subset B_1(0) \subset \mathbb{R}^P$.
$($**Break**$)_2$*:* *The vector field $\partial_q \Theta$ is unbounded on $[t_0, t_1) \times \mathcal{Q}$.*

Part 2. *Suppose in addition that $\mathbf{M}^3 \geq 0$ on $\mathcal{A}$, and $h(q, 0) = 0$ when $q^3 < \frac{1}{2}$. Then the solution Θ of Part 1 vanishes identically for $q^3 < \frac{1}{2}$.*

Proof (of Part 1). Fix a smooth transition function $\psi = \psi(|\Theta|) : \mathbb{R} \to [0, 1]$ which is equal to 1 on $(-\infty, \frac{4}{3})$ and equal to 0 on $(\frac{5}{3}, \infty)$. It is for this reason that $B_2(0)$ appears in **(EB0)**. Set

$$\mathbf{N} = \psi\, \mathbf{M} + (1 - \psi)\, \not{\mathbf{M}}, \qquad g = \psi\, h + (1 - \psi)\, \not{H}\, \Theta.$$

By construction, g and the symmetric matrix $\mathbf{N}^\mu$ are smooth on $\mathcal{B} = \mathcal{U} \times \mathbb{R}^P$, and their derivatives of all orders extend continuously to $\overline{\mathcal{B}}$. Note that $\mathbf{N}^0 \geq \frac{1}{2}$ on $\mathcal{B}$ and there is a constant $b > 0$ such that $-b \leq \mathbf{N}^i \leq b$, $i = 1, 2, 3$ on $[t_0, T] \times \mathbb{R}^3 \times \mathbb{R}^P$. The latter statement follows from the fact that $\mathbf{N}^\mu$ is constant $(= \not{\mathbf{M}}^\mu)$ on the complement, in $[t_0, T] \times \mathbb{R}^3 \times \mathbb{R}^P$, of the compact set $[t_0, T] \times \mathcal{K} \times \overline{B_2(0)}$. Fix the velocity $v^* = 2\sqrt{3}\, b$ (see Proposition 7.2).

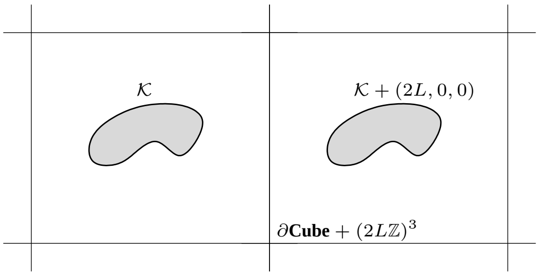
We want to reduce our existence / breakdown theorem to [Tay]. To do this, fix $L > 0$ big enough so that

$$\begin{aligned} \mathcal{K} \;&\subset\; \textbf{Cube} \overset{\text{def}}{=} [-L, L]^3 \\ \operatorname{dist}_{\mathbb{R}^3}\big(\partial\, \textbf{Cube}, \mathcal{K}\big) \;&>\; 1 + v^*\, |T - t_0| \end{aligned}$$

and smoothly extend $\mathbf{N}$ and $\mathbf{g}$ from $(-\infty, T) \times$ **Cube** $\times \mathbb{R}^P$ to spatially periodic matrix and vector valued functions on $(-\infty, T) \times \mathbb{R}^3 \times \mathbb{R}^P$. With these preliminaries, the hypotheses of Proposition 2.1 on page 370, Proposition 1.5 on page 365 and Corollary 1.6 on page 366 in [Tay] are all satisfied, and there is a $\tau \in (t_0, T]$ and a spatially periodic smooth solution $\Theta : [t_0, \tau) \times \mathbb{R}^3 \to \mathbb{R}^P$ with trivial initial data at $t = t_0$ to the symmetric hyperbolic system corresponding to the spatially periodic extension of $\mathbf{N}$ and $\mathbf{g}$, such that, if $\tau \neq T$, then the vector

$$\big(\Theta,\, \partial_q \Theta\big) \ \in\ \mathbb{R}^P \oplus \big(\mathbb{R}^4 \otimes \mathbb{R}^P\big)$$

is unbounded on $[t_0, \tau) \times \mathbb{R}^3$. (There is one caveat: [Tay], for convenience, considers systems defined for all time. By direct inspection, his argument applies to any open subinterval of $\mathbb{R}$.)



Let $\mathcal{J} = \mathcal{K} + (2L\mathbb{Z})^3$. By construction, the spatially periodic system introduced in the last paragraph reduces to $\not{\mathbf{M}}\Theta = \not{H}(t)\Theta$, on $(-\infty, T) \times (\mathbb{R}^3 \setminus \mathcal{J}) \times \mathbb{R}^P$, and admits the trivial solution. Intuitively, "signals can travel at most a distance $v^*|T - t_0|$", which is less than the distance between $\mathcal{K}$ and ∂ **Cube**. This intuition is formalized by applying Proposition 7.2 to the open set $(t_0, \tau - \epsilon) \times (\mathbb{R}^3 \setminus \mathcal{J})$ for arbitrarily small $\epsilon > 0$. Consequently, Θ vanishes at every point $(t, \mathbf{q}) \in [t_0, \tau) \times \mathbb{R}^3$ with $\mathrm{dist}_{\mathbb{R}^3}(\mathcal{J}, \mathbf{q}) > \frac{1}{2} + v^*|T - t_0|$, because $\Theta|_{t=t_0} = 0$. It follows from our choice of L that the periodic solution Θ vanishes in a neighborhood of $[t_0, \tau) \times (\partial$ **Cube**$)$. For this reason, and because of **(EB4)**, the modified field

$$[t_0, \tau) \times \mathbb{R}^3 \ \ni\ q \ \mapsto \begin{cases} 0, & \text{if } \mathbf{q} \in (\mathbb{R}^3 \setminus \textbf{Cube}) \\ \Theta(q), & \text{if } \mathbf{q} \in \textbf{Cube} \end{cases} \quad \in\ \mathbb{R}^P \tag{7.11}$$

which we continue to call Θ, is a smooth solution to the non-periodic system $\mathbf{N}\Theta = g$ with trivial initial data. Moreover, if $\tau \neq T$, then the vector $(\Theta,\, \partial_q \Theta)$ is unbounded.

Suppose $\tau \neq T$. We show that $(\Theta,\, \partial_q \Theta)$ is bounded on $\mathcal{V} = [t_0, \tau) \times (\mathbb{R}^3 \setminus \mathcal{Q})$, and consequently, unbounded on $[t_0, \tau) \times \mathcal{Q}$. For any time $t_2 \in (t_0, \tau)$, decompose

$$\begin{aligned} [t_0, \tau) \times \mathbb{R}^3 &= \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \\ \mathcal{V}_1 &= [t_0, \tau) \times (\mathbb{R}^3 \setminus \textbf{Cube}) \\ \mathcal{V}_2 &= [t_0, t_2] \times \textbf{Cube} \\ \mathcal{V}_3 &= (t_2, \tau) \times \textbf{Cube} \end{aligned}$$

By (7.11), the vector $(\Theta,\, \partial_q \Theta)$ is bounded on $\mathcal{V} \cap \mathcal{V}_1$ and, by compactness, also on $\mathcal{V} \cap \mathcal{V}_2$.

Let

$$d \;=\; \mathrm{dist}_{\mathbb{R}^3}(\mathcal{K},\, \mathbb{R}^3 \setminus \mathcal{Q}) \;>\; 0.$$

To verify that $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V}\cap\mathcal{V}_3$, we choose $t_2 \in (t_0, \tau)$ with $v^*\,|\tau - t_2| \leq d/2$. Intuitively, this choice means that "signals can travel at most a distance $d/2$ in the time interval (t_2, τ)", which is less than the distance between $\mathcal{K}$ and $\mathbb{R}^3 \setminus \mathcal{Q}$. There exists (see, for instance, [Tay2]) a unique smooth solution $\Theta_2 \,:\, [t_2, T) \times \mathbb{R}^3 \to \mathbb{R}^P$ to the linear system $\not{\mathbf{M}}\Theta_2 \,=\, \not{H}(t)\Theta_2$ with initial condition $(\Theta - \Theta_2)|_{t=t_2} \,=\, 0$. In particular, $(\Theta_2,\, \partial_q\, \Theta_2)$ is bounded on the compact set $\overline{\mathcal{V}\cap\mathcal{V}_3}$. By **(EB4)**, Θ is a solution to the same system on $[t_0, \tau) \times (\mathbb{R}^3 \setminus \mathcal{K})$. As in the last paragraph, Proposition 7.2 implies that $\Theta = \Theta_2$ on $[t_2, \tau) \times (\mathbb{R}^3 \setminus \mathcal{Q}) \;\supset\; \mathcal{V}\cap\mathcal{V}_3$. Consequently, $(\Theta,\, \partial_q\, \Theta)$ is bounded on $\mathcal{V}\cap\mathcal{V}_3$, and we are done.

The final step is to remove the transition function ψ. Let

$$\mathcal{I} = \Big\{ t \in [t_0, \tau) \;\Big|\; \overline{\Theta\big([t_0, t] \times \mathcal{Q}\big)} \subset B_1(0) \subset \mathbb{R}^P \Big\}.$$

We show that $\mathcal{I} \,=\, [t_0, t_1)$, where $t_1 \,\in\, (t_0, \tau]$. First, $t_0 \,\in\, \mathcal{I}$ since the initial data vanishes. Second, if $t' \in \mathcal{I}$, then $[t_0, t'] \in \mathcal{I}$. Let $t_1 = \sup \mathcal{I}$. If $t_1 \,=\, \tau$, then $t_1 \neq \mathcal{I}$. If $t_1 \,<\, \tau$, assume by contradiction $t_1 \,\in\, \mathcal{I}$. Then, the compact set $\overline{\Theta([t_0, t_1] \times \mathcal{Q})}$ is contained in a ball $B_r(0) \,\subset\, \mathbb{R}^P$ of radius $r \,<\, 1$. However, $\partial_t \Theta$ is bounded on $[t_0, t_1]\times\mathbb{R}^3$, since $\mathrm{supp}_{\mathbb{R}^3}(\partial_t\Theta)(t,\,\cdot\,) \subset$ **Cube** is compact for all $t \in [t_0, t_1]$. Therefore, $t_1 + \epsilon \in \mathcal{I}$ for all sufficiently small $\epsilon > 0$.

The smooth solution of (7.9) that we are looking for is $\Theta|_{[t_0,t_1)\times\mathbb{R}^3}$. Indeed, it has trivial initial data, support contained in $[t_0, t_1) \times$ **Cube** and satisfies (7.10). If $q \in [t_0, t_1) \times \mathcal{Q}$, then $|\Theta(q)| \,\leq\, 1$ and $\psi(|\Theta(q)|) = 1$, by the definition of t_1. In this case, the system $\mathbf{N}\Theta = g$ reduces to (7.9). On $[t_0, t_1) \times (\mathbb{R}^3 \setminus \mathcal{K})$, **(EB4)** directly implies that the system also reduces to (7.9).

If $t_1 \neq T$, there are two alternatives: $t_1 < \tau \leq T$ and $t_1 = \tau < T$. For the first, we use the continuity of Θ on $\overline{[t_0, t_1] \times \mathcal{Q}}$ to conclude that

$$\overline{\Theta([t_0, t_1) \times \mathcal{Q})} \;=\; \overline{\Theta([t_0, t_1] \times \mathcal{Q})} \;\not\subset\; B_1(0)$$

since $t_1 \,\notin\, \mathcal{I}$. That is, we have $(\textbf{Break})_1$. For the second, $\tau \,\neq\, T$, and $(\Theta,\, \partial_q\, \Theta)$ is unbounded on $[t_0, t_1) \times \mathcal{Q}$. Since Θ is bounded, $(\textbf{Break})_2$ applies. The proof of Part 1 is complete. $\square$

Proof (of Part 2). Let **Half** $= \{\mathbf{q} \in \mathbb{R}^3 \,|\, q^3 \,<\, \frac{1}{2}\}$. The assumption $h(q, 0) = 0$, when $q^3 < \frac{1}{2}$, implies that $\Theta_1 = 0$ is a solution to $\mathbf{N}\Theta = g$ on $(t_0, t_1) \times$ **Half**. Also, $\Theta_2 = \Theta$ is a smooth solution, and $(\Theta_2 - \Theta_1)|_{\{t_0\}\times\textbf{Half}} \,=\, 0$. The assumption $\mathbf{M}^3 \,\geq\, 0$ on $\mathcal{A}$ implies $\not{\mathbf{M}}^3 \geq 0$ and consequently, $\mathbf{N}^3 \geq 0$ on $\mathcal{U} \times \mathbb{R}^P$. At last, Part 2 of Proposition 7.2, applied to the open set $(t_0, t_1 - \epsilon) \times$ **Half** with arbitrarily small $\epsilon \,>\, 0$, forces $\Theta_2 - \Theta_1 = \Theta$ to vanish on $[t_0, t_1) \times$ **Half**. $\square$

7.4. Energy Estimate.
Assumptions for the energy estimate: All the quantities here are real.

(E0) $\mathcal{U} \,=\, \mathcal{I} \times \mathcal{O}(b)$ where $\mathcal{O}(b) \,=\, \mathbb{R}^2 \times (0, b) \,\subset\, \mathbb{R}^3,\;\; b \,\in\, [1, 2]$ and $\mathcal{I} \,=\, (t_0, t^*)$, $-\infty < t_0 < t^* \leq -1$.

(E1) $\mathbf{M}^\mu(q)$ is a symmetric $P\times P$ matrix on $\mathcal{U}$. Particularly, for all $q\in\mathcal{U}$, $\frac{1}{2}\le \mathbf{M}^0\le 2$ and $\mathbf{M}^3\ge 0$. We assume the integer P is less than some big absolute constant, say $P\le 10^9$.
(E2) $H(q)$ is a $P\times P$ matrix on $\mathcal{U}$.
(E3) **Src**(q) is an $\mathbb{R}^P$ valued function on $\mathcal{U}$.
(E4) $\Theta(q)$ is an $\mathbb{R}^P$ valued function on $\mathcal{U}$, which is a solution to the linear, inhomogeneous, symmetric hyperbolic system

$$\mathbf{M}(q)\Theta = H(q)\Theta + \mathbf{Src}(q), \qquad \mathbf{M}=\mathbf{M}^\mu \tfrac{\partial}{\partial q^\mu}. \tag{7.12}$$

(E5) Fix a non-negative integer R. Then, $\mathbf{M}^\mu$, Θ (resp. H, **Src**) are C^{R+1} (resp. C^R) functions on $\mathcal{U}$, all of whose derivatives of order $\le R+1$ (resp. $\le R$) extend continuously to $\overline{\mathcal{U}}$.
(E6) $\operatorname{supp}\Theta(t,\,\cdot\,)$ and $\operatorname{supp}\mathbf{Src}(t,\,\cdot\,)$ are contained in a ball in $\mathbb{R}^3$ with radius independent of t. Moreover, Θ and **Src** vanish identically when $q^3<\frac{1}{2}$.

Let $\mathbb{R}^P=\mathbb{R}^{P_1}\oplus\mathbb{R}^{P_2}\oplus\mathbb{R}^{P_3}$. We decompose

$$\Theta=(\Theta_1,\,\Theta_2,\,\Theta_3), \qquad\qquad \mathbf{Src}=(\mathbf{Src}_1,\,\mathbf{Src}_2,\,\mathbf{Src}_3).$$

Each $P\times P$ matrix is decomposed into nine blocks of size $P_m\times P_n$, where $m,n=1,2,3$. Especially,

$$\mathbf{M}^\mu=(M^\mu_{mn})_{m,n=1,2,3}, \qquad\qquad H=(H_{mn})_{m,n=1,2,3}. \tag{7.13}$$

(E7) The matrix $\mathbf{M}^\mu$ is block-diagonal, $\mathbf{M}^\mu=\operatorname{diag}(M_1^\mu,\,M_2^\mu,\,M_3^\mu)$, and $(M_2)_{ij}=\mu(q)\,\delta_{ij}\,(\frac{\partial}{\partial q^0}+\frac{\partial}{\partial q^3})$, $i,j=1,\dots,P_2$, for some function μ.
(E8) $\not{\mathbf{M}}^\mu$ are constant symmetric $P\times P$ matrices, with $\frac{1}{2}\le\not{\mathbf{M}}^0\le 2$, $\not{\mathbf{M}}^1=0$, $\not{\mathbf{M}}^2=0$ and $\not{\mathbf{M}}^3\ge 0$.
(E9) $\not{H}(t)$ is a $P\times P$ matrix depending only on t with $\mathbb{R}^{P_1}\oplus\mathbb{R}^{P_2}\oplus\mathbb{R}^{P_3}$ block-form

$$\not{H}=\left(\not{H}_{mn}\right)_{m,n=1,2,3}=\begin{pmatrix}0 & 0 & 0\\ \not{H}_1 & 0 & 0\\ 0 & |t|^{-1}\not{H}_2 & |t|^{-1}\not{H}_3\end{pmatrix}$$

where $\not{H}_1$, $\not{H}_2$, $\not{H}_3$ are constant matrices, $\not{H}_3$ is symmetric and $\not{H}_3\le 0$.

Definition 7.1. *For every open $\mathcal{X}\subset\mathbb{R}^3$, the energy of f contained in $\mathcal{X}$ at time t is*

$$E^k_{\mathcal{X}}\{f\}(t)\overset{\text{def}}{=}\sum_{\substack{|\alpha|\le k\\ \alpha\in\mathbb{N}_0^4}}\int_{\mathcal{X}}\mathrm{d}^3\mathbf{q}\,|\partial^\alpha f(t,\mathbf{q})|^2 \tag{7.14}$$

and the supremum norm

$$\mathbf{Sup}^{(k)}_{\mathcal{X}}\{f\}(t)\overset{\text{def}}{=}\sup_{\substack{|\alpha|\le k\\ \alpha\in\mathbb{N}_0^4}}\ \sup_{\mathbf{q}\in\mathcal{X}}|\partial^\alpha f(t,\mathbf{q})| \tag{7.15}$$

for any scalar, vector or matrix valued C^k-function f. As usual, we denote $\partial^\alpha=\prod_{\mu=0}^3(\partial_\mu)^{\alpha_\mu}$ where $\partial_\mu=\frac{\partial}{\partial q^\mu}$, for any $\alpha=(\alpha_\mu)_{\mu=0,1,2,3}\in\mathbb{N}_0^4$. The pointwise norm $|\cdot|$ is always the Euclidean norm (for matrices, $|A|^2=\operatorname{tr}(A^TA)$).

(E10) There are constants $\mathbf{c}_1 \geq 0$ and $J > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|^{2J+2}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_1\}(t) \\ |t|^{2J}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_2\}(t) \\ |t|^{2J+2}\, E^R_{\mathcal{O}(b)}\{\mathbf{Src}_3\}(t) \end{array}\right\} \leq \mathbf{c}_1^2,$$

(E11a) Assume $R \geq 4$. There is a constant $\mathbf{c}_2 > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|^2\, E^R_{\mathcal{O}(b)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}(t) \\ |t|^2\, E^R_{\mathcal{O}(b)}\{H_{1n} - \not{H}_{1n}\}(t) \\ E^R_{\mathcal{O}(b)}\{H_{2n} - \not{H}_{2n}\}(t) \\ |t|^2\, E^R_{\mathcal{O}(b)}\{H_{3n} - \not{H}_{3n}\}(t) \end{array}\right\} \leq \mathbf{c}_2^2.$$

(E11b) Assume $R \geq 0$. There is a constant $\mathbf{c}_2 > 0$ such that for all $t \in \mathcal{I}$:

$$\left.\begin{array}{r} |t|\, \mathbf{Sup}^{(\max\{1,R\})}_{\mathcal{O}(b)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}(t) \\ |t|\, \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{1n} - \not{H}_{1n}\}(t) \\ \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{2n} - \not{H}_{2n}\}(t) \\ |t|\, \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}\{H_{3n} - \not{H}_{3n}\}(t) \end{array}\right\} \leq \mathbf{c}_2.$$

Proposition 7.4 (Energy Estimate). *Suppose the hypotheses* **(E0)** *through* **(E10)** *hold, and, also, either* **(E11a)** *or* **(E11b)** *holds. Let $J_0 > 0$ and assume $J \geq J_0$, see* **(E10)**. *Then, there are constants $\mathbf{c}_3(X) \in (0,1)$, $\mathbf{c}_4(X) > 0$ depending only on $X = \big(R, J_0, |\not{H}_1|, |\not{H}_2|, |\not{H}_3|\big)$, such that $\mathbf{c}_2 \leq \mathbf{c}_3(X)$ and $|t^*|^{-1} \leq \mathbf{c}_3(X)$ imply that*

$$\sqrt{E^R_{\mathcal{O}(b)}\{\Theta\}(\tau)} \;\leq \mathbf{c}_4(X)\, \frac{|t_0|^J \sqrt{E^R_{\mathcal{O}(b)}\{\Theta\}(t_0)} \,+\, \mathbf{c}_1}{|\tau|^J} \tag{7.16}$$

for all $\tau \in \mathcal{I}$ (see, **(E0)** *for the definition of $\mathcal{I}$).*

Proof. In the proof, we denote $E^R = E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$.

Preliminaries 1: For a function f with values in $\mathbb{R}^{Pi}$, $i = 1,2,3$, we define the energy naturally associated to the linear symmetric hyperbolic system (7.12)

$$\mathbf{E}^0_i\{f\}(t) = \int_{\mathcal{O}(b)} \mathrm{d}^3\mathbf{q}\, \big(f^T M^0_i f\big)(t,\mathbf{q}) \quad , \quad \mathbf{E}^R_i\{f\}(t) = \sum_{\substack{|\alpha|\leq R \\ \alpha\in\mathbb{N}^4_0}} \mathbf{E}^0_i\{\partial^\alpha f\}(t) \tag{7.17}$$

See, (7.13). This energy is comparable, by **(E1)**, to the one defined in (7.14). Namely,

$$E^R\{f\}(t) \leq 2\, \mathbf{E}^R_i\{f\}(t), \qquad \mathbf{E}^R_i\{f\}(t) \leq 2\, E^R\{f\}(t). \tag{7.18}$$

If $R \geq 2$ and f is a vector or matrix valued C^R function, Lemma 7.2 implies:

$$\mathbf{Sup}^{(R-2)}\{f\}(t) \lesssim_R \sqrt{E^R\{f\}(t)}. \tag{7.19}$$

If $\alpha \in \mathbb{N}_0^4,\ |\alpha| \le R$ and $R \ge 4$ (to be used in the case **(E11a)**) implies

$$E^R\{f_1 f_2\}(t) \lesssim_R E^R\{f_1\}(t)\, E^R\{f_2\}(t) \tag{7.20}$$
$$E^0\big\{\, [\partial^\alpha, f_1] f_2 \,\big\}(t) \lesssim_R \sum_{|\beta|=1} E^{R-1}\{\partial^\beta f_1\}(t)\, E^{R-1}\{f_2\}(t) \tag{7.21}$$

whereas $R \ge 0$ (to be used in the case **(E11b)**) implies

$$E^R\{f_1 f_2\}(t) \lesssim_R \big(\mathbf{Sup}^{(R)}\{f_1\}(t)\big)^2\, E^R\{f_2\}(t) \tag{7.22}$$
$$E^0\big\{\, [\partial^\alpha, f_1] f_2 \,\big\}(t) \lesssim_R \sum_{|\beta|=1} \big(\mathbf{Sup}^{(R-1)}\{\partial^\beta f_1\}(t)\big)^2\, E^{R-1}\{f_2\}(t) \tag{7.23}$$

Inequalities (7.22) and (7.23) are direct consequences of the product rule. The inequalities (7.20) and (7.21), require, in addition, $R \ge 4$ and the Sobolev inequality (7.19). In fact, by the product rule,

$$E^R\{f_1 f_2\}(t) \lesssim_R \sum_{|\alpha|+|\beta| \le R} \int_{\mathcal{O}(b)} \mathrm{d}^3\mathbf{q}\, \big|\partial^\alpha f_1(t,\mathbf{q})\big|^2 \big|\partial^\beta f_2(t,\mathbf{q})\big|^2$$

For each pair of multiindices (α,β) with $|\alpha| + |\beta| \le R$, at least one of $|\alpha|$ or $|\beta|$ is less than or equal to $R-2$, say α. Then, by the Sobolev inequality,

$$\sup_{\mathbf{q} \in \mathcal{O}(b)} |\partial^\alpha f_1(t,\mathbf{q})|^2 \lesssim_R E^R\{f_1\}(t)$$

Inequality (7.20) follows at once. This argument works for $R \ge 3$. An entirely similar argument gives (7.21), but with $R \ge 4$.

Preliminaries 2: In this subsection, $t \in \mathcal{I}$ and $\alpha \in \mathbb{N}_0^4$, $|\alpha| \le R$, are arbitrary. We apply ∂^α to (7.12) and obtain (all the derivatives make sense classically)

$$\mathbf{M}(\partial^\alpha \Theta) = \not{H}\, \partial^\alpha \Theta + (S_1^\alpha,\, S_2^\alpha,\, S_3^\alpha) + \partial^\alpha \mathbf{Src} \tag{7.24}$$
$$(S_1^\alpha,\, S_2^\alpha,\, S_3^\alpha) \overset{\text{def}}{=} \partial^\alpha\big((H - \not{H})\Theta\big) + [\partial^\alpha, \not{H}]\Theta + [\mathbf{M}^\mu - \not{\mathbf{M}}^\mu, \partial^\alpha]\partial_\mu \Theta.$$

If $R \ge 4$, (7.20), (7.21) and **(E9)** imply that

$$E^0\{S_i^\alpha\} \lesssim_R \Big\{ \sum_{j=1}^3 E^R\{H_{ij} - \not{H}_{ij}\} + \frac{|\not{H}_2|^2 + |\not{H}_3|^2}{|t|^4} + \sum_{\mu=0}^3 E^R\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\} \Big\}\, E^R\{\Theta\}.$$

If $R \ge 0$, (7.22), (7.23) and **(E9)** imply that

$$E^0\{S_i^\alpha\} \lesssim_R \Big\{ \sum_{j=1}^3 \big(\mathbf{Sup}^{(R)}\{H_{ij} - \not{H}_{ij}\}\big)^2 + \frac{|\not{H}_2|^2 + |\not{H}_3|^2}{|t|^4}$$
$$+ \sum_{\mu=0}^3 \big(\mathbf{Sup}^{(R)}\{\mathbf{M}^\mu - \not{\mathbf{M}}^\mu\}\big)^2 \Big\}\, E^R\{\Theta\}.$$

If **(E11a)** or **(E11b)**, it follows from the inequalities just above that

$$\left.\begin{aligned} |t|^2\, E^0\{S_1^\alpha\}(t) \\ E^0\{S_2^\alpha\}(t) \\ |t|^2\, E^0\{S_3^\alpha\}(t) \end{aligned}\right\} \lesssim_R \ \mathbf{c}_*^2\, E^R\{\Theta\}(t) \tag{7.25}$$

where $\mathbf{c}_* = \max\big\{\mathbf{c}_2,\ |t^*|^{-1}(|\not{H}_2|^2 + |\not{H}_3|^2)^{1/2}\big\}$.

Estimates: We derive the energy inequalities 7.33, stated below. For $i = 1, 2, 3$, define "energy currents" associated to $\Theta = (\Theta_1,\, \Theta_2,\, \Theta_3)$ (see **(E4)**)

$$j_i^\mu[\Theta_i](q) = \big(\Theta_i^T\, M_i^\mu\, \Theta_i\big)(q). \tag{7.26}$$

(Warning: we never sum over repeated "lower" indices.) The important current identity

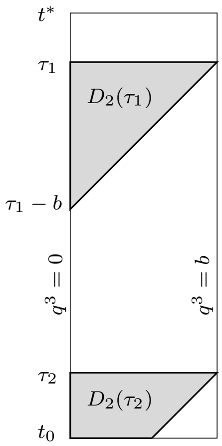
$$\partial_\mu j_i^\mu[\Theta_i] = \Theta_i^T\, (\partial_\mu M_i^\mu)\, \Theta_i + 2\, \Theta_i^T \big(M_i^\mu\, \partial_\mu \Theta_i\big), \tag{7.27}$$

follows from $(M_i^\mu)^T = M_i^\mu$, see **(E1)**. For $\tau \in \overline{\mathcal{I}}$, let $\tau_- = \max\{t_0,\, \tau - 2\}$. Let

$$\mathcal{D}_1(\tau) = \mathcal{D}_3(\tau) = \big\{(t, \mathbf{q}) \in \mathcal{U} \ \big|\ t \in (\tau_-, \tau)\big\} \tag{7.28}$$

$$\mathcal{D}_2(\tau) = \mathcal{D}_1(\tau) \cap \{\, q \mid q^3 - q^0 < b - \tau\}. \tag{7.29}$$

For the case $i = 2$, see the nearby figure, where $t_0 < \tau_2 < t_0 + b < \tau_1 < t^*$. Energy estimates are obtained by integrating (7.27) over $\mathcal{D}_i(\tau) \subset \mathcal{U} = \mathcal{I} \times \mathcal{O}$ and applying the Euclidean divergence theorem. The divergence theorem generates integrals over the boundary $\partial\mathcal{D}_i(\tau)$, which we now discuss. Recall **(E6)**. The $q^0 = \tau$ boundary contributes $\mathbf{E}_i^0\{\Theta_i\}(\tau)$. There is no contribution from the $q^3 = 0$, by **(E6)**. For $i = 1, 3$, the $q^0 = \tau_-$ boundary contributes $-\mathbf{E}_i^0\{\Theta_i\}(\tau_-)$, and the contribution from $q^3 = b$ is non-negative, by **(E1)**. If $i = 2$, there is always a boundary contribution from $q^3 - q^0 = b - \tau$ and it vanishes by **(E7)**. If $i = 2$ and $\tau < t_0 + b$, there is an additional boundary contribution at $q^0 = t_0$ which is $\geq -\mathbf{E}_2^0\{\Theta_2\}(t_0)$.



The discussion of the last paragraph literally transposes from Θ_i and $j_i^\mu[\Theta_i]$ to $\partial^\alpha\Theta_i$ and $j_i^\mu[\partial^\alpha\Theta_i]$, for $|\alpha| \leq R$. The current $j_i^\mu[\partial^\alpha\Theta_i]$ is C^1 and extends, with its derivatives, continuously to $\overline{\mathcal{U}}$. The preceding analysis of the boundary terms gives the general inequalities

$$\mathbf{E}_i^0\{\partial^\alpha\Theta_i\}(\tau) - k_i(\tau)\, \mathbf{E}_i^0\{\partial^\alpha\Theta_i\}(\tau_-) \leq \int_{\mathcal{D}_i(\tau)} \mathrm{d}^4 q\ \ \partial_\mu j_i^\mu[\partial^\alpha\Theta_i](q). \tag{7.30}$$

for $i = 1, 2, 3$. Here, by definition, $k_1(\tau) = k_3(\tau) = 1$ for all τ, whereas $k_2(\tau)$ vanishes when $\tau_- > t_0$ and is equal to 1 when $\tau_- = t_0$. Summing over $|\alpha| \leq R$,

$$\mathbf{E}_i^R\{\Theta_i\}(\tau) - k_i(\tau)\, \mathbf{E}_i^R\{\Theta_i\}(\tau_-) \leq \int_{\mathcal{D}_i(\tau)} \mathrm{d}^4 q \sum_{|\alpha| \leq R} \partial_\mu j_i^\mu[\partial^\alpha\Theta_i](q). \tag{7.31}$$

The current identity (7.27), with $\partial^\alpha \Theta_i$ in the role of Θ_i, and (7.24) imply

$$\begin{aligned}
&\partial_\mu j_i^\mu[\partial^\alpha \Theta_i](q) \qquad (7.32)\\
&= 2(\partial^\alpha \Theta_i)^T \Big\{ \Big(\sum_{j=1}^{3} \not{H}_{ij}\, (\partial^\alpha \Theta_j) \Big) + \tfrac{1}{2}\, (\partial_\mu M_i^\mu)(\partial^\alpha \Theta_i) + (\partial^\alpha \mathbf{Src}_i) + S_i^\alpha \Big\}.
\end{aligned}$$

For $i = 1, 2$, we directly estimate the right hand side of (7.31), by using Schwarz's inequality for the spatial part of the integral, and **(E6)**, **(E10)**, (7.25) and (7.18). For $i = 3$, we first exploit $\not{H}_3 \leq 0$ (see **(E9)**) to drop the term $2(\partial^\alpha \Theta_3)^T \not{H}_{33}(\partial^\alpha \Theta_3)$, and then go on as before. We also use the estimate $|\partial_\mu \mathbf{M}^\mu| = |\partial_\mu (\mathbf{M}^\mu - \not{\mathbf{M}}^\mu)| \lesssim_R \mathbf{c}_2\, |t|^{-1} \leq \mathbf{c}_*\, |t|^{-1}$ that holds when either **(E11a)** or **(E11b)** is assumed (in the first case, we use (7.19)). Abbreviating $\mathbf{E}_i = \mathbf{E}_i^R\{\Theta_i\}$ and $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$, we have for all $\tau \in \overline{\mathcal{I}}$:

$$\begin{aligned}
\mathbf{E}_1(\tau) - \mathbf{E}_1(\tau_-) &\lesssim_X \int_{\tau_-}^{\tau} \frac{\mathrm{d}t}{|t|}\, \sqrt{\mathbf{E}_1(t)} \Big(\mathbf{c}_*\, \sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J} \Big)\\
\mathbf{E}_2(\tau) - \mathbf{E}_2(t_0) &\lesssim_X \int_{\tau_-}^{\tau} \mathrm{d}t\, \sqrt{\mathbf{E}_2(t)} \Big(\sqrt{\mathbf{E}_1(t)} + \mathbf{c}_*\, \sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J} \Big) \qquad (7.33)\\
\mathbf{E}_3(\tau) - \mathbf{E}_3(\tau_-) &\lesssim_X \int_{\tau_-}^{\tau} \frac{\mathrm{d}t}{|t|}\, \sqrt{\mathbf{E}_3(t)} \Big(\sqrt{\mathbf{E}_2(t)} + \mathbf{c}_*\, \sqrt{\mathbf{E}(t)} + \frac{\mathbf{c}_1}{|t|^J} \Big)
\end{aligned}$$

where X is defined as in the proposition.

For each $A = (A_1, A_2, A_3) \in (0, \infty)^3$, define

$$\mathcal{J}(A) = \Big\{ t \in \overline{\mathcal{I}} \;\Big|\; \sup_{\tau \in [t_0, t]} |\tau|^{2J} \mathbf{E}_i(\tau) \leq A_i^2, \quad i = 1, 2, 3 \Big\}$$

Assume A satisfies (recall that $J \geq J_0 > 0$, by assumption)

$$\begin{array}{lll}
A_1 > |t_0|^J \sqrt{\mathbf{E}_1(t_0)} & A_1 > \frac{C}{J_0}\, \mathbf{c}_1 & A_1 > \frac{C}{J_0} \mathbf{c}_* |A| \\
A_2 > 2|t_0|^J \sqrt{\mathbf{E}_2(t_0)} & A_2 > 8C\mathbf{c}_1 & A_2 > 8C(A_1 + \mathbf{c}_* |A|) \qquad (7.34)\\
A_3 > |t_0|^J \sqrt{\mathbf{E}_3(t_0)} & A_3 > \frac{C}{J_0} \mathbf{c}_1 & A_3 > \frac{C}{J_0}(A_2 + \mathbf{c}_* |A|)
\end{array}$$

where $|A|^2 = A_1^2 + A_2^2 + A_3^2$ and where $C = C(X) > 0$ is the maximum of the three constants of proportionality in the inequalities (7.33). It is a direct consequence of the inequalities (7.33), (7.34) and the continuity of $\overline{\mathcal{I}} \ni \tau \mapsto \mathbf{E}_i(\tau)$ that $\mathcal{J}(A)$ is an open and closed sub-interval of $\overline{\mathcal{I}}$ which contains t_0. Therefore, $\mathcal{J}(A) = \overline{\mathcal{I}}$. To see that $\mathcal{J}(A)$ is open in $\overline{\mathcal{I}}$, first observe that for every $\tau \in \mathcal{J}(A)$, the inequalities (7.33), (7.34) imply the strict inequalities $\mathbf{E}_i(\tau) < (A_i |\tau|^{-J})^2$, and then use continuity.

For each $\lambda \geq 0$, set

$$A(\lambda) = \lambda \Big(1\;,\; 1 + 8C\;,\; 1 + \tfrac{C}{J_0}(1 + 8C) \Big).$$

The three rightmost inequalities in (7.34) are homogeneous (degree 1) in A, and hold for $A(\lambda)$, $\lambda > 0$, if and only if they hold for $A(1)$, which is the case if $\mathbf{c}_* > 0$ is sufficiently small depending only on X, because it is true for $\mathbf{c}_* = 0$. The definition of $\mathbf{c}_*$ right after (7.25) implies $\mathbf{c}_* \leq (1 + |\not{H}_2|^2 + |\not{H}_3|^2)^{1/2}\, \mathbf{c}_3(X)$. Consequently, the condition on $\mathbf{c}_*$ holds if $\mathbf{c}_3(X)$ is suitably small. If

$$\lambda \,>\, \lambda_0 \stackrel{\text{def}}{=} 2|t_0|^J \sqrt{\mathbf{E}(t_0)} + \max\{8,\, J_0^{-1}\}\, C\, \mathbf{c}_1 \,\geq\, 0$$

then the remaining inequalities in (7.34) hold for $A(\lambda)$, that is $\mathcal{J}(A(\lambda)) = \overline{\mathcal{I}}$. By the definition of $\mathcal{J}(A)$, we have $\mathcal{J}(A(\lambda_0)) = \overline{\mathcal{I}}$. By (7.18), inequality (7.16) follows if $\mathbf{c}_4(X)$ is sufficiently big. $\square$

Remark 7.1. Once the system of inequalities (7.33) has been established, the rest of the argument is abstract, in the sense that it holds for any three functions $\mathbf{E}_i$, $i = 1, 2, 3$, satisfying (7.33).

7.5. Refined energy estimate. We proved a finite speed of propagation theorem for formal power series vacuum fields $[\varPsi]$, see Proposition 6.1. The refined energy estimate obtained in this subsection plays a similar role for a classical vacuum field $\varPsi$.
To make the last statement more precise, recall that the energy estimate for the symmetric hyperbolic system (7.12) was obtained by integrating the divergence current identity (7.27) over appropriate open subsets $\mathcal{C} \subset \mathbb{R}^4$. We now construct more refined sets $\mathcal{C}$ which allow us to estimate the energy "localized in the (ξ^1, ξ^2) plane". We will be guided by the basic requirement that the boundary integrals in the divergence theorem have definite signs. That is, the boundary of $\mathcal{C}$ must be non-timelike (with respect to the symmetric hyperbolic system).

Convention 7.2. Until further notice, we use the coordinates

$$x = (x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u).$$

Recall the matrix differential operators $\mathbf{A}(\varPhi)$ and $\widehat{\mathbf{A}}(\varPhi)$ associated to $\varPhi = (e, \gamma, w)$. See, (2.5), (2.8). Suppose θ is a one-form and suppose (see, (2.2))

$$\theta_\mu L^\mu \geq 0, \qquad \theta_\mu N^\mu \geq 0, \qquad \theta_\mu \begin{pmatrix} N & D \\ \bar{D} & L \end{pmatrix}^\mu \geq 0 \tag{7.35}$$

The last inequality is in the sense of Hermitian matrices. Then

$$\theta_\mu \mathbf{A}^\mu(\varPhi) \geq 0, \qquad \theta_\mu \widehat{\mathbf{A}}^\mu(\varPhi) \geq 0. \tag{7.36}$$

For each $x_0 = (\xi_0, \underline{u}_0, u_0) \in \mathbb{R}^2 \times (0,\infty) \times (-\infty, 0)$ and choice of constants $k_0, k_1, \mathfrak{d} > 0$, where $\mathfrak{d} < \underline{u}_0$ and $\mathfrak{d} < |u_0|^{-1}$, set

$$\mathcal{C} = \bigcup_{(\underline{u},u)\,\in\,\mathcal{B}} \Big(D_{r(\underline{u},u)}(\xi_0) \times \{(\underline{u}, u)\} \Big)$$
$$\mathcal{F} = \bigcup_{(\underline{u},u)\,\in\,\mathcal{B}} \Big(\partial D_{r(\underline{u},u)}(\xi_0) \times \{(\underline{u}, u)\} \Big)$$

where

$$\begin{aligned} \mathcal{B} &= (0, \underline{u}_0 - \mathfrak{d}) \times \big(-\infty, -\tfrac{1}{|u_0|^{-1} - \mathfrak{d}} \big) \subset \mathbb{R}^2, \\ r(\underline{u}, u) &= k_0 + k_1\, |\underline{u}_0 - \underline{u}| \cdot \big| |u_0|^{-1} - |u|^{-1} \big|. \end{aligned} \tag{7.37}$$

More geometrically, $\mathcal{C}$ is a disk bundle over the base $\mathcal{B}$, and $\mathcal{F}$ the corresponding circle bundle. The set $\mathcal{C}$ is an open subset of $\mathbb{R}^4$. Note that, $r : \mathcal{B} \to (k_0,\ k_0 + k_1|\underline{u}_0|/|u_0|)$, a bounded set. The set $\mathcal{C}$ has piecewise smooth boundary. We concentrate on the smooth piece $\mathcal{F} \subset \partial\mathcal{C}$ here. Let θ be a 1-form along $\mathcal{F}$ whose kernel coincides with the tangent space to $\mathcal{F}$ and for which $\theta(X) > 0$ if X is a vector pointing out of $\mathcal{C}$. We choose

$$\theta = \sum_{i=1}^{2} \widehat{\xi^i}\,\mathrm{d}\xi^i + k_1 \left| \frac{1}{|u|} - \frac{1}{|u_0|} \right| \mathrm{d}\underline{u} + k_1 \frac{|\underline{u}_0 - \underline{u}|}{|u|^2}\,\mathrm{d}u, \qquad \widehat{\xi^i} = \frac{\xi^i - \xi^i_0}{|\xi - \xi_0|}. \quad (7.38)$$

Proposition 7.5. *Let $x \in \mathcal{F}$. If $e_3(x) > 0$ and the inequality*

$$k_1\,\mathfrak{d} \geq\ 2\ \max \left\{ \frac{|u|}{\sqrt{e_3}} \sqrt{|e_1|^2 + |e_2|^2},\ \ |u|^2 \sqrt{|e_4|^2 + |e_5|^2} \right\} \quad (7.39)$$

holds at x, then (7.36) *holds at x with θ given by* (7.38)*.*

Proof. If $\theta_\mu L^\mu > 0$ and $\theta_\mu N^\mu > 0$ and $\det \theta_\mu \left(\begin{smallmatrix} N & D \\ D & L \end{smallmatrix}\right)^\mu > 0$, then (7.35) and therefore (7.36) hold. The condition $e_3 > 0$ implies $\theta_\mu L^\mu > 0$. By (7.39),

$$\widehat{\xi}^1 e_4 + \widehat{\xi}^2 e_5 + k_1\,|u|^{-2}\,|\underline{u}_0 - \underline{u}|\ \geq\ \frac{k_1 \mathfrak{d}}{2|u|^2} > 0$$

which implies $\theta_\mu N^\mu > 0$. Finally, $e_3 > 0$ and (7.39) imply

$$e_3\,k_1 \left| |u|^{-1} - |u_0|^{-1} \right| \left(\widehat{\xi}^1 e_4 + \widehat{\xi}^2 e_5 + k_1\,|u|^{-2}\,|\underline{u}_0 - \underline{u}| \right) - \left| \widehat{\xi}^1 e_1 + \widehat{\xi}^2 e_2 \right|^2 > 0$$

and therefore $\det \theta_\mu \left(\begin{smallmatrix} N & D \\ D & L \end{smallmatrix}\right)^\mu > 0$. □

Remark 7.2. Proposition 7.5 will be applied as follows. Fix Φ, and consider symmetric hyperbolic systems with differential operators given by $\mathbf{A}(\Phi)$ or $\widehat{\mathbf{A}}(\Phi)$. Then, if the assumptions of Proposition 7.5 are satisfied for all points $x \in \mathcal{F}' \subset \mathcal{F}$, the boundary integral $\int_{\mathcal{F}'} \langle j, \nu \rangle$, where j is the energy current vector field, is non-negative.

Convention 7.3. Observe that the definitions of $\mathcal{C}$ and $\mathcal{F}$ depend only on the parameters k_0, k_1, $\mathfrak{d}$, ξ_0, $\underline{u}_0$, u_0. **For the rest of this paper**, the sets $\mathcal{C}$ and $\mathcal{F}$ are determined by the specific choice of parameters

$$\mathfrak{d} = 10^{-3}, \qquad k_0 = \tfrac{1}{4}, \qquad k_1 = \tfrac{1}{18}\mathfrak{d}^{-1}, \qquad u_0 = -\tfrac{1}{2}\mathfrak{d}^{-1}, \qquad \underline{u}_0 = b + \mathfrak{d} \quad (7.40)$$

For each $b \in [1, 2]$ and $\xi_0 \in \mathbb{R}^2$, we denote the corresponding sets by $\mathcal{C}(\xi_0, b)$ and $\mathcal{F}(\xi_0, b)$. The base $\mathcal{B}$ is given by $\mathcal{B} = (0, b) \times (-\infty, -\mathfrak{d}^{-1})$ and the radius function $r(\underline{u}, u)$ takes values in $(\frac{1}{4}, \frac{1}{2})$ on $\mathcal{B}$.

Recall the far field ansatz (see, Section 5) $\Phi = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\,\Psi$, where $\Psi = (f, \omega, z)$. The ansatz depends on the scaling parameters a and $\mathfrak{A}$.

Convention 7.4. **For the rest of this paper**, the parameters $a, \mathfrak{A} \in \mathbb{R}$ are restricted by

$$0 < |\mathfrak{A}|\ \leq\ |a|\ \leq\ 10^{-3} \quad (7.41)$$

Proposition 7.6. *Let* $\mathfrak{d}$ *be fixed as in* (7.40)*. Let* $\xi_0 \in \mathbb{R}^2$ *and* $b \in [1,2]$*. Assume* $a, \mathfrak{A} \in \mathbb{R}$ *satisfy* (7.41)*. If, in addition,*

$$|\Psi(x)| \;\leq\; 5$$

at $x = (\xi, \underline{u}, u) \in \mathcal{F}(\xi_0, b)$ *and* $|\xi| < 4\,|\frac{a}{\mathfrak{A}}|$*, then the assumptions of Proposition 7.5 hold at* x*.*

Proof. The first five components of Ψ satisfy $|f_i| \leq |\Psi| \leq 5$. Consequently, the first five components of Φ satisfy

$$\begin{aligned}
|u|\,|e_i| &\leq |u|\big|\rho^{-1}_{a,\mathfrak{A}}\mathbf{e}_{a,\mathfrak{A}}\big| + |u|\,\big|\tfrac{1}{u^2}f_i\big| \leq |\mathbf{e}_{a,\mathfrak{A}}| + \tfrac{1}{|u|}|f_i| \leq \tfrac{17}{2}\,|a| + 5\mathfrak{d} && i = 1,2\\
e_3 &= 1 + \tfrac{1}{u^2}f_3 \geq 1 - \tfrac{1}{|u|^2}|f_3| \geq 1 - 5\mathfrak{d}^2 \geq (\tfrac{3}{4})^2 &&\\
|u|^2\,|e_i| &= \tfrac{1}{|u|}|f_i| \leq \tfrac{5}{|u|} \leq 5\mathfrak{d} && i = 4,5
\end{aligned}$$

In the first line, we use $|\xi| < 4\,|\frac{a}{\mathfrak{A}}|$. The proposition follows by direct inspection. $\square$

Convention 7.5. For the rest of this subsection, we use coordinates

$$q = (q^0,\, q^1,\, q^2,\, q^3) = \big(t = u + \underline{u}\,,\ \xi^1,\, \xi^2,\, \underline{u}\big)$$

Assumptions for the refined energy estimate. $\mathfrak{d}$ is defined in (7.40).

(RE0) $\mathcal{I} = (t_0, t^*)$ where $-\infty < t_0 < t^* < -\mathfrak{d}^{-1}$, $\ \xi_0 \in \mathbb{R}^2$, $\ b \in [1,2]$,

$$\begin{aligned}
\mathcal{U} &= \bigcup_{t\in\mathcal{I}} \Big(\{t\} \times \mathcal{O}(\xi_0, b, t)\Big) \;\subset\; \mathbb{R}^4\\
\mathcal{O}(\xi_0, b, t) &= \bigcup_{\underline{u}\in(0,b)} \Big(D_{r'(t,\underline{u})}(\xi_0) \times \{\underline{u}\}\Big),\\
r'(t,\underline{u}) &= \tfrac{1}{4} + \tfrac{1}{18\mathfrak{d}}|b + \mathfrak{d} - \underline{u}| \cdot \big|2\mathfrak{d} - \tfrac{1}{\underline{u}+|t|}\big|.
\end{aligned}$$

(RE1) - (RE9) are formulated identically to **(E1) - (E9)**.

(RE10), (RE11a), (RE11b) are formulated identically to **(E10), (E11a), (E11b)** with the understanding that $E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$ are replaced by $E^R_{\mathcal{O}(\xi_0,b,t)}$ and $\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}$, see (7.14) and (7.15).

(RE12) Let the 1-form θ be as in (7.38). Then, $\theta_\mu\, \mathbf{M}^\mu \geq 0$ on

$$(\partial\mathcal{U}) \cap \big(\mathcal{I} \times \mathbb{R}^2 \times (0,b)\big)$$

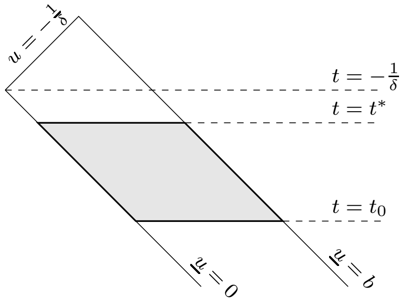
Remark 7.3. $\mathcal{U}$ is a bundle over $\mathcal{I}$ with fiber $\mathcal{O}(\xi_0, b, t) \subset \mathbb{R}^3$ at $t \in \mathcal{I}$. The fiber is an open disk bundle over the $\underline{u}$-interval $(0, b)$. An equivalent description of the fiber is

$$\mathcal{O}(\xi_0, b, t) = \Big\{\mathbf{q} = (\xi^1,\, \xi^2,\, \underline{u}) \in \mathbb{R}^3 \;\Big|\; (\xi^1,\, \xi^2,\, \underline{u},\, t - \underline{u}) \in \mathcal{C}(\xi_0, b)\Big\}$$

for each $t \in \mathcal{I}$. It is important that

$$(\partial\mathcal{U}) \cap \big(\mathcal{I} \times \mathbb{R}^2 \times (0,b)\big) \;\subset\; \mathcal{F}(\xi_0, b).$$

For each $t \in \mathcal{I}$, the map $r'(t,\,\cdot\,):\ (0,b) \to (\frac{1}{4}, \frac{1}{2})$ is decreasing.



Proposition 7.7 (Refined Energy Estimate). *Suppose that the refined hypotheses* **(RE0)** *through* **(RE10)** *and* **(RE12)** *hold, and, also, either* **(RE11a)** *or* **(RE11b)** *holds. Let $J_0 > 0$ and assume $J \geq J_0$, see* **(RE10)**. *Then, there are constants $\mathbf{c}_3(X) \in (0,1)$, $\mathbf{c}_4(X) > 0$ depending only on $X = \big(R, J_0, |\not{H}_1|, |\not{H}_2|, |\not{H}_3|\big)$, such that $\mathbf{c}_2 \leq \mathbf{c}_3(X)$ and $|t^*|^{-1} \leq \mathbf{c}_3(X)$ imply that*

$$\sqrt{E^R_{\mathcal{O}(\xi_0,b,\tau)}\{\Theta\}(\tau)} \;\leq \mathbf{c}_4(X)\, \frac{|t_0|^J \sqrt{E^R_{\mathcal{O}(\xi_0,b,t_0)}\{\Theta\}(t_0)} \,+\, \mathbf{c}_1}{|\tau|^J}. \tag{7.42}$$

for all $\tau \in \mathcal{I}$ (see, **(E0)** *for the definition of $\mathcal{I}$).*

Remark 7.4. We use the same names for the constants in the assumptions and statements of both energy estimates, Propositions 7.4 and 7.7. This was done for convenience, and does not imply that there is any relationship between them.

Proof. This proof completely mimics the proof of Proposition 7.4 with a few modifications. First of all, our previous conventions that $E^R = E^R_{\mathcal{O}(b)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(b)}$ are replaced by the conventions $E^R = E^R_{\mathcal{O}(\xi_0,b,t)}$ and $\mathbf{Sup}^{(R)} = \mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}$. Also, the definitions (7.17) are replaced by

$$\mathbf{E}^0_i\{f\}(t) = \int_{\mathcal{O}(\xi_0,b,t)} \mathrm{d}^3\mathbf{q}\,\big(f^T M^0_i f\big)(t,\mathbf{q}) \quad , \quad \mathbf{E}^R_i\{f\}(t) = \sum_{\substack{|\alpha|\leq R\\ \alpha\in\mathbb{N}^4_0}} \mathbf{E}^0_i\{\partial^\alpha f\}(t)$$

The inequalities (7.18), (7.19), (7.20), (7.21), (7.22), (7.23) still hold with these modifications. The only one that requires discussion is the Sobolev inequality (7.19). For this purpose, let $\mathbf{CYL} = D_{\frac{1}{4}}(0) \times (0,b)$ and $\phi:\ \mathbf{CYL} \to \mathcal{O}(\xi_0,b,t)$ be the diffeomorphism $\phi(\xi,\underline{u}) = \big(\xi_0 + 4\,r'(t,\underline{u})\,\xi,\ \underline{u}\big)$. Then,

$$\begin{aligned}\mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) &\lesssim_R \mathbf{Sup}^{(R-2)}_{\mathbf{CYL}}\{f\circ\phi\}(t)\\ &\lesssim_R \sqrt{E^R_{\mathbf{CYL}}\{f\circ\phi\}(t)} \lesssim_R \sqrt{E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)}.\end{aligned} \tag{7.43}$$

The second inequality follows from Lemma 7.2. The first and third inequalities are direct consequences of the chain rule, because all derivatives of order up to $R-1$ of the Jacobians of ϕ and ϕ^{-1} have finite sup-norms on their domains of definition depending *only* on R, especially, independent of ξ_0, b and t.

Observe that in (7.28), (7.29), the set $\mathcal{U}$ is now given as in **(RE0)**. Estimate (7.30) still holds. By construction, $D_1(\tau)$ is a disk bundle over the $(t,\underline{u})$-**rectangle** $(\tau,\tau_-)\times(0,b)$.

The boundary $\partial D_1(\tau)$ has five components, four of them arising as disk bundles over the boundary of the **rectangle**, the fifth is a circle bundle over the interior. The treatment of the first four components is unchanged. The fifth is accounted for by **(RE12)**. The domains $D_2(\tau)$ and $D_3(\tau)$ are handled in the same way.

The rest of the proof is completely unchanged. $\square$

8. Classical Vacuum Fields

In Section 6 we have constructed formal power series vacuum fields $[\,\Psi\,]$ by solving an initial value problem for (5.4a). The goal of this section is to prove the existence of an actual, classical, vacuum field Ψ, for which $[\,\Psi\,]$ is rigorously an asymptotic expansion. This is accomplished in Theorem 8.1.

Convention 8.1. We adopt the conventions of Section 7. We use the coordinates $q = (t, \mathbf{q})$ (see, Convention 7.1). Keep in mind that $u = u(q) = q^0 - q^3$. To conveniently translate between the x coordinate system (Sections 2 through 6) and the q coordinate system (Sections 7 and 8), we abuse notation and write $f(q)$ instead of $f(x(q))$, for any function f. It is also implicit that partial derivatives are adapted to the new coordinate system. For example, the matrix differential operator $\mathbf{A}^\mu(x(q), \Psi)\frac{\partial}{\partial x^\mu}$ is abbreviated as $\mathbf{A}^\mu(q, \Psi)\frac{\partial}{\partial q^\mu}$.

8.1. *Preparatory Definitions and Estimates.*

The goal of this subsection is to make the necessary definitions and estimates so that the Existence/Breakdown Theorem and the Refined Energy Estimate can be applied to (5.4a) and (5.4b).

Convention 8.2. (5.4a) and (5.4b) are equivalent to *real* symmetric hyperbolic systems for $\mathcal{R} \cong \mathbb{R}^{31}$ and $\widehat{\mathcal{R}} \cong \mathbb{R}^{32}$ valued fields, respectively. See, Remarks 2.8 and 5.1. This equivalence will be implicit each time the Existence/Breakdown theorem and the Refined Energy Estimate are applied to (5.4a) and (5.4b), or to equivalent systems.

Convention 8.3. In this section, $\mathbb{C}^m$ is a vector space over $\mathbb{R}$ with dimension $2m$. A linear map from $\mathbb{C}^m$ to $\mathbb{C}^n$ is, by convention, linear over $\mathbb{R}$. It can be represented either as a $2n \times 2m$ real matrix, or as an $n \times m$ complex matrix which may have the complex conjugation operator C as matrix elements. We adopt similar conventions for the real subspaces $\mathcal{R} \subset \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5$ and $\widehat{\mathcal{R}} \subset \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3$.

Convention 8.4. The notation $F(q, f, \partial_q f, \ldots)$ displays the explicit pointwise dependence of F on q, $f(q)$, $\partial_q f(q)$, $\ldots$

To put (5.4a) in the form required by Propositions 7.3 and 7.7, we use

(S1) $a, \mathfrak{A} \in \mathbb{R}$ satisfy Convention 7.4.
(S2) $[\,\Psi\,] = \sum_{k=0}^\infty (\frac{1}{u})^k\,\Psi(k)$ is the formal power series solution in Proposition 6.1 corresponding to **DATA**$(\xi, \underline{u}) =$ **DATA**$(\mathbf{q})$ which vanishes for $q^3 < \frac{1}{2}$. Therefore, $[\,\Psi\,]$ vanishes when $q^3 < \frac{1}{2}$ by Proposition 6.1. Fix an integer $K \geq 0$, and set $\Psi_K = \sum_{k=0}^{K+1} (\frac{1}{u})^k\,\Psi(k)$.

(S3) The field Ψ is expressed in terms of (h, σ, ℓ) by

$$\Psi \;=\; (f, \omega, z) \;=\; \Psi_K + (h, \sigma, \ell). \tag{8.1}$$

Let $\Xi = (\Xi_1, \Xi_2, \Xi_3)$ be the field given by

$$\begin{aligned}&(\Xi_1,\ \Xi_2,\ \Xi_3)\\&= \big(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\big) \oplus \big(h_1, h_2, h_4, h_5, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_8\big) \oplus \big(h_3, \sigma_5, \sigma_6, \sigma_7\big)\end{aligned}$$

There is a permutation matrix π so that

$$(h, \sigma, \ell) \;=\; \pi\,(\Xi_1, \Xi_2, \Xi_3) \tag{8.2}$$

The field $\Xi = (\Xi_1, \Xi_2, \Xi_3)$ takes values in $\pi^{-1}\mathcal{R} \;\subset\; \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$. The permutation is required for Proposition 7.7, see **(S6)**.

(S4) System (5.4a) is abbreviated as $\mathbf{A}(q, \Psi)\,\Psi = \mathbf{f}(q, \Psi)$ (see, Convention 8.1). Some of its properties are discussed in Remark 5.1. System (5.4a) is equivalent to

$$\mathbf{B}(q, \Xi)\Xi \;=\; Q(q, \Xi)\,\Xi + \mathbf{Src}(q) \quad , \quad \mathbf{B} = \mathbf{B}^\mu \tfrac{\partial}{\partial q^\mu} \tag{8.3a}$$

$$\mathbf{B}(q, \Xi) = \pi^{-1}\mathbf{A}(q, \Psi_K + \pi\,\Xi)\,\pi \tag{8.3b}$$

$$Q(q, \Xi)\Pi = \pi^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\int_0^1 \mathrm{d}s'\Big(- \mathbf{A}(q, s\pi\Pi)\Psi_K + \mathbf{f}(q, \Psi_K + s'\,\pi\Xi + s\,\pi\Pi)\Big) \tag{8.3c}$$

(here Π is a dummy variable for a field like Ξ) with the source term

$$\mathbf{Src}(q) = \pi^{-1}\left(\mathbf{f}(q, \Psi_K) - \mathbf{A}(q, \Psi_K)\Psi_K\right).$$

The transformation $Q(q, \Xi)$ acting on $\pi^{-1}\mathcal{R}$ is linear over $\mathbb{R}$. Note that the bracketed expression in (8.3c) is a quadratic polynomial in s and s'. The operator $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\int_0^1 \mathrm{d}s'$ selects certain combinations of its coefficients.

(S5) The matrices $\mathbf{B}^\mu$ and Q are affine linear (over $\mathbb{R}$) in Ξ. Let $\dot{\mathbf{B}}^\mu(q)$ and $\dot{Q}(q)$ be the $\mathbb{R}$ linear maps given by

$$\dot{\mathbf{B}}^\mu(q)\Pi = \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,\mathbf{B}^\mu(q, s\Pi) \quad , \quad \dot{Q}(q)\Pi = \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,Q(q, s\Pi)$$

We have, $\mathbf{B}^\mu(q, \Xi) = \mathbf{B}^\mu(q, 0) + \dot{\mathbf{B}}^\mu(q)\,\Xi$. Similarly for Q.

(S6) The three by three $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block-decomposition of $\mathbf{B}$ is

$$\mathbf{B} = \mathrm{diag}(B_1, B_2, B_3), \qquad B_2 = \mathbb{1}_9\,L, \qquad B_3 = \mathbb{1}_4\,N,$$

and B_1 is the 5×5 Hermitian matrix operator on the left hand side of (5.7c). The block-decomposition of Q is denoted $Q = (Q_{mn})_{m,n=1,2,3}$.

(S7) $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ are constant 9×5, 4×9, 4×4 matrices. Their nonzero entries are (C is the complex conjugation operator):

$$\begin{array}{lll}(\mathcal{Q}_1)_{51} = -1 & (\mathcal{Q}_1)_{72} = -1 & (\mathcal{Q}_1)_{93} = 1\\(\mathcal{Q}_2)_{19} = -1 - C & (\mathcal{Q}_2)_{27} = C & (\mathcal{Q}_2)_{28} = -1\\(\mathcal{Q}_3)_{11} = -2 & (\mathcal{Q}_3)_{22} = -1 & \end{array}$$

Observe that $\not{Q}_3$ is symmetric and $\not{Q}_3 \leq 0$. Let $\not{Q}(t) = (\not{Q}_{mn})_{m,n=1,2,3}$ be the $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block matrix

$$\not{Q}(t) = \begin{pmatrix} 0 & 0 & 0 \\ \not{Q}_1 & 0 & 0 \\ 0 & |t|^{-1}\not{Q}_2 & |t|^{-1}\not{Q}_3 \end{pmatrix}$$

(S8) $\not{\mathbf{B}}^\mu$ are the constant, diagonal, $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block matrices

$$\not{\mathbf{B}} = \operatorname{diag}(U,U,U,U,V) \,\oplus\, \mathbb{1}_9\, V \,\oplus\, \mathbb{1}_4\, U \quad , \quad \not{\mathbf{B}} = \not{\mathbf{B}}^\mu \tfrac{\partial}{\partial q^\mu}$$

where $U = \frac{\partial}{\partial q^0}$ and $V = \frac{\partial}{\partial q^0} + \frac{\partial}{\partial q^3}$. Note that $\not{\mathbf{B}}^0 = \mathbb{1}_{18}$, $\not{\mathbf{B}}^1 = \not{\mathbf{B}}^2 = 0$, $\not{\mathbf{B}}^3 \geq 0$.

(S9) Let $s(x)$ be the smooth function that vanishes when $x \leq 0$ and is equal to $e^{-1/x}$ when $x > 0$. Let $\psi = \psi(\mathbf{q}) : \mathbb{R}^3 \to [0,1]$ be the smooth cutoff function

$$\psi(\mathbf{q}) = \frac{s\big(3 - |\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2}\big)}{s\big(3 - |\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2}\big) + s\big(|\frac{\mathfrak{A}}{a}\xi|_{\mathbb{R}^2} - \frac{5}{2}\big)}\; \frac{s\big(\frac{3}{4} - |q^3 - 1|\big)}{s\big(\frac{3}{4} - |q^3-1|\big) + s\big(|q^3 - 1| - \frac{2}{3}\big)}$$

where $\xi = (\xi^1, \xi^2) = (q^1, q^2)$. Let

$$\mathcal{K} = \overline{D_{3|\frac{a}{\mathfrak{A}}|}(0) \times (\tfrac{1}{4}, \tfrac{7}{4})} \qquad \subset \qquad \mathcal{Q} = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0,2)$$

By construction, $\operatorname{supp}_{\mathbb{R}^3} \psi \subset \mathcal{K}$ and ψ is equal to 1 on $D_{\frac{5}{2}|\frac{a}{\mathfrak{A}}|}(0) \times (\frac{1}{3}, \frac{5}{3})$. For each integer $R \geq 0$, the bound $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$ is independent of a and $\mathfrak{A}$, see **(S1)**.

(S10) Define

$$\begin{aligned} \mathbf{M}^\mu(q,\Xi) &= \psi\, \mathbf{B}^\mu(q,\Xi) + (1-\psi)\, \not{\mathbf{B}}^\mu \\ H(q,\Xi) &= \psi\, Q(q,\Xi) + (1-\psi)\, \not{Q}(t) \\ h(q,\Xi) &= H(q,\Xi)\Xi + \psi\, \mathbf{Src}(q) \end{aligned}$$

where $\psi = \psi(\mathbf{q})$ is given in **(S9)**.

(S11) If $\Xi^{(1)}$ and $\Xi^{(2)}$ are both smooth solutions to $\mathbf{B}\,\Xi = Q\,\Xi + \mathbf{Src}$, see **(S4)**, then their difference $\Upsilon = \Xi^{(2)} - \Xi^{(1)}$ is a solution to

$$\begin{aligned} &\mathbf{B}(q,\Xi^{(1)})\Upsilon \;=\; G\,\Upsilon \\ &G\,\Pi \;\stackrel{\text{def}}{=}\; \tfrac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(Q(q,\Xi^{(1)})(s\Pi) - \mathbf{B}^\mu(q,s\Pi)\,\tfrac{\partial \Xi^{(2)}}{\partial q^\mu} + Q(q,s\Pi)\,\Xi^{(2)}\Big) \end{aligned} \tag{8.4}$$

where $G\big(q,\Xi^{(1)},\Xi^{(2)},\partial_q\Xi^{(2)}\big)$ acts on $\pi^{-1}\mathcal{R}$ linearly over $\mathbb{R}$. The bracketed expression in (8.4) is affine linear in s. The operator $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}$ selects the coefficient of s.

Definition 8.1. *Each entry to the left of the vertical bar is a generic symbol for a polynomial (with complex coefficients) in the (components of the) quantities to the right and* *their complex conjugates.*

$$\begin{array}{r|l} \mathcal{J} & u^{-1} \\ \boldsymbol{\mathcal{J}} & 1 \quad \textit{(That is, a generic symbol for a complex number)} \\ \mathcal{H} & \textit{linear over } \mathbb{R} \textit{ in } \Psi(0) \\ \boldsymbol{\mathcal{H}} & \textit{linear over } \mathbb{R} \textit{ in } \mathfrak{A},\ \mathbf{e},\ \boldsymbol{\lambda} \\ \mathcal{G}_K & u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(k)_{k=0\dots K+1},\ \textit{and their first derivatives} \\ \boldsymbol{\mathcal{G}}_K & u^{-1},\ \mathfrak{A},\ \underline{u},\ S_K,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(k)_{k=0\dots K+1},\ \textit{and their first derivatives.} \\ & \textit{It has no constant term as a polynomial in } \Psi(k) \textit{ and its derivatives.} \end{array}$$

There is no subscript K on the generic symbols $\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ because they represent polynomials that are required, *by definition, to be independent of K. Precisely, neither their coefficients nor their degrees depend on K. By contrast, the presence of the subscript K on the generic symbols $\mathcal{G}_K$, $\boldsymbol{\mathcal{G}}_K$ indicates that they represent polynomials that are allowed, by definition, to depend on K in an arbitrary manner. Precisely, their coefficients and degrees may be functions of K.*
Above, S_K is defined by $S = -\sum_{k=0}^{K}(\frac{1}{u})^k\, \mathfrak{A}^{2(k+1)}\underline{u}^{k+1} + \frac{1}{u^{K+1}} S_K$, where as before $\frac{1}{\rho} = -\frac{1}{u} + \frac{S}{u^2}$, see (4.4) *and* (6.5)*.*

Proposition 8.1.

$$\begin{aligned} \mathbf{B}^\mu(q,0) &= \not{\mathbf{B}}^\mu && + && u^{-1}\boldsymbol{\mathcal{H}} && + && u^{-2}\mathcal{G}_K && \text{(8.5a)} \\ Q_{1n}(q,0) &= \not{Q}_{1n}(q) && + && u^{-1}\mathcal{H} + u^{-1}\boldsymbol{\mathcal{H}} && + && u^{-2}\mathcal{G}_K && \text{(8.5b)} \\ Q_{2n}(q,0) &= \not{Q}_{2n}(q) && + && \mathcal{H} + \boldsymbol{\mathcal{H}} && + && u^{-1}\mathcal{G}_K && \text{(8.5c)} \\ Q_{3n}(q,0) &= \not{Q}_{3n}(q) && + && (|t|^{-1} + u^{-1})\boldsymbol{\mathcal{J}} && + && u^{-2}\mathcal{G}_K && \text{(8.5d)} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Src}_1(q) &= u^{-(K+2)}\boldsymbol{\mathcal{G}}_K && \text{(8.6a)} \\ \mathbf{Src}_2(q) &= u^{-(K+2)}\boldsymbol{\mathcal{G}}_K && \text{(8.6b)} \\ \mathbf{Src}_3(q) &= u^{-(K+3)}\boldsymbol{\mathcal{G}}_K && \text{(8.6c)} \end{aligned}$$

(here $\mathbf{Src} = (\mathbf{Src}_1, \mathbf{Src}_2, \mathbf{Src}_3)$ is the $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ decomposition) and

$$\begin{aligned} \overset{\bullet}{\mathbf{B}}{}^\mu(q) &= u^{-2}\mathcal{J} && \text{(8.7a)} \\ \overset{\bullet}{Q}_{1n}(q) &= u^{-1}\mathcal{J} && \text{(8.7b)} \\ \overset{\bullet}{Q}_{2n}(q) &= \mathcal{J} && \text{(8.7c)} \\ \overset{\bullet}{Q}_{3n}(q) &= u^{-2}\mathcal{J} && \text{(8.7d)} \end{aligned}$$

Remark 8.1. This proposition is a detailed examination of large $|u|$ behavior of the constituents of the symmetric hyperbolic system (8.3a). To convey its significance, it is helpful to suppress all but the $\frac{\partial}{\partial q^0}$ derivatives in (8.3a) and analyze the caricature scalar ordinary differential equation

$$\mathbf{b}\big(u,f\big)\, \tfrac{\mathrm{d}}{\mathrm{d}u} f = \mathbf{q}\big(u,f\big)\, f + \mathbf{s}(u) \qquad \text{(8.8)}$$

In this remark, u plays the role of q^0. Suppose $f(u)$ is a solution to this equation on $(-\infty, T)$, $T < 0$, with asymptotic data $\lim_{u\to-\infty} f(u) = 0$ and $\mathbf{b}(u, f(u)) > 0$. How can we estimate $f(u)$? For all $u_1 < T$,

$$f(u_1) = \int_{-\infty}^{u_1} \mathrm{d}u \, \exp\left(\int_u^{u_1} \mathrm{d}s \, \frac{\mathbf{q}(s, f(s))}{\mathbf{b}(s, f(s))} \right) \frac{\mathbf{s}(u)}{\mathbf{b}(u, f(u))} \tag{8.9}$$

If there were constants $A > 0$ and $b > c > 0$ with

$$\mathbf{b}(u, f(u)) \geq b \qquad \mathbf{q}(u, f(u)) \leq c|u|^{-1} \qquad |\mathbf{s}(u)| \leq A|u|^{-2} \tag{8.10}$$

for all $u \in (-\infty, T)$, then (8.9) would imply

$$\sup_{u\in(-\infty,T)} |f(u)| \leq \frac{A}{b-c}\,\frac{1}{|T|} \tag{8.11}$$

The apparent difficulty is that the functions in (8.10) depend on the solution $f(u)$. However, if it can be shown that a *strictly weaker* bound than (8.11), say (8.11) with A replaced by $2A$, implies (8.10), then an open-closed argument justifies (8.11). More precisely, one would first cutoff $-\infty$ by a finite value, argue by continuity, and then remove the cutoff.
To apply this reasoning, assume, in analogy with (8.3a), that $\mathbf{b}$, $\mathbf{q}$ are affine linear in f:

$$\begin{aligned} \mathbf{b}(u, f) &= \mathbf{b}(u, 0) + \overset{\bullet}{\mathbf{b}}(u)\, f & \qquad \overset{\bullet}{\mathbf{b}}(u) &= (\tfrac{\partial}{\partial f}\mathbf{b})(u, 0) \\ \mathbf{q}(u, f) &= \mathbf{q}(u, 0) + \overset{\bullet}{\mathbf{q}}(u)\, f & \qquad \overset{\bullet}{\mathbf{q}}(u) &= (\tfrac{\partial}{\partial f}\mathbf{q})(u, 0) \end{aligned}$$

Also, in analogy with (8.5a), (8.5b), (8.7a), (8.7b), assume that there is a constant $\epsilon > 0$, so that

$$\big|\mathbf{b}(u,0) - \not{b}\big| \leq \epsilon|u|^{-1} \qquad \big|\mathbf{q}(u,0)\big| \leq \epsilon|u|^{-1} \qquad \big|\overset{\bullet}{\mathbf{b}}(u)\big| \leq |u|^{-1} \qquad \big|\overset{\bullet}{\mathbf{q}}(u)\big| \leq |u|^{-1}$$

For convenience, suppose $\not{b} = 1$. The last inequality in (8.10) is an analog of (8.6a). If

$$\epsilon,\ A,\ |T|^{-1} \quad \text{are sufficiently small,} \tag{8.12}$$

then (8.11), with A replaced by $2A$, implies (8.10), with $b = \frac{1}{2}$ and $c = \frac{1}{4}$. It follows from an open-closed argument that (8.11) is a genuine estimate for $f(u)$.
To interpret (8.12) in the light of our analogy, observe that the generic symbols $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ in the second column in (8.5a), (8.5b) can be made small by making $\Psi(0)$ (equivalently, **DATA**) and the angular scaling parameter a small.
We conclude the present discussion with the following remarks:

- The analog of the step from (8.10) to (8.11) for the system (8.3a) is provided by the energy estimate.
- Neglecting $\not{Q}$ for the moment, (8.5d), (8.6c), (8.7d) are similar to (8.5b), (8.6a), (8.7b), since $|t|^{-1} + u^{-1}$ is $\mathcal{O}(u^{-2})$ as $u \to -\infty$ uniformly for $\underline{u}$ in a compact set. The interpretation of (8.5c), (8.6b), (8.7c) is different, because (8.8) is not the appropriate toy model problem for the equation satisfied by Ξ_2. In fact, Ξ_2 satisfies an ordinary differential equation along the *short* integral curves of L, so that less u decay is required.

- The inequality for $\mathbf{q}$ in (8.10) remains true if we add any non-positive constant to $\mathbf{q}$. Analogously, the matrix $\not{Q}(t)$, see **(S7)**, appearing in (8.5c), (8.5d), has only non-positive eigenvalues. This is implicitly exploited in the proof of the energy estimate.

Remark 8.2. In (8.6a) all but the last component of $\mathbf{Src}_1$ are actually $u^{-(K+3)}\boldsymbol{\mathcal{G}}_K$. It is to accommodate the last component that we truncated the formal power series $[\,\Psi\,]$ at $K+1$ rather than K, see **(S2)**.

Proof (of Proposition 8.1). The proof is by direct verification, using **(S4)**, **(S6)**, **(S7)**, Proposition 5.2, Remark 5.2 and Definition 5.1. See the Supplement to Proposition 8.1 (Appendix G). To give the flavor, we schematize the calculations for a few representative cases. Let C be the complex conjugation operator. Now

matrix	component	
$\mathbf{B}^1(q,0) - \not{\mathbf{B}}^1$	$(4,5)$	$-\frac{1}{u}\mathbf{e} + \frac{1}{u^2}\mathcal{G}_K$
$Q_{22}(q,0) - \not{Q}_{22}(q)$	$(6,5)$	$-\omega_1(0)\,C - \overline{\omega_1(0)} + \frac{1}{u}\mathcal{G}_K$
$Q_{33}(q,0) - \not{Q}_{33}(q)$	$(1,1)$	$(\frac{2}{u} + \frac{1}{u^2}\mathcal{G}_K) - (-\frac{2}{\lvert t\rvert})$

in agreement with (8.5a), (8.5c) and (8.5d).

To verify (8.6a), note that $\mathbf{Src}_1 = \mathcal{G}_K$ has no constant term as a polynomial in $\Psi(k)$ and its first derivatives. This follows directly from the definition of **Src** and the properties of f given in **(S4)**. If S is replaced by S_K, see definition (8.1), then $\mathbf{Src}_1 = \boldsymbol{\mathcal{G}}_K$. There is an overall $u^{-(K+2)}$, by construction of the formal power series solution $[\,\Psi\,]$. This implies (8.6a). $\square$

To put (5.4b) in the form required by Proposition 7.7, we use

$\widehat{\text{S1}}$ Let $\varXi^\sharp = (\varXi^\sharp{}_1,\, \varXi^\sharp{}_2,\, \varXi^\sharp{}_3)$ be the field given by

$$
\begin{aligned}
&(\varXi^\sharp{}_1,\, \varXi^\sharp{}_2,\, \varXi^\sharp{}_3)\\
&= (y_1,y_2,y_3) \oplus (s_1,s_2,s_4,s_5,p_1,p_2,p_3,p_4,p_7,p_8) \oplus (s_3,p_5,p_6,p_9)
\end{aligned}
$$

where $\Psi^\sharp = (s,p,y)$ is the constraint field. There is a permutation matrix $\widehat{\pi}$ so that

$$(s,p,y) \;=\; \widehat{\pi}\,(\varXi^\sharp{}_1,\, \varXi^\sharp{}_2,\, \varXi^\sharp{}_3).$$

The field $\varXi^\sharp = (\varXi^\sharp{}_1, \varXi^\sharp{}_2, \varXi^\sharp{}_3)$ takes values in $\widehat{\pi}^{-1}\widehat{\mathcal{R}} \;\subset\; \mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$.

$\widehat{\text{S2}}$ System (5.4b) is abbreviated as $\widehat{\mathbf{A}}(q,\Psi)\Psi^\sharp \;=\; \widehat{\mathbf{f}}(q,\Psi,\partial_q\Psi)\Psi^\sharp$ (see, Convention 8.1). Some of its properties are discussed in Remark 5.1. System (5.4b) is equivalent to the linear, homogeneous symmetric hyperbolic system

$$
\begin{aligned}
\widehat{\mathbf{B}}(q,\Psi)\,\varXi^\sharp = \widehat{Q}(q,\Psi,\partial_q\Psi)\,\varXi^\sharp \quad &, \quad \widehat{\mathbf{B}} = \widehat{\mathbf{B}}^\mu \tfrac{\partial}{\partial q^\mu}\\
\widehat{\mathbf{B}}^\mu(q,\Psi) = \widehat{\pi}^{-1}\,\widehat{\mathbf{A}}^\mu(q,\Psi)\,\widehat{\pi}&\\
\widehat{Q}(q,\Psi,\partial_q\Psi) = \widehat{\pi}^{-1}\,\widehat{\mathbf{f}}(q,\Psi,\partial_q\Psi)\,\widehat{\pi}&
\end{aligned}
$$

The transformation $\widehat{Q}$ acting on $\widehat{\pi}^{-1}\widehat{\mathcal{R}}$ is linear over $\mathbb{R}$. Moreover, $\widehat{\mathbf{B}}^\mu$ depends affine linearly over $\mathbb{R}$ on Ψ, and $\widehat{Q}$ depends affine linearly over $\mathbb{R}$ on $\Psi \oplus \partial_q\Psi$.

$\widehat{\textbf{(S3)}}$ The three by three $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block-decomposition of $\widehat{\mathbf{B}}$ is

$$\widehat{\mathbf{B}} = \mathrm{diag}\big(\widehat{B}_1, \widehat{B}_2, \widehat{B}_3\big), \qquad \widehat{B}_2 = \mathbb{1}_{10}\, L, \qquad \widehat{B}_3 = \mathbb{1}_4\, N,$$

and $\widehat{B}_1$ is the 3×3 Hermitian matrix operator on the left hand side of (5.9c).

$\widehat{\textbf{(S4)}}$ $\widehat{\not{Q}}_1, \widehat{\not{Q}}_2, \widehat{\not{Q}}_3$ are constant 10×3, 4×10, 4×4 matrices. Their nonzero entries are (C is the complex conjugation operator):

$$\begin{array}{ll} (\widehat{\not{Q}}_1)_{5,1} = 1 & (\widehat{\not{Q}}_2)_{1,9} = (\widehat{\not{Q}}_2)_{4,10} = -1 \\ (\widehat{\not{Q}}_3)_{1,1} = -1 & (\widehat{\not{Q}}_2)_{1,10} = (\widehat{\not{Q}}_2)_{2,8} = (\widehat{\not{Q}}_2)_{3,7} = (\widehat{\not{Q}}_2)_{4,9} = -C \end{array}$$

Observe that $\widehat{\not{Q}}_3$ is symmetric and $\widehat{\not{Q}}_3 \le 0$. Let $\widehat{\not{Q}}(t) = (\widehat{\not{Q}}_{mn})_{m,n=1,2,3}$ be the $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block matrix

$$\widehat{\not{Q}}(t) = \begin{pmatrix} 0 & 0 & 0 \\ \widehat{\not{Q}}_1 & 0 & 0 \\ 0 & |t|^{-1}\widehat{\not{Q}}_2 & |t|^{-1}\widehat{\not{Q}}_3 \end{pmatrix}.$$

$\widehat{\textbf{(S5)}}$ $\widehat{\not{\mathbf{B}}}^\mu$ are the constant, diagonal, $\mathbb{C}^3 \oplus \mathbb{C}^{10} \oplus \mathbb{C}^4$ block matrices

$$\widehat{\not{\mathbf{B}}} = \mathbb{1}_3\, U \oplus \mathbb{1}_{10}\, V \oplus \mathbb{1}_4\, U \quad , \quad \widehat{\not{\mathbf{B}}} = \widehat{\not{\mathbf{B}}}^\mu \tfrac{\partial}{\partial q^\mu}$$

with U, V as in **(S8)**. Note that $\widehat{\not{\mathbf{B}}}^0 = \mathbb{1}_{17}$, $\widehat{\not{\mathbf{B}}}^1 = \widehat{\not{\mathbf{B}}}^2 = 0$, $\widehat{\not{\mathbf{B}}}^3 \ge 0$.

Definition 8.2. *Each entry to the left of the vertical bar is a generic symbol for a polynomial (with complex coefficients) in the (components of the) quantities to the right and their complex conjugates.*

$$\begin{array}{l} \mathcal{G}^\sharp\big|\; u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),\ \Psi - \Psi(0),\ \textit{and first derivatives} \\ \mathcal{G}^\sharp_0\big|\; u^{-1},\ \mathfrak{A},\ S,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),\ \Psi - \Psi(0) \\ \mathcal{G}^\sharp_1\big|\; \textit{like } \mathcal{G}^\sharp\textit{, but it has no constant term as a polynomial in } \Psi - \Psi(0),\ \partial_q\big(\Psi - \Psi(0)\big) \\ \boldsymbol{\mathcal{G}}^\sharp\big|\; u^{-1},\ \mathfrak{A},\ \underline{u},\ S_0,\ \mathbf{e},\ \boldsymbol{\lambda},\ \Psi(0),\ \textit{and first derivatives} \end{array}$$

where S_0 is defined by $S = -\mathfrak{A}^2\underline{u} + u^{-1}S_0$, see (4.4) *and* (6.5).

Proposition 8.2. *Suppose* $\widehat{\textbf{(S1)}}$ *to* $\widehat{\textbf{(S5)}}$. *Then*

$$\begin{aligned} \widehat{\mathbf{B}}^\mu(q,\Psi) &= \widehat{\not{\mathbf{B}}}^\mu && + u^{-2}\mathcal{G}^\sharp_0 && (8.13a) \\ \widehat{Q}_{1n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{1n}(q) && + u^{-2}\mathcal{G}^\sharp && (8.13b) \\ \widehat{Q}_{2n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{2n}(q) \quad + \quad \boldsymbol{\mathcal{H}} && + u^{-1}\mathcal{G}^\sharp && (8.13c) \\ \widehat{Q}_{3n}(q,\Psi,\partial_q\Psi) &= \widehat{\not{Q}}_{3n}(q) \quad + (|t|^{-1} + u^{-1})\boldsymbol{\mathcal{J}} && + u^{-2}\mathcal{G}^\sharp && (8.13d) \end{aligned}$$

Moreover,

$$\Xi^\sharp{}_1,\ \Xi^\sharp{}_3,\ s_1,\ s_2,\ p_1,\ p_2,\ p_3 = u^{-1}\boldsymbol{\mathcal{G}}^\sharp + \quad \mathcal{G}^\sharp_1 \tag{8.14a}$$

$$s_4,\ s_5,\ p_4,\ p_7,\ p_8 = u^{-1}\boldsymbol{\mathcal{G}}^\sharp + u\,\mathcal{G}^\sharp_1 \tag{8.14b}$$

Finally, it is a consequence of (5.4b) *that*

$$L \begin{pmatrix} s_4 \\ s_5 \\ p_4 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} \mathbf{e}(\overline{p}_4 - p_5) + \ \ \mathbf{e}(\overline{p}_6 - p_3) \\ i\,\mathbf{e}(\overline{p}_4 - p_5) - i\,\mathbf{e}(\overline{p}_6 - p_3) \\ 0 \\ \boldsymbol{\lambda}(p_6 - \overline{p}_3) - \overline{\boldsymbol{\lambda}}(p_4 - \overline{p}_5) \\ -\overline{\boldsymbol{\lambda}}(\overline{p}_6 - p_3) + \boldsymbol{\lambda}(\overline{p}_4 - p_5) \end{pmatrix} + u^{-1}\,\mathcal{G}^\sharp\,\Xi^\sharp \tag{8.15}$$

Proof. The first part, (8.13), and the last part, (8.15), follow directly from Proposition 5.4, Remark 5.2 and Definition 5.1. It is entirely similar to the proof of Proposition 8.1.

To prove (8.14), write $\Psi = \Psi(0) + (\Psi - \Psi(0))$, and consider each object on the left hand side of (8.14) as a polynomial in $\Psi - \Psi(0)$ and $\partial_q(\Psi - \Psi(0))$, with coefficients possibly depending on $\Psi(0)$ and $\partial_q\Psi(0)$ (see, Proposition 5.3). The idea is that the constant term of this polynomial is of the generic form $u^{-1}\boldsymbol{\mathcal{G}}^\sharp$. Everything else is of the form $\mathcal{G}_1^\sharp$ or $u\,\mathcal{G}_1^\sharp$, respectively. The fact that the constant term is of the generic form $u^{-1}\boldsymbol{\mathcal{G}}^\sharp$ is an essential part of the construction. It is the fact that $\Psi(0)$ is built so that the first term in the formal power series of the constraint field, $\Psi^\sharp(0)$, vanishes (this follows from the vanishing of the formal constraint field and Remark 6.1). $\square$

Everything we have done in this section so far was to prepare for the next proposition that provides a list of sufficient conditions under which the abstract propositions of Section 7 can be applied to the various symmetric hyperbolic systems that are required for the proof of Theorem 8.1.

Proposition 8.3 (Main Technical Proposition). *Fix $K \geq 0$ as in* **(S2)**. *Recall* **(S1)** *through* **(S11)** *and* $\widehat{\textbf{(S1)}}$ *through* $\widehat{\textbf{(S5)}}$. *Let $R \geq 4$ be an integer. Set*

$$Y = \Big(\, R,\ K,\ \|\mathbf{DATA}\|_{C^{R+2K+6}(\mathcal{Q})} \Big) \tag{8.16}$$

Let $\mathfrak{d} = 10^{-3}$ be as in (7.40)*. Fix $\mathbf{c}_2' \in (0,1)$ and $T \in (-\infty, -\mathfrak{d}^{-1})$. There are constants $\mathbf{c}_6(R) \in (0,1)$ and $\mathbf{c}_7(Y) \in (0,1)$, non-increasing in all their arguments, such that Parts 1, 2 and 3 below hold whenever*

$$|a| \leq \mathbf{c}_6(R)\,\mathbf{c}_2' \quad , \quad \|\mathbf{DATA}\|_{C^{R+4}(\mathcal{Q})} \leq \mathbf{c}_6(R)\,\mathbf{c}_2' \quad , \quad |T|^{-1} \leq \mathbf{c}_7(Y)\,\mathbf{c}_2' \tag{8.17}$$

Part 1. *The system $\mathbf{M}(q,\Xi)\,\Xi = h(q,\Xi)$ in* **(S10)** *satisfies* **(EB0)** *through* **(EB4)** *of Subsection 7.3 and the assumptions of Part 2 of Proposition 7.3, with:*

(EB0) - (EB4)	T	P	$\mathbf{M}^\mu(q,\Theta)$	$h(q,\Theta)$	$\not{\mathbf{M}}^\mu$	$\not{H}(t)$	$\mathcal{Q}$	$\mathcal{K}$
(S1) - (S10)	T	31	$\mathbf{M}^\mu(q,\Xi)$	$h(q,\Xi)$	$\not{\mathbf{B}}^\mu$	$\not{\mathcal{Q}}(t)$	$\mathcal{Q}$	$\mathcal{K}$

The table indicates that the symbols in the first row, appearing in the general **(EB0)** *through* **(EB4)***, are given by the specific objects in the second row, appearing in* **(S1)** *through* **(S10)**.

Part 2. *If $t_0 < T$ and $\Xi : [t_0, t_0 + \epsilon) \times \mathbb{R}^3 \to \pi^{-1}\mathcal{R}$ ($\epsilon > 0$) is a C^∞ solution to $\mathbf{M}(q,\Xi)\,\Xi = h(q,\Xi)$ which vanishes identically at t_0, then*

$$|t_0|^{K+1} \sup_{\xi_0 \in \mathbb{R}^2} \sqrt{E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\Xi\}(t_0)} \ \ \leq \ \ (\mathbf{c}_7(Y))^{-1} \tag{8.18}$$

Here, the energy $E^R_{\mathcal{O}(\xi,b,t)}$ is defined as in the Refined Energy Estimate, Proposition 7.7.

Assumptions of Part 3. *We distinguish three alternative systems, denoted by* **(Sys1)**, **(Sys2)** *and* **(Sys3)**, *that are given in the columns of the table below. This part of the Proposition applies to each of these systems individually. To evaluate the entries in the table, we require the following information. First,*

(Sys1):	$b = 2$	$\xi_0 \in \mathbb{R}^2$
(Sys2):	$b = 1$	$\xi_0 \in D_{2\lvert\frac{a}{\mathfrak{A}}\rvert}(0)$
(Sys3):	$b = 1$	$\xi_0 \in D_{2\lvert\frac{a}{\mathfrak{A}}\rvert}(0)$

Second, fix $t_0 < T$ and define the open set

$$\mathcal{V} \;=\; \bigcup_{t\in(t_0,T)} \{t\} \times \mathcal{O}(\xi_0, b, t) \;\;\subset\;\; \mathbb{R}^4.$$

For **(Sys1)** *and* **(Sys3)** *there is a single field $\varXi$ defined on $\mathcal{V}$ taking values in $\pi^{-1}\mathcal{R}$. For system* **(Sys2)**, *there are two fields, $\varXi^{(1)}$ and $\varXi^{(2)}$, of this kind. The various fields satisfy the conditions:*

(i) They are C^p and their derivatives of order $\le p$ extend continuously to $\overline{\mathcal{V}}$. Here $p = \infty$ for **(Sys1)**, **(Sys3)** *and $p = 1$ for* **(Sys2)**.
(ii) They are solutions to

$$\begin{cases} \mathbf{M}(q,\varXi)\varXi = h(q,\varXi) & \textit{for } \textbf{(Sys1)} \\ \mathbf{B}(q,\varXi^{(i)})\varXi^{(i)} = Q(q,\varXi^{(i)}) + \mathbf{Src}(q) & \textit{for } \textbf{(Sys2)} \\ \mathbf{B}(q,\varXi)\varXi = Q(q,\varXi) + \mathbf{Src}(q) & \textit{for } \textbf{(Sys3)} \end{cases}$$

See **(S4)** *and* **(S10)**.
(iii) They vanish when $q^3 < \frac{1}{2}$.
(iv) For all $t \in (t_0,T)$, they satisfy

$$\begin{cases} E^R_{\mathcal{O}(\xi_0,2,t)}\{\varXi\}(t) \;\le\; (\mathbf{c}_6(R)\mathbf{c}_2')^2 & \textit{for } \textbf{(Sys1)} \\ \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi^{(i)}\}(t) \;\le\; \mathbf{c}_6(R)\mathbf{c}_2' & \textit{for } \textbf{(Sys2)} \\ \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\varXi\}(t) \;\le\; \mathbf{c}_6(R)\mathbf{c}_2' & \textit{for } \textbf{(Sys3)} \end{cases}$$

Conclusions of Part 3. *To state the conclusions, recall the notation $\varUpsilon = \varXi^{(2)} - \varXi^{(1)}$ and the usage $\varPsi^\sharp$ for the constraint field associated to $\varPsi = \varPsi_K + \pi\,\varXi$, see* **(S2)**. *Finally, $\varPsi^\sharp = \widehat{\pi}\,\varXi^\sharp$, as in* $\widehat{\textbf{(S1)}}$.

Conclusion 1: $\varXi(\mathcal{V}),\ \varXi^{(1)}(\mathcal{V}),\ \varXi^{(2)}(\mathcal{V}) \;\subset\; \overline{B_{1/2}(0)} \;\subset\; \pi^{-1}\mathcal{R} \;\cong\; \mathbb{R}^{31}$.

Conclusion 2: *The assumptions* **(RE0)** *through* **(RE12)** *hold, with* **(RE11a)** *for* **(Sys1)** *and* **(RE11b)** *for* **(Sys2)** *and* **(Sys3)**, *provided that the symbols in the first column of the table* *below (appearing in the general* **(RE0)** *through* **(RE12)***) are given by the specific objects in the other three columns.*

(RE0) - (RE12)	**(Sys1)**	**(Sys2)**	**(Sys3)**
$\mathcal{I} = (t_0, t^*)$	(t_0, T)	(t_0, T)	(t_0, T)
b	2	1	1
(P_1, P_2, P_3)	$(10, 15, 6)$	$(10, 15, 6)$	$(6, 18, 8)$
$\mathbf{M}^\mu(q)$	$\mathbf{M}^\mu(q, \Xi(q))$	$\mathbf{B}^\mu(q, \Xi^{(1)}(q))$	$\widehat{\mathbf{B}}^\mu(q, \Psi(q))$
$H(q)$	$H(q, \Xi(q))$	$G\big(q, \Xi^{(1)}, \Xi^{(2)}, \partial_q \Xi^{(2)}\big)$	$\widehat{Q}(q, \Psi, \partial_q \Psi)$
$\mathbf{Src}(q)$	$\psi\, \mathbf{Src}(q)$	0	0
$\Theta(q)$	$\Xi(q)$	$\Upsilon(q)$	$\Xi^\sharp(q)$
$\not{\mathbf{M}}^\mu$	$\not{\mathbf{B}}^\mu$	$\not{\mathbf{B}}^\mu$	$\widehat{\not{\mathbf{B}}}^\mu$
$\not{H}(t)$	$\not{Q}(t)$	$\not{Q}(t)$	$\widehat{\not{Q}}(t)$
R	R	0	0
$\mathbf{c}_1$	$(\mathbf{c}_7(Y))^{-1}$	0	0
$\mathbf{c}_2$	$\mathbf{c}_2'$	$\mathbf{c}_2'$	$\mathbf{c}_2'$
J	$K+1$	>0	>0
ξ_0	ξ_0	ξ_0	ξ_0

Conclusion 3: *For* **(Sys3)***, if in addition* $t_0 + 1 < T$*, then*

$$\sup_{t \in (t_0+1, T)} |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi^\sharp\}(t) \ \lesssim_Y \ \Big(1 + \sup_{t \in (t_0, T)} |t|\, \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi\}(t)\Big) \tag{8.19}$$

Proof. We begin with a warning.

First Warning. In the course of this proof, we produce a finite chain of smallness assumptions on $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$. It is *essential*, for the purpose of showing that the far field expansion is truly an asymptotic expansion to a classical solution of (5.4a), that these smallness assumptions depend only on R and Y, respectively. To give a representative example, suppose **quantity** $\lesssim_R$ $\mathbf{c}_6(R)$. Then there is a legitimate smallness assumption on $\mathbf{c}_6(R)$ making, say, **quantity** ≤ 1. By contrast, there is no legitimate smallness assumption associated to **quantity** $\lesssim_Y$ $\mathbf{c}_6(R)$. We can take a more relaxed attitude to the system (5.4b), because it is only necessary to demonstrate uniqueness.

Convention 8.5. In this proof, the constants of proportionality in $\lesssim_R$ and $\lesssim_Y$ are always non-decreasing in R and the components of Y, respectively.

Overall Preliminaries. For all $n \geq 0$ and $0 \leq k \leq K+1$ and $\beta \in \mathbb{N}_0^4$ with $|\beta| \leq 1+R$, the following estimates hold on $(-\infty, T) \times \mathcal{Q}$:

$$\begin{array}{lll} |\partial^\beta u^{-n}| \lesssim_{(R,n)} |t|^{-n} & |\partial^\beta \mathfrak{A}| = |\mathfrak{A}|\, \delta_{\beta 0} \leq |a| & |\partial^\beta \underline{u}| \leq 2 \\ |\partial^\beta \Psi(0)| \lesssim_R \mathbf{c}_6(R)\, \mathbf{c}_2' & |\partial^\beta \mathbf{e}| \leq \frac{17}{2}\, |a| & |\partial^\beta \boldsymbol{\lambda}| \leq \frac{17}{2}\, |a| \\ |\partial^\beta \Psi(k)| \lesssim_Y 1 & |\partial^\beta S| \lesssim_R 1 & |\partial^\beta S_K| \lesssim_Y 1 \end{array}$$

Only the estimates on $\Psi(0)$ and $\Psi(k)$ require discussion. They follow from Proposition 6.3, and (8.16) and (8.17). See Definition 8.1 for S_K.

It follows from the estimates just above, the product rule and (8.17), that for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le R$,

$$
\begin{aligned}
&n = 0,1 && |t|^n\,|\partial^\alpha(u^{-(n+1)}\mathcal{G}_K)| \;\lesssim_Y\; |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}_2' && \text{(8.20a)}\\
&n = 1,2 && |t|^{K+n}|\partial^\alpha(u^{-(K+n)}\boldsymbol{\mathcal{G}}_K)| \;\lesssim_Y\; 1 && \text{(8.20b)}\\
&n = 0,1 && |t|^n\,|\partial^\alpha(u^{-n}\mathcal{H})| \;\lesssim_R\; \mathbf{c}_6(R)\,\mathbf{c}_2' && \text{(8.20c)}\\
&n = 0,1 && |t|^n\,|\partial^\alpha(u^{-n}\boldsymbol{\mathcal{H}})| \;\lesssim_R\; |a| \lesssim_R \mathbf{c}_6(R)\,\mathbf{c}_2' && \text{(8.20d)}\\
&n = 0,1,2 && |t|^n\,|\partial^\alpha(u^{-n}\mathcal{J})| \;\lesssim_R 1 && \text{(8.20e)}\\
& && |t|\,|\partial^\alpha(|t|^{-1}+u^{-1})\boldsymbol{\mathcal{J}}| \;\lesssim_R\; |T|^{-1} \lesssim_R \mathbf{c}_7(Y)\,\mathbf{c}_2' && \text{(8.20f)}
\end{aligned}
$$

at every point of $(-\infty, T) \times \mathcal{Q}$. In this instance, the constants also depend on the particular polynomial represented by the generic symbols. Observe that in the second inequality, one does not use the property that $\boldsymbol{\mathcal{G}}_K$ has no constant term as a polynomial in $\Psi(k)$ and its derivatives.

Second Warning. It is crucially important that whenever $\lesssim_R$ appears in an estimate (for example, (8.20c), (8.20d), (8.20e), (8.20f)) that the generic symbol on the left hand side has no subindex K, see Definition 8.1. On the other hand, whenever $\lesssim_Y$ appears (for example, (8.20a), (8.20b)), the generic symbol on the left hand side is allowed to carry a subindex K.

Preliminaries for Part 3. For Part 3, it is necessary to supplement the Overall Preliminaries. Let $(\mathcal{V}_1, \mathcal{V}_2)$ be the open cover of $\mathcal{V}$ given by

$$\mathcal{V}_1 = \mathcal{V} \cap \big((t_0, T) \times \mathcal{Q}\big), \qquad \mathcal{V}_2 = \mathcal{V} \cap \big((t_0, T) \times (\mathbb{R}^3 \setminus \mathcal{K})\big).$$

The sets $\mathcal{Q}$, $\mathcal{K}$ are defined in **(S9)**.

- For **(Sys1)**, observe that the Overall Preliminaries apply to $\mathcal{V}_1$. On $\mathcal{V}_2$, we have $\psi = 0$, and the equations simplify, see **(S10)**. The estimate $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$, see **(S9)**, will be used on the transition region for ψ.
- For **(Sys2)** we have $\mathcal{V} = \mathcal{V}_1$. In this case, the Overall Preliminaries will suffice.
- For **(Sys3)** we also have $\mathcal{V} = \mathcal{V}_1$. However, in addition to the Overall Preliminaries, we require the estimates

$$
\begin{aligned}
& & |\Psi - \Psi(0)| \;,\; \big|\partial_q\big(\Psi - \Psi(0)\big)\big| \;&\lesssim_Y\; 1 && \text{(8.21)}\\
& & |t|\,|u^{-2}\mathcal{G}_0^\sharp| \;,\; |t|\,|\partial_q(u^{-2}\mathcal{G}_0^\sharp)| \;&\lesssim_Y\; |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}_2'\\
&n = 0,1: & |t|^n\,|u^{-(n+1)}\mathcal{G}^\sharp| \;&\lesssim_Y\; |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}_2',\\
& & |\boldsymbol{\mathcal{H}}| \;&\lesssim\; |a| \lesssim \mathbf{c}_6(R)\,\mathbf{c}_2'\\
& & |t|\,|(|t|^{-1}+u^{-1})\boldsymbol{\mathcal{J}}| \;&\lesssim_Y\; |T|^{-1} \lesssim_Y \mathbf{c}_7(Y)\,\mathbf{c}_2'.
\end{aligned}
$$

on $\mathcal{V}$. Estimate (8.21) follows from

$$\Psi - \Psi(0) = \sum_{k=1}^{K+1} (\tfrac{1}{u})^k\,\Psi(k) + \pi\,\Xi \qquad \text{(8.22)}$$

and condition (iv) in the Proposition. The rest are consequences of (8.21) and the Overall Preliminaries estimates.

Proof of Part 1. One verifies **(EB0)** through **(EB4)** by direct inspection, apart from the inequality in **(EB1)**. However,

$$\tfrac{1}{2} \le \mathbf{M}^0(q,\Xi) \le 2, \qquad \mathbf{M}^3(q,\Xi) \ge 0, \tag{8.23}$$

for all $(q,\Xi) \in (-\infty,T) \times \mathbb{R}^3 \times B_2(0)$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are small enough. Here $B_2(0) \subset \mathbb{R}^P$ as in **(EB0)**, with $P = 31$. The first inequality in (8.23) completes the verification of **(EB1)**. The second inequality in (8.23) is used for the first supplemental hypothesis in Part 2 of Proposition 7.3.

To check (8.23), recall from **(S10)** that

$$\mathbf{M}^\mu(q,\Xi) = \psi\,\mathbf{B}^\mu(q,\Xi) + (1-\psi)\,\not{\mathbf{B}}^\mu \tag{8.24}$$

is a convex combination of $\mathbf{B}^\mu$ and $\not{\mathbf{B}}^\mu$ on $(-\infty,T) \times \mathbb{R}^3 \times B_2(0)$. It suffices to verify (8.23) for $\mathbf{B}^\mu$ and $\not{\mathbf{B}}^\mu$ separately. For $\not{\mathbf{B}}^\mu$, see **(S8)**. For $\mathbf{B}^\mu$ it suffices to verify

$$\tfrac{1}{2} \le 1 + \tfrac{1}{u^2} f_3 \le 2, \qquad \tfrac{1}{2} \le 1 + \tfrac{1}{u^2}(1 + \tfrac{1}{u^2} f_3) \le 2 \tag{8.25}$$

for $\mathbf{q} \in \operatorname{supp}_{\mathbb{R}^3} \psi \subset \mathcal{Q}$, see **(S6)** and Remark 5.2. Here, f_3 is one of the components of $\Psi = \Psi_K + \pi\,\Xi = (f,\omega,z)$, see **(S3)**. To check (8.25), note that

$$|\Psi| \;\le\; |\Psi(0)| + \tfrac{1}{|u|} \sum_{k=1}^{K+1} \tfrac{1}{|u|^{k-1}}\,|\Psi(k)| + |\Xi|.$$

The three terms are respectively $\lesssim_R \mathbf{c}_6(R)$ and $\lesssim_Y \frac{1}{|T|} \lesssim_Y \mathbf{c}_7(Y)$ (see, Overall Preliminaries) and ≤ 2. By the choice of $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ (see, the First Warning), we can make

$$|\Psi| \le 3 \;\;,\;\text{when}\quad (q,\Xi) \in (-\infty,T) \times \mathcal{Q} \times B_2(0) \tag{8.26}$$

Consequently, $|f_3| \le 3$, and therefore, (8.25) holds because $\frac{1}{|u|} \le \frac{1}{|T|} \le \mathfrak{d}$, see (7.40).

To validate the second supplemental hypothesis in Part 2 of Proposition 7.3, it is necessary to show that $h(q,0) = \psi\,\mathbf{Src}(q) = 0$ for all $q \in (-\infty,T) \times \mathbb{R}^3$ with $q^3 < \frac{1}{2}$. To do this, observe that $\Psi_K = 0$ there, see **(S2)**.

Remark 8.3. Later on, in the proof of Part 3, we need the analogous inequalities

$$\tfrac{1}{2} \le \widehat{\mathbf{B}}^0(q,\Psi) \le 2, \qquad \widehat{\mathbf{B}}^3(q,\Psi) \ge 0$$

for all $(q,\Xi) \in (-\infty,T) \times \mathbb{R}^3 \times B_2(0)$, with the same smallness assumptions. These inequalities can again be reduced to (8.25).

Proof of Part 2. To prove (8.18), rewrite $\mathbf{M}(q,\Xi)\,\Xi = h(q,\Xi)$ as

$$\partial_t \Xi(q) = \big(\mathbf{M}^0(q,\Xi)\big)^{-1}\Big(-\sum_{i=1,2,3} \mathbf{M}^i(q,\Xi)\partial_i \Xi + H(q,\Xi)\,\Xi + \psi\,\mathbf{Src}\Big) \tag{8.27}$$

Here, $\mathbf{M}^0(\,\cdot\,,\Xi)$ is invertible on an open neighborhood of $\{t_0\} \times \mathbb{R}^3$ in the set $[t_0, t_0 + \epsilon) \times \mathbb{R}^3$. This is a consequence of (8.23), and the assumption $\Xi(t_0,\,\cdot\,) \equiv 0$.

By repeated differentiation of (8.27) with respect to t, we obtain an expression for $\partial_t^m \Xi(q)$, for any $m \ge 1$. Restrict the result to $\{t_0\} \times \mathbb{R}^3$ and simplify it using the assumption $\partial^\beta \Xi(t_0,\,\cdot\,) \equiv 0$ for all $\beta \in \mathbb{N}_0^4$ with $\beta_0 = 0$. In every surviving multi-derivative $\partial^\beta \Xi(t_0,\,\cdot\,)$ we must have $1 \le \beta_0 \le m-1$, and each one is recursively expressed using $\partial_t^n \Xi(t_0,\,\cdot\,)$ with $1 \le n \le m-1$. This procedure generates an explicit expression for $\partial_t^m \Xi(t_0,\,\cdot\,)$ in terms of the quantities in (8.28) just below, and their derivatives.

By differentiation with respect to the remaining coordinates $\mathbf{q}$, we find (inductively) that for every $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \le R$, the function $\partial^\alpha \Xi(t_0,\,\cdot\,)$ is a polynomial in $(\mathbf{M}^0((t_0,\,\cdot\,),0))^{-1}$ as well as derivatives of order $\le R-1$ of

$$\mathbf{M}^\mu\big((t_0,\,\cdot\,),0\big)\ ,\ H\big((t_0,\,\cdot\,),0\big)\ ,\ \overset{\bullet}{\mathbf{M}}{}^\mu(t_0,\,\cdot\,)\ ,\ \overset{\bullet}{H}(t_0,\,\cdot\,)\ ,\ \psi\,\mathbf{Src}(t_0,\,\cdot\,) \tag{8.28}$$

This polynomial has no constant term as a polynomial in $\psi\,\mathbf{Src}(t_0,\,\cdot\,)$ and its derivatives. This has two consequences. Consequence A, $\partial^\alpha \Xi(t_0,\,\cdot\,)$ vanishes on $\mathbb{R}^3 \setminus \mathcal{K}$, by the support of ψ. Consequence B,

$$|t_0|^{K+1}\,\big|\partial^\alpha \Xi(t_0,\mathbf{q})\big| \lesssim_Y 1$$

for all $\mathbf{q} \in \mathcal{Q}$. To verify this inequality, check (using $\|\psi\|_{C^R(\mathbb{R}^3)} \lesssim_R 1$ in **(S9)**, the first inequality in (8.23), Proposition 8.1 and the Overall Preliminaries) that the matrix $(\mathbf{M}^0((t_0,\,\cdot\,),0))^{-1}$ and the derivatives of order $\le R-1$ of all the terms in (8.28) are bounded in absolute value on $\mathcal{Q}$ by $\lesssim_Y 1$. At this point, we have $\big|\partial^\alpha \Xi(t_0,\mathbf{q})\big| \lesssim_Y 1$. To get the stated decay, use (8.6) (even though one can get a better result). It is here that one exploits the fact that the expression for $\partial^\alpha \Xi(t_0,\,\cdot\,)$ has no constant term as a polynomial in $\psi\,\mathbf{Src}(t_0,\,\cdot\,)$ and its derivatives.

The proof of Part 2 is completed by combining Consequences A and B with

$$E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\Xi\}(t_0) \lesssim_R \big(\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,2,t_0)}\{\Xi\}(t_0)\big)^2$$

and making a suitable choice of $\mathbf{c}_7(Y)$ (see, the First Warning).

Proof of Part 3, Conclusion 1. Follows from condition (iv) in the Proposition, by suitable choice of $\mathbf{c}_6(R)$. For **(Sys1)**, we also use $R \ge 2$ and the Sobolev inequality (7.43).

Proof of Part 3, Conclusion 2. To start with, we check that **(RE10)** and **(RE11a)** or **(RE11b)** hold, when $\mathbf{c}_6(R)$, $\mathbf{c}_7(Y)$ are made sufficiently small. We will freely use the Overall Preliminaries, the Preliminaries for Part 3, Propositions 8.1, 8.2, and the inequality

$$E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) \lesssim_R \big(\mathbf{Sup}^{(R)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)\big)^2. \tag{8.29}$$

- **(Sys1)**: Let $t \in (t_0,T)$. For **(RE10)**, we have

  $$|t|^{2K+4}E^R_{\mathcal{O}(\xi_0,2,t)}\{\psi\,\mathbf{Src}_1\}(t) = |t|^{2K+4}E^R_{\mathcal{O}(\xi_0,2,t)}\{\psi\,u^{-(K+2)}\boldsymbol{\mathcal{G}}_K\}(t) \lesssim_Y 1$$

  Therefore, the left hand side is $\le (\mathbf{c}_7(Y))^{-2}$, when $\mathbf{c}_7(Y) > 0$ is small enough (see, the First Warning). Similarly, for $\psi\,\mathbf{Src}_2$ and $\psi\,\mathbf{Src}_3$. In these two cases, one could get a better decay estimate, but we don't need it.
  For **(RE11a)**, we verify the first and second inequalities, the other two are similar. The second goes

  $$\begin{aligned}
  &|t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{H_{1n}(q,\Xi) - \mathcal{Q}_{1n}\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(Q_{1n}(q,0) - \mathcal{Q}_{1n}\big) + \psi\,\overset{\bullet}{Q}_{1n}(q)\Xi\big\}(t)\\
  &= |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K\big) + \psi\,\tfrac{1}{u}\mathcal{J}\,\Xi\big\}(t)\\
  &\lesssim_R |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\big(\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u}\mathcal{J}\,\Xi\big)\big\}(t) + |t|^2\,E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\,\tfrac{1}{u^2}\mathcal{G}_K\big\}(t)
  \end{aligned}$$

The first term is $\lesssim_R (\mathbf{c}_6(R)\mathbf{c}_2')^2$, the second is $\lesssim_Y (\mathbf{c}_7(Y)\mathbf{c}_2')^2$. Here, we use condition (iv) in the Proposition. By the First Warning, the result is $\leq (\mathbf{c}_2')^2$ if $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are small enough. The first goes

$$\begin{aligned}
&|t|^2\, E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\mathbf{M}^\mu(q,\Xi) - \not{\mathbf{B}}^\mu\big\}(t)\\
&= |t|^2\, E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi(\mathbf{B}^\mu(q,0) - \not{\mathbf{B}}^\mu) + \psi\, \dot{\mathbf{B}}^\mu(q)\,\Xi\big\}(t)\\
&= |t|^2\, E^R_{\mathcal{O}(\xi_0,2,t)}\big\{\psi\,(\tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K) + \psi\,\tfrac{1}{u^2}\mathcal{J}\,\Xi\big\}(t)
\end{aligned}$$

The estimate is completed just as above.

- **(Sys2)**: There is nothing to check for **(RE10)**. For **(RE11b)**,

$$\begin{aligned}
&|t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\mathbf{B}^\mu(q,\Xi^{(1)}) - \not{\mathbf{B}}^\mu\big\}(t)\\
&= |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\big(\mathbf{B}^\mu(q,0) - \not{\mathbf{B}}^\mu\big) + \dot{\mathbf{B}}^\mu(q)\,\Xi^{(1)}\big\}(t)\\
&= |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K + \tfrac{1}{u^2}\mathcal{J}\,\Xi^{(1)}\big\}(t)
\end{aligned}$$

which is $\leq \mathbf{c}_2'$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. It is important here that $\mathcal{V} = \mathcal{V}_1$, see, the Preliminaries for Part 3. For the second, third and fourth parts of **(RE11b)**, observe that, by (8.4),

$$\begin{aligned}
(G - \not{Q})\,\Pi \;&= \big(Q(q,0) - \not{Q}(t)\big)\Pi + \big(\dot{Q}(q)\,\Xi^{(1)}\big)\Pi\\
&\quad + \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\Big(-\mathbf{B}^\mu(q,s\Pi)\,\tfrac{\partial\Xi^{(2)}}{\partial q^\mu} + Q(q,s\Pi)\,\Xi^{(2)}\Big).
\end{aligned}$$

Therefore, for the second,

$$\begin{aligned}
&|t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{G_{1n}\big(q,\Xi^{(1)},\Xi^{(2)},\partial_q\Xi^{(2)}\big) - \not{Q}_{1n}(t)\big\}(t)\\
&= |t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u}\mathcal{H} + \tfrac{1}{u}\boldsymbol{\mathcal{H}} + \tfrac{1}{u^2}\mathcal{G}_K + \tfrac{1}{u}\mathcal{J}\Xi^{(1)} + \tfrac{1}{u^2}\mathcal{J}\,\partial_q\Xi^{(2)} + \tfrac{1}{u}\mathcal{J}\Xi^{(2)}\big\}(t)
\end{aligned}$$

which is $\leq \mathbf{c}_2'$ when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. The third an fourth inequalities in **(RE11b)** are checked in the same way.

- **(Sys3)**: There is nothing to check for **(RE10)**. For **(RE11b)**,

$$\begin{aligned}
|t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{\mathbf{B}}^\mu(q,\Psi) - \widehat{\not{\mathbf{B}}}^\mu\big\}(t) &= |t|\,\mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u^2}\mathcal{G}_0^\sharp\big\}(t)\\
|t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{1n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{1n}\big\}(t) &= |t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\tfrac{1}{u^2}\mathcal{G}^\sharp\big\}(t)\\
\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{2n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{2n}\big\}(t) &= \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\boldsymbol{\mathcal{H}} + \tfrac{1}{u}\,\mathcal{G}^\sharp\big\}(t)\\
|t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{\widehat{Q}_{3n}(q,\Psi,\partial_q\Psi) - \widehat{\not{Q}}_{3n}\big\}(t) &\\
&= |t|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\big\{(\tfrac{1}{|t|} + \tfrac{1}{u})\boldsymbol{\mathcal{J}} + \tfrac{1}{u^2}\mathcal{G}^\sharp\big\}(t)
\end{aligned}$$

which are all $\leq \mathbf{c}_2'$, when $\mathbf{c}_6(R)$ and $\mathbf{c}_7(Y)$ are made sufficiently small. It is important here that $\mathcal{V} = \mathcal{V}_1$, see, the Preliminaries for Part 3.

We are now finished checking **(RE10)** and **(RE11a)** or **(RE11b)**.

Next, we check **(RE1)**. In order, **(RE1)** follows from

- **(Sys1)**: inequalities (8.23), since $\Xi(q) \in \overline{B_{\frac{1}{2}}(0)} \subset \mathbb{R}^{31}$ for all $q \in \mathcal{V}$ by Conclusion 1.

- **(Sys2)**: the discussion of (8.25), Conclusion 1 and $\mathcal{V} = \mathcal{V}_1$.
- **(Sys3)**: Remark 8.3, Conclusion 1 and $\mathcal{V} = \mathcal{V}_1$.

To check **(RE12)**, we can assume $|\Psi| \leq 3$ on $\mathcal{V}_1$, by (8.26) and Conclusion 1. By Proposition 7.6 and **(S1)**, this implies **(RE12)** for **(Sys2)** and **(Sys3)**, because $\mathcal{V} = \mathcal{V}_1$. For **(Sys1)**, we use convexity, see (8.24). The $\psi\mathbf{B}$ term is again handled by Proposition 7.6, using the support properties of ψ. It therefore suffices to verify that

$$\theta_\mu \not{\mathbf{B}}^\mu \geq 0 \qquad \text{on} \qquad (\partial\mathcal{V}) \cap ((t_0, T) \times \mathbb{R}^2 \times (0, 2))$$

with θ defined by (7.38). This is a consequence of $\theta(U) = k_1|\underline{u}_0 - \underline{u}|/|u|^2 \geq 0$ and $\theta(V) = k_1\big||u|^{-1} - |u_0|^{-1}\big| \geq 0$, see **(S8)** and (7.40).

(RE5) holds, by condition (i) in the Proposition, for **(Sys1)**, **(Sys2)** and **(Sys3)**. In particular, for **(Sys2)**, $G\big(q, \Xi^{(1)}, \Xi^{(2)}, \partial_q \Xi^{(2)}\big)$ is C^0.

(RE4) holds, by condition (ii) in the Proposition, for **(Sys1)**, **(Sys2)** and **(Sys3)**. For **(Sys2)**, see **(S11)**. For **(Sys3)**, recall that $\Psi^\sharp$ solves (5.4b) because Ψ solves (5.4a).

Note that condition (iii) in the Proposition and **(S2)** imply that $\Psi = \Psi_K + \pi\,\Xi = 0$ and $\Psi^\sharp = 0$ when $q^3 < \frac{1}{2}$. In particular, $\mathbf{Src} = 0$ there. These facts imply **(RE6)** for **(Sys1)**, **(Sys2)** and **(Sys3)**.

The remaining assumptions, **(RE0)**, **(RE2)**, **(RE3)**, **(RE7)**, **(RE8)**, **(RE9)** are verified by direct inspection.

Proof of Part 3, Conclusion 3. We have to prove

$$\sup_{t \in (t_0+1, T)} |t|\, \mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi^\sharp\}(t) \;\lesssim_Y\; \Big(1 + \sup_{t \in (t_0, T)} |t|\, \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,t)}\{\Xi\}(t)\Big) \;\overset{\text{def}}{=}\; \kappa$$

It follows from the Overall Preliminaries, the Preliminaries for Part 3, as well as (8.14) and (8.15) (see, Proposition 8.2) that on $\mathcal{V} = \mathcal{V}_1$,

(A)	$\lvert\boldsymbol{\mathcal{G}}^\sharp\rvert \lesssim_R 1$	by Definition 8.2
(B)	$\lvert\Psi - \Psi(0)\rvert,\ \big\lvert\partial_q(\Psi - \Psi(0))\big\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (8.22)
(C)	$\lvert\mathcal{G}_1^\sharp\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (B), Definition 8.2
(D)	$\lvert{\Xi^\sharp}_1\rvert, \lvert{\Xi^\sharp}_3\rvert, \lvert s_1\rvert, \lvert s_2\rvert, \lvert p_1\rvert, \lvert p_2\rvert, \lvert p_3\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (A), (C), (8.14a)
(E)	$\lvert\Xi^\sharp\rvert \lesssim_Y \kappa$	by (A), (C), (8.14a), (8.14b)
(F)	$\lvert u^{-1}\mathcal{G}^\sharp\Xi^\sharp\rvert \lesssim_Y \kappa\lvert t\rvert^{-1}$	by (E), Definition 8.2

For each point $(t_1, \mathbf{q}_1) \in \mathcal{V}$ with $t_1 > t_0 + 1$, consider the line segment

$$\mathbf{Seg} \;=\; \big((t_1, \mathbf{q}_1) - \mathbb{R}_+(1,0,0,1)\big) \cap \{q \in \mathbb{R}^4 \mid q^3 > 0\}.$$

We have $\mathbf{Seg} \subset \mathcal{V}$. To see this, view $\mathcal{V}$ as an open disk bundle over the $(t, \underline{u})$-rectangle $(t_0, T) \times (0, 1)$. The projection of **Seg** to the $(t, \underline{u})$ plane is injective and contained in the base, because $t_1 \in (t_0 + 1, T)$. At each point in the image of the projection of **Seg**, the corresponding point on **Seg** is contained in the fiber, because the radius function $r(\underline{u}, u)$ (see (7.37)) is a decreasing function of $\underline{u}$ on the base for fixed $u = t - \underline{u}$, and because the endpoint $(t_1, \mathbf{q}_1)$ is contained in the fiber, by assumption.

By Conclusion 2, $\Xi^\sharp$ is a C^∞ solution to $\widehat{\mathbf{B}}\Xi^\sharp = \widehat{Q}\Xi^\sharp$ which vanishes when $q^3 < \frac{1}{2}$. In particular (8.15) holds. Recall $L = e_3\big(\frac{\partial}{\partial q^0} + \frac{\partial}{\partial q^3}\big)$, where $\Phi = (e, \gamma, w)$, and

$\frac{1}{2} \le e_3 \le 2$ (see **(RE1)**), and that **Seg** is an integral curve of L. The last three sentences, **Seg** $\subset \mathcal{V}$ and (F) imply, by integrating the equation for p_4 in (8.15) along **Seg**, that

$$|p_4| \lesssim_Y \kappa\,|t|^{-1} \tag{8.30}$$

on **Seg** for each endpoint $(t_1, \mathbf{q}_1) \in \mathcal{V}$ with $t_1 > t_0 + 1$.

Finally, integrating the remaining equations in (8.15) along **Seg**, and using (8.30) and (D), we obtain $|s_4|$, $|s_5|$, $|p_7|$, $|p_8| \lesssim_Y \kappa|t|^{-1}$ on all admissible segments **Seg**. Therefore, $|\Xi^\sharp| \lesssim_Y \kappa|t|^{-1}$ on $\mathcal{V}$ when $t \in (t_0+1, T)$. $\square$

8.2. Construction of classical vacuum fields.

Theorem 8.1. *Let $(\xi, \underline{u}, u)$ be the usual coordinates on the truncated strip*

$$\textbf{Strip}(1,\lambda) \;=\; \mathbb{R}^2 \times (0,1) \times \big(-\infty\,,\, -\lambda^{-1}\big)$$

of width 1, for each $\lambda > 0$. Suppose $0 < |\mathfrak{A}| \le |a|$. Assume the functions

$$\textbf{DATA}^\sigma(\xi, \underline{u}) :\; \mathbb{R}^2 \times (0,\infty) \to \mathbb{C}$$

$\sigma \in \{-,+\}$, are smooth, vanish when $\underline{u} < \frac{1}{2}$, and are Pole-Flip compatible

$$\textbf{DATA}^\sigma = \textbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \textbf{DATA}^{-\sigma} \tag{8.31}$$

for all $(\xi, \underline{u}) \in (\mathbb{R}^2 \setminus \{0\}) \times (0,\infty)$, see Definition 6.3. Let $[\,\Psi^\sigma\,]$ be the formal power series solution corresponding to $\textbf{DATA}^\sigma$. Fix integers $R \ge 4$, $K \ge 0$ and an $\epsilon \in (0, \frac{1}{2})$. Set

$$B = (R, \epsilon) \quad , \quad C = \Big(R,\ \epsilon,\ K,\ \max_{\sigma \in \{-,+\}} \|\textbf{DATA}^\sigma\|_{C^{R+2K+6}\left(\mathcal{C}(a,\,\mathfrak{A},\,2)\right)}\Big)$$

where $\mathcal{C}(a, \mathfrak{A}, b) = D_{4|\frac{a}{\mathfrak{A}}|}(0) \times (0, b)$ for each $b > 0$.

Let

$$\mathbf{b} = \mathbf{b}(B) \qquad\qquad \mathbf{c} = \mathbf{c}(C)$$

be constants in $(0,1)$. If $\mathbf{b}$ and $\mathbf{c}$ are made sufficiently small depending only on B and C, respectively, then the Existence and Uniqueness statements below hold whenever

$$0 < |\mathfrak{A}| \le |a| \le \mathbf{b}, \qquad \max_{\sigma \in \{-,+\}} \|\textbf{DATA}^\sigma\|_{C^{R+4}(\mathcal{C}(a,\mathfrak{A},2))} \;\le\; \mathbf{b} \tag{8.32}$$

Existence:

Part 1: *There exists a pair (Ψ^-, Ψ^+) of Pole-Flip compatible C^1-fields*

$$\Psi^\sigma : \textbf{Strip}(1, \mathbf{c}) \;\to\; \mathcal{R},$$

which are both solutions to (5.4a)*, vanish when $\underline{u} < \frac{1}{2}$, extend with their first derivatives continuously to $\overline{\textbf{Strip}(1,\mathbf{c})}$ and satisfy*

$$\lim_{u \to -\infty} |u|^\epsilon \sup_{\alpha \in \mathbb{N}_0^4:\ |\alpha| \le 1} \Big\| \partial^\alpha \big(\Psi^\sigma - \Psi^\sigma(0)\big)(\,\cdot\,, u)\,\Big\|_{C^0\left(\mathcal{C}(a,\mathfrak{A},1)\right)} = 0 \tag{8.33}$$

Part 2: *The constraint fields $(\Psi^-)^\sharp$, $(\Psi^+)^\sharp$ associated to the fields in Part 1 vanish, and*

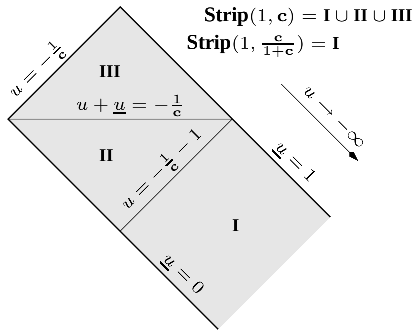
$$(\Phi^-, \Phi^+) \;=\; \big(\mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^-,\ \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^+\big)$$

are a pair of Pole-Flip compatible vacuum fields (see, Definition 2.2) with initial data $\Psi^\sigma(0)$.
Part 3: *The fields in Part 1 are actually of class* C^{R-3}*, and extend with their derivatives of order* $\le R-3$ *continuously to* $\overline{\textbf{Strip}(1,\mathbf{c})}$. *Moreover*

$$\sup_{u<-\mathbf{c}^{-1}} |u|^{K+1} \sup_{\substack{\alpha\in\mathbb{N}_0^4\\ |\alpha|\le R-3}} \left\| \partial^\alpha \Big(\Psi^\sigma(\,\cdot\,,u) - \sum_{k=0}^{K} \frac{\Psi^\sigma(k)(\,\cdot\,)}{u^k} \Big) \right\|_{C^0\left(\mathcal{C}(a,\mathfrak{A},1)\right)} \le \frac{1}{\mathbf{c}} \tag{8.34}$$

Uniqueness: *Assume* (Ψ^-,Ψ^+) *and* $(\widetilde{\Psi}^-,\widetilde{\Psi}^+)$ *have all the properties listed in Part 1. Then they coincide on* $\textbf{Strip}(1,\frac{\mathbf{c}}{1+\mathbf{c}}) \subset \textbf{Strip}(1,\mathbf{c})$.

Remark 8.4. Part 1 asserts the existence of a solution on $\mathbf{I}\cup\mathbf{II}\cup\mathbf{III}$. However, Uniqueness in Theorem 8.1 refers only to $\mathbf{I}$. We actually prove uniqueness on $\mathbf{I}\cup\mathbf{II}$. By a standard finite speed of propagation argument, which we do not carry out, the domain of uniqueness can be extended to $\mathbf{III}$.



Remark 8.5. The $\textbf{DATA}^\sigma$ are given for $\underline{u} \in (0,\infty)$, just for convenience. By construction, the restriction of $\Psi^\sigma(0)$ to $\underline{u}\in(0,1)$ depends only on the restriction of $\textbf{DATA}^\sigma$ to $\underline{u}\in(0,1)$, see equations (6.3). It now follows from Uniqueness in Theorem 8.1 that Ψ^-, Ψ^+ are determined on $\textbf{Strip}(1,\frac{\mathbf{c}}{1+\mathbf{c}})$ by the restriction of $\textbf{DATA}^\sigma$ to $\underline{u}\in(0,1)$.

Proof. Theorem 8.1 is formulated in the coordinate system $(\xi,\underline{u},u)$. Almost the entire proof, however, is given in the coordinate system $q=(t,\xi,\underline{u})$, where $t=u+\underline{u}$.

We assume **(S1)** through **(S11)** and $\widehat{\textbf{(S1)}}$ through $\widehat{\textbf{(S5)}}$, where a, $\mathfrak{A}$, K in **(S1)** through **(S11)** are identified with their occurrences in the statement of Theorem 8.1 and where **DATA** in **(S2)** is identified with either one of $\textbf{DATA}^\sigma$, $\sigma=-,+$. By direct inspection, our assumptions and identifications are consistent, when we make the legitimate smallness assumption $\mathbf{b}<10^{-3}$, that ensures that a, $\mathfrak{A}$ satisfy **(S1)**. This condition is subsumed in (8.37) below.

Convention 8.6. For the entire proof, $\mathbf{c}_3(\,\cdot\,)$ and $\mathbf{c}_4(\,\cdot\,)$, as well as $\mathbf{c}_6(\,\cdot\,)$ and $\mathbf{c}_7(\,\cdot\,)$ are defined as in Proposition 7.7 (Refined Energy Estimate) and Proposition 8.3. Furthermore, $\mathbf{c}_8(R)>1$ will always denote a constant, such that the Sobolev inequality

$$\textbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,b,t)}\{f\}(t) \le \mathbf{c}_8(R)\sqrt{E^R_{\mathcal{O}(\xi_0,b,t)}\{f\}(t)} \tag{8.35}$$

holds for all $(\xi_0,b,t)\in\mathbb{R}^2\times[1,2]\times(-\infty,-\mathfrak{d}^{-1})$ and all vector valued C^R functions f. See (7.43) for the Sobolev inequality and (7.40) for the definition of $\mathfrak{d}$.

The smallness condition on $\mathbf{b}$. Set $J_0 = \epsilon$, with ϵ as in Theorem 8.1, and

$$X = \big(R, J_0, |\mathcal{Q}_1|, |\mathcal{Q}_2|, |\mathcal{Q}_3|\big)$$
$$X^* = \big(\,0, J_0, |\mathcal{Q}_1|, |\mathcal{Q}_2|, |\mathcal{Q}_3|\big)$$
$$\widehat{X} = \big(\,0, J_0, |\widehat{\mathcal{Q}}_1|, |\widehat{\mathcal{Q}}_2|, |\widehat{\mathcal{Q}}_3|\big)$$

where $\mathcal{Q}_i$ and $\widehat{\mathcal{Q}}_i$ are fixed as in **(S7)** and $\widehat{\textbf{(S4)}}$, respectively. Also set

$$\mathbf{c}_2'(B) = \tfrac{1}{2}\,\min\big\{\mathbf{c}_3(X),\ \mathbf{c}_3(X^*),\ \mathbf{c}_3(\widehat{X})\big\} \in (0,1) \tag{8.36}$$

The right hand side of (8.36) only depends on B, which justifies the notation $\mathbf{c}_2'(B)$. We impose the legitimate smallness condition

$$\mathbf{b}\ <\ \min\big\{10^{-3},\ \mathbf{c}_6(R)\,\mathbf{c}_2'(B)\big\} \tag{8.37}$$

It is the only smallness condition on $\mathbf{b}$ in the entire proof.

The first smallness condition on $\mathbf{c}$. Set

$$Y = \Big(R, K, \max_{\sigma\in\{-,+\}} \|\mathbf{DATA}^\sigma\|_{C^{R+2K+6}(\mathcal{Q})}\Big)$$
$$\mathbf{T}(C) = -1 - \max\left\{\frac{1}{\mathfrak{d}},\ \ \frac{1}{\mathbf{c}_7(Y)\mathbf{c}_2'(B)},\ \ \left(\frac{4\,\mathbf{c}_8(R)\,(\mathbf{c}_4(X)+1)}{\mathbf{c}_2'(B)\mathbf{c}_6(R)\mathbf{c}_7(Y)}\right)^{\frac{1}{K+1}}\right\} \tag{8.38}$$

Observe that $\mathcal{C}(a,\mathfrak{A},2) = \mathcal{Q}$ where $\mathcal{Q}$ is defined in **(S9)**. Therefore, Y depends only on C, and so does the right hand side of (8.38), justifying the notation $\mathbf{T}(C)$. We impose the legitimate smallness condition

$$\mathbf{c}\ <\ \frac{1}{|\mathbf{T}(C)|+2} \tag{8.39}$$

There will be one more smallness condition on $\mathbf{c}$, later in the proof.

Convention 8.7. The system (8.3a) corresponding to $\mathbf{DATA}^\sigma$ will be denoted (8.3a)$^\sigma$. We sometimes suppress the superscript σ and simply write **DATA** and (8.3a), in which case the discussion applies equally to $\mathbf{DATA}^\sigma$ and (8.3a)$^\sigma$ for $\sigma = -, +$.

Convention 8.8. In every application of Proposition 8.3, the $\mathbf{c}_2'$ of Proposition 8.3 will be the $\mathbf{c}_2'(B)$ of (8.36).

Remark 8.6. We have $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\big(\mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi\big) = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\left(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\Psi\right)$ since the transformation $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is linear, maps $\mathcal{M}_{a,\mathfrak{A}}$ to itself and commutes with the matrix u^{-M}. In particular, if Ψ solves (5.4a), then $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\Psi$ solves (5.4a).
Observe that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\Psi_K^\sigma = \Psi_K^{-\sigma}$, because $\mathbf{DATA}^\sigma$ are Pole-Flip compatible, by assumption. Therefore, $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\big(\Psi_K^\sigma + \pi\,\Xi\big) = \Psi_K^{-\sigma} + \mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot(\pi\,\Xi)$. In particular, if Ξ solves (8.3a)$^\sigma$, then $\pi^{-1}\,\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot(\pi\Xi)$ solves (8.3a)$^{-\sigma}$.

Convention 8.9. For the rest of this proof, we consciously abuse notation and write $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot\Xi$ for $\pi^{-1}\,\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot(\pi\Xi)$. See, **(S3)** for the definition of π.

Guide. The proof now proceeds through a sequence of 10 steps. Each step begins with a statement (in italics), that is then proven.

Step 1. The assumptions of Proposition 8.3 up to and including (8.17) *are satisfied for all* $T \leq \mathbf{T}(C)$, *if* $\textbf{DATA} = \textbf{DATA}^{\sigma}$ *for* $\sigma = -, +$.
By direct inspection. Recall that $\mathbf{c}_7(\,\cdot\,)$ is non-increasing in all its arguments.

Step 2. For each $t_0 < \mathbf{T}(C)$ *there is a* $t_1(t_0) \in (t_0, \mathbf{T}(C)]$ *and a smooth solution*

$$\varXi_{t_0} : [t_0, t_1(t_0)) \times \mathbb{R}^3 \;\;\to\;\; \pi^{-1}\mathcal{R} \;\cong\; \mathbb{R}^{31}$$

to the system $\mathbf{M}(q,\varXi)\varXi = h(q,\varXi)$, *with trivial initial data,* $\varXi_{t_0}(t_0,\,\cdot\,) = 0$, *vanishing identically for* $q^3 < \frac{1}{2}$, *so that* $t_1(t_0) \neq \mathbf{T}(C)$ *implies either one or both of* **(Break)**$_1$ *or* **(Break)**$_2$ *(see Proposition 7.3).*
Apply Parts 1 and 2 of Proposition 7.3, in the context of Part 1 of Proposition 8.3, with $T = \mathbf{T}(C)$.

Remark 8.7. $\varXi_{t_0}$ is a 1-parameter family of solutions, parametrized by $t_0 < \mathbf{T}(C)$.

Step 3. $t_1(t_0) = \mathbf{T}(C)$, *for all* $t_0 < \mathbf{T}(C)$.
For each $\xi_0 \in \mathbb{R}^2$, we introduce the set

$$\mathcal{J}(\xi_0, t_0) = \Big\{ t \in [t_0, t_1(t_0)) \;\Big|\; \sup_{\tau \in [t_0, t]} E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\varXi_{t_0}\}(\tau) \leq \big(\mathbf{c}_6(R)\mathbf{c}_2'(B)\big)^2 \Big\}$$

It is an interval and closed as a subset of $[t_0, t_1(t_0))$. By Part 2 of Proposition 8.3 and by (8.38),

$$\sqrt{E^R_{\mathcal{O}(\xi_0,2,t_0)}\{\varXi_{t_0}\}(t_0)} \leq \frac{1}{\mathbf{c}_7(Y)\,|\mathbf{T}(C)|^{K+1}} \leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{4}.$$

Therefore, $t_0 \in \mathcal{J}(\xi_0, t_0)$. By continuity of the energy, $\mathcal{J}(\xi_0, t_0)$ contains at least one point different from t_0. For every $t^* \in \mathcal{J}(\xi_0, t_0)$, $t^* > t_0$, the assumptions of Proposition 8.3, Part 3, **(Sys1)** are satisfied with $T = t^*$. It follows from Conclusion 1 that

$$\varXi_{t_0}(q) \in \overline{B_{1/2}(0)} \subset \mathbb{R}^{31} \qquad \text{for all} \qquad q \in \textstyle\bigcup_{t \in (t_0, t^*)} \{t\} \times \mathcal{O}(\xi_0, 2, t) \tag{8.40}$$

By Conclusion 2 and

$$K + 1 \geq J_0, \qquad \mathbf{c}_2'(B) \leq \mathbf{c}_3(X), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}_2'(B) \leq -1/\mathbf{c}_3(X)$$

we can apply the Refined Energy Estimate (7.42) in Proposition 7.7 in the context of **(Sys1)**. Combining (7.42) with Part 2 of Proposition 8.3, one obtains

$$\begin{aligned} \sqrt{E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\varXi_{t_0}\}(\tau)} \leq \frac{2\mathbf{c}_4(X)}{\mathbf{c}_7(Y)\,|\tau|^{K+1}} &\leq \frac{2\,\mathbf{c}_4(X)}{\mathbf{c}_7(Y)\,|\mathbf{T}(C)|^{K+1}} \\ &\leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{2\,\mathbf{c}_8(R)} \leq \frac{\mathbf{c}_6(R)\,\mathbf{c}_2'(B)}{2} \end{aligned} \tag{8.41}$$

for all $\tau \in (t_0, t^*)$. The second inequality is self-evident. For the third, use (8.38) again.

The continuity of the energy $E^R_{\mathcal{O}(\xi_0,2,\tau)}\{\varXi_{t_0}\}(\tau)$ for $\tau \in [t_0, t_1(t_0))$ implies that (8.41) holds for $\tau = t^*$, and, consequently, for all $\tau \in \mathcal{J}(\xi_0, t_0)$. It follows that $\mathcal{J}(\xi_0, t_0)$ is also open as a subset of $[t_0, t_1(t_0))$. The set $\mathcal{J}(\xi_0, t_0)$ is nonempty, open and closed as a subset of $[t_0, t_1(t_0))$, and we conclude $\mathcal{J}(\xi_0, t_0) = [t_0, t_1(t_0))$. The upshot is that (8.40) holds with t^* replaced by $t_1(t_0)$, for all $\xi_0 \in \mathbb{R}^2$, and therefore $\Xi_{t_0}\big([t_0, t_1(t_0)) \times \mathcal{Q}\big) \subset \overline{B_{1/2}(0)} \subset \mathbb{R}^{31}$ because

$$[t_0, t_1(t_0)) \times \mathcal{Q} \;\subset\; [t_0, t_1(t_0)) \times \mathbb{R}^2 \times (0,2) \;=\; \bigcup_{\xi_0 \in \mathbb{R}^2} \; \bigcup_{t \in [t_0, t_1(t_0))} \{t\} \times \mathcal{O}(\xi_0, 2, t). \tag{8.42}$$

Now, $(\textbf{Break})_1$ (see Step 2) is excluded. The inclusion (8.42), the fact that $\mathcal{J}(\xi_0, t_0) = [t_0, t_1(t_0))$ and the Sobolev inequality, (8.35), exclude $(\textbf{Break})_2$. By Step 2, we conclude that $t_1(t_0) = \mathbf{T}(C)$, for all $t_0 < \mathbf{T}(C)$.

Remark 8.8. A byproduct of the proof of Step 3 is:

$$\text{The inequality (8.41) holds for all } \tau \in [t_0, \mathbf{T}(C)) \text{ and } \xi_0 \in \mathbb{R}^2. \tag{8.43}$$

Convention 8.10 (for Steps 4 and 5). We introduce a new field that is used in the next two steps. It is the restriction of Ξ_{t_0} to $[t_0, \mathbf{T}(C)) \times \mathcal{W}$, where

$$\mathcal{W} \;=\; D_{\frac{5}{2}|\frac{a}{\mathfrak{A}}|}(0) \times (0,1) \;\subset\; \mathbb{R}^3$$

Consciously abusing notation, we will denote this new field by the same symbol Ξ_{t_0}. The new field Ξ_{t_0} is smooth and extends, with all its derivatives, continuously to $[t_0, \mathbf{T}(C)) \times \overline{\mathcal{W}} \subset \mathbb{R}^4$.

Step 4. Ξ_{t_0} is a solution to (8.3a)*, for each $t_0 < \mathbf{T}(C)$.*
Define the open cover $(\mathcal{W}_1, \mathcal{W}_2)$ of $\mathcal{W}$,

$$\mathcal{W}_1 = \mathcal{W} \cap \big(\mathbb{R}^2 \times (0, \tfrac{1}{2})\big) \qquad\qquad \mathcal{W}_2 = \mathcal{W} \cap \big(\mathbb{R}^2 \times (\tfrac{1}{3}, 1)\big)$$

If $\mathbf{q} \in \mathcal{W}_1$, then $\Xi_{t_0}(q) = 0$ (see, Step 2) and $\mathbf{Src}(q) = 0$ (see, **(S2)** and **(S4)**). In this case, Ξ_{t_0} is self-evidently a solution to (8.3a). On the other hand, if $\mathbf{q} \in \mathcal{W}_2$, then $\psi(q) = 1$ (see, **(S9)**). By direct inspection of **(S10)**, the equation $\mathbf{M}(q, \Xi)\Xi = h(q, \Xi)$ collapses to (8.3a).

Remark 8.9. Recall from Remark 7.3 that, for all $(\xi_0, b, t) \in \mathbb{R}^2 \times [1,2] \times (-\infty, -\frac{1}{\mathfrak{d}})$, the set $\mathcal{O}(\xi_0, b, t)$ is a bundle over the $\underline{u}$-interval $(0, b)$ whose fibers are disks centered at ξ_0, with radii $< \frac{1}{2}$.

Step 5. For each $t_0 < \mathbf{T}(C)$, let $\Xi^{\sigma}_{t_0}$ be the solution to (8.3a)$^{\sigma}$*. Let*

$$\begin{aligned} \mathcal{Y}(t_0) &= [t_0, \mathbf{T}(C)) \times \big\{ \tfrac{2}{5}|\tfrac{a}{\mathfrak{A}}| < |\xi| < \tfrac{5}{2}|\tfrac{a}{\mathfrak{A}}| \big\} \times (0,1) \\ \mathcal{Z}(t_0) &= \bigcup_{\tau \in [t_0, \mathbf{T}(C))} \; \bigcup_{|\frac{a}{\mathfrak{A}}| \le |\xi_0| < 2|\frac{a}{\mathfrak{A}}|} \{\tau\} \times \mathcal{O}(\xi_0, 1, \tau) \;\subset\; \mathcal{Y}(t_0) \end{aligned}$$

Note that both $\Xi^{\sigma}_{t_0}$ and $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$ (see, Convention 8.9) are defined on $\mathcal{Y}(t_0)$. We claim that $\Xi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$ on $\mathcal{Z}(t_0) \subset \mathcal{Y}(t_0)$.
The argument is by finite speed of propagation. For any $|\frac{a}{\mathfrak{A}}| \le |\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, let

$$\begin{aligned} &\mathcal{I}(\xi_0, t_0) = \\ &\Big\{ t \in [t_0, \mathbf{T}(C)) \;\Big|\; \Xi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0} \quad \text{on} \quad \bigcup_{\tau \in [t_0, t]} \{\tau\} \times \mathcal{O}(\xi_0, 1, \tau) \;\subset\; \mathcal{Z}(t_0) \Big\} \end{aligned}$$

Our goal is $\mathcal{I}(\xi_0, t_0) = [t_0, \mathbf{T}(C))$, for each $|\frac{a}{\mathfrak{A}}| \leq |\xi_0| < 2|\frac{a}{\mathfrak{A}}|$. First of all, $\mathcal{I}(\xi_0, t_0)$ is an interval that contains t_0, because $\Xi^{\sigma}_{t_0}$ and $\Xi^{-\sigma}_{t_0}$ have trivial initial data. By continuity, it is closed as a subset of $[t_0, \mathbf{T}(C))$. We now show that $\mathcal{I}(\xi_0, t_0)$ is also open as a subset of $[t_0, \mathbf{T}(C))$.
By the second to last inequality in (8.43), and by (8.35),

$$\sup_{\tau\in[t_0,\mathbf{T}(C))} \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^{\sigma}_{t_0}\}(\tau) \;\leq\; \tfrac{1}{2}\,\mathbf{c}_6(R)\mathbf{c}_2'(B).$$

Let $t' \in \mathcal{I}(\xi_0, t_0)$. By the definition of $\mathcal{I}(\xi_0, t_0)$, and by continuity

$$\sup_{\tau\in[t_0,t^*)} \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,\tau)}\{\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}\cdot \Xi^{-\sigma}_{t_0}\}(\tau) \;\leq\; \tfrac{3}{4}\,\mathbf{c}_6(R)\mathbf{c}_2'(B)$$

for some $t^* \in (t', \mathbf{T}(C))$. The assumptions of Proposition 8.3, Part 3, **(Sys2)** are satisfied with $T = t^*$, $\Xi^{(1)} = \Xi^{\sigma}_{t_0}$, $\Xi^{(2)} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$. Conclusion 2 of Proposition 8.3 enables us to apply Proposition 7.7 (Refined Energy Estimate) for **(Sys2)**, with $J = \frac{1}{2}$. The assumptions of Proposition 7.7 are satisfied because

$$J = \tfrac{1}{2} \geq J_0, \qquad \mathbf{c}_2'(B) \leq \mathbf{c}_3(X^*), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}_2'(B) \leq -1/\mathbf{c}_3(X^*). \tag{8.44}$$

In the present case, the Refined Energy Estimate (7.42) becomes

$$\sqrt{E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\Upsilon\}(\tau)} \;\leq \mathbf{c}_4(X^*)\;\frac{|t_0|^{1/2}\sqrt{E^0_{\mathcal{O}(\xi_0,1,t_0)}\{\Upsilon\}(t_0)}\;+\;0}{|\tau|^{1/2}}$$

for all $\tau \in (t_0, t^*)$, where $\Upsilon = \Xi^{\sigma}_{t_0} - \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}$. Furthermore, the energy on the right hand side is zero. The vanishing of the energy on the left hand side implies $[t_0, t^*) \subset \mathcal{I}(\xi_0, t_0)$. Hence, $\mathcal{I}(\xi_0, t_0)$ is an open subset of $[t_0, \mathbf{T}(C))$.

Convention 8.11 (for the remaining steps). We introduce a new field for the remaining steps. For each $t_0 < \mathbf{T}(C)$ and $\sigma \in \{-,+\}$, it is the map (see, Convention 8.9)

$$\begin{aligned}
[t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \;\;&\to\;\; \pi^{-1}\mathcal{R} \;\cong\; \mathbb{R}^{31} \\
q = (t, \xi, \underline{u}) \;\;&\mapsto\;\; \begin{cases} \Xi^{\sigma}_{t_0}(q) & \text{if } |\xi| < 2|\frac{a}{\mathfrak{A}}| \\ \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0}(q) & \text{if } |\xi| > \frac{1}{2}|\frac{a}{\mathfrak{A}}| \end{cases}
\end{aligned} \tag{8.45}$$

It is well defined on the flip-invariant $[t_0, \mathbf{T}(C)) \times \left\{\frac{1}{2}|\frac{a}{\mathfrak{A}}| < |\xi| < 2|\frac{a}{\mathfrak{A}}|\right\} \times (0,1)$, which is contained in $\mathcal{Z}(t_0) \cup \left(\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{Z}(t_0)\right)$ by Step 5. It coincides with $\Xi^{\sigma}_{t_0}$ on the set $\mathcal{Z}(t_0)$ of Step 5. Consciously abusing notation, we will denote this new field by the same symbol $\Xi^{\sigma}_{t_0}$.

Step 6. For each $t_0 < \mathbf{T}(C)$ and $\sigma \in \{-,+\}$,

$$\Xi^{\sigma}_{t_0} = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}_{t_0} \qquad \textit{on} \qquad [t_0, \mathbf{T}(C)) \times (\mathbb{R}^2 \setminus \{0\}) \times (0,1)$$

The field $\Xi^{\sigma}_{t_0} : [t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}$ is smooth, vanishes when $q^3 < \frac{1}{2}$, and extends, with its derivatives of all orders, continuously to $[t_0, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$.

Moreover, it is a solution to (8.3a)$^\sigma$ *on its entire domain of definition and satisfies*

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau\in(t_0,\mathbf{T}(C))} \mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_0}\}(\tau) \le \tfrac{1}{2}\,\mathbf{c}_6(R)\mathbf{c}'_2(B) \tag{8.46a}$$

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau\in(t_0,\mathbf{T}(C))} |\tau|^{K+1}\,\mathbf{Sup}^{(R-2)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_0}\}(\tau) \le \frac{2\mathbf{c}_4(X)\mathbf{c}_8(R)}{\mathbf{c}_7(Y)} \tag{8.46b}$$

The main point before (8.46a) is that $\mathbf{Flip}_{\frac{a}{\mathfrak{A}}}$ is a field configuration symmetry. The inequalities (8.46a), (8.46b) are consequences of (8.43) and the Sobolev inequality (8.35). Be aware that that the Ξ_{t_0} in (8.43) is related to the present $\Xi^\sigma_{t_0}$ by (8.45).

Step 7. For each $t_0 < \mathbf{T}(C) - 1$ *and* $\sigma \in \{-,+\}$,

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau\in(t_0+1,\mathbf{T}(C))} E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(\tau)\ \lesssim_{(Y,J_0)}\ \frac{1}{|t_0|}.$$

where $\big(\Xi^\sigma_{t_0}\big)^\sharp$ *is the constraint field associated to the field* (8.45).
For any $|\xi_0| < 2\,|\frac{a}{\mathfrak{A}}|$ and any $t^* \in (t_0+1, \mathbf{T}(C))$, the assumptions of Proposition 8.3, Part 3, **(Sys3)** are satisfied with $T = t^*$. By Conclusion 3, $R \ge 4$, $K \ge 0$, and (8.46b),

$$\sup_{\tau\in(t_0+1,t^*)} |\tau|\,\mathbf{Sup}^{(0)}_{\mathcal{O}(\xi_0,1,\tau)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(\tau)\ \lesssim_{(Y,J_0)}\ 1.$$

By continuity, this holds for $t = t_0 + 1$ as well. Therefore, the energy satisfies

$$|t_0+1|^2\,E^0_{\mathcal{O}(\xi_0,1,t_0+1)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(t_0+1)\ \lesssim_{(Y,J_0)}\ 1. \tag{8.47}$$

By Conclusion 2, we can apply Proposition 7.7 (Refined Energy Estimate) for **(Sys3)**, with $J = \frac{1}{2}$, $\mathcal{I} = (t_0+1, t^*)$. The assumptions of Proposition 7.7 are satisfied, because

$$J = \tfrac{1}{2} \ge J_0, \qquad \mathbf{c}'_2(B) \le \mathbf{c}_3(\widehat{X}), \qquad t^* < \mathbf{T}(C) < -1/\mathbf{c}'_2(B) \le -1/\mathbf{c}_3(\widehat{X}).$$

The Refined Energy Estimate (7.42) and (8.47) imply

$$\sup_{\tau\in(t_0+1,t^*)} E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\big(\Xi^\sigma_{t_0}\big)^\sharp\big\}(\tau)\ \lesssim_{(Y,J_0)}\ \frac{1}{|t_0+1|}\ \lesssim_{(Y,J_0)}\ \frac{1}{|t_0|}.$$

Step 8. For all $\sigma \in \{-,+\}$ *and all* $t_1 \le t_2 < \mathbf{T}(C)$,

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|}\ \sup_{\tau\in(t_2,\mathbf{T}(C))} E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_2} - \Xi^\sigma_{t_1}\}(\tau)\ \lesssim_{(Y,J_0)}\ \frac{1}{|t_2|}$$

For any $t^* \in (t_2, \mathbf{T}(C))$ and $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, the assumptions in Proposition 8.3, Part 3, for **(Sys2)**, are satisfied with $t_0 = t_2$, $T = t^*$ and $\Xi^{(1)} = \Xi^\sigma_{t_1}$, $\Xi^{(2)} = \Xi^\sigma_{t_2}$. By Conclusion 2, we can apply Proposition 7.7 (Refined Energy Estimate) with $J = \frac{1}{2}$, see (8.44). The Refined Energy Estimate (7.42) implies

$$E^0_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma_{t_2} - \Xi^\sigma_{t_1}\}(\tau)\ \le\ \big(\mathbf{c}_4(X^*)\big)^2\,\frac{|t_2|\,E^0_{\mathcal{O}(\xi_0,1,t_2)}\{\Xi^\sigma_{t_1}\}(t_2)}{|\tau|}\ \lesssim_{(Y,J_0)}\ \frac{1}{|t_2|^{2K+1}}$$

for all $\tau \in (t_2, t^*)$. For the second inequality, see (8.46b).

Step 9. There exist

$$\Xi^\sigma : (-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}, \qquad \sigma \in \{-,+\}$$

with $\Xi^\sigma = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}$ *(see, Convention 8.9), that vanish when* $q^3 < \frac{1}{2}$*, are* C^{R-3}*, and extend, with their derivatives of all orders* $\leq R-3$*, continuously to* $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$*. Moreover, they are solutions to both* (8.3a)$^\sigma$ *and* $(\Xi^\sigma)^\sharp = 0$*, and*

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|} \sup_{\tau \in (-\infty, \mathbf{T}(C))} \mathbf{Sup}^{(R-3)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma\}(\tau) \leq \tfrac{1}{2}\mathbf{c}_6(R)\mathbf{c}_2'(B) \tag{8.48a}$$

$$\sup_{|\xi_0|<2|\frac{a}{\mathfrak{A}}|} \sup_{\tau \in (-\infty, \mathbf{T}(C))} |\tau|^{K+1} \mathbf{Sup}^{(R-3)}_{\mathcal{O}(\xi_0,1,\tau)}\{\Xi^\sigma\}(\tau) \lesssim_{(Y,J_0)} 1 \tag{8.48b}$$

$$\sup_{\tau \in (-\infty, \mathbf{T}(C))} |\tau|^{K+1} \mathbf{Sup}^{(R-3)}_{D_{4|\frac{a}{\mathfrak{A}}|}(0)\times(0,1)}\{\Xi^\sigma\}(\tau) \lesssim_{(Y,J_0)} 1 \tag{8.48c}$$

$$\tfrac{1}{2} \leq \mathbf{B}^0(q, \Xi(q)) \leq 2 \text{ for all } q \in (-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1) \tag{8.48d}$$

For each $\beta \in (0,1)$, introduce the compact set

$$\mathcal{X}_\beta = [\mathbf{T}(C) - \beta^{-1}, \mathbf{T}(C) - \beta] \times \overline{D_{2|\frac{a}{\mathfrak{A}}|}(0)} \times [0,1]$$

For every sequence $t_n \to -\infty$, with $t_n < \mathbf{T}(C) - \beta^{-1}$, the sequence of fields $\Xi^\sigma_{t_n}$ is, by Step 8, a Cauchy sequence in $L^2(\mathcal{X}_\beta)$. Set $\Xi^\sigma|_{\mathcal{X}_\beta} = L^2\text{-}\lim_{t\to-\infty} \Xi^\sigma_t$. By (8.46a), the 1-parameter family Ξ^σ_t with $t < \mathbf{T}(C) - \beta^{-1}$ is a bounded subset of $C^{R-2}(\mathcal{X}_\beta)$, the space of $\pi^{-1}\mathcal{R} \cong \mathbb{R}^{31}$ valued functions of class C^{R-2} on the interior of $\mathcal{X}_\beta$, that extend continuously, with their derivatives of all orders $\leq R-2$, to the boundary. By Arzela-Ascoli, there is a subsequence that converges in $C^{R-3}(\mathcal{X}_\beta)$. Therefore, $\Xi^\sigma|_{\mathcal{X}_\beta}$ is in $C^{R-3}(\mathcal{X}_\beta)$. It follows that Ξ^σ is C^{R-3} on the interior of $\bigcup_{\beta\in(0,1)} \mathcal{X}_\beta$, and extends with its derivatives of all orders $\leq R-3$ continuously to $\bigcup_{\beta\in(0,1)} \mathcal{X}_\beta = (-\infty, \mathbf{T}(C)) \times \overline{D_{2|\frac{a}{\mathfrak{A}}|}(0)} \times [0,1]$. By construction, $\Xi^\sigma = \mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \Xi^{-\sigma}$ on $\mathcal{X}_\beta \cap (\mathbf{Flip}_{\frac{a}{\mathfrak{A}}} \cdot \mathcal{X}_\beta)$. Hence, the pair of fields Ξ^σ have unique C^{R-3} Pole-Flip compatible extensions to $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1)$, which extend with their derivatives of all orders $\leq R-3$ continuously to $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times [0,1]$, as required by Step 9.
It follows directly from Step 6, that the pair Ξ^σ has all the desired properties, including the bounds (8.48a), (8.48b) (recall that $R-3 \geq 1$), with the exception of $(\Xi^\sigma)^\sharp = 0$, (8.48c) and (8.48d). It is implicit in our construction that for each $\beta \in (0,1)$, there is a sequence $t_n \to -\infty$ so that $\Xi^\sigma_{t_n} \to \Xi^\sigma$ in $C^1(\mathcal{X}_\beta)$, and therefore $(\Xi^\sigma_{t_n})^\sharp \to (\Xi^\sigma)^\sharp$ in $C^0(\mathcal{X}_\beta)$. Now, by step 7, $(\Xi^\sigma)^\sharp|_{\mathcal{X}_\beta} = 0$, for all $\beta \in (0,1)$. By Pole-Flip compatibility, $(\Xi^\sigma)^\sharp = 0$ everywhere. The estimate (8.48c) follows from (8.48b) when $|\xi| < 2|\frac{a}{\mathfrak{A}}|$. For $\frac{1}{2}|\frac{a}{\mathfrak{A}}| < |\xi| < 4|\frac{a}{\mathfrak{A}}|$, it also follows from (8.48b), by using Pole-Flip compatibility and Lemma F.1 in Appendix F. To verify (8.48d), observe that the assumptions of Proposition 8.3, Part 3, for **(Sys2)**, are satisfied for any $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, $t_0 < T < \mathbf{T}(C)$, $\Xi^{(1)} = \Xi^{(2)} = \Xi^\sigma$. By Conclusion 2, **(RE1)** holds in the context of **(Sys2)**, which implies (8.48d) for $q \in (-\infty, \mathbf{T}(C)) \times D_{2|\frac{a}{\mathfrak{A}}|}(0) \times (0,1)$, and for general q by Pole-Flip compatibility.

Step 10. The fields Ξ^σ *in Step 9 are unique in the following sense: Suppose, for some* $t_1 < \mathbf{T}(C)$*, the* C^1*-fields* $\widetilde{\Xi}^\sigma : (-\infty, t_1) \times \mathbb{R}^2 \times (0,1) \to \pi^{-1}\mathcal{R}$ *are Pole-Flip* *compatible, extend with their first derivatives continuously to* $(-\infty, t_1] \times \mathbb{R}^2 \times [0,1]$, *are solutions to* (8.3a)$^\sigma$, *vanish when* $q^3 < \frac{1}{2}$, *and satisfy*

$$\lim_{\tau \to -\infty} \sup_{|\xi_0| < 2|\frac{a}{\mathfrak{A}}|} |\tau|^{J_0} \, \mathbf{Sup}^{(1)}_{\mathcal{O}(\xi_0,1,\tau)} \{\widetilde{\Xi}^\sigma\}(\tau) \;=\; 0 \tag{8.49}$$

Then, $\widetilde{\Xi}^\sigma = \Xi^\sigma$ *on* $(-\infty, t_1) \times \mathbb{R}^2 \times (0,1)$.

For every sufficiently negative $\tau < t_1$, the assumptions of Proposition 8.3, Part 3, for **(Sys2)**, are satisfied for any $|\xi_0| < 2|\frac{a}{\mathfrak{A}}|$, $t_0 < T = \tau$, $\Xi^{(1)} = \Xi^\sigma$, $\Xi^{(2)} = \widetilde{\Xi}^\sigma$. The condition that τ is sufficiently negative is used to verify hypothesis (iv) of Part 3 of Proposition 8.3. By Conclusion 2, we can apply (checking, the additional hypothesis similarly to (8.44)) Proposition 7.7 with $J = J_0$. It follows from (7.42) that

$$E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(\tau) \;\le\; \big(\mathbf{c}_4(X^*)\big)^2 \, \frac{|t_0|^{2J_0} \, E^0_{\mathcal{O}(\xi_0,1,t_0)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(t_0)}{|\tau|^{2J_0}}.$$

We take the limit $t_0 \to -\infty$, keeping τ fixed. By (8.48b) and (8.49), and the fact that $2J_0 < 1$, we conclude that the energy $E^0_{\mathcal{O}(\xi_0,1,\tau)}\big\{\Xi^\sigma - \widetilde{\Xi}^\sigma\big\}(\tau) = 0$. Exploiting the Pole-Flip compatibility, $\Xi^\sigma(\tau, \cdot\,) = \widetilde{\Xi}^\sigma(\tau, \cdot\,)$ for all sufficiently negative τ. To demonstrate that $\Xi^\sigma = \widetilde{\Xi}^\sigma$ on $(-\infty, t_1) \times \mathbb{R}^2 \times (0,1)$, we make a closed-open argument almost identical to the one in the proof of Step 5.

We finally return from the $q = (t, \xi, \underline{u})$ to the $x = (\xi, \underline{u}, u)$ coordinate system, and complete the proof of Theorem 8.1. The x-set **Strip**$(1, \mathbf{c})$ is contained in the q-set $(-\infty, \mathbf{T}(C)) \times \mathbb{R}^2 \times (0,1)$, by the smallness condition (8.39).

Existence in Theorem 8.1 follows from Step 9, with $\Psi^\sigma = \Psi^\sigma_K + \pi\, \Xi^\sigma$. We only have to check (8.33), (8.34) and conditions $(\star)$ and $(\star\star)$ (see, Definitions 2.1 and 2.2), and apply Proposition 2.2. Write

$$\Psi^\sigma(\,\cdot\,, u) - \Psi^\sigma(0)(\,\cdot\,, u) = \tfrac{1}{u} \sum_{k=0}^{K} (\tfrac{1}{u})^k \, \Psi^\sigma(k+1)(\,\cdot\,) + \pi\, \Xi^\sigma(\,\cdot\,, u)$$

$$\Psi^\sigma(\,\cdot\,, u) - \sum_{k=0}^{K} (\tfrac{1}{u})^k \, \Psi^\sigma(k)(\,\cdot\,) = \tfrac{1}{u^{K+1}} \, \Psi^\sigma(K+1)(\,\cdot\,) + \pi\, \Xi^\sigma(\,\cdot\,, u)$$

The coefficient functions $\Psi^\sigma(k+1)$, appearing on the right hand sides, are estimated using $\|\Psi^\sigma(k+1)\|_{C^{R+1}(\mathcal{C}(a,\mathfrak{A},2))} \lesssim_Y 1$ (see, the Overall Preliminaries in the proof of Proposition 8.3) and Ξ^σ is estimated using (8.48c). Now, (8.33) follows. Also (8.34) follows, with an additional legitimate smallness condition on $\mathbf{c}$ depending only on (Y, J_0). Condition $(\star)$ is a consequence of the inequality $\frac{1}{2} \le e_3 \le 2$ on **Strip**$(1, \mathbf{c})$, which follows from (8.48d). Here, e_3 is a component of $\Phi^\sigma = (e, \gamma, w) = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^\sigma$. Finally, the equation $L\big(e_1\overline{e}_2 - \overline{e}_1 e_2\big) = -2\gamma_2\, \big(e_1\overline{e}_2 - \overline{e}_1 e_2\big)$ (a consequence of the first two lines of (2.4)) implies that $\Im\big(e_1\overline{e}_2\big)$ cannot change sign along the integral curves of L, and therefore $\Im\big(e_1\overline{e}_2\big) < 0$ on **Strip**$(1, \mathbf{c})$ because it is negative when $\underline{u} < \frac{1}{2}$. This implies $(\star\star)$.

Uniqueness in Theorem 8.1 follows from Step 10. □

9. Conclusions

We assume, without further comment, the definitions and conventions of Section 2 through Section 6 and Theorem 8.1. However, see the Index of Notation, Appendix A.

Proposition 9.1 (Asymptotic expansion). *Let $\Phi^\sigma = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^\sigma$ be the pair of Pole-Flip compatible vacuum fields of Theorem 8.1 for $K=0$. For each $L \geq 0$,*

$$\sup_{\substack{\alpha\in\mathbb{N}_0^4\\ |\alpha|\leq R-3}} \Big\|\partial^\alpha\Big(\Psi^\sigma(\,\cdot\,,u) - \sum_{k=0}^{L}\frac{\Psi^\sigma(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(\mathcal{C}(a,\mathfrak{A},1)\right)} \;=\; \mathcal{O}\Big(\frac{1}{|u|^{L+1}}\Big)$$

as $u \to -\infty$. In other words, the far field formal power series $[\,\Psi^\sigma\,]$ is an asymptotic expansion for Ψ^σ.

Proof. Observe that the conditions imposed on the data a, $\mathfrak{A}$, **DATA**$^\sigma$, R, ϵ in Theorem 8.1 are independent of K. Therefore, Theorem 8.1 can be applied with the same data for all $K \geq 0$. For each $K \geq 0$, we obtain a pair of Pole-Flip compatible vacuum fields on **Strip**$(1,\lambda_K) \subset \mathbb{R}^4$, where $\lambda_K > 0$ depends on K through the vector C in Theorem 8.1. The vacuum fields corresponding to any pair $K,\ K' \geq 0$ coincide for sufficiently negative u, by the uniqueness statement of Theorem 8.1. In particular, this is true for $K=0$, $K'=L$. The bound (8.34) for the $K'=L$ vacuum field implies the proposition. $\square$

9.1. Three Points of View. It is helpful to consider the focusing of gravitational waves from three perspectives, that yield three different pictures.

Regularized Picture (R)

$\uparrow$ Field transformation $\mathfrak{C}\circ\mathfrak{A}$ where $\mathfrak{C}(\xi) = a\,\xi$

High Amplitude Picture (H)

$\downarrow$ Isotropic scaling transformation $\mathfrak{J}$ with scaling constant $\mathfrak{J} = \mathfrak{A}^4$

Finite Mass Picture (F)

First, recall from Section 3 that the Isotropic Scaling $\mathfrak{J}$, the Anisotropic Scaling $\mathfrak{A}$ and the Angular Coordinate Transformation $\mathfrak{C}$ are field symmetries (see, Definition 3.1). Their isotropic respectively anisotropic character refers to their action on the frame and the coordinate system. Both scalings are, at the level of the Lorentzian metric, global conformal transformations.

The pictures are fixed by the table

	Regularized	**High Amplitude**	**Finite Mass**
Background	$\mathcal{M}_{a,\mathfrak{A}}$	$\mathcal{M}_{1,1}$	$\mathcal{M}_{1,1}$
Data	$\eta^\sigma\big(\xi,\,\underline{u}\big)$	$\mathfrak{A}^{-2}\,\eta^\sigma\big(\frac{a}{\mathfrak{A}}\xi,\,\underline{u}\big)$	$\mathfrak{A}^{-2}\,\eta^\sigma\big(\frac{a}{\mathfrak{A}}\xi,\,\mathfrak{A}^{-4}\underline{u}\big)$
Domain	**Strip**$\big(1,\mathbf{c}\big)$	**Strip**$\big(1,\mathbf{c}\,\mathfrak{A}^2\big)$	**Strip**$\big(\mathfrak{A}^4,\mathbf{c}\,\mathfrak{A}^{-2}\big)$
Hemisphere	$\lvert\xi\rvert < \lvert\frac{a}{\mathfrak{A}}\rvert$	$\lvert\xi\rvert < 1$	$\lvert\xi\rvert < 1$

The Regularized Picture is the arena of Theorem 8.1. The other two are obtained, as indicated above, by scaling. Row 1 displays the Minkowski background fields. Row 2 gives the functional form of the initial data at past null infinity. More precisely,

$$\mathbf{DATA}^{\sigma}_{\mathrm{R}}(\xi,\underline{u}) = \eta^{\sigma}(\xi,\underline{u}) \qquad \Phi^{\sigma}_{R} = \mathcal{M}_{a,\mathfrak{A}} + u^{-M}\Psi^{\sigma}_{R} \tag{9.1a}$$

$$\mathbf{DATA}^{\sigma}_{\mathrm{H}}(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^{\sigma}\big(\tfrac{a}{\mathfrak{A}}\xi,\ \underline{u}\big) \qquad \Phi^{\sigma}_{H} = \mathcal{M}_{1,1} + u^{-M}\Psi^{\sigma}_{H} \tag{9.1b}$$

$$\mathbf{DATA}^{\sigma}_{\mathrm{F}}(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^{\sigma}\big(\tfrac{a}{\mathfrak{A}}\xi,\ \mathfrak{A}^{-4}\underline{u}\big) \qquad \Phi^{\sigma}_{F} = \mathcal{M}_{1,1} + u^{-M}\Psi^{\sigma}_{F} \tag{9.1c}$$

The first equation gives two new names for the $\mathbf{DATA}^{\sigma}(\xi,\underline{u})$ of Theorem 8.1. Row 3 displays the functional dependence of the domains on the scaling parameter $\mathfrak{A}$, with the notation $\mathbf{Strip}(\mu,\lambda) = \mathbb{R}^2\times(0,\mu)\times(-\infty,-\lambda^{-1})$. Row 4 gives the size of the ξ-disk which, under stereographic projection, corresponds to one hemisphere of S^2.

9.2. Two Physical Regimes. There are two natural physical regimes. Informally, both appear as limits $\mathfrak{A}\downarrow 0$ in the Regularized Picture, keeping the data η^{σ} fixed. They are distinguished by:

- *2D (Scaling) Limit*: $a = \mathfrak{A}\downarrow 0$.
- *4D (Scaling) Limit*: a fixed, $\mathfrak{A}\downarrow 0$.

(The 4D Limit breaks Pole-Flip compatibility of η^{σ}. This will be discussed below.)

Definition 9.1. *The* 2D Limit Assumptions *are the hypotheses, with $a=\mathfrak{A}$ and $\epsilon = \frac{1}{4}$, of Theorem 8.1 on $\mathfrak{A}$, $\mathbf{DATA}^{\sigma} = \eta^{\sigma}$, R and K up to and including condition* (8.32)*.*

Remark 9.1. Explicitly, the 2D Limit Assumptions are: $R\geq 4,\ K\geq 0,\ \eta^{\sigma}=0$ when $\underline{u}<\frac{1}{2}$, and

$$0<|\mathfrak{A}|<\mathbf{b}, \qquad \eta^{\sigma} = \mathbf{Flip}_1\cdot\eta^{-\sigma}, \qquad \max_{\sigma\in\{-,+\}}\|\eta^{\sigma}\|_{C^{R+4}(D_4(0)\times(0,2))}\leq \mathbf{b}$$

Here, $\mathbf{b}\in(0,1)$ depends only on R. The constant $\mathbf{c}\in(0,1)$ in Theorem 8.1 depends only on R, K and $\max_{\sigma\in\{-,+\}}\|\eta^{\sigma}\|_{C^{R+2K+6}(D_4(0)\times(0,2))}$.
The conditions on $\eta^{\sigma}:\mathbb{R}^2\times(0,\infty)\to\mathbb{C}$ are independent of $\mathfrak{A}$. Also the domain of definition $\mathbf{Strip}(1,\mathbf{c})$ of the vacuum field in Theorem 8.1 is independent of $\mathfrak{A}$. Therefore, Theorem 8.1 is consistent with the 2D Limit. The choice $\epsilon=\frac{1}{4}$ is just for concreteness.

Remark 9.2. The intuition behind the designations Regularized Picture and High Amplitude Picture is immediately clear in the context of the 2D Limit. For the Regularized picture, see Remark 6.3. In the High Amplitude Picture, the initial data at past null infinity for the corresponding family of vacuum fields is $\mathbf{DATA}^{\sigma}_{H}(\xi,\underline{u}) = \mathfrak{A}^{-2}\,\eta^{\sigma}(\xi,\underline{u})$. It grows unboundedly as $\mathfrak{A}\downarrow 0$. The Finite Mass Picture will be discussed momentarily.

Remark 9.3. From our perspective, [Chr] investigates the 2D Limit ($a=\mathfrak{A}$) in the Finite Mass Picture. Christodoulou's small parameter $\delta>0$ is to be identified with our $\mathfrak{A}^4$. With this translation, the first equation in (9.1c) is precisely Christodoulou's "short pulse ansatz". For [Chr], the "short pulse hierarchy" plays a central role (see equation (24) on page 20 in [Chr], and the following discussion). In our approach, this hierarchy plays no role at all. However, it can be recovered through the scaling transformations required to go from the Regularized Picture to the Finite Mass Picture, see (9.5c) and (9.6c) below. By contrast, our working picture, the Regularized Picture, merely contains a dichotomy: the $\mathfrak{P}$-even components display one behavior, the $\mathfrak{P}$-odd components another, see Remark 6.3 or (9.8). This dichotomy disappears in our 4D Limit.

9.3. Trapped Spheres.

Proposition 9.2. *Make the 2D Limit Assumptions (Definition 9.1) with $K = 0$. Set*

$$\Lambda(\underline{u}_1) \;=\; \min_{\sigma\in\{-,+\}} \inf_{\xi\in D_4(0)} \int_0^{\underline{u}_1} \mathrm{d}\underline{u}\, |\eta^\sigma(\xi,\underline{u})|^2$$

Suppose $\Lambda(\underline{u}_1) > 0$ for some $\underline{u}_1 \in (0,1)$. Then γ_2^σ and γ_6^σ are everywhere negative on the sphere

$$S_{\underline{u},u}: \qquad (\underline{u},u) = \big(\underline{u}_1,\; -\tfrac{1}{2}\,\Lambda(\underline{u}_1)\,\mathfrak{A}^{-2}\big)$$

whenever $\mathfrak{A} \in (0,\mathbf{b})$ is sufficiently small depending only on $\Lambda(\underline{u}_1)$ and $\mathbf{c}$. For instance, $\mathfrak{A} < \frac{1}{4}\sqrt{\mathbf{c}}\,\min\{1,\Lambda(\underline{u}_1)\}$ will do. In other words, $S_{\underline{u},u}$ is a trapped sphere (see, Remark 2.5).

Remark 9.4. Clearly, there is an infinite dimensional family of pairs η^σ, $\mathfrak{A}$ satisfying the assumptions of Proposition 9.2.

Proof. By (8.34), the components γ_2^σ, γ_6^σ of $\Phi^\sigma = \mathcal{M}_{\mathfrak{A},\mathfrak{A}} + u^{-M}\Psi^\sigma$ satisfy:

$$\begin{aligned}
\Big| u^2\gamma_2^\sigma(\xi,\underline{u},u) - \Big(+ \frac{\mathfrak{A}^2u^2}{\mathfrak{A}^2\underline{u}-u} + \omega_2^\sigma(0)(\xi,\underline{u})\Big)\Big| \;&\leq\; \frac{1}{\mathbf{c}\,|u|}\\
\Big| u^2\gamma_6^\sigma(\xi,\underline{u},u) - \Big(- \frac{u^2}{\mathfrak{A}^2\underline{u}-u} + \omega_6^\sigma(0)(\xi,\underline{u})\Big)\Big| \;&\leq\; \frac{1}{\mathbf{c}\,|u|}
\end{aligned}$$

for all $(\xi,\underline{u},u) \in D_4(0)\times(0,1)\times(-\infty,-\mathbf{c}^{-1})$, where (see equations (6.3))

$$\omega_2^\sigma(0)(\xi,\underline{u}) \;=\; -\int_0^{\underline{u}} \mathrm{d}s\,|\eta^\sigma(\xi,s)|^2 \qquad , \qquad \omega_6^\sigma(0)(\xi,\underline{u}) = 0$$

To find a trapped sphere, let $u = -\lambda\mathfrak{A}^{-2}$, where $\lambda > 0$. Now, for all $(\xi,\underline{u}) \in D_4(0)\times(0,1)$ and $\mathbf{c}\lambda > \mathfrak{A}^2$:

$$\begin{aligned}
&\text{If} \quad \lambda - \int_0^{\underline{u}} \mathrm{d}s\,|\eta^\sigma(\xi,s)|^2 + \frac{\mathfrak{A}^2}{\mathbf{c}\lambda} \;<\; 0 \quad , \quad \text{then} \quad \gamma_2^\sigma(\xi,\underline{u},-\lambda\mathfrak{A}^{-2}) \;<\; 0.\\
&\text{If} \qquad\qquad -\frac{\lambda^2}{\mathfrak{A}^2(\mathfrak{A}^4+\lambda)} + \frac{\mathfrak{A}^2}{\mathbf{c}\lambda} \;<\; 0 \quad , \quad \text{then} \quad \gamma_6^\sigma(\xi,\underline{u},-\lambda\mathfrak{A}^{-2}) \;<\; 0.
\end{aligned} \tag{9.2}$$

Proposition 9.2 is a direct consequence of (9.2) with $\lambda = \frac{1}{2}\Lambda(\underline{u}_1)$, if we also recall that $\mathbf{Flip}_1$ does not change the sign of γ_2^σ and γ_6^σ. $\square$

9.4. The 2D Limit in the Finite Mass Picture for a Finite Duration Pulse. We make the 2D Limit Assumptions (Definition 9.1), and the assumptions for a finite duration pulse:

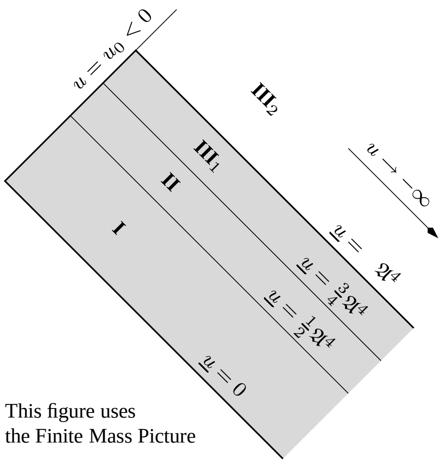
$$\eta^\sigma : \mathbb{R}^2\times(0,\infty)\to\mathbb{C} \text{ has support contained in } \mathbb{R}^2\times(\tfrac{1}{2},\tfrac{3}{4}). \tag{9.3a}$$

$$\int_{1/2}^{3/4} \mathrm{d}\underline{u}\,\eta^\sigma(\xi,\underline{u}) = 0 \text{ for all } \xi \text{ and } \sigma. \tag{9.3b}$$

Remark 9.5. Clearly, there is an infinite dimensional family of Pole-Flip compatible η^σ satisfying (9.3). The conditions (9.3) are equivalent to the condition that $\eta^\sigma(\xi, \underline{u}) = \frac{\partial}{\partial \underline{u}} \mathbf{G}^\sigma(\xi, \underline{u})$ for a function $\mathbf{G}^\sigma : \mathbb{R}^2 \times (0, \infty) \to \mathbb{C}$ with $\operatorname{supp} \mathbf{G}^\sigma \subset \mathbb{R}^2 \times (\frac{1}{2}, \frac{3}{4})$.

Remark 9.6. One can relax the conditions (9.3) by imposing suitable smallness conditions for η^σ on $\mathbb{R}^2 \times (\frac{3}{4}, \infty)$ and for $\int_{1/2}^{3/4} \mathrm{d}\underline{u}\, \eta^\sigma(\xi, \underline{u})$, depending on $\mathfrak{A}$. In this case, contrary to our current convention, η^σ itself would have to be allowed to depend on $\mathfrak{A}$, say polynomially. A smallness condition depending on $\mathfrak{A}$ is an "open" condition, as opposed to (9.3).

We begin with an informal discussion of the 2D Limit $a = \mathfrak{A} \downarrow 0$ in the Finite Mass Picture. Fix $u_0 < 0$. The shaded region in the figure (just below) is $\textbf{Strip}(\mathfrak{A}^4, |u_0|^{-1})$. For $|\mathfrak{A}|$ small enough, it is contained in $\textbf{Strip}(\mathfrak{A}^4, \mathbf{c}\mathfrak{A}^{-2})$, the domain, in the Finite Mass Picture, on which the solution to Theorem 8.1 exists. Region **I** is Minkowski space $\mathcal{M}_{1,1}$. In the "pulse region" **II**, the solution grows unboundedly as $\mathfrak{A} \downarrow 0$. Nevertheless, as we will show, (9.3) implies that on each compact $K \subset \textbf{Strip}(\infty, |u_0|^{-1})$ the far field formal solution converges as $\mathfrak{A} \downarrow 0$. Note that as $\mathfrak{A} \downarrow 0$, the compact K is eventually contained in the "after the pulse region" $\mathbf{III} = \mathbf{III}_1 \cup \mathbf{III}_2$. One would like to have an analogous result for classical solutions. In this paper we take a step towards such a result, by controlling the solution in $\mathbf{III}_1$, a strip moving and shrinking with $\mathfrak{A}$.



If $\int_{1/2}^{3/4} \mathrm{d}\underline{u}\, |\eta^\sigma(\xi, \underline{u})|^2$ is positive and independent of ξ and σ, the limit $\mathfrak{A} \downarrow 0$ of the formal power series solution is the field corresponding to a Schwarzschild spacetime, whose future horizon is a level set of u, with $u < u_0$, when $|u_0| > 0$ is sufficiently small.

We use the notation

$$\Psi^\sigma_{\text{Picture}} \qquad \text{and} \qquad [\,\Psi^\sigma_{\text{Picture}}\,] = \sum_{k=0}^{\infty} \frac{1}{u^k}\, \Psi^\sigma_{\text{Picture}}(k)(\xi, \underline{u})$$

for the field and for the formal power series, where Picture $=$ R, H, F. Let RH, HR, FH, HF, FR, RF be the "transition matrices" between the different pictures, given by

$$\mathrm{RH} = -\mathrm{HR} = \mathrm{diag}\,(3,3,4,4,4,\quad 2,4,3,3,3,2,2,4,\quad 2,\quad 3,4,5,4) \tag{9.4a}$$
$$\mathrm{FH} = -\mathrm{HF} = \mathrm{diag}\,(4,4,8,8,8,\quad 0,4,4,4,4,4,4,8,\ -4,\quad 0,4,8,8) \tag{9.4b}$$
$$\mathrm{FR} = -\mathrm{RF} = \mathrm{diag}\,(1,1,4,4,4,\ -2,0,1,1,1,2,2,4,\ -6,-3,0,3,4) \tag{9.4c}$$

Observe that FR $=$ FH $+$ HR. Then (use: the figure in Subsection 9.1, equations (9.1), $a = \mathfrak{A}$ and Definitions 3.2, 3.4, 3.5)

$$\Psi^\sigma_{\mathrm{H}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\mathrm{HR}}\,\Psi^\sigma_{\mathrm{R}}(\xi,\underline{u},\mathfrak{A}^2 u) \tag{9.5a}$$
$$\Psi^\sigma_{\mathrm{F}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\mathrm{FH}}\,\Psi^\sigma_{\mathrm{H}}(\xi,\mathfrak{A}^{-4}\underline{u},\mathfrak{A}^{-4} u) \tag{9.5b}$$
$$\Psi^\sigma_{\mathrm{F}}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{\mathrm{FR}}\,\Psi^\sigma_{\mathrm{R}}(\xi,\mathfrak{A}^{-4}\underline{u},\mathfrak{A}^{-2} u) \tag{9.5c}$$

The coefficient functions of the formal power series transform according to

$$\Psi^\sigma_{\mathrm{H}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\mathrm{HR}-2k}\,\Psi^\sigma_{\mathrm{R}}(k)(\xi,\underline{u}) \tag{9.6a}$$
$$\Psi^\sigma_{\mathrm{F}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\mathrm{FH}+4k}\,\Psi^\sigma_{\mathrm{H}}(k)(\xi,\mathfrak{A}^{-4}\underline{u}) \tag{9.6b}$$
$$\Psi^\sigma_{\mathrm{F}}(k)(\xi,\underline{u}) \;=\; \mathfrak{A}^{\mathrm{FR}+2k}\,\Psi^\sigma_{\mathrm{R}}(k)(\xi,\mathfrak{A}^{-4}\underline{u}) \tag{9.6c}$$

For all $k \geq 0$, we have:

$$\Psi^\sigma_{\mathrm{R}}(k)(\xi,\underline{u}) \text{ is a polynomial in } \mathfrak{A} \tag{9.7a}$$
$$\Psi^\sigma_{\mathrm{H}}(k)(\xi,\underline{u}) \text{ is a polynomial in } \mathfrak{A}^{-2} \text{ without constant term} \tag{9.7b}$$

Statement (9.7) is verified by induction over k. Just follow the construction of $[\,\Psi^\sigma_{\mathrm{R}}\,]$ and $[\,\Psi^\sigma_{\mathrm{H}}\,]$ in the proof of Lemma 6.1, keeping in mind that the coefficient functions, in the Regularized Picture, of the Minkowski background $[\,\mathcal{M}_{\mathfrak{A},\mathfrak{A}}\,]$ depend polynomially on $\mathfrak{A}$, and that $\mathbf{DATA}^\sigma_R$ is independent of $\mathfrak{A}$. On the other hand, in the High Amplitude Picture, the Minkowski background $[\,\mathcal{M}_{1,1}\,]$ is independent of $\mathfrak{A}$, while $\mathbf{DATA}^\sigma_H$ is proportional to $\mathfrak{A}^{-2}$. Incidentally, (9.7a) has already been shown in Remark 6.3. There, it was also shown that

$$\mathfrak{P}\text{-even } (\mathfrak{P}\text{-odd}) \text{ components of } \Psi^\sigma_{\mathrm{R}}(k)(\xi,\underline{u}) \text{ are even (odd) polynomials in } \mathfrak{A} \tag{9.8}$$

The statement (9.8) also follows from (9.7a), (9.7b) and (9.6a), because the i-th component of $\Psi^\sigma_R(k)(\xi,\underline{u})$ is $\mathfrak{P}$-even ($\mathfrak{P}$-odd) if $(\mathrm{RH})_{ii}$ is even (odd). Furthermore,

$$\text{the } i\text{-th component of } \Psi^\sigma_R(k)(\xi,\underline{u}) \text{ has degree} \leq (\mathrm{RH})_{ii} - 2 + 2k \tag{9.9}$$

as a polynomial in $\mathfrak{A}$.

Lemma 9.1. *The 18 components of each $\Psi^\sigma_F(k)(\xi,\mathfrak{A}^4\underline{u})$, $k \geq 0$, are Laurent polynomials in $\mathfrak{A}^2$. If a component does not appear on the list*

$$\begin{array}{ll} (\omega_1)^\sigma_F(k)(\xi,\mathfrak{A}^4\underline{u}) & k = 0,1 \\ (\omega_2)^\sigma_F(0)(\xi,\mathfrak{A}^4\underline{u}) & \\ (z_1)^\sigma_F(k)(\xi,\mathfrak{A}^4\underline{u}) & k = 0,1,2,3 \\ (z_2)^\sigma_F(k)(\xi,\mathfrak{A}^4\underline{u}) & k = 0,1 \\ (z_3)^\sigma_F(0)(\xi,\mathfrak{A}^4\underline{u}) & \end{array} \tag{9.10}$$

then it is an actual polynomial in $\mathfrak{A}^2$ without constant term.

Proof. By (9.4b), (9.6b), (9.7b), all the components are Laurent polynomials in $\mathfrak{A}^2$. By (9.4c), (9.6c), (9.7a), all those not in (9.10) are polynomials without constant term. □

Lemma 9.2. *Suppose* (9.3)*. Define* $U = \mathbb{R}^2 \times (\frac{3}{4}, \infty)$ *and* $\Lambda^\sigma(\xi) = \int_{1/2}^{3/4} \mathrm{d}\underline{u} |\eta^\sigma(\xi, \underline{u})|^2$. *For* $(\xi, \underline{u}) \in U$*, the functions in* (9.10) *are polynomials in* $\mathfrak{A}^2$*, and:*

$$\begin{aligned}
&(\omega_1)^\sigma_F(0)(\xi, \mathfrak{A}^4\underline{u}) = 0 && (z_1)^\sigma_F(2)(\xi, \mathfrak{A}^4\underline{u}) = 2\,(\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda}_{1,1})\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma \\
&(\omega_1)^\sigma_F(1)(\xi, \mathfrak{A}^4\underline{u}) = 0 && (z_1)^\sigma_F(3)(\xi, \mathfrak{A}^4\underline{u}) = \mathcal{O}(\mathfrak{A}^2) \\
&(\omega_2)^\sigma_F(0)(\xi, \mathfrak{A}^4\underline{u}) = -\Lambda^\sigma && (z_2)^\sigma_F(0)(\xi, \mathfrak{A}^4\underline{u}) = 0 \\
&(z_1)^\sigma_F(0)(\xi, \mathfrak{A}^4\underline{u}) = 0 && (z_2)^\sigma_F(1)(\xi, \mathfrak{A}^4\underline{u}) = -2\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\overline{\xi}}\Lambda^\sigma \\
&(z_1)^\sigma_F(1)(\xi, \mathfrak{A}^4\underline{u}) = 0 && (z_3)^\sigma_F(0)(\xi, \mathfrak{A}^4\underline{u}) = \Lambda^\sigma
\end{aligned} \tag{9.11}$$

where $2\frac{\partial}{\partial\overline{\xi}} = \frac{\partial}{\partial\xi^1} + i\frac{\partial}{\partial\xi^2}$ *and* $\mathbf{e}_{1,1} = \frac{1}{2}(1+|\xi|^2)$ *and* $\boldsymbol{\lambda}_{1,1} = -\frac{1}{2}(\xi^1 + i\xi^2)$.

Proof. By (9.6c), the equations (9.11) can be translated from the Finite Mass to the Regularized Picture. In this proof, we work exclusively in the Regularized Picture. For convenience, we suppress the R and σ indices as well as the argument $(\xi, \underline{u})$. For example, $\omega_1(k)$ means $(\omega_1)^\sigma_{\mathrm{R}}(k)(\xi, \underline{u})$. We use the shorthands $\mathbf{e} = \mathbf{e}_{\mathfrak{A},\mathfrak{A}}$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathfrak{A},\mathfrak{A}}$ (see, (4.3)). Equivalently, $\mathbf{e} = \mathfrak{A}\mathbf{e}_{1,1}$ and $\boldsymbol{\lambda} = \mathfrak{A}\boldsymbol{\lambda}_{1,1}$. For all the equations in (9.11) concerning *zeroth* order coefficient functions, use equations (6.3) and (9.6c). We only note that on $\mathbb{R}^2 \times (0, \infty)$,

$$z_3(0) \;=\; -4(\mathbf{e}\tfrac{\partial}{\partial\xi} + \overline{\boldsymbol{\lambda}})(\mathbf{e}\tfrac{\partial}{\partial\xi} + 2\overline{\boldsymbol{\lambda}})\,\partial_{\underline{u}}^{-1}\eta - \eta\,\partial_{\underline{u}}^{-1}\overline{\eta} + \partial_{\underline{u}}^{-1}|\eta|^2$$

which reduces to $z_3(0) = \Lambda^\sigma$ on U. For the rest of (9.11), we use the equations

$$\begin{aligned}
N(z_1) + \tfrac{1}{u}D(z_2) &= \tfrac{1}{u^2}\big(Sz_1 - 2\boldsymbol{\lambda}z_2 - \omega_6 z_1\big) \\
&\qquad + \tfrac{1}{u^3}\big(2S\boldsymbol{\lambda}z_2 - 3\omega_1 z_3 + 6\omega_3 z_2 + 4\omega_8 z_1 - 4\overline{\omega}_4 z_2\big) \\
N(z_2) + \tfrac{1}{u}D(z_3) &= \tfrac{1}{u^2}\big(+2Sz_2 - 2\omega_6 z_2\big) \\
&\qquad + \tfrac{1}{u^3}\big(-2\omega_1 z_4 + 3\omega_3 z_3 + 2\omega_8 z_2 - 3\overline{\omega}_4 z_3\big) \\
L(\omega_1) &= -z_1 + \tfrac{1}{u}\,2\,\mathfrak{A}^2\,\omega_1 + \tfrac{1}{u^2}\big(-2\,S\,\mathfrak{A}^2\,\omega_1 - 2\omega_1\omega_2\big)
\end{aligned}$$

The first and third appear (5.4a). For the second, we use (5.4a) and the constraint equation $y_1 = 0$. For the vector fields D, N, L, see (5.6). We obtain, in succession,

	on $\mathbb{R}^2 \times (0,\infty)$	on U
$z_1(1)$	$= -\mathfrak{A}^2\underline{u}\partial_{\underline{u}}\eta - 2(\mathbf{e}\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})z_2(0)$	$= 0$
$z_2(1)$	$= 2\,\mathfrak{A}^2\,\underline{u}\,z_2(0) - 2\,\mathbf{e}\frac{\partial}{\partial\overline{\xi}}\,z_3(0)$	$= -2\,\mathbf{e}\frac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma$
$\omega_1(1)$	$= -4(\mathbf{e}\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})(\mathbf{e}\frac{\partial}{\partial\xi} + 2\overline{\boldsymbol{\lambda}})\partial_{\underline{u}}^{-1}\eta$ $+ \mathfrak{A}^2\,\underline{u}\,\eta + \mathfrak{A}^2\,\partial_{\underline{u}}^{-1}\eta$	$= 0$
$z_1(2)$	= (not needed)	$= 2\,(\mathbf{e}\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})\,\mathbf{e}\frac{\partial}{\partial\overline{\xi}}\,\Lambda^\sigma$
$z_1(3)$	= (not needed)	$= \mathfrak{A}^2\underline{u}\,z_1(2) - \frac{2}{3}(\mathbf{e}\frac{\partial}{\partial\overline{\xi}} - \boldsymbol{\lambda})z_2(2)$

For $z_1(3)$, we have also used that $\omega_3(0)$, $\omega_4(0)$, $f_1(0)$, $f_2(0)$ all vanish on U (see equations (6.3)). We know from (9.8) that $z_1(3)$ is a polynomial in $\mathfrak{A}^2$. It has no constant term on U, because $\mathbf{e}$, $\boldsymbol{\lambda}$ and $z_2(2)$ are odd polynomials in $\mathfrak{A}$. Now, use (9.6c). □

Lemma 9.3. *Suppose* (9.3)*. Each component of each* $\Psi_F^\sigma(k)(\xi,\underline{u})$*,* $k\geq 0$*, is a Laurent polynomial in* $\mathfrak{A}^2$ *and a polynomial in* $\underline{u}$ *when* $\underline{u}>\frac{3}{4}\mathfrak{A}^4$*.*

Proof. By (9.3a) and the construction of the formal power series in the proof of Lemma 6.1, each $\Psi_{\mathrm{R}}^\sigma(k)(\xi,\underline{u})$ is a polynomial in $\underline{u}$ on $U=\mathbb{R}^2\times(\frac{3}{4},\infty)$. Also recall (9.7a) and (9.8). The lemma now follows from (9.6c). □

Let $\widehat{\Psi}_{\mathrm{F}}^\sigma(k)(\xi,\underline{u})$ be the polynomial extension in $\underline{u}$ of $\Psi_{\mathrm{F}}^\sigma(k)(\xi,\underline{u})$ from $\underline{u}>\frac{3}{4}\mathfrak{A}^4$ to $\underline{u}>0$. Then, $[\,\widehat{\Psi}_{\mathrm{F}}^\sigma\,]$ is a Pole-Flip compatible pair of formal solutions to (5.4a) on $\mathbf{Strip}_\infty$ with Minkowski background $[\,\mathcal{M}_{1,1}\,]$, and $[\,\widehat{\Psi}_{\mathrm{F}}^{\sigma\,\sharp}\,]=0$.

Lemma 9.4. *Suppose* (9.3)*. Then,* $\widehat{\Psi}_F^\sigma(0)(\xi,\underline{u})$ *is a polynomial in* $\mathfrak{A}^2$*. More precisely,*

$$\widehat{\Psi}_F^\sigma(0)(\xi,\underline{u})= \tag{9.12}$$
$$\big(0,0,-\underline{u}\Lambda^\sigma,0,0,\ \ 0,-\Lambda^\sigma,0,0,0,0,0,\underline{u}\Lambda^\sigma,\ \ 0,0,\Lambda^\sigma,2\,\underline{u}\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\xi}\Lambda^\sigma,0\big)+\mathcal{O}(\mathfrak{A}^2)$$

Proof. By direct calculation. □

Proposition 9.3. *Suppose* (9.3)*. Each component of each* $\widehat{\Psi}_F^\sigma(k)(\xi,\underline{u})$*,* $k\geq 0$*, is simultaneously a polynomial in* $\underline{u}$ *and* $\mathfrak{A}^2$*, for all* $(\xi,\underline{u})\in\mathbb{R}^2\times(0,\infty)$*.*

Proof. They are polynomials in $\underline{u}$ by definition. The case $k=0$ is covered by Lemma 9.4. The general case is shown by induction over k, using the fact that the equations (6.6) hold with Minkowski background $[\,\mathcal{M}_{1,1}\,]$. In the present case, (6.6) are equations for polynomials in $\underline{u}$. The generic terms $\mathcal{P}_k$ on the right hand sides in (6.6) are, by the inductive hypothesis, polynomials in $\mathfrak{A}^2$. When using (6.6a) through (6.6r) in this order to determine the components of $\widehat{\Psi}_F^\sigma(k)$, only polynomials in $\mathfrak{A}^2$ are generated. If $\frac{\partial}{\partial\underline{u}}$ appears on the left hand side, then the non-constant terms as a $\underline{u}$-polynomial of the corresponding component of $\widehat{\Psi}_F^\sigma(k)$ are determined uniquely by the right hand side. The constant term of integration is determined by the restriction of $\widehat{\Psi}_F^\sigma$ to $\underline{u}=\mathfrak{A}^4$, that is $\widehat{\Psi}_F^\sigma(k)(\xi,\mathfrak{A}^4)$, which is itself a polynomial in $\mathfrak{A}^2$, by Lemmas 9.1 and 9.2. □

Proposition 9.4. *Suppose* (9.3)*. For each* $k\geq 0$*, let* $\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma(k)$ *be the constant term of* $\widehat{\Psi}_F^\sigma(k)$ *as a polynomial in* $\mathfrak{A}^2$*. Then* $[\,\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma\,]$ *is the unique formal solution to* (5.4a) *with Minkowski background* $[\,\mathcal{M}_{1,1}\,]$ *and characteristic initial data*

$$\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma(0)(\xi,\underline{u})=$$
$$\big(0,0,-\underline{u}\Lambda^\sigma,0,0,\ \ 0,-\Lambda^\sigma,0,0,0,0,0,\underline{u}\Lambda^\sigma,\ \ 0,0,\Lambda^\sigma,2\,\underline{u}\,\mathbf{e}_{1,1}\tfrac{\partial}{\partial\xi}\Lambda^\sigma,0\big) \tag{9.13a}$$

$$[\,\widehat{\Psi}_{F,\mathfrak{A}=0}^\sigma\,](\xi,0,u)=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}\oplus\begin{pmatrix}0\\-\Lambda^\sigma\\0\\\vdots\\0\end{pmatrix}\oplus\begin{pmatrix}\frac{2}{u^2}(\mathbf{e}_{1,1}\frac{\partial}{\partial\xi}-\boldsymbol{\lambda}_{1,1})\mathbf{e}_{1,1}\frac{\partial}{\partial\xi}\Lambda^\sigma\\-\frac{2}{u}\mathbf{e}_{1,1}\frac{\partial}{\partial\xi}\Lambda^\sigma\\\Lambda^\sigma\\0\\0\end{pmatrix} \tag{9.13b}$$

Its coefficient functions are polynomials in $\underline{u}$*. Moreover,* $[\,\widehat{\Psi}_{F,\mathfrak{A}=0}^{\sigma\,\sharp}\,]=0$*.*
Particularly, if $\Lambda^\sigma(\xi)\equiv\Lambda$ *is independent of* ξ *and* σ*, then it represents Schwarzschild spacetime, with mass* $m=2^{-3/2}\Lambda$*.*

Proof. The first part follows from Lemma 9.1, 9.2, 9.3, 9.4 and Proposition 9.3. In particular, (9.13a) follows from (9.12), and (9.13b) follows from (9.11).
If $\varLambda^\sigma(\xi) \equiv \varLambda$ is independent of ξ and σ, then $\widehat{\varPsi}^\sigma_{\mathrm{F},\mathfrak{A}=0}(0)(\xi,\underline{u})$ and $[\,\widehat{\varPsi}^\sigma_{\mathrm{F},\mathfrak{A}=0}\,](\xi,0,u)$ in (9.13a) and (9.13b) correspond to spherically symmetric initial data for (5.4a) with Minkowski background $[\,\mathcal{M}_{1,1}\,]$. Therefore, $[\,\widehat{\varPsi}^\sigma_{\mathrm{F},\mathfrak{A}=0}\,]$ is spherically symmetric, and therefore, by a formal Birkhoff Theorem, a formal expansion of a Schwarzschild vacuum field. The component γ_2 of $[\,\mathcal{M}_{1,1}\,] + u^{-M}\,[\,\widehat{\varPsi}^\sigma_{\mathrm{F},\mathfrak{A}=0}\,]$ vanishes on the sphere $(u,\underline{u}) = (-\varLambda, 0)$, and therefore this sphere is a section of the Schwarzschild horizon. Its area is equal to $2\pi\varLambda^2$, which gives the formula for the mass m. $\square$

The discussion of the formal solution $[\,\varPsi^\sigma_{\mathrm{F}}\,]$ is finished. We now turn to classical solutions. The bound (8.34) implies that for all $u < -\mathbf{c}^{-1}$,

$$\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^\alpha\Big(\varPsi^\sigma_{\mathrm{R}}(\,\cdot\,,u) - \sum_{k=0}^{K}\frac{\widehat{\varPsi}^\sigma_{\mathrm{R}}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4},1)\right)} \;\le\; \frac{1}{\mathbf{c}\,|u|^{K+1}}$$

This bound, in turn, implies the Finite Mass Picture bound (use (9.5c), (9.6c))

$$\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^\alpha\Big(\varPsi^\sigma_{\mathrm{F}}(\,\cdot\,,u) - \sum_{k=0}^{K}\frac{\widehat{\varPsi}^\sigma_{\mathrm{F}}(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)\right)} \;\le\; \frac{\mathfrak{A}^{2K-4R+8}}{\mathbf{c}\,|u|^{K+1}}$$

when $u < -\mathfrak{A}^2\mathbf{c}^{-1}$. The power of $\mathfrak{A}$ on the right hand side arises as $2K - 4R + 8 = 2(K+1) - 4(R-3) - 6$. Given $R \ge 4$, we choose $K = 2R-3$. (Then, the constant $\mathbf{c}$ depends only on R and $\max_{\sigma\in\{-,+\}}\|\eta^\sigma\|_{C^{5R}(D_4(0)\times(0,2))}$.) Altogether, we obtain:

Proposition 9.5. *Suppose* (9.3). *For each $u_0 < 0$ and each $R \ge 4$, the limit as $\mathfrak{A}\downarrow 0$ of*

$$\sup_{u<u_0}\,|u|^{2R-2}\sup_{\substack{\alpha\in\mathbb{N}_0^4\\|\alpha|\le R-3}} \Big\|\partial^\alpha\Big(\varPsi^\sigma_F(\,\cdot\,,u) - \sum_{k=0}^{2R-3}\frac{\widehat{\varPsi}^\sigma_F(k)(\,\cdot\,)}{u^k}\Big)\Big\|_{C^0\left(D_4(0)\times(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)\right)}$$

is zero. Here, the solution $\varPsi^\sigma_F$, the functions $\widehat{\varPsi}^\sigma_F(k)$, and the $\underline{u}$-interval $(\frac{3}{4}\mathfrak{A}^4,\mathfrak{A}^4)$ depend on $\mathfrak{A}$. Under appropriate conditions (see, Proposition 9.4), the Schwarzschild vacuum field can be approximated arbitrarily closely on the strip $\mathbf{III}_1$.

Remark 9.7. To obtain the last result, we had to explicitly calculate the first four orders of the far field expansion, in particular for the component z_1.

So far, we have provided complete, detailed arguments for each of our statements. At this point of the paper, the character of our discussion changes. For the rest of Section 9, we sketch additional applications of our overall hybrid method and give informal arguments to support our informal assertions. We will give rigorous discussions in another place.

9.5. 2D Limit in the Regularized Picture: beyond the far field regime. Theorem 8.1 produces vacuum fields on $\mathbf{Strip}(1,\mathbf{c})$, with $\mathbf{c}>0$ independent of $\mathfrak{A}$ (see, Remark 9.1). In this subsection, we informally answer the question: How can one control these vacuum fields when $u>-\frac{1}{\mathbf{c}}$ and get closer to the (expected) singularity? The strategy, as before, is to generate an appropriate formal solution to the initial value problem (see Section 6), and then to construct a classical solution by estimating its deviation from a truncation of the formal solution. The far field expansion has run its course. We need an expansion in $\mathfrak{A}^2$.

In this subsection, we make the 2D Limit Assumptions (see, Definition 9.1), work exclusively in the Regularized Picture, and suppress the R and σ indices. For example, Φ means Φ^σ_R.

In analogy with (5.2a), (5.2c), (5.3a), (5.3b), set

$$P \;=\; \mathrm{diag}(1,1,0,2,2)\oplus\mathrm{diag}(0,0,1,1,1,0,0,0)\oplus\mathrm{diag}(0,1,0,1,0) \tag{9.14a}$$

$$P^\sharp \;=\; \mathrm{diag}(2,2,1,1,1)\oplus\mathrm{diag}(1,0,0,0,0,0,1,1,1)\oplus\mathrm{diag}(1,0,1) \tag{9.14b}$$

$$\Phi \;=\; \mathfrak{A}^P\left(\mathfrak{A}^{-P}\Phi\right) \tag{9.14c}$$

$$\Phi^\sharp \;=\; \mathfrak{A}^{P^\sharp}\left(\mathfrak{A}^{-P^\sharp}\Phi^\sharp\right) \tag{9.14d}$$

The third line indicates that we wish to write Φ as $\mathfrak{A}^P$ times a *new field*. In order not to introduce yet another name, we write the *new field* as $\mathfrak{A}^{-P}\Phi$. Similar for $\mathfrak{A}^{-P^\sharp}\Phi^\sharp$. We make formal expansions of $\mathfrak{A}^{-P}\Phi$ in powers of $\mathfrak{A}^2$. The properties of the far field expansion, for instance (9.7a) and (9.8), suggest that the ansatz (9.14) is consistent. However, this must be checked.

A formal power series $\{\,f\,\}$ in $\mathfrak{A}^2$ on an open subset $\mathcal{U}\subset\mathbf{Strip}_\infty$ with values in a vector space X is a formal sum

$$\{\,f\,\} \;=\; \sum_{\ell=0}^{\infty}\left(\mathfrak{A}^2\right)^\ell f\{\ell\}(x) \tag{9.15}$$

For each $\ell\geq 0$, the coefficient $f\{\ell\}:\mathcal{U}\to X$ is smooth and independent of $\mathfrak{A}$.

Let $\{\,\mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}\,\}$ be the formal expansion of $\mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ in powers of $\mathfrak{A}^2$ (see, Definition 4.1). It is defined on $\mathbf{Strip}_\infty$ and takes values in $\mathcal{R}$.

Our ansatz is to write the field $\mathfrak{A}^{-P}\Phi$ as a formal series $\{\,\mathfrak{A}^{-P}\Phi\,\}$ on some open set $\mathcal{U}\subset\mathbf{Strip}_\infty$ with values in $\mathcal{R}$. In this context, the far field ansatz (Section 5) becomes

$$\left\{\,\mathfrak{A}^{-P}\Phi\,\right\} \;=\; \left\{\,\mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}\,\right\}+u^{-M}\left\{\,\mathfrak{A}^{-P}\Psi\,\right\} \tag{9.16}$$

To define the associated formal constraint field, see Definition 2.4 and (5.3b), we fix the weight functions $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ by (5.3c), as before. Then, by direct inspection, $\{\,\mathfrak{A}^{-P^\sharp}\Phi^\sharp\,\}$ or, equivalently, $\{\,\mathfrak{A}^{-P^\sharp}\Psi^\sharp\,\}$ are also formal power series of the form (9.15). They are defined on $\mathcal{U}$ and take values in $\widehat{\mathcal{R}}$. For each $\ell\geq 0$, the coefficient $(\mathfrak{A}^{-P^\sharp}\Phi^\sharp)\{\ell\}$ is determined by $(\mathfrak{A}^{-P}\Phi)\{m\}$, $0\leq m\leq\ell$. Similar for $(\mathfrak{A}^{-P^\sharp}\Psi^\sharp)\{\ell\}$.

We want to formally solve the same *characteristic initial value problem* as before:

- (5.4a) with Ψ and $\mathcal{M}_{\mathfrak{A},\mathfrak{A}}$ replaced by $\mathfrak{A}^P\{\,\mathfrak{A}^{-P}\Psi\,\}$ and $\mathfrak{A}^P\{\,\mathfrak{A}^{-P}\mathcal{M}_{\mathfrak{A},\mathfrak{A}}\,\}$,
- $\{\,\mathfrak{A}^{-P^\sharp}\Psi^\sharp\,\}=0$,
- formal asymptotic initial conditions (6.8a), (6.8b) with, say, $\underline{u}_0=\frac{1}{2}$.

It is understood that $\text{DATA} = \eta$ (see equations (6.3)) is fixed and independent of $\mathfrak{A}$.

Remark 9.8. The meaning of the formal asymptotic initial condition (6.8a) is

$$\sum_{\ell=0}^{\infty} \left(\mathfrak{A}^2\right)^{\ell} \lim_{u\to-\infty} \left(\mathfrak{A}^{-P}\Psi\right)\{\ell\}(\xi,\underline{u},u) \;=\; \mathfrak{A}^{-P}\,\Psi(0)(\xi,\underline{u}).$$

Equality is in the sense of $\mathcal{R}$-valued formal power series in $\mathfrak{A}^2$. The right hand side is actually a polynomial in $\mathfrak{A}^2$ of degree 1, see equations (6.3).

Constructing a solution $\{\,\mathfrak{A}^{-P}\Psi\,\}$ to this formal initial value problem requires solving an infinite family of differential equations. All but a finite number of them are linear. It is possible to arrange these equations so that, when they are solved step by step, the “angular derivatives” $\frac{\partial}{\partial\xi^1}$, $\frac{\partial}{\partial\xi^2}$ are only applied to functions that have already been constructed. An essential ingredient is

$$D = \mathcal{O}(\mathfrak{A}) \qquad N = \tfrac{\partial}{\partial u} + \mathcal{O}(\mathfrak{A}^2) \qquad L = (1 + \tfrac{1}{u^2} f_3)\,\tfrac{\partial}{\partial \underline{u}} \tag{9.17}$$

as $\mathfrak{A} \to 0$. Here, $\mathcal{O}(\mathfrak{A}^k)$ stands for $\mathcal{O}(\mathfrak{A}^k)\frac{\partial}{\partial\xi^1} + \mathcal{O}(\mathfrak{A}^k)\frac{\partial}{\partial\xi^2}$, when $k = 1, 2$. In this sense, one only has to solve 2-dimensional problems in the $(\underline{u}, u)$ plane.

Observe that property (9.9) of the $\frac{1}{u}$ expansion implies that the coefficient function $(\mathfrak{A}^{-P}\Psi)\{\ell\}$, $\ell \geq 1$ is of the order $\mathcal{O}(|u|^{-\ell+1})$ as $u \to -\infty$. For this reason, we expect that all the arguments in Section 8 can be applied, with minor modifications, when the function Ψ_K in **(S2)** is replaced by the truncation

$$\mathfrak{A}^P \sum_{\ell=0}^{K+2} \left(\mathfrak{A}^2\right)^{\ell} \left(\mathfrak{A}^{-P}\Psi\right)\{\ell\}$$

of $\mathfrak{A}^P\,\{\,\mathfrak{A}^{-P}\Psi\,\}$. One should be able to conclude, in analogy with Theorem 8.1, that both a classical solution Ψ and a formal power series solution $\mathfrak{A}^P\{\,\mathfrak{A}^{-P}\Psi\,\}$ exist on $\textbf{Strip}(1,\mathbf{c})$, and that

$$\mathfrak{A}^{-P}\Psi \;\;-\;\; \sum_{\ell=0}^{K+1} \left(\mathfrak{A}^2\right)^{\ell} \left(\mathfrak{A}^{-P}\Psi\right)\{\ell\}, \tag{9.18}$$

and all its partial derivatives up to some finite order, are estimated, in absolute value, by $\leq \mathbf{c}^{-1}\,\mathfrak{A}^{2K+4}\,|u|^{-K-1}$ on $\textbf{Strip}(1,\mathbf{c})$. Here, smallness conditions similar to those in Theorem 8.1 must be made. In particular, $\mathbf{c} > 0$ has to be sufficiently small.

The fact that the difference (9.18) goes to zero as $\mathfrak{A} \downarrow 0$, uniformly on $\textbf{Strip}(1,\mathbf{c})$, means that the formal $\mathfrak{A}^2$ expansion “has not yet been exhausted”. To better understand what happens, let us examine the formal expansion in just a little more detail. We only discuss the zeroth coefficient, $(\mathfrak{A}^{-P}\Psi)\{0\}$ or, equivalently, $(\mathfrak{A}^{-P}\Phi)\{0\}$, see (9.16). Set $a = \gamma_2\{0\}/e_3\{0\}$ and $b = \gamma_6\{0\}$. The constraint equations $u_2\{0\} = u_3\{0\} = u_6\{0\} = 0$ and the equation $\frac{\partial}{\partial u}e_3\{0\} = 2e_3\{0\}\,\Re\gamma_8\{0\}$ derived from (5.4a) yield the system $\frac{\partial}{\partial u}a = -2ab$ and $\frac{\partial}{\partial \underline{u}}b = -2ab$. The initial conditions (see, Remark 9.8)

	$e_3\{0\}$	$\omega_2\{0\} = u^2\,\gamma_2\{0\}$	$\omega_6\{0\} = u^2(\gamma_6\{0\} - \frac{1}{u})$
$\underline{u} < \frac{1}{2}$	1	0	0
$u \to -\infty$	1	$-\partial_{\underline{u}}^{-1}\lvert\text{DATA}\rvert^2$	0

select the unique solution

$$\tfrac{\gamma_2\{0\}}{e_3\{0\}} = \tfrac{1}{2h}\,\tfrac{\partial}{\partial \underline{u}} h \qquad \gamma_6\{0\} = \tfrac{1}{2h}\,\tfrac{\partial}{\partial u} h \qquad h(\xi,\underline{u},u) = u^2 - \big(\varphi(\xi,\underline{u})\big)^2 \tag{9.19}$$

where $\varphi \geq 0$ and $\varphi^2 = 2\,\partial_{\underline{u}}^{-1}\partial_{\underline{u}}^{-1}|\textbf{DATA}|^2$. The solution (9.19) is defined on $\mathcal{U}_0$, where

$$\mathcal{U}_\epsilon \;=\; \big\{(\xi,\underline{u},u) \in \textbf{Strip}_\infty \;\big|\; u < -\epsilon - \varphi(\xi,\underline{u})\big\}$$

for every $\epsilon \geq 0$. The solution (9.19) and therefore the formal solution $\{\,\mathfrak{A}^{-P}\Phi\,\}$ break down at $u = -\varphi(\xi,\underline{u}) < 0$, for example because $\gamma_6\{0\}$ diverges. Conversely, it can be shown that the whole formal solution $\{\,\mathfrak{A}^{-P}\Phi\,\}$ to the initial value problem exists on $\mathcal{U}_0$, that is, *no "earlier" breakdown occurs*. At the formal level, the scalar curvature invariants are in general unbounded as $u \uparrow -\varphi(\xi,\underline{u}) < 0$. We refer to $u = -\varphi(\xi,\underline{u}) < 0$ as the *formal (naive) singularity*. Observe that $\gamma_2\{0\} \leq 0$ and $\gamma_6\{0\} < 0$ on $\mathcal{U}_0$.

It follows from the structure of the matrix P, in particular its nonzero entries, that the components $e_3\{0\}$, $\gamma_1\{0\}$, $\gamma_2\{0\}$, $\gamma_6\{0\}$, $\gamma_7\{0\}$, $\gamma_8\{0\}$, $w_1\{0\}$, $w_3\{0\}$, $w_5\{0\}$ of the coefficient function $(\mathfrak{A}^{-P}\Phi)\{0\}$ satisfy the quasilinear symmetric hyperbolic system and the constraints in Proposition 2.4. This system has been investigated in situations with higher symmetry, for example in [Sze]. In our present context, however, the fields depend on all four coordinates. The collapse of the frame as $\mathfrak{A} \to 0$, see (9.17), is responsible for reducing the four-dimensional system to a family of two-dimensional systems, one for each ξ. It is possible to quasi-explicitly solve these two-dimensional systems near the formal singularity. That is, there is a formal solution given by an appropriate expansion in the "distance" from the formal singularity which is an asymptotic expansion to the true classical solution (of the two-dimensional system). The behavior of the solution to this two-dimensional system leads us to speculate that the $\mathfrak{A}^2$ expansion exhibits an instability close to the formal singularity. This instability appears to drive the full four dimensional system into a new regime in which the classical vacuum solution may display features of the BKL scenario. See, [BKL] and references therein.

We conclude this subsection with a further discussion of the figure that appears at the end of Section 1 (Introduction):

- Christodoulou [Chr] constructs strongly focused gravitational wave solutions on **I** (see, Remark 9.3). Recall that $u \sim -\frac{1}{\mathfrak{A}^2}$ is the place where trapped spheres first form, see Proposition 9.2.
- In this paper, the far field expansion has been used to construct vacuum fields on the larger $\textbf{I} \cup \textbf{II} = \textbf{Strip}(1,\mathbf{c})$, where $\mathbf{c} > 0$ is sufficiently small. See, Theorem 8.1.
- The $\mathfrak{A}^2$ expansion outlined in this subsection allows one, using appropriate energy estimates, to construct classical vacuum fields on at least $\textbf{I}\cup\textbf{II}\cup\textbf{III} = \textbf{Strip}(1,\epsilon'^{-1}) \cap \mathcal{U}_\epsilon$. Here, $\epsilon,\epsilon' > 0$ are arbitrary constants (in the figure, $0 < \epsilon < \epsilon'$), and $|\mathfrak{A}|$ is sufficiently small, depending on ϵ, ϵ'. Moreover, the formal $\mathfrak{A}^2$ power series solution is an asymptotic expansion to the true classical solution as $\mathfrak{A} \to 0$, uniformly on $\textbf{I} \cup \textbf{II} \cup \textbf{III}$. In other words, the $\mathfrak{A}^2$ expansion allows one to construct and control the solution up to any "finite distance" from the formal singularity.
  The justification of the last statement relies on the fact that a suitable truncation of the $\mathfrak{A}^2$ expansion is an approximate vacuum field, with error terms going to zero uniformly on $\textbf{I} \cup \textbf{II} \cup \textbf{III}$ as $\mathfrak{A} \to 0$ (by a compactness argument). Furthermore, these error terms decay quickly enough as $u \to -\infty$ to be "integrable". See, the discussion of (9.18).

Remark 9.9. Motivated by [BK] and [AnRe], we expect that coupling the gravitational field to a massless scalar field will make it possible to construct, under suitable generic conditions, strongly focused solutions from past null infinity all the way into a piece of the singularity.

Remark 9.10. Observe that:

- One only has to solve *linear* equations to inductively construct the far field expansion $[\,\Psi\,]$ in Lemma 6.1.
- By contrast, the construction of the leading term of the $\mathfrak{A}^2$ expansion $\{\,\mathfrak{A}^{-P}\Psi\,\}$ requires the solution of a *nonlinear* (effectively two-dimensional) system. In other words, one is expanding around a non-trivial "background".

9.6. 4D Limit. Recall that in the 2D Limit, the $\mathfrak{P}$-odd components of the coefficients of the far field formal solution $[\,\Phi_R\,]$ go to zero as $\mathfrak{A} \to 0$ (in the Regularized Picture). On the other hand, the $\mathfrak{P}$-even components do not in general go to zero. See, Remark 6.3 or (9.8). This asymmetric feature of the 2D Limit disappears in the more general 4D Limit (see, Subsection 9.2), as one can see by looking at the far field expansion. It is for the purpose of taking the 4D Limit that the *two* scaling parameters, a and $\mathfrak{A}$, have been carried along through the whole paper.

To compare the 4D Limit with the 2D Limit, it is useful to formulate the second smallness condition in (8.32) in a picture in which the Minkowski background and the stereographic coordinates ξ are fixed (independent of a and $\mathfrak{A}$), for example the High Amplitude Picture. The smallness condition becomes

$$\max_{\sigma\in\{-,+\}} \big\|\, \partial^{\alpha}\, \mathbf{DATA}^{\sigma}_{H} \,\big\|_{C^0(D_4(0)\times(0,1))} \;\leq\; \frac{\mathbf{b}}{\mathfrak{A}^2}\left|\frac{a}{\mathfrak{A}}\right|^{\alpha_1+\alpha_2} .$$

for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| \leq R + 4$. Notice that in the 4D Limit, $|\frac{a}{\mathfrak{A}}|$ is a large factor, and there is one factor for each "angular derivative".

A. Index of Notation

This is a partial list of symbols used in this paper. *It refers to their main / typical usage. Warning: These symbols can have different meanings. However, these meanings will always be made clear in each particular context.* (For example, the entry for the symbol S in the index below refers to equation (4.4), which corresponds to the main / typical meaning of the symbol S in many sections of this paper. Nevertheless, the symbol S stands for a field transformation in Section 3, and it stands for a set in the local context of Proposition 7.1.) Symbols which only appear in the Appendices are not listed. In the third column, a selected reference is given.

Symbol	Typical Meaning	See
$(\star)$, $(\star\star)$	frame nondegeneracy conditions	Definition 2.1
$[A,B] = AB - BA$	commutator of operators	
$\lesssim_p$	parameter p dependent bound	Convention 7.1
$\mathfrak{A}$, a	scaling parameters	Definitions 3.5, 4.1
$\mathfrak{A}$, $\mathfrak{I}$, $\mathfrak{Z}$, $\mathfrak{C}$	field transformations	Section 3
$\mathbf{A}(\Phi)$, $\mathbf{f}(\Phi)$	constituents of **(SHS)**	Definition 2.3

Symbol	Typical Meaning	See
$\mathbf{A}(\Psi, x)$, $\mathbf{f}(\Psi, x)$	constituents of (5.4a)	Proposition 5.1
$\widehat{\mathbf{A}}(\Phi)$, $\widehat{\mathbf{f}}(\Phi, \partial_x\Phi)$	constituents of $\widehat{\textbf{(SHS)}}$	Proposition 2.3
$\widehat{\mathbf{A}}(\Psi, x)$, $\widehat{\mathbf{f}}(\Psi, \partial_x\Psi, x)$	constituents of (5.4b)	Proposition 5.1
$\mathbf{B}\Xi = Q\Xi + \mathbf{Src}$		**(S4)** in Section 8
$\widehat{\mathbf{B}}\Xi^\sharp = \widehat{Q}\Xi^\sharp$		$\widehat{\textbf{(S2)}}$ in Section 8
b, **c**	small constants	Theorem 8.1
C	complex conjugation operator	Convention 8.3
$\mathfrak{d}$		Convention 7.3
DATA	data at past null infinity	Proposition 6.1
$D, \overline{D}, N, L$	complex frame vector fields	Definition 2.1
D_r, B_r	open disk and open ball	Convention 7.1
e, $\boldsymbol{\lambda}$, ρ		Definition 4.1
$E^k_{\mathcal{X}}\{f\}(t)$	energy	Definition 7.1
$\Phi = (e, \gamma, w)$	$\mathcal{R}$ valued field	Section 2
$\Phi^\sharp = (t, u, v)$	$\widehat{\mathcal{R}}$ valued constraint field	Definition 2.4
$\mathbf{Flip}_\alpha$	Pole-Flip transformation	Definition 3.6
$\mathcal{I}$, $\mathcal{J}$	intervals	
$\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\ldots$, $\mathcal{G}^\sharp$,$\ldots$	generic symbols	Definitions 8.1, 8.2
j^μ	energy current vector field	Section 7
K	truncation index, see also 'Ψ_K'	Theorem 8.1
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	weight functions	Definition 2.3
M	far field ansatz matrix	Section 5
$\mathcal{M}_{a,\mathfrak{a}}$, $[\,\mathcal{M}_{a,\mathfrak{a}}\,]$	doubly scaled Minkowski field	Definitions 4.1, 6.1
$\mathbf{M}(q,\Theta)\Theta = h(q,\Theta)$	general symm. hyp. system	Section 7
$\mathbf{M}(q,\Xi)\Xi = h(q,\Xi)$	a particular symm. hyp. system	**(S10)** in Section 8
$\mathbb{N}$	set of integers > 0	
$\mathbb{N}_0$	set of integers ≥ 0	
$\mathcal{O}(b)$, $\mathcal{O}(\xi_0, b, t)$	families of subsets of $\mathbb{R}^3$	**(E0)**, **(RE0)** in Sec. 7
π, $\widehat{\pi}$	certain permutation matrices	**(S3)**, $\widehat{\textbf{(S1)}}$ in Sec. 8
$\mathfrak{P}$	parity field transformation	Remark 2.9
$\mathcal{P}$, $\mathcal{P}^\sharp$, $\mathcal{P}_k$	generic symbols	Definitions 5.1, 6.2
∂^α	multi-derivative	Definition 7.1
$\partial_{\underline{u}}^{-1}$	an integration operator	(6.4)
∂_x, ∂_q, $\partial_{\mathbf{q}}$	gradient operator w.r.t. x, q, $\mathbf{q}$	
$\frac{\partial}{\partial \xi}$	$= \frac{1}{2}(\frac{\partial}{\partial \xi^1} - i\frac{\partial}{\partial \xi^2})$	Proposition 6.1
$q = (q^0, q^1, q^2, q^3)$ $= (t, \xi^1, \xi^2, \underline{u})$	coordinates	Convention 7.1
$\mathbf{q} = (q^1, q^2, q^3)$	spatial components of q	Convention 7.1
$\mathcal{Q}$, $\mathcal{K}$	general subsets of $\mathbb{R}^3$	**(EB4)** in Section 7
$\mathcal{Q}$, $\mathcal{K}$	particular subsets of $\mathbb{R}^3$	**(S9)** in Section 8
R, H, F	three pictures	Section 9
R	differentiability index	Theorem 8.1
$\mathcal{R}$, $\widehat{\mathcal{R}}$	real vector spaces	(2.1), (2.6)
$\Re$, $\Im$	real / imaginary part operators	
$\mathbb{R}$, $\mathbb{C}$	the real and complex numbers	
S		(4.4)

Symbol	Typical Meaning	See
$\mathbf{Sup}^{(k)}_{\mathcal{X}}\{f\}(t)$	supremum norm	Definition 7.1
(SHS) , $\widehat{\textbf{(SHS)}}$, **(subSHS)**	symmetric hyperbolic systems	Section 2
$\mathbf{Strip}(\mu,\lambda)$	family of open subsets of $\mathbb{R}^4$	(4.1)
σ	stereographic chart superscript	Proposition 6.4
t	time coordinate, equal to $u+\underline{u}$	Convention 7.1
$\mathcal{U}$	general open subset of $\mathbb{R}^4$	Section 2
Ξ, $\Xi^\sharp$	$\pi^{-1}\mathcal{R}$ and $\widehat{\pi}^{-1}\widehat{\mathcal{R}}$ valued fields	**(S3)**, $\widehat{\textbf{(S1)}}$ in Section 8
$x=(x^1,x^2,x^3,x^4)$	coordinates	Section 2
$\quad=(\xi^1,\xi^2,\underline{u},u)$	coordinates	Section 2
ξ	either (ξ^1,ξ^2) or $\xi^1+i\xi^2$	
$\overline{z}=\Re z-i\,\Im z$	complex conjugation	
$\Psi=(f,\omega,z)$	$\mathcal{R}$ valued field	(5.1)
$[\Psi]$, $\Psi(k)$	formal power series, coefficients	(6.1)
Ψ_K	truncated formal power series	**(S2)** in Section 8
$\Psi^\sharp=(s,p,y)$	$\widehat{\mathcal{R}}$ valued constraint field	Proposition 5.1
$[\Psi^\sharp]$, $\Psi^\sharp(k)$	formal power series, coefficients	Section 6

B. Generalized Vacuum Equations

Our main reference for this appendix is [Fr].

The vacuum Einstein equations, written in local coordinates x^1, x^2, x^3, x^4 on a connected open set $\mathcal{U}$ in $\mathbb{R}^4$, are a nonlinear system of partial differential equations for the ten metric tensor fields $g_{\mu\nu}$. Namely,

$$R^{\gamma}{}_{\alpha\gamma\beta} \;=\; 0 \tag{B.1}$$

where,

$$R^{\delta}{}_{\alpha\gamma\beta} \;=\; \tfrac{\partial}{\partial x^\gamma}\Gamma^\delta_{\beta\alpha} - \tfrac{\partial}{\partial x^\beta}\Gamma^\delta_{\gamma\alpha} + \Gamma^\mu_{\beta\alpha}\Gamma^\delta_{\gamma\mu} - \Gamma^\mu_{\gamma\alpha}\Gamma^\delta_{\beta\mu} - \big(\Gamma^\mu_{\gamma\beta} - \Gamma^\mu_{\beta\gamma}\big)\Gamma^\delta_{\mu\alpha}$$

are the components of the Riemann curvature tensor for the Levi-Civita connection $\Gamma^\gamma_{\mu\nu} = g^{\gamma\lambda}\Gamma_{\mu\nu\lambda}$:

$$\Gamma_{\mu\nu\lambda} \;=\; \tfrac{1}{2}\big(\tfrac{\partial}{\partial x^\mu}g_{\nu\lambda} + \tfrac{\partial}{\partial x^\nu}g_{\mu\lambda} - \tfrac{\partial}{\partial x^\lambda}g_{\mu\nu}\big)$$

associated to the metric $g_{\mu\nu}$.

There are patent mathematical advantages to introducing more fields and equations that, in the presence of appropriate constraints, collapse to the vacuum Einstein equations. The purpose of this appendix is to introduce a particular generalized system of vacuum equations and explain how it will be used. In this appendix and in Appendix C, we work with real quantities. In Appendix D we employ a complex tetrad formalism, see [NP], as in the main body of this paper.

Definition B.1. *A* **generalized spacetime** *is an open subset $\mathcal{U}$ of $\mathbb{R}^4$ together with*

- *16 frame fields ${E_a}^\mu$ and the associated vector fields*

$$E_a \;=\; {E_a}^\mu \tfrac{\partial}{\partial x^\mu}$$

It is assumed that E_1, E_2, E_3, E_4 are frame vector fields. That is, they are linearly independent tangent vectors at every point of $\mathcal{U}$.

• *A constant, symmetric matrix g_{ab} with three positive and one negative eigenvalues. For our purposes, the matrix g_{ab} and its inverse g^{ab} are*

$$\big(\, g_{ab} \,\big) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \big(\, g^{ab} \,\big) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{B.2}$$

• *24 connection fields $\varGamma_{abc}$ that are antisymmetric in the indices b and c.*
• *10 Weyl fields $W_{abk\ell}$ characterized by*

$$\begin{aligned} W_{abk\ell} &= -W_{bak\ell} & \qquad W_{ajk\ell} + W_{a\ell jk} + W_{ak\ell j} &= 0 \\ W_{abk\ell} &= -W_{ab\ell k} & g^{ak} W_{abk\ell} &= 0 \\ W_{abk\ell} &= W_{k\ell ab} \end{aligned}$$

Convention B.1. Small Latin indices a, b, c ... are frame indices and always run from one to four. Small Greek indices λ, μ, ν ... are coordinate indices and also always run from one to four. Frame indices are raised and lowered with the constant tensor g_{ab}.

We associate to every generalized spacetime

- A Lorentzian metric g determined by $g(E_a, E_b) = g_{ab}$.
- A connection ∇ specified by

$$g\big(\, \nabla_{E_a} E_b \,,\, E_c \,\big) = \varGamma_{abc} \qquad \text{or, equivalently,} \qquad \nabla_{E_a} E_b = \varGamma_{ab}{}^c E_c$$

  where $\varGamma_{ab}{}^c = g^{cd}\varGamma_{abd}$. The antisymmetry of $\varGamma_{abc}$ in the last two indices, expresses the property that the connection ∇ is compatible with the metric.
- 24 connection torsion fields

$$T_{ab}{}^{\mu} \;=\; \varGamma_{ab}{}^c E_c{}^{\mu} - \varGamma_{ba}{}^c E_c{}^{\mu} - E_a\big(E_b{}^{\mu}\big) + E_b\big(E_a{}^{\mu}\big) \tag{B.3}$$

  They measure the deviation of ∇ from the Levi-Civita connection for the metric g. That is, $T_{ab}{}^{\mu}$ vanishes if and only if

$$\varGamma_{abc} \;=\; \tfrac{1}{2}\Big(- g\big(E_a, [E_b, E_c]\big) + g\big(E_c, [E_a, E_b]\big) + g\big(E_b, [E_c, E_a]\big)\Big)$$

- 36 curvature torsion fields

$$\begin{aligned} U_{k\ell ab} \;&= E_a\big(\varGamma_{b\ell k}\big) - E_b\big(\varGamma_{a\ell k}\big) \\ &\quad + \varGamma_{b\ell}{}^m \varGamma_{amk} - \varGamma_{a\ell}{}^m \varGamma_{b\,m\,k} - \big(\varGamma_{a\,b}{}^m - \varGamma_{ba}{}^m\big)\varGamma_{m\ell k} \;-\; W_{k\ell ab} \end{aligned}$$

  The curvature tensor $R^k{}_{\ell ab}$ for the connection ∇ is given by

$$\begin{aligned} R^k{}_{\ell ab} \;&=\; E_a\big(\varGamma_{b\ell}{}^k\big) - E_b\big(\varGamma_{a\ell}{}^k\big) \\ &\quad + \varGamma_{b\ell}{}^m \varGamma_{am}{}^k - \varGamma_{a\ell}{}^m \varGamma_{bm}{}^k - \big(\varGamma_{ab}{}^m - \varGamma_{ba}{}^m - T_{ab}{}^m\big)\varGamma_{m\ell}{}^k \\ &=\; T_{ab}{}^m \varGamma_{m\ell}{}^k + U^k{}_{\ell ab} + W^k{}_{\ell ab} \end{aligned} \tag{B.4}$$

where ${T_{ab}}^m E_m^\mu = {T_{ab}}^\mu$. In the event that ${T_{ab}}^\mu$ vanishes, the curvature torsion fields $U_{k\ell ab}$ measure the deviation of the Riemann curvature for the Levi - Civita connection from the Weyl tensor $W_{k\ell ab}$.
The curvature tensor $R_{k\ell ab}$ for the connection ∇ has the symmetries

$$R_{k\ell ab} = -R_{\ell kab} \qquad R_{k\ell ab} = -R_{k\ell ba}$$

Warning. The customary pair exchange symmetry and cyclic identity do not necessarily hold when the torsion ${T_{ab}}^\mu$ is not zero.

- 16 Bianchi fields

$$V_{abijk} \;=\; \nabla_i W_{abjk} + \nabla_k W_{abij} + \nabla_j W_{abki}$$

or, equivalently, contracted Bianchi fields

$$V_{bjk} \;=\; g^{ai} V_{abijk} \;=\; \nabla^a W_{abjk}$$

For the definition of $\nabla_i W_{abjk}$, see (B.5) below. The fields V_{abijk} vanish if and only if the Weyl tensor W_{abki} satisfies the Bianchi identities with respect to the connection ∇. Similarly, the fields V_{bjk} vanish if and only if the Weyl tensor satisfies the contracted Bianchi identities with respect to the connection ∇.

Remark B.1. The contracted Bianchi fields are equivalent to the Bianchi fields. This fact is an immediate consequence of the following algebraic identity. Suppose, A_{abijk} is antisymmetric in the first two indices and totally antisymmetric in the last three. Set

$$A_{aij} \;=\; g^{bk} A_{abijk} \qquad A^\sharp_{ak\ell} \;=\; \tfrac{1}{2}\,\epsilon_{k\ell ij} {A_a}^{ij}$$

Then $A_{abijk} = \frac{1}{2}\,{\epsilon_{ijk}}^\ell \big(A^\sharp_{a\ell b} - A^\sharp_{b\ell a} + A^\sharp_{\ell ab}\big)$.

Convention B.2. We define E, Γ, W, T, U, V to be covariant tensors (vector field valued in the case of E and T) on $\mathcal{U}$ whose components with respect to the **fixed** frame E_a are given by

$$\begin{aligned} E_a &= {E_a}^\mu \tfrac{\partial}{\partial x^\mu} & T(E_a,E_b) &= {T_{ab}}^\mu \tfrac{\partial}{\partial x^\mu} \\ \Gamma(E_a,E_b,E_c) &= \Gamma_{abc} & U(E_a,E_b,E_c,E_d) &= U_{abcd} \\ W(E_a,E_b,E_c,E_d) &= W_{abcd} & V(E_a,E_b,E_c) &= V_{abc} \end{aligned}$$

From this perspective:

- If X_1, X_2, X_3 are vector fields on $\mathcal{U}$, then $\Gamma(X_1,X_2,X_3) \;=\; X_1^a\,X_2^b\,X_3^c\,\Gamma_{abc}$. Here $X_i = X_i^a E_a$ for $i = 1,2,3$. In general, $\Gamma(X_1,X_2,X_3) \neq g(\nabla_{X_1} X_2, X_3)$.
- Covariant derivatives of all these tensors are well defined. For example,

$$\nabla_i W_{abcd} = E_i(W_{abcd}) - {\Gamma_{ia}}^m W_{mbcd} - {\Gamma_{ib}}^m W_{amcd} - {\Gamma_{ic}}^m W_{abmd} - {\Gamma_{id}}^m W_{abcm} \tag{B.5}$$

Definition B.2. *We refer to the system*

$$\big(\,T,\,U,\,V\,\big) \;=\; 0 \tag{B.6}$$

as the **generalized vacuum field equations***.*

Proposition B.1. *A generalized spacetime $E_a{}^\mu$, Γ_{abc}, $W_{abk\ell}$ on an open subset $\mathcal{U}$ of $\mathbb{R}^4$ is a solution to the generalized vacuum field equations* (B.6) *if and only if ∇ is the Levi-Civita connection for the metric g, and the associated Riemann curvature tensor coincides with the Weyl tensor W. In this event, g is a solution to the vacuum Einstein equations* (B.1) *on $\mathcal{U}$.*

Proposition B.2. *The tensors T, U, V have the algebraic symmetries:*

$$T_{ab}{}^\mu = -T_{ba}{}^\mu \qquad V_{kab} = -V_{kba} \tag{B.7a}$$
$$U_{abk\ell} = -U_{bak\ell} \qquad V^k{}_{kb} = 0 \tag{B.7b}$$
$$U_{abk\ell} = -U_{ab\ell k} \qquad V_{abc} + V_{bca} + V_{cab} = 0 \tag{B.7c}$$

They satisfy the **generalized Bianchi equations**

$$\big(\mathfrak{T}, \mathfrak{U}, \mathfrak{V}\big) = 0 \tag{B.8}$$

where, by definition,

$$\mathfrak{T}_a{}^\mu \ = \ \epsilon_a{}^{ijk}\left(\widehat{\nabla}_i T_{jk}{}^\mu - U^m{}_{kij}\, E_m{}^\mu - T_{jk}{}^\nu \tfrac{\partial}{\partial x^\nu}\, E_i{}^\mu\right) \tag{B.9a}$$
$$\mathfrak{U}_{cab} \ = \ \epsilon_c{}^{ijk}\left(\nabla_i U_{abjk} - U^m{}_{ijk}\, \Gamma_{mab} + \tfrac{1}{3} V_{abijk} - T_{ij}{}^\mu \tfrac{\partial}{\partial x^\mu}\, \Gamma_{kab}\right) \tag{B.9b}$$
$$\begin{aligned}\mathfrak{V}_{jk} \ &= \ \nabla_b V^b{}_{jk} + U^a{}_{mab}\, W^{mb}{}_{jk} - \tfrac{1}{2} U_{mjab}\, W^{abm}{}_k \\ &\quad + \tfrac{1}{2} U_{mkab}\, W^{abm}{}_j - \tfrac{1}{2} T_{ab}{}^\mu \tfrac{\partial}{\partial x^\mu} W^{ab}{}_{jk}\end{aligned} \tag{B.9c}$$

Here ϵ_{abcd} is totally antisymmetric and $\epsilon_{1234} = -1$. Furthermore, $\widehat{\nabla}_i$ is the tensor derivation that acts on frame indices as ∇_i and ignores coordinate indices. Explicitly, $\widehat{\nabla}_i T_{jk}{}^\mu = E_i(T_{jk}{}^\mu) - \Gamma_{ij}{}^m T_{mk}{}^\mu - \Gamma_{ik}{}^m T_{jm}{}^\mu$.

Remark B.2. The generalized Bianchi equations (B.8) are *identities*: they hold for all generalized spacetimes. Both (B.6) and (B.8) are quadratically nonlinear. Each, has exactly one linear term. Respectively, $-W_{k\ell ab}$ and $-V_{abijk}$ in the equations $U = 0$ and $\mathfrak{U} = 0$. The only coordinate index appears in the $T = 0$ and $\mathfrak{T} = 0$ equations. Observe that, for fixed $E_a{}^\mu$, Γ_{abc}, W_{abjk}, the equations (B.8) are linear and homogeneous in $T_{ab}{}^\mu$, U_{abjk} and V_{abijk}.

Our goal is to construct physically interesting vacuum spacetimes. In this appendix we have traded in the 10 traditional metric tensor fields for 50 frame, connection and Weyl fields and an additional 76 connection torsion, curvature torsion and Bianchi fields. How can this formalism be of any practical use? Not only are there 126 fields, but both the generalized vacuum and Bianchi equations are overdetermined, since the tensor V vanishes whenever T and U both vanish.

Here is a rough outline of our strategy. Regard the frame $E_a{}^\mu$ and general connection Γ_{ijk} as vector fields with values in $\mathbb{R}^{16}$ and $\mathbb{R}^{24}$ respectively. We conceptualize abstract gauge conditions as fixed affine linear subspaces $\mathcal{E} \subset \mathbb{R}^{16}$ and $\mathcal{G} \subset \mathbb{R}^{24}$. The frame and connection are gauge fixed when $E_a{}^\mu(p) \in \mathcal{E}$ and $\Gamma_{ijk}(p) \in \mathcal{G}$ for all points $p \in \mathcal{U}$. No conditions are imposed on the Weyl tensor. There are

$$\dim \mathcal{E} + \dim \mathcal{G} + 10 \ \le \ 50$$

independent gauge fixed frame, gauge fixed connection and Weyl fields. An abstract gauge fixed, generalized spacetime is summarized by a field Φ on $\mathcal{U}$ taking values in $\mathcal{E} \oplus \mathcal{G} \oplus \mathbb{R}^{10}$.

Abstract gauge conditions should have three properties:

- *Property B.1.* They are always "locally realizable". That is, near each point in every spacetime there is a coordinate system and a frame such that the components of the frame, with respect to the coordinate vector fields, and the components of the Levi - Civita connection, with respect to the frame, lie in the gauge subspaces $\mathcal{E}$ and $\mathcal{G}$. In other words, abstract gauge conditions should not exclude, a priori, any spacetimes.
- *Property B.2.* They are "symmetric hyperbolic". In the present context, a symmetric hyperbolic system of partial differential equations for the column vector v is a system

  $$A^a \, E_a(v) \; = \; f$$

  where A^a is a symmetric matrix and $A^3 + A^4$ is strictly positive definite. Now, pick bases for $\mathcal{E}$ and $\mathcal{G}$, and rewrite the 60 equations $T = 0$ and $U = 0$ explicitly in terms of the components of the field Φ. It is required that one can select exactly $\dim \mathcal{E}$ linear combinations of the 24 connection torsion equations ($T = 0$) and exactly $\dim \mathcal{G}$ linear combinations of the 36 curvature torsion equations ($U = 0$) equations and 10 linear combinations of the 16 Bianchi equations ($V = 0$) which taken together comprise a (quasilinear) symmetric hyperbolic system for Φ which we will refer to as **(SHS)** .
  Property B.2 is not just wishful thinking, as it may first appear. Only the principal parts of (B.6),

  $$\begin{aligned}
  &E_a({E_b}^{\mu}) - E_b({E_a}^{\mu}) \\
  &E_j(\Gamma_{kba}) - E_k(\Gamma_{jab}) \\
  &E_i(W_{abjk}) + E_k(W_{abij}) + E_j(W_{abki})
  \end{aligned}$$

  have to be considered in the quest for symmetric hyperbolic equations. Furthermore, only the frame and connection fields in the principal parts have to be written out in terms of Φ. It is unnecessary to open up and look at the occurrences of the frame fields inside the first order differential operators E_a. At this level, it is required that there are, in turn, linear combinations of the principal parts that are symmetric hyperbolic. In principle, the field Φ, that contains all information about the generalized spacetime, is now uniquely determined, given appropriate data, by **(SHS)** . However, there is an important catch. **(SHS)** and the abstract gauge conditions imply that some part of the tensors T, U and V vanish, but not all. The remaining components are summarized in the *constraint field* $\Phi^\sharp$. If **(SHS)** is satisfied and $\Phi^\sharp$ vanishes, then ${E_a}^{\mu}$, Γ_{abc}, W_{abjk} is a solution to the generalized vacuum field equations.
- *Property B.3.* They are "dual symmetric hyperbolic". If **(SHS)** is satisfied, it is required that, in an entirely similar way, judicious linear combinations of the generalized Bianchi equations (B.8) can be brought into the form of a *linear, homogeneous* symmetric hyperbolic system for $\Phi^\sharp$ which we refer to as $\widehat{\textbf{(SHS)}}$. In particular, if the data for any well posed problem for the system $\widehat{\textbf{(SHS)}}$ vanishes, then the constraint field $\Phi^\sharp$ vanishes everywhere.

It is much simpler to carry out this general methodology in practice than to formulate it in broad conceptual terms. Different problems require different gauges and symmetric hyperbolic systems. In Appendix C, we introduce the wavefront gauge for Lorentzian manifolds. In Appendix D, we fix the abstract wavefront gauge and select symmetric hyperbolic subsystems from the generalized vacuum and Bianchi equations that are particularly suited to the problem we are solving in this paper.

Proof (of the Generalized Bianchi Equations (B.8)*).*

- $\mathfrak{T} = 0$: Write $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. Repeatedly exploiting the total antisymmetry of ϵ^{aijk} and then using the Jacobi identity,

$$\epsilon^{aijk}\big(\nabla_i \nabla_j E_k - \nabla_j \nabla_i E_k - \nabla_{[E_i, E_j]} E_k\big) \\ = \epsilon^{aijk}\Big(\nabla_i\big(T(E_j, E_k)\big) + T\big(E_i, [E_j, E_k]\big)\Big)$$

  It follows that $\epsilon^{aijk} R^b{}_{ijk} E^\mu_b = \epsilon^{aijk}\big(\widehat{\nabla}_i T_{jk}{}^\mu + T_{jk}{}^b \Gamma_{bi}{}^c E_c{}^\mu - T_{jk}{}^\nu \frac{\partial}{\partial x^\nu} E_i{}^\mu\big)$. Substituting (B.4) for the curvature $R^b{}_{ijk}$, the identity $\mathfrak{T} = 0$ follows.
- $\mathfrak{U} = 0$: Apply the operator $\epsilon^{aijk}\nabla_i$ to the identity (B.4) in the form

$$R_{bcjk} \;=\; U_{bcjk} + T_{jk}{}^m \Gamma_{mcb} + W_{bcjk}$$

  and use the standard Bianchi identity for the curvature tensor corresponding to a connection with torsion, $\epsilon^{aijk}\big(\nabla_i R_{bcjk} + T_{ij}{}^\ell R_{bc\ell k}\big) = 0$.
- $\mathfrak{V} = 0$: The divergence $\nabla_b V^b{}_{jk} = -\frac{1}{2}\big(\nabla_a \nabla_b - \nabla_b \nabla_a\big) W^{ab}{}_{jk}$. Express the commutators in terms of the curvature tensor:

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) W_{abjk} \\ = -T_{cd}{}^\ell \nabla_\ell W_{abjk} - R^\ell{}_{acd} W_{\ell bjk} - R^\ell{}_{bcd} W_{a\ell jk} - R^\ell{}_{jcd} W_{ab\ell k} - R^\ell{}_{kcd} W_{abj\ell}$$

  Substitute (B.4) for the curvature, contract indices, write out the covariant derivatives, rearrange, collect terms and cancel to obtain $\mathfrak{V} = 0$. □

C. The Wavefront Gauge for Lorentzian Manifolds

Here, we introduce the *geometric wavefront gauge* in the language of Lorentzian geometry. The *abstract wavefront gauge*, in the language of generalized spacetimes, is introduced in Appendix D.

Proposition C.1 (Geometric wavefront gauge). *Every point on any Lorentzian* 4*-manifold* (M, g) *has an open neighborhood on which there are coordinates*

$$(x^1, x^2, x^3, x^4) = (\xi^1, \xi^2, \underline{u}, u)$$

and an oriented frame (E_1, E_2, E_3, E_4) *such that* $g(E_a,\, E_b) = g_{ab}$*, see* (B.2)*, such that* E_3 *and* E_4 *are both future directed vector fields, and such that*

(a) the coordinate functions u *and* $\underline{u}$ *are solutions to the eikonal equation, that is,* $g^{ab} E_a(u) E_b(u) = 0$ *and* $g^{ab} E_a(\underline{u}) E_b(\underline{u}) = 0$*.*
(b) the vector field E_4 *is minus the gradient of* u*, that is* $g(E_4,\, \cdot\,) = -\mathrm{d}u$*.*
(c) the coordinates ξ^1*,* ξ^2 *are constant along the integral curves of* E_4*.*
(d) the function $e_3 = E_4(\underline{u})$ *is strictly positive and the vector field* $e_3\, E_3$ *is minus the gradient of* $\underline{u}$*, that is* $g(e_3 E_3,\, \cdot\,) = -\mathrm{d}\underline{u}$*.*
(e) E_4 *and* $e_3 E_3$ *are null geodesic vector fields.*
(f) the frame vector fields E_1 *and* E_2 *satisfy*

$$g\big(\nabla_{E_4} E_1,\, E_2\big) \;=\; 0$$

where ∇ *is the Levi-Civita connection.*

Proof (Informal). To start with, suppose M is Minkowski space. Let $X = (X^0, \mathbf{X}) = (X^0, X^1, X^2, X^3) \in \mathbb{R} \times \mathbb{R}^3$ be standard Cartesian coordinates. For every fixed $\epsilon > 0$, define $u, \underline{u} : \{X \in \mathbb{R}^4 :\ X^0 > -\epsilon\} \to \mathbb{R}$ by

$$u(p) = \sup_{(-\epsilon,\mathbf{X})\in I^-(p)} X^1 \quad , \quad \underline{u}(p) = -\inf_{(-\epsilon,\mathbf{X})\in I^-(p)} X^1.$$

Here, $I^-(p)$ $\big($resp., $I^+(p)\big)$ is the chronological past (future) of p, that is, all points that can be reached from p by traveling along past (future) directed, piecewise smooth, timelike curves. Note that

- by construction, $u(p) \le u(q)$ and $\underline{u}(p) \le \underline{u}(q)$ for all $q \in I^+(p)$,
- there is a sufficiently small $\epsilon > 0$ and an open neighborhood $U \subset M$ of the origin $X = 0$, on which u, $\underline{u}$ are smooth and $\mathrm{d}u$, $\mathrm{d}\underline{u}$ are linearly independent.

It follows that, $\mathrm{d}u$ and $\mathrm{d}\underline{u}$ are timelike or null on U.

Suppose $\mathrm{d}u$ is timelike at a point $p \in U$. Then, the level set Σ of u passing through p would be (locally) a smooth spacelike hypersurface. If $q \in I^+(p)$ is sufficiently close to p, then every past directed timelike curve from q intersects Σ, and

$$u(q) = \sup_{(-\epsilon,\mathbf{X})\in I^-(q)} X^1 = \sup_{p'\in\Sigma\cap I^-(q)} \ \sup_{(-\epsilon,\mathbf{X})\in I^-(p')} X^1 = \sup_{p'\in\Sigma\cap I^-(q)} u(p') = u(p).$$

because $I^-(q) \cap H = \bigcup_{p'\in\Sigma\cap I^-(q)} I^-(p') \cap H$, with $H = \{X \in \mathbb{R}^4 :\ X^0 = -\epsilon\}$. This contradicts the assumption that $\mathrm{d}u(p)$ is timelike. Therefore, $\mathrm{d}u$ is null. Similarly, $\mathrm{d}\underline{u}$ is null.

Now, fix a point p_0 on any Lorentzian manifold M, and let $(X^0,\ \mathbf{X})$ be smooth local coordinates that vanish at p, with $-\mathrm{d}X^0$ a future directed 1-form. Precisely the same construction for u and $\underline{u}$ works on a suitably small neighborhood $U \subset M$ of p_0.

Define vector fields L and $\underline{L}$ by $g(L,\,\cdot\,) = -\mathrm{d}u$ and $g(\underline{L},\,\cdot\,) = -\mathrm{d}\underline{u}$. They are future null. Define $E_4 = L$, $e_3 = L(\underline{u}) > 0$ and $E_3 = e_3^{-1}\underline{L}$. In particular, $g(E_3, E_4) = -1$.

Condition (e) is equivalent to $\nabla_L L = 0$ and $\nabla_{\underline{L}}\underline{L} = 0$. These are consequences of the general fact that for any function w, the acceleration $\nabla_W W$ of its gradient field W is the gradient field of the function $\frac{1}{2}g(W, W)$.

Let K_1 and K_2 be spacelike, orthonormal vector fields, perpendicular to E_3 and E_4. Define $\left(\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right)\left(\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right)$ where α satisfies the differential equation $E_4(\alpha) = -\,g\big(\nabla_{E_4}K_1,\ K_2\big)$ along the integral curves of E_4.

Let ξ^1, ξ^2 be functions on the level set of $\underline{u}$ that goes through p_0, such that (ξ^1, ξ^2, u) are local coordinates for this level set. Since E_4 is transverse to the level set, $E_4(\underline{u}) > 0$, there is a unique extension of ξ^1, ξ^2 to a neighborhood of p_0 that satisfies the transport equations $E_4(\xi^1) = E_4(\xi^2) = 0$. Moreover, $(\xi^1, \xi^2, \underline{u}, u)$ are local coordinates. $\square$

Remark C.1. The spacelike vector fields E_1 and E_2 and the null geodesic vector field E_4 are tangent to each level set of u. Each level set of u is a union of null geodesics, the lines of constant ξ^1, ξ^2 and u.
Similarly, E_1, E_2 and the null geodesic vector field $e_3\,E_3$ are tangent to each level set of $\underline{u}$. Each level set of $\underline{u}$ is a union of null geodesics. They are, in general, not given by the lines of constant ξ^1, ξ^2 and $\underline{u}$.
We refer to the intersections of level sets of u and $\underline{u}$ as *wavefronts*. They are spacelike and their tangent space is spanned by E_1 and E_2.

Proposition C.2. *Fix a coordinate system and frame as in the geometric wavefront gauge of Proposition C.1. Let ${E_a}^\mu$ be the components of the frame field with respect to the coordinate system and let Γ_{abc} be the components of the Levi-Civita connection ∇,*

$$E_a = {E_a}^\mu \tfrac{\partial}{\partial x^\mu}, \qquad \Gamma_{abc} = g\big(\nabla_{E_a} E_b,\, E_c\big).$$

Then

$$\left({E_a}^\mu\right) = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 1 \\ 0 & 0 & e_3 & 0 \end{pmatrix}$$

where $*$ is a generic symbol. Moreover,

$$\left(\Gamma_{a(bc)}\right) = \begin{pmatrix} * & \underline{\chi}_{11} & \underline{\chi}_{12} & \chi_{11} & \chi_{12} & \bigstar \\ * & \underline{\chi}_{21} & \underline{\chi}_{22} & \chi_{21} & \chi_{22} & \bigstar \\ * & 0 & 0 & \bigstar & \bigstar & * \\ 0 & * & * & 0 & 0 & 0 \end{pmatrix}$$

where the matrix indices (bc) run over the ordered sequence

$$(12), \quad (31), \quad (32), \quad (41), \quad (42), \quad (34).$$

Here, χ_{AB} and $\underline{\chi}_{AB}$ are the second fundamental forms of the wavefronts in the normal null directions E_4 and E_3. As such,

$$\chi_{AB} = \chi_{BA} \qquad \underline{\chi}_{AB} = \underline{\chi}_{BA}.$$

Moreover, the entries filled with the generic symbol $\bigstar$ satisfy

$$\Gamma_{A34} = \Gamma_{3A4} \qquad A = 1, 2.$$

Proof. We first verify the 0's and 1's in these matrices. The entries of $({E_a}^\mu)$ follow directly from Proposition C.1, for example, $E_3(u) = \mathrm{d}u(E_3) = -g(E_4, E_3) = 1$ by (b). The zeros in (Γ_{abc}) are accounted for by (f), by $\nabla_{E_4} E_4 = 0$, see (e), and by the fact that $\nabla_{E_3} E_3$ is proportional to E_3, see (e). Next, $\Gamma_{123} - \Gamma_{213} = 0$ ($\underline{\chi}_{12} = \underline{\chi}_{21}$) and $\Gamma_{124} - \Gamma_{214} = 0$ ($\chi_{12} = \chi_{21}$) follow, by (d) and (b), from $[E_1, E_2](\underline{u}) = 0$ and $[E_1, E_2](u) = 0$, respectively. Finally, $\Gamma_{A34} - \Gamma_{3A4} = 0$ follows from $[E_A, E_3](u) = 0$. □

D. The Abstract Wavefront Gauge

In this Appendix, we leave the realm of Lorentz 4-manifolds, and speak exclusively in the language of generalized spacetimes, as in Appendix B.

We now define the *abstract wavefront gauge* and show that it has the Properties B.1 through B.3.

It is convenient to introduce the complex frame

$$(F_1, F_2, F_3, F_4) = (D, \overline{D}, N, L), \qquad F_a = {\mathbf{F}_a}^\mu \tfrac{\partial}{\partial x^\mu}$$

where

$$D = 2^{-\frac{1}{2}}(E_1 + iE_2), \qquad \overline{D} = 2^{-\frac{1}{2}}(E_1 - iE_2), \qquad N = E_3, \qquad L = E_4.$$

These fields are sections of the complexified tangent bundle.

Convention D.1. Let t be the constant matrix for which $\mathbf{F}_a{}^\mu = t_a{}^b E_b{}^\mu$, and t^{-1} its inverse. For every tensor field S in Appendix B with components $S_{a_1\ldots a_n}{}^{b_1\ldots b_m}$ with respect to the real frame (E_a), the corresponding components with respect to the complex frame (F_a) are distinguished by boldface letters and defined by

$$\mathbf{S}_{a_1\ldots a_n}{}^{b_1\ldots b_m} = t_{a_1}{}^{i_1}\cdots t_{a_n}{}^{i_n} S_{i_1\ldots i_n}{}^{j_1\ldots j_m}(t^{-1})_{j_1}{}^{b_1}\cdots(t^{-1})_{j_m}{}^{b_m} \tag{D.1}$$

If the tensor field S also carries coordinate indices, they are unaffected by (D.1). An equivalent statement to (D.1) is: if S is the tensor field for which

$$S(E_{a_1},\ldots,E_{a_n}) = S_{a_1\ldots a_n}{}^{b_1\ldots b_m} E_{b_1}\otimes_{\mathbb{R}}\cdots\otimes_{\mathbb{R}} E_{b_m}$$

and S is extended complex linearly in its n arguments, then

$$S(F_{a_1},\ldots,F_{a_n}) = \mathbf{S}_{a_1\ldots a_n}{}^{b_1\ldots b_m} F_{b_1}\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} F_{b_m}$$

The transformation (D.1) commutes with contraction of indices. Accordingly, the indices of boldface fields have to be raised and lowered with

$$\mathbf{g}^{ab} = g^{ij}(t^{-1})_i{}^a(t^{-1})_j{}^b \qquad \mathbf{g}_{ab} = t_a{}^i t_b{}^j g_{ij}$$

see (B.2) and (2.3). The complex components $\mathbf{S}_{a_1\ldots a_n}{}^{b_1\ldots b_m}$ are only introduced for notational convenience. The corresponding tensor field S will, however, always be real, in the sense that if all its n arguments are real, then the result is real.

For the particular covariant tensors E, Γ, W, T, U, V (see, Convention B.2), the transformation (D.1) becomes

$$\mathbf{F}_a{}^\mu \frac{\partial}{\partial x^\mu} = F_a \qquad \mathbf{T}_{ab}{}^\mu \frac{\partial}{\partial x^\mu} = T(F_a, F_b) \tag{D.2a}$$
$$\boldsymbol{\Gamma}_{abc} = \Gamma(F_a, F_b, F_c) \qquad \mathbf{U}_{abcd} = U(F_a, F_b, F_c, F_d) \tag{D.2b}$$
$$\mathbf{W}_{abcd} = W(F_a, F_b, F_c, F_d) \qquad \mathbf{V}_{abc} = V(F_a, F_b, F_c) \tag{D.2c}$$

Definition D.1. *Let*

$$\Phi = (e, \gamma, w) : \ \mathcal{U} \to \mathcal{R} \subset \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 \qquad \textit{see } (2.1)$$

be a sufficiently differentiable field satisfying conditions $(\star)$ *and* $(\star\star)$ *in Definition 2.1. The* **Abstract Wavefront Gauge Spacetime**

$$M_\Phi = (\mathbf{F}_a{}^\mu,\ \boldsymbol{\Gamma}_{abc},\ \mathbf{W}_{abcd})$$

is defined just as in Definition 2.1.

Remark D.1. Definition D.1 implicitly fixes abstract gauge conditions in the sense of Appendix B. The affine spaces $\mathcal{E}$ and $\mathcal{G}$ have real dimensions, respectively, 7 and 14. That is, the field Φ has 31 real components.

Remark D.2. Observe that Proposition C.2 is the statement that the abstract wavefront gauge in Definition D.1 is locally realizable in the sense of Appendix B (Property B.1).

Proposition D.1. *Let M_Φ be an abstract wavefront gauge spacetime. Let λ_1, λ_2, λ_3, λ_4 be strictly positive weight functions on $\mathcal{U}$. Then, there are unique fields*

$$
\begin{aligned}
(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}): \quad & \mathcal{U} \to \mathcal{R} \subset \mathbb{C}^5 \oplus \mathbb{C}^8 \oplus \mathbb{C}^5 \\
(t, u, v): \quad & \mathcal{U} \to \widehat{\mathcal{R}} \subset \mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^3
\end{aligned}
$$

(where $\mathcal{R}$ and $\widehat{\mathcal{R}}$ are defined in (2.1) *and* (2.6)*) and such that*

$$
\left(\mathbf{T}_{(ab)}{}^{\mu}\right) = \begin{pmatrix} i\,t_1 & i\,t_2 & 0 & 0 \\ \overline{t}_4 & \overline{t}_5 & 0 & 0 \\ t_4 & t_5 & 0 & 0 \\ -\,\mathfrak{t}_1 & -\,\mathfrak{t}_2 & t_3 & 0 \\ -\,\overline{\mathfrak{t}}_1 & -\,\overline{\mathfrak{t}}_2 & \overline{t}_3 & 0 \\ \mathfrak{t}_4 & \mathfrak{t}_5 & -\,\mathfrak{t}_3 & 0 \end{pmatrix}
$$

$$
\left(\mathbf{U}_{(ab)(jk)}\right) = \begin{pmatrix} u_2 + \overline{u}_2 & u_7 - \overline{u}_8 & u_8 - \overline{u}_7 & -\mathfrak{u}_3 - \overline{\mathfrak{u}}_4 & \mathfrak{u}_4 + \overline{\mathfrak{u}}_3 & \mathfrak{u}_8 - \overline{\mathfrak{u}}_8 \\ \overline{u}_9 & -\overline{\mathfrak{u}}_7 & -\mathfrak{u}_6 & \overline{u}_5 & -\overline{u}_6 & -\overline{\mathfrak{u}}_5 \\ -u_9 & -\mathfrak{u}_6 & -\mathfrak{u}_7 & -u_6 & u_5 & -\mathfrak{u}_5 \\ -u_1 & u_4 & -u_3 & -\mathfrak{u}_1 & -\mathfrak{u}_2 & -\mathfrak{u}_3 + \overline{\mathfrak{u}}_4 \\ \overline{u}_1 & -\overline{u}_3 & \overline{u}_4 & -\mathfrak{u}_2 & -\overline{\mathfrak{u}}_1 & \mathfrak{u}_4 - \overline{\mathfrak{u}}_3 \\ u_2 - \overline{u}_2 & u_7 + \overline{u}_8 & u_8 + \overline{u}_7 & -\mathfrak{u}_3 + \overline{\mathfrak{u}}_4 & \mathfrak{u}_4 - \overline{\mathfrak{u}}_3 & \mathfrak{u}_8 + \overline{\mathfrak{u}}_8 \end{pmatrix}
$$

$$
\left(\mathbf{V}_{a(jk)}\right) = \begin{pmatrix} \frac{1}{\lambda_2}(-\mathfrak{v}_2 + v_1) - \frac{1}{\lambda_3}\overline{v}_3 & -\frac{1}{\lambda_4}\overline{\mathfrak{v}}_5 & \frac{1}{\lambda_3}(\mathfrak{v}_3 - v_2) & -\frac{1}{\lambda_1}\mathfrak{v}_1 & \frac{1}{\lambda_2}\overline{v}_2 & \frac{1}{\lambda_2}(-\mathfrak{v}_2 + v_1) + \frac{1}{\lambda_3}\overline{v}_3 \\ \frac{1}{\lambda_2}(\overline{\mathfrak{v}}_2 - \overline{v}_1) + \frac{1}{\lambda_3}v_3 & \frac{1}{\lambda_3}(\overline{\mathfrak{v}}_3 - \overline{v}_2) & -\frac{1}{\lambda_4}\mathfrak{v}_5 & \frac{1}{\lambda_2}v_2 & -\frac{1}{\lambda_1}\overline{\mathfrak{v}}_1 & \frac{1}{\lambda_2}(-\overline{\mathfrak{v}}_2 + \overline{v}_1) + \frac{1}{\lambda_3}v_3 \\ \frac{1}{\lambda_3}(\mathfrak{v}_3 - \overline{\mathfrak{v}}_3 - v_2 + \overline{v}_2) & \frac{1}{\lambda_4}(-\overline{\mathfrak{v}}_4 + \overline{v}_3) & \frac{1}{\lambda_4}(-\mathfrak{v}_4 + v_3) & \frac{1}{\lambda_2}(\mathfrak{v}_2 - v_1) & \frac{1}{\lambda_2}(\overline{\mathfrak{v}}_2 - \overline{v}_1) & \frac{1}{\lambda_3}(\mathfrak{v}_3 + \overline{\mathfrak{v}}_3 - v_2 - \overline{v}_2) \\ \frac{1}{\lambda_2}(-v_2 + \overline{v}_2) & \frac{1}{\lambda_3}\overline{v}_3 & \frac{1}{\lambda_3}v_3 & -\frac{1}{\lambda_1}v_1 & -\frac{1}{\lambda_1}\overline{v}_1 & -\frac{1}{\lambda_2}(v_2 + \overline{v}_2) \end{pmatrix}
$$

The matrix indices (ab), (jk) run over the ordered sequence

$$
(12) \quad (31) \quad (32) \quad (41) \quad (42) \quad (34)
$$

Proof. In general, the tensors T, U and V lie (pointwise) in spaces of real dimension 24, 36 and 16, respectively (see, equations (B.7)). By direct inspection, the following equations hold for every field Φ: $\mathbf{T}_{12}{}^{3} = 0$ (1 real equation); $\mathbf{T}_{31}{}^{3} = 0$ (2 real equations); $\mathbf{T}_{ab}{}^{4} = 0$ (6 real equations); $\mathbf{U}_{3441} = \mathbf{U}_{4134}$ (2 real equations); $\Im\,\mathbf{U}_{3132} = 0$ (1 real equation); $\Im\,\mathbf{U}_{4142} = 0$ (1 real equation). Consequently, the associated tensors T, U and V lie in subspaces of real dimension $24 - 9 = 15,\ 36 - 4 = 32$ and $16 - 0 = 16$, respectively. By construction, the matrices on the right hand sides of the equations above lie in these subspaces. The linear map from $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \oplus (t, u, v)$ to these matrices has maximal rank $\dim_{\mathbb{R}} \mathcal{R} + \dim_{\mathbb{R}} \widehat{\mathcal{R}} = 31 + 32 = 63$. Since this is equal to $15 + 32 + 16$, the fields $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and (t, u, v) exist and are unique. $\square$

Proposition D.1 defines a unique "splitting" of the nonzero components of (T, U, V) into two sets, $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and (t, u, v). This splitting and the role of the weight functions λ_i is clarified by the following proposition and the subsequent remarks.

Proposition D.2. *For every choice of strictly positive weight functions* λ_1, λ_2, λ_3, λ_4 *on* $\mathcal{U}$, *the system of equations*

$$(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$$

is equivalent to the system **(SHS)** , $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$, *in Definition 2.3, which is a (quasi-linear) symmetric hyperbolic system for the field* $\Phi = (e, \gamma, w)$ *provided* $e_3 > 0$. *In particular, the abstract wavefront gauge of Definition D.1 has Property B.2.*

Proof. The equivalence of $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$ with the symmetric hyperbolic system **(SHS)** in Definition 2.3 is by direct (machine) verification. For a sample calculation, see the proof of Proposition D.3. □

Remark D.3. The term "symmetric" is a slight misnomer, in the sense that the matrices $\mathbf{A}^\mu(\Phi)$ determining the principal part $\mathbf{A}^\mu(\Phi)\frac{\partial}{\partial x^\mu}$ are complex Hermitian rather than real symmetric.

Remark D.4. **(SHS)** is of a form which is particularly suited to constructing solutions. The reason is that the first two blocks $\mathbf{A}_1(\Phi)$, $\mathbf{A}_2(\Phi)$ of the principal part, corresponding to the principal parts of $\mathfrak{t} = 0$ and $\mathfrak{u} = 0$, are diagonal and only L or N appear.

Remark D.5. Note that

$$\Big(\textbf{Principal Part Operator}\Big)\begin{pmatrix}\mathfrak{v}_1\\ \mathfrak{v}_2\\ \mathfrak{v}_3\\ \mathfrak{v}_4\\ \mathfrak{v}_5\end{pmatrix} = \begin{pmatrix} \lambda_1\times\boxed{\begin{matrix}N & D\\ \overline{D} & L\end{matrix}} & & & \\ & \lambda_2\times\boxed{\begin{matrix}N & D\\ \overline{D} & L\end{matrix}} & & \\ & & \lambda_3\times\boxed{\begin{matrix}N & D\\ \overline{D} & L\end{matrix}} & \\ & & & \lambda_4\times\boxed{\begin{matrix}N & D\\ \overline{D} & L\end{matrix}} \end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\\ w_4\\ w_5\end{pmatrix}$$

The dotted lines in the schematic diagram for the 5×5 matrix on the right hand side indicate that the overlapping entries are the sums $\lambda_i L + \lambda_{i+1} N$. Each $\left(\begin{smallmatrix}N & D\\ \overline{D} & L\end{smallmatrix}\right)$ block is symmetric hyperbolic, and consequently, so is the 5×5 matrix for any choice of the strictly positive weight functions.

Remark D.6. The weights have a natural interpretation in terms of energies. The energy current naturally associated to **(SHS)** , $\mathbf{A}(\Phi)\Phi = \mathbf{f}(\Phi)$, is the vector field $j^\mu = \Phi^\dagger \mathbf{A}^\mu(\Phi)\,\Phi$. Estimates are obtained by applying the divergence theorem to

$$\partial_\mu j^\mu = \Phi^\dagger(\partial_\mu \mathbf{A}^\mu)\Phi + 2\Re\big(\Phi^\dagger \mathbf{f}(\Phi)\big).$$

The energies are integrals over the spacelike components of the boundary. The functions λ_1, λ_2, λ_3, λ_4 appear in the boundary integrals and play the role of weights for the components w_1, w_2, w_3, w_4, w_5.

Proposition D.3. *Let* λ_1, λ_2, λ_3, λ_4 *be strictly positive weight functions on* $\mathcal{U}$. *The components of* (t, u, v) *are given by* (2.7) *in Definition 2.4. The field* $\Phi^\sharp = (t, u, v) : \mathcal{U} \to \widehat{\mathcal{R}}$ *is called the* **constraint field** *associated to* $\Phi = (e, \gamma, w)$.

Proof. By direct (machine) calculation. We make a sample calculation:

$$
\begin{aligned}
it_1 &\overset{(1)}{=} \mathbf{T}_{12}{}^{1} \\
&\overset{(2)}{=} \boldsymbol{\Gamma}_{12}{}^{c}\mathbf{F}_c{}^{1} - \boldsymbol{\Gamma}_{21}{}^{c}\mathbf{F}_c{}^{1} - \mathbf{F}_1(\mathbf{F}_2{}^{1}) + \mathbf{F}_2(\mathbf{F}_1{}^{1}) \\
&\overset{(3)}{=} \boldsymbol{\Gamma}_{12}{}^{1}\mathbf{F}_1{}^{1} + \boldsymbol{\Gamma}_{12}{}^{2}\mathbf{F}_2{}^{1} + \boldsymbol{\Gamma}_{12}{}^{3}\mathbf{F}_3{}^{1} + \boldsymbol{\Gamma}_{12}{}^{4}\mathbf{F}_4{}^{1} \\
&\qquad - \boldsymbol{\Gamma}_{21}{}^{1}\mathbf{F}_1{}^{1} - \boldsymbol{\Gamma}_{21}{}^{2}\mathbf{F}_2{}^{1} - \boldsymbol{\Gamma}_{21}{}^{3}\mathbf{F}_3{}^{1} - \boldsymbol{\Gamma}_{21}{}^{4}\mathbf{F}_4{}^{1} - \mathbf{F}_1(\mathbf{F}_2{}^{1}) + \mathbf{F}_2(\mathbf{F}_1{}^{1}) \\
&\overset{(4)}{=} \boldsymbol{\Gamma}_{122}\mathbf{F}_1{}^{1} + \boldsymbol{\Gamma}_{121}\mathbf{F}_2{}^{1} - \boldsymbol{\Gamma}_{124}\mathbf{F}_3{}^{1} - \boldsymbol{\Gamma}_{123}\mathbf{F}_4{}^{1} \\
&\qquad - \boldsymbol{\Gamma}_{212}\mathbf{F}_1{}^{1} - \boldsymbol{\Gamma}_{211}\mathbf{F}_2{}^{1} + \boldsymbol{\Gamma}_{214}\mathbf{F}_3{}^{1} + \boldsymbol{\Gamma}_{213}\mathbf{F}_4{}^{1} - \mathbf{F}_1(\mathbf{F}_2{}^{1}) + \mathbf{F}_2(\mathbf{F}_1{}^{1}) \\
&\overset{(5)}{=} -\boldsymbol{\Gamma}_{112}\mathbf{F}_2{}^{1} + \boldsymbol{\Gamma}_{142}\mathbf{F}_3{}^{1} - \boldsymbol{\Gamma}_{212}\mathbf{F}_1{}^{1} - \boldsymbol{\Gamma}_{241}\mathbf{F}_3{}^{1} - \mathbf{F}_1(\mathbf{F}_2{}^{1}) + \mathbf{F}_2(\mathbf{F}_1{}^{1}) \\
&\overset{(6)}{=} -(\gamma_3 + \overline{\gamma}_4)\overline{e}_1 + \gamma_2 e_4 - (-\gamma_4 - \overline{\gamma}_3)e_1 - \gamma_2 e_4 - D(\overline{e}_1) + \overline{D}(e_1)
\end{aligned}
$$

(1) by Proposition D.1; (2) by equations (D.2a) and (B.3); (3) by the Einstein summation convention; (4) by lowering frame indices, using the first matrix in equation (2.3); (5) by antisymmetry of $(\boldsymbol{\Gamma}_{ajk})$ in the last two indices and $\mathbf{F}_4{}^1 = 0$, see Definition 2.1; (6) by Definition 2.1. The result of this sample calculation matches with Definition 2.4. □

The generalized vacuum field equations (B.6) reduce, in the abstract wavefront gauge, to **(SHS)** and $\Phi^\sharp = 0$, see Proposition D.1. How can we ensure that a solution to **(SHS)** also satisfies $\Phi^\sharp = 0$? The answer is given in Proposition D.4.

Proposition D.4. *Assume that* $\Phi = (e, \gamma, w)$ *satisfies* $e_3 > 0$ *and solves* **(SHS)** *or, equivalently,* $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$*. Let the Latin indices in* $\boldsymbol{\mathfrak{T}}_a{}^\mu$, $\boldsymbol{\mathfrak{U}}_{abc}$, $\boldsymbol{\mathfrak{V}}_{ab}$ *denote components of the fields* (B.9) *with respect to the complex frame field* F_a *(see, Convention D.1). The subsystem of the generalized Bianchi equations* (B.8) *given by*

$$
\begin{pmatrix} \boldsymbol{\mathfrak{T}}_4{}^1 \\ \boldsymbol{\mathfrak{T}}_4{}^2 \\ \boldsymbol{\mathfrak{T}}_1{}^3 \\ \boldsymbol{\mathfrak{T}}_2{}^1 \\ \boldsymbol{\mathfrak{T}}_2{}^2 \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{\mathfrak{U}}_{414} \\ \frac{1}{2}(\boldsymbol{\mathfrak{U}}_{412} + \boldsymbol{\mathfrak{U}}_{434}) \\ \boldsymbol{\mathfrak{U}}_{214} \\ \boldsymbol{\mathfrak{U}}_{114} \\ \boldsymbol{\mathfrak{U}}_{223} \\ \boldsymbol{\mathfrak{U}}_{123} \\ \frac{1}{2}(\boldsymbol{\mathfrak{U}}_{112} + \boldsymbol{\mathfrak{U}}_{134}) \\ \frac{1}{2}(\boldsymbol{\mathfrak{U}}_{212} + \boldsymbol{\mathfrak{U}}_{234}) \\ \boldsymbol{\mathfrak{U}}_{332} \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{\mathfrak{V}}_{41} \\ \frac{1}{2}(\boldsymbol{\mathfrak{V}}_{12} + \boldsymbol{\mathfrak{V}}_{34}) \\ \boldsymbol{\mathfrak{V}}_{23} \end{pmatrix} = 0 \tag{D.3}
$$

is equivalent to the system $\widehat{\textbf{(SHS)}}$, $\widehat{\mathbf{A}}(\Phi)\Phi^\sharp = \widehat{\mathbf{f}}(\Phi, \partial_x\Phi)\Phi^\sharp$*, in Proposition 2.3, which is a linear homogeneous symmetric hyperbolic system for* $\Phi^\sharp = (t, u, v)$*. In particular, the abstract wavefront gauge of Definition D.1 has Property B.3.*

Proof. By assumption, $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$. The equivalence of (D.3) with the linear, homogeneous symmetric hyperbolic system $\widehat{\textbf{(SHS)}}$ is by direct (machine) verification. □

E. Symmetries: Proofs

In this section, we prove that the field transformations introduced in Section 3 are field symmetries (see, Definition 3.1).

Recall from Section 3 the definition of a field transformation S. In this Appendix we take a slightly different perspective and regard $x \in \mathcal{U}$ and $x' \in \mathcal{U}'$ as two sets of (global) coordinates on the same 4-dimensional manifold. Similarly, we regard Φ, Λ, F_a, ∇, W and their primed counterparts as objects on this 4-manifold. Here, ∇ is the connection associated with Γ, and W is viewed as a 4-covariant tensor (see, Appendix B and Convention D.1). Field transformations are defined, in this Appendix, through their action on the coordinates x, the complex frame vector fields F_a, the connection ∇, the 4-covariant tensor W, and the strictly positive weight functions Λ. The definitions of $\mathfrak{J}$, $\mathfrak{A}$ given below are equivalent to the corresponding definitions in Section 3. Those of $\mathfrak{C}$, $\mathfrak{Z}$ are slight generalizations, because $\mathfrak{C}^1$, $\mathfrak{C}^2$ and ζ are allowed to depend on (x^1, x^2, x^4).

Convention E.1. For the rest of this appendix, it is implicitly assumed that $x' = S \cdot x$ is a local diffeomorphism on $\mathbb{R}^4$. With this understanding, it is unnecessary to specify the ranges of the x and x', because the discussion is purely algebraic. A dot, $\cdot$, always denotes a group action.

Angular coordinate transformation $\mathfrak{C}$. Let $\mathfrak{C}^1(x^1, x^2, x^4), \mathfrak{C}^2(x^1, x^2, x^4)$ be functions.

$$x' = \mathfrak{C} \cdot x = \big(\mathfrak{C}^1(x^1, x^2, x^4),\ \mathfrak{C}^2(x^1, x^2, x^4),\ x^3,\ x^4\big) \qquad \mathfrak{C} \cdot \nabla = \nabla$$
$$\big(\mathfrak{C} \cdot \mathbf{F}\big)_a{}^{\mu} \tfrac{\partial}{\partial (x')^\mu} = \mathbf{F}_a{}^{\mu} \tfrac{\partial}{\partial x^\mu} \qquad \mathfrak{C} \cdot W = W$$
$$\mathfrak{C} \cdot \Lambda = \Lambda$$

$U(1)$ *transformation* $\mathfrak{Z}$. Let $\zeta = \zeta(x^1, x^2, x^4) \in U(1)$.

$$x' = \mathfrak{Z} \cdot x = x \qquad \mathfrak{Z} \cdot \nabla = \nabla$$
$$\big(\mathfrak{Z} \cdot \mathbf{F}\big)_a{}^{\mu} \tfrac{\partial}{\partial (x')^\mu} = \big(\zeta\, \mathbf{F}_1{}^{\mu},\ \zeta^{-1}\, \mathbf{F}_2{}^{\mu},\ \mathbf{F}_3{}^{\mu},\ \mathbf{F}_4{}^{\mu}\big) \tfrac{\partial}{\partial x^\mu} \qquad \mathfrak{Z} \cdot W = W$$
$$\mathfrak{Z} \cdot \Lambda = \Lambda$$

Global Isotropic Scaling $\mathfrak{J}$. Let $\mathfrak{J} > 0$ be a constant.

$$x' = \mathfrak{J} \cdot x = (x^1,\ x^2,\ \mathfrak{J} x^3,\ \mathfrak{J} x^4) \qquad \mathfrak{J} \cdot \nabla = \nabla$$
$$\big(\mathfrak{J} \cdot \mathbf{F}\big)_a{}^{\mu} \tfrac{\partial}{\partial (x')^\mu} = \mathfrak{J}^{-1}\, \mathbf{F}_a{}^{\mu} \tfrac{\partial}{\partial x^\mu} \qquad \mathfrak{J} \cdot W = \mathfrak{J}^2 W$$
$$\mathfrak{J} \cdot \Lambda = \Lambda$$

Global Anisotropic Scaling $\mathfrak{A}$. Let $\mathfrak{A} \neq 0$ be a constant.

$$x' = \mathfrak{A} \cdot x = (\tfrac{1}{\mathfrak{A}} x^1,\ \tfrac{1}{\mathfrak{A}} x^2,\ x^3,\ \mathfrak{A}^2 x^4) \qquad \mathfrak{A} \cdot \nabla = \nabla$$
$$\big(\mathfrak{A} \cdot \mathbf{F}\big)_a{}^{\mu} \tfrac{\partial}{\partial (x')^\mu} = \big(\tfrac{1}{\mathfrak{A}}\, \mathbf{F}_1{}^{\mu},\ \tfrac{1}{\mathfrak{A}}\, \mathbf{F}_2{}^{\mu},\ \tfrac{1}{\mathfrak{A}^2}\, \mathbf{F}_3{}^{\mu},\ \mathbf{F}_4{}^{\mu}\big) \tfrac{\partial}{\partial x^\mu} \qquad \mathfrak{A} \cdot W = \mathfrak{A}^2 W$$
$$\mathfrak{A} \cdot \Lambda = \mathrm{diag}(1, \mathfrak{A}^2, \mathfrak{A}^4, \mathfrak{A}^6)\, \Lambda$$

Remark E.1. The action on the frame induces a global conformal transformation of the associated metric: $\mathfrak{C} \cdot g = g,\ \mathfrak{Z} \cdot g = g,\ \mathfrak{J} \cdot g = \mathfrak{J}^2 g,\ \mathfrak{A} \cdot g = \mathfrak{A}^2 g.$

Remark E.2. $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ preserve the wavefront gauge and, consequently, induce an action on $\Phi = (e, \gamma, w)$. We illustrate this important fact by three examples. First,

$$\begin{aligned}\big(\mathfrak{A} \cdot \mathbf{F}\big)_3{}^{\mu} \tfrac{\partial}{\partial (x')^\mu} &= \mathfrak{A}^{-2}\, \mathbf{F}_3{}^{\mu}\ \tfrac{\partial}{\partial x^\mu} = \mathfrak{A}^{-2}\,\big(e_4\, \tfrac{\partial}{\partial x^1} + e_5\, \tfrac{\partial}{\partial x^2} + \tfrac{\partial}{\partial x^4}\big) \\ &= \mathfrak{A}^{-2}\,\big(e_4\, \mathfrak{A}^{-1}\, \tfrac{\partial}{\partial (x')^1} + e_5\, \mathfrak{A}^{-1}\, \tfrac{\partial}{\partial (x')^2} + \mathfrak{A}^2\, \tfrac{\partial}{\partial (x')^4}\big)\end{aligned}$$

compatible with the wavefront gauge. Necessarily, $(\mathfrak{A} \cdot e)_i = \mathfrak{A}^{-3}\, e_i$ for $i = 4, 5$. Second, abbreviating $F_a' = \mathfrak{A} \cdot F_a$, we have

$$\begin{aligned}(\mathfrak{A} \cdot w)_5 = (\mathfrak{A} \cdot W)\big(F_3',\, F_2',\, F_3',\, F_2'\big) &= \mathfrak{A}^{-6}\, (\mathfrak{A} \cdot W)\big(F_3,\, F_2,\, F_3,\, F_2\big)\\ &= \mathfrak{A}^{-6}\, \mathfrak{A}^{2}\, W\big(F_3,\, F_2,\, F_3,\, F_2\big) = \mathfrak{A}^{-4}\, w_5.\end{aligned}$$

Third, abbreviating $F_a' = \mathfrak{Z} \cdot F_a$

$$\begin{aligned}(\mathfrak{Z} \cdot \mathit{\Gamma})(F_1',\, F_1',\, F_2') &= (\mathfrak{Z} \cdot g)\big((\mathfrak{Z} \cdot \nabla)_{F_1'} F_1', F_2'\big) = \zeta\, g\big(\nabla_{F_1} F_1, F_2\big) + F_1(\zeta)\\ (\mathfrak{Z} \cdot \mathit{\Gamma})(F_1',\, F_3',\, F_4') &= (\mathfrak{Z} \cdot g)\big((\mathfrak{Z} \cdot \nabla)_{F_1'} F_3', F_4'\big) = \zeta\, g\big(\nabla_{F_1} F_3, F_4\big)\end{aligned}$$

which is consistent with the wave front gauge if and only if $(\mathfrak{Z} \cdot \gamma)_3 = \zeta \gamma_3 + \frac{1}{2}\, F_1(\zeta)$ and $(\mathfrak{Z} \cdot \gamma)_4 = \zeta^{-1} \gamma_4 + \frac{1}{2}\, F_2(\zeta^{-1})$. By direct calculation, the present definitions of $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$ are seen to be equivalent to those in Section 3 (generalizations in the case of $\mathfrak{C}$ and $\mathfrak{Z}$).

Proposition E.1. *Let S be one of $\mathfrak{C}$, $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$. Then, separately:*

- $(\star)$ *and* $(\star\star)$ *are preserved (see, Definition 2.1).*
- $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) = 0$ *if and only if* $(S \cdot \mathfrak{t},\, S \cdot \mathfrak{u},\, S \cdot \mathfrak{v}) = 0$.
- $(t, u, v) = 0$ *if and only if* $(S \cdot t,\, S \cdot u,\, S \cdot v) = 0$.

In particular, S is a field symmetry *in the sense of Definition 3.1.*

Proof. Let Riem be the Riemann curvature tensor associated to g, considered as a 4-covariant tensor. For each S, there are complex functions κ_1, κ_2, κ_3, κ_4 and a constant $\mathit{\Omega} > 0$ such that $\kappa_1 \kappa_2 \mathit{\Omega}^2 = \kappa_3 \kappa_4 \mathit{\Omega}^2 = 1$ and

$$(S \cdot \mathbf{F}_a)^\mu \tfrac{\partial}{\partial (x')^\mu} = \kappa_a\, \mathbf{F}_a{}^\mu \tfrac{\partial}{\partial x^\mu} \qquad a = 1, 2, 3, 4$$

$$S \cdot \nabla = \nabla \qquad S \cdot g = \mathit{\Omega}^2 g \qquad S \cdot \mathrm{Riem} = \mathit{\Omega}^2\, \mathrm{Riem} \qquad S \cdot W = \mathit{\Omega}^2\, W$$

We abbreviate $F_a' = S \cdot F_a = \kappa_a\, F_a$. By the definition of T, U, V in Appendix B,

$$(S \cdot T)^\mu (F_a', F_b') \tfrac{\partial}{\partial (x')^\mu} = \kappa_a \kappa_b\, T^\mu (F_a, F_b) \tfrac{\partial}{\partial x^\mu} \tag{E.1}$$

$$\begin{aligned}(S \cdot U)(F_a', F_b', F_c', F_d') = \mathit{\Omega}^2 \kappa_a \kappa_b \kappa_c \kappa_d\, \Big(& U(F_a, F_b, F_c, F_d)\\ & + g(F_a, F_b)\, T^\mu (F_c, F_d) \tfrac{1}{\kappa_a} \tfrac{\partial \kappa_a}{\partial x^\mu} \Big)\end{aligned} \tag{E.2}$$

$$(S \cdot V)(F_a', F_b', F_c') = \kappa_a \kappa_b \kappa_c\, V(F_a, F_b, F_c) \tag{E.3}$$

By the definition of $(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})$ and (t, u, v) in Proposition D.1,

	$\mathfrak{C}$	$\mathfrak{Z}$	$\mathfrak{J}$	$\mathfrak{A}$
$\mathfrak{t} = 0 \iff S \cdot \mathfrak{t} = 0$ and $t = 0 \iff S \cdot t = 0$		true	true	true
$\mathfrak{u} = 0 \iff S \cdot \mathfrak{u} = 0$ and $t = 0 \iff S \cdot t = 0$	true		true	true
$\mathfrak{v} = 0 \iff S \cdot \mathfrak{v} = 0$ and $v = 0 \iff S \cdot v = 0$	true	true	true	true

where

- in the case of $\mathfrak{t}$ and t we use (E.1), observing that $\frac{\partial}{\partial (x')^\mu}$ is proportional to $\frac{\partial}{\partial x^\mu}$ for $\mu = 1, 2, 3, 4$ if S is one of $\mathfrak{Z}$, $\mathfrak{J}$, $\mathfrak{A}$,

- in the case of $\mathfrak{u}$ and u we use (E.2), observing that $\frac{\partial \kappa_a}{\partial x^\mu} = 0$ for $a, \mu = 1,2,3,4$ if S is one of $\mathfrak{C}$, $\mathfrak{J}$, $\mathfrak{A}$,
- in the case of $\mathfrak{v}$ and v we use (E.3) and the transformation law for Λ.

The remaining cases are discussed separately. *Case 1:* If $S = \mathfrak{C}$, we have

$$\frac{\partial}{\partial x^a} = \frac{\partial \mathfrak{C}^1}{\partial x^a}\frac{\partial}{\partial (x')^1} + \frac{\partial \mathfrak{C}^2}{\partial x^a}\frac{\partial}{\partial (x')^2} \qquad (a = 1,2) \qquad\qquad \frac{\partial}{\partial x^3} = \frac{\partial}{\partial (x')^3}$$

such that $\mathfrak{t} = 0 \iff \mathfrak{C} \cdot \mathfrak{t} = 0$ and $t = 0 \iff \mathfrak{C} \cdot t = 0$ follow from Proposition D.1. *Case 2:* If $S = \mathfrak{Z}$, note that for all a, b, c, d,

$$g(F_a, F_b)\, T^\mu(F_c, F_d) \tfrac{1}{\kappa_a}\tfrac{\partial \kappa_a}{\partial x^\mu} = g(F_a, F_b)\, \textstyle\sum_{i=1,2} T^i(F_c, F_d) \tfrac{1}{\kappa_a}\tfrac{\partial \kappa_a}{\partial x^i}$$

by the structure of the torsion-matrix (last column vanishes) and $\frac{\partial \kappa_a}{\partial x^3} = 0$. Because of $\kappa_3 = \kappa_4 = 1$ and the term $g(F_a, F_b)$, the expression vanishes unless $(a,b) = (1,2),\ (2,1)$. At this point, one verifies directly that

$$\begin{aligned} (\mathfrak{t}, \mathfrak{u}) = 0 \quad &\Longrightarrow \quad \mathfrak{Z} \cdot \mathfrak{u} = 0, & (\mathfrak{Z} \cdot \mathfrak{t}, \mathfrak{Z} \cdot \mathfrak{u}) = 0 \quad &\Longrightarrow \quad \mathfrak{u} = 0, \\ (t, u) = 0 \quad &\Longrightarrow \quad \mathfrak{Z} \cdot u = 0, & (\mathfrak{Z} \cdot t, \mathfrak{Z} \cdot u) = 0 \quad &\Longrightarrow \quad u = 0 \end{aligned}$$

This concludes the proof. □

F. An Estimate for Pole-Flip

Lemma F.1. *Let $B \subset \mathbb{R}^2$ be open. For all $0 < r_1 < r_2$ let*

$$A(r_1, r_2) = \big\{(\xi, \underline{u}, u) \in \mathbb{R}^4 \ :\ (\underline{u}, u) \in B,\ r_1 < |\xi| < r_2\big\}.$$

For all $|\alpha| \geq 1$, all integers $R \geq 0$, and all C^R-fields $\Phi : A(|\alpha| r_1, |\alpha| r_2) \to \mathcal{R}$,

$$\big\|\mathbf{Flip}_\alpha \cdot \Phi\big\|_{C^R(A(\frac{|\alpha|}{r_2}, \frac{|\alpha|}{r_1}))} \lesssim_{(R, r_1, r_2)} \|\Phi\|_{C^R(A(|\alpha| r_1, |\alpha| r_2))} \tag{F.1}$$

For $\mathbf{Flip}_\alpha$, see Definition 3.6. The same estimate holds for the C^R norms on the image of the sets $A(\frac{|\alpha|}{r_2}, \frac{|\alpha|}{r_1})$ and $A(|\alpha| r_1, |\alpha| r_2)$ under the change of coordinates from $x = (\xi, \underline{u}, u)$ to $q = (t, \xi, \underline{u})$, see Convention 7.1.

Remark F.1. The point here is that the constant in (F.1) is independent of $|\alpha| \geq 1$ and B. Lemma F.1 is used in Step 9 of the proof of Theorem 8.1. If Theorem 8.1 would be stated with the additional condition $a = \mathfrak{A}$, then Lemma F.1 would not appear in its proof.

Proof. Without loss of generality, we can assume $\alpha \geq 1$, because $\mathbf{Flip}_\alpha = \mathbf{Flip}_{-\alpha}$. Let $\mathfrak{C}_\alpha$ be the angular coordinate transformation (Definition 3.2) given by $\mathfrak{C}(\xi) = \alpha\xi$. Let $\mathfrak{F}$ be the angular coordinate transformation with $\mathfrak{C}(\xi) = \xi^{-1}$. Let $\mathfrak{Z}$ be the $U(1)$ transformation (Definition 3.3) with $\zeta(\xi) = -\xi/\overline{\xi}$. We have $\mathbf{Flip}_\alpha = \mathfrak{Z} \circ \mathfrak{C}_\alpha \circ \mathfrak{F} \circ \mathfrak{C}_{1/\alpha} = \mathfrak{C}_\alpha \circ \mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{1/\alpha}$. Decompose $\Phi = (e, \gamma, w) = \Phi_1 \oplus \Phi_2$ where $\Phi_1 = (e_1, e_2, e_4, e_5)$ and $\Phi_2 = (e_3, \gamma_1, \ldots, \gamma_8, w_1, \ldots, w_5)$. Introduce the notation $\epsilon(1) = 1$ and $\epsilon(2) = 0$.

For all $\beta = (\beta_1, \beta_2, 0, 0) \in \mathbb{N}_0^4$ with $|\beta| \leq R$, all $0 < a < b$, all $\lambda > 0$, all $0 \leq r \leq R$, all $N = 1, 2$, and all fields Φ,

$$\begin{aligned}
\|\partial^\beta (\mathfrak{C}_\lambda \cdot \Phi)_N\|_{C^0(A(\lambda a, \lambda b))} &= \lambda^{\epsilon(N)-|\beta|} \, \|\partial^\beta \Phi_N\|_{C^0(A(a,b))} \\
\|(\mathfrak{C}_\lambda \cdot \Phi)_N\|_{C^r(A(\lambda a, \lambda b))} &\leq \lambda^{\epsilon(N)-r} \, \|\Phi_N\|_{C^r(A(a,b))} \quad \text{for} \quad 0 < \lambda \leq 1 \\
\|(\mathfrak{F} \cdot \Phi)_N\|_{C^r(A(b^{-1}, a^{-1}))} &\lesssim_{(R,a,b)} \|\Phi_N\|_{C^r(A(a,b))} \\
\|(\mathfrak{Z} \cdot \Phi)_1\|_{C^r(A(a,b))} &\lesssim_{(R,a,b)} \|\Phi_1\|_{C^r(A(a,b))} \\
\|(\mathfrak{Z} \cdot \Phi)_2\|_{C^r(A(a,b))} &\lesssim_{(R,a,b)} \|\Phi_1\|_{C^r(A(a,b))} + \|\Phi_2\|_{C^r(A(a,b))}
\end{aligned}$$

Set $X = (R, r_1, r_2)$. The above estimates imply

$$\begin{aligned}
&\|\partial^\beta (\mathbf{Flip}_\alpha \cdot \Phi)_2\|_{C^0(A(\frac{\alpha}{r_2}, \frac{\alpha}{r_1}))} \\
&= \|\partial^\beta ((\mathfrak{C}_\alpha \circ \mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^0(A(\frac{\alpha}{r_2}, \frac{\alpha}{r_1}))} \\
&= \alpha^{-|\beta|} \|\partial^\beta ((\mathfrak{Z} \circ \mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^0(A(\frac{1}{r_2}, \frac{1}{r_1}))} \\
&\lesssim_X \alpha^{-|\beta|} \|((\mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_1\|_{C^{|\beta|}(A(\frac{1}{r_2}, \frac{1}{r_1}))} + \alpha^{-|\beta|} \|((\mathfrak{F} \circ \mathfrak{C}_{\frac{1}{\alpha}}) \cdot \Phi)_2\|_{C^{|\beta|}(A(\frac{1}{r_2}, \frac{1}{r_1}))} \\
&\lesssim_X \alpha^{-|\beta|} \|(\mathfrak{C}_{\frac{1}{\alpha}} \cdot \Phi)_1\|_{C^{|\beta|}(A(r_1, r_2))} + \alpha^{-|\beta|} \|(\mathfrak{C}_{\frac{1}{\alpha}} \cdot \Phi)_2\|_{C^{|\beta|}(A(r_1, r_2))} \\
&\lesssim_X \alpha^{-1} \|\Phi_1\|_{C^{|\beta|}(A(\alpha r_1, \alpha r_2))} + \|\Phi_2\|_{C^{|\beta|}(A(\alpha r_1, \alpha r_2))} \\
&\lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1, \alpha r_2))}
\end{aligned}$$

Similarly, $\|\partial^\beta (\mathbf{Flip}_\alpha \cdot \Phi)_1\|_{C^0(A(\frac{\alpha}{r_2}, \frac{\alpha}{r_1}))} \lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1, \alpha r_2))}$. Together,

$$\|\partial^\beta (\mathbf{Flip}_\alpha \cdot \Phi)\|_{C^0(A(\frac{\alpha}{r_2}, \frac{\alpha}{r_1}))} \lesssim_X \|\Phi\|_{C^{|\beta|}(A(\alpha r_1, \alpha r_2))}$$

The last estimate, and the fact that $\frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^4}$ both commute with $\mathbf{Flip}_\alpha$, imply Lemma F.1. $\square$

G. Supplement to Proposition 8.1.

Here we make explicit all the polynomials $\mathcal{J}$, $\boldsymbol{\mathcal{J}}$, $\mathcal{H}$, $\boldsymbol{\mathcal{H}}$ (see, Definition 8.1) that appear in Proposition 8.1. In particular, it will be clear, by inspection, that these polynomials are independent of K, as required. We use the $\mathbb{C}^5 \oplus \mathbb{C}^9 \oplus \mathbb{C}^4$ block-notation, as in **(S6)** and **(S7)**, and the complex conjugation operator C.

- Recall that, for each μ, the square matrices $B_1^\mu(q,0)$, $B_2^\mu(q,0)$ and $B_3^\mu(q,0)$ are 5×5, 9×9 and 4×4, respectively. See **(S6)**.
  In equation (8.5a), we have $B_i^\mu(q,0) - \not{B}_i^\mu = u^{-2} \mathcal{G}_K$ (that is, $\boldsymbol{\mathcal{H}} = 0$) for all $i = 1, 2, 3$ and $\mu = 0, 1, 2, 3$, except in the cases $(i, \mu) = (1, A)$ with $A = 1, 2$, when

$$B_1^1(q,0) - \not{B}_1^1 = \frac{1}{u} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{e} \\ 0 & 0 & 0 & -\mathbf{e} & 0 \end{pmatrix} + \frac{1}{u^2} \mathcal{G}_K = \frac{1}{u} \boldsymbol{\mathcal{H}} + \frac{1}{u^2} \mathcal{G}_K$$

  and similar for $B_1^2(q,0) - \not{B}_1^2$.

- For (8.5b),

$$\Big(Q_{11}-\not{Q}_{11}\Big|\,Q_{12}-\not{Q}_{12}\Big|\,Q_{13}-\not{Q}_{13}\Big)\;-\;\tfrac{1}{u^2}\mathcal{G}_K$$
$$=\tfrac{1}{u}\left(\begin{array}{ccccc|ccc|cccc}
0&0&0&0&0&0&\cdots&0&0&0&0&0\\
0&0&0&0&0&0&\cdots&0&0&0&0&0\\
0&0&0&0&0&0&\cdots&0&0&0&0&0\\
0&0&0&0&4\boldsymbol{\lambda}&0&\cdots&0&0&0&0&0\\
0&0&-3\,\omega_7(0)&-2\overline{\boldsymbol{\lambda}}&\mathfrak{A}^2&0&\cdots&0&0&0&0&-3\,z_3(0)
\end{array}\right)$$

- For (8.5c),

$$\Big(Q_{21}-\not{Q}_{21}\Big|\,Q_{22}-\not{Q}_{22}\Big|\,Q_{23}-\not{Q}_{23}\Big)\;-\;\tfrac{1}{u}\mathcal{G}_K$$
$$=\left(\begin{array}{cc|ccccccc|cccc}
0&\cdots&\cdots&0&\mathbf{e}&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&-i\mathbf{e}&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&0&0&-\mathbf{e}\,\mathfrak{C}_+&\mathbf{e}\,\mathfrak{C}_+&0&0&-\mathbf{e}\,\mathfrak{C}_+&0&0\\
0&\cdots&\cdots&0&0&0&\mathbf{e}\,\mathfrak{C}_-&\mathbf{e}\,\mathfrak{C}_-&0&0&-\mathbf{e}\,\mathfrak{C}_-&0&0\\
0&\cdots&\cdots&0&0&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&-\mathfrak{C}_+\,\overline{\omega_1(0)}&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&-\overline{\boldsymbol{\lambda}}&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&-\boldsymbol{\lambda}\,C&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&0&0&\mathfrak{C}_-\,\overline{\boldsymbol{\lambda}}&\mathfrak{C}_-\,\boldsymbol{\lambda}&0&0&\mathfrak{C}_-\,\boldsymbol{\lambda}&0&0
\end{array}\right)$$

where the operators $\mathfrak{C}_+\,\varphi$ and $\mathfrak{C}_-\,\varphi$ are defined by $\varphi+\overline{\varphi}\,C$ and $i(\varphi-\overline{\varphi}\,C)$, respectively, where φ is any complex valued function.

- For (8.5d)

$$\Big(Q_{31}-\not{Q}_{31}\Big|\,Q_{32}-\not{Q}_{32}\Big|\,Q_{33}-\not{Q}_{33}\Big)\;-\;\tfrac{1}{u^2}\mathcal{G}_K$$
$$=\big(\tfrac{1}{|t|}+\tfrac{1}{u}\big)\left(\begin{array}{cc|ccccc|cccc}
0&\cdots&\cdots&0&0&0&1+C&2&0&0&0\\
0&\cdots&\cdots&0&-C&1&0&0&1&0&0\\
0&\cdots&\cdots&0&0&0&0&0&0&0&0\\
0&\cdots&\cdots&0&0&0&0&0&0&0&0
\end{array}\right)$$

- For (8.7b), (8.7c), (8.7d), recall from Remark 5.1 that $\mathbf{f}(q,\Psi)$ is a quadratic polynomial in Ψ, $\overline{\Psi}$ without constant term. Let $\mathbf{f}(q,\Psi)=\mathbf{f}_{(1)}(q,\Psi)+\mathbf{f}_{(2)}(q,\Psi)$ be its decomposition into homogeneous (over $\mathbb{R}$) parts. By definition (8.3c),

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,Q(q,s\Xi)\,\Pi=\tfrac{\mathrm{d}^2}{\mathrm{d}s_1\,\mathrm{d}s_2}\big|_{(s_1,s_2)=0}\;\tfrac{1}{2}\,\pi^{-1}\,\mathbf{f}_{(2)}\big(q,\,\pi(s_1\Xi+s_2\Pi)\big)$$

It follows from direct inspection of (5.5b) that $\mathbf{f}_{(2)}(q,\Psi)$ is a polynomial in Ψ, $\overline{\Psi}$ whose coefficients are Laurent polynomials in $\frac{1}{u}$, with complex coefficients. Now one reads off from (5.7) that the Laurent polynomials have the structure recorded in (8.7b), (8.7c) and (8.7d).

- For (8.7a), recall from Remark 5.1 that $\mathbf{A}(q,\Psi)$ is affine linear (over $\mathbb{R}$) in Ψ. Let $\mathbf{A}^\mu(q,\Psi)=\mathbf{A}^\mu_{(0)}(q,\Psi)+\mathbf{A}^\mu_{(1)}(q,\Psi)$ be its decomposition into homogeneous parts. By (8.3b), we have $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,\mathbf{B}^\mu(q,s\Xi)=\pi^{-1}\,\mathbf{A}^\mu_{(1)}(q,\pi\,\Xi)\pi$. Writing out the result in the notation $(h,\sigma,\ell)=\pi\,(\Xi_1,\Xi_2,\Xi_3)$ of (8.2), we obtain for $\mu=0$ and $\mu=3$,

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\,\mathbf{B}^\mu(q,s\Xi)=\mathrm{diag}\big(0,\,\tfrac{1}{u^4}h_3,\,\tfrac{1}{u^4}h_3,\,\tfrac{1}{u^4}h_3,\,\tfrac{1}{u^2}h_3\big)\oplus\big(\tfrac{1}{u^2}h_3\,\mathbb{1}_9\big)\oplus 0_{4\times4}$$

and for $\mu = A = 1, 2$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathbf{B}^A(q, s\Xi)$$
$$= \begin{pmatrix} \frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0 & 0 & 0 \\ \frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0 & 0 \\ 0 & \frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^3}h_A & 0 \\ 0 & 0 & \frac{1}{u^3}\overline{h}_A & \frac{1}{u^3}h_{A+3} & \frac{1}{u^2}h_A \\ 0 & 0 & 0 & \frac{1}{u^2}\overline{h}_A & 0 \end{pmatrix} \oplus 0_{9\times 9} \oplus \left(\frac{1}{u^3}h_{A+3}\, \mathbb{1}_4\right)$$

References

[AnRe] Anderson, L. and Rendall, A.D., Commun. Math. Phys. **218**, (2001) 479-511
[BK] Belinskii, V.A., Khalatnikov, I.M., Sov. Phys. JETP **36**, (1973) 591-597
[BKL] Belinskii, V.A., Khalatnikov, I.M., Lifshitz, E.M., Adv. in Phys. **31**, (1982) 639-667
[BBM] Bondi, H., van der Burg, M.G.J. and Metzner, A.W.K., Proc. Roy. Soc. Lond. A **269**, (1962) 21-52
[Cha] Chandrasekhar, S., *The Mathematical Theory of Black Holes* (Oxford U., 1983)
[Chr] Christodoulou, D., *The Formation of Black Holes in General Relativity* (EMS, 2009)
[ChrKl] Christodoulou, D. and Klainerman, S.,
The Global Nonlinear Stability of the Minkowski Space, (Princeton U., 1993)
[Fr] Friedrich, H., Proc. R. Soc. Lond. A **375**, (1981) 169-184
[John] John, F., *Partial Differential Equations* (Springer, 1978) 139
[NP] Newman, E. and Penrose, R., J. Math. Phys. **3**, (1962) 566-578
[Pen] Penrose, R., Phys. Rev. Lett. **14**, (1965) 57-59
[Sze] Szekeres, P., J. Math. Phys. **13**, (1972) 286-294
[Tay] Taylor, M. E., *Partial Differential Equations III* (Springer, 1997) 360-370
[Tay2] Taylor, M. E., *Partial Differential Equations I* (Springer, 1996) 434-439

The BKL Conjectures for Spatially Homogeneous Spacetimes

Michael Reiterer, Eugene Trubowitz

Department of Mathematics, ETH Zurich, Switzerland

Abstract: We rigorously construct and control a generic class of spatially homogeneous (Bianchi VIII and Bianchi IX) vacuum spacetimes that exhibit the oscillatory BKL phenomenology. We investigate the causal structure of these spacetimes and show that there is a "particle horizon".

1. Introduction

The goal of this paper is to rigorously construct and explicitly control a generic class of solutions $\Phi = \alpha \oplus \beta : [0, \infty) \to \mathbb{R}^3 \oplus \mathbb{R}^3$, with independent variable $\tau \in [0, \infty)$ and with[1] $(\alpha_1 + \alpha_2 + \alpha_3)|_{\tau=0} < 0$, to the autonomous system of six ordinary differential equations

$$0 = -\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{i}} - (\beta_{\mathbf{i}})^2 + (\beta_{\mathbf{j}})^2 + (\beta_{\mathbf{k}})^2 - 2\beta_{\mathbf{j}}\beta_{\mathbf{k}} \tag{1.1a}$$

$$0 = -\tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_{\mathbf{i}} + \beta_{\mathbf{i}}\alpha_{\mathbf{i}} \tag{1.1b}$$

for all $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathcal{C} \stackrel{\text{def}}{=} \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, subject to the quadratic constraint[2]

$$0 = \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 - (\beta_1)^2 - (\beta_2)^2 - (\beta_3)^2 + 2\beta_2\beta_3 + 2\beta_3\beta_1 + 2\beta_1\beta_2 \tag{1.1c}$$

Here, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$. The system (1.1) are the vacuum Einstein equations for spatially homogeneous (Bianchi) spacetimes, see Proposition 2.1.

The pioneering calculations and heuristic picture of Belinskii, Khalatnikov, Lifshitz[3] [BKL1] and Misner [Mis] suggest that a generic class of solutions to (1.1) are oscillatory as $\tau \to +\infty$ and that the dynamics of one degree of freedom is closely related to the discrete dynamics of the Gauss map $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$, a non-invertible map

[1] If $\tau \mapsto \Phi(\tau)$ is a solution to (1.1), so is $\tau \mapsto -\Phi(-\tau)$. The condition $(\alpha_1 + \alpha_2 + \alpha_3)|_{\tau=0} < 0$ breaks this symmetry. Solutions to (1.1) with $(\alpha_1 + \alpha_2 + \alpha_3)|_{\tau=0} < 0$ do not break down in finite positive time, that is, they extend to $[0, \infty)$. A proof of this fact is given later in this introduction.

[2] As a quadratic form on $\mathbb{R}^3 \oplus \mathbb{R}^3$, the right hand side of (1.1c) has signature $(+, +, -, -, -, -)$.

[3] The work of Belinskii, Khalatnikov, Lifshitz concerns general (inhomogeneous) spacetimes, but relies on intuition about the homogeneous case.

from $(0,1)\setminus\mathbb{Q}$ to itself. Every element of $(0,1)\setminus\mathbb{Q}$ admits a unique infinite continued fraction expansion

$$\langle k_1, k_2, k_3, \ldots\rangle \;=\; \cfrac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \ldots}}} \tag{1.2}$$

where $(k_n)_{n\geq 1}$ are strictly positive integers. The Gauss map is the left-shift,

$$G\big(\langle k_1, k_2, k_3, \ldots\rangle\big) = \langle k_2, k_3, k_4, \ldots\rangle \tag{1.3}$$

Rigorous results about spatially homogeneous spacetimes have been obtained by Rendall [Ren] and Ringström [Ri1], [Ri2]. See also Heinzle and Uggla [HU2]. We refer to the very readable paper [HU1] for a detailed discussion.

The first rigorous proofs that there exist spatially homogeneous vacuum spacetimes whose asymptotic behavior is related, in a precise sense, to iterates of the Gauss map, have been obtained recently by Béguin [Be] and by Liebscher, Härterich, Webster and Georgi [LHWG]. These theorems apply to a dense subset of $(0,1)\setminus\mathbb{Q}$. A basic restriction of both these works is that the sequence $(k_n)_{n\geq 1}$ has to be bounded, a condition fulfilled only by a Lebesgue measure zero subset of $(0,1)\setminus\mathbb{Q}$. The results of the present paper apply to any sequence $(k_n)_{n\geq 1}$ that grows at most polynomially. The corresponding subset of $(0,1)\setminus\mathbb{Q}$ has full Lebesgue measure one.

We point out some properties of the system (1.1a), (1.1b), not assuming (1.1c):

(i) The right hand side of (1.1c) is a conserved quantity.
(ii) If $\tau \mapsto \varPhi(\tau)$ is a solution, so is $\tau \mapsto p\,\varPhi(p\tau + q)$, for all $p, q \in \mathbb{R}$.
(iii) The signatures $(\operatorname{sgn}\beta_1, \operatorname{sgn}\beta_2, \operatorname{sgn}\beta_3)$ are constant.
(iv) $\frac{\mathrm{d}}{\mathrm{d}\tau}|\beta_1\beta_2\beta_3|^2 = 2(\alpha_1+\alpha_2+\alpha_3)|\beta_1\beta_2\beta_3|^2$.
(v) We have[4] $\frac{\mathrm{d}}{\mathrm{d}\tau}(\alpha_1+\alpha_2+\alpha_3) \geq -3|\beta_1\beta_2\beta_3|^{2/3}$.

If in addition we assume (1.1c), then:

(vi) $\frac{\mathrm{d}}{\mathrm{d}\tau}(\alpha_1+\alpha_2+\alpha_3) = \alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 \leq \frac{1}{3}(\alpha_1+\alpha_2+\alpha_3)^2$.

Let $\varPhi = \alpha \oplus \beta$ be any solution to (1.1), that is (1.1a), (1.1b), (1.1c), on the half-open interval $[0,\tau_1)$ with $0 < \tau_1 < \infty$. Set $\not{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and suppose $\not{\alpha}(0) < 0$. Then

$$\not{\alpha}(\tau) \leq -|\not{\alpha}(0)|/\big(1 + \tfrac{1}{3}|\not{\alpha}(0)|\tau\big) < 0 \qquad \text{for all } \tau \in [0,\tau_1) \tag{1.4}$$

by (vi). Consequently, $|\beta_1\beta_2\beta_3|$ is bounded, by (iv), and $\not{\alpha}$ is bounded below, by (v), on $[0,\tau_1)$. The constraint (1.1c) implies that[5] $(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 \leq 6\,|\beta_1\beta_2\beta_3|^{2/3} + \not{\alpha}^2$ is bounded. Now (1.1b) implies that $(\beta_1)^2 + (\beta_2)^2 + (\beta_3)^2$ is bounded. Therefore, solutions to (1.1) with $\not{\alpha}(0) < 0$ can be extended to $[0,\infty)$. The solutions considered in this paper belong to this general class. We are especially interested in their $\tau \to +\infty$ asymptotics.

---

[4] $(\beta_1)^2 + (\beta_2)^2 + (\beta_3)^2 - 2\beta_2\beta_3 - 2\beta_3\beta_1 - 2\beta_1\beta_2 + 3|\beta_1\beta_2\beta_3|^{2/3} \geq 0$ holds for all $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, see [HU1]. The only nontrivial cases are $\beta_1, \beta_2, \beta_3 > 0$ or $\beta_1, \beta_2, \beta_3 < 0$. In these cases, the inequality is a direct consequence of the polynomial identity

$$\begin{aligned}&x^6 + y^6 + z^6 - 2y^3z^3 - 2z^3x^3 - 2x^3y^3 + 3x^2y^2z^2 = \\ &\tfrac{1}{2}\left(x^2+y^2+z^2+yz+zx+xy\right)\Big((y-z)^2(y+z-x)^2 + (z-x)^2(z+x-y)^2 + (x-y)^2(x+y-z)^2\Big)\end{aligned}$$

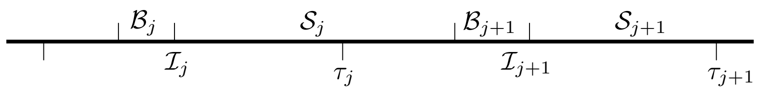
[5] Use $2(\alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2) = \not{\alpha}^2 - (\alpha_1)^2 - (\alpha_2)^2 - (\alpha_3)^2$.

For every solution to (1.1) with $\not{\alpha}(0) < 0$, as in the last paragraph, the half-infinite interval $[0, \infty)$ actually corresponds to a *finite* physical duration of the associated spatially homogeneous vacuum spacetime (given in Proposition 2.1). In fact, an increasing affine parameter along the timelike geodesics orthogonal to the level sets of τ is given by $\tau \mapsto \int_0^\tau \exp(\frac{1}{2}\int_0^s \not{\alpha})\mathrm{d}s$, with uniform upper bound $6|\not{\alpha}(0)|^{-1}$, by (1.4).

In this paper, we consider only solutions to (1.1) for which $\beta_1, \beta_2, \beta_3 \neq 0$ (also called Bianchi VIII or IX models). We now give an informal description of the solutions that we construct, the phenomenological picture of [BKL1]. The structure of each of these solutions is described by three sequences of compact subintervals $(\mathcal{I}_j)_{j\geq 1}$, $(\mathcal{B}_j)_{j\geq 1}$, $(\mathcal{S}_j)_{j\geq 1}$ of $[0, \infty)$, for which:

(a.1) The left endpoint of $\mathcal{I}_1$ is the origin, and the right endpoint of $\mathcal{I}_j$, henceforth denoted τ_j, coincides with the left endpoint of $\mathcal{I}_{j+1}$, for all $j \geq 1$. Set $\tau_0 = 0$.
(a.2) $\bigcup_{j\geq 1} \mathcal{I}_j = [0, \infty)$, that is, $\lim_{j\to+\infty} \tau_j = +\infty$.
(a.3) $\mathcal{B}_j$ is contained in the interior of $\mathcal{I}_j$, and $0 < |\mathcal{B}_j| \ll |\mathcal{I}_j|$, for all $j \geq 1$.
(a.4) $\mathcal{S}_j$ is the closed interval of all points between $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$, for all $j \geq 1$.

Here is a picture:



Let S_3 be the set of all permutations $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of the triple $(1, 2, 3)$. The solution is further described by a sequence $(\pi_j)_{j\geq 1}$ in S_3, with $\pi_j = (\mathbf{a}(j), \mathbf{b}(j), \mathbf{c}(j))$, so that:

(b.1) On $\mathcal{I}_j$, the components $\beta_{\mathbf{b}(j)}$, $\beta_{\mathbf{c}(j)}$ are so small in absolute value that the local dynamics of $\Phi = \alpha \oplus \beta$ is essentially unaffected if $\beta_{\mathbf{b}(j)}$, $\beta_{\mathbf{c}(j)}$ are set equal to zero in the four equations (1.1a) and (1.1c).
(b.2) On $\mathcal{I}_j \setminus \mathcal{B}_j$, the component $\beta_{\mathbf{a}(j)}$ is so small in absolute value that the local dynamics of $\Phi = \alpha \oplus \beta$ is essentially unaffected if $\beta_{\mathbf{a}(j)}$ is set equal to zero in the four equations (1.1a) and (1.1c). The component $\beta_{\mathbf{a}(j)}$ is *not small* on $\mathcal{B}_j$, but the mixed products $\beta_{\mathbf{a}(j)}\beta_{\mathbf{b}(j)}$ and $\beta_{\mathbf{a}(j)}\beta_{\mathbf{c}(j)}$ are still small.
(b.3) Items (b.1) and (b.2) imply that mixed products of components of β are small on all of $[0, \infty)$, and that all three components of β are small on $\bigcup_{j\geq 1} \mathcal{S}_j$.
(b.4) $\mathbf{a}(j) \neq \mathbf{a}(j + 1)$ for all $j \geq 1$.
(b.5) None of the properties listed so far distinguishes $\mathbf{b}(j)$ from $\mathbf{c}(j)$. By (b.4), this ambiguity can be consistently eliminated by stipulating $\mathbf{b}(j) = \mathbf{a}(j + 1)$.

We can draw the following heuristic consequences from the eight heuristic properties above. *Separately on each interval $\mathcal{S}_j$, $j \geq 1$:*

(c.1) The components of α are essentially constant, by (1.1a) and (b.3), and $\log|\beta_1|$, $\log|\beta_2|$, $\log|\beta_3|$ are essentially linear functions with slopes α_1, α_2, α_3, by (1.1b).
(c.2) The constraint (1.1c) essentially reduces to $\alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2 = 0$. As before, we require $\not{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 < 0$. Furthermore, we make the generic assumption that all components of α are nonzero. These conditions imply that two components of α are negative, one component of α is positive, and the sum of any two is negative.
(c.3) The single positive component of α has to be $\alpha_{\mathbf{b}(j)} = \alpha_{\mathbf{a}(j+1)}$. In fact, we know that $|\beta_{\mathbf{a}(j+1)}|$ is very small on $\mathcal{S}_j$ but is not small on $\mathcal{B}_{j+1}$. Therefore, the slope of $\log|\beta_{\mathbf{a}(j+1)}|$, which is $\alpha_{\mathbf{a}(j+1)}$ by (c.1), has to be positive on $\mathcal{S}_j$.

(c.4) By the last three items and (b.4), there is at most one point in $\mathcal{S}_j$ where $|\beta_{\mathbf{a}(j)}| = |\beta_{\mathbf{a}(j+1)}|$. By (b.1), (b.2), there is such a point, because $|\beta_{\mathbf{a}(j)}|$ is going from not small to small on $\mathcal{S}_j$, and $|\beta_{\mathbf{a}(j+1)}|$ is going from small to not small on $\mathcal{S}_j$. By convention, this point is τ_j.

Separately on each interval $\mathcal{I}_j$, $j \geq 1$ (in particular on $\mathcal{B}_j \subset \mathcal{I}_j$):

(d.1) $\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{b}(j)}$ and $\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{c}(j)}$ are essentially constant, by (1.1a), (b.1), and they are both negative, by (c.1), (c.2). Also, $\log|\beta_{\mathbf{a}(j)}\beta_{\mathbf{b}(j)}|$, $\log|\beta_{\mathbf{a}(j)}\beta_{\mathbf{c}(j)}|$ are essentially linear functions with slopes $\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{b}(j)}$ and $\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{c}(j)}$, by (1.1b).

(d.2) Essentially $(\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{b}(j)})(\alpha_{\mathbf{a}(j)} + \alpha_{\mathbf{c}(j)}) = (\alpha_{\mathbf{a}(j)})^2 + (\beta_{\mathbf{a}(j)})^2$, by (1.1c). Since the left hand side is essentially constant by (d.1), so is the right hand side.

(d.3) By (d.1), it only remains to understand the behavior of $\alpha_{\mathbf{a}(j)}$, $\beta_{\mathbf{a}(j)}$. By (1.1a), we essentially have

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{a}(j)} = -(\beta_{\mathbf{a}(j)})^2 \qquad \tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_{\mathbf{a}(j)} = \alpha_{\mathbf{a}(j)}\beta_{\mathbf{a}(j)} \tag{1.5}$$

A special solution is $\alpha_{\mathbf{a}(j)} = -\tanh\tau$ and $\beta_{\mathbf{a}(j)} = \pm\operatorname{sech}\tau = \pm(\cosh\tau)^{-1}$. The general solution is obtained from the special solution by applying the affine symmetry transformation (ii) above, with $p > 0$. Since $\mathcal{B}_j$ is essentially the interval on which $|\beta_{\mathbf{a}(j)}|$ is not small, see (b.2), we must have $p \sim |\mathcal{B}_j|^{-1}$ (here $\sim$ means "same order of magnitude"). See [BKL1], Section 3, in particular pages 534 and 535.

(d.4) Recall (c.1). By (d.3), we have $\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_{j-1}} = -\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_j}$, since the hyperbolic tangent just flips the sign. Therefore, by (d.1), the net change across $\mathcal{B}_j$ of the components of α, from right to left, is given by

$$\begin{aligned}
\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_{j-1}} &= \alpha_{\mathbf{a}(j)}|_{\mathcal{S}_j} - 2\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_j}\\
\alpha_{\mathbf{b}(j)}|_{\mathcal{S}_{j-1}} &= \alpha_{\mathbf{b}(j)}|_{\mathcal{S}_j} + 2\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_j}\\
\alpha_{\mathbf{c}(j)}|_{\mathcal{S}_{j-1}} &= \alpha_{\mathbf{c}(j)}|_{\mathcal{S}_j} + 2\alpha_{\mathbf{a}(j)}|_{\mathcal{S}_j}
\end{aligned}$$

These equations make sense only for $j \geq 2$, since $\mathcal{S}_0$ has not been defined.

In this paper, we turn the heuristic picture of [BKL1], sketched above, into a mathematically rigorous one, globally on $[0,\infty)$, for a generic class of solutions. The first step is to construct a discrete dynamical system, that maps the state $\Phi(\tau_j)$ to the state $\Phi(\tau_{j-1})$ at the earlier time $\tau_{j-1} < \tau_j$, for all $j \geq 1$. That is, the construction proceeds from right-to-left, beginning at $\tau = +\infty$. We refer to the discrete dynamical system maps as transfer maps.

For each $j \geq 0$, two components of $\beta(\tau_j)$ have the same absolute value, see (c.4), and $\Phi(\tau_j)$ satisfies the constraint (1.1c). Therefore, the states of the discrete dynamical system have 4 continuous degrees of freedom. By the symmetry (ii), the transfer maps commute with rescalings. Taking the quotient, one obtains a 3-dimensional discrete dynamical system. The three "dimensionless" quantities that we use to parametrize the discrete states are denoted $\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j)$. Morally, they are interpreted as follows:

- $\mathbf{h}_j \sim |\mathcal{B}_j|/|\mathcal{I}_j| > 0$. In the billiard picture of [Mis], it is the dimensionless ratio of the collision and free-motion times. By (a.3), one has $0 < \mathbf{h}_j \ll 1$. In fact, $\mathbf{h}_j$ is the all-important small parameter in our construction. It goes to zero rapidly as $j \to \infty$. This is necessary for us to make a global construction on $[0,\infty)$. The precise rate depends on the sequence $(k_n)_{n\geq 1}$. The rate is the same as in Proposition 4.4, up to even smaller corrections.

- The components of α are essentially constant on $\mathcal{S}_j$ and subject to the reduced constraint equation in (c.2). Thus, modulo the scaling symmetry (ii), only one degree of freedom is required to parametrize $\alpha|_{\mathcal{S}_j}$. We use $w_j \approx -(\alpha_{\mathbf{b}(j)}/(\alpha_{\mathbf{a}(j)}+\alpha_{\mathbf{b}(j)}))|_{\mathcal{S}_j}$. By (c.2) and (c.3), we have $w_j > 0$. The left-to-right discrete dynamics of w_j (which is opposite to the right-to-left direction of our transfer maps) is closely related to a variant of the Gauss map, sometimes referred to as the *BKL map* or *Kasner map*.
- The meaning of q_j will be explained in a more indirect way. As pointed out above, the left-to-right dynamics of w_j is related to the Gauss map, which is a non-invertible left-shift, see (1.3). The non-invertibility of the Gauss map seems to be at odds with the invertible dynamics of the system of ordinary differential equations (1.1). The parameter q_j is introduced so that the *joint* left-to-right discrete dynamics of (w_j, q_j) is closely related to the left-shift on *two-sided* sequences $(k_n)_{n\in\mathbb{Z}}$ of strictly positive integers, which is invertible. Accordingly, the right-to-left transfer maps are related to the right shift on two-sided sequences $(k_n)_{n\in\mathbb{Z}}$.

This concludes the informal discussion. We emphasize that the notation used above is specific to the introduction. In particular, $(\mathcal{I}_j)_{j\geq 1}$, $(\mathcal{B}_j)_{j\geq 1}$, $(\mathcal{S}_j)_{j\geq 1}$ do not appear in the main text. Starting from Section 2, all the notation is introduced from scratch.

We now state simplified, self-contained versions of our results. References to their stronger counterparts are given. Here is a short guide:

Definition 1.1 (equivalent to Definition 3.12). Introduces the state vectors $\Phi_\star(\pi, \mathbf{f}, \sigma_*)$ of the 3-dimensional discrete dynamical system. The dynamics of the signature vector σ_* is trivial, by (iii), but it affects the dynamics of $(\pi, \mathbf{f})$ in a non-trivial way.

Definition 1.2 (this is Definition 3.16). Introduces explicit maps $\mathcal{P}_L$, $\mathcal{Q}_L$, λ_L that turn out to be very good approximations to the transfer maps. It is shown in Section 4 that iterates of $\mathcal{Q}_L$ can be understood in terms of the Gauss map / continued fractions and, by a change of variables, in terms of solutions to certain linear equations.

Definition 3.19 (only in the main text). The essential smallness condition on $\mathbf{h} > 0$ is quantitatively encoded in an open subset $\mathcal{F} \subset (0,1) \times (0,\infty) \times ((0,\infty) \setminus \{1\})$. It determines the domain of definition of the transfer maps.

Proposition 1.1 (slimmed-down version of Proposition 3.3). It asserts the existence of transfer maps. The pair $(\mathcal{P}_L, \Pi)$ and the triple $(\mathcal{P}_L, \Pi, \Lambda)$ constitute the transfer maps for the 3-dimensional and 4-dimensional systems, respectively, and they are very close to $(\mathcal{P}_L, \mathcal{Q}_L)$ and $(\mathcal{P}_L, \mathcal{Q}_L, \lambda_L)$. Explicit error bounds and precise estimates for the transfer solution appear only in the full version, Proposition 3.3.

Theorem 1.1 (simplified version of Theorems 6.2, 6.3). Gives a generic class of iterates to $(\mathcal{P}_L, \Pi)$ that are super-exponentially close to iterates of $(\mathcal{P}_L, \mathcal{Q}_L)$. That is, it asserts the existence of solutions to the 3-dimensional discrete dynamical system.

The overview is as follows. Every solution to the 3-dimensional discrete dynamical system as in Theorem 1.1 can be lifted to a unique solution to the 4-dimensional discrete dynamical system, up to an overall scale, through the map Λ in Proposition 1.1. This solution corresponds to the sequence of states $(\Phi(\tau_j))_{j\geq 0}$ in the informal discussion. Proposition 1.1 gives solutions to (1.1) on compact intervals that connect next-neighbor states. Symmetry (ii) is used to translate these compact intervals and place them next to each other, beginning at $\tau = 0$, just like the $(\mathcal{I}_j)_{j\geq 1}$ in the informal discussion. As in (a.2) of the informal discussion, the union of these intervals is indeed $[0,\infty)$, and a semi-global solution to (1.1) is obtained. To see this, denote the states by $\lambda_j \Phi_\star(\pi_j, \mathbf{g}_j, \sigma_*)$ with $\lambda_j > 0$ and $\pi_j \in S_3$ and $\mathbf{g}_j = (\mathbf{h}'_j, w'_j, q'_j) \in \mathcal{F}$, where $j \geq 0$. One has $\lambda_j = \Lambda[\pi_{j+1}, \sigma_*](\mathbf{g}_{j+1})\lambda_{j+1} \geq \lambda_{j+1}$ by the definition of Λ and

$\mathbf{h}'_j \in (0,1)$ by the definition of $\mathcal{F}$. In particular, the sequence of products $(\lambda_j \mathbf{h}'_j)_{j\geq 0}$ is bounded from above by $\lambda_0 > 0$. By Proposition 1.1, the length of each of the intervals is bounded from below by $(2\lambda_0)^{-1} > 0$.

Definition 1.1 (State vectors). *Let* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$ *and* $\sigma_* \in \{-1,+1\}^3$ *and* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^2 \times \mathbb{R}$. *Let* $\Phi_\star = \Phi_\star(\pi, \mathbf{f}, \sigma_*) = \alpha \oplus \beta \in \mathbb{R}^3 \oplus \mathbb{R}^3$ *be the vector given by* $(\operatorname{sgn}\beta_1, \operatorname{sgn}\beta_2, \operatorname{sgn}\beta_3) = \sigma_*$ *and by*

$$\begin{aligned}
\alpha_{\mathbf{a}} &= -1 & \mathbf{h}\log|\tfrac{1}{2}\beta_{\mathbf{a}}| &= -\tfrac{1+w}{1+2w}(1+\mathbf{h}\log 2)\\
\alpha_{\mathbf{b}} &= \tfrac{w}{1+w} & \mathbf{h}\log|\tfrac{1}{2}\beta_{\mathbf{b}}| &= -\tfrac{1+w}{1+2w}(1+\mathbf{h}\log 2)\\
\alpha_{\mathbf{c}} &= -w-\mu & \mathbf{h}\log|\tfrac{1}{2}\beta_{\mathbf{c}}| &= -(1+w)q - \tfrac{w(1+w)}{1+2w} - \tfrac{1+3w+w^2}{1+2w}\mathbf{h}\log 2
\end{aligned}$$

where $\mu = \mu(\pi, \mathbf{f}, \sigma_*) \in \mathbb{R}$ *is uniquely determined by requiring that* (1.1c) *holds.*

Definition 1.2 (Approximate transfer maps). *Introduce three maps*

$$\begin{aligned}
\mathcal{P}_L : \quad & S_3 \times (0,\infty)^3 \to S_3 & ((\mathbf{a},\mathbf{b},\mathbf{c}),\mathbf{f}) &\mapsto (\mathbf{a}',\mathbf{b}',\mathbf{c}')\\
\mathcal{Q}_L : \quad & (0,\infty)^3 \to (0,\infty)^2 \times \mathbb{R} & \mathbf{f} &\mapsto (\mathbf{h}_L, w_L, q_L)\\
\lambda_L : \quad & (0,\infty)^3 \to (0,\infty) & \mathbf{f} &\mapsto \lambda_L
\end{aligned}$$

where $\mathbf{f} = (\mathbf{h}, w, q)$ *and* $q_L = \mathrm{num1}_L/\mathrm{den}_L$ *and* $\mathbf{h}_L = \mathrm{num2}_L/\mathrm{den}_L$*, and:*

- *if* $q \leq 1$*:*

$$\begin{aligned}
(\mathbf{a}',\mathbf{b}',\mathbf{c}') &= (\mathbf{c},\mathbf{a},\mathbf{b}) & \mathrm{num1}_L &= (1+w)(1-q) - \mathbf{h}\log 2 + \mathbf{h}w\log(2+w)\\
w_L &= \tfrac{1}{1+w} & \mathrm{num2}_L &= \mathbf{h}(2+w)\\
\lambda_L &= 2+w & \mathrm{den}_L &= (1+w)(q-\mathbf{h}\log 2) + \mathbf{h}(3+w)\log(2+w)
\end{aligned}$$

- *if* $q > 1$*:*

$$\begin{aligned}
(\mathbf{a}',\mathbf{b}',\mathbf{c}') &= (\mathbf{b},\mathbf{a},\mathbf{c}) & \mathrm{num1}_L &= (1+w)(q-1-\mathbf{h}\log 2) - \mathbf{h}w\log\tfrac{2+w}{1+w}\\
w_L &= 1+w & \mathrm{num2}_L &= \mathbf{h}(2+w)\\
\lambda_L &= \tfrac{2+w}{1+w} & \mathrm{den}_L &= (1+w) - \mathbf{h}\log 2 + \mathbf{h}(3+2w)\log\tfrac{2+w}{1+w}
\end{aligned}$$

Observe that $\mathrm{den}_L > 0$.

Proposition 1.1 (Transfer maps). *Fix* $\sigma_* \in \{-1,+1\}^3$ *and* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$*. There exist maps*[6]

$$\mathit{\Pi}[\pi,\sigma_*] : \ \mathcal{F} \to (0,\infty)^2 \times \mathbb{R} \qquad \textit{and} \qquad \mathit{\Lambda}[\pi,\sigma_*] : \ \mathcal{F} \to [1,\infty)$$

such that for every $\lambda > 0$ *and* $\mathbf{f} = (\mathbf{h}, w, q) \in \mathcal{F}$*, the solution to* (1.1) *starting at* $\lambda\,\Phi_\star(\pi,\mathbf{f},\sigma_*)$ *at time* 0 *passes through* $\lambda'\,\Phi_\star(\pi',\mathbf{f}',\sigma_*)$ *at an earlier time* $\tau' < 0$*, with* $\frac{1}{2} \leq \mathbf{h}\,\lambda\,|\tau'| \leq 3$*. Here* $\mathbf{f}' = \mathit{\Pi}[\pi,\sigma_*](\mathbf{f})$ *and* $\lambda' = \lambda\,\mathit{\Lambda}[\pi,\sigma_*](\mathbf{f})$ *and* $\pi' = \mathcal{P}_L(\pi,\mathbf{f})$*. Schematically, the transition is*

$$\lambda\,\mathit{\Lambda}[\pi,\sigma_*](\mathbf{f})\;\Phi_\star\Big(\mathcal{P}_L(\pi,\mathbf{f}),\ \mathit{\Pi}[\pi,\sigma_*](\mathbf{f}),\ \sigma_*\Big) \qquad \longleftarrow \qquad \lambda\,\Phi_\star(\pi,\mathbf{f},\sigma_*)$$

Furthermore (informal): $\mathit{\Pi}$ *and* $\mathit{\Lambda}$ *are approximated by the maps* $\mathcal{Q}_L$ *and* λ_L*, with errors that go to zero exponentially as* $\mathbf{h} \downarrow 0$ *(for fixed* w*,* q*). See Proposition 3.3*

[6] Caution: The maps $\mathit{\Pi}$ cannot immediately be iterated / composed, because $(0,\infty)^2 \times \mathbb{R} \not\subset \mathcal{F}$.

Theorem 1.1. *Fix $\sigma_* \in \{-1,+1\}^3$ and $\pi_0 \in S_3$. Fix constants $\mathbf{D} \geq 1$, $\gamma \geq 0$. Suppose the vector $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0) \in (0,\infty)^3$ satisfies*

(i) $w_0 \in (0,1) \setminus \mathbb{Q}$ *and* $q_0 \in (0,\infty) \setminus \mathbb{Q}$.

(ii) $k_n \leq \mathbf{D} \max\{1,n\}^\gamma$ *for all* $n \geq -2$, *where the two-sided sequence of strictly positive integers* $(k_n)_{n\in\mathbb{Z}}$ *is given by*

$$(1+q_0)^{-1} = \langle k_0, k_{-1}, k_{-2}, \ldots \rangle \qquad w_0 = \langle k_1, k_2, k_3, \ldots \rangle$$

(iii) $0 < \mathbf{h}_0 < \mathbf{A}^\sharp$ *where* $\mathbf{A}^\sharp = \mathbf{A}^\sharp(\mathbf{D},\gamma) = 2^{-56}\mathbf{D}^{-4}(4(\gamma+1))^{-4(\gamma+1)}$.

Then $\mathbf{f}_0$ and π_0 are the first elements of a unique sequence $(\mathbf{f}_j)_{j\geq 0}$ in $\mathcal{F}$ and a unique sequence $(\pi_j)_{j\geq 0}$ in S_3, respectively, with $\pi_j = \mathcal{P}_L(\pi_{j+1}, \mathbf{f}_{j+1})$ and $\mathbf{f}_j = \mathcal{Q}_L(\mathbf{f}_{j+1})$ for all $j \geq 0$. Furthermore, there exists a sequence $(\mathbf{g}_j)_{j\geq 0}$ in $\mathcal{F}$ such that for all $j \geq 0$,

$$\mathbf{g}_j = \mathit{\Pi}[\pi_{j+1}, \sigma_*](\mathbf{g}_{j+1}) \qquad \text{and} \qquad \pi_j = \mathcal{P}_L(\pi_{j+1}, \mathbf{g}_{j+1})$$

and, with $\rho_+ = \frac{1}{2}(1+\sqrt{5})$,

$$\|\mathbf{g}_j - \mathbf{f}_j\|_{\mathbb{R}^3} \leq \exp\Big(- \tfrac{1}{\mathbf{h}_0} \mathbf{A}^\sharp \, \rho_+^{((\mathbf{D}^{-1}j)^{1/(\gamma+1)})} \Big)$$

If $\gamma > 1$ and $\mathbf{D} > \frac{1}{\log 2}\frac{\gamma}{\gamma-1}$, then the set of all vectors $\mathbf{f}_0 \in (0,\infty)^3$ that satisfy (i), (ii), (iii) has positive Lebesgue measure.

The class of solutions that we construct is generic in the sense of the last sentence of Theorem 1.1. It would be desirable to have a stronger genericity statement, namely a genericity statement for "the $\mathbf{g}_0$ rather than the $\mathbf{f}_0$".

For the causal structure and particle horizons, see Proposition 2.2 and Section 7.

It is a pleasure to thank J. Fröhlich, G.M. Graf and T. Spencer for their support and encouragement.

2. Spatially homogeneous vacuum spacetimes

Proposition 2.1. *Let $\alpha \oplus \beta : (\tau_0, \tau_1) \to \mathbb{R}^3 \oplus \mathbb{R}^3$ be a solution to (1.1) and let $\mathit{\Omega} \subset \mathbb{R}^3$ be open, with Cartesian coordinates $\mathbf{x} = (x^1, x^2, x^3)$. Fix any $\tau_* \in (\tau_0, \tau_1)$ and let*

$$v_1 = \textstyle\sum_{\mu=1}^3 v_1{}^\mu(\mathbf{x}) \frac{\partial}{\partial x^\mu} \qquad v_2 = \sum_{\mu=1}^3 v_2{}^\mu(\mathbf{x}) \frac{\partial}{\partial x^\mu} \qquad v_3 = \sum_{\mu=1}^3 v_3{}^\mu(\mathbf{x}) \frac{\partial}{\partial x^\mu}$$

be three smooth vector fields on $\mathit{\Omega}$ that are a frame at each point and satisfy

$$[v_\mathbf{j}, v_\mathbf{k}] = \beta_\mathbf{i}(\tau_*)\, v_\mathbf{i} \qquad \text{on } \mathit{\Omega}$$

for all $(\mathbf{i},\mathbf{j},\mathbf{k}) \in \mathcal{C} \overset{def}{=} \{(1,2,3),(2,3,1),(3,1,2)\}$. Introduce

$$\begin{aligned} e_0 &= e^{\zeta(\tau)} \tfrac{\partial}{\partial \tau} & e_\mathbf{i} &= e^{\zeta_\mathbf{i}(\tau)} v_\mathbf{i} & \mathbf{i} &= 1,2,3 \\ \zeta(\tau) &= \zeta_1(\tau) + \zeta_2(\tau) + \zeta_3(\tau) & \zeta_\mathbf{i}(\tau) &= -\tfrac{1}{2}\textstyle\int_{\tau_*}^{\tau} \mathrm{d}s\, \alpha_\mathbf{i}(s) & \mathbf{i} &= 1,2,3 \end{aligned}$$

on the domain $(\tau_0,\tau_1) \times \mathit{\Omega} \subset \mathbb{R}^4$. Then, the Lorentzian metric g with inverse

$$g^{-1} = -e_0 \otimes e_0 + e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

is a solution to the vacuum Einstein equations $\mathrm{Ric}(g) = 0$ on $(\tau_0,\tau_1) \times \mathit{\Omega}$.

Proof. In this proof, everywhere $(\mathbf{i},\mathbf{j},\mathbf{k}) \in \mathcal{C}$. It follows from $\frac{\mathrm{d}}{\mathrm{d}\tau} e^{-2\zeta_{\mathbf{i}}} = \alpha_{\mathbf{i}} e^{-2\zeta_{\mathbf{i}}}$ and $\zeta_{\mathbf{i}}(\tau_*) = 0$ and (1.1b) that $\beta_{\mathbf{i}}(\tau) = \beta_{\mathbf{i}}(\tau_*) e^{-2\zeta_{\mathbf{i}}(\tau)}$. Now, by direct calculation,

$$[e_0, e_{\mathbf{i}}] = -\tfrac{1}{2} e^{\zeta} \alpha_{\mathbf{i}} e_{\mathbf{i}} \qquad\qquad [e_{\mathbf{j}}, e_{\mathbf{k}}] = e^{\zeta} \beta_{\mathbf{i}} e_{\mathbf{i}}$$

Let ∇ be the Levi-Civita connection associated to g. Then, for all $a,b,c = 0,1,2,3$,

$$g\big(\nabla_{e_a} e_b,\, e_c\big) = \tfrac{1}{2}\Big(g\big([e_a,e_b],e_c\big) - g\big([e_b,e_c],e_a\big) + g\big([e_c,e_a],e_b\big)\Big)$$

By direct calculation,

$$\begin{aligned}
\nabla_{e_0} e_0 &= 0 & \nabla_{e_{\mathbf{i}}} e_{\mathbf{i}} &= \tfrac{1}{2} e^{\zeta} \alpha_{\mathbf{i}} e_0 \\
\nabla_{e_0} e_{\mathbf{i}} &= 0 & \nabla_{e_{\mathbf{j}}} e_{\mathbf{k}} &= \tfrac{1}{2} e^{\zeta} (+\beta_{\mathbf{i}} - \beta_{\mathbf{j}} + \beta_{\mathbf{k}}) e_{\mathbf{i}} \\
\nabla_{e_{\mathbf{i}}} e_0 &= \tfrac{1}{2} e^{\zeta} \alpha_{\mathbf{i}} e_{\mathbf{i}} & \nabla_{e_{\mathbf{k}}} e_{\mathbf{j}} &= \tfrac{1}{2} e^{\zeta} (-\beta_{\mathbf{i}} - \beta_{\mathbf{j}} + \beta_{\mathbf{k}}) e_{\mathbf{i}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Riem}(e_{\mathbf{i}}, e_{\mathbf{j}}, e_{\mathbf{i}}, e_{\mathbf{j}}) \\
&= \tfrac{1}{4} e^{2\zeta} \Big((+\beta_{\mathbf{i}} - \beta_{\mathbf{j}} - \beta_{\mathbf{k}})(+\beta_{\mathbf{i}} - \beta_{\mathbf{j}} + \beta_{\mathbf{k}}) + 2\beta_{\mathbf{k}}(+\beta_{\mathbf{i}} + \beta_{\mathbf{j}} - \beta_{\mathbf{k}}) + \alpha_{\mathbf{i}}\alpha_{\mathbf{j}} \Big) \\
&\mathrm{Riem}(e_0, e_{\mathbf{a}}, e_{\mathbf{i}}, e_{\mathbf{j}}) \\
&= \tfrac{1}{4} e^{2\zeta} \delta_{\mathbf{ak}} \Big((-\beta_{\mathbf{i}} + \beta_{\mathbf{j}} - \beta_{\mathbf{k}})\alpha_{\mathbf{i}} + (+\beta_{\mathbf{i}} - \beta_{\mathbf{j}} - \beta_{\mathbf{k}})\alpha_{\mathbf{j}} + 2\alpha_{\mathbf{k}}\beta_{\mathbf{k}} \Big) \\
&\mathrm{Riem}(e_0, e_{\mathbf{a}}, e_0, e_{\mathbf{i}}) \\
&= -\tfrac{1}{4} e^{2\zeta}\, \delta_{\mathbf{ai}} \Big(2\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{i}} - (\alpha_{\mathbf{j}} + \alpha_{\mathbf{k}})\alpha_{\mathbf{i}} \Big)
\end{aligned}$$

Furthermore, $\mathrm{Riem}(e_{\mathbf{a}}, e_{\mathbf{b}}, e_{\mathbf{c}}, e_{\mathbf{d}}) = 0$ unless $\{\mathbf{a},\mathbf{b}\} = \{\mathbf{c},\mathbf{d}\}$ with $\mathbf{a} \neq \mathbf{b}$. The Riemann curvature tensor is completely specified by these identities and by its algebraic symmetries. It follows that

$$\begin{aligned}
\mathrm{Ric}(e_0, e_0) &= -\tfrac{1}{2} e^{2\zeta}\, \tfrac{\mathrm{d}}{\mathrm{d}\tau}(\alpha_1 + \alpha_2 + \alpha_3) + \tfrac{1}{2} e^{2\zeta} (\alpha_2\alpha_3 + \alpha_3\alpha_1 + \alpha_1\alpha_2) \\
\mathrm{Ric}(e_0, e_{\mathbf{i}}) &= 0 \\
\mathrm{Ric}(e_{\mathbf{i}}, e_{\mathbf{i}}) &= +\tfrac{1}{2} e^{2\zeta} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{i}} + \tfrac{1}{2} e^{2\zeta} \big(+(\beta_{\mathbf{i}})^2 - (\beta_{\mathbf{j}})^2 - (\beta_{\mathbf{k}})^2 + 2\beta_{\mathbf{j}}\beta_{\mathbf{k}} \big) \\
\mathrm{Ric}(e_{\mathbf{j}}, e_{\mathbf{k}}) &= 0
\end{aligned}$$

The right hand sides of the first and third equation vanish by (1.1a) and (1.1c). □

Proposition 2.2. *In the context of Proposition 2.1, let $\gamma : (\tau_0', \tau_1') \to (\tau_0, \tau_1) \times \Omega$ be a smooth curve given by $\gamma(\tau) = (\tau, \gamma^\sharp(\tau))$, where $\gamma^\sharp$ is a curve on Ω. Let $g^\sharp$ be the Riemannian metric on Ω defined by $g^\sharp(v_{\mathbf{a}}, v_{\mathbf{b}}) = \delta_{\mathbf{ab}}$ for all $\mathbf{a},\mathbf{b} = 1,2,3$. If γ is non-spacelike with respect to g, then the length of $\gamma^\sharp$ with respect to $g^\sharp$ is bounded by*

$$\mathrm{Length}_{g^\sharp}(\gamma^\sharp) \;\leq\; \int_{\tau_0'}^{\tau_1'} \mathrm{d}\tau \max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} e^{-\zeta_{\mathbf{j}} - \zeta_{\mathbf{k}}} \tag{2.1}$$

The integral on the right hand side may be divergent.

Proof. Write the velocity $\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma$ as

$$\tfrac{\partial}{\partial\tau} + \textstyle\sum_{\mathbf{i}=1}^{3} X^{\mathbf{i}} v_{\mathbf{i}} = e^{-\zeta} e_0 + \sum_{\mathbf{i}=1}^{3} X^{\mathbf{i}} e^{-\zeta_{\mathbf{i}}} e_{\mathbf{i}}$$

with smooth coefficients $X^{\mathbf{i}} = X^{\mathbf{i}}(\tau)$. By assumption, γ is non-spacelike:

$$0 \geq g(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\gamma, \tfrac{\mathrm{d}}{\mathrm{d}\tau}\gamma) = -e^{-2\zeta} + \textstyle\sum_{\mathbf{i}=1}^{3} (X^{\mathbf{i}})^2 e^{-2\zeta_{\mathbf{i}}}$$

Consequently, $\sum_{\mathbf{i}=1}^{3} (X^{\mathbf{i}})^2 \leq \max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} e^{-2\zeta_{\mathbf{j}}-2\zeta_{\mathbf{k}}}$. Now, the claim follows from

$$\mathrm{Length}_{g^\sharp}(\gamma^\sharp) = \int_{\tau_0'}^{\tau_1'} \mathrm{d}\tau \, \sqrt{g^\sharp(\tfrac{\mathrm{d}}{\mathrm{d}\tau}\gamma^\sharp, \tfrac{\mathrm{d}}{\mathrm{d}\tau}\gamma^\sharp)} = \int_{\tau_0'}^{\tau_1'} \mathrm{d}\tau \, \sqrt{\textstyle\sum_{\mathbf{i}=1}^{3} (X^{\mathbf{i}})^2}$$

□

3. Construction of the transfer maps

Let $(\tau_0, \tau_1) \subset \mathbb{R}$ be a finite or infinite open interval, parametrized by $\tau \in (\tau_0, \tau_1)$ ("time"). In this paper, the unknown field is a vector valued map $\Phi \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6)$:

$$\Phi = \alpha[\Phi] \oplus \beta[\Phi] \; : \; (\tau_0, \tau_1) \to \mathbb{R}^3 \oplus \mathbb{R}^3 \tag{3.1}$$

If no confusion can arise, we just write $\Phi = \alpha \oplus \beta$.

Definition 3.1. *To every field $\Phi = \alpha \oplus \beta \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6)$, every constant $\mathbf{h} > 0$ and every $n \in \mathbb{R}^3$, associate a field*

$$\mathfrak{a}[\Phi, \mathbf{h}, n] \oplus \mathfrak{b}[\Phi, \mathbf{h}, n] \oplus c[\Phi, \mathbf{h}, n] \; : \; (\tau_0, \tau_1) \to \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}$$

by

$$\mathfrak{a}_{\mathbf{i}}[\Phi, \mathbf{h}, n] = -\mathbf{h}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{i}} - (n_{\mathbf{i}}\beta_{\mathbf{i}})^2 + (n_{\mathbf{j}}\beta_{\mathbf{j}} - n_{\mathbf{k}}\beta_{\mathbf{k}})^2 \tag{3.2a}$$

$$\mathfrak{b}_{\mathbf{i}}[\Phi, \mathbf{h}, n] = -\mathbf{h}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\beta_{\mathbf{i}} + \beta_{\mathbf{i}}\alpha_{\mathbf{i}} \tag{3.2b}$$

$$c[\Phi, \mathbf{h}, n] = \textstyle\sum_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} \left(\alpha_{\mathbf{j}}\alpha_{\mathbf{k}} - (n_{\mathbf{i}}\beta_{\mathbf{i}})^2 + 2n_{\mathbf{j}}n_{\mathbf{k}}\beta_{\mathbf{j}}\beta_{\mathbf{k}}\right) \tag{3.2c}$$

for all $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in \mathcal{C}$. For later use, it is convenient to introduce, for all $m, n \in \mathbb{R}^3$,

$$\begin{aligned} \mathfrak{a}_{\mathbf{i}}[\Phi, \mathbf{h}, n, m] &= \mathfrak{a}_{\mathbf{i}}[\Phi, \mathbf{h}, n] - \mathfrak{a}_{\mathbf{i}}[\Phi, \mathbf{h}, m] \\ &= -(n_{\mathbf{i}}\beta_{\mathbf{i}})^2 + (n_{\mathbf{j}}\beta_{\mathbf{j}} - n_{\mathbf{k}}\beta_{\mathbf{k}})^2 + (m_{\mathbf{i}}\beta_{\mathbf{i}})^2 - (m_{\mathbf{j}}\beta_{\mathbf{j}} - m_{\mathbf{k}}\beta_{\mathbf{k}})^2 \end{aligned} \tag{3.3}$$

Definition 3.2.

$$B_1 = (1, 0, 0) \qquad B_2 = (0, 1, 0) \qquad B_3 = (0, 0, 1) \qquad Z = (1, 1, 1)$$

These vectors will play the role of the vector $n \in \mathbb{R}^3$ that appears in Definition 3.1.

Proposition 3.1 (Global symmetries). *Let $\chi : (\tau_0, \tau_1) \to (\tau_0', \tau_1')$ be a linear diffeomorphism between finite or infinite intervals, $\chi(\tau) = p\tau + q$ with $p > 0$, and let $A > 0$ be a constant. Then*

$$(\mathfrak{a}, \mathfrak{b}, c)\Big[A\,(\Phi \circ \chi), \; \tfrac{1}{p}A\mathbf{h}, \; n\Big] = A^2\left((\mathfrak{a}, \mathfrak{b}, c)[\Phi, \mathbf{h}, n] \circ \chi\right)$$

for all fields $\Phi = \alpha \oplus \beta \in C^\infty((\tau_0', \tau_1'), \mathbb{R}^6)$, all constants $\mathbf{h} > 0$ and all $n \in \mathbb{R}^3$.

Corollary 3.1. *In Proposition 3.1, the field* $(\mathfrak{a}, \mathfrak{b}, c)[A\,(\Phi \circ \chi),\ \frac{1}{p}A\mathbf{h},\ n]$ *vanishes identically on* (τ_0, τ_1) *if and only if* $(\mathfrak{a}, \mathfrak{b}, c)[\Phi, \mathbf{h}, n]$ *vanishes identically on* (τ_0', τ_1').

Remark 3.1. The equations $(\mathfrak{a}, \mathfrak{b}, c)[\Phi, 1, Z] = 0$ are identical to (1.1). The equations $(\mathfrak{a}, \mathfrak{b}, c)[\Phi, \mathbf{h}, Z] = 0$ are equivalent to (1.1), for any $\mathbf{h} > 0$, by Corollary 3.1.

Proposition 3.2. *Recall Definition 3.1. For all* $\Phi = \alpha \oplus \beta \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6)$, *all* $\mathbf{h} > 0$ *and all* $n \in \mathbb{R}^3$, *we have*

$$0 = -\,\mathbf{h}\tfrac{\mathrm{d}}{\mathrm{d}\tau}c + \sum_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} \Big(-\,\alpha_{\mathbf{j}}\mathfrak{a}_{\mathbf{k}} - \alpha_{\mathbf{k}}\mathfrak{a}_{\mathbf{j}} + 2(n_{\mathbf{i}})^2\beta_{\mathbf{i}}\mathfrak{b}_{\mathbf{i}} - 2n_{\mathbf{j}}n_{\mathbf{k}}\beta_{\mathbf{j}}\mathfrak{b}_{\mathbf{k}} - 2n_{\mathbf{j}}n_{\mathbf{k}}\beta_{\mathbf{k}}\mathfrak{b}_{\mathbf{j}} \Big) \tag{3.4}$$

with $(\mathfrak{a}, \mathfrak{b}, c) = (\mathfrak{a}, \mathfrak{b}, c)[\Phi, \mathbf{h}, n]$. *In particular, if* $(\mathfrak{a}, \mathfrak{b}) = 0$ *identically on* (τ_0, τ_1), *then* c *vanishes identically on* (τ_0, τ_1) *if and only if* c *vanishes at one point of* (τ_0, τ_1).

Proof. Replace all occurrences of $\mathfrak{a}$, $\mathfrak{b}$ and c on the right hand side of (3.4) by the respective right hand sides of (3.2). Then, verify that everything cancels. □

Definition 3.3. *For all* $\mathbf{h} > 0$ *and all vectors* $\Phi = \alpha \oplus \beta \in \mathbb{R}^3 \oplus \mathbb{R}^3$ *with* $\beta_1, \beta_2, \beta_3 \neq 0$, *define* $A_{\mathbf{m}}[\Phi] \in (0, \infty)$ *and* $\varphi_{\mathbf{m}}[\Phi] \in \mathbb{R}$ *by*

$$\begin{aligned} A_{\mathbf{m}}[\Phi] &= \sqrt{|\alpha_{\mathbf{m}}|^2 + |\beta_{\mathbf{m}}|^2} \ > \ |\alpha_{\mathbf{m}}| \ \geq \ 0 \\ \varphi_{\mathbf{m}}[\Phi] &= -\operatorname{arcsinh} \tfrac{\alpha_{\mathbf{m}}}{|\beta_{\mathbf{m}}|} \end{aligned}$$

for all $\mathbf{m} = 1, 2, 3$. *Equivalently,*

$$\alpha_{\mathbf{m}} = \qquad\qquad -\,A_{\mathbf{m}}[\Phi]\ \tanh\varphi_{\mathbf{m}}[\Phi] \tag{3.5a}$$

$$\beta_{\mathbf{m}} = (\operatorname{sgn}\beta_{\mathbf{m}})\ A_{\mathbf{m}}[\Phi]\ \operatorname{sech}\,\varphi_{\mathbf{m}}[\Phi] \tag{3.5b}$$

Furthermore, define $\xi_{\mathbf{m}}[\Phi, \mathbf{h}] \in \mathbb{R}$ *by*

$$\xi_{\mathbf{m}}[\Phi, \mathbf{h}] = \mathbf{h}\ \log\big|\tfrac{1}{2}\beta_{\mathbf{m}}\big|$$

for all $\mathbf{m} = 1, 2, 3$. *Furthermore, for all* $\mathbf{m}, \mathbf{n} = 1, 2, 3$, *introduce the abbreviations*

$$\alpha_{\mathbf{m},\mathbf{n}}[\Phi] = \alpha_{\mathbf{m}} + \alpha_{\mathbf{n}} \qquad\qquad \xi_{\mathbf{m},\mathbf{n}}[\Phi, \mathbf{h}] = \xi_{\mathbf{m}}[\Phi, \mathbf{h}] + \xi_{\mathbf{n}}[\Phi, \mathbf{h}]$$

If no confusion can arise, we drop the explicit dependence $[\Phi]$ *or* $[\Phi, \mathbf{h}]$. *For instance, we write* $A_{\mathbf{m}} = A_{\mathbf{m}}[\Phi]$. *If* Φ *is not an element of* $\mathbb{R}^3 \oplus \mathbb{R}^3$, *but rather a function of the real variable* τ *with values in* $\mathbb{R}^3 \oplus \mathbb{R}^3$, *with* $\beta_1, \beta_2, \beta_3 \neq 0$ *everywhere, then* $A_{\mathbf{m}}, \varphi_{\mathbf{m}}$, $\xi_{\mathbf{m}}$, $\xi_{\mathbf{m},\mathbf{n}}$, $\alpha_{\mathbf{m},\mathbf{n}}$, *with* $\mathbf{m}, \mathbf{n} = 1, 2, 3$, *are functions of* τ, *too. In this case, we define the additional functions* $\theta_{\mathbf{m}}[\Phi, \mathbf{h}]$, $\mathbf{m} = 1, 2, 3$, *through*

$$\varphi_{\mathbf{m}}[\Phi](\tau) = \tfrac{1}{\mathbf{h}}\big(\tau - \theta_{\mathbf{m}}[\Phi, \mathbf{h}](\tau)\big)\ A_{\mathbf{m}}[\Phi](\tau)$$

Remark 3.2. In the context of Definition 3.3, we have, for all $\mathbf{m} = 1, 2, 3$:

$$\mathbf{h}\,|\varphi_{\mathbf{m}}| = -\xi_{\mathbf{m}} + \mathbf{h}\ \log\Big(|\tfrac{1}{2}\alpha_{\mathbf{m}}| + \sqrt{|\tfrac{1}{2}\alpha_{\mathbf{m}}|^2 + \exp(\tfrac{1}{\mathbf{h}}2\xi_{\mathbf{m}})}\ \Big)$$

Lemma 3.1. *Recall Definitions 3.1, 3.2, 3.3. For all* $\mathbf{h} > 0$ *and all* $\varPhi = \alpha \oplus \beta \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6)$ *such that* β_1, β_2, β_3 *never vanish on* (τ_0, τ_1), *we have*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} A_{\mathbf{i}} \\ \theta_{\mathbf{i}} \end{pmatrix} = \frac{1}{(A_{\mathbf{i}})^2}\begin{pmatrix} \frac{1}{\mathbf{h}}(A_{\mathbf{i}})^2 \tanh\varphi_{\mathbf{i}} & \frac{1}{\mathbf{h}}(A_{\mathbf{i}})^2 \operatorname{sech}\varphi_{\mathbf{i}} \\ \varphi_{\mathbf{i}} \tanh\varphi_{\mathbf{i}} - 1 & \sinh\varphi_{\mathbf{i}} + \varphi_{\mathbf{i}} \operatorname{sech}\varphi_{\mathbf{i}} \end{pmatrix}\begin{pmatrix} \mathfrak{a}_{\mathbf{i}}[\varPhi, \mathbf{h}, B_{\mathbf{i}}] \\ -\sigma_{\mathbf{i}}\, \mathfrak{b}_{\mathbf{i}}[\varPhi, \mathbf{h}, B_{\mathbf{i}}] \end{pmatrix}$$

for $\mathbf{i} = 1, 2, 3$ *and* $\sigma_{\mathbf{i}} = \operatorname{sgn}\beta_{\mathbf{i}} \in \{-1, +1\}$. *The matrix on the right hand side has determinant* $\frac{1}{\mathbf{h}}(A_{\mathbf{i}})^2 \cosh\varphi_{\mathbf{i}} \neq 0$.

Proof. We have $\mathfrak{a}_{\mathbf{i}}[\varPhi, \mathbf{h}, B_{\mathbf{i}}] = -\mathbf{h}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{i}} - (\beta_{\mathbf{i}})^2$ and $\mathfrak{b}_{\mathbf{i}}[\varPhi, \mathbf{h}, B_{\mathbf{i}}] = -\mathbf{h}\frac{\mathrm{d}}{\mathrm{d}\tau}\beta_{\mathbf{i}} + \alpha_{\mathbf{i}}\beta_{\mathbf{i}}$. Replace all occurrences of $\alpha_{\mathbf{i}}$ and $\beta_{\mathbf{i}}$ by the right hand sides of (3.5), respectively. Use $\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_{\mathbf{i}} = \frac{1}{A_{\mathbf{i}}}(\frac{\mathrm{d}}{\mathrm{d}\tau}A_{\mathbf{i}})\varphi_{\mathbf{i}} + \frac{1}{\mathbf{h}}A_{\mathbf{i}}(1 - \frac{\mathrm{d}}{\mathrm{d}\tau}\theta_{\mathbf{i}})$. Now, solve for $\frac{\mathrm{d}}{\mathrm{d}\tau}A_{\mathbf{i}}$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\theta_{\mathbf{i}}$. □

Remark 3.3. So far, we have stated all definitions and propositions for a C^∞-field $\varPhi = \alpha \oplus \beta$, defined on an open interval. This was just for convenience. We will, from now on, use these definitions and propositions even when the C^∞-requirement is not met, or when the field is defined on, say, a closed interval rather than an open interval. It will be clear in each case, that the respective definition or proposition still makes sense.

Definition 3.4. *Set* $S_3 = \{(1,2,3), (2,3,1), (3,1,2), (3,2,1), (1,3,2), (2,1,3)\}$, *the set of all permutations of* $(1, 2, 3)$.

Definition 3.5. *For all* $\sigma_* \in \{-1, +1\}^3$ *let* $\mathcal{D}(\sigma_*)$ *be the set of all* $\varPhi = \alpha \oplus \beta \in \mathbb{R}^3 \oplus \mathbb{R}^3$ *with* $\big(\operatorname{sgn}\beta_1, \operatorname{sgn}\beta_2, \operatorname{sgn}\beta_3\big) = \sigma_*$. *For all* $\tau_0, \tau_1 \in \mathbb{R}$ *with* $\tau_0 < \tau_1$ *let* $\mathcal{E}(\sigma_*; \tau_0, \tau_1)$ *be the set of all continuous maps* $\varPhi : [\tau_0, \tau_1] \to \mathcal{D}(\sigma_*)$.

Definition 3.6. *For all* $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$ *and* $\mathbf{h} > 0$ *and* $\sigma_* \in \{-1, +1\}^3$ *define two functions* $\mathcal{D}(\sigma_*) \times \mathcal{D}(\sigma_*) \to [0, \infty)$ *by*

$$\begin{aligned} d_{\mathcal{D}(\sigma_*),(\pi,\mathbf{h})}(\varPhi, \varPsi) = \max\Big\{ & \big|A_{\mathbf{a}}[\varPhi] - A_{\mathbf{a}}[\varPsi]\big| \;,\; \Big|\mathbf{h}\tfrac{\varphi_{\mathbf{a}}[\varPhi]}{A_{\mathbf{a}}[\varPhi]} - \mathbf{h}\tfrac{\varphi_{\mathbf{a}}[\varPsi]}{A_{\mathbf{a}}[\varPsi]}\Big| \;, \\ & \big|\alpha_{\mathbf{b},\mathbf{a}}[\varPhi] - \alpha_{\mathbf{b},\mathbf{a}}[\varPsi]\big| \;,\; \big|\xi_{\mathbf{b},\mathbf{a}}[\varPhi, \mathbf{h}] - \xi_{\mathbf{b},\mathbf{a}}[\varPsi, \mathbf{h}]\big|, \\ & \big|\alpha_{\mathbf{c},\mathbf{a}}[\varPhi] - \alpha_{\mathbf{c},\mathbf{a}}[\varPsi]\big| \;,\; \big|\xi_{\mathbf{c},\mathbf{a}}[\varPhi, \mathbf{h}] - \xi_{\mathbf{c},\mathbf{a}}[\varPsi, \mathbf{h}]\big| \;\Big\} \end{aligned}$$

and

$$\not{d}_{\mathcal{D}(\sigma_*),\mathbf{h}}(\varPhi, \varPsi) = \max_{\mathbf{i}=1,2,3}\Big\{\big|\alpha_{\mathbf{i}}[\varPhi] - \alpha_{\mathbf{i}}[\varPsi]\big|,\ \big|\xi_{\mathbf{i}}[\varPhi, \mathbf{h}] - \xi_{\mathbf{i}}[\varPsi, \mathbf{h}]\big|\Big\}$$

Then $(\mathcal{D}(\sigma_*), d_{\mathcal{D}(\sigma_*),(\pi,\mathbf{h})})$ *and* $(\mathcal{D}(\sigma_*), \not{d}_{\mathcal{D}(\sigma_*),\mathbf{h}})$ *are metric spaces.*

Definition 3.7. *For all* $\pi \in S_3$ *and* $\mathbf{h} > 0$ *and* $\sigma_* \in \{-1, +1\}^3$ *and* $\tau_0, \tau_1 \in \mathbb{R}$ *with* $\tau_0 < \tau_1$ *define a function* $\mathcal{E}(\sigma_*; \tau_0, \tau_1) \times \mathcal{E}(\sigma_*; \tau_0, \tau_1) \to [0, \infty)$ *by*

$$d_{\mathcal{E}(\sigma_*;\tau_0,\tau_1),(\pi,\mathbf{h})}(\varPhi, \varPsi) = \sup_{\tau \in [\tau_0, \tau_1]} d_{\mathcal{D}(\sigma_*),(\pi,\mathbf{h})}(\varPhi(\tau), \varPsi(\tau))$$

Then $(\mathcal{E}(\sigma_*; \tau_0, \tau_1), d_{\mathcal{E}(\sigma_*;\tau_0,\tau_1),(\pi,\mathbf{h})})$ *is a metric space.*

Lemma 3.2. *Let* $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$ *and* $\mathbf{h} > 0$ *and* $\sigma_* \in \{-1, +1\}^3$. *Suppose* $\mathbf{h} \leq 1$. *Let* $C, D \geq 1$ *be constants. Then, for all* $\varPhi, \varPsi \in \mathcal{D}(\sigma_*)$ *such that*

$$C^{-1} \leq A_{\mathbf{a}}[X] \leq C \qquad\qquad D^{-1} \leq \mathbf{h}\,|\varphi_{\mathbf{a}}[X]| \leq D$$

for both $X = \varPhi$ *and* $X = \varPsi$ *and such that* $\operatorname{sgn}\varphi_{\mathbf{a}}[\varPhi] = \operatorname{sgn}\varphi_{\mathbf{a}}[\varPsi]$, *we have:*

(a) $\not{d}_{\mathcal{D}}(\Phi,\Psi) \leq 2^3 C^2 D\, d_{\mathcal{D}}(\Phi,\Psi)$
(b) If $\exp(-\frac{1}{\mathbf{h}} C^{-2} D^{-1}) \leq 2^{-6} C^{-4} D^{-2}$*, then* $d_{\mathcal{D}}(\Phi,\Psi) \leq 2^5 C^3 D\, \not{d}_{\mathcal{D}}(\Phi,\Psi)$

Here, $d_{\mathcal{D}} = d_{\mathcal{D}(\sigma_*),(\pi,\mathbf{h})}$ *and* $\not{d}_{\mathcal{D}} = \not{d}_{\mathcal{D}(\sigma_*),\mathbf{h}}$.

Proof. In this proof, A, B, α, ξ play the roles of $A_{\mathbf{a}}$, $\mathbf{h}\varphi_{\mathbf{a}}/A_{\mathbf{a}}$, $\alpha_{\mathbf{a}}$, $\xi_{\mathbf{a}}$, respectively. To show (a), let $P : (0,\infty)\times\mathbb{R} \to \mathbb{R}^2,\ (A,B) \mapsto (\alpha(A,B), \xi(A,B))$, where

$$\alpha(A,B) = -A\tanh(\tfrac{1}{\mathbf{h}}AB) \qquad \xi(A,B) = \mathbf{h}\log(\tfrac{1}{2}A\,\mathrm{sech}(\tfrac{1}{\mathbf{h}}AB))$$

This is a diffeomorphism. The Jacobian J of P is given by

$$J = \begin{pmatrix} \frac{\partial\alpha}{\partial A} & \frac{\partial\alpha}{\partial B} \\ \frac{\partial\xi}{\partial A} & \frac{\partial\xi}{\partial B} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\mathbf{h}}AB\,\mathrm{sech}^2(\frac{1}{\mathbf{h}}AB) - \tanh(\frac{1}{\mathbf{h}}AB) & -\frac{1}{\mathbf{h}}A^2\,\mathrm{sech}^2(\frac{1}{\mathbf{h}}AB) \\ \frac{\mathbf{h}}{A} - B\tanh(\frac{1}{\mathbf{h}}AB) & -A\tanh(\frac{1}{\mathbf{h}}AB) \end{pmatrix}$$

Let $p_i = (A_i, B_i) \in (0,\infty)\times\mathbb{R}$ and set $(\alpha_i,\xi_i) = P(p_i)$, where $i = 0,1$. Set $\gamma(t) = (A(t), B(t)) = (1-t)p_0 + tp_1$ where $t \in [0,1]$. We have

$$\left(\begin{smallmatrix} \alpha_1-\alpha_0 \\ \xi_1-\xi_0 \end{smallmatrix}\right) = M\left(\begin{smallmatrix} A_1-A_0 \\ B_1-B_0 \end{smallmatrix}\right) \qquad \text{with} \qquad M = \left(\begin{smallmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{smallmatrix}\right) = \int_0^1 \mathrm{d}t\, J(\gamma(t))$$

Suppose $C^{-1} \leq A_i \leq C$ and $(CD)^{-1} \leq |B_i| \leq CD$ and $\mathrm{sgn}\, B_0 = \mathrm{sgn}\, B_1$. Then, $C^{-1} \leq A(t) \leq C$ and $(CD)^{-1} \leq |B(t)| \leq CD$ for all $t \in [0,1]$. Observe that $|\varphi\,\mathrm{sech}^2\varphi| \leq \frac{1}{2}$ for all $\varphi \in \mathbb{R}$. We have $|M_{ij}| \leq 2C^2 D$ for all $i,j \in \{0,1\}$. This implies (a).
We show that under the assumptions of (b), we have $|\det M| \geq 2^{-3}C^{-1}$, and therefore $|(M^{-1})_{ij}| \leq 2^4 C^3 D$ for all $i,j \in \{0,1\}$. This would imply (b). We have $|\det M| \geq |M_{00}M_{11}| - |M_{01}M_{10}|$. Set $\varphi(t) = \frac{1}{\mathbf{h}}A(t)B(t)$. We have $|\varphi(t)| \geq \frac{1}{\mathbf{h}}C^{-2}D^{-1}$. By the assumption of (b), we have $e^{-|\varphi(t)|} \leq 2^{-6}C^{-4}D^{-2}$, for all $t \in [0,1]$. We will also use the general inequalities $0 \leq 1 - \tanh|\varphi| \leq 2e^{-2|\varphi|}$ and $|\varphi\,\mathrm{sech}^2\varphi| \leq 4|\varphi|e^{-2|\varphi|} \leq 4e^{-|\varphi|}$. We have $|-\varphi\,\mathrm{sech}^2\varphi - \tanh\varphi| \geq \tanh|\varphi| = 1 - (1-\tanh|\varphi|) \geq 2^{-1}$. The last inequality holds for all $t \in [0,1]$ and implies $|M_{00}| \geq 2^{-1}$, because φ has constant sign. We have $|M_{11}| \geq 2^{-1}C^{-1}$ and $|M_{10}| \leq 2CD$ and $|M_{01}| \leq 2^{-4}C^{-2}D^{-1}$. This implies $|\det M| \geq 2^{-3}C^{-1}$. □

Definition 3.8. *Let* $\mathcal{X} = \mathcal{D}(\sigma_*)$ *or* $\mathcal{X} = \mathcal{E}(\sigma_*;\tau_0,\tau_1)$. *For all* $\delta \geq 0$ *and* $\Phi \in \mathcal{X}$ *and* $\pi \in S_3$ *and* $\mathbf{h} > 0$, *set* $B_{\mathcal{X},(\pi,\mathbf{h})}[\delta,\Phi] = \{\Psi \in \mathcal{X} \mid d_{\mathcal{X},(\pi,\mathbf{h})}(\Phi,\Psi) \leq \delta\}$.

Definition 3.9 (The reference field Φ_0). *For all* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3,\ \mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3,\ \sigma_* \in \{-1,+1\}^3$ *let* $\Phi_0 = \Phi_0(\pi,\mathbf{f},\sigma_*) : \mathbb{R} \to \mathcal{D}(\sigma_*)$ *be given by*

$$A_{\mathbf{a}}[\Phi_0](\tau) = 1 \tag{3.6a}$$
$$\theta_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau) = 0 \tag{3.6b}$$
$$\alpha_{\mathbf{b},\mathbf{a}}[\Phi_0](\tau) = -(1+w)^{-1} \tag{3.6c}$$
$$\alpha_{\mathbf{c},\mathbf{a}}[\Phi_0](\tau) = -(1+w) \tag{3.6d}$$
$$\xi_{\mathbf{b},\mathbf{a}}[\Phi_0,\mathbf{h}](\tau) = -1 - \mathbf{h}\log 2 - (1+w)^{-1}\,\tau \tag{3.6e}$$
$$\xi_{\mathbf{c},\mathbf{a}}[\Phi_0,\mathbf{h}](\tau) = -(1+w)q - \mathbf{h}\log 2 - (1+w)\,\tau \tag{3.6f}$$

(see Definition 3.3) for all $\tau \in \mathbb{R}$.

Remark 3.4. The field Φ_0 is, up to renaming, given by equation (3.12) in [BKL1].

Lemma 3.3. *Let Φ_0 be as in Definition 3.9. Then* $(\mathfrak{a}, \mathfrak{b}, c)[\Phi_0, \mathbf{h}, B_\mathbf{a}] = 0$ *on* $\mathbb{R}$.

Proof. Let $\alpha = \alpha[\Phi_0]$, $\beta = \beta[\Phi_0]$, $\xi = \xi[\Phi_0, \mathbf{h}]$. We have $(\mathfrak{a}_\mathbf{a}, \mathfrak{b}_\mathbf{a})[\Phi_0, \mathbf{h}, B_\mathbf{a}] = 0$ by Lemma 3.1. For $\mathbf{p} \in \{\mathbf{b}, \mathbf{c}\}$, we have $\mathfrak{a}_\mathbf{a}[\Phi_0, \mathbf{h}, B_\mathbf{a}] + \mathfrak{a}_\mathbf{p}[\Phi_0, \mathbf{h}, B_\mathbf{a}] = -\mathbf{h}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{a},\mathbf{p}} = 0$, that is $\mathfrak{a}_\mathbf{p}[\Phi_0, \mathbf{h}, B_\mathbf{a}] = 0$. We also have $\beta_\mathbf{a}^{-1}\mathfrak{b}_\mathbf{a}[\Phi_0, \mathbf{h}, B_\mathbf{a}] + \beta_\mathbf{p}^{-1}\mathfrak{b}_\mathbf{p}[\Phi_0, \mathbf{h}, B_\mathbf{a}] = -\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_{\mathbf{a},\mathbf{p}} + \alpha_{\mathbf{a},\mathbf{p}} = 0$, that is $\mathfrak{b}_\mathbf{p}[\Phi_0, \mathbf{h}, B_\mathbf{a}] = 0$. Finally, $c[\Phi_0, \mathbf{h}, B_\mathbf{a}] = -\alpha_\mathbf{a}^2 - \beta_\mathbf{a}^2 + \alpha_{\mathbf{a},\mathbf{b}}\alpha_{\mathbf{a},\mathbf{c}} = -A_\mathbf{a}^2 + \alpha_{\mathbf{a},\mathbf{b}}\alpha_{\mathbf{a},\mathbf{c}} = 0$. Here, $A_\mathbf{a} = A_\mathbf{a}[\Phi_0]$. □

Definition 3.10. *For all* $\mathbf{f} = (\mathbf{h}, w, q) \in (0, \infty)^3$ *set*

$$\begin{aligned} \tau_-(\mathbf{f}) &= -\big(1 - \tfrac{1}{2+w}\big)\, \min\{1, q\} \qquad &< 0 \\ \tau_+(\mathbf{f}) &= 1 + \tfrac{1}{w} &> 0 \end{aligned}$$

Lemma 3.4 (Technical Lemma 1). *Let* $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$, $\mathbf{f} = (\mathbf{h}, w, q) \in (0, \infty)^3$, $\sigma_* \in \{-1, +1\}^3$ *and fix* $\delta > 0$, $\epsilon_- \in (0, -\tau_-)$, $\epsilon_+ \in (0, \tau_+)$ *where* $\tau_\pm = \tau_\pm(\mathbf{f})$. *Set*

$$\tau_{0-} = \tau_- + \epsilon_- < 0 \qquad \Phi_0 = \Phi_0(\pi, \mathbf{f}, \sigma_*)\big|_{[\tau_{0-}, \tau_{0+}]} \tag{3.7a}$$

$$\tau_{0+} = \tau_+ - \epsilon_+ > 0 \qquad \mathcal{E} = \mathcal{E}(\sigma_*; \tau_{0-}, \tau_{0+}) \tag{3.7b}$$

Then $\Phi_0 \in \mathcal{E}$. *Furthermore, if the inequality*

$$\delta \le 2^{-4} \min\big\{1,\, w,\, \epsilon_-,\, \tfrac{\epsilon_+}{\tau_+\tau_{0+}}\big\} \tag{3.8}$$

holds, then for all $\Phi = \alpha \oplus \beta \in B_{\mathcal{E},(\pi,\mathbf{h})}[\delta, \Phi_0]$ *the estimates*

$$\begin{aligned} \max\big\{|\beta_\mathbf{b}|^2,\, |\beta_\mathbf{c}|^2,\, |\beta_\mathbf{b}\beta_\mathbf{a}|,\, |\beta_\mathbf{c}\beta_\mathbf{a}|\big\} &\le 2^4 \exp\big(-\tfrac{1}{4\mathbf{h}}\min\{1,\, \epsilon_-,\, \tfrac{\epsilon_+}{\tau_+}\}\big) \\ |A_\mathbf{a}[\Phi] - 1| &\le 2^{-1} \\ |\varphi_\mathbf{a}[\Phi]| &\le \tfrac{1}{\mathbf{h}}2(1 + |\tau|) \\ |\beta_\mathbf{a}| &\le 2 \end{aligned}$$

hold on $[\tau_{0-}, \tau_{0+}]$.

Proof. The following estimates hold for the components of Φ, for all $\tau \in [\tau_{0-}, \tau_{0+}]$:

$$\begin{aligned} |\beta_\mathbf{b}\beta_\mathbf{a}| &= 4\exp\big(\tfrac{1}{\mathbf{h}}\xi_{\mathbf{b},\mathbf{a}}\big) \\ &\le 4\exp\big(\tfrac{1}{\mathbf{h}}\xi_{\mathbf{b},\mathbf{a}}[\Phi_0, \mathbf{h}] + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}} - \tfrac{1}{\mathbf{h}}(1+w)^{-1}\tau + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}} - \tfrac{1}{\mathbf{h}}(1+w)^{-1}\tau_- + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}} + \tfrac{1}{\mathbf{h}}(2+w)^{-1} + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{4\mathbf{h}}\big) \\ |\beta_\mathbf{c}\beta_\mathbf{a}| &= 4\exp\big(\tfrac{1}{\mathbf{h}}\xi_{\mathbf{c},\mathbf{a}}\big) \\ &\le 4\exp\big(\tfrac{1}{\mathbf{h}}\xi_{\mathbf{c},\mathbf{a}}[\Phi_0, \mathbf{h}] + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q - \tfrac{1}{\mathbf{h}}(1+w)\tau + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q - \tfrac{1}{\mathbf{h}}(1+w)(\tau_- + \epsilon_-) + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q + \tfrac{1}{\mathbf{h}}\tfrac{(1+w)^2}{2+w}q - \tfrac{1}{\mathbf{h}}\epsilon_- + \tfrac{1}{\mathbf{h}}\delta\big) \\ &\le 2\exp\big(-\tfrac{1}{2\mathbf{h}}\epsilon_-\big) \end{aligned}$$

$$
\begin{aligned}
|\varphi_{\mathbf{a}}| &= \tfrac{1}{\mathbf{h}} A_{\mathbf{a}} |\tau - \theta_{\mathbf{a}}| \\
&\le \tfrac{1}{\mathbf{h}} (1+\delta)\big(|\tau| + \delta\big) \\
&\le \tfrac{1}{\mathbf{h}} \big(|\tau| + \delta|\tau| + 2\delta\big) \\
&\le \tfrac{1}{\mathbf{h}} 2(1+|\tau|) \\
|\beta_{\mathbf{a}}|^{-1} &= |A_{\mathbf{a}}|^{-1} \cosh \varphi_{\mathbf{a}} \\
&\le 2\exp\big(|\varphi_{\mathbf{a}}|\big) \\
&\le 2\exp\big(\tfrac{1}{\mathbf{h}}|\tau| + \tfrac{1}{\mathbf{h}}\delta|\tau| + \tfrac{1}{\mathbf{h}} 2\delta\big) \\
|\beta_{\mathbf{b}}| &= |\beta_{\mathbf{b}}\beta_{\mathbf{a}}| \cdot |\beta_{\mathbf{a}}|^{-1} \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}} - \tfrac{1}{\mathbf{h}}(1+w)^{-1}\tau + \tfrac{1}{\mathbf{h}}|\tau| + \tfrac{1}{\mathbf{h}}\delta|\tau| + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(\tfrac{1}{\mathbf{h}}\max\big\{-1 - \tfrac{2+w}{1+w}\tau_{0-} - \delta\tau_{0-},\, -1 + \tfrac{w}{1+w}\tau_{0+} + \delta\tau_{0+}\big\} + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(\tfrac{1}{\mathbf{h}}\max\{-\epsilon_- - \delta\tau_{0-}, -\tfrac{\epsilon_+}{\tau_+} + \delta\tau_{0+}\} + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}\tfrac{15}{16}\min\{\epsilon_-, \tfrac{\epsilon_+}{\tau_+}\} + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{2\mathbf{h}}\min\{\epsilon_-, \tfrac{\epsilon_+}{\tau_+}\}\big)
\end{aligned}
$$

The last step uses $\delta \le 2^{-3}\frac{\epsilon_+}{\tau_+}$. In the case $\epsilon_+ \ge \frac{1}{2}\tau_+$, this follows from $\delta \le 2^{-4}$. If $\epsilon_+ \le \frac{1}{2}\tau_+$, then this follows from $\delta \le 2^{-4}\frac{\epsilon_+}{\tau_+\tau_{0+}}$, because $\tau_{0+} = \tau_+ - \epsilon_+ \ge \frac{1}{2}\tau_+ \ge \frac{1}{2}$.

$$
\begin{aligned}
|\beta_{\mathbf{c}}| &= |\beta_{\mathbf{c}}\beta_{\mathbf{a}}| \cdot |\beta_{\mathbf{a}}|^{-1} \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q - \tfrac{1}{\mathbf{h}}(1+w)\tau + \tfrac{1}{\mathbf{h}}|\tau| + \tfrac{1}{\mathbf{h}}\delta|\tau| + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q \\
&\qquad\qquad + \tfrac{1}{\mathbf{h}}\max\big\{-(2+w+\delta)\tau_{0-}, -(w-\delta)\tau_{0+}\big\} + \tfrac{1}{\mathbf{h}}3\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}(1+w)q + \tfrac{1}{\mathbf{h}}(2+w)|\tau_-| - \tfrac{1}{\mathbf{h}}(2+w+\delta)\epsilon_- + \tfrac{1}{\mathbf{h}}4\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}2\epsilon_- + \tfrac{1}{\mathbf{h}}4\delta\big) \\
&\le 4\exp\big(-\tfrac{1}{\mathbf{h}}\epsilon_-\big)
\end{aligned}
$$

This concludes the proof. □

Lemma 3.5. *Recall Definitions 3.1 and 3.2. We have*

$$
\begin{aligned}
\mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, Z, B_{\mathbf{a}}] &= +\beta_{\mathbf{b}}^2 + \beta_{\mathbf{c}}^2 - 2\beta_{\mathbf{b}}\beta_{\mathbf{c}} \\
\mathfrak{a}_{\mathbf{b}}[\Phi, \mathbf{h}, Z, B_{\mathbf{a}}] &= -\beta_{\mathbf{b}}^2 + \beta_{\mathbf{c}}^2 - 2\beta_{\mathbf{a}}\beta_{\mathbf{c}} \\
\mathfrak{a}_{\mathbf{c}}[\Phi, \mathbf{h}, Z, B_{\mathbf{a}}] &= +\beta_{\mathbf{b}}^2 - \beta_{\mathbf{c}}^2 - 2\beta_{\mathbf{a}}\beta_{\mathbf{b}}
\end{aligned}
$$

for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$.

Remark 3.5. Lemma 3.5 displays the differences between the equations $\mathfrak{a}[\Phi, \mathbf{h}, Z] = 0$ and $\mathfrak{a}[\Phi, \mathbf{h}, B_{\mathbf{a}}] = 0$. Lemma 3.4 gives bounds for the terms that appear in these differences. Informally, they tend exponentially to zero as $\mathbf{h} \downarrow 0$. This quantifies a basic guiding intuition of [BKL1].

Definition 3.11. *For all vectors $\mathit{\Phi} = \alpha \oplus \beta \in \mathbb{R}^3 \oplus \mathbb{R}^3$ with $\beta_1, \beta_2, \beta_3 \neq 0$, all $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$ and all $\mathbf{h} > 0$, define four real numbers by*

$$\begin{aligned} \mathbf{I}_1[\mathit{\Phi}, \mathbf{h}, \pi] &= -\tfrac{1}{\mathbf{h}}\, \mathfrak{a}_{\mathbf{a}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}]\ \tanh \varphi_{\mathbf{a}}[\mathit{\Phi}] \\ \mathbf{I}_2[\mathit{\Phi}, \mathbf{h}, \pi] &= \big(A_{\mathbf{a}}[\mathit{\Phi}]\big)^{-2}\, \mathfrak{a}_{\mathbf{a}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}]\ \Big(1 - \varphi_{\mathbf{a}}[\mathit{\Phi}] \tanh \varphi_{\mathbf{a}}[\mathit{\Phi}]\Big) \\ \mathbf{I}_{(3,\mathbf{p})}[\mathit{\Phi}, \mathbf{h}, \pi] &= \tfrac{1}{\mathbf{h}} \mathfrak{a}_{\mathbf{p}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}] + \tfrac{1}{\mathbf{h}} \mathfrak{a}_{\mathbf{a}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}] \end{aligned}$$

where $\mathbf{p} \in \{\mathbf{b}, \mathbf{c}\}$. If $\mathit{\Phi}$ is not an element of $\mathbb{R}^3 \oplus \mathbb{R}^3$, but rather a function with values in $\mathbb{R}^3 \oplus \mathbb{R}^3$, with $\beta_1, \beta_2, \beta_3 \neq 0$ everywhere, then $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_{(3,\mathbf{b})}$, $\mathbf{I}_{(3,\mathbf{c})}$ are functions, too.

Lemma 3.6 (Technical Lemma 2). *In the context of Lemma 3.4, if $\delta > 0$ satisfies* (3.8), *then, for all $\mathit{\Phi}, \mathit{\Psi} \in B_{\mathcal{E},(\pi,\mathbf{h})}[\delta, \mathit{\Phi}_0]$ and all $S \in \{1, 2, (3, \mathbf{b}), (3, \mathbf{c})\}$, the estimates*

$$\big|\mathbf{I}_S[\mathit{\Phi}]\big| \ \leq\ 2^{11} \max\{1, \tfrac{1}{\mathbf{h}}, \tfrac{1}{\mathbf{h}}|\tau|\}\ \exp\big(- \tfrac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \tfrac{\epsilon_+}{\tau_+}\}\big) \tag{3.9a}$$

$$\big|\mathbf{I}_S[\mathit{\Phi}] - \mathbf{I}_S[\mathit{\Psi}]\big| \ \leq\ 2^{17}\big(\max\{1, \tfrac{1}{\mathbf{h}}, \tfrac{1}{\mathbf{h}}|\tau|\}\big)^2 \exp\big(- \tfrac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \tfrac{\epsilon_+}{\tau_+}\}\big) d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \tag{3.9b}$$

hold on $[\tau_{0-}, \tau_{0+}]$. Here, $\mathbf{I}_S[\mathit{\Phi}] = \mathbf{I}_S[\mathit{\Phi}, \mathbf{h}, \pi]$, $\mathbf{I}_S[\mathit{\Psi}] = \mathbf{I}_S[\mathit{\Psi}, \mathbf{h}, \pi]$ and $d_{\mathcal{E}} = d_{\mathcal{E},(\pi,\mathbf{h})}$.

Proof. In this proof, we simplify the notation by suppressing $\mathbf{h} > 0$ and abbreviating

$$M = \exp(-\tfrac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \tfrac{\epsilon_+}{\tau_+}\}) \qquad M_1 = \max\{1, \tfrac{1}{\mathbf{h}}, \tfrac{1}{\mathbf{h}}|\tau|\}$$

Lemmas 3.4, 3.5 imply $\big|\mathfrak{a}_{\mathbf{i}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}]\big| \leq 2^6 M$, $\mathbf{i} = 1, 2, 3$, and $|\varphi_{\mathbf{a}}[\mathit{\Phi}]| \leq 2^2 M_1$ and $(A_{\mathbf{a}}[\mathit{\Phi}])^{-2} \leq 2^2$. This implies (3.9a). To show (3.9b), observe that (here $\mathbf{p}, \mathbf{q} \in \{\mathbf{b}, \mathbf{c}\}$)

$$\begin{aligned} \big|\varphi_{\mathbf{a}}[\mathit{\Phi}] - \varphi_{\mathbf{a}}[\mathit{\Psi}]\big| &\leq \tfrac{1}{\mathbf{h}} |A_{\mathbf{a}}[\mathit{\Phi}] - A_{\mathbf{a}}[\mathit{\Psi}]|\, |\tau| + \tfrac{1}{\mathbf{h}} |A_{\mathbf{a}}[\mathit{\Phi}] \theta_{\mathbf{a}}[\mathit{\Phi}] - A_{\mathbf{a}}[\mathit{\Psi}] \theta_{\mathbf{a}}[\mathit{\Psi}]| \\ &\leq \tfrac{1}{\mathbf{h}} (1 + |\tau|) |A_{\mathbf{a}}[\mathit{\Phi}] - A_{\mathbf{a}}[\mathit{\Psi}]| + \tfrac{1}{\mathbf{h}} 2 |\theta_{\mathbf{a}}[\mathit{\Phi}] - \theta_{\mathbf{a}}[\mathit{\Psi}]| \\ &\leq 2^2 M_1\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \\ \big|\xi_{\mathbf{a}}[\mathit{\Phi}] - \xi_{\mathbf{a}}[\mathit{\Psi}]\big| &\leq \mathbf{h} |\log A_{\mathbf{a}}[\mathit{\Phi}] - \log A_{\mathbf{a}}[\mathit{\Psi}]| \\ &\qquad\qquad + \mathbf{h}\, |\log\cosh \varphi_{\mathbf{a}}[\mathit{\Phi}] - \log\cosh \varphi_{\mathbf{a}}[\mathit{\Psi}]| \\ &\leq 2^3 \mathbf{h}\, M_1\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \\ \big|\beta_{\mathbf{p}}[\mathit{\Phi}] \beta_{\mathbf{a}}[\mathit{\Phi}] - \beta_{\mathbf{p}}[\mathit{\Psi}] \beta_{\mathbf{a}}[\mathit{\Psi}]\big| &\leq \tfrac{1}{\mathbf{h}}\ \max\big\{|\beta_{\mathbf{p}}[\mathit{\Phi}] \beta_{\mathbf{a}}[\mathit{\Phi}]|, |\beta_{\mathbf{p}}[\mathit{\Psi}] \beta_{\mathbf{a}}[\mathit{\Psi}]|\big\} \\ &\qquad\qquad \times \big|\xi_{\mathbf{a},\mathbf{p}}[\mathit{\Phi}] - \xi_{\mathbf{a},\mathbf{p}}[\mathit{\Psi}]\big| \\ &\leq 2^4 M_1 M\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \\ \big|\beta_{\mathbf{p}}[\mathit{\Phi}] - \beta_{\mathbf{p}}[\mathit{\Psi}]\big| &\leq \tfrac{1}{\mathbf{h}}\ \max\big\{|\beta_{\mathbf{p}}[\mathit{\Phi}]|, |\beta_{\mathbf{p}}[\mathit{\Psi}]|\big\} \big|\xi_{\mathbf{p}}[\mathit{\Phi}] - \xi_{\mathbf{p}}[\mathit{\Psi}]\big| \\ &\leq \tfrac{1}{\mathbf{h}} 2^2 M^{1/2}\, \big(\big|\xi_{\mathbf{a},\mathbf{p}}[\mathit{\Phi}] - \xi_{\mathbf{a},\mathbf{p}}[\mathit{\Psi}]\big| + \big|\xi_{\mathbf{a}}[\mathit{\Phi}] - \xi_{\mathbf{a}}[\mathit{\Psi}]\big| \big) \\ &\leq 2^6 M_1 M^{1/2}\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \\ \big|\beta_{\mathbf{p}}[\mathit{\Phi}] \beta_{\mathbf{q}}[\mathit{\Phi}] - \beta_{\mathbf{p}}[\mathit{\Psi}] \beta_{\mathbf{q}}[\mathit{\Psi}]\big| &\leq 2^9 M_1 M\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi}) \end{aligned}$$

Consequently, for $\mathbf{i} = 1, 2, 3$,

$$\big|\mathfrak{a}_{\mathbf{i}}[\mathit{\Phi}, \mathbf{h}, Z, B_{\mathbf{a}}] - \mathfrak{a}_{\mathbf{i}}[\mathit{\Psi}, \mathbf{h}, Z, B_{\mathbf{a}}]\big| \ \leq\ 2^{11} M_1 M\, d_{\mathcal{E}}(\mathit{\Phi}, \mathit{\Psi})$$

With these estimates, (3.9b) follows. Observe that $\mathbb{R} \to \mathbb{R}, x \mapsto x \tanh x$ is Lipschitz with Lipschitz-constant $L > 0$ determined by $L \tanh L = 1$, in particular $L < 2$. $\square$

Definition 3.12. *For all* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$ *and* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^2 \times \mathbb{R}$ *(we don't require* $q > 0$ *here) and* $\sigma_* \in \{-1,+1\}^3$*, let* $\Phi_\star = \Phi_\star(\pi, \mathbf{f}, \sigma_*) \in \mathcal{D}(\sigma_*)$ *be given by*

$$\begin{aligned}
\alpha_{\mathbf{a}}[\Phi_\star] &= -1 & \xi_{\mathbf{a}}[\Phi_\star, \mathbf{h}] &= -\tfrac{1+w}{1+2w}(1 + \mathbf{h}\log 2)\\
\alpha_{\mathbf{b}}[\Phi_\star] &= \tfrac{w}{1+w} & \xi_{\mathbf{b}}[\Phi_\star, \mathbf{h}] &= -\tfrac{1+w}{1+2w}(1 + \mathbf{h}\log 2)\\
\alpha_{\mathbf{c}}[\Phi_\star] &= -w - \mu & \xi_{\mathbf{c}}[\Phi_\star, \mathbf{h}] &= -(1+w)q - \tfrac{w(1+w)}{1+2w} - \tfrac{1+3w+w^2}{1+2w}\mathbf{h}\log 2
\end{aligned}$$

and

$$\mu = (1+w)\big(\beta_1^2 + \beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 - 2\beta_3\beta_1 - 2\beta_1\beta_2\big)|_{\beta=\beta[\Phi_\star]} \tag{3.10}$$

Definition 3.13. *For all* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$, $\sigma_* \in \{-1,+1\}^3$ *let* $\mathcal{H}(\pi,\sigma_*) \subset \mathcal{D}(\sigma_*)$ *be the set of all vectors* $\Phi = \alpha \oplus \beta \in \mathcal{D}(\sigma_*)$ *with*

$$|\beta_{\mathbf{a}}| = |\beta_{\mathbf{b}}| \qquad \textstyle\sum_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} \big(\alpha_{\mathbf{j}}\alpha_{\mathbf{k}} - (\beta_{\mathbf{i}})^2 + 2\beta_{\mathbf{j}}\beta_{\mathbf{k}}\big) = 0 \tag{3.11a}$$

$$0 < \alpha_{\mathbf{b}} < -\alpha_{\mathbf{a}} \qquad \big(\alpha_{\mathbf{b}} + |\alpha_{\mathbf{a}}|\big)\, \log|\beta_{\mathbf{a}}/\alpha_{\mathbf{a}}| < \alpha_{\mathbf{b}}\log 2 \tag{3.11b}$$

Lemma 3.7. *Let* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$ *and* $\sigma_* \in \{-1,+1\}^3$*. The set* $\mathcal{H}(\pi,\sigma_*) \subset \mathcal{D}(\sigma_*)$ *is a smooth 4-dimensional submanifold. The map*

$$\begin{aligned}
(0,\infty)^3 \times \mathbb{R} &\to \mathcal{H}(\pi,\sigma_*)\\
(\lambda, \mathbf{h}, w, q) &\mapsto \lambda\,\Phi_\star(\pi, (\mathbf{h},w,q), \sigma_*)
\end{aligned} \tag{3.12}$$

is a diffeomorphism. Its inverse is given by

$$w = -\alpha_{\mathbf{b}}/(\alpha_{\mathbf{a}} + \alpha_{\mathbf{b}}) \qquad \tfrac{1}{\mathbf{h}} = -\tfrac{1+2w}{1+w}\log|\beta_{\mathbf{a}}/\alpha_{\mathbf{a}}| + \tfrac{w}{1+w}\log 2 \tag{3.13a}$$

$$\lambda = -\alpha_{\mathbf{a}} \qquad q = -\tfrac{1}{1+w}\,\mathbf{h}\log|\beta_{\mathbf{c}}/\alpha_{\mathbf{a}}| - \tfrac{w}{1+2w}\big(1 + \mathbf{h}\log 2\big) \tag{3.13b}$$

Proof. $\mathcal{H}(\pi,\sigma_*)$ is the graph of a smooth map from an open subset of $\mathbb{R}^4$ to $\mathbb{R}^2$. Namely the map given by solving (3.11a) for $(\alpha_{\mathbf{c}}, \beta_{\mathbf{b}})$ in terms of $(\alpha_{\mathbf{a}}, \alpha_{\mathbf{b}}, \beta_{\mathbf{a}}, \beta_{\mathbf{c}})$, whose domain is given by (3.11b) and appropriate sign conditions inherited from $\mathcal{D}(\sigma_*)$. The map (3.12) is well-defined, i.e. $\lambda\Phi_\star(\pi, (\mathbf{h},w,q), \sigma_*) \in \mathcal{H}(\pi,\sigma_*)$. The map (3.13) is well-defined, because the two right hand sides in (3.13a) and the first right hand side in (3.13b) are positive, by (3.11b). By direct calculation, the two maps are inverses. □

Definition 3.14. *For all* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ *set*

$$\begin{aligned}
\tau_{1-}(\mathbf{f}) &= \begin{cases} -\frac{1+w}{3+w}\,q - \frac{1}{3+w}\,\mathbf{h}\log 2 & \text{if } q \le 1\\ -\frac{1+w}{3+2w} - \frac{1+w}{3+2w}\,\mathbf{h}\log 2 & \text{if } q > 1\end{cases} & &< 0\\
\tau_{1+}(\mathbf{f}) &= (1 + \mathbf{h}\log 2)\tfrac{1+w}{1+2w} & &> 0
\end{aligned}$$

Definition 3.15. *For all* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3$ *and* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ *and* $\sigma_* \in \{-1,+1\}^3$ *let* $\Phi_1 = \Phi_1(\pi,\mathbf{f},\sigma_*) :\ \mathbb{R} \to \mathcal{D}(\sigma_*)$ *be given by*

$$\begin{aligned}
A_{\mathbf{a}}[\Phi_1](\tau) &= A_{\mathbf{a}}[\Phi_\star] & \alpha_{\mathbf{p},\mathbf{a}}[\Phi_1](\tau) &= \alpha_{\mathbf{p},\mathbf{a}}[\Phi_\star]\\
\theta_{\mathbf{a}}[\Phi_1, \mathbf{h}](\tau) &= \theta_{\mathbf{a}}[\Phi_\star, \mathbf{h}] & \xi_{\mathbf{p},\mathbf{a}}[\Phi_1, \mathbf{h}](\tau) &= \xi_{\mathbf{p},\mathbf{a}}[\Phi_\star, \mathbf{h}] + (\tau - \tau_{1+})\alpha_{\mathbf{p},\mathbf{a}}[\Phi_\star]
\end{aligned}$$

for all $\tau \in \mathbb{R}$ *and* $\mathbf{p} \in \{\mathbf{b},\mathbf{c}\}$*. Here,* $\tau_{1+} = \tau_{1+}(\mathbf{f})$ *and* $\Phi_\star = \Phi_\star(\pi,\mathbf{f},\sigma_*)$.

Lemma 3.8. *For all* $\pi = (\mathbf{a},\mathbf{b},\mathbf{c}) \in S_3,\ \mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3,\ \sigma_* \in \{-1,+1\}^3$, *set* $\Phi_0 = \Phi_0(\pi,\mathbf{f},\sigma_*)$ *and* $\Phi_1 = \Phi_1(\pi,\mathbf{f},\sigma_*)$ *and* $\tau_{1+} = \tau_{1+}(\mathbf{f})$ *and* $d_{\mathcal{D}} = d_{\mathcal{D}(\sigma_*),(\pi,\mathbf{h})}$ *and* $\not{d}_{\mathcal{D}} = \not{d}_{\mathcal{D}(\sigma_*),\mathbf{h}}$. *Then*

(a) $|\beta_{\mathbf{a}}[\Phi_1](\tau_{1+})| = |\beta_{\mathbf{b}}[\Phi_1](\tau_{1+})|$
(b) $c[\Phi_1,\mathbf{h},Z](\tau_{1+}) = 0$, *see Definitions 3.1 and 3.2 for* c *and* Z, *respectively*
(c) $\not{d}_{\mathcal{D}}(\Phi_0(\tau_{1+}),\Phi_1(\tau_{1+}))\ \leq\ 2^7 \max\{1+w,\mathbf{h}\}\exp(-\tfrac{1}{2\mathbf{h}}\min\{1,w+q\})$
(d) $d_{\mathcal{D}}(\Phi_0(\tau),\Phi_1(\tau)) \leq \big(1+|\tau-\tau_{1+}|\big)\, d_{\mathcal{D}}(\Phi_0(\tau_{1+}),\Phi_1(\tau_{1+}))$ *for all* $\tau \in \mathbb{R}$

Proof. We discuss (c) only. By direct calculation,

$$\begin{aligned}
\alpha_{\mathbf{a}}[\Phi_0](\tau_{1+}) - \alpha_{\mathbf{a}}[\Phi_1](\tau_{1+}) &= -X & \xi_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau_{1+}) - \xi_{\mathbf{a}}[\Phi_1,\mathbf{h}](\tau_{1+}) &= -Y\\
\alpha_{\mathbf{b}}[\Phi_0](\tau_{1+}) - \alpha_{\mathbf{b}}[\Phi_1](\tau_{1+}) &= +X & \xi_{\mathbf{b}}[\Phi_0,\mathbf{h}](\tau_{1+}) - \xi_{\mathbf{b}}[\Phi_1,\mathbf{h}](\tau_{1+}) &= +Y\\
\alpha_{\mathbf{c}}[\Phi_0](\tau_{1+}) - \alpha_{\mathbf{c}}[\Phi_1](\tau_{1+}) &= +X+\mu & \xi_{\mathbf{c}}[\Phi_0,\mathbf{h}](\tau_{1+}) - \xi_{\mathbf{c}}[\Phi_1,\mathbf{h}](\tau_{1+}) &= +Y
\end{aligned}$$

with $X = -1 + \tanh\big(\tfrac{1}{\mathbf{h}}\tau_{1+}\big)$ and $Y = \mathbf{h}\log\big(1+\exp(-2\tfrac{1}{\mathbf{h}}\tau_{1+})\big)$. The estimates

$$\begin{aligned}
|X| &\leq 2\exp(-2\tfrac{1}{\mathbf{h}}\tau_{1+}) \leq 2\exp(-\tfrac{1}{\mathbf{h}})\\
|Y| &\leq \mathbf{h}\exp(-2\tfrac{1}{\mathbf{h}}\tau_{1+}) \leq \mathbf{h}\exp(-\tfrac{1}{\mathbf{h}})\\
|\mu| &\leq (1+w)2^6\exp(-\tfrac{1}{2\mathbf{h}}\min\{1,w+q\})
\end{aligned}$$

imply (c). $\square$

Definition 3.16. *This is, verbatim, Definition 1.2 in the Introduction.*

Lemma 3.9. *In the context of Definition 3.16, the identities*

$$\lambda_L = 1 - \alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{1-}) = 1 - \alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau) \tag{3.14a}$$

$$w_L = -\big(\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{1-})\big)^{-1} = -\big(\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau)\big)^{-1} \tag{3.14b}$$

$$\tfrac{\mathbf{h}}{\mathbf{h}_L} = \tfrac{1+2w_L}{1+w_L}\big(-\tau_{1-} + \mathbf{h}\log\lambda_L\big) - \mathbf{h}\ \log 2 \tag{3.14c}$$

$$\begin{aligned}q_L = \tfrac{1}{1+w_L}\Big(&\mathbf{h}_L\log\lambda_L - \tfrac{\mathbf{h}_L}{\mathbf{h}}\xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{1-}) + \tfrac{\mathbf{h}_L}{\mathbf{h}}\tau_{1-} - \tfrac{w_L(1+w_L)}{1+2w_L}\\ &- \tfrac{1+3w_L+(w_L)^2}{1+2w_L}\,\mathbf{h}_L\log 2\Big)\end{aligned} \tag{3.14d}$$

hold, where $\Phi_0 = \Phi_0(\pi,\mathbf{f},\sigma_*)$ *and* $\tau_{1-} = \tau_{1-}(\mathbf{f})$ *and* $\tau \in \mathbb{R}$. *Furthermore,*

$$\big(\xi_{\mathbf{a}}[\Phi_0,\mathbf{h}] - \xi_{\mathbf{a}'}[\Phi_0,\mathbf{h}]\big)\,F = \tau - \tau_{1-} - 2\mathbf{h}\log\big(1+e^{2\tau/\mathbf{h}}\big)\,F \tag{3.15}$$

for all $\tau \in \mathbb{R}$, *where*

$$F = \begin{cases}\frac{1}{3+w} & q \leq 1\\ \frac{1+w}{3+2w} & q > 1\end{cases} \tag{3.16}$$

Proof. By direct calculation. In each case, distinguish $q \leq 1$ and $q > 1$. $\square$

Definition 3.17. *For all* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ *set*

$$\tau_*(\mathbf{f}) = \begin{cases}\frac{q}{1+w} & \textit{if } q \leq 1\\ 1 & \textit{if } q > 1\end{cases}$$

Definition 3.18. *For all* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ *set*

$$\mathbf{K}(\mathbf{f}) = 2^{40}\big(\tfrac{1}{\mathbf{h}}\big)^2 \max\{(\tfrac{1}{w})^2, w^3\}\ \max\{(\tfrac{1}{q})^2, q\}\ \exp\big(-\tfrac{1}{\mathbf{h}}2^{-7}\tau_*(\mathbf{f})\big) \tag{3.17}$$

Definition 3.19. *Let* $\mathcal{F}$ *be the open set of all* $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ *for which*

$$q \neq 1 \qquad\qquad \mathbf{K}(\mathbf{f}) < 1 \qquad\qquad \mathbf{h} < 2^{-7}\tau_*(\mathbf{f}) \tag{3.18}$$

Proposition 3.3. *For all* $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3,\ \sigma_* \in \{-1,+1\}^3$, *there are* **unique** *maps*

$$\begin{aligned} \varPi = \varPi[\pi,\sigma_*] : &\qquad \mathcal{F} \to (0,\infty)^2 \times \mathbb{R} \\ \varLambda = \varLambda[\pi,\sigma_*] : &\qquad \mathcal{F} \to [1,\infty) \\ \tau_{2-} = \tau_{2-}[\pi,\sigma_*] : &\qquad \mathcal{F} \to (-\infty, 0) \end{aligned}$$

so that for all $\mathbf{f} = (\mathbf{h}, w, q) \in \mathcal{F}$ *(see Definitions 3.5, 3.7, 3.9, 3.10, 3.12, 3.14, 3.16)*

(a) $\|\varPi(\mathbf{f}) - \mathcal{Q}_L(\mathbf{f})\|_{\mathbb{R}^3} \le \mathbf{K}(\mathbf{f})$
(b) $|\varLambda(\mathbf{f}) - \lambda_L(\mathbf{f})| \le \mathbf{K}(\mathbf{f})$
(c) $\tau_-(\mathbf{f}) < \tau_{2-}(\mathbf{f}) < \frac{1}{2}\tau_{1-}(\mathbf{f})$ *and* $|\tau_{2-}(\mathbf{f}) - \tau_{1-}(\mathbf{f})| \le \mathbf{K}(\mathbf{f})$
(d) $\varPi$, $\varLambda$ *and* τ_{2-} *are continuous*
(e) if we set $\tau_{2-} = \tau_{2-}(\mathbf{f}),\ \tau_{2+} = \tau_{1+}(\mathbf{f}),\ \pi' = (\mathbf{a}', \mathbf{b}', \mathbf{c}') = \mathcal{P}_L(\pi, \mathbf{f}),\ \lambda = \varLambda(\mathbf{f})$ *and* $\mathbf{f}' = (\mathbf{h}', w', q') = \varPi(\mathbf{f})$, *then* $\frac{1}{2} \le \tau_{2+} - \tau_{2-} \le 3$ *and there is a smooth field*

$$\varPhi = \alpha \oplus \beta \ \in\ \mathcal{E} = \mathcal{E}(\sigma_*; \tau_{2-}, \tau_{2+})$$

that satisfies
(e.1) $(\mathfrak{a}, \mathfrak{b}, c)[\varPhi, \mathbf{h}, Z] = 0$ *on* $[\tau_{2-}, \tau_{2+}]$
(e.2) $\varPhi(\tau_{2+}) = \varPhi_\star(\pi, \mathbf{f}, \sigma_*)$ *and* $\varPhi(\tau_{2-}) = \lambda\,\varPhi_\star(\pi', \mathbf{f}', \sigma_*)$, *in particular*

$$\varPhi(\tau_{2+}) \in \mathcal{H}(\pi, \sigma_*) \qquad \textit{and} \qquad \varPhi(\tau_{2-}) \in \mathcal{H}(\pi', \sigma_*)$$

(e.3) $|\beta_{\mathbf{a}}[\varPhi](\tau)| \ge |\beta_{\mathbf{a}'}[\varPhi](\tau)|$ *for all* $\tau \in [\tau_{2-}, \frac{1}{2}\tau_{1-}(\mathbf{f})]$ *with equality iff* $\tau = \tau_{2-}$
(e.4) $d_{\mathcal{E},(\pi,\mathbf{h})}(\varPhi, \varPhi_0) \le \mathbf{K}(\mathbf{f})$, *where* $\varPhi_0 = \varPhi_0(\pi, \mathbf{f}, \sigma_*)|_{[\tau_{2-},\tau_{2+}]}$
(e.5) $\sup_{\tau \in [\tau_{2-},\tau_{2+}]} \max\{\alpha_{\mathbf{b},\mathbf{c}}[\varPhi], \alpha_{\mathbf{c},\mathbf{a}}[\varPhi], \alpha_{\mathbf{a},\mathbf{b}}[\varPhi]\}(\tau) \le -2^{-2}\min\{w^2, w^{-1}\}$

Proof. The main part of this proof is the construction of the field $\varPhi$ that appears in (e). To make the proof more transparent, we replace some numerical constants in (3.17) and (3.18) by the components of a parameter vector $\ell = (\ell_1, \ldots, \ell_8) \in \mathbb{R}^8$. In the course of the construction of $\varPhi$, we require a finite number of inequalities of the form $\ell \ge \ell'$. Each inequality of this kind is marked by (•) and is *assumed to hold for the rest of the proof*, once it has been stated. At the end of the construction, we check that the particular parameters appearing in (3.17) and (3.18) satisfy all these inequalities.
Let $\pi = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in S_3$ and $\sigma_* \in \{-1,+1\}^3$. Fix any $\mathbf{f} = (\mathbf{h}, w, q) \in (0,\infty)^3$ with $q \neq 1$ and $\mathbf{h} \le 1$. Set $\tau_* = \tau_*(\mathbf{f})$. For any $s = (s_1, \ldots, s_7) \in \mathbb{R}^7$, set

$$\mathbf{X}(s) = \mathbf{X}(s_1, \ldots, s_7) = \\ 2^{s_1}\big(\tfrac{1}{\mathbf{h}}\big)^{s_2} \times \begin{Bmatrix} (\frac{1}{w})^{s_3} & \text{if } w \le 1 \\ w^{s_4} & \text{if } w > 1 \end{Bmatrix} \times \begin{Bmatrix} (\frac{1}{q})^{s_5} & \text{if } q \le 1 \\ q^{s_6} & \text{if } q > 1 \end{Bmatrix} \times \exp\big(\tfrac{1}{\mathbf{h}} s_7 \tau_*\big)$$

Basic properties of $\mathbf{X}(s)$. The quantity $\mathbf{X}(s)$ is positive, non-decreasing in each of its seven arguments (recall $0 < \mathbf{h} \le 1$), and $\mathbf{X}(s)\mathbf{X}(s') = \mathbf{X}(s+s')$ for all $s, s' \in \mathbb{R}^7$, and $\mathbf{X}(0,\dots,0) = 1$. Also, we have $\tau_* \ge \mathbf{X}(-1,0,0,-1,-1,0,0)$.
Basic smallness assumptions. Introduce a parameter vector $\ell = (\ell_1,\dots,\ell_8) \in \mathbb{R}^8$ with

$$(\ell_1,\dots,\ell_7) \ge (0,0,0,0,0,0,-\infty) \qquad \text{and} \qquad \ell_8 \ge 0 \qquad (\bullet)_1$$

Our basic assumptions on the vector $\mathbf{f} = (\mathbf{h}, w, q)$ are:

$$q \neq 1 \qquad \mathbf{K} \overset{\text{def}}{=} \mathbf{X}(\ell_1,\dots,\ell_7) < 1 \qquad \mathbf{h} < 2^{-\ell_8}\tau_* \tag{3.19}$$

Observe that our previous assumptions $q \neq 1$ and $\mathbf{h} \le 1$ are subsumed in (3.19). *Abbreviations.* $\tau_\pm = \tau_\pm(\mathbf{f})$ and $\tau_{1\pm} = \tau_{1\pm}(\mathbf{f})$ and $\tau_{2+} = \tau_{1+}(\mathbf{f})$ and

$$\tau_{0-} = \tfrac{1}{2}\tau_{1-} + \tfrac{1}{2}\tau_- < 0 \qquad\qquad \tau_{0+} = \tau_{1+} > 0$$

and $\mathcal{D} = \mathcal{D}(\sigma_*)$ and $\mathcal{E} = \mathcal{E}(\sigma_*;\tau_{0-},\tau_{0+})$ and $\Phi_0 = \Phi_0(\pi,\mathbf{f},\sigma_*)|_{[\tau_{0-},\tau_{0+}]}$ and $\Phi_1 = \Phi_1(\pi,\mathbf{f},\sigma_*)|_{[\tau_{0-},\tau_{0+}]}$ and $\Phi_\star = \Phi_\star(\pi,\mathbf{f},\sigma_*)$ and $d_{\mathcal{E}} = d_{\mathcal{E},(\pi,\mathbf{h})}$ and $d_{\mathcal{D}} = d_{\mathcal{D},(\pi,\mathbf{h})}$ and $\not{d}_{\mathcal{D}} = \not{d}_{\mathcal{D},\mathbf{h}}$ and $B_{\mathcal{E}}[\,\cdot\,,\cdot\,] = B_{\mathcal{E},(\pi,\mathbf{h})}[\,\cdot\,,\cdot\,]$ and $\pi' = (\mathbf{a}',\mathbf{b}',\mathbf{c}') = \mathcal{P}_L(\pi,\mathbf{f})$.
Preliminaries 1. Introduce ϵ_- and ϵ_+ by $\tau_{0-} = \tau_- + \epsilon_-$ and $\tau_{0+} = \tau_+ - \epsilon_+$, just as in Lemma 3.4. We have

$$\begin{aligned}
\epsilon_+ &= \tau_+ - \tau_{1+} = \tfrac{1+w}{1+2w}\big(1 + \tfrac{1}{w} - \mathbf{h}\log 2\big)\\
\epsilon_- &= \tfrac{1}{2}(\tau_{1-} - \tau_-) = \begin{cases} \frac{1}{2(3+w)}\big(\frac{1+w}{2+w}\,q - \mathbf{h}\log 2\big) & \text{if } q < 1\\ \frac{1+w}{2(3+2w)}\big(\frac{1+w}{2+w} - \mathbf{h}\log 2\big) & \text{if } q > 1\end{cases}
\end{aligned}$$

Require $\ell_8 \ge 2$ $(\bullet)_2$. Then $\mathbf{h}\log 2 \le \mathbf{h} \le 2^{-2}\min\{1,q\}$, and (recall that $\tau_+ = 1 + \frac{1}{w}$)

$$2^{-2} \le \epsilon_+/\tau_+ \le 1 \qquad\qquad 2^{-5} \le \epsilon_-/\tau_* \le 2^{-1}$$

and $\epsilon_- \in (0,-\tau_-)$ and $\epsilon_+ \in (0,\tau_+)$, as required by Lemma 3.4. We have

$$-1 < \tau_- < \tau_{0-} < \tau_{1-} < 0 < \tfrac{1}{2} < \tau_{0+} = \tau_{1+} = \tau_{2+} < \min\{2,\tau_+\}$$

Set

$$\delta \overset{\text{def}}{=} 2^{-9}\min\{1,w\}\,\tau_* = \mathbf{X}(-9,0,-1,0,0,0,0)\,\tau_* \ge \mathbf{X}(-10,0,-1,-1,-1,0,0) \tag{3.20}$$

This implies $\delta \le 2^{-4}\min\{1,w,\epsilon_-,\frac{\epsilon_+}{\tau_+\tau_{0+}}\}$, the main hypothesis of Lemma 3.4. This lemma will be applied later.
Preliminaries 2. Require $\ell_8 \ge 7$ $(\bullet)_3$. Then

$$\begin{aligned}
d_{\mathcal{E}}(\Phi_0,\Phi_1) &\le 2^2 d_{\mathcal{D}}(\Phi_0(\tau_{1+}),\Phi_\star) \le 2^{11}\not{d}_{\mathcal{D}}(\Phi_0(\tau_{1+}),\Phi_\star)\\
&\le 2^{18}(1+w)\exp\big(-\tfrac{1}{2\mathbf{h}}\min\{1,q\}\big) \le 2^{18}(1+w)\exp\big(-\tfrac{1}{2\mathbf{h}}\tau_*\big)\\
&\le \mathbf{X}(19,0,0,1,0,0,-2^{-1}) \le 2^{-2}\delta\,\mathbf{X}(31,0,1,2,1,0,-2^{-1})
\end{aligned} \tag{3.21}$$

The first and third inequality follow from (d) and (c) in Lemma 3.8, respectively, using $\sup_{\tau\in[\tau_{0-},\tau_{0+}]}(1 + |\tau - \tau_{1+}|) \le 2^2$. The second inequality follows from Lemma 3.2 (b), with $C = D = 2$. Its assumptions are satisfied, because $\mathbf{h} \le 2^{-7}$ and $A_{\mathbf{a}}[\Phi_0](\tau_{1+}) = 1$ and $\mathbf{h}\,\varphi_{\mathbf{a}}[\Phi_0](\tau_{1+}) = \tau_{1+} \in [\frac{1}{2},2]$ and $\xi_{\mathbf{a}}[\Phi_\star,\mathbf{h}] \in [-\frac{3}{2},-\frac{1}{2}]$ and $|\beta_{\mathbf{a}}[\Phi_\star]| \le 2\exp(-\frac{1}{2\mathbf{h}}) < 1$ and $A_{\mathbf{a}}[\Phi_\star] \in [1,2]$ and $0 \le \mathbf{h}|\varphi_{\mathbf{a}}[\Phi_\star]| + \xi_{\mathbf{a}}[\Phi_\star,\mathbf{h}] \le 2^{-3}$

(see Remark 3.2) and $\mathbf{h}\,|\varphi_{\mathbf{a}}[\Phi_\star]| \in [\frac{1}{2}, 2]$ and $\operatorname{sgn}\varphi_{\mathbf{a}}[\Phi_\star] = -\operatorname{sgn}\alpha_{\mathbf{a}}[\Phi_\star] = +1$, and because $\ell_8 \geq 7$ implies $\mathbf{h} \leq 2^{-7}$ and therefore $\exp(-\frac{1}{\mathbf{h}}2^{-3}) \leq 2^{-12}$.
Require $(\ell_1, \ldots, \ell_7) \geq (31, 0, 1, 2, 1, 0, -2^{-1})$ $(\bullet)_4$. Then, by (3.19),

$$\Phi_1 \in B_{\mathcal{E}}[2^{-2}\delta, \Phi_0] \tag{3.22}$$

Construction of Φ. Define a map $P : B_{\mathcal{E}}[\delta, \Phi_0] \to B_{\mathcal{E}}[\delta, \Phi_0], \Psi \mapsto P(\Psi)$ by

$$A_{\mathbf{a}}[P(\Psi)](\tau) - A_{\mathbf{a}}[\Phi_1](\tau) = \int_{\tau_{0+}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_1[\Psi, \mathbf{h}, \pi](\tau') \tag{3.23a}$$

$$\theta_{\mathbf{a}}[P(\Psi), \mathbf{h}](\tau) - \theta_{\mathbf{a}}[\Phi_1, \mathbf{h}](\tau) = \int_{\tau_{0+}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_2[\Psi, \mathbf{h}, \pi](\tau') \tag{3.23b}$$

$$\alpha_{\mathbf{p},\mathbf{a}}[P(\Psi)](\tau) - \alpha_{\mathbf{p},\mathbf{a}}[\Phi_1](\tau) = \int_{\tau_{0+}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_{(3,\mathbf{p})}[\Psi, \mathbf{h}, \pi](\tau') \tag{3.23c}$$

$$\xi_{\mathbf{p},\mathbf{a}}[P(\Psi), \mathbf{h}](\tau) - \xi_{\mathbf{p},\mathbf{a}}[\Phi_1, \mathbf{h}](\tau) = \int_{\tau_{0+}}^{\tau} \mathrm{d}\tau'' \int_{\tau_{0+}}^{\tau''} \mathrm{d}\tau'\, \mathbf{I}_{(3,\mathbf{p})}[\Psi, \mathbf{h}, \pi](\tau') \tag{3.23d}$$

for all $\mathbf{p} \in \{\mathbf{b}, \mathbf{c}\}$ and $\tau \in [\tau_{0-}, \tau_{0+}]$. To make sure that P is well defined, we require $(\ell_1, \ldots, \ell_7) \geq (28, 2, 1, 1, 1, 0, -2^{-7})$ $(\bullet)_5$, in which case Lemma 3.6 (see Preliminaries 1) implies the uniform estimates

$$|\mathbf{I}_S[\Psi, \mathbf{h}, \pi]| \leq \mathbf{X}(12, 1, 0, 0, 0, 0, -2^{-7}) \leq 2^{-6}\delta\, \mathbf{X}(28, 1, 1, 1, 1, 0, -2^{-7}) \leq 2^{-6}\delta \tag{3.24a}$$

$$|\mathbf{I}_S[\Psi, \mathbf{h}, \pi] - \mathbf{I}_S[\Psi', \mathbf{h}, \pi]| \leq 2^{-5}\mathbf{X}(24, 2, 0, 0, 0, 0, -2^{-7}) d_{\mathcal{E}}(\Psi, \Psi') \leq 2^{-5} d_{\mathcal{E}}(\Psi, \Psi') \tag{3.24b}$$

on the interval $[\tau_{0-}, \tau_{0+}]$, for all $\Psi, \Psi' \in B_{\mathcal{E}}[\delta, \Phi_0]$ and all $S \in \{1, 2, (3, \mathbf{b}), (3, \mathbf{c})\}$. Since $\sup_{\tau \in [\tau_{0-}, \tau_{0+}]} |\tau - \tau_{0+}| \leq 4$, we have:

- $A_{\mathbf{a}}[P(\Psi)] > \frac{1}{2}$ on $[\tau_{0-}, \tau_{0+}]$, which makes $P(\Psi)$ a well defined element of $\mathcal{E}$.
- Each right hand side of (3.23) is $\leq 2^{-2}\delta$, hence $P(\Psi) \in B_{\mathcal{E}}[\frac{1}{2}\delta, \Phi_0]$.
- The map P is Lipschitz-continuous with constant $\leq \frac{1}{2}$.

The metric space $B_{\mathcal{E}}[\delta, \Phi_0]$ is nonempty and complete. By the Banach Fixed Point Theorem, the contraction P admits a unique fixed point

$$\Phi \in B_{\mathcal{E}}[\tfrac{1}{2}\delta, \Phi_0] \tag{3.25}$$

Proof that the fixed point satisfies $(\mathfrak{a}, \mathfrak{b}, c)[\Phi, \mathbf{h}, Z] = 0$. The fixed point Φ is smooth. We have $\Phi(\tau_{0+}) = \Phi_1(\tau_{0+}) = \Phi_\star$ and $c[\Phi, \mathbf{h}, Z](\tau_{0+}) = 0$, by Lemma 3.8 (b), and because $\tau_{0+} = \tau_{1+}$. Set $\Psi = P(\Psi) = \Phi$ in (3.23) and differentiate with respect to τ. The result of differentiating (3.23a) and (3.23b) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} A_{\mathbf{a}} \\ \theta_{\mathbf{a}} \end{pmatrix} = \frac{1}{(A_{\mathbf{a}})^2}\begin{pmatrix} \frac{1}{\mathbf{h}}(A_{\mathbf{a}})^2 \tanh\varphi_{\mathbf{a}} & \frac{1}{\mathbf{h}}(A_{\mathbf{a}})^2 \operatorname{sech}\varphi_{\mathbf{a}} \\ \varphi_{\mathbf{a}}\tanh\varphi_{\mathbf{a}} - 1 & \sinh\varphi_{\mathbf{a}} + \varphi_{\mathbf{a}}\operatorname{sech}\varphi_{\mathbf{a}} \end{pmatrix} \times \begin{pmatrix} \mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, B_{\mathbf{a}}] - \mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, Z] \\ -(\sigma_*)_{\mathbf{a}}\,\mathfrak{b}_{\mathbf{a}}[\Phi, \mathbf{h}, B_{\mathbf{a}}] + (\sigma_*)_{\mathbf{a}}\,\mathfrak{b}_{\mathbf{a}}[\Phi, \mathbf{h}, Z] \end{pmatrix}$$

where $A_{\mathbf{a}} = A_{\mathbf{a}}[\Phi]$, $\theta_{\mathbf{a}} = \theta_{\mathbf{a}}[\Phi, \mathbf{h}]$, $\varphi_{\mathbf{a}} = \varphi_{\mathbf{a}}[\Phi]$, because $\mathfrak{b}_{\mathbf{a}}[\Phi, \mathbf{h}, B_{\mathbf{a}}] = \mathfrak{b}_{\mathbf{a}}[\Phi, \mathbf{h}, Z]$. Now, Lemma 3.1 implies $\mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, Z] = \mathfrak{b}_{\mathbf{a}}[\Phi, \mathbf{h}, Z] = 0$. Differentiation of (3.23c) gives $\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{p},\mathbf{a}}[\Phi] = \frac{1}{\mathbf{h}}\mathfrak{a}_{\mathbf{p}}[\Phi, \mathbf{h}, Z, B_{\mathbf{a}}] + \frac{1}{\mathbf{h}}\mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, Z, B_{\mathbf{a}}]$. Together with $\mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, Z] = 0$ and the general identity $\mathfrak{a}_{\mathbf{p}}[\Phi, \mathbf{h}, B_{\mathbf{a}}] + \mathfrak{a}_{\mathbf{a}}[\Phi, \mathbf{h}, B_{\mathbf{a}}] = -\mathbf{h}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{\mathbf{p},\mathbf{a}}[\Phi]$, we obtain $\mathfrak{a}_{\mathbf{p}}[\Phi, \mathbf{h}, Z] = 0$. Differentiating (3.23d) and simplifying the result with (3.23c) gives

$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_{\mathbf{p},\mathbf{a}}[\Phi,\mathbf{h}] = \alpha_{\mathbf{p},\mathbf{a}}[\Phi]$ which, by $\mathfrak{b}_{\mathbf{a}}[\Phi,\mathbf{h},Z] = 0$, implies $\mathfrak{b}_{\mathbf{p}}[\Phi,\mathbf{h},Z] = 0$. Now, Proposition 3.2 and the fact that $c[\Phi,\mathbf{h},Z](\tau_{0+}) = 0$ imply that $c[\Phi,\mathbf{h},Z] = 0$ identically on $[\tau_{0-},\tau_{0+}]$.
Estimates on Φ. By the fixed point equation $P(\Phi) = \Phi$ and by (3.21) and (3.24a),

$$\begin{aligned} d_{\mathcal{E}}(\Phi_0,\Phi) &\leq d_{\mathcal{E}}(\Phi_0,\Phi_1) + d_{\mathcal{E}}(\Phi_1,P(\Phi)) \\ &\leq \mathbf{X}(19,0,0,1,0,0,-2^{-1}) + \mathbf{X}(16,1,0,0,0,0,-2^{-7}) \\ &\leq \mathbf{X}(20,1,0,1,0,0,-2^{-7}) \end{aligned} \tag{3.26}$$

We require $(\ell_1,\ldots,\ell_7) \geq (20,1,0,1,0,0,-2^{-7})\ (\bullet)_6$, which implies $d_{\mathcal{E}}(\Phi_0,\Phi) \leq \mathbf{K}$. To apply Lemma 3.2 (a), set $\mathcal{J} = [\tau_{0-}, \frac{1}{2}\tau_{1-}] \subset [\tau_{0-},\tau_{0+}]$ and $C = 2$ and $D = 12\max\{1,q^{-1}\}$. We check the assumptions of Lemma 3.2. The inequalities

$$C^{-1} \leq A_{\mathbf{a}}[X] \leq C \qquad\qquad D^{-1} \leq \mathbf{h}\,|\varphi_{\mathbf{a}}[X]| \leq D$$

hold for both $X = \Phi_0(\tau)$ and $X = \Phi(\tau)$, for all $\tau \in \mathcal{J}$. The inequality for $A_{\mathbf{a}}$ follows from $A_{\mathbf{a}}[\Phi_0](\tau) = 1$ and the bound $d_{\mathcal{E}}(\Phi_0,\Phi) \leq \delta \leq 2^{-9}$. To check the inequality for $\varphi_{\mathbf{a}}$, observe that $\mathbf{h}\varphi_{\mathbf{a}}[\Phi_0](\tau) = \tau \in \mathcal{J} \subset [-(D/2), -(D/2)^{-1}]$, see the definitions of τ_- and τ_{1-}. Furthermore, for all $\tau \in \mathcal{J}$, we have

$$|\mathbf{h}\varphi_{\mathbf{a}}[\Phi] - \mathbf{h}\varphi_{\mathbf{a}}[\Phi_0]| \leq |\tau|\,|A_{\mathbf{a}}[\Phi] - 1| + A_{\mathbf{a}}[\Phi]\,|\theta_{\mathbf{a}}[\Phi,\mathbf{h}]| \leq 4\delta \leq (2D)^{-1}$$

This implies $\mathbf{h}\,\varphi_{\mathbf{a}}[\Phi](\tau) \in [-D,-D^{-1}]$ and $\operatorname{sgn}\varphi_{\mathbf{a}}[\Phi_0](\tau) = \operatorname{sgn}\varphi_{\mathbf{a}}[\Phi](\tau) = -1$ for all $\tau \in \mathcal{J}$. Now, Lemma 3.2 (a) and $2^3C^2D \leq \mathbf{X}(9,0,0,0,1,0,0)$ imply for all $\tau \in \mathcal{J}$

$$\not{d}_{\mathcal{D}}(\Phi_0(\tau),\Phi(\tau)) \leq 2^3C^2D\,d_{\mathcal{E}}(\Phi_0,\Phi) \leq \mathbf{X}(29,1,0,1,1,0,-2^{-7}) \overset{\text{def}}{=} \mathbf{M} \tag{3.27}$$

Construction of τ_{2-}. Recall that $\mathbf{a}' = \mathbf{c}$ if $q < 1$ and $\mathbf{a}' = \mathbf{b}$ if $q > 1$. By (3.15),

$$\big(\xi_{\mathbf{a}}[\Phi_0,\mathbf{h}] - \xi_{\mathbf{a}'}[\Phi_0,\mathbf{h}]\big)\,F = \tau - \tau_{1-} - T_1 \tag{3.28a}$$
$$\big(\xi_{\mathbf{a}}[\Phi,\mathbf{h}] - \xi_{\mathbf{a}'}[\Phi,\mathbf{h}]\big)\,F = \tau - \tau_{1-} - T_2 \tag{3.28b}$$

for all $\tau \in \mathcal{J}$, where F is given by (3.16), and

$$\begin{aligned} T_1 &= T_1(\tau) = 2\mathbf{h}\log\big(1 + e^{2\tau/\mathbf{h}}\big)\,F \\ T_2 &= T_2(\tau) = T_1 - \big(\xi_{\mathbf{a}}[\Phi,\mathbf{h}] - \xi_{\mathbf{a}}[\Phi_0,\mathbf{h}]\big)\,F + \big(\xi_{\mathbf{a}'}[\Phi,\mathbf{h}] - \xi_{\mathbf{a}'}[\Phi_0,\mathbf{h}]\big)\,F \end{aligned}$$

For all $\tau \in \mathcal{J}$ we have $0 < T_1 \leq 2\mathbf{h}\,e^{2\tau/\mathbf{h}}F \leq 2\mathbf{h}\,e^{\tau_{1-}/\mathbf{h}}F \leq \mathbf{M}F$ and therefore $|T_2| \leq 3\mathbf{M}F \leq \frac{3}{2}\mathbf{M}$. Estimate

$$\operatorname{dist}_{\mathbb{R}}\big(\tau_{1-},\,\mathbb{R}\setminus\mathcal{J}\big) = \min\big\{\tfrac{1}{2}|\tau_{1-}|, \epsilon_-\big\} \geq 2^{-5}\tau_* \geq \mathbf{X}(-6,0,0,-1,-1,0,0)$$

Therefore, the condition $(\ell_1,\ldots,\ell_7) \geq (37,1,0,2,2,0,-2^{-7})\ (\bullet)_7$ yields

$$|T_2| \leq \tfrac{1}{2}\operatorname{dist}_{\mathbb{R}}\big(\tau_{1-},\,\mathbb{R}\setminus\mathcal{J}\big) \tag{3.29}$$

for all $\tau \in \mathcal{J}$. Set

$$\tau_{2-} = \sup\big\{\,\tau \in \mathcal{J} \;\big|\; \xi_{\mathbf{a}}[\Phi,\mathbf{h}](\tau) \leq \xi_{\mathbf{a}'}[\Phi,\mathbf{h}](\tau)\,\big\} \tag{3.30}$$

The set on the right is nonempty, by (3.28b) and (3.29), it contains τ_{0-}. We have $\tau_{2-} \in (\tau_{0-},\frac{1}{2}\tau_{1-}) \subset \mathcal{J}$ and, by continuity, $\xi_{\mathbf{a}}[\Phi,\mathbf{h}](\tau_{2-}) = \xi_{\mathbf{a}'}[\Phi,\mathbf{h}](\tau_{2-})$, and $|\tau_{2-} -$

$\tau_{1-}| \leq \frac{3}{2}\mathbf{M}$. For all $\tau \in [\tau_{2-}, \frac{1}{2}\tau_{1-}]$, we have $|\beta_{\mathbf{a}}[\Phi](\tau)| \geq |\beta_{\mathbf{a}'}[\Phi](\tau)|$ with equality iff $\tau = \tau_{2-}$. The condition $(\ell_1, \ldots, \ell_7) \geq (31, 1, 0, 1, 1, 0, -2^{-7})$ $(\bullet)_8$ implies $|\tau_{2-} - \tau_{1-}| \leq \mathbf{K}$.

Estimates on Φ_0. For all $\tau \in \mathcal{J}$, we have

$$\begin{aligned} |\alpha_{\mathbf{a}}[\Phi_0](\tau) - 1| &= |\tanh \tfrac{1}{\mathbf{h}}|\tau| - 1| \leq 2\exp(-\tfrac{2}{\mathbf{h}}|\tau|) \\ |\xi_{\mathbf{a}}[\Phi_0, \mathbf{h}](\tau) - \tau| &= \big|\mathbf{h}\log(2\cosh\tfrac{1}{\mathbf{h}}|\tau|) - |\tau|\big| \leq \mathbf{h}\exp\big(-\tfrac{2}{\mathbf{h}}|\tau|\big) \\ \exp(-\tfrac{2}{\mathbf{h}}|\tau|) &\leq \exp(-\tfrac{1}{\mathbf{h}}|\tau_{1-}|) \leq \exp(-\tfrac{1}{\mathbf{h}}2^{-2}\tau_*) \leq 2^{-29}\mathbf{M} \end{aligned}$$

These estimates will be used without further comment.

Construction of λ. Set $\lambda_L = \lambda_L(\mathbf{f})$ and recall (3.14a). Set

$$\lambda = -\alpha_{\mathbf{a}'}[\Phi](\tau_{2-}) \tag{3.31}$$

Then,

$$\begin{aligned} |\lambda - \lambda_L| &\leq \big|\alpha_{\mathbf{a}'}[\Phi](\tau_{2-}) - \alpha_{\mathbf{a}'}[\Phi_0](\tau_{2-})\big| + \big|\alpha_{\mathbf{a}'}[\Phi_0](\tau_{2-}) - \alpha_{\mathbf{a}'}[\Phi_0](\tau_{1-})\big| \\ &\quad + \big|\alpha_{\mathbf{a}'}[\Phi_0](\tau_{1-}) + \big(1 - \alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{1-})\big)\big| \\ &\leq \big|\alpha_{\mathbf{a}'}[\Phi](\tau_{2-}) - \alpha_{\mathbf{a}'}[\Phi_0](\tau_{2-})\big| \\ &\quad + \Big(\big|\alpha_{\mathbf{a}}[\Phi_0](\tau_{2-}) - 1\big| + \big|\alpha_{\mathbf{a}}[\Phi_0](\tau_{1-}) - 1\big|\Big) + \big|1 - \alpha_{\mathbf{a}}[\Phi_0](\tau_{1-})\big| \leq 2\mathbf{M} \end{aligned}$$

See (3.27). Require $(\ell_1, \ldots, \ell_7) \geq (32, 1, 0, 2, 1, 0, -2^{-7})$ $(\bullet)_9$. Then $4\mathbf{M}(1+w) \leq \mathbf{K} \leq 1$ and $|\lambda - \lambda_L| \leq \frac{1}{2}(1+w)^{-1}\mathbf{K}$. In particular $\lambda \geq \lambda_L - (1+w)^{-1} \geq 1$.

We now construct the components of $\mathbf{f}' = (\mathbf{h}', w', q')$.

Construction of w'. Require $(\ell_1, \ldots, \ell_7) \geq (32, 1, 0, 2, 1, 0, -2^{-7})$ $(\bullet)_{10}$ and set

$$w' = \frac{\alpha_{\mathbf{a}}[\Phi](\tau_{2-})}{-\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau_{2-})} > 0 \tag{3.32}$$

To check that the denominator is nonzero and that $w' > 0$, note that for all $\tau \in \mathcal{J}$:

$$\begin{aligned} &\big|\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau) - \alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau)\big| \leq 2\mathbf{M} \\ &\qquad |\alpha_{\mathbf{a}}[\Phi](\tau) - 1| \leq \big|\alpha_{\mathbf{a}}[\Phi](\tau) - \alpha_{\mathbf{a}}[\Phi_0](\tau)\big| + \big|\alpha_{\mathbf{a}}[\Phi_0](\tau) - 1\big| \leq 2\mathbf{M} \end{aligned}$$

and $4\mathbf{M} \leq \mathbf{X}(-1, 0, 0, -1, 0, 0, 0) \leq \frac{1}{1+w} \leq |\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau)|$ and $4\mathbf{M} \leq 1$. Hence,

$$|\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau) - \alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau)| \leq \tfrac{1}{2}|\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau)| \qquad |\alpha_{\mathbf{a}}[\Phi](\tau) - 1| \leq \tfrac{1}{2}$$

In particular, $\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau_{2-}) \leq \frac{1}{2}\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{2-}) < 0$ and $\alpha_{\mathbf{a}}[\Phi](\tau_{2-}) > 0$. Consequently, w' is well defined and positive. Recall (3.14b) and estimate

$$\begin{aligned} |w' - w_L| &\leq \Big|\frac{\alpha_{\mathbf{a}}[\Phi](\tau_{2-})}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau_{2-})} - \frac{\alpha_{\mathbf{a}}[\Phi_0](\tau_{2-})}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau_{2-})}\Big| + \Big|\frac{\alpha_{\mathbf{a}}[\Phi_0](\tau_{2-})}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi](\tau_{2-})} - \frac{\alpha_{\mathbf{a}}[\Phi_0](\tau_{2-})}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{2-})}\Big| \\ &\quad + \Big|\frac{\alpha_{\mathbf{a}}[\Phi_0](\tau_{2-})}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{2-})} - \frac{1}{\alpha_{\mathbf{a},\mathbf{a}'}[\Phi_0](\tau_{2-})}\Big| \\ &\leq 2w_L\mathbf{M} + 4w_L^2\mathbf{M} + w_L\mathbf{M} \leq 2^3(1+w)^2\mathbf{M} \leq \tfrac{1}{2}\mathbf{X}(6, 0, 0, 2, 0, 0, 0)\,\mathbf{M} \end{aligned}$$

We require $(\ell_1,\dots,\ell_7) \geq (35,1,0,3,1,0,-2^{-7})$ $(\bullet)_{11}$. Hence $|w'-w_L| \leq \frac{1}{2}\mathbf{K} \leq \frac{1}{2}$.
Construction of $\mathbf{h}'$. Let λ and w' be given by (3.31) and (3.32). Set

$$\mu = \tfrac{1+2w'}{1+w'}\big(-\xi_{\mathbf{a}}[\Phi,\mathbf{h}](\tau_{2-}) + \mathbf{h}\ \log\lambda\big) - \mathbf{h}\log 2$$

Recall (3.14c) and estimate

$$\begin{aligned}
\big|\mu - \tfrac{\mathbf{h}}{\mathbf{h}_L}\big| &\leq \tfrac{1+2w'}{1+w'}\big| - \xi_{\mathbf{a}}[\Phi,\mathbf{h}](\tau_{2-}) + \mathbf{h}\ \log\lambda + \tau_{1-} - \mathbf{h}\ \log\lambda_L\big| \\
&\qquad + \big|\tfrac{1+2w'}{1+w'} - \tfrac{1+2w_L}{1+w_L}\big|\,\big|\tau_{1-} - \mathbf{h}\ \log\lambda_L\big| \\
&\leq 2\big|\tau_{1-}-\tau_{2-}\big| + 2\big|\tau_{2-} - \xi_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau_{2-})\big| \\
&\qquad + 2\big|\xi_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau_{2-}) - \xi_{\mathbf{a}}[\Phi,\mathbf{h}](\tau_{2-})\big| + 4\mathbf{h}\,|\lambda-\lambda_L| + 4\,\tfrac{|w'-w_L|}{(1+w_L)^2} \\
&\leq 2^2\mathbf{M} + \mathbf{M} + 2\mathbf{M} + 2^3\mathbf{M} + 2^4\mathbf{M} \leq 2^5\mathbf{M}
\end{aligned}$$

For the second inequality, use $(1+2w') \leq 2(1+w')$ and $\lambda,\lambda_L \geq \frac{1}{2}$ and $|\tau_{1-}| \leq 1$ and $|\mathbf{h}\log\lambda_L| \leq |\tau_*\log\lambda_L| \leq 1$, see (3.19), and $1+w' \geq \frac{1}{2}(1+w_L)$. By inspection,

$$\tfrac{\mathbf{h}}{\mathbf{h}_L} \geq \tfrac{1+w}{2+w}\ \min\{1,q\} \geq \mathbf{X}(-1,0,0,0,-1,0,0)$$

To make sure that $\mu > 0$, we require $(\ell_1,\dots,\ell_7) \geq (36,1,0,1,2,0,-2^{-7})$ $(\bullet)_{12}$, so that $2^5\mathbf{M} \leq \frac{1}{2}\mathbf{X}(-1,0,0,0,-1,0,0)\,\mathbf{K} \leq \frac{1}{2}\frac{\mathbf{h}}{\mathbf{h}_L}$, that is $|\mu - \frac{\mathbf{h}}{\mathbf{h}_L}| \leq \frac{1}{2}\frac{\mathbf{h}}{\mathbf{h}_L}$ and $\mu > 0$. Set

$$\mathbf{h}' = \mathbf{h}/\mu > 0 \tag{3.33}$$

Require $\ell_8 \geq 7$ $(\bullet)_{13}$, so that $\mathbf{h} \leq \mathbf{X}(-7,0,0,0,-1,0,0)$ and

$$\big|\mathbf{h}'-\mathbf{h}_L\big| = \mathbf{h}\tfrac{\mathbf{h}/\mathbf{h}_L}{\mu}\,\big(\tfrac{\mathbf{h}_L}{\mathbf{h}}\big)^2\,\big|\mu - \tfrac{\mathbf{h}}{\mathbf{h}_L}\big| \leq \tfrac{1}{2}\,\mathbf{X}(2,0,0,0,1,0,0)\,\mathbf{M}$$

We require $(\ell_1,\dots,\ell_7) \geq (31,1,0,1,2,0,-2^{-7})$ $(\bullet)_{14}$. Then $|\mathbf{h}'-\mathbf{h}_L| \leq \frac{1}{2}\mathbf{K} \leq \frac{1}{2}$.
Construction of q'. Set

$$q' = \tfrac{1}{1+w'}\Big(\mathbf{h}'\log\lambda - \tfrac{\mathbf{h}'}{\mathbf{h}}\xi_{\mathbf{c}'}[\Phi,\mathbf{h}](\tau_{2-}) - \tfrac{w'(1+w')}{1+2w'} - \tfrac{1+3w'+(w')^2}{1+2w'}\,\mathbf{h}'\log 2\Big) \tag{3.34}$$

Recall (3.14d) and estimate

$$\begin{aligned}
&|q'-q_L| \\
&\leq \big|\tfrac{1}{1+w'}\mathbf{h}'\log\lambda - \tfrac{1}{1+w_L}\mathbf{h}_L\log\lambda_L\big| \\
&\quad + \big|\tfrac{1}{1+w'}\tfrac{\mathbf{h}'}{\mathbf{h}}\xi_{\mathbf{c}'}[\Phi,\mathbf{h}](\tau_{2-}) - \tfrac{1}{1+w_L}\tfrac{\mathbf{h}_L}{\mathbf{h}}\,\big(\xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{1-}) - \tau_{1-}\big)\big| \\
&\quad + \big|\tfrac{w'}{1+2w'} - \tfrac{w_L}{1+2w_L}\big| + \big|\tfrac{1+3w'+(w')^2}{(1+w')(1+2w')}\,\mathbf{h}' - \tfrac{1+3w_L+(w_L)^2}{(1+w_L)(1+2w_L)}\,\mathbf{h}_L\big| \\
&\leq \tfrac{1}{1+w'}\mathbf{h}'\big|\log\lambda - \log\lambda_L\big| + \tfrac{1}{1+w'}\big|\mathbf{h}'-\mathbf{h}_L\big|\log\lambda_L + \big|\tfrac{1}{1+w'} - \tfrac{1}{1+w_L}\big|\mathbf{h}_L\log\lambda_L \\
&\quad + \tfrac{1}{1+w'}\tfrac{\mathbf{h}'}{\mathbf{h}}\big|\xi_{\mathbf{c}'}[\Phi,\mathbf{h}](\tau_{2-}) - \xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| + \tfrac{1}{1+w'}\big|\tfrac{\mathbf{h}'}{\mathbf{h}} - \tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\,\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| \\
&\quad + \big|\tfrac{1}{1+w'} - \tfrac{1}{1+w_L}\big|\,\tfrac{\mathbf{h}_L}{\mathbf{h}}\,\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| \\
&\quad + \tfrac{1}{1+w_L}\tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-}) - \xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{1-}) + \tau_{1-}\big| \\
&\quad + \big|\tfrac{w'}{1+2w'} - \tfrac{w_L}{1+2w_L}\big| + \big|\tfrac{1+3w'+(w')^2}{(1+w')(1+2w')} - \tfrac{1+3w_L+(w_L)^2}{(1+w_L)(1+2w_L)}\big|\,\mathbf{h}' + \big|\mathbf{h}'-\mathbf{h}_L\big|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^3\big|\lambda-\lambda_L\big| + \big|\mathbf{h}'-\mathbf{h}_L\big|(1+w) + 4\tfrac{|w'-w_L|}{(1+w_L)^2}(1+w)\\
&\quad + 2\tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\xi_{\mathbf{c}'}[\Phi,\mathbf{h}](\tau_{2-}) - \xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| + \big|\tfrac{\mathbf{h}'}{\mathbf{h}} - \tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\,\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big|\\
&\quad + 2\tfrac{|w'-w_L|}{(1+w_L)^2}\,\tfrac{\mathbf{h}_L}{\mathbf{h}}\,\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| + \tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-}) - \xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{1-})\big|\\
&\quad + \tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\xi_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau_{2-}) - \tau_{2-}\big| + \tfrac{\mathbf{h}_L}{\mathbf{h}}\big|\tau_{2-}-\tau_{1-}\big|\\
&\quad + 2\tfrac{|w'-w_L|}{(1+w_L)^2} + 2^3\tfrac{|w'-w_L|}{(1+w_L)^2} + \big|\mathbf{h}'-\mathbf{h}_L\big|\\
&\leq 2^4\mathbf{M} + \mathbf{X}(2,0,0,1,1,0,0)\,\mathbf{M} + \mathbf{X}(5,0,0,1,0,0,0)\,\mathbf{M}\\
&\quad + \mathbf{X}(2,0,0,0,1,0,0)\,\mathbf{M} + \mathbf{X}(6,1,0,1,1,1,0)\,\mathbf{M}\\
&\quad + \mathbf{X}(9,0,0,1,1,1,0)\,\mathbf{M} + \mathbf{X}(3,0,0,1,1,0,0)\,\mathbf{M}\\
&\quad + \mathbf{X}(1,0,0,0,1,0,0)\,\mathbf{M} + \mathbf{X}(2,0,0,0,1,0,0)\,\mathbf{M}\\
&\quad + 2^3\mathbf{M} + 2^5\mathbf{M} + \mathbf{X}(1,0,0,0,1,0,0)\,\mathbf{M}\\
&\leq \tfrac{1}{2}\mathbf{X}(11,1,0,1,1,1,0)\,\mathbf{M}
\end{aligned}
$$

For the third inequality, use $\mathbf{h}_L \leq \mathbf{X}(1,0,0,0,1,0,0)\,\mathbf{h} \leq 2$ and $\mathbf{h}' = \mathbf{h}/\mu \leq 2\mathbf{h}_L \leq 2^2$ and $\lambda,\lambda_L \geq \frac{1}{2}$ and $|\mathbf{h}\log\lambda_L| \leq 1$ and $\log\lambda_L \leq 1+w$ and $(1+w') \geq \frac{1}{2}(1+w_L)$. For the fourth inequality, use $(1+w) \leq \mathbf{X}(1,0,0,1,0,0,0)$ and $\frac{|w'-w_L|}{(1+w_L)^2} \leq 2^2\mathbf{M}$ and

$$
\begin{aligned}
\big|\xi_{\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| &\leq \big|\xi_{\mathbf{a},\mathbf{c}'}[\Phi_0,\mathbf{h}](\tau_{2-})\big| + \big|\xi_{\mathbf{a}}[\Phi_0,\mathbf{h}](\tau_{2-}) - \tau_{2-}\big| + |\tau_{2-}|\\
&\leq \mathbf{X}(3,0,0,1,0,1,0) + 1 + 1 \leq \mathbf{X}(5,0,0,1,0,1,0)
\end{aligned}
$$

We require $(\ell_1,\ldots,\ell_7) \geq (40,2,0,2,2,1,-2^{-7})$ $(\bullet)_{15}$, such that $|q'-q_L| \leq \frac{1}{2}\mathbf{K} \leq \frac{1}{2}$.
The maximum of $\alpha_{\mathbf{b},\mathbf{c}}$, $\alpha_{\mathbf{c},\mathbf{a}}$, $\alpha_{\mathbf{a},\mathbf{b}}$. Require $(\ell_1,\ldots,\ell_7) \geq (24,1,2,2,0,0,-2^{-7})$ $(\bullet)_{16}$. Then $\mathbf{X}(20,1,0,1,0,0,-2^{-7}) \leq 2^{-3}(1+w)^{-1}\min\{w^2,1\}$. By the inequality (3.26), we have $d_{\mathcal{D}}(\Phi_0(\tau),\Phi(\tau)) \leq 2^{-3}(1+w)^{-1}\min\{w^2,1\}$, for all $\tau \in [\tau_{0-},\tau_{0+}]$. Hence,

$$
\begin{aligned}
&\alpha_{\mathbf{a},\mathbf{p}}[\Phi] \leq \alpha_{\mathbf{a},\mathbf{p}}[\Phi_0] + d_{\mathcal{D}}(\Phi_0,\Phi) \leq -(1+w)^{-1} + d_{\mathcal{D}}(\Phi_0,\Phi) \leq -2^{-1}(1+w)^{-1}\\
&\alpha_{\mathbf{b},\mathbf{c}}[\Phi] \leq \alpha_{\mathbf{a},\mathbf{b}}[\Phi] + \alpha_{\mathbf{a},\mathbf{c}}[\Phi] - 2\alpha_{\mathbf{a}}[\Phi] \leq \alpha_{\mathbf{a},\mathbf{b}}[\Phi] + \alpha_{\mathbf{a},\mathbf{c}}[\Phi] + 2A_{\mathbf{a}}[\Phi]\\
&\qquad\quad \leq 2^2 d_{\mathcal{D}}(\Phi_0,\Phi) + \alpha_{\mathbf{a},\mathbf{b}}[\Phi_0] + \alpha_{\mathbf{a},\mathbf{c}}[\Phi_0] + 2A_{\mathbf{a}}[\Phi_0] \leq -2^{-1}(1+w)^{-1}w^2
\end{aligned}
$$

for all $\tau \in [\tau_{0-},\tau_{0+}]$ and all $\mathbf{p} \in \{\mathbf{b},\mathbf{c}\}$.
Definition of the maps Π, Λ *and* τ_{2-}. Set $(\ell_1,\ldots,\ell_7) = (40,2,2,3,2,1,-2^{-7})$ and $\ell_8 = 7$. With this choice, all inequalities $(\bullet)$ hold. The constant $\mathbf{K}$ defined by (3.19) coincides with $\mathbf{K}(\mathbf{f})$, defined by (3.17). Furthermore, a vector $\mathbf{f} = (\mathbf{h},w,q) \in (0,\infty)^3$ satisfies our basic assumption (3.19) if and only if $\mathbf{f} \in \mathcal{F}$. Therefore, we can set

$$
\begin{aligned}
\Pi[\pi,\sigma_*]:&\quad \mathcal{F} \to (0,\infty)^2\times\mathbb{R} && \mathbf{f} \mapsto \text{right hand sides of ((3.33), (3.32), (3.34))}\\
\Lambda[\pi,\sigma_*]:&\quad \mathcal{F} \to [1,\infty) && \mathbf{f} \mapsto \text{right hand side of (3.31)}\\
\tau_{2-}[\pi,\sigma_*]:&\quad \mathcal{F} \to (-\infty,0) && \mathbf{f} \mapsto \text{right hand side of (3.30)}
\end{aligned}
$$

Properties (a), (b), (c) and (e) in Proposition 3.3 are by construction, where it is understood that the fixed point Φ of the map P, whose domain of definition is $[\tau_{0-},\tau_{0+}]$, has to be restricted to the subinterval $[\tau_{2-},\tau_{2+}]$ to comply with the statement in Proposition 3.3 (e). The statements of (e.1), (e.3), (e.4), (e.5) have already been discussed in this proof. Equation $\Phi(\tau_{2+}) = \Phi_\star$ in (e.2), with $\Phi_\star = \Phi_\star(\pi,\mathbf{f},\sigma_*)$, follows from the fixed point equation $P(\Phi) = \Phi$, see (3.23), and from $\Phi_1(\tau_{1+}) = \Phi_\star$ and $\tau_{0+} = \tau_{1+} = \tau_{2+}$.

Equation $\Phi(\tau_{2-}) = \lambda\,\Phi_\star(\pi', \mathbf{f}', \sigma_*)$ in (e.2) with $\mathbf{f}' = (\mathbf{h}', w', q')$ is equivalent to (recall $\mathbf{b}' = \mathbf{a}$)

$$\alpha_{\mathbf{a}'}[\Phi](\tau_{2-}) = -\lambda \tag{3.35a}$$

$$\alpha_{\mathbf{a}}[\Phi](\tau_{2-}) = \lambda\,\tfrac{w'}{1+w'} \tag{3.35b}$$

$$\alpha_{\mathbf{c}'}[\Phi](\tau_{2-}) = \lambda\,(-w' - \mu') \tag{3.35c}$$

$$\tfrac{1}{\mathbf{h}}\xi_{\mathbf{a}'}[\Phi, \mathbf{h}](\tau_{2-}) = \log\lambda + \tfrac{1}{\mathbf{h}'}\big\{ -\tfrac{1+w'}{1+2w'}(1 + \mathbf{h}'\log 2)\big\} \tag{3.35d}$$

$$\tfrac{1}{\mathbf{h}}\xi_{\mathbf{a}}[\Phi, \mathbf{h}](\tau_{2-}) = \log\lambda + \tfrac{1}{\mathbf{h}'}\big\{ -\tfrac{1+w'}{1+2w'}(1 + \mathbf{h}'\log 2)\big\} \tag{3.35e}$$

$$\tfrac{1}{\mathbf{h}}\xi_{\mathbf{c}'}[\Phi, \mathbf{h}](\tau_{2-}) = \log\lambda + \tfrac{1}{\mathbf{h}'}\big\{ -(1+w')q' - \tfrac{w'(1+w')}{1+2w'} - \tfrac{1+3w'+(w')^2}{1+2w'}\mathbf{h}'\log 2\big\} \tag{3.35f}$$

with $\mu' = (1+w')\big(\beta_1^2 + \beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 - 2\beta_3\beta_1 - 2\beta_1\beta_2\big)|_{\beta=\beta[\Phi_\star(\pi',\mathbf{f}',\sigma_*)]}$. By inspection: (3.35a) follows from (3.31); (3.35b) follows from (3.31) and (3.32); (3.35e) follows from (3.33); (3.35f) follows from (3.34); (3.35d) follows from (3.35e) and the discussion following (3.30). These five equations and $c[\Phi, \mathbf{h}, Z](\tau_{2-}) = 0$ imply (3.35c). We have now checked (e.2). We now discuss (d).

Continuity of the maps Π, Λ and τ_{2-}. Fix $\mathbf{f}^\Psi = (\mathbf{h}^\Psi, w^\Psi, q^\Psi) \in \mathcal{F}$. Let $r > 0$ and let $\mathbf{f}^\Upsilon = (\mathbf{h}^\Upsilon, w^\Upsilon, q^\Upsilon) \in \mathcal{F}$ with $\|\mathbf{f}^\Psi - \mathbf{f}^\Upsilon\|_{\mathbb{R}^3} \leq r$. All the objects and abbreviations that have been introduced for a single element of $\mathcal{F}$ before, now come in two versions, one associated to each of $\mathbf{f}^B \in \mathcal{F}$ with $B = \Psi, \Upsilon$. By convention, these two versions are distinguished by a superscript B. For instance, $\tau_{0+}^B = \tau_{+1}^B = \tau_{2+}^B = \tau_{+1}(\mathbf{f}^B)$ and $\Phi_0^B = \Phi_0(\pi, \mathbf{f}^B, \sigma_*)|_{[\tau_{0-}^B, \tau_{0+}^B]}$ and $\mathcal{E}^B = \mathcal{E}(\sigma_*; \tau_{0-}^B, \tau_{0+}^B)$ and so forth. Following this convention, the contraction mapping fixed points are denoted $\Phi^B \in \mathcal{E}^B$. However, we also write $\Phi^\Psi = \Psi$ and $\Phi^\Upsilon = \Upsilon$. Suppose $r \leq \frac{1}{2}|q^\Psi - 1|$. Then

$$0 \neq \operatorname{sgn}(q^\Psi - 1) = \operatorname{sgn}(q^\Upsilon - 1) \tag{3.36}$$

Define $\chi : \mathbb{R} \to \mathbb{R}$ by $\chi(\tau) = \frac{\mathbf{h}^\Upsilon}{\mathbf{h}^\Psi}(\tau - \tau_{0+}^\Psi) + \tau_{0+}^\Upsilon$. Introduce four closed intervals $\mathcal{I}^B = [\tau_{0-}^B, \tau_{0+}^B]$, $B = \Psi, \Upsilon$, and $\mathcal{I}^\Xi = [\chi^{-1}(\tau_{0-}^\Upsilon), \tau_{0+}^\Psi]$ and $\mathcal{I} = \mathcal{I}^\Psi \cap \mathcal{I}^\Xi$. Observe that $\chi(\mathcal{I}^\Xi) = \mathcal{I}^\Upsilon$. By Proposition 3.1, the field $\Xi = \Upsilon \circ (\chi|_{\mathcal{I}^\Xi})$ satisfies $(\mathfrak{a}, \mathfrak{b}, c)[\Xi, \mathbf{h}^\Psi, Z] = 0$ on $\mathcal{I}^\Xi$. Recall $\mathcal{J}^B = [\tau_{0-}^B, \frac{1}{2}\tau_{1-}^B] \subset \mathcal{I}^B$ and $|\tau_{2-}^B - \tau_{1-}^B| \leq \frac{1}{2}\operatorname{dist}_{\mathbb{R}}(\tau_{1-}^B, \mathbb{R} \setminus \mathcal{J}^B)$, see (3.28b) and (3.29). Set $\mathcal{J} = \mathcal{J}^\Psi \cap \mathcal{J}^\Xi \subset \mathcal{I}$ with $\mathcal{J}^\Xi = \chi^{-1}(\mathcal{J}^\Upsilon)$. If $r > 0$ is sufficiently small, then

$$\tau_{2-}^\Psi \in \mathcal{J} \qquad \text{and} \qquad \chi^{-1}(\tau_{2-}^\Upsilon) \in \mathcal{J}$$

These inclusions have similar proofs. We only verify $\tau_{2-}^\Psi \in \mathcal{J}$. We have $\tau_{2-}^\Psi \in \mathcal{J}^\Psi$ and

$$\begin{aligned} |\chi(\tau_{2-}^\Psi) - \tau_{1-}^\Upsilon| &\leq |\chi(\tau_{2-}^\Psi) - \chi(\tau_{1-}^\Psi)| + |\chi(\tau_{1-}^\Psi) - \tau_{1-}^\Upsilon| \\ &\leq \tfrac{\mathbf{h}^\Upsilon}{\mathbf{h}^\Psi}\tfrac{1}{2}\operatorname{dist}_{\mathbb{R}}(\tau_{1-}^\Psi, \mathbb{R} \setminus \mathcal{J}^\Psi) + |\chi(\tau_{1-}^\Psi) - \tau_{1-}^\Upsilon| \end{aligned} \tag{3.37}$$

The right hand side of (3.37) is a continuous function of $\mathbf{f}^\Upsilon \in \mathcal{F}$ (with $\mathbf{f}^\Psi$ fixed) and is equal to $\frac{1}{2}\operatorname{dist}_{\mathbb{R}}(\tau_{1-}^\Upsilon, \mathbb{R} \setminus \mathcal{J}^\Upsilon) > 0$ when $\mathbf{f}^\Upsilon = \mathbf{f}^\Psi$. Therefore (3.37) is $<$ $\operatorname{dist}_{\mathbb{R}}(\tau_{1-}^\Upsilon, \mathbb{R} \setminus \mathcal{J}^\Upsilon)$ if $r > 0$ is small enough. Hence $\chi(\tau_{2-}^\Psi) \in \mathcal{J}^\Upsilon$, that is $\tau_{2-}^\Psi \in \mathcal{J}^\Xi$. Set $\mathcal{D} = \mathcal{D}^\Psi = \mathcal{D}^\Upsilon = \mathcal{D}(\sigma_*)$ and $\mathcal{E} = \mathcal{E}(\sigma_*; \mathcal{I})$ and $\Phi_0 = \Phi_0(\pi, \mathbf{f}^\Psi, \sigma_*)|_{\mathcal{I}}$. Equivalently, $\Phi_0 = \Phi_0^\Psi|_{\mathcal{I}}$. Abbreviate $d_{\mathcal{X}} = d_{\mathcal{X},(\pi,\mathbf{h}^\Psi)}$ for $\mathcal{X} = \mathcal{E}, \mathcal{D}$ and $d_{\mathcal{X}^B} = d_{\mathcal{X}^B,(\pi,\mathbf{h}^B)}$

for $B = \Psi, \Upsilon$. By (3.25), we have $d_{\mathcal{E}^B}(B, \Phi_0^B) \leq \frac{1}{2}\delta^B$ for $B = \Psi, \Upsilon$. If $r > 0$ is sufficiently small, then

$$d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Phi_0) \leq \delta^{\Psi} \qquad \text{and} \qquad d_{\mathcal{E}}(\Xi|_{\mathcal{I}}, \Phi_0) \leq \delta^{\Psi} \tag{3.38}$$

The first follows from $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Phi_0) \leq d_{\mathcal{E}^{\Psi}}(\Psi, \Phi_0^{\Psi}) \leq \frac{1}{2}\delta^{\Psi}$. The second follows from

$$\begin{aligned} d_{\mathcal{E}}(\Xi|_{\mathcal{I}}, \Phi_0) &\leq d_{\mathcal{E}}(\Upsilon \circ \chi|_{\mathcal{I}}, \Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}) + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0) \\ &\leq \max\{1, \tfrac{\mathbf{h}^{\Psi}}{\mathbf{h}^{\Upsilon}}\}\, d_{\mathcal{E}^{\Upsilon}}(\Upsilon, \Phi_0^{\Upsilon}) + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0) \\ &\leq \max\{1, \tfrac{\mathbf{h}^{\Psi}}{\mathbf{h}^{\Upsilon}}\}\, \tfrac{1}{2}\delta^{\Upsilon} + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0) \end{aligned} \tag{3.39}$$

and because the right hand side of (3.39) is a continuous function of $\mathbf{f}^{\Upsilon} \in \mathcal{F}$ (with $\mathbf{f}^{\Psi}$ fixed), see (3.20), that is equal to $\frac{1}{2}\delta^{\Psi}$ when $\mathbf{f}^{\Upsilon} = \mathbf{f}^{\Psi}$.
Both $X = \Psi|_{\mathcal{I}}$ and $X = \Xi|_{\mathcal{I}}$ satisfy $(\mathfrak{a}, \mathfrak{b}, c)[X, \mathbf{h}^{\Psi}, Z] = 0$ on $\mathcal{I}$,

$$A_{\mathbf{a}}[X](\tau) = A_{\mathbf{a}}[X](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_1[X, \mathbf{h}^{\Psi}, \pi](\tau') \tag{3.40a}$$

$$\theta_{\mathbf{a}}[X, \mathbf{h}^{\Psi}](\tau) = \theta_{\mathbf{a}}[X, \mathbf{h}^{\Psi}](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_2[X, \mathbf{h}^{\Psi}, \pi](\tau') \tag{3.40b}$$

$$\alpha_{\mathbf{p},\mathbf{a}}[X](\tau) = \alpha_{\mathbf{p},\mathbf{a}}[X](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} \mathrm{d}\tau'\, \mathbf{I}_{(3,\mathbf{p})}[X, \mathbf{h}^{\Psi}, \pi](\tau') \tag{3.40c}$$

$$\begin{aligned} \xi_{\mathbf{p},\mathbf{a}}[X, \mathbf{h}^{\Psi}](\tau) = {}& \xi_{\mathbf{p},\mathbf{a}}[X, \mathbf{h}^{\Psi}](\tau_{0+}^{\Psi}) + \alpha_{\mathbf{p},\mathbf{a}}[X](\tau_{0+}^{\Psi})\,(\tau - \tau_{0+}^{\Psi}) \\ &+ \int_{\tau_{0+}^{\Psi}}^{\tau} \mathrm{d}\tau'' \int_{\tau_{0+}^{\Psi}}^{\tau''} \mathrm{d}\tau'\, \mathbf{I}_{(3,\mathbf{p})}[X, \mathbf{h}^{\Psi}, \pi](\tau') \end{aligned} \tag{3.40d}$$

for all $\mathbf{p} \in \{\mathbf{b}, \mathbf{c}\}$ and $\tau \in \mathcal{I}$. By (3.24b), (3.38), (3.40) and by $\sup_{\tau\in\mathcal{I}} |\tau - \tau_{0+}^{\Psi}| \leq 4$, we have $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \leq 2^3 d_{\mathcal{D}}(\Psi(\tau_{0+}^{\Psi}), \Xi(\tau_{0+}^{\Psi})) + 2^{-1} d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}})$, and consequently

$$d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \leq 2^4 d_{\mathcal{D}}(\Psi(\tau_{0+}^{\Psi}), \Xi(\tau_{0+}^{\Psi})) = 2^4\, d_{\mathcal{D}}\big(\Phi_\star(\pi, \mathbf{f}^{\Psi}, \sigma_*), \Phi_\star(\pi, \mathbf{f}^{\Upsilon}, \sigma_*)\big)$$

In particular, $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \to 0$ as $\mathbf{f}^{\Upsilon} \to \mathbf{f}^{\Psi}$. Furthermore,

$$\begin{aligned} & d_{\mathcal{D}}\big(\lambda^{\Psi}\, \Phi_\star(\pi', \mathbf{f}'^{\Psi}, \sigma_*), \lambda^{\Upsilon}\, \Phi_\star(\pi', \mathbf{f}'^{\Upsilon}, \sigma_*)\big) \\ &= d_{\mathcal{D}}\big(\Psi(\tau_{2-}^{\Psi}), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))\big) \\ &\leq d_{\mathcal{D}}\big(\Psi(\tau_{2-}^{\Psi}), \Psi(\chi^{-1}(\tau_{2-}^{\Upsilon}))\big) + d_{\mathcal{D}}\big(\Psi(\chi^{-1}(\tau_{2-}^{\Upsilon})), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))\big) \\ &\leq d_{\mathcal{D}}\big(\Psi(\tau_{2-}^{\Psi}), \Psi(\chi^{-1}(\tau_{2-}^{\Upsilon}))\big) + 2^4\, d_{\mathcal{D}}\big(\Phi_\star(\pi, \mathbf{f}^{\Psi}, \sigma_*), \Phi_\star(\pi, \mathbf{f}^{\Upsilon}, \sigma_*)\big) \end{aligned}$$

By the last inequality, if we can show that $\chi^{-1}(\tau_{2-}^{\Upsilon}) \to \tau_{2-}^{\Psi}$ as $\mathbf{f}^{\Upsilon} \to \mathbf{f}^{\Psi}$, then $\tau_{2-}^{\Upsilon} \to \tau_{2-}^{\Psi}$ and $\lambda^{\Upsilon} \to \lambda^{\Psi}$ and $\mathbf{f}'^{\Upsilon} \to \mathbf{f}'^{\Psi}$. In other words, to show that Π, Λ and τ_{2-} are continuous, it suffices to show that $\chi^{-1}(\tau_{2-}^{\Upsilon}) \to \tau_{2-}^{\Psi}$ as $\mathbf{f}^{\Upsilon} \to \mathbf{f}^{\Psi}$.
By the discussion after (3.32), we have $\alpha_{\mathbf{a}}[\Psi](\tau) \geq \frac{1}{2}$ and $\alpha_{\mathbf{a},\mathbf{a}'}[\Psi](\tau) \leq 0$ for all $\tau \in \mathcal{J} \subset \mathcal{J}^{\Psi}$. Hence, for all $\tau \in \mathcal{J}$,

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big(\xi_{\mathbf{a}}[\Psi, \mathbf{h}^{\Psi}] - \xi_{\mathbf{a}'}[\Psi, \mathbf{h}^{\Psi}]\big) = \alpha_{\mathbf{a}}[\Psi] - \alpha_{\mathbf{a}'}[\Psi] = 2\alpha_{\mathbf{a}}[\Psi] - \alpha_{\mathbf{a},\mathbf{a}'}[\Psi] \geq 1$$

Hence, $|\tau - \tau_{2-}^{\Psi}| \leq \big|\xi_{\mathbf{a}}[\Psi, \mathbf{h}^{\Psi}](\tau) - \xi_{\mathbf{a}'}[\Psi, \mathbf{h}^{\Psi}](\tau)\big|$ if $\tau \in \mathcal{J}$. Set $\tau = \chi^{-1}(\tau_{2-}^{\Upsilon}) \in \mathcal{J}$:

$$\begin{aligned} |\chi^{-1}(\tau_{2-}^{\Upsilon}) - \tau_{2-}^{\Psi}| &\leq \big|\xi_{\mathbf{a}}[\Psi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon})) - \xi_{\mathbf{a}'}[\Psi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon}))\big| \\ &\leq 2\, \not{d}_{\mathcal{D},\mathbf{h}^{\Psi}}\big(\Psi(\chi^{-1}(\tau_{2-}^{\Upsilon})), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))\big) \end{aligned} \tag{3.41}$$

The last inequality follows from $\xi_{\mathbf{a}}[\Xi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon})) = \xi_{\mathbf{a}'}[\Xi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon}))$ and the triangle inequality. Since $\mathbf{f}^{\Upsilon} \to \mathbf{f}^{\Psi}$ implies $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \to 0$, also the right hand side of (3.41) goes to zero, that is, $|\chi^{-1}(\tau_{2-}^{\Upsilon}) - \tau_{2-}^{\Psi}| \to 0$, as required.
Uniqueness of Π, Λ and τ_{2-}. Suppose we have two triples Π_i, Λ_i, $\tau_{2-,i}$ with $i = 1, 2$. Let $\mathbf{f} \in \mathcal{F}$ and let Φ_i be the corresponding fields in (e). By (e.1) and (e.2) and the local uniqueness for solutions to ODE's, we have $\Phi_1 = \Phi_2$ on the intersection of their domains of definition $[\max\{\tau_{2-,1}(\mathbf{f}), \tau_{2-,2}(\mathbf{f})\}, \tau_{2+}]$. Observe that $\tau_{-}(\mathbf{f}) < \tau_{2-,1}(\mathbf{f}), \tau_{2-,2}(\mathbf{f}) < \frac{1}{2}\tau_{1-}(\mathbf{f})$, by (c). By (e.3), we have $\tau_{2-,1}(\mathbf{f}) = \tau_{2-,2}(\mathbf{f})$. By (e.2), we have $\Pi_1(\mathbf{f}) = \Pi_2(\mathbf{f})$, $\Lambda_1(\mathbf{f}) = \Lambda_2(\mathbf{f})$. $\square$

Remark 3.6. In Proposition 3.3, the signature vector σ_* appears to play a passive role. However, observe that $\Phi_\star = \Phi_\star(\pi, \mathbf{f}, \sigma_*)$ in (e.2) depends on it in a crucial way, see Definition 3.12. For instance, while $\alpha_{\mathbf{a}}[\Phi_\star]$ and $\alpha_{\mathbf{b}}[\Phi_\star]$ do not depend on σ_* at all, and $\beta_{\mathbf{i}}[\Phi_\star]$, $i = 1, 2, 3$ only in a trivial way through their signs, the component $\alpha_{\mathbf{c}}[\Phi_\star]$ does depend on σ_* in a more important way, because the right hand side of (3.10) does. That σ_* plays a role is not surprising, after all it distinguishes Bianchi VIII and IX.

4. The approximate epoch-to-epoch and era-to-era maps

This section is logically self-contained, and the notation is introduced from scratch. Its goal is to study two maps, denoted $\mathcal{Q}_R$ and $\mathcal{E}_R$, that we informally refer to (following [BKL1]) as the *epoch-to-epoch* and *era-to-era* maps. The two maps are related, the second is some iterate of the first. The subscript R is for *right* (as opposed to *left*). For the moment, the definition of $\mathcal{Q}_R$ is taken for granted without motivation. To understand its role, see Part 3 of Proposition 4.4 and its proof.

Definition 4.1 (Epoch-to-epoch map). *Set*

$$\mathcal{Q}_R : \quad (0,\infty) \setminus \mathbb{Q} \to (0,\infty) \setminus \mathbb{Q}$$
$$w \mapsto \mathcal{Q}_R(w) = \begin{cases} \frac{1}{w} - 1 & \text{if } w < 1 \\ w - 1 & \text{if } w > 1 \end{cases}$$

For every $w \in (0,\infty) \setminus \mathbb{Q}$, set

$$\boldsymbol{\mathcal{Q}}_R\{w\}(q, \mathbf{h}) = \left(\frac{\mathrm{num1}}{\mathrm{den}}, \frac{\mathrm{num2}}{\mathrm{den}} \right)$$

where, if $w < 1$,

$$\mathrm{num1} = 1 + w + \mathbf{h}\log 2 - \mathbf{h}(1+2w)\log(1+\tfrac{1}{w}) \tag{4.1a}$$
$$\mathrm{num2} = \mathbf{h} \tag{4.1b}$$
$$\mathrm{den} = (1+w)(1+q+\mathbf{h}\log 2) - \mathbf{h}(2+w)\log(1+\tfrac{1}{w}) \tag{4.1c}$$

and, if $w > 1$,

$$\mathrm{num1} = (1+w)(1+q+\mathbf{h}\log 2) - \mathbf{h}(2+w)\log(1+\tfrac{1}{w}) \tag{4.2a}$$
$$\mathrm{num2} = \mathbf{h}w \tag{4.2b}$$
$$\mathrm{den} = 1 + w + \mathbf{h}\log 2 - \mathbf{h}(1+2w)\log(1+\tfrac{1}{w}) \tag{4.2c}$$

Here, we regard $\boldsymbol{\mathcal{Q}}_R\{w\}$ as a pair of rational functions over $\mathbb{R}$ of degree one in the pair of abstract variables $(q, \mathbf{h})$. Finally, for all $w \in (0,\infty) \setminus \mathbb{Q}$ and all integers $n \geq 0$, set

$$\mathcal{Q}_R^n(w) = \big(\underbrace{\mathcal{Q}_R \circ \cdots \circ \mathcal{Q}_R}_{n}\big)(w)$$

$$\boldsymbol{\mathcal{Q}}_R^n\{w\} = \boldsymbol{\mathcal{Q}}_R\{\mathcal{Q}_R^{n-1}(w)\} \circ \cdots \circ \boldsymbol{\mathcal{Q}}_R\{\mathcal{Q}_R^2(w)\} \circ \boldsymbol{\mathcal{Q}}_R\{\mathcal{Q}_R(w)\} \circ \boldsymbol{\mathcal{Q}}_R\{w\}$$

Warning: $\boldsymbol{\mathcal{Q}}_R^n\{w\}$ is not the n-fold composition of $\boldsymbol{\mathcal{Q}}_R\{w\}$ with itself.

The goal of this section is to understand the bulk behavior of $\boldsymbol{\mathcal{Q}}_R^n\{w\}$ for large $n \geq 0$.

Definition 4.2. *The floor function is $\mathbb{R} \ni x \mapsto \lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, n \leq x\}$.*

Definition 4.3 (Era-to-era map). *Define $\mathcal{E}_R : (0,1) \setminus \mathbb{Q} \to (0,1) \setminus \mathbb{Q}$ by $\mathcal{E}_R(w) = \mathcal{Q}_R^{\lfloor 1/w \rfloor}(w)$. For every $w \in (0,1) \setminus \mathbb{Q}$, denote by $\boldsymbol{\mathcal{E}}_R\{w\}$ the pair of rational functions over $\mathbb{R}$ given by $\boldsymbol{\mathcal{E}}_R\{w\} = \boldsymbol{\mathcal{Q}}_R^{\lfloor 1/w \rfloor}\{w\}$. Finally, for all $w \in (0,1) \setminus \mathbb{Q}$ and all integers $n \geq 0$, set*

$$\mathcal{E}_R^n(w) = \big(\underbrace{\mathcal{E}_R \circ \cdots \circ \mathcal{E}_R}_{n}\big)(w)$$

$$\boldsymbol{\mathcal{E}}_R^n\{w\} = \boldsymbol{\mathcal{E}}_R\{\mathcal{E}_R^{n-1}(w)\} \circ \cdots \circ \boldsymbol{\mathcal{E}}_R\{\mathcal{E}_R^2(w)\} \circ \boldsymbol{\mathcal{E}}_R\{\mathcal{E}_R(w)\} \circ \boldsymbol{\mathcal{E}}_R\{w\}$$

Lemma 4.1. *For all integers $m, n \geq 0$,*

- $\boldsymbol{\mathcal{Q}}_R^{m+n}\{w\} = \boldsymbol{\mathcal{Q}}_R^m\{\mathcal{Q}_R^n(w)\} \circ \boldsymbol{\mathcal{Q}}_R^n\{w\}$ *for* $w \in (0,\infty) \setminus \mathbb{Q}$
- $\boldsymbol{\mathcal{E}}_R^{m+n}\{w\} = \boldsymbol{\mathcal{E}}_R^m\{\mathcal{E}_R^n(w)\} \circ \boldsymbol{\mathcal{E}}_R^n\{w\}$ *for* $w \in (0,1) \setminus \mathbb{Q}$

Proposition 4.1. *Let $w \in (0,1) \setminus \mathbb{Q}$. Then, for every integer $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$,*

$$\boldsymbol{\mathcal{Q}}_R^r\{w\}(q,\mathbf{h}) = \left(\frac{\mathrm{num1}_r}{\mathrm{den}_r}, \frac{\mathrm{num2}_r}{\mathrm{den}_r}\right) \tag{4.3}$$

where

$$\mathrm{num1}_r = (1+w)\big(r + rq - q\big) + \mathbf{h}\, A_1(w,r) \tag{4.4a}$$
$$\mathrm{num2}_r = \mathbf{h}\big(w + 1 - wr\big) \tag{4.4b}$$
$$\mathrm{den}_r = (1+w)(1+q) + \mathbf{h}\, A_2(w,r) \tag{4.4c}$$

and where

$$\begin{aligned} A_1(w,r) = {} & \big(2r - 1 + wr - w\big)\log 2 - \big(2r - 1 + wr + w\big)\log(1 + \tfrac{1}{w}) \\ & + \textstyle\sum_{k=1}^{r-1} \big(1 + 2k - 2k^2 w - w\big)\log\big(1 + \frac{w}{1-kw}\big) \\ & + r\,\textstyle\sum_{k=1}^{r-1} \big((2k-1)w - 2\big)\log\big(1 + \frac{w}{1-kw}\big) \end{aligned} \tag{4.5a}$$

$$\begin{aligned} A_2(w,r) = {} & (1 + wr)\log 2 - (2 + w)\log(1 + \tfrac{1}{w}) \\ & + \textstyle\sum_{k=1}^{r-1} \big((2k-1)w - 2\big)\log\big(1 + \frac{w}{1-kw}\big) \end{aligned} \tag{4.5b}$$

Furthermore, $\mathcal{E}_R(w) = \frac{1}{w} - \lfloor \frac{1}{w} \rfloor$, that is, $\mathcal{E}_R$ is the Gauss map, and

$$\boldsymbol{\mathcal{E}}_R\{w\}(q,\mathbf{h}) = \left(\frac{\mathrm{num1}_{\lfloor 1/w \rfloor}}{\mathrm{den}_{\lfloor 1/w \rfloor}}, \frac{\mathrm{num2}_{\lfloor 1/w \rfloor}}{\mathrm{den}_{\lfloor 1/w \rfloor}}\right)$$

Remark 4.1. In equation (4.5), we have $0 \le \frac{w}{1-kw} \le 1$ for all $1 \le k \le r-1$.

Proof. Let $w \in (0,1) \setminus \mathbb{Q}$. We show (4.3) by induction over $1 \le r \le \lfloor \frac{1}{w} \rfloor$. The $r=1$ base case of the induction argument, $\boldsymbol{\mathcal{Q}}_R\{w\}(q,\mathbf{h}) = (\mathrm{num1}_1/\mathrm{den}_1, \mathrm{num2}_1/\mathrm{den}_1)$, is by direct inspection, using (4.1). The induction step becomes the identity

$$\boldsymbol{\mathcal{Q}}_R\{\mathcal{Q}_R^{r-1}(w)\}\left(\frac{\mathrm{num1}_{r-1}}{\mathrm{den}_{r-1}}, \frac{\mathrm{num2}_{r-1}}{\mathrm{den}_{r-1}}\right) = \left(\frac{\mathrm{num1}_r}{\mathrm{den}_r}, \frac{\mathrm{num2}_r}{\mathrm{den}_r}\right) \tag{4.6}$$

for all $2 \le r \le \lfloor \frac{1}{w} \rfloor$. To calculate $\boldsymbol{\mathcal{Q}}_R\{\mathcal{Q}_R^{r-1}(w)\}(\cdot)$, use formulas (4.2), since $\mathcal{Q}_R^{r-1}(w) = \frac{1}{w} - r + 1 > 1$. Observe that (4.6) follows from the identities

$$\begin{aligned}
\lambda\, \mathrm{num1}_r &= \big(1 + (\tfrac{1}{w} - r + 1)\big)\big(\mathrm{den}_{r-1} + \mathrm{num1}_{r-1} + \mathrm{num2}_{r-1}\log 2\big) \\
&\quad - \mathrm{num2}_{r-1}\big(2 + (\tfrac{1}{w} - r + 1)\big)\log(1 + \tfrac{w}{1-(r-1)w}) \\
\lambda\, \mathrm{num2}_r &= \mathrm{num2}_{r-1}\left(\tfrac{1}{w} - r + 1\right) \\
\lambda\, \mathrm{den}_r &= \mathrm{den}_{r-1}\big(1 + (\tfrac{1}{w} - r + 1)\big) + \mathrm{num2}_{r-1}\log 2 \\
&\quad - \mathrm{num2}_{r-1}\big(1 + 2(\tfrac{1}{w} - r + 1)\big)\log(1 + \tfrac{w}{1-(r-1)w})
\end{aligned}$$

where $\lambda = 2 + \frac{1}{w} - r > 2$. To verify each of these identities, divide both sides by λ, and use $\mathrm{num2}_{r-1} = \mathbf{h}w\lambda$, to obtain the equivalent identities

$$\begin{aligned}
\mathrm{num1}_r &= \mathrm{den}_{r-1} + \mathrm{num1}_{r-1} + \mathrm{num2}_{r-1}\log 2 \\
&\quad - \mathbf{h}(3w + 1 - rw)\log(1 + \tfrac{w}{1-(r-1)w}) \\
\mathrm{num2}_r &= \mathbf{h}(1 - rw + w) \\
\mathrm{den}_r &= \mathrm{den}_{r-1} + \mathbf{h}w\log 2 - \mathbf{h}(3w + 2 - 2rw)\log(1 + \tfrac{w}{1-(r-1)w})
\end{aligned}$$

The last three identities are verified directly. □

The following lemma will be used later.

Lemma 4.2. *For every $w \in (0,1) \setminus \mathbb{Q}$ and every integer r with $1 \le r \le \lfloor \frac{1}{w} \rfloor$,*

$$\begin{array}{rcccl}
0 & \le & A_1(w,r) - rA_2(w,r) + \log 2 & \le & 6\frac{1}{w} \\
-8\log(1 + \frac{1}{w}) & \le & A_2(w,r) & \le & 0
\end{array}$$

Here, A_1 and A_2 are defined by (4.5)*.*

Proof. Observe that $rw \le 1$. Calculate

$$\begin{aligned}
\big\{A_1(w,r) - rA_2(w,r) + \log 2\big\}\, w \;&=\; w^2(r-1)\log 2 + rw(1-rw)\log 2 \\
+ (1-w)\, w\log(1 + \tfrac{1}{w}) &+ w\textstyle\sum_{k=1}^{r-1}\big(2k(1-kw) + (1-w)\big)\,\log(1 + \tfrac{w}{1-kw})
\end{aligned}$$

By inspection, the right hand side is non-negative, and bounded by

$$\le 3 + \tfrac{1}{r}\textstyle\sum_{k=1}^{r-1}\big(2k(1-kw) + (1-w)\big)\tfrac{w}{1-kw} \le 6$$

We have $A_2(w,r) \leq 0$, because $rw \leq 1$, the sum of the first two terms on the right hand side of (4.5b) is non-positive, and the third term is non-positive. Estimate

$$
\begin{aligned}
|A_2(w,r)| &\leq (2+w)\log(1+\tfrac{1}{w}) + 2\textstyle\sum_{k=1}^{r-1}\log(1+\frac{w}{1-kw}) \\
&\leq 3\log(1+\tfrac{1}{w}) + 2\textstyle\sum_{k=1}^{r-1}\frac{w}{1-kw} \leq 3\log(1+\tfrac{1}{w}) + 2\big(1+\int_0^{r-1}\mathrm{d}k\,\frac{w}{1-kw}\big) \\
&\leq 3\log(1+\tfrac{1}{w}) + 2\big(1-\log(1-(r-1)w)\big) \leq 8\log(1+\tfrac{1}{w})
\end{aligned}
$$

since $2 < 3\log 2 \leq 3\log(1+\frac{1}{w})$ and $-\log(1-rw+w) \leq -\log w = \log\frac{1}{w} \leq \log(1+\frac{1}{w})$. □

Proposition 4.2. *For every $w \in (0,1)\setminus\mathbb{Q}$, every $p>0$ and every integer $1 \leq r \leq \lfloor\frac{1}{w}\rfloor$, let (μ',ν') be the pair of rational functions over $\mathbb{R}$ in the pair of abstract variables (μ,ν) given implicitly by*

$$
\Big(p' + \frac{\mu'}{\nu'},\ \frac{1+w'}{\nu'}\Big) = \boldsymbol{\mathcal{Q}}_R^r\{w\}\,\Big(p+\frac{\mu}{\nu},\ \frac{1+w}{\nu}\Big)
$$

where $w' = \mathcal{Q}_R^r(w) = \frac{1}{w} - r$ and $p' = r - p/(1+p)$, that is $(p',0) = \boldsymbol{\mathcal{Q}}_R^r\{w\}(p,0)$. Then μ' is actually a linear polynomial over $\mathbb{R}$ in μ, and ν' is actually a linear polynomial over $\mathbb{R}$ in the pair (μ,ν). Explicitly

$$
\begin{pmatrix}\mu'\\ \nu'\end{pmatrix} = \frac{1}{w}\begin{pmatrix}-\frac{1}{1+p} & 0\\ 1 & 1+p\end{pmatrix}\begin{pmatrix}\mu\\ \nu\end{pmatrix} + \frac{1}{w}\begin{pmatrix}A_1(w,r) - p'\,A_2(w,r)\\ A_2(w,r)\end{pmatrix} \tag{4.7}
$$

The first and second entries of the vector

$$
\frac{1}{w}\begin{pmatrix}A_1(w,r) - p'\,A_2(w,r)\\ A_2(w,r)\end{pmatrix} \tag{4.8}
$$

are bounded in absolute value by $\leq 2^4(\frac{1}{w})^2$ and $\leq 2^3\frac{1}{w}\log(1+\frac{1}{w})$, respectively.

Proof. Equation (4.7) follows from equation (4.3). To check the bounds, observe that

$$
A_1(w,r) - p'A_2(w,r) = \Big(A_1(w,r) - r\,A_2(w,r) + \log 2\Big) - \log 2 + \tfrac{p}{1+p}\,A_2(w,r)
$$

Now, use Lemma 4.2 and $\log 2 \leq \frac{1}{w}$ and $\log(1+\frac{1}{w}) \leq \frac{1}{w}$. . □

Definition 4.4. *For every sequence of strictly positive integers $(k_n)_{n\geq 0}$, we denote the associated infinite continued fraction by*

$$
\langle k_0, k_1, \ldots\rangle \;=\; \frac{1}{k_0 + \frac{1}{k_1+\ldots}} \qquad \in \qquad \big(\tfrac{1}{k_0+1}, \tfrac{1}{k_0}\big)\setminus\mathbb{Q}
$$

Every element of $(0,1)\setminus\mathbb{Q}$ has a unique continued fraction expansion of this form.

We now show that when $\mathbf{h} = 0$, the era-to-era maps can be realized as a left-shift operator on two-sided sequences of positive integers.

Proposition 4.3. *Fix any two-sided sequence $(k_n)_{n\in\mathbb{Z}}$ of strictly positive integers and define two-sided sequences $(p_n)_{n\in\mathbb{Z}}$ and $(w_n)_{n\in\mathbb{Z}}$ by*

$$\tfrac{1}{1+p_n} = \langle k_n, k_{n-1}, k_{n-2}, \ldots\rangle \qquad w_n = \langle k_{n+1}, k_{n+2}, k_{n+3}\ldots\rangle \tag{4.9}$$

Then $w_{n+1} = \mathcal{E}_R(w_n)$ and $(p_{n+1},0) = \boldsymbol{\mathcal{E}}_R\{w_n\}(p_n,0)$ for all $n\in\mathbb{Z}$, and $\mathcal{E}_R^n(w_0) = w_n$ and $\boldsymbol{\mathcal{E}}_R^n\{w_0\}(p_0,0) = (p_n,0)$ for all $n\geq 0$.

Proof. Use $\mathcal{E}_R(w) = \frac{1}{w} - \lfloor\frac{1}{w}\rfloor$ and $\boldsymbol{\mathcal{E}}_R\{w\}(p,0) = (\lfloor\frac{1}{w}\rfloor - 1 + \frac{1}{1+p},\, 0)$. □

Definition 4.5. *Fix any two-sided sequence $(k_n)_{n\in\mathbb{Z}}$ of strictly positive integers and define $(p_n)_{n\in\mathbb{Z}}$ and $(w_n)_{n\in\mathbb{Z}}$ by* (4.9)*. For every integer $n\geq 0$, let (μ_n,ν_n) be the pair of linear polynomials over $\mathbb{R}$ in the abstract variables (μ_0,ν_0), with coefficients depending only on the fixed sequence $(k_n)_{n\in\mathbb{Z}}$, given implicitly by*

$$\Big(p_n + \frac{\mu_n}{\nu_n},\, \frac{1+w_n}{\nu_n}\Big) = \boldsymbol{\mathcal{E}}_R^n\{w_0\}\Big(p_0 + \frac{\mu_0}{\nu_0},\, \frac{1+w_0}{\nu_0}\Big) \tag{4.10a}$$

or by the equivalent recursive prescription

$$\Big(p_{n+1} + \frac{\mu_{n+1}}{\nu_{n+1}},\, \frac{1+w_{n+1}}{\nu_{n+1}}\Big) = \boldsymbol{\mathcal{E}}_R\{w_n\}\Big(p_n + \frac{\mu_n}{\nu_n},\, \frac{1+w_n}{\nu_n}\Big) \tag{4.10b}$$

By Proposition 4.2, equation (4.10b) *is $V_{n+1} = X_nV_n + Y_n$, where $V_n = (\mu_n,\nu_n)^T$ and*

$$X_n = \frac{1}{w_n}\begin{pmatrix} -\frac{1}{1+p_n} & 0 \\ 1 & 1+p_n\end{pmatrix} \qquad Y_n = \frac{1}{w_n}\begin{pmatrix} A_1(w_n) - p_{n+1}A_2(w_n) \\ A_2(w_n)\end{pmatrix}$$

Here, $A_1(w) = A_1(w,\lfloor\frac{1}{w}\rfloor)$ and $A_2(w) = A_2(w,\lfloor\frac{1}{w}\rfloor)$, see equations (4.5)*.*

Example 4.1. We consider Definition 4.5 when $k_n = 1$ for all $n\in\mathbb{Z}$. Then $w_n = p_n = w$ for all $n\in\mathbb{Z}$, where $w = \frac{1}{2}(\sqrt{5}-1)\in(0,1)\setminus\mathbb{Q}$. We have $w^2+w-1=0$ and $\lfloor\frac{1}{w}\rfloor = 1$ and

$$X_n = \begin{pmatrix} -1 & 0 \\ 1+w & 2+w\end{pmatrix} \qquad Y_n = \begin{pmatrix} -2\log(1+w) \\ (2+w)\log 2 - (6+4w)\log(1+w)\end{pmatrix}$$

for all $n\geq 0$. It follows that $\mu_{n+2} = \mu_n$ for all $n\geq 0$, that is, $\mu_{2n} = \mu_0$ and $\mu_{2n+1} = -\mu_0 - 2\log(1+w)$. There are unique $\lambda_1 = \lambda_1(\mu_0)$ and $\lambda_2 = \lambda_2(\mu_0)$, depending only on μ_0, such that $\nu_{2n+2}-\lambda_1 = (2+w)^2(\nu_{2n}-\lambda_1)$ and $\nu_{2n+3}-\lambda_2 = (2+w)^2(\nu_{2n+1}-\lambda_2)$. That is, $\nu_{2n} = (2+w)^{2n}(\nu_0-\lambda_1)+\lambda_1$ and $\nu_{2n+1} = (2+w)^{2n}(\nu_1-\lambda_2)+\lambda_2$. Here, $\nu_1 = (2+w)\nu_0 + (1+w)\mu_0 + (2+w)\log 2 - (6+4w)\log(1+w)$.

Definition 4.6 (Propagator). *Let $(p_n)_{n\in\mathbb{Z}}$, $(w_n)_{n\in\mathbb{Z}}$, $(X_n)_{n\geq 0}$ be as in Definition 4.5. Then for all integers $n\geq m\geq 0$, let $P_{n,m} = X_{n-1}\cdots X_m$. Explicitly,*

$$P_{n,m} = \begin{pmatrix} a_{n-1}\cdots a_m & 0 \\ \sum_{\ell=m}^{n-1} x_\ell & c_{n-1}\cdots c_m\end{pmatrix}$$

where $x_\ell = c_{n-1}\cdots c_{\ell+1}b_\ell a_{\ell-1}\cdots a_m$ whenever $n-1\geq\ell\geq m$, and for all $\ell\geq 0$,

$$X_\ell = \begin{pmatrix} a_\ell & 0 \\ b_\ell & c_\ell\end{pmatrix} \qquad a_\ell = \frac{-1}{w_\ell(1+p_\ell)} \qquad b_\ell = \frac{1}{w_\ell} \qquad c_\ell = \frac{1+p_\ell}{w_\ell}$$

In this definition, a sequence of dots $\cdots$ indicates that indices increase towards the left, one by one. A product of the form $F_k \cdots F_j$ is equal to one if $k = j-1$. In particular, $P_{n,n} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 4.3. *In the context of Definition 4.5, we have $V_n = P_{n,0}V_0 + \sum_{\ell=0}^{n-1} P_{n,\ell+1}Y_\ell$.*

Lemma 4.4. *Recall Definition 4.6. For all integers $n \geq m \geq 0$, we have*

$$\tfrac{1}{2} \leq \tfrac{w_{n-1}}{w_{m-1}}(-1)^{m+n}a_{n-1}\cdots a_m \leq 2 \tag{4.11a}$$

$$(1-\delta_{mn})\,\tfrac{1}{4} \leq \tfrac{w_{n-1}}{w_{m-1}^2}\big(w_{n-2}\cdots w_{m-1}\big)^2 \textstyle\sum_{\ell=m}^{n-1} x_\ell \leq 2 \tag{4.11b}$$

$$\tfrac{1}{2} \leq \tfrac{w_{n-1}}{w_{m-1}}\big(w_{n-2}\cdots w_{m-1}\big)^2 c_{n-1}\cdots c_m \leq 2 \tag{4.11c}$$

Moreover,

$$w_{n-2}\cdots w_{m-1} \leq |\rho_-|^{n-m-1} = \rho_+^{-n+m+1} \qquad \textit{when} \qquad n \geq m \geq 0 \tag{4.12}$$

Here, $\rho_\pm = \frac{1}{2}(1\pm\sqrt{5})$ are the roots of the polynomial $\rho^2-\rho-1$. Observe that $|\rho_-| < 1$. In this lemma, a sequence of dots $\cdots$ indicates that indices increase towards the left, one by one. A product of the form $F_k \cdots F_j$ is equal to one if $k = j-1$.

Proof. In this proof, abbreviate $v_\ell = 1/(1+p_{\ell+1}) = \langle k_{\ell+1}, k_\ell, k_{\ell-1}\ldots\rangle$. We have

$$(-1)^{m+n}a_{n-1}\cdots a_m = \frac{v_{n-2}\cdots v_{m-1}}{w_{n-2}\cdots w_{m-1}} \cdot \frac{w_{m-1}}{w_{n-1}} \tag{4.13a}$$

$$x_m = c_{n-1}\cdots c_{m+1}b_m = \frac{w_{n-2}\cdots w_m}{v_{n-2}\cdots v_m} \cdot \frac{w_{m-1}^2}{w_{n-1}} \cdot \left(\frac{1}{w_{n-2}\cdots w_{m-1}}\right)^2 \tag{4.13b}$$

$$c_{n-1}\cdots c_m = \frac{w_{n-2}\cdots w_{m-1}}{v_{n-2}\cdots v_{m-1}} \cdot \frac{w_{m-1}}{w_{n-1}} \cdot \left(\frac{1}{w_{n-2}\cdots w_{m-1}}\right)^2 \tag{4.13c}$$

where $n \geq m$ in (4.13a) and (4.13c) and $n > m$ in (4.13b). Each right hand side is written as a product of positive quotients, whose first factor is contained in the closed interval $[\frac{1}{2}, 2]$, see Proposition A.1 (a) of Appendix A. This implies (4.11a) and (4.11c). If $n = m$, the sum in (4.11b) vanishes and the estimate is trivial. Suppose $n > m$. We have $\operatorname{sgn} x_\ell = (-1)^{\ell+m}$. If, in addition, ℓ satisfies $n-2 \geq \ell \geq m$, we have $|x_{\ell+1}|/|x_\ell| = v_\ell v_{\ell-1} = v_\ell(v_\ell^{-1} - \lfloor v_\ell^{-1}\rfloor) \leq \frac{1}{2}$, that is $\frac{1}{2}|x_\ell| \geq |x_{\ell+1}|$. Therefore, the alternating sum in (4.11b) is non-negative and bounded from above by its first summand $x_m > 0$ and from below by $x_m + x_{m+1} \geq x_m - |x_{m+1}| \geq \frac{1}{2}x_m$. Actually, x_{m+1} is only defined when $n \geq m+2$, but $\frac{1}{2}x_m$ is a lower bound for all $n \geq m+1$. Now, estimate x_m, which is the left hand side of (4.13b).
Inequality (4.12) is a consequence of Proposition A.1 (b). □

Warning: In the next proposition, the sequences $(w_j)_{j\in\mathbb{Z}}$ and $(p_j)_{j\in\mathbb{Z}}$ do not have the property that w_j and $(1+p_j)^{-1}$ always lie in $(0,1)\setminus\mathbb{Q}$. Rather, they lie in $(0,\infty)\setminus\mathbb{Q}$. However, in the proof of Proposition 4.4, the auxiliary sequences $(w_n^*)_{n\in\mathbb{Z}}$ and $(p_n^*)_{n\in\mathbb{Z}}$ do have the property that w_n^* and $(1+p_n^*)^{-1}$ always lie in $(0,1)\setminus\mathbb{Q}$. The discussion beginning with Proposition 4.3 and ending just above will be applied to the auxiliary sequences.

Proposition 4.4. *For all $w_0 \in (0,1)\setminus\mathbb{Q}$ and $q_0 \in (0,\infty)\setminus\mathbb{Q}$, introduce*

- *a two sided sequence of strictly positive integers* $(k_n)_{n\in\mathbb{Z}}$ *by*

$$(1+q_0)^{-1} = \langle k_0, k_{-1}, k_{-2}, \ldots\rangle \qquad w_0 = \langle k_1, k_2, k_3, \ldots\rangle$$

- *(Era Pointer)* $J : \mathbb{Z} \to \mathbb{Z}$ *by* $J(0) = 0$ *and* $J(n+1) = J(n) + k_{n+1}$
- *(Era Counter)* $N : \mathbb{Z} \to \mathbb{Z}$ *by* $N(0) = 0$ *and* $N(j+1) = N(j) + \chi_{J(\mathbb{Z})}(j)$*, where* $\chi_{J(\mathbb{Z})}$ *is the characteristic function of the image* $J(\mathbb{Z}) \subset \mathbb{Z}$*; equivalently*

$$N(j) = \min\{n \in \mathbb{Z} \,|\, J(n) \geq j\} \tag{4.14}$$

- *sequences* $(w_j)_{j\in\mathbb{Z}}$ *and* $(p_j)_{j\in\mathbb{Z}}$ *by (observe that* w_0 *is defined consistently)*

$$w_j = \langle k_{N(j)+1}, k_{N(j)+2}, \ldots\rangle + J\big(N(j)\big) - j \tag{4.15a}$$

$$p_j = \langle k_{N(j)-1}, k_{N(j)-2}, \ldots\rangle + k_{N(j)} + j - J\big(N(j)\big) - 1 \tag{4.15b}$$

Part 1. *Then* $p_0 = q_0$ *and* $w_j, p_j > 0$ *and* $\mathcal{Q}_R(w_j) = w_{j+1}$ *and* $\boldsymbol{\mathcal{Q}}_R\{w_j\}(p_j, 0) = (p_{j+1}, 0)$ *for all* $j \in \mathbb{Z}$*, and* $\mathcal{Q}_R^j(w_0) = w_j$ *and* $\boldsymbol{\mathcal{Q}}_R^j\{w_0\}(q_0, 0) = (p_j, 0)$ *for all* $j \geq 0$*.*
Part 2. *Let* $\rho_+ = \frac{1}{2}\big(1+\sqrt{5}\big)$ *and set*

$$\mathbf{C}(w_0, q_0) = \sup\nolimits_{n\geq 0} (n+1)\rho_+^{-2n} k_n \, \max\{k_{n-1}, k_{n-2}\} \quad \in \quad [1, \infty]$$

Suppose $\mathbf{C}(w_0, q_0) < \infty$*. Fix any* $0 < \mathbf{h}_0 \leq 2^{-14}(\mathbf{C}(w_0, q_0))^{-1}$*. Then, there are sequences* $(q_j)_{j\geq 0}$*,* $(\mathbf{h}_j)_{j\geq 0}$ *of real numbers such that for every* $j \geq 0$*, the denominator appearing in the pair of rational functions* $\boldsymbol{\mathcal{Q}}_R\{w_j\}$*, given by* (4.1c) *or* (4.2c)*, is strictly positive at* $(q_j, \mathbf{h}_j)$*, and*

$$(q_{j+1}, \mathbf{h}_{j+1}) = \boldsymbol{\mathcal{Q}}_R\{w_j\}(q_j, \mathbf{h}_j)$$

or $(q_j, \mathbf{h}_j) = \boldsymbol{\mathcal{Q}}_R^j\{w_0\}(q_0, \mathbf{h}_0)$*. For all* $j \geq 0$*,*

- $0 < \mathbf{h}_j \leq 2^6\, \mathbf{h}_0\, \rho_+^{-2N(j)}$ *and*

$$\frac{1}{4} \quad \leq \quad \frac{\mathbf{h}_j}{\mathbf{h}_0}\, \frac{1+w_0}{1+w_j} \prod_{\ell=0}^{N(j)-1} \frac{1}{w_{J(\ell)} w_{J(\ell-1)}} \quad \leq \quad 4$$

- $q_j \in (0,\infty) \setminus \mathbb{Z}$ *and* $|q_j - p_j| \leq 2^{12}\, \mathbf{h}_0\, N(j)\, \rho_+^{-2N(j)} k_{N(j)}$
- $q_j \in (0,1)$ *if and only if* $p_j \in (0,1)$ *if and only if* $j - 1 \in J(\mathbb{Z})$
- $\max\{\frac{1}{w_j}, w_j, \frac{1}{q_j}, \frac{1}{|q_j-1|}, q_j\} \leq 2^4 \max\{k_{N(j)-2}, k_{N(j)-1}, k_{N(j)}, k_{N(j)+1}\}$

Part 3. *Let the map* $\mathcal{Q}_L : (0,\infty)^3 \to (0,\infty)^2 \times \mathbb{R}$ *be given as in Definition 3.16. Then the sequences* $(\mathbf{h}_j)_{j\geq 0}$*,* $(w_j)_{j\geq 0}$*,* $(q_j)_{j\geq 0}$ *in Part 2 satisfy for all* $j \geq 0$*:*

$$(\mathbf{h}_j, w_j, q_j) = \mathcal{Q}_L(\mathbf{h}_{j+1}, w_{j+1}, q_{j+1})$$

Example 4.2. In Part 1 of Proposition 4.4, suppose the continued fraction expansions begin as follows: $(1+q_0)^{-1} = \langle 1, 2, \ldots\rangle$ and $w_0 = \langle 3, 1, 2, 4 \ldots\rangle$. Then,

j	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$\chi_{J(\mathbb{Z})}(j)$	1	0	1	1	0	0	1	1	0	1	0	0	0	1
$N(j)$	-2	-1	-1	0	1	1	1	2	3	3	4	4	4	4
$J(N(j))$	-3	-1	-1	0	3	3	3	4	6	6	10	10	10	10

Proof (of Proposition 4.4). Two basic properties of J and N are, for all $j \in \mathbb{Z}$:

- $N \circ J$ is the identity; consequently $J(N(j)) = j$ if and only if $j \in J(\mathbb{Z})$
- $J(N(j)) \geq j$ and $J(N(j) - 1) \leq j - 1$ by (4.14); consequently

$$j \leq J(N(j)) \leq k_{N(j)} + j - 1 \tag{4.16}$$

The second bullet implies $w_j > 0$ and $p_j > 0$, for all $j \in \mathbb{Z}$. The first bullet implies that $w_j \in (0,1)$ if and only if $j \in J(\mathbb{Z})$. Therefore, we have

$$\mathcal{Q}_R(w_j) = \begin{cases} \frac{1}{w_j} - 1 & \text{if } j \in J(\mathbb{Z}) \\ w_j - 1 & \text{if } j \notin J(\mathbb{Z}) \end{cases} \qquad \boldsymbol{\mathcal{Q}}_R\{w_j\}(p_j, 0) = \begin{cases} (\frac{1}{1+p_j}, 0) & \text{if } j \in J(\mathbb{Z}) \\ (1 + p_j, 0) & \text{if } j \notin J(\mathbb{Z}) \end{cases}$$

In the case $j \notin J(\mathbb{Z})$, we have $N(j+1) = N(j)$, and therefore $w_{j+1} = w_j - 1$ and $p_{j+1} = p_j + 1$, as required. In the case $j \in J(\mathbb{Z})$, we have $N(j+1) = N(j) + 1$ and $J(N(j+1)) = J(N(j)+1) = J(N(j)) + k_{N(j)+1} = j + k_{N(j)+1}$, which implies

$$\begin{aligned} w_{j+1} &= \langle k_{N(j)+2}, k_{N(j)+3}, \ldots \rangle + k_{N(j)+1} - 1 = \tfrac{1}{w_j} - 1 \\ p_{j+1} &= \langle k_{N(j)}, k_{N(j)-1}, \ldots \rangle = \tfrac{1}{1+p_j} \end{aligned}$$

as required. **Part 1** is checked.
To prove **Part 2**, we first construct two sequences $(q_j)_{j \geq 0}$ and $(\mathbf{h}_j)_{j \geq 0}$. Then we verify that they have the desired properties. Below, a sequence of dots $\cdots$ in any product of the form $F_m \cdots F_n$ indicates that indices increase towards the left, one by one. The product is equal to one if $m = n - 1$. Define sequences $(w_n^*)_{n \in \mathbb{Z}}$ and $(p_n^*)_{n \in \mathbb{Z}}$ by $w_n^* = w_{J(n)} \in (0,1) \setminus \mathbb{Q}$ and $p_n^* = p_{J(n)} \in (0, \infty) \setminus \mathbb{Q}$. Equivalently,

$$\tfrac{1}{1+p_n^*} = \langle k_n, k_{n-1}, k_{n-2}, \ldots \rangle \qquad w_n^* = \langle k_{n+1}, k_{n+2}, k_{n+3}, \ldots \rangle$$

so that $w_{n+1}^* = \mathcal{E}_R(w_n^*)$ and $(p_{n+1}^*, 0) = \boldsymbol{\mathcal{E}}_R\{w_n^*\}(p_n^*, 0)$, by Proposition 4.3. Let $(V_n^*)_{n \geq 0}$, with $V_n^* = (\mu_n^*, \nu_n^*)^T$, as in Definition 4.5, be the solution to $V_{n+1}^* = X_n^* V_n^* + Y_n^*$ for all $n \geq 0$ with $\mu_0^* = 0$ and $\nu_0^* = (1 + w_0^*)/\mathbf{h}_0 > 0$, where

$$X_n^* = \frac{1}{w_n^*} \begin{pmatrix} -\frac{1}{1+p_n^*} & 0 \\ 1 & 1 + p_n^* \end{pmatrix} \qquad Y_n^* = \frac{1}{w_n^*} \begin{pmatrix} A_1(w_n^*) - p_{n+1}^* A_2(w_n^*) \\ A_2(w_n^*) \end{pmatrix}$$

Let $(V_j)_{j \geq 0}$, with $V_j = (\mu_j, \nu_j)^T$, be given by $V_0 = V_0^*$ and for all $j \geq 1$ by $V_j = X_{N(j)-1}^* V_{N(j)-1}^* + Y_j$, where

$$Y_j = \frac{1}{w_s^*} \begin{pmatrix} A_1(w_s^*, j - J(s)) - p_j A_2(w_s^*, j - J(s)) \\ A_2(w_s^*, j - J(s)) \end{pmatrix} \Bigg|_{s = N(j) - 1}$$

The functions A_1 and A_2, on the right hand side, are well defined at $(w_s, j - J(s))$, where $s = N(j) - 1$, because $1 \leq j - J(s) \leq k_{N(j)} = \lfloor 1/w_s^* \rfloor$. The following two observations will be used later on:

- Recall (4.15b). For all $j \geq 1$, $s = N(j) - 1$, we have $p_j = (j - J(s)) - p_s^*/(1 + p_s^*)$, and consequently the estimates after (4.8) apply to Y_j, $j \geq 1$. They also apply to Y_n^*, $n \geq 0$, because $p_{n+1}^* = \lfloor 1/w_n^* \rfloor - p_n^*/(1 + p_n^*)$.
- $Y_{J(n)} = Y_{n-1}^*$ for all $n \geq 1$, and consequently $V_{J(n)} = V_n^*$. The last identity is also true when $n = 0$, because $J(0) = 0$.

As in Definition 4.6, set $P^*_{n,m} = X^*_{n-1} \cdots X^*_m$ for all $n \geq m \geq 0$. For all $j \geq 1$, $s = N(j) - 1$, Lemma 4.3 implies

$$V_j = X^*_s \big(P^*_{s,0} V^*_0 + \textstyle\sum_{\ell=0}^{s-1} P^*_{s,\ell+1} Y^*_\ell \big) + Y_j = P^*_{s+1,0} V^*_0 + \textstyle\sum_{\ell=0}^{s-1} P^*_{s+1,\ell+1} Y^*_\ell + Y_j$$

The last equation, the estimates after (4.8), and the estimates in Lemma 4.4 imply

$$|\mu_j| \leq \frac{2^5}{w^*_s} \sum_{\ell=0}^{s} \frac{1}{w^*_\ell} \tag{4.17a}$$

$$\nu_j \geq \frac{1}{2w^*_s} \Big(\frac{1}{w^*_{s-1} \cdots w^*_{-1}} \Big)^2 \Big(w^*_{-1} \nu_0 - 2^8 \sum_{\ell=0}^{s} \big(w^*_{\ell-1} \cdots w^*_{-1} \big)^2 \ \log \Big(1 + \frac{1}{w^*_\ell} \Big) \Big) \tag{4.17b}$$

$$\nu_j \leq \frac{2}{w^*_s} \Big(\frac{1}{w^*_{s-1} \cdots w^*_{-1}} \Big)^2 \Big(w^*_{-1} \nu_0 + 2^6 \sum_{\ell=0}^{s} \big(w^*_{\ell-1} \cdots w^*_{-1} \big)^2 \ \log \Big(1 + \frac{1}{w^*_\ell} \Big) \Big) \tag{4.17c}$$

for all $j \geq 1$ and $s = N(j) - 1$. All three estimates are also true when $j = 0$, $s = -1$. Abbreviate $\mathbf{C} = \mathbf{C}(w_0, q_0) \geq 1$. We have $k_n \leq \mathbf{C} \rho_+^{2n}$ for all $n \geq 0$. Estimate

$$\begin{aligned}
&2^8 \textstyle\sum_{\ell=0}^{s} \big(w^*_{\ell-1} \cdots w^*_{-1} \big)^2 \ \log \big(1 + 1/w^*_\ell \big) \\
&\quad \leq \ 2^8 w^*_{-1} \textstyle\sum_{\ell=0}^{\infty} (\rho_+)^{-2\ell+2} \log(2 + k_{\ell+1}) \qquad \qquad \text{see inequality (4.12)} \\
&\quad \leq \ 2^8 w^*_{-1} \textstyle\sum_{\ell=0}^{\infty} (\rho_+)^{-2\ell+2} (2 + \log k_{\ell+1}) \\
&\quad \leq \ 2^{13} w^*_{-1} \big(1 + \log \mathbf{C} \big) \ \leq \ 2^{13} w^*_{-1} \mathbf{C} \ \leq \ 2^{-1} w^*_{-1} \tfrac{1}{\mathbf{h}_0} \ \leq \ 2^{-1} w^*_{-1} \nu_0
\end{aligned}$$

Hence, for all $j \geq 0$,

$$\frac{1}{4} \ \leq \ \frac{w^*_{N(j)-1}}{w^*_{-1}} \big(w^*_{N(j)-2} \cdots w^*_{-1} \big)^2 \frac{\mathbf{h}_0}{1 + w^*_0} \ \nu_j \ \leq \ 4 \tag{4.18}$$

Define sequences $(\mathbf{h}_j)_{j \geq 0}$ and $(q_j)_{j \geq 0}$ by $\mathbf{h}_j = (1 + w_j)/\nu_j > 0$ and $q_j = p_j + \mu_j/\nu_j$. These definitions are consistent when $j = 0$. Observe that $1 + w_j \leq 2 + J(N(j)) - j \leq 1 + k_{N(j)} \leq 2/w^*_{N(j)-1}$. Therefore, the estimates (4.12), (4.17), (4.18) imply for $j \geq 1$:

$$\frac{1}{4} \leq \frac{\mathbf{h}_j}{\mathbf{H}_j} \leq 4 \quad \text{where} \quad \mathbf{H}_j = \mathbf{h}_0 \, \frac{1 + w_j}{1 + w^*_0} \prod_{\ell=0}^{N(j)-1} \big(w^*_\ell w^*_{\ell-1} \big) \tag{4.19a}$$

$$\mathbf{H}_j \leq 2 \, \mathbf{h}_0 \, \big(\textstyle\prod_{\ell=0}^{N(j)-2} w^*_\ell \big) \big(\textstyle\prod_{\ell=-1}^{N(j)-2} w^*_\ell \big) \leq 2^4 \, \mathbf{h}_0 \, \rho_+^{-2N(j)} \tag{4.19b}$$

$$|q_j - p_j| \leq \frac{2^7 \mathbf{h}_0}{w^*_{N(j)-1}} \Big(\sum_{\ell=0}^{N(j)-1} \frac{1}{w^*_\ell} \Big) \prod_{\ell=0}^{N(j)-1} (w^*_\ell w^*_{\ell-1}) \leq 2^{12} \, \mathbf{h}_0 \, N(j) \, \rho_+^{-2N(j)} k_{N(j)}$$

The left hand sides are also less than or equal to the right hand sides when $j = 0$. Using (4.15b), one estimates

$$\begin{aligned}
\mathrm{dist}_{\mathbb{R}}(p_j, \mathbb{Z}) &= \mathrm{dist}_{\mathbb{R}} \big(\langle k_{N(j)-1}, k_{N(j)-2}, \ldots \rangle, \{0, 1\} \big) \\
&\geq \min \Big\{ \frac{1}{k_{N(j)-1} + 1}, \frac{1}{k_{N(j)-2} + 2} \Big\} \geq \frac{1}{3 \max\{k_{N(j)-1}, k_{N(j)-2}\}}
\end{aligned}$$

By the definition of $\mathbf{C}$ and by the assumption $\mathbf{h}_0 \leq 2^{-14}\mathbf{C}^{-1}$, we have $|q_j - p_j| \leq \frac{3}{4}\mathrm{dist}_{\mathbb{R}}(p_j, \mathbb{Z}) < \mathrm{dist}_{\mathbb{R}}(p_j, \mathbb{Z})$ for all $j \geq 0$. Therefore, $q_j \in (0, \infty) \setminus \mathbb{Z}$. Moreover, $q_j \in (0,1)$ iff $p_j \in (0,1)$ iff $k_{N(j)} + j - J(N(j)) - 1 = 0$ iff $J(N(j) - 1) = j - 1$ iff $N(j) - 1 = N(j-1)$ iff $j - 1 \in J(\mathbb{Z})$.
For every $j \geq 0$,

$$\begin{aligned} w_j &\leq J(N(j)) - j + 1 \leq k_{N(j)} \\ 1/w_j &\leq k_{N(j)+1} + 1 \\ q_j &\leq p_j + 1 \leq k_{N(j)} + j - J(N(j)) + 1 \leq k_{N(j)} + 1 \\ \big(\mathrm{dist}_{\mathbb{R}}(q_j, \mathbb{Z})\big)^{-1} &\leq 4\left(\mathrm{dist}_{\mathbb{R}}(p_j, \mathbb{Z})\right)^{-1} \leq 12 \max\{k_{N(j)-1}, k_{N(j)-2}\} \end{aligned}$$

Finally, we show that for all $j \geq 0$,

(a) the denominator of $\boldsymbol{\mathcal{Q}}_R\{w_j\}$, given by (4.1c) or (4.2c), is strictly positive at $(q_j, \mathbf{h}_j)$
(b) $(q_{j+1}, \mathbf{h}_{j+1}) = \boldsymbol{\mathcal{Q}}_R\{w_j\}(q_j, \mathbf{h}_j)$

For all $j \geq 0$, we have

$$2\mathbf{h}_j k_{N(j)+1} \leq 2(2^6 \mathbf{h}_0 \rho_+^{-2N(j)})(\mathbf{C}\rho_+^{2N(j)+2}) \leq 2^9 \mathbf{h}_0 \mathbf{C} \leq 2^{-5}$$

This implies $\mathbf{h}_j \log(1 + 1/w_j) \leq 2\mathbf{h}_j k_{N(j)+1} < 2^{-1}$, which by inspection of (4.1c) and (4.2c) implies (a). To show (b), observe that by construction of $(V_n^*)_{n\geq 0}$,

$$\Big(p_{n+1}^* + \frac{\mu_{n+1}^*}{\nu_{n+1}^*},\ \frac{1 + w_{n+1}^*}{\nu_{n+1}^*}\Big) = \boldsymbol{\mathcal{E}}_R\{w_n^*\}\Big(p_n^* + \frac{\mu_n^*}{\nu_n^*},\ \frac{1 + w_n^*}{\nu_n^*}\Big)$$

for all $n \geq 0$, see Definition 4.5 and Proposition 4.2. Since $V_{J(n)} = V_n^*$ for all $n \geq 0$ and since $\lfloor 1/w_n^* \rfloor = k_{n+1} = J(n+1) - J(n)$, the last equation is equivalent to

$$(q_{J(n+1)}, \mathbf{h}_{J(n+1)}) = \boldsymbol{\mathcal{Q}}_R^{J(n+1)-J(n)}\{w_{J(n)}\}(q_{J(n)}, \mathbf{h}_{J(n)}) \tag{4.20}$$

By Proposition 4.2 and by the construction of $(V_j)_{j\geq 0}$, for all $j \geq 1$, $s = N(j) - 1$,

$$\Big(p_j + \frac{\mu_j}{\nu_j},\ \frac{1 + 1/w_s^* - j + J(s)}{\nu_j}\Big) = \boldsymbol{\mathcal{Q}}_R^{j-J(s)}\{w_s^*\}\Big(p_s^* + \frac{\mu_s^*}{\nu_s^*},\ \frac{1 + w_s^*}{\nu_s^*}\Big)$$

Since $1/w_s^* - j + J(s) = w_j$, this implies $(q_j, \mathbf{h}_j) = \boldsymbol{\mathcal{Q}}_R^{j-J(s)}\{w_{J(s)}\}(q_{J(s)}, \mathbf{h}_{J(s)})$, for all $j \geq 1$, $s = N(j) - 1$. The last identity and (4.20) imply $(q_j, \mathbf{h}_j) = \boldsymbol{\mathcal{Q}}_R^j\{w_0\}(q_0, \mathbf{h}_0)$ for all $j \geq 0$, which is equivalent to (b).
To prove **Part 3**, check that for all $j \geq 0$ the following implication holds:

$$\left.\begin{aligned} w_{j+1} &= \mathcal{Q}_R(w_j) \\ (q_{j+1}, \mathbf{h}_{j+1}) &= \boldsymbol{\mathcal{Q}}_R\{w_j\}(q_j, \mathbf{h}_j) \end{aligned}\right\} \quad \Longrightarrow \quad \mathcal{Q}_L(\mathbf{h}_{j+1}, w_{j+1}, q_{j+1}) = (\mathbf{h}_j, w_j, q_j)$$

To make this calculation, distinguish the cases $j \in J(\mathbb{Z})$ and $j \notin J(\mathbb{Z})$, and recall $w_j, q_j \in (0,\infty) \setminus \mathbb{Z}$ and that $w_j \in (0,1)$ iff $j \in J(\mathbb{Z})$ iff $q_{j+1} \in (0,1)$. $\square$

5. An abstract semi-global existence theorem

This section is logically self-contained, and the notation is introduced from scratch. The objects in this section are abstractions of concrete objects that appear in other sections of this paper. This relationship is reflected in the choice of notation: abstract objects are named after their concrete counterparts, whenever possible. *This section is an independent unit. Definitions in other sections are irrelevant here and must be ignored.*

Definition 5.1. *For every integer $d \geq 1$, denote by $\|\cdot\|$ the Euclidean distance in $\mathbb{R}^d$. Set $B[\delta, \mathbf{f}] = \{\mathbf{g} \in \mathbb{R}^d \mid \|\mathbf{g} - \mathbf{f}\| \leq \delta\}$ for every $\delta \geq 0$ and every $\mathbf{f} \in \mathbb{R}^d$.*

Proposition 5.1. *Fix an integer $d \geq 1$. Suppose:*

(a) $\mathcal{F} \subset \mathbb{R}^d$ is a nonempty open subset and $B\mathcal{F} = \{(\delta, \mathbf{f}) \in [0,\infty) \times \mathcal{F} \,\big|\, B[\delta, \mathbf{f}] \subset \mathcal{F}\}$.
(b) $\mathit{\Pi}_j : \mathcal{F} \to \mathbb{R}^d$ is a continuous map, for every integer $j \geq 1$.
(c) $\mathcal{Q}_L : \mathcal{F} \to \mathbb{R}^d$ and $\mathrm{Err} : B\mathcal{F} \to [0,\infty)$ and $\mathrm{Lip} : B\mathcal{F} \to [0,\infty)$ are maps such that for all $(\delta, \mathbf{f}) \in B\mathcal{F}$:

$$\sup_{j\geq 1} \sup_{\mathbf{g}\in B[\delta,\mathbf{f}]} \|\mathit{\Pi}_j(\mathbf{g}) - \mathcal{Q}_L(\mathbf{g})\| \leq \mathrm{Err}(\delta, \mathbf{f}) \tag{5.1a}$$

$$\sup_{\mathbf{g},\mathbf{g}'\in B[\delta,\mathbf{f}],\ \mathbf{g}\neq\mathbf{g}'} \frac{\|\mathcal{Q}_L(\mathbf{g}) - \mathcal{Q}_L(\mathbf{g}')\|}{\|\mathbf{g} - \mathbf{g}'\|} \leq \mathrm{Lip}(\delta, \mathbf{f}) \tag{5.1b}$$

(d) $(\delta_j, \mathbf{f}_j)_{j\geq 0}$ is a sequence in $B\mathcal{F}$ so that $\mathbf{f}_{j-1} = \mathcal{Q}_L(\mathbf{f}_j)$ for all $j \geq 1$, and so that

$$\sum_{n=j+1}^{\infty} \Big\{ \prod_{k=j+1}^{n-1} \mathrm{Lip}(\delta_k, \mathbf{f}_k) \Big\} \mathrm{Err}(\delta_n, \mathbf{f}_n) \quad\leq\quad \delta_j \tag{5.2}$$

for all $j \geq 0$.

Then, there exists a sequence $(\mathbf{g}_j)_{j\geq 0}$ with $\mathbf{g}_j \in B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ such that for all $j \geq 1$:

$$\mathbf{g}_{j-1} = \mathit{\Pi}_j(\mathbf{g}_j)$$

Proof. For all integers $0 \leq j \leq \ell$, set

$$E_j^\ell = \sum_{n=j+1}^{\ell} \Big\{ \prod_{k=j+1}^{n-1} \mathrm{Lip}(\delta_k, \mathbf{f}_k) \Big\} \mathrm{Err}(\delta_n, \mathbf{f}_n) \quad\in\quad [0,\infty)$$

Then $E_j = \lim_{\ell\to\infty} E_j^\ell$ is the left hand side of (5.2). Observe that $E_j^j = 0$ and $E_j^\ell \leq E_j \leq \delta_j$ by (d). Moreover, $E_{j-1}^\ell = \mathrm{Lip}(\delta_j, \mathbf{f}_j) E_j^\ell + \mathrm{Err}(\delta_j, \mathbf{f}_j)$ when $1 \leq j \leq \ell$.

For all integers $0 \leq m \leq \ell$, let (A)$^{m,\ell}$ be the statement: *There is a finite sequence $\mathbf{g}^{m,\ell} = (\mathbf{g}_j^{m,\ell})_{m\leq j\leq\ell}$ with $\mathbf{g}_j^{m,\ell} \in B[E_j^\ell, \mathbf{f}_j] \subset B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ for all $m \leq j \leq \ell$, such that $\mathbf{g}_\ell^{m,\ell} = \mathbf{f}_\ell$ and $\mathbf{g}_{j-1}^{m,\ell} = \mathit{\Pi}_j(\mathbf{g}_j^{m,\ell})$ when $m+1 \leq j \leq \ell$.* Observe that if (A)$^{m,\ell}$ is true, then the sequence $\mathbf{g}^{m,\ell}$ is unique.

For every fixed $\ell \geq 0$, we show by induction over m, one-by-one from $m = \ell$ down to $m = 0$, that (A)$^{m,\ell}$ is true. The base case (A)$^{\ell,\ell}$ is trivial. For the induction step, let $1 \leq m \leq \ell$ and suppose (A)$^{m,\ell}$ is true. Define $\mathbf{g}^{m-1,\ell}$ by $\mathbf{g}_j^{m-1,\ell} = \mathbf{g}_j^{m,\ell} \in B[E_j^\ell, \mathbf{f}_j] \subset B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ when $m \leq j \leq \ell$, and set $\mathbf{g}_{m-1}^{m-1,\ell} = \mathit{\Pi}_m(\mathbf{g}_m^{m-1,\ell}) = \mathit{\Pi}_m(\mathbf{g}_m^{m,\ell}) \in \mathbb{R}^d$. The statement (A)$^{m-1,\ell}$ is true, if $\mathbf{g}_{m-1}^{m-1,\ell} \in B[E_{m-1}^\ell, \mathbf{f}_{m-1}]$, which follows from

$$\begin{aligned}
\|\mathbf{g}_{m-1}^{m-1,\ell} - \mathbf{f}_{m-1}\| &= \|\mathit{\Pi}_m(\mathbf{g}_m^{m,\ell}) - \mathcal{Q}_L(\mathbf{f}_m)\| \\
&\leq \|\mathit{\Pi}_m(\mathbf{g}_m^{m,\ell}) - \mathcal{Q}_L(\mathbf{g}_m^{m,\ell})\| + \|\mathcal{Q}_L(\mathbf{g}_m^{m,\ell}) - \mathcal{Q}_L(\mathbf{f}_m)\| \\
&\leq \mathrm{Err}(\delta_m, \mathbf{f}_m) + \mathrm{Lip}(\delta_m, \mathbf{f}_m) E_m^\ell = E_{m-1}^\ell
\end{aligned}$$

We have shown that (A)$^{m,\ell}$ is true for all $0 \le m \le \ell$. For all integers $0 \le j \le \ell$, set $\mathbf{g}_j^\ell = \mathbf{g}_j^{0,\ell} \in B[\delta_j, \mathbf{f}_j]$, where $\mathbf{g}^{0,\ell} = (\mathbf{g}_j^{0,\ell})_{0 \le j \le \ell}$ is the sequence in (A)$^{0,\ell}$. For every fixed $j \ge 0$, the sequence $\mathbf{g}_j = (\mathbf{g}_j^\ell)_{\ell \ge j}$ in the compact $B[\delta_j, \mathbf{f}_j]$ has a convergent subsequence $(\mathbf{g}_j^\ell)_{\ell \in \mathcal{L}_j}$, where $\mathcal{L}_j \subset [j, \infty) \cap \mathbb{Z}$ is infinite. One may choose $\mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots$, that is $\mathcal{L}_{j-1} \supset \mathcal{L}_j$ for all $j \ge 1$. Pick a sequence $(\ell_j)_{j \ge 0}$ with $\ell_j \in \mathcal{L}_j$ for all $j \ge 0$, such that $\ell_{j-1} < \ell_j$ for all $j \ge 1$. Set $\mathcal{L} = \{\ell_j \mid j \ge 0\}$. By construction, all but a finite number of elements of $\mathcal{L}$ are in $\mathcal{L}_j$, for every $j \ge 0$. That is, $(\mathbf{g}_j^\ell)_{\ell \in \mathcal{L} \cap [j,\infty)}$ converges. Set $\mathbf{g}_j = \lim_{\ell \to \infty,\ \ell \in \mathcal{L} \cap [j,\infty)} \mathbf{g}_j^\ell \in B[\delta_j, \mathbf{f}_j]$. For all $j \ge 1$,

$$\begin{aligned} \Pi_j(\mathbf{g}_j) &= \lim_{\ell \to \infty,\ \ell \in \mathcal{L} \cap [j,\infty)} \Pi_j(\mathbf{g}_j^\ell) \\ &= \lim_{\ell \to \infty,\ \ell \in \mathcal{L} \cap [j,\infty)} \mathbf{g}_{j-1}^\ell = \mathbf{g}_{j-1} \end{aligned}$$

because Π_j is continuous by (b). $\square$

6. Main Theorems

In this section, τ_*, $\mathbf{K}$, $\mathcal{F}$ are given just as in Definitions 3.17, 3.18, 3.19, and $\mathcal{Q}_L$ is the map in Definition 3.16.

Definition 6.1. *Let $\|\cdot\|$ be the Euclidean distance in $\mathbb{R}^3$. For every $\delta \ge 0$ and every $\mathbf{f} \in \mathbb{R}^3$, set $B[\delta, \mathbf{f}] = \{\mathbf{g} \in \mathbb{R}^3 \mid \|\mathbf{g} - \mathbf{f}\| \le \delta\}$.*

Definition 6.2. *Let $\mathcal{F} \subset (0,\infty)^3$ be as in Definition 3.19. For all $\zeta \ge 1$ set*

$$B_\zeta \mathcal{F} = \big\{(\delta, \mathbf{f}) \in [0,\infty) \times \mathcal{F} \mid B[\zeta\delta, \mathbf{f}] \subset \mathcal{F}\big\} \qquad \textit{and} \qquad B\mathcal{F} = B_1 \mathcal{F}$$

Lemma 6.1. *For all $(\delta, \mathbf{f}) \in B\mathcal{F}$ set*

$$\begin{aligned} W(\delta, \mathbf{f}) &= \max\{\tfrac{1}{w-\delta}, w+\delta, \tfrac{1}{q-\delta}, \tfrac{1}{|q-1|-\delta}, q+\delta\} && \in [1,\infty) \\ W(\mathbf{f}) = W(0, \mathbf{f}) &= \max\{\tfrac{1}{w}, w, \tfrac{1}{q}, \tfrac{1}{|q-1|}, q\} && \in [1,\infty) \end{aligned}$$

where $\mathbf{f} = (\mathbf{h}, w, q)$. Then:

(a) $W(\mathbf{g}) \le W(\delta, \mathbf{f})$ for all $\mathbf{g} \in B[\delta, \mathbf{f}]$.
(b) If $(\delta, \mathbf{f}) \in B_2\mathcal{F} \subset B\mathcal{F}$ then $W(\delta, \mathbf{f}) \le 2W(\mathbf{f})$.

Lemma 6.2. *Let $\mathrm{Err} : B\mathcal{F} \to [0,\infty)$ be given by*

$$\mathrm{Err}(\delta, \mathbf{f}) = 2^{40}\big(\tfrac{1}{\mathbf{h}-\delta}\big)^2 W(\delta, \mathbf{f})^5 \exp(-\tfrac{1}{\mathbf{h}} 2^{-9} W(\delta, \mathbf{f})^{-2})$$

where $\mathbf{f} = (\mathbf{h}, w, q)$. Then for all $(\delta, \mathbf{f}) \in B\mathcal{F}$, we have $\mathbf{K}(\mathbf{g}) \le \mathrm{Err}(\delta, \mathbf{f})$ for all $\mathbf{g} \in B[\delta, \mathbf{f}] \subset \mathcal{F}$ (see Definition 3.18).

Proof. Let $\mathbf{g} = (\mathbf{h}', w', q') \in B[\delta, \mathbf{f}]$. Then $\tau_*(\mathbf{g}) \ge \frac{1}{2} W(\mathbf{g})^{-2}$ and $0 < \mathbf{h} - \delta \le \mathbf{h}' \le \mathbf{h} + \delta \le 2\mathbf{h}$ and $\frac{1}{\mathbf{h}'} \ge \frac{1}{2\mathbf{h}}$. Hence, $\mathbf{K}(\mathbf{g}) \le 2^{40}(\frac{1}{\mathbf{h}-\delta})^2 W(\mathbf{g})^5 \exp(-\frac{1}{\mathbf{h}} 2^{-9} W(\mathbf{g})^{-2})$. Now use Lemma 6.1 (a). $\square$

Lemma 6.3. *Let $\mathcal{Q}_L$ be as in Definition 3.16. Set $\mathrm{Lip} : B\mathcal{F} \to [0,\infty)$, $\mathrm{Lip}(\delta, \mathbf{f}) = 2^{13} W(\delta, \mathbf{f})^3$. Then $\|\mathcal{Q}_L(\mathbf{g}) - \mathcal{Q}_L(\mathbf{g}')\| \le \mathrm{Lip}(\delta, \mathbf{f})\, \|\mathbf{g} - \mathbf{g}'\|$ for all $\mathbf{g}, \mathbf{g}' \in B[\delta, \mathbf{f}]$.*

Proof. Let $\mathbf{f} = (\mathbf{h}, w, q)$. If $\mathbf{g} = \mathbf{g}'$, there is nothing to prove. Suppose $\mathbf{g} \neq \mathbf{g}'$. In Lemma B.1 of Appendix B, set $\mathbf{f}_1 = (\mathbf{h}_1, w_1, q_1) = \mathbf{g}$ and $\mathbf{f}_2 = (\mathbf{h}_2, w_2, q_2) = \mathbf{g}'$. Observe that $0 < \mathbf{h}_i \leq 1$ by $\mathbf{g}, \mathbf{g}' \in B[\delta, \mathbf{f}] \subset \mathcal{F}$. Since $\delta < |q-1|$, either $q, q_1, q_2 < 1$ or $q, q_1, q_2 > 1$. We have $w_{\max} \leq \max\{W(\mathbf{g}), W(\mathbf{g}')\}$ and $q_{\max} \leq \max\{W(\mathbf{g}), W(\mathbf{g}')\}$ and $q_{\min}^{-1} = \max\{q_1^{-1}, q_2^{-1}\} \leq \max\{W(\mathbf{g}), W(\mathbf{g}')\}$. Now use $\log(2 + w_{\max}) \leq 1 + w_{\max}$ and Lemma 6.1 (a). $\square$

Theorem 6.1 (Main Theorem 1). *Recall the definitions of $\mathcal{P}_L$ and $\mathcal{Q}_L$ (Definition 3.16), $\mathcal{F}$ (Definition 3.19), Π (Proposition 3.3), $B_\varsigma\mathcal{F}$ (Definition 6.2), W (Lemma 6.1),* Err *(Lemma 6.2),* Lip *(Lemma 6.3). Suppose:*

(a) $(\mathbf{f}_j)_{j\geq 0}$, with $\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j) \in \mathcal{F}$, satisfies $\mathbf{f}_{j-1} = \mathcal{Q}_L(\mathbf{f}_j)$ for all $j \geq 1$.
(b) The sequence $(\delta_j)_{j\geq 0}$ given by

$$\delta_j = \sum_{\ell=j+1}^{\infty} \Big\{ \prod_{k=j+1}^{\ell-1} 2^{16} W(\mathbf{f}_k)^3 \Big\}\, 2^{47} \big(\tfrac{1}{\mathbf{h}_\ell}\big)^2\, W(\mathbf{f}_\ell)^5 \exp\Big(-\tfrac{1}{\mathbf{h}_\ell} 2^{-11} W(\mathbf{f}_\ell)^{-2} \Big)$$

satisfies $\delta_j < \infty$ and $(\delta_j, \mathbf{f}_j) \in B_2\mathcal{F}$ for all $j \geq 0$.
(c) $\pi_0 \in S_3$ and $(\pi_j)_{j\geq 0}$ is the unique sequence in S_3 that satisfies $\pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{f}_j)$ for all $j \geq 1$.
(d) $\sigma_* \in \{-1, +1\}^3$.

Then, there exists a sequence $(\mathbf{g}_j)_{j\geq 0}$ with $\mathbf{g}_j \in B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ such that for all $j \geq 1$:

$$\mathbf{g}_{j-1} = \Pi[\pi_j, \sigma_*](\mathbf{g}_j) \qquad \textit{and} \qquad \pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{g}_j)$$

Proof. We use Proposition 5.1, with the understanding that the abstract objects of Proposition 5.1 in the left column are given by the special objects in the right column:

d	3
$\mathcal{F}$	$\mathcal{F}$ as in Definition 3.19
Π_j	$\Pi[\pi_j, \sigma_*]$, see Proposition 3.3 and the hypotheses Theorem 6.1 (c), (d)
$\mathcal{Q}_L$	$\mathcal{Q}_L\|_{\mathcal{F}}$, with $\mathcal{Q}_L$ as in Definition 3.16
Err	Err as in Lemma 6.2
Lip	Lip as in Lemma 6.3
$(\delta_j, \mathbf{f}_j)$	$(\delta_j, \mathbf{f}_j)$ as in hypotheses Theorem 6.1 (a) and (b)

We check that the assumptions (a), (b), (c), (d) of Proposition 5.1 are satisfied:

(a) The definitions of $B\mathcal{F}$ in Proposition 5.1 and in Definition 6.2 are consistent.
(b) $\Pi[\pi_j, \sigma_*] : \mathcal{F} \to (0, \infty)^2 \times \mathbb{R} \subset \mathbb{R}^3$ is continuous, by Proposition 3.3.
(c) The domains of definition of $\mathcal{Q}_L|_{\mathcal{F}}$ and Err and Lip are just as required by Proposition 5.1 (c). For all $(\delta, \mathbf{f}) \in B\mathcal{F}$ and $\mathbf{g}, \mathbf{g}' \in B[\delta, \mathbf{f}] \subset \mathcal{F}$ and $j \geq 1$,

$$\|\Pi[\pi_j, \sigma_*](\mathbf{g}) - \mathcal{Q}_L|_{\mathcal{F}}(\mathbf{g})\| \leq \mathbf{K}(\mathbf{g}) \leq \mathrm{Err}(\delta, \mathbf{f})$$
$$\|\mathcal{Q}_L|_{\mathcal{F}}(\mathbf{g}) - \mathcal{Q}_L|_{\mathcal{F}}(\mathbf{g}')\| \leq \mathrm{Lip}(\delta, \mathbf{f})\, \|\mathbf{g} - \mathbf{g}'\|$$

by Proposition 3.3 (a) and by Lemmas 6.2 and 6.3. That is, (5.1a) and (5.1b) hold.

(d) By assumption, $(\delta_j, \mathbf{f}_j) \in B_2\mathcal{F} \subset B\mathcal{F}$ for all $j \geq 0$. Hence $\frac{1}{2}\mathbf{h}_j \leq \mathbf{h}_j - \delta_j$ and, by Lemma 6.1 (b), we have $W(\delta_j, \mathbf{f}_j) \leq 2W(\mathbf{f}_j)$. Consequently, for all $j \geq 0$,

$$\begin{aligned}
&\sum_{\ell=j+1}^{\infty} \big\{ \prod_{k=j+1}^{\ell-1} \mathrm{Lip}(\delta_k, \mathbf{f}_k) \big\} \,\mathrm{Err}(\delta_\ell, \mathbf{f}_\ell) \\
&= \sum_{\ell=j+1}^{\infty} \big\{ \prod_{k=j+1}^{\ell-1} 2^{13} W(\delta_k, \mathbf{f}_k)^3 \big\} \;\; 2^{40} \big(\tfrac{1}{\mathbf{h}_\ell - \delta_\ell}\big)^2 W(\delta_\ell, \mathbf{f}_\ell)^5 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\big(- \tfrac{1}{\mathbf{h}_\ell} 2^{-9} W(\delta_\ell, \mathbf{f}_\ell)^{-2} \big) \\
&\leq \sum_{\ell=j+1}^{\infty} \big\{ \prod_{k=j+1}^{\ell-1} 2^{16} W(\mathbf{f}_k)^3 \big\} \, 2^{47} \big(\tfrac{1}{\mathbf{h}_\ell}\big)^2 W(\mathbf{f}_\ell)^5 \exp\big(- \tfrac{1}{\mathbf{h}_\ell} 2^{-11} W(\mathbf{f}_\ell)^{-2} \big)
\end{aligned}$$

The last expression is equal to δ_j, and (5.2) is checked.

Now, Theorem 6.1 follows from Proposition 5.1. □

Theorem 6.2 (Main Theorem 2). *Suppose the vector $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0)$ satisfies the assumptions of Proposition 4.4, that is*

$$w_0 \in (0,1) \setminus \mathbb{Q} \qquad\qquad \mathbf{C}(w_0, q_0) < \infty \tag{6.1a}$$

$$q_0 \in (0,\infty) \setminus \mathbb{Q} \qquad\qquad 0 < \mathbf{h}_0 \leq 2^{-14} (\mathbf{C}(w_0, q_0))^{-1} \tag{6.1b}$$

Let $(k_n)_{n\in\mathbb{Z}}$ and $J : \mathbb{Z} \to \mathbb{Z}$ (Era Pointer) and $N : \mathbb{Z} \to \mathbb{Z}$ (Era Counter) and $(w_j)_{j\in\mathbb{Z}}$, $(q_j)_{j\geq 0}$, $(\mathbf{h}_j)_{j\geq 0}$ be just as in Proposition 4.4. Introduce the sequence $(\mathbf{f}_j)_{j\geq 0}$ by

$$\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j) \quad \in \quad (0,\infty)^3$$

Introduce sequences $(\mathbf{H}_j)_{j\geq 0}$ and $(K_j)_{j\geq 0}$ by

$$\begin{aligned}
\mathbf{H}_j &= \mathbf{h}_0 \, \tfrac{1+w_j}{1+w_0} \prod_{\ell=0}^{N(j)-1} w_{J(\ell)} w_{J(\ell-1)} && > 0 \\
K_j &= \max\{k_{N(j)-2}, k_{N(j)-1}, k_{N(j)}, k_{N(j)+1}\} && \geq 1
\end{aligned}$$

Suppose:

(a) $\mathbf{H}_j < 2^{-21} (K_j)^{-2}$ *for all* $j \geq 0$.
(b) $2^{71} \big(\frac{1}{\mathbf{H}_j}\big)^2 (K_j)^5 \exp\big(- \frac{1}{\mathbf{H}_j} 2^{-21} (K_j)^{-2} \big) < 1$ *for all* $j \geq 0$.
(c) The sequence $(\not\delta_j)_{j\geq 0}$ given by

$$\not\delta_j = \sum_{\ell=j+1}^{\infty} \Big\{ \prod_{k=j+1}^{\ell-1} 2^{28} (K_k)^3 \Big\} \, 2^{71} \big(\tfrac{1}{\mathbf{H}_\ell}\big)^2 \, (K_\ell)^5 \exp\Big(- \tfrac{1}{\mathbf{H}_\ell} 2^{-21} (K_\ell)^{-2} \Big) \qquad > 0$$

satisfies $\not\delta_j \leq 2^{-4} \mathbf{H}_j < \infty$.
(d) $\pi_0 \in S_3$ *and* $(\pi_j)_{j\geq 0}$ *is the unique sequence in* S_3 *that satisfies* $\pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{f}_j)$ *for all* $j \geq 1$.
(e) $\sigma_* \in \{-1, +1\}^3$.

Then $(\not\delta_j, \mathbf{f}_j) \in B_2\mathcal{F}$ for all $j \geq 0$ and there exists a sequence $(\mathbf{g}_j)_{j\geq 0}$ with $\mathbf{g}_j \in B[\not\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ such that for all $j \geq 1$:

$$\mathbf{g}_{j-1} = \Pi[\pi_j, \sigma_*](\mathbf{g}_j) \qquad \textit{and} \qquad \pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{g}_j)$$

Proof. By Proposition 4.4 and by hypotheses (a), (c) in Theorem 6.2, for all $j \geq 0$:

$$2^{-2}\mathbf{H}_j \leq \mathbf{h}_j \leq 2^2\mathbf{H}_j \tag{6.2a}$$

$$\max\{\tfrac{1}{w_j}, w_j, \tfrac{1}{q_j}, \tfrac{1}{|q_j-1|}, q_j\} \leq 2^4 K_j \tag{6.2b}$$

$$2^{-4}(K_j)^{-1} \leq \min\{w_j, q_j, |q_j - 1|\}$$

$$2\not{\delta}_j \leq 2^{-1}\min\{w_j, q_j, |q_j-1|, \mathbf{h}_j\}$$

Hence, $B[2\not{\delta}_j, \mathbf{f}_j] \subset (0,\infty)^3$ for every $j \geq 0$. Furthermore, for all $j \geq 0$ and all $(\mathbf{h}', w', q') \in B[2\not{\delta}_j, \mathbf{f}_j] \subset (0,\infty)^3$, we have $q' \neq 1$ and

$$2^{-3}\mathbf{H}_j \leq 2^{-1}\mathbf{h}_j \leq \mathbf{h}_j - 2\not{\delta}_j \leq \mathbf{h}' \leq \mathbf{h}_j + 2\not{\delta}_j \leq 2\mathbf{h}_j \leq 2^3\mathbf{H}_j \tag{6.3a}$$

and

$$\begin{aligned} &\max\left\{\tfrac{1}{w'}, w', \tfrac{1}{q'}, \tfrac{1}{|q'-1|}, q'\right\} \\ &\leq \max\left\{\tfrac{1}{w_j - 2\not{\delta}_j}, w_j + 2\not{\delta}_j, \tfrac{1}{q_j - 2\not{\delta}_j}, \tfrac{1}{|q_j-1| - 2\not{\delta}_j}, q_j + 2\not{\delta}_j\right\} \\ &\leq 2\,\max\left\{\tfrac{1}{w_j}, w_j, \tfrac{1}{q_j}, \tfrac{1}{|q_j-1|}, q_j\right\} \ \leq 2^5\, K_j \end{aligned} \tag{6.3b}$$

The last two estimates (6.3) imply $\tau_*(\mathbf{h}', w', q') \geq 2^{-11}(K_j)^{-2}$ and

$$\mathbf{K}(\mathbf{h}', w', q') \leq 2^{71}\big(\tfrac{1}{\mathbf{H}_j}\big)^2 (K_j)^5 \exp\big(-\tfrac{1}{\mathbf{H}_j}2^{-21}(K_j)^{-2}\big) < 1$$

The last inequality is hypothesis (b) in Theorem 6.2. Furthermore,

$$\mathbf{h}' \leq 2^3\mathbf{H}_j < 2^{-18}(K_j)^{-2} \leq 2^{-7}\tau_*(\mathbf{h}', w', q')$$

The second inequality is hypothesis (a) in Theorem 6.2. These estimates are true for all $(\mathbf{h}', w', q') \in B[2\not{\delta}_j, \mathbf{f}_j]$, and therefore $B[2\not{\delta}_j, \mathbf{f}_j] \subset \mathcal{F}$ for all $j \geq 0$, in particular $\mathbf{f}_j \in \mathcal{F}$ (see Definition 3.19). In other words, $(\not{\delta}_j, \mathbf{f}_j) \in B_2\mathcal{F}$.
The last result and the fact that $\mathbf{f}_{j-1} = \mathcal{Q}_L(\mathbf{f}_j)$ for all $j \geq 1$ (see Proposition 4.4) imply that Theorem 6.2 follows from Theorem 6.1, if we can show that $\delta_j \leq \not{\delta}_j$ for all $j \geq 0$, where δ_j is given as in Theorem 6.1. The inequality $\delta_j \leq \not{\delta}_j$ is a consequence of $W(\mathbf{f}_\ell) \leq 2^4 K_\ell$ and $2^{-2}\mathbf{H}_\ell \leq \mathbf{h}_\ell \leq 2^2\mathbf{H}_\ell$, where $j, \ell \geq 0$. $\square$

Theorem 6.3 (Main Theorem 3). *Fix constants* $\mathbf{D} \geq 1$, $\gamma \geq 0$. *Suppose the vector* $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0) \in (0,\infty)^3$ *satisfies*

(i) $w_0 \in (0,1) \setminus \mathbb{Q}$ *and* $q_0 \in (0,\infty) \setminus \mathbb{Q}$.
(ii) $k_n \leq \mathbf{D}\,\max\{1, n\}^\gamma$ *for all* $n \geq -2$, *with* $(k_n)_{n\in\mathbb{Z}}$ *as in Proposition 4.4, that is*

$$(1+q_0)^{-1} = \langle k_0, k_{-1}, k_{-2}, \ldots\rangle \qquad w_0 = \langle k_1, k_2, k_3, \ldots\rangle$$

(iii) $0 < \mathbf{h}_0 < \mathbf{A}^\sharp$ *where* $\mathbf{A}^\sharp = \mathbf{A}^\sharp(\mathbf{D}, \gamma) = 2^{-56}\mathbf{D}^{-4}(4(\gamma+1))^{-4(\gamma+1)}$.

Then

- *The assumptions* (6.1) *and (a), (b), (c) of Theorem 6.2 hold.*
- *Set* $\rho_+ = \frac{1}{2}(1+\sqrt{5})$. *The sequence* $(\not{\delta}_j)_{j\geq 0}$ *in Theorem 6.2 satisfies for all* $j \geq 0$*:*

$$\not{\delta}_j \leq \exp\big(-\tfrac{1}{\mathbf{h}_0}\mathbf{A}^\sharp\rho_+^{N(j)}\big) \qquad \textit{and} \qquad N(j) \geq \big(\mathbf{D}^{-1}j\big)^{1/(\gamma+1)} \tag{6.4}$$

where $N : \mathbb{Z} \to \mathbb{Z}$ *(Era Counter) is the map in Proposition 4.4.*

If $\gamma > 1$ and $\mathbf{D} > \frac{1}{\log 2}\frac{\gamma}{\gamma-1}$, then the set of all vectors $\mathbf{f}_0 \in (0,\infty)^3$ that satisfy (i), (ii), (iii) has positive Lebesgue measure.

Proof. Preliminaries. The following facts will be used without further comment:

- $a^{-x}x^b \leq \left(\frac{b}{e\,\log a}\right)^b$ for all real numbers $a > 1$, $b > 0$, $x \geq 0$ where $e = \exp(1)$.
- $a^b \leq c^d$ for all real numbers $1 \leq a \leq c$ and $0 \leq b \leq d$.
- $1 < \rho_+ < 2$ and $1 < e \log \rho_+ < 2$ where $e = \exp(1)$ and $\rho_+ = \frac{1}{2}(1+\sqrt{5})$.

Fix $\mathbf{D} \geq 1$ and $\gamma \geq 0$ as in Theorem 6.3. For all 5-tuples of real numbers $s = (s_1, s_2, s_3, s_4, s_5) \geq (0,0,0,1,0)$, set $\mathbf{A}(s) = 2^{-s_1 - s_2\gamma}\mathbf{D}^{-s_3}(s_4(\gamma+1))^{-s_5(\gamma+1)}$. Observe that $0 < \mathbf{A}(s) \leq 2^{-s_1} \leq 1$ and $\mathbf{A}(s) \leq \mathbf{A}(s')$ if $s \geq s'$.
Basic smallness assumptions. $k_n \leq \mathbf{D}\,\max\{1,n\}^\gamma$ for all $n \geq -2$ and $\mathbf{h}_0 < \mathbf{A}(\kappa)$. The vector $\kappa = (\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5) \geq (0,0,0,1,0)$ will be fixed during the proof.
Estimates 1. Recall Proposition 4.4 and $\rho_+ = \frac{1}{2}(1+\sqrt{5})$. For all $j \geq 0$, $n \geq 0$:

$$\begin{aligned}
\mathbf{C}(w_0,q_0) &= \sup\nolimits_{n\geq 0}(n+1)\rho_+^{-2n}k_n\max\{k_{n-1},k_{n-2}\}\\
&\leq 2\mathbf{D}^2\sup\nolimits_{n\geq 0}\rho_+^{-2n}\max\{1,n\}^{2(\gamma+1)}\\
&\leq 2\mathbf{D}^2(\gamma+1)^{2(\gamma+1)} = \mathbf{A}(1,0,2,1,2)^{-1}\\
J(n) &= \textstyle\sum_{\ell=1}^n k_\ell \leq \mathbf{D}\sum_{\ell=1}^n \ell^\gamma \leq \mathbf{D}n^{\gamma+1}\\
j &\leq J(N(j)) \leq \mathbf{D}N(j)^{\gamma+1}\\
N(j) &\geq (\mathbf{D}^{-1}j)^{1/(\gamma+1)}\\
\mathbf{H}_j &\leq 2^4\,\mathbf{h}_0\,\rho_+^{-2N(j)} \qquad \text{see (4.19)}\\
\mathbf{H}_j &\geq 2^{-1}\mathbf{h}_0\textstyle\prod_{\ell=0}^{N(j)-1}(k_\ell+1)^{-1}(k_{\ell+1}+1)^{-1}\\
&\geq 2^{-1}\mathbf{h}_0\textstyle\prod_{\ell=0}^{N(j)-1}(2\mathbf{D}(\ell+1)^\gamma)^{-2} \geq 2^{-1}\mathbf{h}_0\,\max\{1,2\mathbf{D}N(j)^\gamma\}^{-2N(j)}\\
K_j &\leq \mathbf{D}(N(j)+1)^\gamma \leq \mathbf{D}2^\gamma\max\{1,N(j)\}^\gamma\\
\mathbf{H}_jK_j^2 &\leq 2^{4+2\gamma}\mathbf{D}^2\mathbf{h}_0\rho_+^{-2N(j)}\max\{1,N(j)\}^{2\gamma}\\
&\leq 2^{4+2\gamma}\mathbf{D}^2\mathbf{h}_0\sup\nolimits_{n\geq 0}\rho_+^{-2n}\max\{1,n\}^{2(\gamma+1)}\\
&\leq 2^{4+2\gamma}\mathbf{D}^2\mathbf{h}_0\,(\gamma+1)^{2(\gamma+1)} = \mathbf{h}_0\,\mathbf{A}(4,2,2,1,2)^{-1}
\end{aligned}$$

Require $\kappa \geq (25,2,2,1,2)$. Then $\mathbf{H}_j < 2^{-21}(K_j)^{-2}$ and $\mathbf{h}_0 \leq 2^{-14}(\mathbf{C}(w_0,q_0))^{-1}$.
Estimates 2. Let $(\oint_j)_{j\geq 0}$ be as in Theorem 6.2. We claim that with proper choice of κ:

(A) $\oint_{J(n)} \leq 2^{-5}\mathbf{h}_0\big(2\mathbf{D}(n+1)^\gamma\big)^{-2(n+1)}\,\exp(-\frac{1}{\mathbf{h}_0}\mathbf{A}(\kappa)\rho_+^{n+1})$ for all $n \geq 0$.
(B) $\oint_j \leq 2^{-4}\mathbf{H}_j$ and $\oint_j \leq \exp(-\frac{1}{\mathbf{h}_0}\mathbf{A}(\kappa)\,\rho_+{}^{N(j)})$ for all $j \geq 0$.

We first check (A) $\Longrightarrow$ (B). Note that $\oint_j \geq \oint_{j+1}$, $j \geq 0$. Fix any $j \geq 0$. Set $n = N(j+1)-1 \geq 0$. By (A), by $j \geq J(n)$ (see the line before (4.16)) and by $n+1 \geq N(j)$,

$$\begin{aligned}
\oint_j \leq \oint_{J(n)} &\leq 2^{-5}\mathbf{h}_0\big(2\mathbf{D}(n+1)^\gamma\big)^{-2(n+1)}\exp(-\tfrac{1}{\mathbf{h}_0}\mathbf{A}(\kappa)\rho_+^{n+1})\\
&\leq \Big(2^{-5}\mathbf{h}_0\max\{1,2\mathbf{D}N(j)^\gamma\}^{-2N(j)}\Big)\exp(-\tfrac{1}{\mathbf{h}_0}\mathbf{A}(\kappa)\rho_+^{N(j)})
\end{aligned}$$

See the second bullet in the preliminaries. On the right hand side, both factors are ≤ 1 (use $\mathbf{h}_0 < \mathbf{A}(\kappa) \leq 1$). By the lower bound on $\mathbf{H}_j$ derived above, claim (B) follows.

We now check (A). For all $n \geq 0$:

$$
\begin{aligned}
&\oint_{J(n)} \\
&= \textstyle\sum_{m=n}^{\infty} \sum_{\ell=J(m)+1}^{J(m+1)} \Big\{ \prod_{k=J(n)+1}^{\ell-1} 2^{28} (K_k)^3 \Big\}\, 2^{71} \big(\tfrac{1}{\mathbf{H}_\ell}\big)^2 (K_\ell)^5 \\
&\qquad\qquad\qquad\qquad \times \exp\Big(- (2^{21}\mathbf{H}_\ell K_\ell^2)^{-1}\Big) \\
&\leq \textstyle\sum_{m=n}^{\infty} \sum_{\ell=J(m)+1}^{J(m+1)} \big(2^{15} \max_{1\leq k\leq \ell} K_k\big)^{3\ell+2} \big(\tfrac{1}{2\mathbf{H}_\ell}\big)^2 \exp\Big(- (2^{21}\mathbf{H}_\ell K_\ell^2)^{-1}\Big) \\
&\leq \textstyle\sum_{m=n}^{\infty} k_{m+1} \big(2^{15+\gamma}\mathbf{D}(m+1)^\gamma\big)^{3J(m+1)+2} \big(\tfrac{1}{\mathbf{h}_0}\big)^2 \big(2\mathbf{D}(m+1)^\gamma\big)^{4(m+1)} \\
&\qquad\qquad\qquad\qquad \times \exp\Big(- 2^{-25-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2(m+1)} (m+1)^{-2\gamma}\Big) \\
&\leq \big(\tfrac{1}{\mathbf{h}_0}\big)^2 \textstyle\sum_{m=n+1}^{\infty} \big(2^{15+\gamma}\mathbf{D} m^\gamma\big)^{(10\mathbf{D} m^{\gamma+1})} \exp\Big(- 2^{-25-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma}\Big)
\end{aligned}
$$

Since $2^5 \frac{1}{\mathbf{h}_0}(2\mathbf{D}(n+1)^\gamma)^{2(n+1)} \leq \frac{1}{\mathbf{h}_0}(2^6\mathbf{D}m^\gamma)^{2m}$ for all $m \geq n+1$, we have

$$
\begin{aligned}
&\mathbf{S}(n) \overset{\text{def}}{=} \oint_{J(n)} 2^5 \tfrac{1}{\mathbf{h}_0}(2\mathbf{D}(n+1)^\gamma)^{2(n+1)} \\
&\leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \textstyle\sum_{m=n+1}^{\infty} \big(2^{15+\gamma}\mathbf{D} m^\gamma\big)^{(12\mathbf{D} m^{\gamma+1})} \exp\Big(- 2^{-25-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma}\Big) \\
&\leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \textstyle\sum_{m=n+1}^{\infty} \exp\Big(12\mathbf{D} m^{\gamma+1} \log\big(2^{15+\gamma}\mathbf{D} m^\gamma\big) - 2^{-25-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma}\Big) \\
&\leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \textstyle\sum_{m=n+1}^{\infty} \exp\Big(2^9\mathbf{D}^2(\gamma+1) m^{\gamma+2} - 2^{-25-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma}\Big)
\end{aligned}
$$

The second term in the argument of the exponential dominates the first term, if we require $\kappa \geq (35, 2, 4, \frac{3}{2}, 3)$. More precisely, the absolute value of the second term is at least twice the absolute value of the first term. In fact,

$$
\begin{aligned}
&2^{35+2\gamma}\mathbf{D}^4(\gamma+1) \sup_{m\geq 1} \rho_+^{-2m} m^{3\gamma+2} \\
&\leq 2^{35+2\gamma}\mathbf{D}^4 \big(\tfrac{3}{2}(\gamma+1)\big)^{3(\gamma+1)} = \mathbf{A}(35, 2, 4, \tfrac{3}{2}, 3)^{-1} \leq \mathbf{A}(\kappa)^{-1} \leq \tfrac{1}{\mathbf{h}_0}
\end{aligned}
$$

Therefore,

$$
\mathbf{S}(n) \leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \textstyle\sum_{m=n+1}^{\infty} \exp\Big(- 2^{-26-2\gamma}\mathbf{D}^{-2} \tfrac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma}\Big)
$$

Moreover, $2^{26+2\gamma}\mathbf{D}^2 \sup_{m\geq 1} \rho_+^{-m} m^{2\gamma} \leq 2^{26+2\gamma}\mathbf{D}^2 (2(\gamma+1))^{2(\gamma+1)} = 2^{-2}\mathbf{A}_*^{-1}$, where $\mathbf{A}_* = \mathbf{A}(28, 2, 2, 2, 2)$. Require $\kappa \geq (28, 2, 2, 2, 2)$. Then $\mathbf{h}_0 \leq \mathbf{A}_*$ and

$$
\begin{aligned}
\mathbf{S}(n) &\leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \textstyle\sum_{m=n+1}^{\infty} \exp\big(- 4\tfrac{1}{\mathbf{h}_0}\mathbf{A}_* \rho_+^m\big) \\
&\leq \exp\big(- \tfrac{1}{\mathbf{h}_0}\mathbf{A}_* \rho_+^{n+1}\big) \Big(\big(\tfrac{1}{\mathbf{h}_0}\big)^3 \exp\big(- \tfrac{1}{\mathbf{h}_0}\mathbf{A}_*\big)\Big) \textstyle\sum_{m=1}^{\infty} \exp\big(- 2\rho_+^m\big)
\end{aligned}
$$

We have $\sum_{m=1}^{\infty} \exp\big(- 2\rho_+^m\big) \leq \frac{1}{2}\sum_{m=1}^{\infty} \rho_+^{-m} = \frac{1}{2}(\rho_+ - 1)^{-1} = \frac{1}{2}\rho_+ \leq 1$. Require $\kappa \geq (56, 4, 4, 2, 4)$. Then $\mathbf{h}_0 \leq \mathbf{A}(\kappa) \leq \mathbf{A}(56, 4, 4, 2, 4) = \mathbf{A}_*^2$, and

$$
\big(\tfrac{1}{\mathbf{h}_0}\big)^3 \exp\big(- \tfrac{1}{\mathbf{h}_0}\mathbf{A}_*\big) \leq \big(\tfrac{1}{\mathbf{h}_0}\big)^3 \exp\big(- (\tfrac{1}{\mathbf{h}_0})^{1/2}\big) \leq 8!\, \mathbf{h}_0 \leq 2^{16}\mathbf{h}_0 \leq 1
$$

Since $\mathbf{A}_* \geq \mathbf{A}(\kappa)$, we have $\mathbf{S}(n) \leq \exp(-\frac{1}{\mathbf{h}_0}\mathbf{A}(\kappa)\rho_+^{n+1})$. Fix $\kappa = (56, 4, 4, 2, 4)$. All the inequalities for κ hold, and claim (A) is proved. Let $\mathbf{A}^\sharp = \mathbf{A}(56, 0, 4, 4, 4)$, as in the statement of Theorem 6.3. Since $\mathbf{A}^\sharp \leq \mathbf{A}(\kappa)$, the condition $\mathbf{h}_0 < \mathbf{A}^\sharp$ in the statement of Theorem 6.3 implies the condition $\mathbf{h}_0 < \mathbf{A}(\kappa)$ used in this proof.
So far, we have verified the estimate (6.4), and we have verified the assumptions Theorem 6.2 (a), (c) and (6.1). In the assumption Theorem 6.2 (b), the cases $j \geq 1$ follow from Theorem 6.2 (a), (c). Since $\mathbf{H}_0 = \mathbf{h}_0$ and $K_0 \leq \mathbf{D}$, the remaining $j = 0$ case in Theorem 6.2 (b) follows from

$$\begin{aligned} 2^{71}(\tfrac{1}{\mathbf{H}_0})^2(K_0)^5 \exp(-\tfrac{1}{\mathbf{H}_0}2^{-21}(K_0)^{-2}) &\leq 2^{71}(\tfrac{1}{\mathbf{h}_0})^2\mathbf{D}^5 \exp(-\tfrac{1}{\mathbf{h}_0}2^{-21}\mathbf{D}^{-2}) \\ &\leq 2^{71}(\tfrac{1}{\mathbf{h}_0})^2\mathbf{D}^5\, 8!\, \big(\mathbf{h}_0 2^{21}\mathbf{D}^2\big)^8 \leq 2^{255}\mathbf{D}^{21}\mathbf{h}_0^6 \leq (\mathbf{h}_0/\mathbf{A}^\sharp)^6 < 1 \end{aligned}$$

Lebesgue measure of the set of admissible $\mathbf{f}_0$. The set of all $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0) \in (0, \infty)^3$ that satisfy (i), (ii), (iii) is a product $(0, \mathbf{A}^\sharp) \times F_w \times F_q$ (depending on $\mathbf{D}$ and γ), where $F_w \subset (0, 1) \setminus \mathbb{Q}$ and $F_q \subset (0, \infty) \setminus \mathbb{Q}$. Both $(0, \mathbf{A}^\sharp)$ and F_q have positive measure, because $\mathbf{A}^\sharp > 0$ and $(\frac{1}{2}, \frac{2}{3}) \setminus \mathbb{Q} \subset F_q$. In fact, if $q_0 \in (\frac{1}{2}, \frac{2}{3})$, then $1/(1 + q_0) = 1/(1 + 1/(1 + 1/(1 + x)))$ with $x = (2q_0 - 1)/(1 - q_0) \in (0, 1)$, that is $k_0 = k_{-1} = k_{-2} = 1 \leq \mathbf{D}$. Suppose $\gamma > 1$ and $\mathbf{D} > (\log 2)^{-1}\gamma/(\gamma - 1)$. Let $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ be the Gauss map from $(0, 1) \setminus \mathbb{Q}$ to itself. We have $k_{n+1} = \lfloor 1/G^n(w_0) \rfloor$ for all $n \geq 0$. For all $n \geq 0$, set

$$X_n = \big\{ w_0 \in (0,1)\backslash\mathbb{Q} \;\big|\; G^n(w_0) < \mathbf{D}^{-1}(n{+}1)^{-\gamma} \big\} = G^{-n}\big(\big(0,\ \mathbf{D}^{-1}(n{+}1)^{-\gamma}\big)\backslash\mathbb{Q} \big)$$

where G^{-n} is the n-th inverse image of sets. Let μ_G be the probability measure on $(0, 1) \setminus \mathbb{Q}$ with density $(\log 2)^{-1}(1 + x)^{-1}$ (with respect to the Lebesgue measure). It is well-known that $\mu_G(X) = \mu_G(G^{-1}(X))$ for all measurable $X \subset (0, 1) \setminus \mathbb{Q}$. Therefore,

$$\mu_G(X_n) = \mu_G\big(\big(0,\ \mathbf{D}^{-1}(n + 1)^{-\gamma}\big) \setminus \mathbb{Q}\big) = \tfrac{1}{\log 2} \log\big(1 + \tfrac{1}{\mathbf{D}(n+1)^\gamma}\big) \leq \tfrac{1}{\log 2} \tfrac{1}{\mathbf{D}(n+1)^\gamma}$$

Let X_n^c be the complement of X_n in $(0, 1) \setminus \mathbb{Q}$. Then $\bigcap_{n \geq 0} X_n^c \subset F_w$, since $w_0 \in X_n^c$ implies $k_{n+1} = \lfloor 1/G^n(w_0) \rfloor \leq 1/G^n(w_0) \leq \mathbf{D}(n + 1)^\gamma$. We have

$$\begin{aligned} &\mu_G(F_w) \geq \mu_G(\textstyle\bigcap_{n \geq 0} X_n^c) = 1 - \mu_G(\bigcup_{n \geq 0} X_n) \geq 1 - \sum_{n \geq 0} \mu_G(X_n) \\ &\geq 1 - \tfrac{1}{\mathbf{D}\log 2} \textstyle\sum_{n \geq 0} \tfrac{1}{(n+1)^\gamma} \geq 1 - \tfrac{1}{\mathbf{D}\log 2}\big(1 + \int_1^\infty x^{-\gamma}\mathrm{d}x\big) = 1 - \tfrac{1}{\mathbf{D}\log 2}\tfrac{\gamma}{\gamma - 1} > 0 \end{aligned}$$

Consequently, also the Lebesgue measure of F_w is positive. □

7. Causal structure and particle horizons

In this section we show that the spatially homogeneous vacuum spacetimes corresponding to those solutions of (1.1) that are obtained by combining Theorems 6.2 and 6.3 and Propositions 3.1 and 3.3, have "particle horizons" (see [Mis] for this notion), contradicting a conjecture in [Mis].

Theorem 7.1. *Let* $\mathbf{D}$, γ, $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0)$ *be as in Theorem 6.3. Let* $\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j)$, π_j, σ_* *and* $\mathbf{g}_j$ *be as in Theorem 6.2. Adopt the remaining notation of Theorems 6.2 and*

6.3. Denote the components of $\mathbf{g}_j \in \mathcal{F}$ *by* $(\mathbf{h}'_j, w'_j, q'_j)$*. Recall that* $\mathbf{h}'_j \in (0,1)$*. Fix a constant* $\lambda_0 > 0$*. Set* $\tau_0 = 0$*. Introduce sequences* $(\lambda_j)_{j\geq 0}$ *and* $(\tau_j)_{j\geq 0}$ *by*

$$\begin{aligned}
\lambda_j &= \lambda_{j-1}\, \big\{\Lambda[\pi_j,\sigma_*](\mathbf{g}_j)\big\}^{-1} \;\in\; (0,\lambda_0] && \textit{for all } j \geq 1 \\
\tau_j &= \tau_{j-1} + (\mathbf{h}'_j\lambda_j)^{-1}\big\{\tau_{1+}(\mathbf{g}_j) - \tau_{2-}[\pi_j,\sigma_*](\mathbf{g}_j)\big\} && \textit{for all } j \geq 1
\end{aligned}$$

Then:

(a) $\tau_j > \tau_{j-1}$ *for all* $j \geq 1$ *and* $\lim_{j\to\infty}\tau_j = +\infty$*.*
(b) The solution to (1.1) *with initial data* $\Phi(0) = \lambda_0\,\Phi_\star(\pi_0,\mathbf{g}_0,\sigma_*)$ *exists for all* $\tau \geq 0$*, that is* $\Phi = \alpha \oplus \beta : [0,\infty) \to \mathcal{D}(\sigma_*)$*, and* $\Phi(\tau_j) = \lambda_j\,\Phi_\star(\pi_j,\mathbf{g}_j,\sigma_*)$ *for all* $j \geq 0$*.*
(c) For all $j \geq 1$ *we have the bound*

$$\mathbf{M}_j \overset{\text{def}}{=} \sup_{\tau\in(\tau_{j-1},\tau_j)} \max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} \;\alpha_{\mathbf{j},\mathbf{k}}[\Phi](\tau) \;\leq\; -2^{-2}\lambda_j \min\{(w'_j)^2,(w'_j)^{-1}\}$$

Set $\zeta_{\mathbf{i}}(\tau) = -\frac{1}{2}\int_0^\tau \mathrm{d}s\,\alpha_{\mathbf{i}}(s)$ *for* $\mathbf{i} = 1,2,3$ *(see Proposition 2.1) and for all* $s \geq 0$ *set*

$$\mathbf{L}(s) \;\overset{\text{def}}{=}\; \int_s^\infty \mathrm{d}\tau\;\max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}} \exp\big(-\zeta_{\mathbf{j}} - \zeta_{\mathbf{k}}\big)$$

(see the right hand side of (2.1) *in Proposition 2.2). Then* $\mathbf{L}(s) \leq \mathbf{L}(0) < \infty$ *and* $\lim_{s\to\infty}\mathbf{L}(s) = 0$*.*

Proof. Proposition 3.3 implies $\frac{1}{2} \leq \mathbf{h}'_j\lambda_j(\tau_j - \tau_{j-1}) \leq 2^2$ and $\tau_j \geq \tau_{j-1} + (2\lambda_0)^{-1}$, which implies (a). Theorems 6.2, 6.3, and Propositions 3.1, 3.3 imply (b). Proposition 3.3 (e.5) implies (c). Estimate

$$\begin{aligned}
\mathbf{L}(0) &\leq \textstyle\sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} \mathrm{d}\tau \exp\Big(\frac{1}{2}\int_0^\tau \mathrm{d}\tau'\;\max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}}\alpha_{\mathbf{j},\mathbf{k}}[\Phi](\tau')\Big) \\
&\leq \textstyle\sum_{\ell=1}^\infty \int_{\tau_{\ell-1}}^{\tau_\ell} \mathrm{d}\tau \exp\Big(\frac{1}{2}\sum_{m=1}^{\ell-1}\int_{\tau_{m-1}}^{\tau_m} \mathrm{d}\tau'\;\max_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{C}}\alpha_{\mathbf{j},\mathbf{k}}[\Phi](\tau')\Big) \\
&\leq \textstyle\sum_{\ell=1}^\infty (\tau_\ell - \tau_{\ell-1}) \exp\Big(\frac{1}{2}\sum_{m=1}^{\ell-1}(\tau_m - \tau_{m-1})\,\mathbf{M}_m\Big) \\
&\leq 2^2 \textstyle\sum_{\ell=1}^\infty (\mathbf{h}'_\ell\lambda_\ell)^{-1} \exp\Big(-2^{-4}\sum_{m=1}^{\ell-1}(\mathbf{h}'_m)^{-1}\;\min\{(w'_m)^2,(w'_m)^{-1}\}\Big)
\end{aligned}$$

By Theorem 6.2, we have $(\not\delta_j,\mathbf{f}_j) \in B_2\mathcal{F}$ and $\mathbf{g}_j \in B[\not\delta_j,\mathbf{f}_j]$ for all $j \geq 0$. Hence, $\frac{1}{2}\mathbf{h}_j \leq \mathbf{h}'_j \leq 2\mathbf{h}_j$ and $\frac{1}{2}w_j \leq w'_j \leq 2w_j$. Proposition 3.3 (b) implies $\Lambda[\pi_j,\sigma_*](\mathbf{g}_j) \leq 1 + \lambda_L(\mathbf{g}_j) \leq 3 + w'_j \leq 3(1+w_j)$ and $\lambda_\ell^{-1} \leq \lambda_0^{-1}\prod_{k=1}^\ell 3(1+w_k)$. Therefore,

$$\begin{aligned}
\mathbf{L}(0) &\leq 2^3(\lambda_0)^{-1}\textstyle\sum_{\ell=1}^\infty(\mathbf{h}_\ell)^{-1}\Big(\prod_{k=1}^\ell 3(1+w_k)\Big) \\
&\qquad\qquad\qquad \times\exp\Big(-2^{-7}\textstyle\sum_{k=1}^{\ell-1}(\mathbf{h}_k)^{-1}\;\min\{(w_k)^2,(w_k)^{-1}\}\Big) \\
&\leq 2^5(\lambda_0)^{-1}\textstyle\sum_{m=0}^\infty\sum_{\ell=J(m)+1}^{J(m+1)}(\mathbf{H}_\ell)^{-1}(2^7\max_{1\leq k\leq\ell}K_k)^\ell \\
&\qquad\qquad\qquad \times\exp\Big(-2^{-17}(1-\delta_{\ell 1})(\mathbf{H}_{\ell-1})^{-1}(K_{\ell-1})^{-2}\Big)
\end{aligned} \tag{7.1}$$

where $\mathbf{H}_j$ and K_j are as in Theorem 6.2, and $\delta_{\ell 1}$ is a Kronecker delta. See (6.2). In the exponential, we have bounded the sum over $k = 1,\ldots,\ell-1$ from below by its $k = \ell-1$ summand if $\ell \geq 2$ and by zero otherwise. The sum over $\ell = J(m)+1,\ldots,J(m+1)$ has $k_{m+1} \leq \mathbf{D}(m+1)^\gamma$ many terms. By the proof of Theorem 6.3, for every $m \geq 0$, the following estimates, uniformly in $\ell = J(m)+1,\ldots,J(m+1)$, hold:

- $(\mathbf{H}_\ell)^{-1} \leq 2(\mathbf{h}_0)^{-1}(2\mathbf{D}(m+1)^\gamma)^{2(m+1)}$
- $(2^7 \max_{1\leq k\leq \ell} K_k)^\ell \leq (2^{7+\gamma}\mathbf{D}(m+1)^\gamma)^{\mathbf{D}(m+1)^{\gamma+1}}$
- $(\mathbf{H}_{\ell-1})^{-1} \geq 2^{-4}(\mathbf{h}_0)^{-1}\rho_+^{2m}$ where $\rho_+ = \frac{1}{2}(1+\sqrt{5})$
- $(K_{\ell-1})^{-2} \geq \mathbf{D}^{-2}2^{-2\gamma}(m+1)^{-2\gamma}$

By these estimates, in particular the fact that $(\mathbf{H}_{\ell-1})^{-1}$ grows at least exponentially in m, the right hand side of (7.1) is finite, and $\mathbf{L}(0) < \infty$. $\square$

A. Bounds for a particular product of continued fractions

This appendix is entirely self-contained, the notation is completely local. Its single purpose is to prove Proposition A.1 below, which is used in the proof of Lemma 4.4.

Definition A.1. *For all integers m and n and all sequences $(x_i)_{i\in\mathcal{I}}$ where $\mathcal{I} \subset \mathbb{Z}$, define $x_{m:n}$ to be the ordered sequence $x_m, x_{m+1}, \ldots, x_{n-1}, x_n$ if $m \leq n$ and the empty sequence if $m > n$. In the first case, it is required that $[m,n]\cap\mathbb{Z} \subset \mathcal{I}$. Similarly, define $x_{m::n}$ to be the ordered sequence $x_m, x_{m-1}, \ldots, x_{n+1}, x_n$ if $m \geq n$ and the empty sequence if $m < n$. In the first case, it is required that $[n,m]\cap\mathbb{Z} \subset \mathcal{I}$.*

Definition A.2 (Continued fractions). *For every integer $n \geq 0$ and every finite sequence of strictly positive integers $(k_i)_{1\leq i\leq n}$ set recursively*

$$\langle k_{1:n}\rangle = \begin{cases} 0 & n=0 \\ \big(k_1 + \langle k_{2:n}\rangle\big)^{-1} & n \geq 1 \end{cases} \qquad \in \qquad [0,1]\cap\mathbb{Q}$$

For every infinite sequence $(k_i)_{i\geq 1}$ of strictly positive integers, set

$$\langle k_1, k_2, \ldots\rangle = \lim_{n\to\infty}\langle k_{1:n}\rangle \quad \in \quad (0,1)\setminus\mathbb{Q}$$

Example A.1. $\langle\,\rangle = \langle k_{1:0}\rangle = 0$ and $\langle k_1\rangle = \langle k_{1:1}\rangle = 1/k_1$ and $\langle k_1, k_2\rangle = \langle k_{1:2}\rangle = 1/(k_1 + 1/k_2)$.

Definition A.3 (Fibonacci numbers). $F_1 = F_2 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$.

Proposition A.1. *For every two-sided sequence of strictly positive integers $(k_i)_{i\in\mathbb{Z}}$, define two-sided sequences $(v_i)_{i\in\mathbb{Z}}$ and $(w_i)_{i\in\mathbb{Z}}$ by $v_i = \langle k_i, k_{i-1}, k_{i-2}, \ldots\rangle$ and $w_i = \langle k_i, k_{i+1}, k_{i+2}, \ldots\rangle$. Then, for all integers $M < N$:*

(a) $\frac{1}{2} \leq \prod_{i=M+1}^{N}(v_i/w_i) \leq 2$
(b) $\prod_{i=M+1}^{N} w_i \leq (F_{N-M+1})^{-1} \leq (\frac{1}{2}(\sqrt{5}-1))^{N-M-1}$

The proof of Proposition A.1 is given at the end of this appendix.

Definition A.4. *Let $P_0(\,) = 1$ and $P_1(x_1) = x_1$ and for all $n \geq 2$, set*

$$P_n(x_{1:n}) = x_1 P_{n-1}(x_{2:n}) + P_{n-2}(x_{3:n}) \tag{A.1}$$

Example A.2. $P_2(x_{1:2}) = 1 + x_1x_2$ and $P_3(x_{1:3}) = x_1 + x_3 + x_1x_2x_3$ and $P_4(x_{1:4}) = 1 + x_1x_2 + x_3x_4 + x_1x_4 + x_1x_2x_3x_4$.

Lemma A.1. *Recall Definition A.4. For all integers $n \geq 0$, we have:*

(a) P_n is a polynomial of degree n, jointly in its n arguments, with coefficients in $\{0,1\}$
(b) P_n is a polynomial of degree 1, separately in each of its n arguments
(c) $P_n(1,\dots,1) = F_{n+1}$
(d) $P_n(x_{1:n}) = P_n(x_{n::1})$ for all $x_1,\dots,x_n \in \mathbb{R}$
(e) $\langle k_{1:n}\rangle = P_{n-1}(k_{2:n})/P_n(k_{1:n})$ for all strictly positive integers $(k_i)_{1\le i\le n}$, $n \ge 1$

Proof. (a) through (e) are all shown by induction, using (A.1). To show (d), observe that (d)$_0$, (d)$_1$, (d)$_2$ and (d)$_3$ hold. For the induction step, let $n \ge 4$ and suppose (d)$_0$ through (d)$_{n-1}$ hold. Then, using only (A.1) and the induction hypothesis,

$$\begin{aligned}
&P_n(x_{1:n}) - P_n(x_{n::1})\\
&= x_1 P_{n-1}(x_{2:n}) + P_{n-2}(x_{3:n}) - x_n P_{n-1}(x_{n-1::1}) - P_{n-2}(x_{n-2::1})\\
&= x_1 P_{n-1}(x_{n::2}) + P_{n-2}(x_{n::3}) - x_n P_{n-1}(x_{1:n-1}) - P_{n-2}(x_{1:n-2})\\
&= x_1\big(x_n P_{n-2}(x_{n-1::2}) + P_{n-3}(x_{n-2::2})\big) + \big(x_n P_{n-3}(x_{n-1::3}) + P_{n-4}(x_{n-2::3})\big)\\
&\quad - x_n\big(x_1 P_{n-2}(x_{2:n-1}) + P_{n-3}(x_{3:n-1})\big) - \big(x_1 P_{n-3}(x_{2:n-2}) + P_{n-4}(x_{3:n-2})\big)
\end{aligned}$$

Verify that all the terms cancel, by the induction hypothesis. This implies (d)$_n$. To show (e), observe that (e)$_1$ holds. Let $n \ge 2$ and suppose (e)$_{n-1}$ holds. Then,

$$\langle k_{1:n}\rangle = \big(k_1 + \langle k_{2:n}\rangle\big)^{-1} = \left(k_1 + \frac{P_{n-2}(k_{3:n})}{P_{n-1}(k_{2:n})}\right)^{-1} = \frac{P_{n-1}(k_{2:n})}{k_1 P_{n-1}(k_{2:n}) + P_{n-2}(k_{3:n})}$$

Now, (A.1) implies (e)$_n$. □

Lemma A.2. *For all integers $m-1 \le M < N \le n$ and all $x_m,\dots,x_n \in [1,\infty)$,*

$$2\,P_{M-m+1}(x_{m:M})P_{n-M}(x_{M+1:n}) - P_{N-m+1}(x_{m:N})P_{n-N}(x_{N+1:n}) \ge 0 \quad \text{(A.2)}$$

Moreover, if $m = M+1$, then the factor 2 on the left hand side can be dropped, that is,

$$P_{n-M}(x_{M+1:n}) - P_{N-M}(x_{M+1:N})P_{n-N}(x_{N+1:n}) \ge 0 \quad \text{(A.3)}$$

Proof. In this proof, we use the recursion relation (A.1) and the reflected recursion relation that is obtained by applying Lemma A.1 (d) to all three terms of (A.1). Fix M and N. Inequality (A.2) is proved by induction over m and n, where $m \le M+1$ and $n \ge N$. Denote the left hand side of (A.2) by $Q_{m,n}$. Then,

$$\begin{aligned}
Q_{M+1,N} &= P_{N-M}(x_{M+1:N}) \ge 0\\
Q_{M+1,N+1} &= 2\,P_{N+1-M}(x_{M+1:N+1}) - P_{N-M}(x_{M+1:N})\,x_{N+1}\\
&= P_{N-M}(x_{M+1:N})\,x_{N+1} + 2\,P_{N-1-M}(x_{M+1:N-1}) \ge 0\\
Q_{M,N} &= 2\,x_M P_{N-M}(x_{M+1:N}) - P_{N-M+1}(x_{M:N})\\
&= x_M P_{N-M}(x_{M+1:N}) - P_{N-M-1}(x_{M+2:N})\\
&\ge x_M x_{M+1} P_{N-M-1}(x_{M+2:N}) - P_{N-M-1}(x_{M+2:N}) \ge 0\\
Q_{M,N+1} &= 2\,x_M P_{N+1-M}(x_{M+1:N+1}) - P_{N-M+1}(x_{M:N})x_{N+1}\\
&= 2\,x_M x_{N+1} P_{N-M}(x_{M+1:N}) + 2\,x_M P_{N-1-M}(x_{M+1:N-1})\\
&\qquad - x_M P_{N-M}(x_{M+1:N})x_{N+1} - P_{N-M-1}(x_{M+2:N})x_{N+1}\\
&\ge x_M x_{N+1} P_{N-M}(x_{M+1:N}) - P_{N-M-1}(x_{M+2:N})x_{N+1}\\
&\ge x_M x_{N+1} x_{M+1} P_{N-M-1}(x_{M+2:N}) - P_{N-M-1}(x_{M+2:N})x_{N+1} \ge 0
\end{aligned}$$

These four cases and the two recursion relations

- $Q_{m,n} = x_m Q_{m+1,n} + Q_{m+2,n}$ when $m \le M-1$ and $n \ge N$
- $Q_{m,n} = x_n Q_{m,n-1} + Q_{m,n-2}$ when $m \le M+1$ and $n \ge N+2$

imply (A.2). Inequality (A.3) is shown in an entirely similar way. □

Proof (of Proposition A.1). Recall (d), (e) in Lemma A.1. Let $m-1 \le M < N \le n$.

$$\prod_{i=M+1}^{N} \frac{\langle k_{i::m}\rangle}{\langle k_{i:n}\rangle} = \prod_{i=M+1}^{N} \frac{P_{i-m}(k_{i-1::m})}{P_{i-m+1}(k_{i::m})} \cdot \frac{P_{n-i+1}(k_{i:n})}{P_{n-i}(k_{i+1:n})}$$
$$= \frac{P_{M-m+1}(k_{M::m})}{P_{N-m+1}(k_{N::m})} \cdot \frac{P_{n-M}(k_{M+1:n})}{P_{n-N}(k_{N+1:n})} = \frac{P_{M-m+1}(k_{m:M})}{P_{N-m+1}(k_{m:N})} \cdot \frac{P_{n-M}(k_{M+1:n})}{P_{n-N}(k_{N+1:n})}$$

The right hand side is $\ge \frac{1}{2}$, by inequality (A.2). Now, let $m \to -\infty$ and $n \to +\infty$ to obtain $\prod_{i=M+1}^{N}(v_i/w_i) \ge \frac{1}{2}$. By symmetry, we also have $\prod_{i=M+1}^{N}(w_i/v_i) \ge \frac{1}{2}$. This implies (a) in Proposition A.1. Similarly, using (A.3),

$$\prod_{i=M+1}^{N} \langle k_{i:n}\rangle = \frac{P_{n-N}(k_{N+1:n})}{P_{n-M}(k_{M+1:n})} \le \frac{1}{P_{N-M}(k_{M+1:N})} \le \frac{1}{P_{N-M}(1,\dots,1)}$$

Let $n \to +\infty$ to obtain $\prod_{i=M+1}^{N} w_i \le 1/F_{N-M+1}$. □

B. The modulus of continuity of the map $\mathcal{Q}_L$ introduced in Definition 3.16

Lemma B.1. *Let $\mathcal{Q}_L : (0,\infty)^3 \to (0,\infty)^2 \times \mathbb{R}$ be the map introduced in Definition 3.16. For all $\mathbf{f}_i = (\mathbf{h}_i, w_i, q_i) \in (0,\infty)^3$ with $0 < \mathbf{h}_i \le 1$, $i = 1, 2$, with $\mathbf{f}_1 \ne \mathbf{f}_2$, such that q_1 and q_2 are either both < 1 or both > 1,*

$$\frac{\|\mathcal{Q}_L(\mathbf{f}_2) - \mathcal{Q}_L(\mathbf{f}_1)\|_{\mathbb{R}^3}}{\|\mathbf{f}_2 - \mathbf{f}_1\|_{\mathbb{R}^3}} \le \begin{cases} 2^{12} q_{\min}^{-2} \log(2 + w_{\max}) & \textit{if } q_1, q_2 < 1 \\ 2^{11} q_{\max} & \textit{if } q_1, q_2 > 1 \end{cases}$$

Here, $w_{\max} = \max\{w_1, w_2\}$ and $q_{\max} = \max\{q_1, q_2\}$ and $q_{\min} = \min\{q_1, q_2\}$.

Proof. We prove the following claim, which implies the Lemma: *Each of the nine partial derivatives of $\mathcal{Q}_L : \mathbf{f} = (\mathbf{h}, w, q) \mapsto \mathcal{Q}_L(\mathbf{f})$ is bounded in absolute value by*

$$\begin{cases} 2^{10} q^{-2} \log(2 + w) & \textit{if } \mathbf{f} \in (0,1] \times (0,\infty) \times (0,1) \\ 2^{9} q & \textit{if } \mathbf{f} \in (0,1] \times (0,\infty) \times (1,\infty) \end{cases} > 1 \tag{B.1}$$

Let $0 < \mathbf{h} \le 1$ and $q \ne 1$. Let $(\mathbf{h}_L, w_L, q_L) = \mathcal{Q}_L(\mathbf{f})$ and let $\mathrm{num1}_L$, $\mathrm{num2}_L$, den_L be as in Definition 3.16. We first estimate the partial derivatives of $q_L = \mathrm{num1}_L/\mathrm{den}_L$ and $\mathbf{h}_L = \mathrm{num2}_L/\mathrm{den}_L$. Each of $\mathrm{num1}_L$, $\mathrm{num2}_L$, den_L is of the form

$$L_1(w,q) + L_2(w,q)q + L_3(w,q)\mathbf{h} + L_4(w,q)\mathbf{h}\log\lambda_L(\mathbf{f})$$

with $\lambda_L(\mathbf{f}) = 1 + 1/w_L(\mathbf{f})$ as in Definition 3.16 and with $L_i(w,q) = a_i(q)w + b_i(q)$ where $a_i(q)$ and $b_i(q)$ are constant separately for $q < 1$ and for $q > 1$ and satisfy $-3 \le a_i(q), b_i(q) \le 3$, where $i = 1,2,3,4$. Let $k = 1,2$ and $\mathrm{num}k_L = L_1 + L_2 q +$

$L_3\mathbf{h} + L_4\mathbf{h}\log\lambda_L$ and $\mathrm{den}_L = L'_1 + L'_2 q + L'_3\mathbf{h} + L'_4\mathbf{h}\log\lambda_L$ with $L_i = a_i w + b_i$ and $L'_i = a'_i w + b'_i$ (*Warning*: the prime does *not* denote a derivative). Then

$$\begin{aligned}&\left(\tfrac{\partial}{\partial x}\mathrm{num}k_L\right)\mathrm{den}_L - \left(\tfrac{\partial}{\partial x}\mathrm{den}_L\right)\mathrm{num}k_L\\ &= \begin{cases} (L_3 + L_4\ \log\lambda_L)(L'_1 + L'_2 q) - (L'_3 + L'_4\log\lambda_L)(L_1 + L_2 q) & \text{if } x = \mathbf{h}\\ \;+(a_1 + a_2 q + a_3\mathbf{h} + a_4\mathbf{h}\log\lambda_L)(b'_1 + b'_2 q + b'_3\mathbf{h} + b'_4\mathbf{h}\log\lambda_L) & \\ \;-(a'_1 + a'_2 q + a'_3\mathbf{h} + a'_4\mathbf{h}\log\lambda_L)(b_1 + b_2 q + b_3\mathbf{h} + b_4\mathbf{h}\log\lambda_L) & \text{if } x = w\\ \;\;+\mathbf{h}\Big\{L_4\big(L'_1 + L'_2 q + L'_3\mathbf{h}\big) - L'_4\big(L_1 + L_2 q + L_3\mathbf{h}\big)\Big\}\tfrac{\partial}{\partial w}\log\lambda_L & \\ L_2(L'_1 + L'_3\,\mathbf{h} + L'_4\mathbf{h}\ \log\lambda_L) - L'_2(L_1 + L_3\,\mathbf{h} + L_4\mathbf{h}\ \log\lambda_L) & \text{if } x = q \end{cases}\end{aligned}$$

Recall that $|\mathbf{h}| \le 1$ and $|a_i|, |a'_i|, |b_i|, |b'_i| \le 3$ and $|L_i|, |L'_i| \le 3(1+w)$ and $\log\lambda_L \ge 0$.

- If $q < 1$, then $|\frac{\partial}{\partial w}\log\lambda_L| \le (1+w)^{-1}$ and

$$\begin{aligned}&\left|\left(\tfrac{\partial}{\partial x}\mathrm{num}k_L\right)\mathrm{den}_L - \left(\tfrac{\partial}{\partial x}\mathrm{den}_L\right)\mathrm{num}k_L\right|\\ &\quad\le \left\{\begin{array}{ll} 36(1+w)^2(1+\log\lambda_L) & \text{if } x = \mathbf{h}\\ 18(3+\log\lambda_L)^2 + 54(1+w) & \text{if } x = w\\ 18(1+w)^2(2+\log\lambda_L) & \text{if } x = q\end{array}\right\} \le 2^{10}(1+w)^2\log(2+w)\end{aligned}$$

  For the second inequality, use $\frac{1}{2} \le \log\lambda_L \le 1 + w$ and $(\log\lambda_L)^2 \le 1 + w$.
- If $q > 1$, then $|\log\lambda_L| \le 1$ and $|\frac{\partial}{\partial w}\log\lambda_L| \le (1+w)^{-2}$ and $a'_2 = b'_2 = 0$ and

$$\begin{aligned}&\left|\left(\tfrac{\partial}{\partial x}\mathrm{num}k_L\right)\mathrm{den}_L - \left(\tfrac{\partial}{\partial x}\mathrm{den}_L\right)\mathrm{num}k_L\right|\\ &\qquad\qquad\le \left\{\begin{array}{ll} 72(1+w)^2\,q & \text{if } x = \mathbf{h}\\ 270\,q & \text{if } x = w\\ 27(1+w)^2 & \text{if } x = q\end{array}\right\} \le 2^9\,(1+w)^2\,q\end{aligned}$$

To finish the proof, observe that $\mathrm{den}_L \ge (1+w)\min\{1, q\} > 0$. Each partial derivative of $q_L = \mathrm{num}1_L/\mathrm{den}_L$ and $\mathbf{h}_L = \mathrm{num}2_L/\mathrm{den}_L$ is bounded in absolute value by (B.1). And so are the partial derivatives of w_L, because $\partial w_L/\partial\mathbf{h} = \partial w_L/\partial q = 0$, and because $\partial w_L/\partial w = -(1+w)^{-2}$ if $q < 1$ and $\partial w_L/\partial w = 1$ if $q > 1$. $\square$

References

[Be] Béguin, F., arXiv 1004.2984 (2010)
[BKL1] Belinskii, V.A., Khalatnikov, I.M., and Lifshitz, E.M., Adv. Phys. **19**, (1970) 525-573
[HU1] Heinzle, J.M. and Uggla, C., arXiv 0901.0776 (2009)
[HU2] Heinzle, J.M. and Uggla, C., arXiv 0901.0806 (2009)
[LHWG] Liebscher, S., Härterich, J., Webster, K. and Georgi, M., arXiv 1004.1989 (2010)
[Mis] Misner, C.W., Phys. Rev. Lett. **22**, (1969) 1071-1074
[Ren] Rendall, A., Class. Quantum Grav. **14**, (1997) 2341-2356
[Ri1] Ringström, H., Class. Quantum Grav. **17** (2000) 713
[Ri2] Ringström, H., Annales Henri Poincaré **2** (2001) 405