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submitted by

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"Running Couplings in the Standard Model and their Implications to possible Physics beyond the Standard Model"

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Abstract

This thesis uses and derives the methods for the calculation of the running couplings in the Standard Model of Particle Physics. On the basis of the running couplings for the three gauge couplings and the Higgs quartic coupling, the phenomena of coupling unification and the instability of the electroweak vacuum are discussed in search of potential implications which hint to beyond the Standard Model Physics. For this, the concepts of Supersymmetric Theories and Grand Unification Theories are discussed in comparison to the predictions of the Standard Model. The effect of adding a pair of fermions charged under SU(3) to the Standard Model is calculated to quantify the change of running couplings by postulating additional particles and to relate this to the previous discussion.

Abstract

Diese Bachelorarbeit leitet die Methoden der Berechnung von laufenden Kopplungen im Standardmodell der Teilchenphysik her und verwendet diese. Vor dem Hintergrund der laufenden Kopplungen der drei Eichkopplungen und der quartischen Higgskopplung werden die Phänomene der Kopplungsvereinheitlichung und der Stabilität des elektroschwachen Vakuums diskutiert um Hinweise auf potentielle Physik hinter dem Standardmodell zu finden. Hierzu werden die Konzepte von Supersymmetrie und großen vereinheitlichenden Theorien im Vergleich zu den Vorhersagen des Standardmodells diskutiert. Der Effekt eines zusätzlichen Paares Fermionen im Standardmodell welches unter SU(3) geladen ist wird berechnet um die Veränderung in den laufenden Kopplungen bei zusätzlichen postulierten Teilchen zu quantifizieren und dies mit der vorherigen Diskussion in Zusammenhang zu bringen.

Contents

1 Introduction

The physicists at the current generation of particle accelerators have been able to very precisely measure scatterings and their cross-sections in the Standard Model of Particle Physics. The last remaining ingredient, the existence of the Higgs boson was recently proved, as in [Aad12]. Therefore the Standard Model is believed to be the most valid description of particle physics available today. The aim of this thesis is to recreate and collect extrapolations of the Standard Model into the regions currently unavailable to experiment and see whether any direct or indirect evidence of beyond the Standard Model physics can be found.
Most of the calculations and derivations have been done and studied by a lot of physicists, as this area of research has been of interest since the derivation of the Standard Model. Therefore this thesis mostly collects and recreates calculations and arguments from sources.

The following paragraphs will give a very short introduction on what the basics of a quantum field theory concerning the running couplings are. For a thorough introduction please refer to [Wei13] or any book on the topic.

The Standard Model of Particle Physics is a Quantum Field Theory, describing the interactions of Fields using the Lagrangian of the theory and the appropriate Feynman rules, which translate Feynman diagrams into mathematical expressions. Since, in calculations of a field theory, infinities are possible to arise, every theory has to undergo a process called renormalization, removing those infinities by absorbing them into the parameters of the Lagrangian, thereby loosing predictability of a number of physical observables. If the number of observables lost is finite, the theory is called renormalizable. A renormalizable theory has to be supplied with the values of the lost predictions and is then able to predict all other interactions in the model. The setting of those parameters is called setting the renormalization conditions. They specify the values of certain Feynman diagrams at some energy scale to the experimentally measured values.
By absorbing the infinities into the unphysical parameters of the Lagrangian and therefore rendering the physical observables finite, the parameters of the Lagrangian become divergent. It is possible to split the divergent parameters of the Lagrangian into the physical observable parameters and the divergent corrections thereof, called the counter-terms. These counter terms ensure that the Feynman diagrams are set to the physical observables, therefore they have to cancel the divergence of the appropriate Feynman diagrams at the energy scale of the renormalization conditions.
There are mainly two different methods to extract the divergences out of the Feynman diagrams and thereby calculate the counter terms. In cutoff-regularisation, the momentum integrals up to arbitrarily high momenta are cut off at the cutoff Λ and after the calculation, the limit $\Lambda \to \infty$ is taken. In dimensional regularisation, the momentum integrals are done in d spacetime dimensions and at the end, the limit $d \to 4$ is taken. Both methods enable to extract the divergences into divergences of either Λ or d.
Since the counterterms only exactly cancel the divergences of the diagrams at the renormalization scale M of the theory, at other energy scales M' the parameters receive finite corrections. This enables to specify a equally well description at this different energy scale M', using new parameters. This comes down to the fact that coupling constants and masses and all other

parameters of the theory are effectively scale dependant. This is done by introducing the effective couplings and other effective parameters which specify the value of the physical observable at an arbitrary energy scale. The behaviour of the effective parameters is specified in the Callan-Symanzik Equation, which has to hold for every Green's function in the theory.

$$\left[M\frac{\partial}{\partial M} + \sum_{\text{couplings}\{g_i\}} \beta(g_i)\frac{\partial}{\partial g_i} + \sum_{\text{fields}\{i\}} n_i\gamma_i(g)\right] G^{(n)}(\{x_i\}; M, \{g_i\}) = 0 \tag{1}$$

This equation governs the behaviour of the effective couplings by solving it for the β-functions. The effective coupling are usually also referred to as the running couplings, as they run with the energy scale p of the theory according to:

$$\frac{\partial}{\partial \ln(p/M)}\overline{g_i} = \beta(\overline{g_i}) \tag{2}$$

$$\text{with: } \overline{g_i}(p = M) = g_i \text{ (taken from experiment at the renormalization scale } M) \tag{3}$$

The running couplings describe the effective strength of the coupling at any energy scale and their behaviour is described by their β-functions.

This thesis will derive the running couplings on one loop level and calculate the running couplings on one and two loop level for the Standard Model of Particle Physics. This will be done for the three gauge couplings, the Top Yukawa coupling and the Higgs quartic coupling to explore the phenomena of coupling unification and the stability of the electroweak vacuum. These phenomena are explored to form a discussion of hints towards beyond the Standard Model Physics in the predictions of the Standard Model. Supersymmetry and Grand Unification are considered and discussed as potential theories, both postulating new particles. Therefore the effect of postulating an additional pair of fermions charged under $SU(3)$ on the running couplings is explored in the last section.

2 Renormalization Conditions and Electroweak Symmetry Breaking

A key ingredient in specifying a theory is to choose a scale M at which to specify the, until then, arbitrary physical parameters of our theory. They are set to experimentally measured values because the prediction of those parameters was lost in the process of renormalization. The relations defining these parameters are therefore called "renormalization conditions". In a general gauge theory, this is usually done by setting the gauge-boson-fermion-vertex, other vertices, the gauge boson propagator and the fermion propagator to the experimental transition amplitudes, measured at some specified energy scale, preferably the same for all renormalization conditions.
Those experimental inputs are vital for the further discussion of the theory because they serve as the initial conditions for the numerical solving of the differential equations for the running couplings.

One problem remains in specifying the renormalization condition for the gauge theories, such as the Standard Model of Particle Physics or the minimal supersymmetric extension thereof. Those theories to be examined in this thesis use the concept of symmetry breaking on their electroweak sector of $SU(2)_L \times U(1)_Y$ to $U(1)_{e.m.}$. Only the symmetry broken $U(1)_{e.m.}$ can be measured experimentally. Therefore it is necessary to understand this concept in order to relate the measured values to the coupling constants of the non-broken theory, which are going to be discussed here. Therefore a short introduction to this process called "The Higgs Mechanism" and derivation of the appropriate equations is necessary.
This will closely follow [Daw98, Chapter 2.1 and 2.2].

2.1 Simple Case of Symmetry Breaking: The Abelian Higgs Model

For the first example of symmetry breaking, a $U(1)$ massless gauge theory together with a single complex scalar field coupled to it is considered. The gauge field has to be massless in order to uphold the $U(1)$ symmetry. The Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi) \tag{4}$$

where: $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative and $V(\phi) = \mu^2 |\phi|^2 + \lambda \left(|\phi|^2\right)^2$, the most general renormalizable potential allowed by the $U(1)$ gauge. In order for a bound state of minimum energy (i.e. a vacuum) to exist, λ has to be positive. Therefore two possible cases remain:
First case: $\mu^2 \geq 0$ Here the state of lowest energy, i.e. the vacuum is the state of $\phi = 0$, the symmetry of the Lagrangian is preserved and no new processes occur.
Second case: $\mu^2 < 0$ Here the potential has the famous "Mexican hat" shape. Therefore the vacuum is not at $\phi = 0$. The vacuum expectation value of the scalar field is:

$$\langle \phi \rangle = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \tag{5}$$

The direction in the complex plane in which the vacuum is chosen is arbitrary, any direction is possible. The convention is to choose it such that ϕ is real. To simplify the following steps, the field ϕ is usually decomposed into the vacuum expectation value and two real fields as the difference from the vacuum. χ and h are real scalar fields with zero vacuum expectation value.

$$\phi \equiv \frac{1}{\sqrt{2}} e^{i\frac{\chi}{v}} (v + h) \tag{6}$$

Inserting this into the Lagrangian of the theory and making a gauge transformation of A_μ to unitary gauge in order to remove the Goldstone Boson χ yields:

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 v^2}{2} A_\mu A^\mu + \frac{1}{2} \left(\partial_\mu h \partial^\mu h + 2\mu^2 h^2\right) \tag{7}$$

This Lagrangian describes a massive gauge boson of Mass proportional to $e \cdot v$, a scalar field h with mass-squared $-2\mu^2 > 0$.
Without going into detail about the choice of gauge and the Goldstone boson, this mechanism breaks the symmetry of the theory using a non-zero vacuum expectation value of a complex scalar field in order to give mass to the gauge boson (and to any fermions, which were not present here). This method of giving fields a mass is called the Higgs Mechanism.

2.2 Electroweak Symmetry Breaking: The Weinberg-Salam Model

The Weinberg-Salam Model of electroweak symmetry breaking uses the principle demonstrated in the sub-chapter above in application to the $SU(2)_L \times U(1)_Y$ gauge theory part of the Standard Model of Particle Physics.
The three W^i_μ, $i = 1, 2, 3$ gauge bosons of the $SU(2)$-symmetry and the one $U(1)$ gauge boson B_μ are symmetry broken into the known $W^\pm$ and Z for the weak interaction and the photon (γ) for the electromagnetic force by a complex scalar $SU(2)$ doublet Φ. The potential of Φ is given in its most general $SU(2)$ invariant form as:

$$V(\Phi) = \mu^2 \left|\Phi^\dagger \Phi\right| + \lambda \left(\left|\Phi^\dagger \Phi\right|\right)^2 \qquad (\lambda > 0) \tag{8}$$

With a non-vanishing vacuum expectation value v of the field Φ for $\mu^2 < 0$. The (symmetry-broken) electromagnetic charge of the scalar field is zero, therefore the desired symmetry breaking $SU(2)_L \times U(1)_Y \to U(1)_{e.m.}$ is achieved.
The physical gauge fields are then, after symmetry breaking with the masses obtained by the Higgs mechanism:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu \mp i W^2_\mu\right) \qquad M_W^2 = \frac{1}{4} g_2^2 v^2 \tag{9}$$

$$Z^\mu = \frac{-g_1 B_\mu + g_2 W^3_\mu}{\sqrt{g_1^2 + g_2^2}} \qquad M_Z^2 = \frac{1}{4}(g_1^2 + g_2^2) v^2 \tag{10}$$

$$A^\mu = \frac{g_2 B_\mu + g_1 W^3_\mu}{\sqrt{g_1^2 + g_2^2}} \qquad M_A^2 = 0 \tag{11}$$

By definition, the massless photon must couple with the electromagnetic coupling constant e, the weak mixing angle θ_W is defined via:

$$e = g_2 \sin\theta_W \tag{12}$$

$$e = g_1 \cos\theta_W \tag{13}$$

$$\cos(\theta_W) = \frac{M_W}{M_Z} \tag{14}$$

Now it is possible to relate the observable electromagnetic coupling e to the non-symmetry broken coupling constants g_1 and g_2.
Two more important parameters of the theory are the Higgs quartic coupling λ, which is closely related to the Higgs mass M_h and the top Yukawa coupling h_t, depending on the top mass m_t.

$$\lambda = \frac{M_h^2}{2v^2} \tag{15}$$

$$h_t = \frac{m_t\sqrt{2}}{v} \tag{16}$$

The value for v can be very accurately measured in the Muon decay and yields the result $v = 246 GeV$ (taken from [Daw98, Eq. 41]).

2.3 Experimental Values for the Renormalization Conditions

The experimental values for the renormalization conditions are taken from [IV13, Eq 3.1-3.4]. They are, measured at a scale of M_Z, the mass of the Z-boson:

$$M_Z = 91.1876GeV, \qquad M_W = 80.385GeV, \qquad m_t = 173.1GeV \tag{17}$$
$$\sin^2(\theta_W)(M_Z) = 0.23126, \qquad \alpha^{-1}_{e.m.}(M_Z) = 127.937, \qquad \alpha_3(M_Z) = 0.1184 \tag{18}$$

Using $\alpha_i = \frac{g_i^2}{4\pi}$, it is now possible to calculate the values of the three coupling constants at the scale of M_Z as:

$$g_1 = \frac{1}{\cos\theta_w}\sqrt{\frac{4\pi}{\alpha^{-1}_{e.m.}}} = 0.3555 \qquad \alpha_1^{-1} = 99.43 \tag{19}$$
$$g_2 = \frac{1}{\sin\theta_w}\sqrt{\frac{4\pi}{\alpha^{-1}_{e.m.}}} = 0.6517 \qquad \alpha_2^{-1} = 29.58 \tag{20}$$
$$g_3 = \sqrt{4\pi\alpha_3} = 1.220 \qquad \alpha_3^{-1} = 8.446 \tag{21}$$

The top Yukawa coupling y_t and the Higgs quartic coupling λ is are necessary for the Renormalization Group Equations on two loop level.
They ware calculated using the above relations to the Higgs and top mass, whose values are taken from [Aad12] for the Higgs mass $M_h = 126.0GeV$ and [JBeaPDG12] for the top mass $m_t = 173.5 \pm 1.4GeV$ (combined statistical and systematic error).

Note: the different normalisation factor of $\frac{3}{5}$ for the $U(1)$ part (as explained in section 4.2) is not included here and SARAH uses a different normalisation for λ by a factors of:

$$\lambda_{\text{SARAH}} = \lambda \cdot 2 \tag{22}$$

These factors are have to be added to the results of the above calculations. Without the different SARAH normalisation's, one gets:

$$h_t(M_Z) = 0.9974 \tag{23}$$
$$\lambda(M_Z) = 0.1312 \tag{24}$$

3 One-Loop Renormalization of Couplings

As a first step in the evaluation of the running couplings, a first explicit calculation of the beta functions of the three interactions that are in the Standard Model will be made, along the lines of [PS07].
The following calculations are done before electroweak symmetry breaking in applied, therefore all particles are massless and the $SU(2) \times U(1)$-symmetry is not broken.
When considering a general, renormalizable Quantum Field Theory of fermions with dimensionless coupling constants, the Green's functions take the general form of (fully connected, up to one loop level) in cutoff-regularisation with cutoff Λ:

$$G^{(n)} = (\text{tree-level}) + (\text{1PI}) + (\text{vertex counterterm}) + (\text{external leg corrections}) \tag{25}$$
$$= \left(\prod_i \frac{i}{p_i^2}\right)\left[-ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i \left(A_i \log\frac{\Lambda^2}{-p^2} - \delta_{Z_i}\right)\right] \tag{26}$$

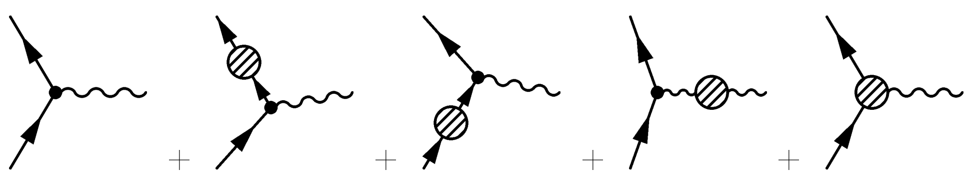
Where the p_i are the external momenta, p^2 is some invariant made out of these, δ_g is the vertex counterterm and δ_{Z_i} is the propagator counterterm.
This can be inserted into the Callan-Symanzik (1) equation and be solved for the β-function. For this the following result from applying the Callan-Symantzik Equation to the two point function is needed:

$$\gamma_i = \frac{1}{2} M \frac{\partial}{\partial M}\delta_{Z_i} \text{ (to first order, for every field involved)} \tag{27}$$

Since, at least to one loop order, the β-function is independent of the exact renormalization scheme, all invariants inside logarithms (like the p^2 above) can be set equal to the renormalization scale M^2 (for a thorough explanation see [PS07, Ch.12.2]). In the counterterm formalism all dependencies on the renormalization scale are kept inside the counterterms, therefore the Callan-Symanzik equation can be solved to:

$$\beta(g) = M\frac{\partial}{\partial M}(-\delta_g + \frac{1}{2} g \sum_i \delta_{Z_i}) \tag{28}$$

where δ_g is the vertex counterterm and the δ_{Z_i} are the propagator counterterms of the involved fields.
Now consider the three point function $\langle\Omega|\, \Psi\Psi A^\mu \,|\Omega\rangle$ with correction terms:



$$= (\text{tree level}) + (\sim \delta_2) + (\sim \delta_2) + (\sim \delta_3) + (\sim (\delta_g = g\delta_1)) \tag{29}$$

Therefore, using (28) gives the result for the one loop β-function of a general, massless, renormalizable gauge theory as:

$$\beta(g) = gM\frac{\partial}{\partial M}(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3) \tag{30}$$

With the computation of the counter terms as specified in chapter 1 this can now be evaluated for specific theories.
The fact that the β-function is independent of the cutoff of the theory follows directly from the renomalizability of the field theory.

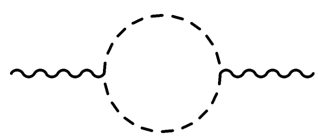
3.1 Contribution of a Scalar on One Loop Level

For the following discussion on the theories behind current particle physics, the Standard Model of Particle Physics and the following discussion going beyond, one key component is the appearance of a complex scalar field that induces spontaneous symmetry breaking. This field is called the Higgs field in the Standard Model, but its true nature is not of significance here, at least for this first discussion.
Due to the fact that the symmetry breaking itself does not change the renormalizability of a theory, the contribution to the β functions can be worked out separately from the other renormalization calculations (those of the fermionic fields and the gauge couplings and fields in the following chapters).
The Lagrangian of the Theory is modified to include a scalar field, which is coupled to the gauge fields via the covariant derivative term.

$$\mathcal{L} \to \mathcal{L} + (D_\mu \phi)^a (D^\mu \phi)^a + (\text{Yukawa couplings, omitted to one loop order}) \tag{31}$$

3.1.1 Scalar Field in the Non-Abelian Case

The contribution can be calculated by extracting the divergence of the gauge boson self energy due to the scalar field, as in [Sre07, Eq. 78.31](using the normalisation of the regularisation of [PS07] for consistency):



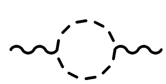
$$= \Pi_{cs}(k^2) = \frac{-g_1^2}{(4\pi)^{(d-2)}} \frac{2}{3} T(R_{cs}) \frac{\Gamma(2-d/2)}{(k^2)^{2-d/2}} + \text{finite} \tag{32}$$

where $T(R_{cs})$ is defined by $Tr(T_a T_b) = T(R)\delta_{ab}$ taking into account the scalar gauge group matrices of their coupling to the gauge field.
To account for this divergence, the term δ_3 has to contain a term of the following type:

$$\delta_3 = -\Pi_{cs}(M^2) + (\text{other contributions}) + (\text{finite}) \tag{33}$$

3.1.2 Scalar Field in the Abelian Case

The diagram contributing to the δ_3 counter term in the Abelian case is just the same as before for the Non-Abelian case. In a theory of Dirac-fermions and complex scalars, the contributions to the counterterm just add, therefore the photon self energy contains a term of the form (as in [Sre07, Eq. 65.8]):



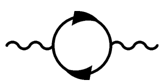
$$\propto -\frac{g_1^2 \cdot Q_{i,SC}^2}{(4\pi)^{(d/2)}} \frac{1}{2} \Gamma(2-d/2) \tag{34}$$

One of these Terms is taking part for every scalar field with charge $Q_{i,sc}$ involved. To cancel this divergence, the counter term must contain the scalar contribution, now for multiple scalar fields:

$$\delta_3 = -\frac{g_1^2 \cdot \sum_i Q_{i,sc}^2}{(4\pi)^2} \frac{1}{3} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} + (\text{other contributions}) + (\text{finite}) \tag{35}$$

3.2 One-Loop β-Function in $U(1)$

When applying the formula for the one-loop-β function to the simplest case of interest in the Standard Model of Particle Physics, the $U(1)$-Symmetry gauge theory, some simplifications can be made to (30).
In this theory in Feynman gauge, the counter terms δ_1 and δ_2 always exactly cancel each other, order by order in perturbation theory, due to the Ward Identity of the $U(1)$-symmetry. Therefore one only has to evaluate δ_3 in this case. To one loop level, this term has to cancel the divergence of the following diagram.

 with the counter term being:  $= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$

This diagram exists for every fermion field in the theory with charge $Q_i \cdot e$. Since no other differences of the fermionic fields play any role in this diagram, it is sufficient to evaluate it for one and then add up the charges for all $U(1)$-charged fermionic fields of the theory.
The loop evaluates to the following in dimensional regularisation:

$$\Pi_2(q^2) = -\frac{g_1^2 \cdot Q_i^2}{(4\pi)^{(d/2)}} \int_0^1 dx \frac{\Gamma(2-d/2)}{(m^2 - x(1-xq^2))^{2-d/2}} (8x(1-x)) \tag{36}$$

To cancel this divergence at the renormalization scale $-M^2$, the counterterm must be:

$$\delta_3 = \Pi(-M^2) = -\frac{g_1^2 \cdot Q_i^2}{(4\pi)^2} \frac{4}{3} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} + (\text{finite}) + (\text{scalar contribution}) \tag{37}$$

Summing up and adding the scalar contribution calculated in (35) and inserting into (30) leads to (as in [Sre07, Eq. 66.29]):

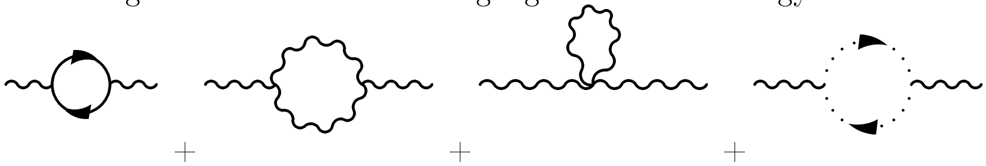
$$\beta(g_1) = \frac{g_1^3}{12\pi^2} \left(\sum_i Q_i^2 + \frac{1}{4} \sum_j Q_{sc,j}^2 \right) \tag{38}$$

3.3 One-Loop β-Function in $SU(2)$ and $SU(3)$

In the two non-abelian gauge symmetries contained in the Standard Model of Particle Physics, the calculation is not as simple. For the calculation of the three counterterms in those theories, several diagrams have to be considered, as the δ_1 and δ_2 counterterms do not cancel here.

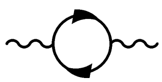
3.3.1 Gauge Boson Self-Energy

Four diagrams contribute to the gauge boson self energy counterterm δ_3 on one-loop level.



$+$(scalar contribution)

The first diagram takes into account all fermionic fields of the theory which are charged under the symmetry of the gauge boson whilst the last three are all pure gauge. The same principle of summing over all fermionic fields as for the $U(1)$ applies here. The fermionic diagram looks just the same as before, but a term of $tr[t^a, t^b] = C(r)\delta^{ab}$ has to be added to account for the fermion gauge group matrices in the representation r. Restricting the theory to n_f fermionic fields in the same representation r yields:



$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{-g^2}{(4\pi)^2}\frac{4}{3}n_f C(r)\Gamma(2-d/2) + \ldots\right) \tag{39}$$

The three pure gauge diagrams add up to the following terms, using the quadratic Casmimir operator $C_2(G)$ of the gauge field representation G (adjoint to r). For an exact derivation please follow [PS07, Ch 16.5 Eq 16.71]. The quadratic Casimir operator can be defined by $t_r^a t_r^a = C_2(G)\delta^{ab}$. Therefore the sum of the three pure gauge diagrams takes the value of:

$$\frac{ig^2}{(4\pi)^{d/2}}C_2(G)\delta^{ab}\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\triangle^{2-d/2}}(g^{\mu\nu}q^2 - q^\mu q^\nu)[(1-d/2)(1-2x)^2 + 2] \tag{40}$$

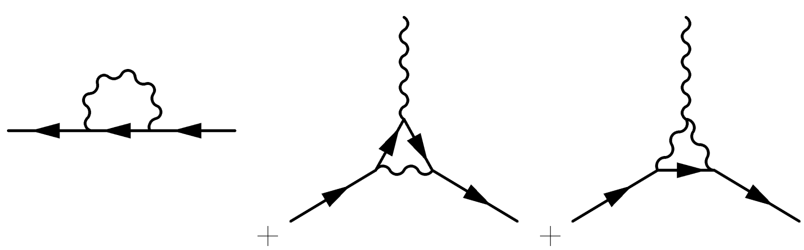
$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{-g^2}{(4\pi)^2}\frac{-5}{3}C_2(G)\Gamma(2-d/2) + \ldots\right) \tag{41}$$

The counterterm δ_3 has to cancel both divergences at the renormalization scale $-M^2$, therefore he has to take the form, to lowest order:

$$\delta_3 = \frac{g^2}{(4\pi)^2}\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] + \text{finite} + \text{scalar contribution} \tag{42}$$

3.3.2 Fermionic Interaction Counterterms

The other two counterterms are defined in the theory to provide the subtraction of the divergence of the following diagrams.



In Feynman-t'Hooft gauge, the fermion propagator one loop diagram can be evaluated as:

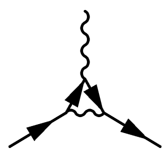
$$\int \frac{dp^4}{(2\pi)^4}(ig)^2\gamma\mu t^a \frac{i(\not p+\not k)}{(p+k)^2}\gamma_\mu t^a \frac{-i}{p^2} \tag{43}$$

$$= \frac{ig^2}{(4\pi)^2}\not k C_2(r)\Gamma(2-d/2)+\dots \tag{44}$$

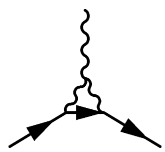
Therefore the fermion propagator counterterm can be calculated as:

$$\delta_2 = \frac{-g^2}{(4\pi)^2}\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}\cdot C_2(r)+\text{finite} \tag{45}$$

The two vertex counterterms both only have a logarithmic superficial degree of freedom, therefore they can be calculated easier in the limit where the loop impulse is much larger than the external momenta. This yields as in [PS07, Ch 16.5 Eq 16.83 and 16.81]:



$$\sim \frac{ig^3}{(4\pi)^2}[C_2(r)-\tfrac{1}{2}C_2(G)]t^a\gamma\mu(\Gamma(2-d/2)+\dots) \tag{46}$$



$$\sim \frac{ig^3}{(4\pi)^2}\tfrac{3}{2}C_2(G)t^a\gamma^\mu(\Gamma(2-d/2)+\dots) \tag{47}$$

Renormalizing the divergences at the scale M^2 yields the counterterm:

$$\delta_1 = \frac{-g^2}{(4\pi)^2}\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}[C_2(r)-C_2(G)]+\text{finite} \tag{48}$$

3.3.3 Combining all Counterterms

After the separate calculation of all counterterms fur the $SU(N)$-gauge symmetry, they can be combined via (30) to get the one loop β function, using the scalar contribution calculated in (33).

$$\beta(g) = \frac{-g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G)-\frac{4}{3}n_f C(r)-n_{sc}T(R_{sc})\right] \tag{49}$$

Including n_{sc} scalar fields in the (assuming all the same) representation R_{sc}, each contributing according to (33).
For n_f Dirac-fermions in the fundamental representation r and $C_2(G)$ the quadratic Casimir operator of the adjoint representation of the group and in an $SU(N)$ gauge theory, the formula can be simplified to:

$$\beta(g) = -\frac{g^3}{(4\pi)^2}[\frac{11}{3}N-\frac{2}{3}n_f-\frac{1}{3}n_{sc}T(R_{sc})] \tag{50}$$

In agreement with [Sre07, Eq. 78.36].

4 The Standard Model of Particle Physics

The Standard Model of Particle Physics is the current theoretical description of the physics of subatomic particles which interact via the electromagnetic, weak and strong nuclear forces. It is quite successful in correctly predicting experimental results, but some phenomena like neutrino-oscillations, dark matter and dark energy are not explained by it.
The Standard Model is a field theory of $SU(3) \times SU(2) \times U(1)$, using three families of fermionic fields and one scalar field, the Higgs field, which is not charged under $SU(3)$ but is an $SU(2)$ doublet. So far no flaws have been discovered in the Standard Model up to the Planck scale of $M_{pl} \approx 10^{19} GeV$, at which point one has to take gravity into account, which is currently not possible.
The cutoff for the Standard Model, being the Planck scale, determines that the model may only be applied to energy scales below it. This corresponds to:

$$\ln\left(\frac{M_{pl}}{M_Z}\right) \approx 40 \qquad \text{or} \qquad \log\left(\frac{M_{pl}}{GeV}\right) \approx 19 \tag{51}$$

Therefore, all runnings of couplings are not done much further than this scale.

4.1 Particle Content of the Standard Model

The missing values for n_f and $\sum_i Q_i^2$ can be extracted from the particle content of the Standard Model as it is stated in the table 1 below (taken from [Wei13]). The usual notation of $(x, y)_z$ is used where x is the representation under $SU(3)$, y is the representation under $SU(2)$ and z is the charge under $U(1)$.

Table 1: Particle Content of the SM

particle type and chirality	contribution to the interactions
Right handed up-type Quarks	$(3,1)_{2/3}$
Right handed down-type Quarks	$(3,1)_{-1/3}$
Left-handed Quarks	$(3,2)_{1/6}$
Right-handed Leptons	$(1,1)_{-1}$
Left-handed Leptons	$(1,2)_{-1/2}$
Right-handed Neutrinos	$(1,1)_0$
Higgs (complex scalar)	$(1,2)_{1/2}$

Therefore for the $U(1)$-Symmetry one gets:

$$\sum_i Q_i^2 = 3 \cdot \left(3 \cdot (\frac{2}{3})^2 + 3 \cdot (\frac{-1}{3})^2 + 6 \cdot (\frac{1}{6})^2 + (-1)^2 + 2 \cdot (\frac{-1}{2})^2\right) = 10 \tag{52}$$

For the derivation of the β functions for the three symmetries, Dirac fermions were considered, with n_f being the number of such fields. The Standard Model of Particle Physics, as a chiral theory, contains Weyl - fermions. A Weyl-fermion contributes only as half a Dirac-fermion to the counterterm, therefore each fermionic field in the Standard Model only contributes half to $n_f = \frac{1}{2} \cdot n_{f,Weyl}$ and $\sum_i Q_i = \frac{1}{2} \sum_i Q_{i,Weyl}$.

Therefore the remaining parameters in the β functions take values using three families for every particle type and including the scalar contribution of one complex scalar field as calculated above:

$$SU(3) \qquad N=3 \qquad n_f = \frac{1}{2}\cdot(6+6)=6 \qquad\qquad \text{no Higgs contribution} \tag{53}$$

$$SU(2) \qquad N=2 \qquad n_f = \frac{1}{2}\cdot(3+9)=6 \qquad\qquad n_{sc}\cdot T(R_{cs}) = \frac{1}{2} \tag{54}$$

$$U(1) \qquad \sum_i Q_i^2 = \frac{1}{2}\cdot 10 \qquad\qquad \sum_i Q_{i,sc}^2 = 2\cdot\frac{1}{4} \tag{55}$$

Therefore the one-loop-β-functions for the Standard Model for the three fundamental forces take the values using the formulas derived above (as in [Mar97, Eq. 6.4.7]):

$$SU(3): \qquad \beta(g_3) = \frac{-7}{16\pi^2}\cdot g_3^3 \tag{56}$$

$$SU(2): \qquad \beta(g_2) = \frac{-19}{96\pi^2}\cdot g_2^3 \tag{57}$$

$$U(1): \qquad \beta(g_1) = \frac{41}{96\pi^2}\cdot g_1^3 \tag{58}$$

Please note the difference of these results from the ones in the source due to the different normalisation of the charges. This will be discussed thoroughly in the next chapter.

4.2 Embedding of $SU(3)\times SU(2)\times U(1)$ into $SU(5)$

In the Standard Model of Particle Physics, there is a certain ambiguity in the normalisation of the weak hypercharge kinetic energy, and equivalently in the normalisation of the charge associated with it because a shift in field strength A_Y^μ can be simply reversed for all physical results by inversely shifting the coupling e_Y, such that $e_Y\cdot A_Y^\mu$ is constant.
One approach to fix this is to consider the minimal unification of $SU(3)\times SU(2)\times U(1)$ into $SU(5)$. This ensures a compatible normalisation of all three gauge couplings needed in order to be able to compare them.
To do this, the generators of $SU(5)$ are written in terms of the generators of the contained symmetries (Pauli matrices σ_i for SU(2), the Dirac or Gamma- Matrices γ_j for SU(3) and simple complex numbers for U(1)).
To achieve the embedding, the generators are combined in the following way, ensuring commutativity and tracelessness of the generators.
For the SU(3) and SU(2) sector it is straight forward to define the 5×5 matrices:

$$\begin{bmatrix} \gamma_i & & \\ & 0 & \\ & & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \sigma_i \end{bmatrix} \tag{59}$$

To do the same with the $U(1)$ sector, one finds that the following matrices commute with the above and create the required symmetry.

$$\begin{bmatrix} \alpha & & & & \\ & \alpha & & & \\ & & \alpha & & \\ & & & \beta & \\ & & & & \beta \end{bmatrix} \tag{60}$$

where α is the charge of a particle in the $SU(3)$ fundamental representation, and β for $SU(2)$. For the Standard Model one gets:

$$\alpha = -\frac{1}{3} \qquad \text{and} \qquad \beta = \frac{1}{2} \tag{61}$$

The normalisation of the U(1) sector is chosen to be such that the trace relation holds:

$$tr(T^a T^b) = C(T) \cdot \delta^{ab} \tag{62}$$

with $C(T) = \frac{1}{2}$ for the Weyl-fermions of the Standard Model. Therefore the $U(1)$ Matrix is re-scaled by a factor of $\sqrt{\frac{3}{5}}$ to get:

$$\sqrt{\frac{3}{5}} \cdot \begin{bmatrix} \frac{-1}{3} & & & & \\ & \frac{-1}{3} & & & \\ & & \frac{-1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{bmatrix} \text{ fulfilling } tr(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{63}$$

These generators all commute and are all traceless. Using this decomposition, it is possible to write the values of the before three separated coupling constants (g_3 g_2 and g_1) now in terms of the new $SU(5)$ coupling constant g_5 (taken from [PS07, Eq. 22.6]) at the point of unification at the scale of $SU(5)$ breaking.

$$g_5 = g_3 = g_2 = \sqrt{\frac{5}{3}} \cdot g_1 \tag{64}$$

To account for this factor of $\sqrt{\frac{5}{3}}$, it is convenient to re-scale the weak hypercharge.

$$g_1 \to g_1 \cdot \sqrt{\frac{3}{5}} \tag{65}$$

This then gives another factor of $\frac{3}{5}$ for the β-function of the weak hypercharge, which then is:

$$\beta(g_1) = \frac{41}{160\pi^2} \cdot g_1^3 \tag{66}$$

This now uses the same normalisation of the charges as [Mar97], [BDG$^+$13] and SARAH (see chapter 7.1).

4.3 One Loop β Functions in the Standard Model

In order to solve the behaviour of the running couplings in the Standard Model of Particle Physics, it is convenient and numerically necessary calculate in terms of α_i^{-1} with $\alpha_i = g_i^2/4\pi$ and not in g_i itself. Here the differential equations are much simpler:

$$U(1): \qquad \frac{\partial}{\partial \ln(p/M)} \alpha_1^{-1} = -\frac{41}{20\pi} \tag{67}$$

$$SU(2): \qquad \frac{\partial}{\partial \ln(p/M)} \alpha_2^{-1} = \frac{19}{12\pi} \tag{68}$$

$$SU(3): \qquad \frac{\partial}{\partial \ln(p/M)} \alpha_3^{-1} = \frac{7}{2\pi} \tag{69}$$

The solutions to this are, using the starting values from 2.3 at $M = M_Z$, now including the normalisation factor for $\alpha_Y = \alpha_1$:

$$\alpha_1^{-1}(p) = \alpha_1^{-1}(M) - \frac{41}{20\pi} \cdot \ln(p/M) \qquad \alpha_1^{-1}(M) \approx \frac{3}{5} \cdot 99.43 = 59.66 \tag{70}$$

$$\alpha_2^{-1}(p) = \alpha_2^{-1}(M) + \frac{19}{12\pi} \cdot \ln(p/M) \qquad \alpha_2^{-1}(M) \approx 29.58 \tag{71}$$

$$\alpha_3^{-1}(p) = \alpha_3^{-1}(M) + \frac{7}{2\pi} \cdot \ln(p/M) \qquad \alpha_3^{-1}(M) \approx 8.446 \tag{72}$$

The results are plotted and discussed later together with the one loop results for the MSSM of the next chapter.

5 The Minimal Supersymmetric Standard Model

The Minimal Supersymmetric Standard Model (MSSM) is an extension on the experimentally verified Standard Model of Particle Physics to try and give explanations on multiple topics, including a possible solution to the hierarchy problem.

In this Model, the particle content of the Standard Model is replaced by Superfields. Several types of Superfields are known, the ones used here are Superfields of one Dirac fermion (or two Weyl fermions) and one complex scalar field (or two real scalar fields) for the matter sector of the Standard Model and vector multiplets, one gauge boson and two Weyl fermions, for the gauge sector.

Each fermion as in table 1 is assigned a supersymmetric, bosonic partner and the other way round in order to be able to write everything in terms of Superfields.

The Higgs-field has to be manipulated a little more to give a reasonable theory because a single scalar field with a chiral Weyl-fermion would not be able to describe the electroweak symmetry braking process of giving particles their mass for all fermions in the MSSM. Therefore two Higgs supermultiplets, H_u and H_d are introduced to the theory, each consisting of one complex scalar field and one Dirac-fermion.

The same normalisation of the $U(1)$-part as in 4.2 is used here.

5.1 Particle Content of the MSSM

A summary of the particle content of the MSSM can be found in the table below, which is taken from [Mar97, Table 1.1 and 1.2]. The same form of denoting charges and representations under the fundamental forces is used as in the Standard Model. The supersymmetry partners added to the Standard Model are marked with a tilde.

Table 2: Particle Content of the MSSM

Names	spin 0	spin1/2	Representation and Charge
squarks, quarks	$(\tilde{u}_L \quad \tilde{d}_L)$	$(u_L \quad d_L)$	$(3,2)_{1/6}$
(three families)	$\tilde{u}_R^*$	$u_R^\dagger$	$(3,1)_{-2/3}$
	$\tilde{d}_R^*$	$d_R^\dagger$	$(3,1)_{1/3}$
sleptons, leptons	$(\tilde{\nu} \quad \tilde{e_L})$	$(\nu \quad e_L)$	$(1,2)_{-1/2}$
(three families)	$\tilde{e}_R^*$	$e_R^\dagger$	$(1,1)_1$
Higgs, higgsinos	$(H_u^+ \quad H_u^0)$	$(\tilde{H}_u^+ \quad \tilde{H}_u^0)$	$(1,2)_{1/2}$
	$(H_d^0 \quad H_d^-)$	$(\tilde{H}_d^0 \quad \tilde{H}_d^-)$	$(1,2)_{-1/2}$
	spin1/2	spin1	
gluino, gluon	$\tilde{g}$	g	$(8,1)_0$
winos, W bosons	$\tilde{W}^\pm \quad \tilde{W}^0$	$W^\pm \quad W^0$	$(1,3)_0$
bino, B boson	$\tilde{B}^0$	B^0	$(1,1)_0$

5.2 One-Loop β-Functions in the MSSM

With this setup of the theory, one now can calculate the one-loop-β-functions for this theory in complete analogy to the derivation done in the standard model. The equations (50) and (38) yield using the same derivation as in the Standard Model with more fields:

$$SU(3) \qquad N=3 \qquad n_f = \frac{1}{2}\cdot(6+6+3) = 7.5 \qquad n_{sc}\cdot T(R_{cs}) = \frac{12}{6} \tag{73}$$

$$SU(2) \qquad N=2 \qquad n_f = \frac{1}{2}\cdot(3+9+4+2) = 9 \qquad n_{sc}\cdot T(R_{cs}) = \frac{(12+2)}{2} \tag{74}$$

$$U(1) \qquad \frac{1}{2}\sum_i Q_i^2 \quad + \quad \frac{1}{4}\sum_i Q_{i,sc}^2 \quad = \quad \frac{33}{5} \tag{75}$$

Using these values, the β-functions can be calculated as, as stated in [Mar97, Eq. 6.4.7]:

$$SU(3): \qquad \beta(g_3) = \frac{1}{16\pi^2}\cdot(-3)\cdot g_3^3 \tag{76}$$

$$SU(2): \qquad \beta(g_2) = \frac{1}{16\pi^2}\cdot 1\cdot g_2^3 \tag{77}$$

$$U(1): \qquad \beta(g_1) = \frac{1}{16\pi^2}\cdot\frac{33}{5}\cdot g_1^3 \tag{78}$$

Now using the same simplification of changing variables to α^{-1}, and solving the differential equation just as in the Standard Model, the result can be written as, using the same starting values as before at $M = M_Z$:

$$\alpha_1^{-1}(p) = \alpha_1^{-1}(M) - \frac{33}{10\pi} \cdot \ln(p/M) \qquad \alpha_1^{-1}(M) \approx \frac{3}{5} \cdot 99.43 = 59.66 \tag{79}$$

$$\alpha_2^{-1}(p) = \alpha_2^{-1}(M) - \frac{1}{2\pi} \cdot \ln(p/M) \qquad \alpha_2^{-1}(M) \approx 29.58 \tag{80}$$

$$\alpha_3^{-1}(p) = \alpha_3^{-1}(M) + \frac{3}{2\pi} \cdot \ln(p/M) \qquad \alpha_3^{-1}(M) \approx 8.446 \tag{81}$$

6 One Loop Running Couplings in SM and MSSM

The following graph 1 shows the results of the one loop β-functions calculations to get the running couplings for the Standard Model of Particle Physics and the Minimal Supersymmetric Standard Model.

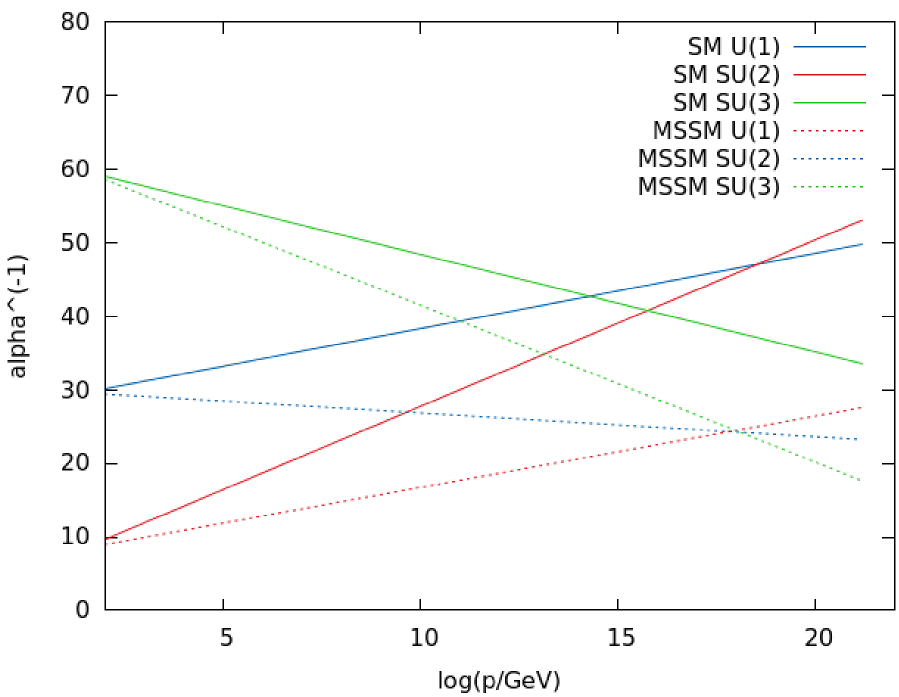


Figure 1: One Loop Running Couplings in the SM and MSSM

The running of the couplings in the Standard Model and the minimal supersymmetric extension thereof show a quite intriguing feature, they run up to a region in which all three fundamental forces have coupling constants of about roughly the same size. For the first approximation of the one loop β-functions for the Standard Model the region in which the couplings are close by each other is quite large, but in the MSSM the couplings unify at almost a single point.
This feature of the running couplings is usually referred to as the "unification of the couplings".

For this unification to happen, a suitable normalisation, as explained in 4.2, has to be done. This is vital because otherwise the couplings are measured in independent scales, which takes the physical meaning out of all potential unifications.
Before discussing potential physical implications of the observed unification of the running couplings an more precise calculation on two loop level will follow. This is done to check whether this feature really is physical or is just a coincidence on this first approximation. In this discussion an measure of how good the unification is will be mentioned and applied to the one loop result too.
The discussion is done in chapter 8.4 .

7 Numerical Methods and SARAH

The two-loop β functions that are discussed in the following chapters are a more complicated set of five first order non-homogeneous differential equations for the SM and four for the MSSM. Their analytical solutions are unknown, therefore they are solved numerically. For this, two different methods are used which are checked against each other.

7.1 SARAH

The first method used in the solving of the two loop β-functions is the program Mathematica (for reference see [Res14]). This program would be able to solve sets of differential equations by itself, but a far more powerful tool is available as a package for Mathematica.
The package is called SARAH, a tool to build and analyse supersymmetric and non-supersymmetric models as quantum field theories. The package is developed by Florian Staub and is available online. The resources and functions that were used in SARAH can be found in [Sta14] and the introductory manual is [Sta08]. The methods of deriving the renormalization group equations is explained in [Sta11].
SARAH uses the normalisation conventions of [LX03]. These match with [BDG+13] up to a factor of 2 for λ.
Using SARAH, the β-functions for the SM and MSSM can be derived, using the predefined models available in the package. These are then solved in Mathematica, where SARAH already writes a solving-function to achieve that. The only input necessary are the values of the couplings that are to be run at a certain energy scale. SARAH then provides interpolating functions of the numerical solutions for the running couplings. These can then be plotted using stock Mathematica routines.
The results that are calculated in SARAH can be exported to a LaTeX-file. The results of the parts required for this thesis, the β-functions for the SM and MSSM, can be found in the Appendix B and C.
The model files that are the basis of the SARAH calculations are textfiles specifying the particle content and gauge fields of the theory, together with various other things like Yukawa-coupling to scalar fields, potentials and super-potentials. During this thesis, the predefined models were used and in chapter 10.1 the model file of the Standard Model was altered only in the definition of the matter sector consisting of fermions and by adding a mass term to the potential of the Lagrangian.

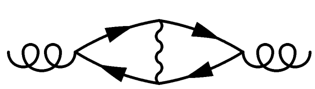
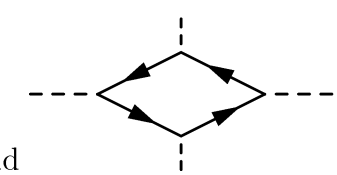
7.2 Numerical Methods

In order to not completely rely on the calculations done in Mathematica using SARAH, the two loop β-functions are taken from [BDG$^+$13] and are run in a C program which can be found in Appendix A. This code was written for this thesis, but relies on the Runge Kutta 4 method from the Numerical Recipes code collection, which is a fourth order explicit one-step solver for general sets of first order differential equations.
Numerical Recipes is an open source collection of a lot of useful functions written in a lot of different coding languages. It is available online ([Co.14]) or as a book ([Vet99]).
The program solves the supplied differential equations using the starting conditions which are written directly into the code and outputs the calculated datapoints into a textfile. This file can then be plotted using any plotting program, here gnuplot (for reference see [TW14]) was used to plot and interpolate the data points.
The Runge Kutta 4 solver was tested with various polynomial-differential equations similar to the β-functions to which analytical solutions are known and no significant error in the numerical solutions was found.
This and the fact that the Runge Kutta 4 results match with the SARAH results, it is possible to neglect numerical errors for the discussions in this thesis as the solutions are sufficiently accurate. The code for this solver was used in many different ways throughout this thesis, mainly in varying the starting conditions and the starting and stopping scale of the solver, but the basic routine stayed the same.

8 Two Loop Running Couplings in SM and MSSM

8.1 Two Loop β-Functions in SM and MSSM

The derivation of the two loop β functions and the two loop running couplings follows the same procedure outlined previously in this thesis for the one loop case. All possible divergent diagrams have to be evaluated and the counterterms computed, which are then plugged into the Callan-Symanzik equation to derive the β functions. In this, a lot more diagrams contribute, especially interesting are the ones of the form:


and


or other mixings of the different couplings, now including Yukawa couplings.
This leads to β-functions not only depending on one running coupling but on all couplings in the system at once. Therefore it is no longer possible to treat them independently, as it was done on one loop level. For the three fundamental forces, the top-Yukawa coupling h_t and the

Scalar quartic coupling λ this is:

$$\begin{aligned}\beta_i(\{g_j\}) =&\beta_{1\text{ loop}}(\{g_i\}) + \beta_{2\text{ loop}}(\{g_i\}) \text{ with } i,j \in \{1,2,3,4,5\} \\ &\text{where} \quad g_4 \equiv y_t \qquad \text{and} \qquad g_5 \equiv \lambda\end{aligned} \tag{82}$$

Up to two loop level, two more couplings have to be considered because they are no longer negligible, compared to the other contributions. The Yukawa coupling of the top quark to the Higgs field y_t (to leading order) and the Higgs quartic coupling λ have to be taken into account. The Higgs quartic coupling is only a free parameter in the Standard Model, the appearance of λ in the MSSM will be briefly explained in section 9.3.
The calculations for this run just as it was done for the one loop level, but a lot more diagrams contribute. Therefore the calculation is not done explicitly here and the results are taken from [BDG+13, Appendix B](neglecting y_d and y_τ, i.e. setting them to zero and using the result only up to two loop). The normalisation of this paper for the $U(1)$ case is in agreement with the previously used normalisation of the weak hypercharge embedded into $SU(5)$.

$$\frac{\partial g_1}{\partial \ln(p/M)} = \frac{1}{(4\pi)^2}\frac{41}{6}\cdot\frac{3}{5}g_1^3 + \frac{g_1^3}{(4\pi)^4}\frac{3}{5}\left(\frac{199}{18}g_1^2\frac{3}{5} + \frac{27}{6}g_2^2 + \frac{44}{3}g_3^3 - \frac{17}{6}h_t^2\right) \tag{83}$$

$$\frac{\partial g_2}{\partial \log(p/M)} = -\frac{1}{(4\pi)^2}\frac{19}{6}g_2^3 + \frac{g_2^3}{(4\pi)^4}\left(\frac{9}{6}\frac{3}{5}g_1^2 + \frac{35}{6}g_2^2 + 12g_3^2 - \frac{3}{2}h_t^2\right) \tag{84}$$

$$\frac{\partial g_3}{\partial \ln(p/M)} = -\frac{1}{(4\pi)^2}7g_3^3 + \frac{g_3^3}{(4\pi)^4}\left(\frac{11}{6}\frac{3}{5}g_1^2 + \frac{9}{2}g_2^2 - 26g_3^2 - 2h_t^2\right) \tag{85}$$

$$\begin{aligned}\frac{\partial y_t}{\partial \ln(p/M)} =& \frac{y_t}{(4\pi)^2}\left[\frac{9y_t^2}{2} - 8g_3^2 - \frac{9g_2^2}{4} - \frac{17g_1^2}{20}\right] + \\ &+ \frac{y_t}{(4\pi)^4}\left[y_t^2\left(-12y_t^2 - 12\lambda + 36g_3^2 + \frac{225g_2^2}{16} + \frac{393g_1^2}{80}\right) + \right. \\ &\left. + 6\lambda^2 - 108g_3^4 - \frac{23g_2^4}{4} + \frac{1187g_1^4}{600} + 9g_3^2g_2^2 + \frac{19}{15}g_3^2g_1^2 - \frac{9}{20}g_2^2g_1^2\right]\end{aligned} \tag{86}$$

$$\begin{aligned}\frac{\partial \lambda}{\partial \ln(p/M)} =& \frac{2}{(4\pi)^2}\left[\lambda\left(12\lambda + 6y_t^2 - \frac{9g_2^2}{2} - \frac{9g_1^2}{10}\right) - 3y_t^4 + \frac{9g_2^4}{16} + \frac{27g_1^4}{400} + \frac{9g_2^2g_1^2}{40}\right] + \\ &+ \frac{2}{(4\pi)^4}\left[\lambda^2\left(-156\lambda - 72y_t^2 + 54g_2^2 + \frac{54g_1^2}{5}\right) + \lambda y_t^2\left(-\frac{3y_t^2}{2} + 40g_3^2 + + \frac{45g_2^2}{4} + \frac{17g_1^2}{4}\right) + \right. \\ &+ \lambda\left(-\frac{73g_2^4}{16} + \frac{1887g_1^4}{400} + \frac{117g_2^2g_1^2}{40}\right) + y_t^4\left(15y_t^2 - 16g_3^2 - \frac{4g_1^2}{5}\right) + \\ &\left. + y_t^2\left(-\frac{9g_2^4}{8} - \frac{171g_1^4}{200} + \frac{63g_2^2g_1^2}{20}\right) + \frac{305g_2^6}{32} - \frac{3411g_1^6}{4000} - \frac{289g_2^4g_1^2}{160} - \frac{1677g_2^2g_1^4}{800}\right]\end{aligned} \tag{87}$$

The two loop β-functions for the MSSM can be found in [IV13]. Due to cancellations they are much simpler and shorter.

$$\frac{\partial g_1}{\partial \ln(p/M)} = \frac{11g_1^3}{(4\pi)^2}\frac{3}{5} + \frac{g_1^3}{(4\pi)^4}\frac{3}{5}\left(\frac{199}{9}\frac{3}{5}g_1^2 + 9g_2^2 + \frac{88}{3}g_3^2 - \frac{26}{3}y_t^2\right) \tag{88}$$

$$\frac{\partial g_2}{\partial \ln(p/M)} = \frac{g_2^3}{(4\pi)^2} + \frac{g_2^3}{(4\pi)^4}\left(3g_1^2\frac{3}{5} + 25g_2^2 + 24g_3^2 - 6y_t^2\right) \tag{89}$$

$$\frac{\partial g_3}{\partial \ln(p/M)} = -\frac{3g_3^3}{(4\pi)^2} + \frac{g_3^3}{(4\pi)^4}\left(\frac{11}{3}\frac{3}{5}g_1^2 + 9g_2^2 + 14g_3^2 - 4y_t^2\right) \tag{90}$$

$$\frac{\partial y_t}{\partial \ln(p/M)} = \frac{h_t}{(4\pi)^2}\left(6y_t^2 - \frac{13g_1^2}{9}\frac{3}{5} - 3g_2^2 - \frac{16g_3^2}{3}\right) \tag{91}$$

Solving these using the Runge Kutta 4 algorithm as explained in section 7.2 yields the same approximate coupling unification behaviour as in the 1 loop case, which can be seen below.

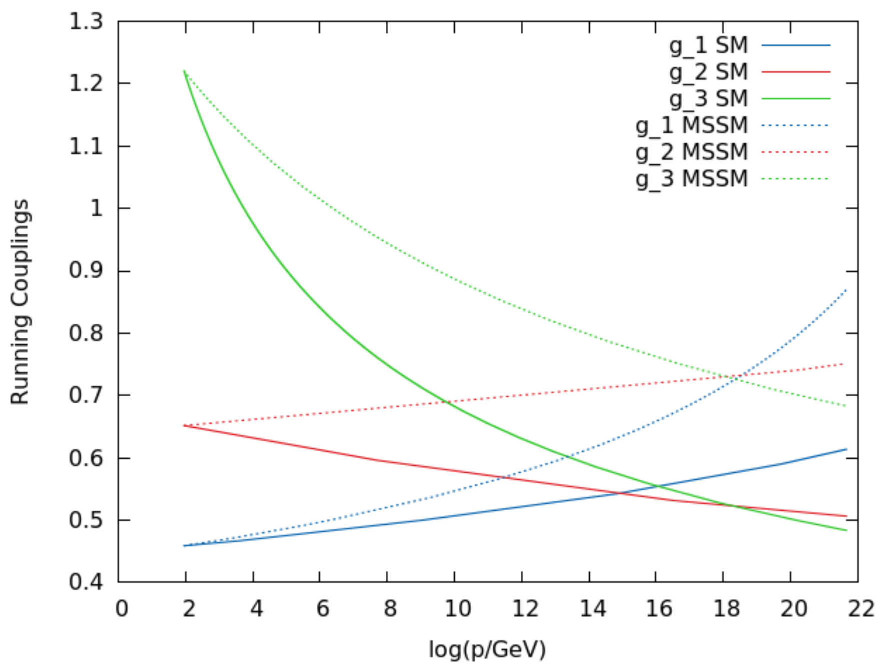
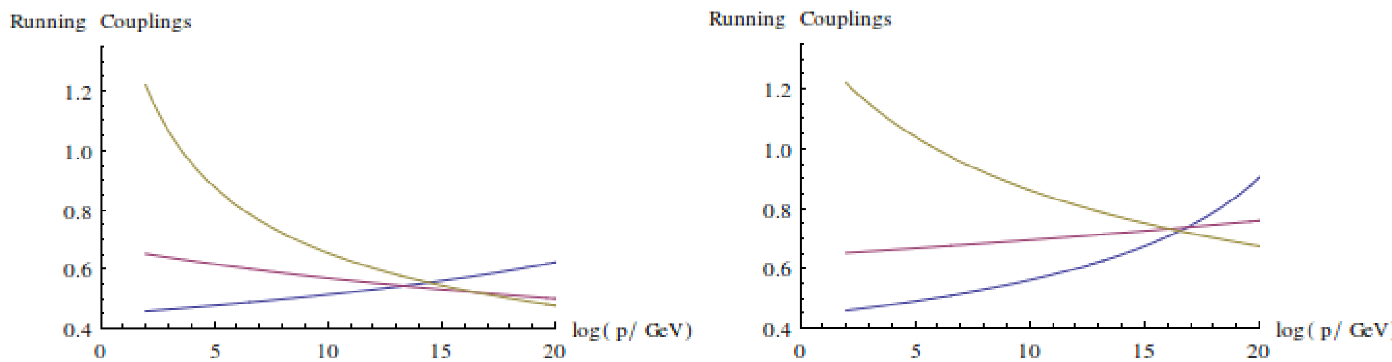


Figure 2: Two Loop Running Couplings in SM and MSSM

8.2 Two Loop Running couplings using SARAH

To verify the previously calculated results, the Mathematica package SARAH is used. For an introduction to SARAH please see section 7.1.

SARAH provides with a numerical solution to the derived Renormalization Group Equations for every model specified in a model file. The Standard Model of Particle Physics and the Minimal Supersymmetric Standard Model are available as predefined models, written by the creator of SARAH: Florian Staub. The solver needs as input the starting values of the three gauge couplings, the top-Yukawa coupling and the Higgs quartic coupling and is then able to produce the following graphs. The β-functions for the three gauge couplings and y_t derived by



(a) Two Loop Running Couplings in SM (b) Two Loop Running Couplings in MSSM

Figure 3: calculated by SARAH (g_3 (yellow), g_2 (purple), g_1 (blue))

SARAH match with the ones plugged into the numerical solver, only in the β- functions for λ do not match completely, but his might be due to the effect the exact renormalization scheme has on the β-functions on two loop level. Even though, the output from SARAH matches the one of the RK4 routine extremely well.

8.3 Comparison of the Unification Behaviour

The results of the calculation of the running coupling on one and two loop level in the SM and the MSSM all show the coupling unification behaviour, but to different degrees. Therefore the measure of unification P_{uni} is usually defined as follows:

$$P_{uni} = \left| \frac{g_3(M_{uni}) - g_2(M_{uni})}{g_2(M_{uni})} \right| \tag{92}$$

$$\text{where} \quad M_{uni} \quad \text{such that} \quad g_3(M_{uni}) = g_1(M_{uni}) \tag{93}$$

The results for this on one and two loop level can be found in the table below. For the one loop results, the coupling constants g_i have to be extracted from the calculated α_i^{-1}.

Table 3: Measures of Coupling Unification

P_{uni}	SM	MSSM
One Loop	4.2%	0.42%
Two Loop	3.5%	0.74%

8.4 Implications of the Coupling Unification

The two loop running of the couplings shows the same coupling unification behaviour in SM and MSSM as it was already seen on one loop level, with the unification in the SM being even better, in the MSSM again almost perfect. It is therefore safe to conclude that this phenomenon was not a coincidence of the one loop calculations, but rather a feature in how our description of nature in the Standard Model of Particle Physics works.

The coupling unification happening in the Standard Model is a consequence of how the β functions and the starting values are set to be in our description of nature. If one of the two were different, the unification behaviour would be different. It is quite likely for two running couplings to intersect each other, i.e to have an energy scale at which their values are the same. In the case of three couplings to intersect that close to each other, and becoming even closer when doing a more exact (two loop) calculation as it is in the Standard Model, this might actually be hinting at some underlying concept that is not contained in the theory itself.
Many different viewpoints on this matter exist but many of them tend to go into one of the following directions.

8.4.1 Coupling Unification as a Hint towards Grand Unifying Theories

The gauge coupling unification, with the suitably normalised couplings as being embedded into $SU(5)$ as explained in chapter 8.4 can be interpreted as a hint that this embedding is not only done for normalisation purposes but has an actual physical meaning. This concept, as described in depth in [DRW91], states that the three gauge groups are just different sectors of one underlying gauge group.
This kind of theory is called a "Grand Unifying Theory" or GUT. Those classes of theories look for fundamental ingredients which are able to explain all three of the fundamental forces of particle physics. The minimal GUT is exactly the embedding of $SU(3) \times SU(2) \times U(1)$ into $SU(5)$. This is stating that the three gauge boson classes are essentially of the same class and that this class is spontaneously broken into the three different forces we observe in our particle accelerators by a process quite similar to the electroweak symmetry breaking. This means that the three coupling constants are just different symmetry broken parts of the same, underlying and more fundamental coupling constant of the $SU(5)$ symmetry and without the symmetry breaking, all three coupling constants should be equal.
This underlying group has to be least $SU(5)$ to be able to account for all twelve known gauge bosons. This means that there would be at least another twelve currently not observed gauge bosons. This can be explained by simply considering those additional gauge bosons to be extremely heavy compared to the known ones. This leads to them not appearing in any experiments available today because of the energy threshold being too low.
The acceptance of a GUT as the replacement for the Standard Model would solve some difficulties in its description by being able to sort gauge bosons and fermions into larger groups which can be handled more easily. But it is currently not possible to prove or disprove this because no implications of this GUT have been measured today. New and vastly bigger and more powerful particle accelerators are needed for that. One of the most discussed implications of a GUT is that in it, the proton is no longer stable, but the currently measured lifetime of a proton is greater than 10^{32} years ([Ryd10]).

8.4.2 Coupling Unification as a Hint towards Supersymmetry

The Standard Model shows the gauge coupling unification nowhere near as close as in the MSSM, but it is still visible. Some theories consider this to be a sign that particle physics should actually be a theory which involves supersymmetry, as this would explain the behaviour of the couplings because coupling unification can be seen as a consequence of supersymmetry. This would include the supersymmetry partners of the currently known gauge bosons and the fermionic matter, which have not been observed yet. To account for this, the same concept of those unobserved particles being of much higher mass is used, just as in the GUT case. Usually, as explained in [Ryd10], the masses of supersymmetric partners should be the same. Therefore to be able to not observe them, the supersymmetry must be broken. The supersymmetry-breaking-process is quite complicated and involves some problems of its own and creates a lot of new parameters in the theory.

But nonetheless if supersymmetry is a part of nature, it would ease up on a couple of problems with persist within the non-supersymmetric standard model, like the hierarchy problem, dark matter and aspects of cosmology. Those are the exact reasons why supersymmetric models were considered in the first place.

Other physicists (like [Ryd10]) consider the possibility of supersymmetry and grand unification working both at the same time, as supersymmetry would solve the problems of the GUT of the not perfect gauge coupling unification and the predicted lifetime of the proton. But again, experimental evidence is lacking due to the extremely high energies an which those theories begin to predict different results than the Standard Model.

8.4.3 Coupling Unification as a coincidence of nature

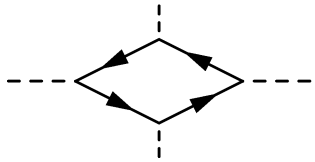
The Standard Model of Particle Physics is only an effective description of the currently known phenomena of particle physics. The Standard Model will certainly break down at the Planck scale where gravity can no longer be neglected. Therefore the search for beyond the Standard Model physics is one way of maybe being able to extend and change the current theory and then maybe find a way to incorporate gravity. Until this is achieved, it is impossible to determine whether the coupling unification is a physical necessity of the theory or just the way nature turned out to be.

This might seem quite unsatisfactory but if the only way to account for the unification is to consider hypothetical particles out of range of experiments, this remains a viable option in the explanation of coupling unification.

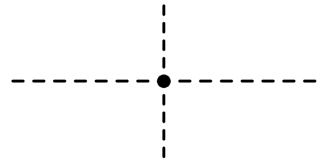
9 The Higgs Quartic Coupling λ

The Higgs quartic coupling λ is one of the parameters of the two loop β functions for the Standard Model of Particle Physics. It is the coupling constant of the scalar field quartic self interaction. The Higgs quartic coupling λ is of special interest in for the evaluation of possible flaws in the Standard Model of Particle Physics due to the fact that in equation (8), the assumption $\lambda > 0$ was made. This assumption is necessary for the potential of the Higgs field to have a stable minimum. In order to explore the implications of this assumption, it is important to know and understand the running of the Higgs quartic coupling λ.

The β function of the Higgs quartic coupling was already used an solved in the discussion of the three gauge couplings on two loop level. The most important term in the running of λ is the one proportional to y_t^4, which is derived from diagrams like the left one below.



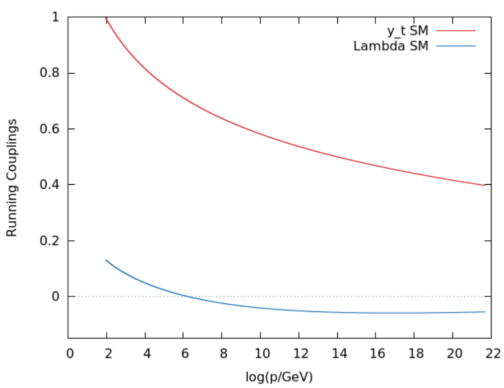
as a correction to:



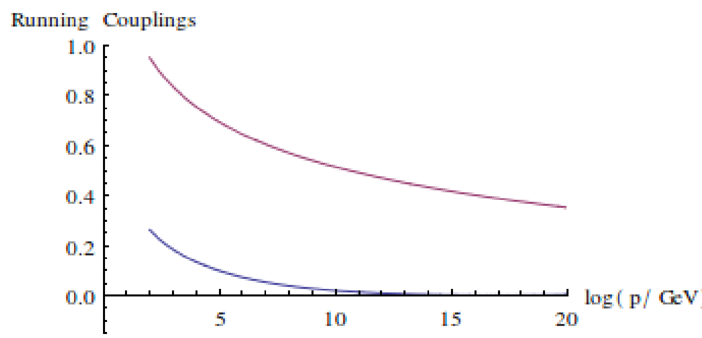
It is unfortunate that exactly on this term, SARAH and [BDG+13] do not agree. The two loop Renormalization Group Equations are dependant on the exact renormalization scheme, and therefore it is possible for two different derivations of the β-functions to not match, as it is the case here. But nonetheless, since exact numerical solutions match to sufficient accuracy for a phenomenological discussion, both will be used in the following analysis.

9.1 The Running of λ in the Standard Model

During the calculations for the running couplings in the previous chapters, the running of λ and y_t was calculated as:



(a) Two Loop Running of λ (blue) and y_t (purple) in the SM, using the RK4 routine



(b) Two Loop Running of λ (blue) and y_t (purple) in the SM, calculated by SARAH

Because of the sensitivity of the running of λ on the starting values, it is necessary to include radiative corrections to the equations which connect physically observable parameters to the starting values, as explained in section 2.3 for the tree-level equations. For the previous discussion, this was not necessary because these changes are small and the differential equations were not as sensitive to them as the two loop β-function for the Higgs quartic coupling is.
The loop-corrected values are taken from [BDG+13, Table 3]. Due to this source, a change of renormalization conditions, now no longer using M_Z but rather $M = M_t$ (top-quark-mass) as the renormalization point is necessary.

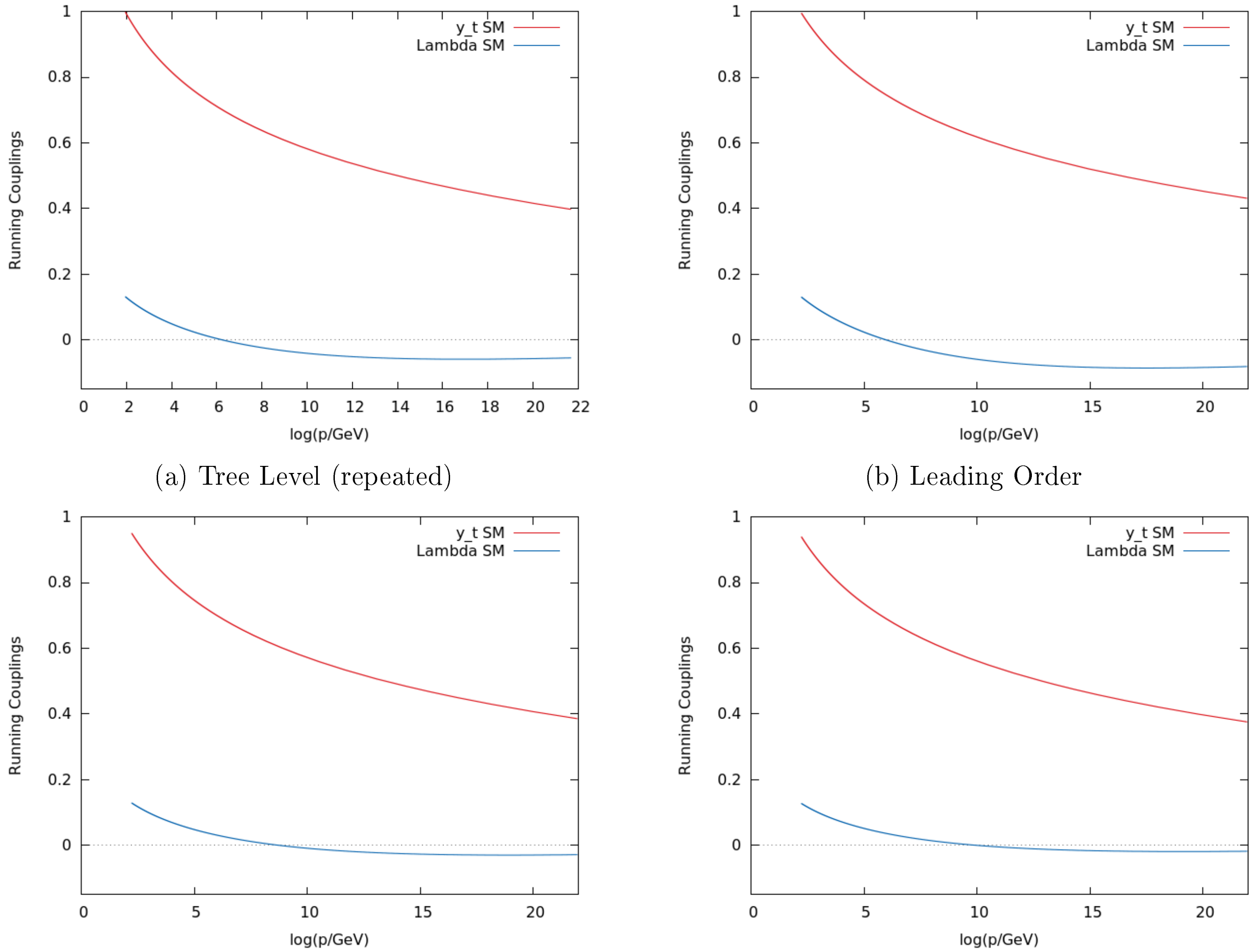
Table 4: Values for the corrected fundamental SM parameters

Equations, corrected to Order:	$\lambda(M_t)$	$y_t(M_t)$	$g_2(M_t)$	$g_Y(M_t) = g_1 \cdot \sqrt{\frac{3}{5}}$
Leading Order (LO, tree-level)	0.13023	0.99425	0.65294	0.34972
Next to Leading Order (NLO)	0.12879	0.94953	0.64755	0.35937
Next to Next to Leading Order (NNLO)	0.12720	0.93849	0.64822	0.35760

The value of g_3 or α_3 is measured directly, and therefore receives no corrections to any equation, it just has to be run from M_Z to M_t. The value is taken from [BDG+13, Eq. 60] and is:

$$g_3(M_t) = 1.1666 \tag{94}$$

The numerical solver was run using all three sets of starting conditions and yields the following results. For comparison, the already discussed tree level result is shown again.



(a) Tree Level (repeated)

(b) Leading Order

(c) Next to Leading Order

(d) Next to Next to Leading Order

Figure 5: Two Loop Running Couplings of λ and y_t, using different sets of starting values

In the above Graphs, it can be seen that the region in which λ is negative is quite sensitive to the starting values. The biggest uncertainty of those values, calculated from experimental data, is the exact value of the mass of the top quark. To analyse this dependency, the following equations show the dependency of the NNLO starting values on the top mass (the equations are taken from [BDG+13], but ignoring all other dependencies).

$$\lambda(M_t) = 0.12711 - 0.00004 \left(\frac{M_t}{GeV} - 173.5 \right) \tag{95}$$

(λ including NNNLO for pure QCD results)

$$y_t(M_t) = 0.93558 + 0.00550 \left(\frac{M_t}{GeV} - 173.5 \right) \tag{96}$$

$$g_2(M_t) = 0.64822 + 0.00004 \left(\frac{M_t}{GeV} - 173.5 \right) \tag{97}$$

$$g_1(M_t) = \sqrt{\frac{5}{3}} \left(0.35761 + 0.00011 \left(\frac{M_t}{GeV} - 173.5 \right) \right) \tag{98}$$

$$g_3(M_t) = 1.1666 - 0.00046 \left(\frac{M_t}{GeV} - 173.5 \right) \tag{99}$$

Using those equations, a new, more exact running of the couplings is done for the value of $M_t = 173.5$ (values from [JBeaPDG12]), and for $M_t \pm 3\sigma = 173.5 \pm 3 \cdot 1.4 GeV$. The results can be seen in the graph below.

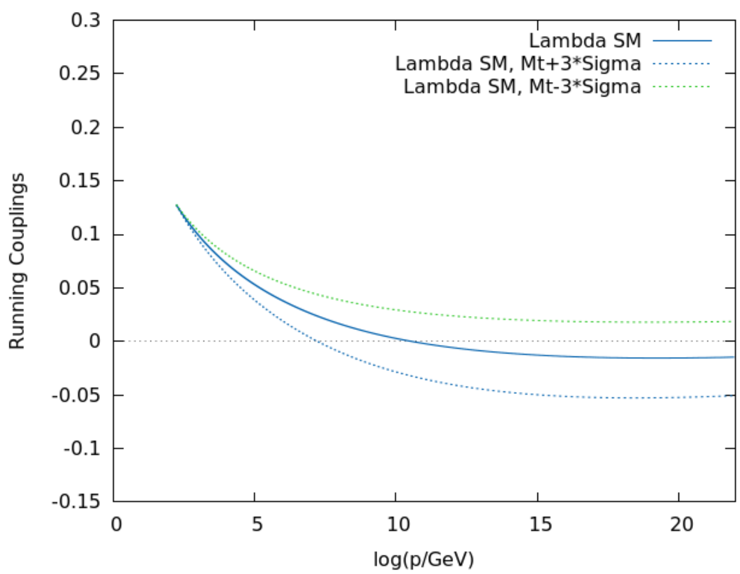


Figure 6: Different Runnings of λ using varying values for M_t

This calculation shows that the crossing of zero for the Higgs quartic coupling not happening at all is within three standard deviations of the currently measured physical parameters. Therefore it is statistically probable that λ remains positive everywhere up to the Planck scale, but it is still worth discussing why the Standard Model is so extremely close to this potential instability which is explained in the next chapter.

9.2 Possible Implications of the Running of λ

Many different and in most cases more exact studies of the running of the Higgs quartic coupling using more exact measurements of the top quark mass, like [BDG+13], show that it is highly

unlikely that the quartic coupling never turns negative up to the Planck scale. Therefore the implication of the running of λ have been a topic for discussion on many occasions, like it was the case for [Esp14].
The running of λ to a point where $\lambda < 0$ would mean that the assumption of a nonzero but finite vacuum expectation value for the Higgs field in the Standard Model would no longer hold. In this region the Higgs potential would no longer have a minimum value. The vacuum of a theory is defined as the minimum of all fields contained in the theory, therefore this poses a problem in the definition of the vacuum. Because of the implications the Higgs field has on the electro-weak symmetry breaking, this is usually called the instability of the electroweak vacuum. As the strong nuclear force, represented by $SU(3)$ does not take any part in the symmetry breaking, it is usually excluded in the discussions of the instability.
Two possible consequences could be the case in nature, which is what any physics model wants to describe.
First, it can be calculated that the electroweak vacuum actually might be unstable, but the decay time is much larger than the current lifetime of the universe, and therefore the vacuum has not decayed yet. Following this, a nonzero but finite vacuum expectation value of the Higgs field and therefore the description in the Standard Model is preserved. This would imply that λ is allowed to turn negative, but its absolute value has to remain small, otherwise the decay time would drop dramatically. This state of the universe is usually called meta-stable. If the universe is meta-stable, this would mean, as a lot of physicists put is:

"The universe is living at the edge."[1].

To many physicists this is counter-intuitive, as one expects nature to be stable and not just completely change at the decay of its vacuum.
The other possibility is that some physics beyond the Standard Model exist that keep the value of λ from getting that close to zero in the first place, ensuring the stability of the electroweak vacuum. What type of physics and the theory behind this is is currently totally unknown, as no experimental data suggests any theories. Two potential candidates are again Supersymmetry and Grand Unification, both providing means to circumvent the instability of the vacuum.

9.3 The Higgs Quartic Coupling in the MSSM

The Minimal Supersymmetric extension of the Standard Model is, by replacing all fields in the SM with superfields, a theory with more symmetry than the SM. Hence the name. Due to this added symmetry, the value of λ is no longer a free parameter to be taken from experiment, but is determined by the electroweak couplings. A thorough explanation can be found in the introduction of [DHHP02] .
This is not the end of the added complexity of the Higgs sector in the MSSM as, to be consistent with super-symmetry, not one but two Higgs scalars exist in the MSSM, posing all kinds of restrictions on Higgs masses, potentials and couplings. Therefore the discussion of the stability of the electroweak vacuum is more complicated, as it is not a simple crossing into the negative for λ.

[1] from [Esp14]

Some discussions see this as a hint towards supersymmetry, as the whole set of problems arising with the values of λ is different there, but supersymmetry has a whole set of problems of its own, as explained in chapter 8.4.2.

10 Further Research and Conclusion

10.1 Modifications to the SM Particle Content

In the discussion of possible beyond the Standard Model physics, many theories consider added particles at an energy scale higher than currently reached by experiment. This enables those theories to have different behaviour in the running of the couplings while still matching the experimental results. It is therefore interesting to try to analyse how the adding of a single particle to the Standard Model at a certain energy scale changes the theory.
The minimal alteration of the Standard Model in adding fermionic matter would be to add a pair of Weyl-fermions charged under only one of the three fundamental forces, in this conjugate representations under $SU(3)$ for the pair (i.e $(3,1)_0$ and $(\bar{3},1)_0$). It is not possible to add just one Weyl Fermion to the Standard Model because this would cause many inconsistencies within the model description.
The β functions for the altered Standard Model are derived by SARAH, using a modified version of the SM model files. A mass parameter for those added fermions is included, independent of the electroweak symmetry breaking for the generation of particle masses. The calculation of the β-functions in SARAH are done in the massless limit, therefore the results are independent of this mass parameter. In order to simulate that the added particles only have an effect above a certain energy scale (i.e. their mass), the β functions are taken from SARAH and used in a modified version of the RK4 numerical solver. The unmodified Standard Model equations are solved up to the energy scale at which the particles are added, using the NNLO starting values of the previous chapter. All generations of new particles are added at the same energy scale. Then the equations are changed to the ones of the modified Standard Model, simulating the effects the added particles have, and the running is finished up to the Planck scale.
This was done in three cases, using one, two or three generations of the added fermion pair, just as there are three generations of every other matter particle in the Standard Model. The SARAH output for the three cases can be found in Appendix D. In each of the three cases, the mass parameter of the added fermions was varied between 10^3 and $10^{17} GeV$ in logarithmic steps of 2. Since the effect of the modifications to the theory are quite small, the effects of the alterations are only visible if the mass parameter of the added fermions is far below the point in which the discussed phenomena happen in the Standard Model. Therefore all cases in which the pair of fermions is added above $10^7 GeV$ are not suited for discussion here, as they are not much different than the Standard Model itself.
The complete set of graphs of these calculations can be found in Appendix E.

10.2 Discussion of the added Fermionic Matter to the SM

The graphs in appendix E confirm that if the generations of the added fermion pairs are added at a lower energy scale, their implications are stronger. The same goes for more generations of the added fermion pairs. Therefore, the maximum impact of the modifications to the Standard

Model can be seen in adding three generations at a scale of $10^3 GeV$, as it is shown below and in appendix E.

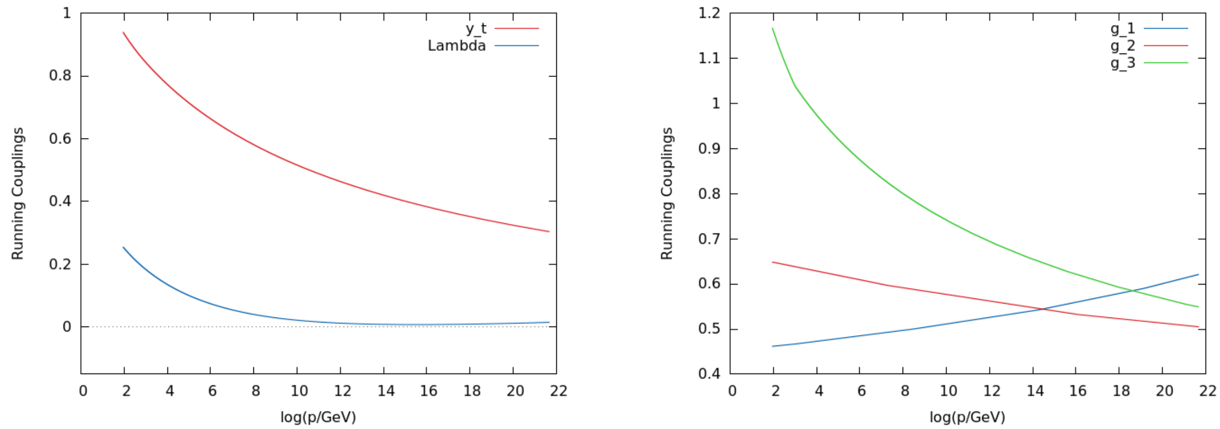


Figure 7: Modified SM, added three generations of the pair of fermions at mass $10^3 GeV$

This is the only graph presented here because all others show the same behaviour, only less strong and at higher energy scales.
The adding of a fermion pair charged only under $SU(3)$ has effects on both, coupling unification and the stability of the electroweak vacuum. In the maximum case, the running of λ does not cross into the negative region any more, therefore ensuring the correctness of the Standard Models description of the electroweak symmetry breaking. At the same time, the coupling unification gets worse if the pair of fermions is added, pushing the unification of g_3 and g_2 far beyond the Planck scale in the maximum modification case.
This leads to two conclusions, both discouraging beyond the Standard Model physics if such a pair of fermions in three generations were to be found in experiment at a scale of $10^3 GeV$. In this case, the altered Standard Model would get rid of the problem of the potential instability of the electroweak vacuum and many hints towards coupling unification. Therefore this would strengthen the description the Standard Model gives of Particle Physics, just with a different matter content.
All of this only holds in the maximum case. But in this case the added fermions should already have shown up in current experiments on the biggest particle accelerators, with the LHC at the CERN already having done experiments at $7 \cdot 10^3 GeV$.
If some of the cases with less than three generations of the added fermions are found to be true, this poses a different question. All other fermions in the Standard Model come in three generations, therefore one that does not is considered strange. It is currently not known why there are exactly three generations for all fermions, but it has been found to be the case in nature.

The next steps in the evaluation of modifications to the Standard Model of Particle Physics would be to consider pairs of fermions charged under $SU(2)$ or $U(1)$ only, and seeing what types of implications these would have on the two phenomena discussed here. But this poses tough problems in the method applied in this thesis, as SARAH ran into problems deriving those β-functions.

10.3 Conclusion

The topic of beyond the Standard Model physics remains a highly interesting one, as many of the predictions of the Standard Model are in regions currently unavailable or only lightly touched upon by experiment. A real measurement of the running of gauge couplings up to the scales of predicted coupling unification is far out of reach.
As soon as experimental evidence is found that is not described in the Standard Model, the search for new models is encouraged into this direction. And at least at the Planck scale this is going to happen. Until such experimental evidence is found, a huge variety of theories are considered and prepared and checked with experimental data, as none of them can be excluded. Even if none of them are later found to be true, advances in the physical description of nature are made as new methods and mathematical descriptions are developed which might be applied to other fields than particle physics.
Taking all of the mentioned hints and possibilities that were discussed in this thesis, and combining them with all other research done on this field, most physicists think it would be too much of a coincidence for all of the discussed phenomena to happen by chance.
Therefore they believe that some physics beyond the Standard Model must exist below the Planck scale. But proof of that has yet to be found in further experiments.

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Appendix

A Runge Kutta 4 Code for Numerical Solutions

The following code is written in C++ and uses the methods and definitions in "nrutil.c" from the Numerical Recipes.

```
#define NRANSI
#include "nrutil.h"
#include <stdio.h>
#include <math.h>
#include <cmath>

const double pi =3.14159265358979323846264338327950288419 71; //taken from: www.pible.de

void rk4(float y[], float dydx[], int n, float x, float h, float yout[],
void (*derivs)(float, float [], float []))
{//Runge Kutta 4 Method, taken from the Numerical Recipies.
int i;
float xh,hh,h6,*dym,*dyt,*yt;
dym=vector(1,n);
dyt=vector(1,n);
yt=vector(1,n);
hh=h*0.5;
h6=h/6.0;
xh=x+hh;
for (i=1;i<=n;i++) yt[i]=y[i]+hh*dydx[i];
(*derivs)(xh,yt,dyt);
for (i=1;i<=n;i++) yt[i]=y[i]+hh*dyt[i];
(*derivs)(xh,yt,dym);
for (i=1;i<=n;i++) {
yt[i]=y[i]+h*dym[i];
dym[i] += dyt[i];
}
(*derivs)(x+h,yt,dyt);
for (i=1;i<=n;i++){
yout[i]=y[i]+h6*(dydx[i]+dyt[i]+2.0*dym[i]);
}
free_vector(yt,1,n);
free_vector(dyt,1,n);
free_vector(dym,1,n);
}


void derivs(float x, float y[], float dydx[]){
```

```
//RGEs taken from the Buttazzo paper.
dydx[1] = 3.0/5.0*1.0/(16.0*pi*pi)*41.0/6.0*y[1]*y[1]*y[1]+y[1]*y[1]*y[1]
*3.0/5.0/(4.0*4.0*4.0*4.0*pi*pi*pi*pi)*(199.0/18.0*3.0/5.0*y[1]*y[1]+
27.0/6.0*y[2]*y[2]+44.0/3.0*y[3]*y[3]-17.0/6.0*y[4]*y[4]);

dydx[2] = -1/(16*pi*pi)*19.0/6.0*y[2]*y[2]*y[2]+y[2]*y[2]*y[2]/(4.0*
4.0*4.0*4.0*pi*pi*pi*pi)*(9.0/6.0*3.0/5.0*y[1]*y[1]+35.0/6.0*y[2]*y[2]
+12.0*y[3]*y[3]-3.0/2.0*y[4]*y[4]);

dydx[3] = -1/(16*pi*pi)*7.0*y[3]*y[3]*y[3]+y[3]*y[3]*y[3]/(4.0*4.0
*4.0*4.0*pi*pi*pi*pi)*(11.0/6.0*3.0/5.0*y[1]*y[1]+9.0/2.0*y[2]+y[2]
-26.0*y[3]*y[3]-2.0*y[4]*y[4]);

        dydx[4] = 1.0/16/pi/pi*y[4]*(9.0/2.0*y[4]*y[4]-17.0/12.0*y[1]*
        y[1]*3.0/5.0-9.0/4.0*y[2]*y[2]-8.0*y[3]*y[3])+
1.0/(4.0*4.0*4.0*4.0*pi*pi*pi*pi)*y[4]*(y[4]*y[4]*(-12.0*y[4]*y[4]
-12.0*y[5]+36.0*y[3]*y[3]+225.0/16.0*y[2]*y[2]+393.0/80.0*y[1]*y[1])
+6.0*y[5]*y[5]-108.0*y[3]*y[3]*y[3]*y[3]-23.0/4.0*y[2]*y[2]*y[2]*y[2]
+1187.0/600.0*y[1]*y[1]*y[1]*y[1]+9.0*y[3]*y[3]*y[2]*y[2]+19.0/15.0
*y[3]*y[3]*y[1]*y[1]-9.0/20.0*y[2]*y[2]*y[1]*y[1]);

        dydx[5]=2.0/16.0/pi/pi*(y[5]*(12*y[5]+6*y[4]*y[4]-9.0/2.0*y[2]
        *y[2]-9.0/10.0*y[1]*y[1])-3.0*y[4]*y[4]*y[4]*y[4]+9.0/16.0*y[2]
        *y[2]*y[2]*y[2]+27.0/400.0*y[1]*y[1]*y[1]*y[1]+9.0/40.0*y[2]
        *y[2]*y[1]*y[1])+2.0/(4.0*4.0*4.0*4.0*pi*pi*pi*pi)*(y[5]*y[5]*
        (-156.0*y[5]-72.0*y[4]*y[4]+54.0*y[2]*y[2]+54.0/5.0*y[1]*y[1])+
        y[5]*y[4]*y[4]*(-3.0/2.0*y[4]*y[4]+40.0*y[3]*y[3]+45.0/4.0*y[2]*
        y[2]+17.0/4.0*y[1]*y[1])+y[5]*(-73.0/16.0*y[2]*y[2]*y[2]*y[2]
        +1887.0/400.0*y[1]*y[1]*y[1]*y[1]+117.0/40.0*y[1]*y[1]*y[2]*y[2])+
        y[4]*y[4]*y[4]*y[4]*(15.0*y[4]*y[5]-16*y[3]*y[3]-4.0/5.0*y[1]*y[1])+
        y[4]*y[4]*(-9.0/8.0*y[2]*y[2]*y[2]*y[2]-171.0/200.0*y[1]*y[1]*y[1]
        *y[1]+63.0/20.0*y[2]*y[2]*y[1]*y[1])+305.0/32.0*y[2]*y[2]*y[2]*y[2]
        *y[2]*y[2]-3411.0/4000.0*y[1]*y[1]*y[1]*y[1]*y[1]*y[1]-289.0/160.0
        *y[1]*y[1]*y[2]*y[2]*y[2]*y[2]-1677.0/800.0*y[2]*y[2]*y[1]*y[1]*y[1]
        *y[1]);

dydx[6]=1.0/16.0/pi/pi*3.0/5.0*11.0*y[6]*y[6]*y[6]+y[6]*y[6]*y[6]
/(4.0*4.0*4.0*4.0*pi*pi*pi*pi)*3.0/5.0*(199.0/9.0*3.0/5.0*y[6]*y[6]+
9.0*y[7]+y[7]+88.0/3.0*y[8]*y[8]-26.0/3.0*y[9]*y[9]);

dydx[7]=1.0/16.0/pi/pi*y[7]*y[7]*y[7]+y[7]*y[7]*y[7]/(4.0*4.0*4.0*4.0
*pi*pi*pi*pi)*(3.0*3.0/5.0*y[6]*y[6]+25.0*y[7]*y[7]+24.0*y[8]*y[8]-6.0*y[9]*y[9]);

dydx[8]=-3.0/16.0/pi/pi*y[8]*y[8]*y[8]+y[8]*y[8]*y[8]/(4.0*4.0*4.0*4.0*pi*pi
*pi*pi)*(11.0/3.0*y[6]*y[6]+9.0*y[7]*y[7]+14.0*y[8]+y[8]-4.0*y[9]*y[9]);
```

```
dydx[9]=y[9]/16.0/pi/pi*(6.0*y[9]*y[9]-13.0/9.0*3.0/5.0*y[6]*y[6]-3.0*y[7]
*y[7]-16.0/3.0*y[8]*y[8]);
}

int main (void) {

FILE * pFile;
float end = 40; //end point for iteration
float h = 0.00005; //stepsize
float x = 0.0; //startpoint
float dydx[10], yout[10], y[10];
 //first 5 values for SM, may be changed to solve for different starting conditions
        y[1] = 0.3555*sqrt(5.0/3.0); //U(1), with GUT normalization
        y[2] = 0.6517; //SU(2)
        y[3] = 1.220;  //SU(3)
        y[4] = 0.9974; //top yukawa coupling
        y[5] = 0.1312; //higgs quartic coupling
        //next four for MSSM
        y[6]=y[1];//U(1)
        y[7]=y[2];//SU(2)
        y[8]=y[3];//SU(3)
        y[9]=y[4];//top yukawa
pFile = fopen ("2loop_LO.txt","w");
fprintf (pFile, "%f %f %f %f %f %f %f %f %f %f \n", x, y[1],
 y[2], y[3], y[4], y[5],y[6],y[7],y[8],y[9]);
while (x<end){
derivs(x, y, dydx);
rk4(y, dydx, 9, x, h, yout, *derivs);
x += h;
//printf("x = %f\n", x);
for (int j = 1; j <= 9; j++){
y[j] = yout[j];
//printf("yout[%d] = %f\n", j, yout[j]);
}
fprintf (pFile, "%f %f %f %f %f %f %f %f %f %f \n", x,
 y[1], y[2], y[3], y[4], y[5],y[6],y[7],y[8],y[9]);
}
fclose (pFile);
return 0;}

#undef NRANSI
/* (C) Copr. 1986-92 Numerical Recipes Software */
```

B SARAH Output for the SM

Gauge Couplings

$$\beta_{g_1}^{(1)} = \frac{41}{10} g_1^3 \tag{100}$$

$$\beta_{g_1}^{(2)} = \frac{1}{50} g_1^3 \Big(135 g_2^2 + 199 g_1^2 - 25 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 440 g_3^2 - 75 \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - 85 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{101}$$

$$\beta_{g_2}^{(1)} = -\frac{19}{6} g_2^3 \tag{102}$$

$$\beta_{g_2}^{(2)} = \frac{1}{30} g_2^3 \Big(- 15 \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 175 g_2^2 + 27 g_1^2 + 360 g_3^2 - 45 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 45 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{103}$$

$$\beta_{g_3}^{(1)} = -7 g_3^3 \tag{104}$$

$$\beta_{g_3}^{(2)} = -\frac{1}{10} g_3^3 \Big(- 11 g_1^2 + 20 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 20 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 260 g_3^2 - 45 g_2^2\Big) \tag{105}$$

Quartic scalar couplings

$$\begin{aligned}\beta_{\lambda}^{(1)} = &+\frac{27}{100} g_1^4 + \frac{9}{10} g_1^2 g_2^2 + \frac{9}{4} g_2^4 - \frac{9}{5} g_1^2 \lambda - 9 g_2^2 \lambda + 12 \lambda^2 + 12 \lambda \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 4 \lambda \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big)\\ &+ 12 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 4 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\end{aligned} \tag{106}$$

$$\begin{aligned}\beta_{\lambda}^{(2)} = &-\frac{3411}{1000} g_1^6 - \frac{1677}{200} g_1^4 g_2^2 - \frac{289}{40} g_1^2 g_2^4 + \frac{305}{8} g_2^6 + \frac{1887}{200} g_1^4 \lambda + \frac{117}{20} g_1^2 g_2^2 \lambda - \frac{73}{8} g_2^4 \lambda + \frac{54}{5} g_1^2 \lambda^2 + 54 g_2^2 \lambda^2\\ &- 78 \lambda^3 + \frac{1}{10} \Big(225 g_2^2 \lambda - 45 g_2^4 + 80 \Big(10 g_3^2 - 9 \lambda \Big)\lambda + 9 g_1^4 + g_1^2 \Big(25 \lambda + 54 g_2^2 \Big)\Big)\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big)\\ &- \frac{3}{10} \Big(15 g_1^4 + 5 \Big(16 \lambda^2 - 5 g_2^2 \lambda + g_2^4 \Big) - g_1^2 \Big(22 g_2^2 + 25 \lambda \Big)\Big)\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - \frac{171}{50} g_1^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\\ &+ \frac{63}{5} g_1^2 g_2^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - \frac{9}{2} g_2^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{17}{2} g_1^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{45}{2} g_2^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\\ &+ 80 g_3^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 72 \lambda^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{8}{5} g_1^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big)\\ &- 3 \lambda \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 42 \lambda \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - \frac{24}{5} g_1^2 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - \lambda \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big)\\ &- \frac{16}{5} g_1^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 3 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) + 60 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger\Big)\\ &+ 12 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - 24 \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger Y_d Y_d^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_u^\dagger Y_u Y_d^\dagger\Big)\\ &+ 20 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) + 60 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\end{aligned} \tag{107}$$

Yukawa Couplings

$$\beta_{Y_u}^{(1)} = -\frac{3}{2} \Big(- Y_u Y_u^\dagger Y_u + Y_u Y_d^\dagger Y_d\Big)$$

$$+ Y_u\Big(3\text{Tr}\Big(Y_d Y_d^\dagger\Big) + 3\text{Tr}\Big(Y_u Y_u^\dagger\Big) - 8 g_3^2 - \frac{17}{20} g_1^2 - \frac{9}{4} g_2^2 + \text{Tr}\Big(Y_e Y_e^\dagger\Big)\Big) \quad (108)$$

$$\begin{aligned}
\beta_{Y_u}^{(2)} = &+\frac{1}{80}\Big(20\Big(11 Y_u Y_d^\dagger Y_d Y_d^\dagger Y_d - 4 Y_u Y_u^\dagger Y_u Y_d^\dagger Y_d + 6 Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u - Y_u Y_d^\dagger Y_d Y_u^\dagger Y_u\Big)\\
&+ Y_u Y_u^\dagger Y_u\Big(1280 g_3^2 - 180\text{Tr}\Big(Y_e Y_e^\dagger\Big) + 223 g_1^2 - 480\lambda - 540\text{Tr}\Big(Y_d Y_d^\dagger\Big) - 540\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 675 g_2^2\Big)\\
&+ Y_u Y_d^\dagger Y_d\Big(100\text{Tr}\Big(Y_e Y_e^\dagger\Big) - 1280 g_3^2 + 300\text{Tr}\Big(Y_d Y_d^\dagger\Big) + 300\text{Tr}\Big(Y_u Y_u^\dagger\Big) - 43 g_1^2 + 45 g_2^2\Big)\Big)\\
&+ \frac{1}{600} Y_u\Big(1187 g_1^4 - 270 g_1^2 g_2^2 - 3450 g_2^4 + 760 g_1^2 g_3^2 + 5400 g_2^2 g_3^2 - 64800 g_3^4 + 900\lambda^2\\
&+ 375\Big(32 g_3^2 + 9 g_2^2 + g_1^2\Big)\text{Tr}\Big(Y_d Y_d^\dagger\Big) + 1125\Big(g_1^2 + g_2^2\Big)\text{Tr}\Big(Y_e Y_e^\dagger\Big) + 1275 g_1^2\text{Tr}\Big(Y_u Y_u^\dagger\Big)\\
&+ 3375 g_2^2\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 12000 g_3^2\text{Tr}\Big(Y_u Y_u^\dagger\Big) - 4050\text{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) + 900\text{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big)\\
&- 1350\text{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 4050\text{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\Big) \quad (109)
\end{aligned}$$

C SARAH Output for the MSSM

Gauge Couplings

$$\beta_{g_1}^{(1)} = \frac{33}{5} g_1^3 \quad (110)$$

$$\beta_{g_1}^{(2)} = \frac{1}{25} g_1^3\Big(- 130\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 135 g_2^2 + 199 g_1^2 + 440 g_3^2 - 70\text{Tr}\Big(Y_d Y_d^\dagger\Big) - 90\text{Tr}\Big(Y_e Y_e^\dagger\Big)\Big) \quad (111)$$

$$\beta_{g_2}^{(1)} = g_2^3 \quad (112)$$

$$\beta_{g_2}^{(2)} = \frac{1}{5} g_2^3\Big(- 10\text{Tr}\Big(Y_e Y_e^\dagger\Big) + 120 g_3^2 + 125 g_2^2 - 30\text{Tr}\Big(Y_d Y_d^\dagger\Big) - 30\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 9 g_1^2\Big) \quad (113)$$

$$\beta_{g_3}^{(1)} = -3 g_3^3 \quad (114)$$

$$\beta_{g_3}^{(2)} = \frac{1}{5} g_3^3\Big(11 g_1^2 - 20\text{Tr}\Big(Y_d Y_d^\dagger\Big) - 20\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 45 g_2^2 + 70 g_3^2\Big) \quad (115)$$

Trilinear Superpotential Parameters

$$\beta_{Y_u}^{(1)} = 3 Y_u Y_u^\dagger Y_u - \frac{1}{15} Y_u\Big(13 g_1^2 + 45 g_2^2 - 45\text{Tr}\Big(Y_u Y_u^\dagger\Big) + 80 g_3^2\Big) + Y_u Y_d^\dagger Y_d \quad (116)$$

$$\begin{aligned}
\beta_{Y_u}^{(2)} = &+\frac{2}{5} g_1^2 Y_u Y_u^\dagger Y_u + 6 g_2^2 Y_u Y_u^\dagger Y_u - 2 Y_u Y_d^\dagger Y_d Y_d^\dagger Y_d - 2 Y_u Y_d^\dagger Y_d Y_u^\dagger Y_u\\
&- 4 Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u + Y_u Y_d^\dagger Y_d\Big(- 3\text{Tr}\Big(Y_d Y_d^\dagger\Big) + \frac{2}{5} g_1^2 - \text{Tr}\Big(Y_e Y_e^\dagger\Big)\Big) - 9 Y_u Y_u^\dagger Y_u\text{Tr}\Big(Y_u Y_u^\dagger\Big)\\
&+ Y_u\Big(\frac{2743}{450} g_1^4 + g_1^2 g_2^2 + \frac{15}{2} g_2^4 + \frac{136}{45} g_1^2 g_3^2 + 8 g_2^2 g_3^2 - \frac{16}{9} g_3^4 + \frac{4}{5}\Big(20 g_3^2 + g_1^2\Big)\text{Tr}\Big(Y_u Y_u^\dagger\Big)
\end{aligned}$$

$$- 3\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - 9\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\Big) \tag{117}$$

D SARAH Output for the modified SM

D.1 One generation of the added fermion pair, charged under SU(3)

Gauge Couplings

$$\beta_{g_1}^{(1)} = \frac{41}{10} g_1^3 \tag{118}$$

$$\beta_{g_1}^{(2)} = \frac{1}{50} g_1^3 \Big(135 g_2^2 + 199 g_1^2 - 25\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 440 g_3^2 - 75\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - 85\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{119}$$

$$\beta_{g_2}^{(1)} = -\frac{19}{6} g_2^3 \tag{120}$$

$$\beta_{g_2}^{(2)} = \frac{1}{30} g_2^3 \Big(- 15\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 175 g_2^2 + 27 g_1^2 + 360 g_3^2 - 45\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 45\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{121}$$

$$\beta_{g_3}^{(1)} = -\frac{19}{3} g_3^3 \tag{122}$$

$$\beta_{g_3}^{(2)} = -\frac{1}{30} g_3^3 \Big(- 135 g_2^2 - 33 g_1^2 + 400 g_3^2 + 60\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 60\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{123}$$

Quartic scalar couplings

$$\begin{aligned}
\beta_\lambda^{(1)} = & +\frac{27}{100} g_1^4 + \frac{9}{10} g_1^2 g_2^2 + \frac{9}{4} g_2^4 - \frac{9}{5} g_1^2 \lambda - 9 g_2^2 \lambda + 12\lambda^2 + 12\lambda\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 4\lambda\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) \\
& + 12\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 12\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 4\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 12\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)
\end{aligned} \tag{124}$$

$$\begin{aligned}
\beta_\lambda^{(2)} = & -\frac{3411}{1000} g_1^6 - \frac{1677}{200} g_1^4 g_2^2 - \frac{289}{40} g_1^2 g_2^4 + \frac{305}{8} g_2^6 + \frac{1887}{200} g_1^4 \lambda + \frac{117}{20} g_1^2 g_2^2 \lambda - \frac{73}{8} g_2^4 \lambda + \frac{54}{5} g_1^2 \lambda^2 + 54 g_2^2 \lambda^2 \\
& - 78\lambda^3 + \frac{1}{10}\Big(225 g_2^2 \lambda - 45 g_2^4 + 80\Big(10 g_3^2 - 9\lambda\Big)\lambda + 9 g_1^4 + g_1^2\Big(25\lambda + 54 g_2^2\Big)\Big)\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) \\
& - \frac{3}{10}\Big(15 g_1^4 + 5\Big(16\lambda^2 - 5 g_2^2 \lambda + g_2^4\Big) - g_1^2\Big(22 g_2^2 + 25\lambda\Big)\Big)\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - \frac{171}{50} g_1^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + \frac{63}{5} g_1^2 g_2^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - \frac{9}{2} g_2^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{17}{2} g_1^2 \lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{45}{2} g_2^2 \lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + 80 g_3^2 \lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 72\lambda^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{8}{5} g_1^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& - 3\lambda\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 42\lambda\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - \frac{24}{5} g_1^2 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - \lambda\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) \\
& - \frac{16}{5} g_1^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 3\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) + 60\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& + 12\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - 24\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger Y_d Y_d^\dagger\Big) - 12\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_u^\dagger Y_u Y_d^\dagger\Big)
\end{aligned}$$

$$+20\mathrm{Tr}\Big(Y_eY_e^\dagger Y_eY_e^\dagger Y_eY_e^\dagger\Big)+60\mathrm{Tr}\Big(Y_uY_u^\dagger Y_uY_u^\dagger Y_uY_u^\dagger\Big) \tag{125}$$

Yukawa Couplings

$$\beta_{Y_u}^{(1)}=-\frac{3}{2}\Big(-Y_uY_u^\dagger Y_u+Y_uY_d^\dagger Y_d\Big)$$

$$+Y_u\Big(3\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)+3\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)-8g_3^2-\frac{17}{20}g_1^2-\frac{9}{4}g_2^2+\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)\Big) \tag{126}$$

$$\beta_{Y_u}^{(2)}=+\frac{1}{80}\Big(20\Big(11Y_uY_d^\dagger Y_dY_d^\dagger Y_d-4Y_uY_u^\dagger Y_uY_d^\dagger Y_d+6Y_uY_u^\dagger Y_uY_u^\dagger Y_u-Y_uY_d^\dagger Y_dY_u^\dagger Y_u\Big)$$

$$+Y_uY_u^\dagger Y_u\Big(1280g_3^2-180\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)+223g_1^2-480\lambda-540\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)-540\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)+675g_2^2\Big)$$

$$+Y_uY_d^\dagger Y_d\Big(100\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)-1280g_3^2+300\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)+300\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)-43g_1^2+45g_2^2\Big)\Big)$$

$$+Y_u\Big(\frac{1187}{600}g_1^4-\frac{9}{20}g_1^2g_2^2-\frac{23}{4}g_2^4+\frac{19}{15}g_1^2g_3^2+9g_2^2g_3^2-\frac{932}{9}g_3^4+\frac{3}{2}\lambda^2+\frac{5}{8}\Big(32g_3^2+9g_2^2+g_1^2\Big)\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)$$

$$+\frac{15}{8}\Big(g_1^2+g_2^2\Big)\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)+\frac{17}{8}g_1^2\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)+\frac{45}{8}g_2^2\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)+20g_3^2\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)$$

$$-\frac{27}{4}\mathrm{Tr}\Big(Y_dY_d^\dagger Y_dY_d^\dagger\Big)+\frac{3}{2}\mathrm{Tr}\Big(Y_dY_u^\dagger Y_uY_d^\dagger\Big)-\frac{9}{4}\mathrm{Tr}\Big(Y_eY_e^\dagger Y_eY_e^\dagger\Big)-\frac{27}{4}\mathrm{Tr}\Big(Y_uY_u^\dagger Y_uY_u^\dagger\Big)\Big) \tag{127}$$

D.2 Two generations of the added fermion pair, charged under SU(3)

Gauge Couplings

$$\beta_{g_1}^{(1)}=\frac{41}{10}g_1^3 \tag{128}$$

$$\beta_{g_1}^{(2)}=\frac{1}{50}g_1^3\Big(135g_2^2+199g_1^2-25\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)+440g_3^2-75\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)-85\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)\Big) \tag{129}$$

$$\beta_{g_2}^{(1)}=-\frac{19}{6}g_2^3 \tag{130}$$

$$\beta_{g_2}^{(2)}=\frac{1}{30}g_2^3\Big(-15\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)+175g_2^2+27g_1^2+360g_3^2-45\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)-45\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)\Big) \tag{131}$$

$$\beta_{g_3}^{(1)}=-\frac{17}{3}g_3^3 \tag{132}$$

$$\beta_{g_3}^{(2)}=-\frac{1}{30}g_3^3\Big(-135g_2^2+20g_3^2-33g_1^2+60\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)+60\mathrm{Tr}\Big(Y_uY_u^\dagger\Big)\Big) \tag{133}$$

Quartic scalar couplings

$$\beta_\lambda^{(1)}=+\frac{27}{100}g_1^4+\frac{9}{10}g_1^2g_2^2+\frac{9}{4}g_2^4-\frac{9}{5}g_1^2\lambda-9g_2^2\lambda+12\lambda^2+12\lambda\mathrm{Tr}\Big(Y_dY_d^\dagger\Big)+4\lambda\mathrm{Tr}\Big(Y_eY_e^\dagger\Big)$$

$$
+ 12\lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 12\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 4\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 12\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) \tag{134}
$$

$$
\begin{aligned}
\beta_\lambda^{(2)} =& -\frac{3411}{1000}g_1^6 - \frac{1677}{200}g_1^4 g_2^2 - \frac{289}{40}g_1^2 g_2^4 + \frac{305}{8}g_2^6 + \frac{1887}{200}g_1^4\lambda + \frac{117}{20}g_1^2 g_2^2\lambda - \frac{73}{8}g_2^4\lambda + \frac{54}{5}g_1^2\lambda^2 + 54 g_2^2\lambda^2 \\
& - 78\lambda^3 + \frac{1}{10}\Big(225 g_2^2\lambda - 45 g_2^4 + 80\Big(10 g_3^2 - 9\lambda\Big)\lambda + 9 g_1^4 + g_1^2\Big(25\lambda + 54 g_2^2\Big)\Big)\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) \\
& - \frac{3}{10}\Big(15 g_1^4 + 5\Big(16\lambda^2 - 5 g_2^2\lambda + g_2^4\Big) - g_1^2\Big(22 g_2^2 + 25\lambda\Big)\Big)\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - \frac{171}{50}g_1^4\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + \frac{63}{5}g_1^2 g_2^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - \frac{9}{2}g_2^4\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{17}{2}g_1^2\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{45}{2}g_2^2\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + 80 g_3^2\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 72\lambda^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{8}{5}g_1^2\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 64 g_3^2\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& - 3\lambda\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 42\lambda\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - \frac{24}{5}g_1^2\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - \lambda\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) \\
& - \frac{16}{5}g_1^2\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 64 g_3^2\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 3\lambda\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) + 60\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& + 12\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - 24\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger Y_d Y_d^\dagger\Big) - 12\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_u^\dagger Y_u Y_d^\dagger\Big) \\
& + 20\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) + 60\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)
\end{aligned} \tag{135}
$$

Yukawa Couplings

$$
\begin{aligned}
\beta_{Y_u}^{(1)} =& -\frac{3}{2}\Big(- Y_u Y_u^\dagger Y_u + Y_u Y_d^\dagger Y_d\Big) \\
& + Y_u\Big(3\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 3\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 8 g_3^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 + \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big)\Big)
\end{aligned} \tag{136}
$$

$$
\begin{aligned}
\beta_{Y_u}^{(2)} =& +\frac{1}{80}\Big(20\Big(11 Y_u Y_d^\dagger Y_d Y_d^\dagger Y_d - 4 Y_u Y_u^\dagger Y_u Y_d^\dagger Y_d + 6 Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u - Y_u Y_d^\dagger Y_d Y_u^\dagger Y_u\Big) \\
& + Y_u Y_u^\dagger Y_u\Big(1280 g_3^2 - 180\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 223 g_1^2 - 480\lambda - 540\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 540\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 675 g_2^2\Big) \\
& + Y_u Y_d^\dagger Y_d\Big(100\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - 1280 g_3^2 + 300\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 300\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 43 g_1^2 + 45 g_2^2\Big)\Big) \\
& + Y_u\Big(\frac{1187}{600}g_1^4 - \frac{9}{20}g_1^2 g_2^2 - \frac{23}{4}g_2^4 + \frac{19}{15}g_1^2 g_3^2 + 9 g_2^2 g_3^2 - \frac{892}{9}g_3^4 + \frac{3}{2}\lambda^2 + \frac{5}{8}\Big(32 g_3^2 + 9 g_2^2 + g_1^2\Big)\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) \\
& + \frac{15}{8}\Big(g_1^2 + g_2^2\Big)\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + \frac{17}{8}g_1^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{45}{8}g_2^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 20 g_3^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& - \frac{27}{4}\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) + \frac{3}{2}\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - \frac{9}{4}\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - \frac{27}{4}\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\Big)
\end{aligned} \tag{137}
$$

D.3 Three Generations of the added fermion pair, charged under SU(3)

Gauge Couplings

$$\beta_{g_1}^{(1)} = \frac{41}{10} g_1^3 \tag{138}$$

$$\beta_{g_1}^{(2)} = \frac{1}{50} g_1^3 \Big(135 g_2^2 + 199 g_1^2 - 25 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 440 g_3^2 - 75 \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - 85 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{139}$$

$$\beta_{g_2}^{(1)} = -\frac{19}{6} g_2^3 \tag{140}$$

$$\beta_{g_2}^{(2)} = \frac{1}{30} g_2^3 \Big(- 15 \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 175 g_2^2 + 27 g_1^2 + 360 g_3^2 - 45 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 45 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) \tag{141}$$

$$\beta_{g_3}^{(1)} = -5 g_3^3 \tag{142}$$

$$\beta_{g_3}^{(2)} = \frac{1}{10} g_3^3 \Big(11 g_1^2 + 120 g_3^2 - 20 \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 20 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 45 g_2^2\Big) \tag{143}$$

Quartic scalar couplings

$$\begin{aligned}
\beta_{\lambda}^{(1)} = & +\frac{27}{100} g_1^4 + \frac{9}{10} g_1^2 g_2^2 + \frac{9}{4} g_2^4 - \frac{9}{5} g_1^2 \lambda - 9 g_2^2 \lambda + 12 \lambda^2 + 12 \lambda \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 4 \lambda \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) \\
& + 12 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 4 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)
\end{aligned} \tag{144}$$

$$\begin{aligned}
\beta_{\lambda}^{(2)} = & -\frac{3411}{1000} g_1^6 - \frac{1677}{200} g_1^4 g_2^2 - \frac{289}{40} g_1^2 g_2^4 + \frac{305}{8} g_2^6 + \frac{1887}{200} g_1^4 \lambda + \frac{117}{20} g_1^2 g_2^2 \lambda - \frac{73}{8} g_2^4 \lambda + \frac{54}{5} g_1^2 \lambda^2 + 54 g_2^2 \lambda^2 \\
& - 78 \lambda^3 + \frac{1}{10} \Big(225 g_2^2 \lambda - 45 g_2^4 + 80 \Big(10 g_3^2 - 9 \lambda \Big)\lambda + 9 g_1^4 + g_1^2 \Big(25 \lambda + 54 g_2^2 \Big)\Big) \mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) \\
& - \frac{3}{10} \Big(15 g_1^4 + 5 \Big(16 \lambda^2 - 5 g_2^2 \lambda + g_2^4 \Big) - g_1^2 \Big(22 g_2^2 + 25 \lambda \Big)\Big) \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - \frac{171}{50} g_1^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + \frac{63}{5} g_1^2 g_2^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - \frac{9}{2} g_2^4 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{17}{2} g_1^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{45}{2} g_2^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) \\
& + 80 g_3^2 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 72 \lambda^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \frac{8}{5} g_1^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& - 3 \lambda \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) - 42 \lambda \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - \frac{24}{5} g_1^2 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - \lambda \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) \\
& - \frac{16}{5} g_1^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 64 g_3^2 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) - 3 \lambda \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big) + 60 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) \\
& + 12 \mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_u^\dagger Y_u Y_d^\dagger\Big) - 24 \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger Y_d Y_d^\dagger\Big) - 12 \mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_u^\dagger Y_u Y_d^\dagger\Big) \\
& + 20 \mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) + 60 \mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)
\end{aligned} \tag{145}$$

Yukawa Couplings

$$\beta_{Y_u}^{(1)} = -\frac{3}{2}\Big(- Y_u Y_u^\dagger Y_u + Y_u Y_d^\dagger Y_d\Big)$$
$$+ Y_u\Big(3\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 3\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 8g_3^2 - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 + \mathrm{Tr}\Big(Y_e Y_e^\dagger\Big)\Big) \tag{146}$$

$$\beta_{Y_u}^{(2)} = +\frac{1}{80}\Big(20\Big(11Y_u Y_d^\dagger Y_d Y_d^\dagger Y_d - 4Y_u Y_u^\dagger Y_u Y_d^\dagger Y_d + 6Y_u Y_u^\dagger Y_u Y_u^\dagger Y_u - Y_u Y_d^\dagger Y_d Y_u^\dagger Y_u\Big)$$
$$+ Y_u Y_u^\dagger Y_u\Big(1280g_3^2 - 180\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 223g_1^2 - 480\lambda - 540\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) - 540\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 675g_2^2\Big)$$
$$+ Y_u Y_d^\dagger Y_d\Big(100\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) - 1280g_3^2 + 300\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 300\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 43g_1^2 + 45g_2^2\Big)\Big)$$
$$+ \frac{1}{600}Y_u\Big(1187g_1^4 - 270g_1^2 g_2^2 - 3450g_2^4 + 760g_1^2 g_3^2 + 5400g_2^2 g_3^2 - 56800g_3^4 + 900\lambda^2$$
$$+ 375\Big(32g_3^2 + 9g_2^2 + g_1^2\Big)\mathrm{Tr}\Big(Y_d Y_d^\dagger\Big) + 1125\Big(g_1^2 + g_2^2\Big)\mathrm{Tr}\Big(Y_e Y_e^\dagger\Big) + 1275g_1^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)$$
$$+ 3375g_2^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + 12000g_3^2\mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) - 4050\mathrm{Tr}\Big(Y_d Y_d^\dagger Y_d Y_d^\dagger\Big) + 900\mathrm{Tr}\Big(Y_d Y_u^\dagger Y_u Y_d^\dagger\Big)$$
$$- 1350\mathrm{Tr}\Big(Y_e Y_e^\dagger Y_e Y_e^\dagger\Big) - 4050\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\Big) \tag{147}$$

E Graphs of the modified Standard Model

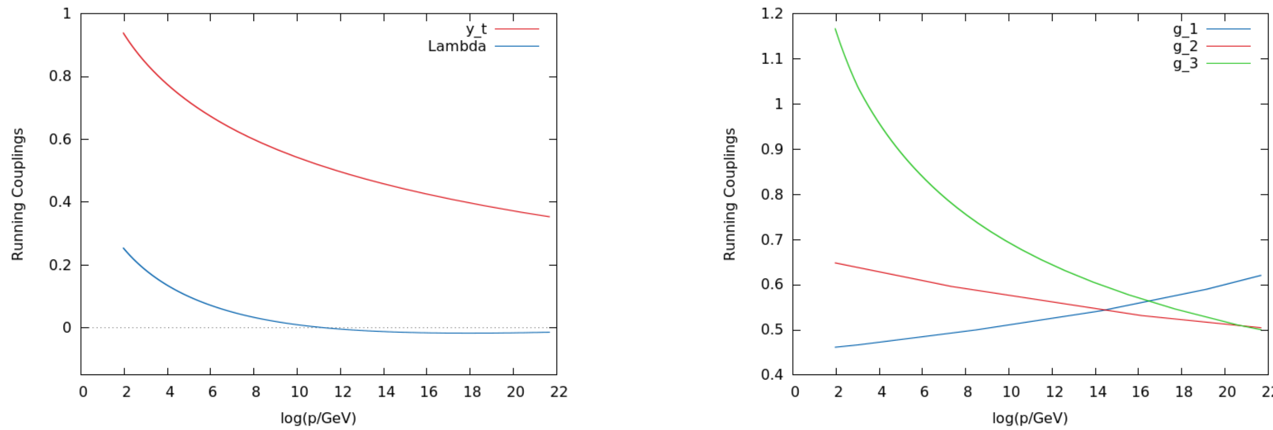

Figure 8: Modified SM, added one generation of the pair of fermions at mass $10^3 GeV$

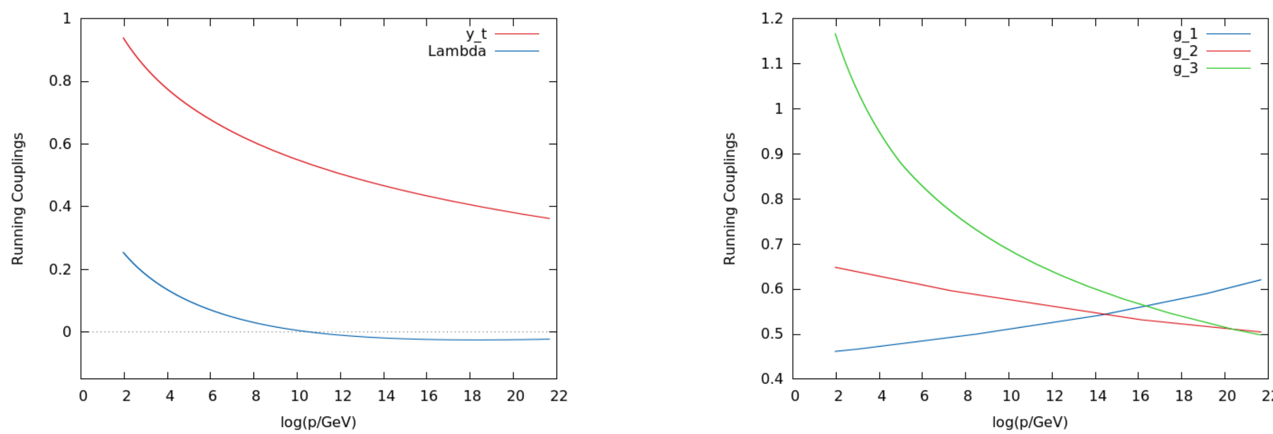

Figure 9: Modified SM, added one generation of the pair of fermions at mass $10^5 GeV$

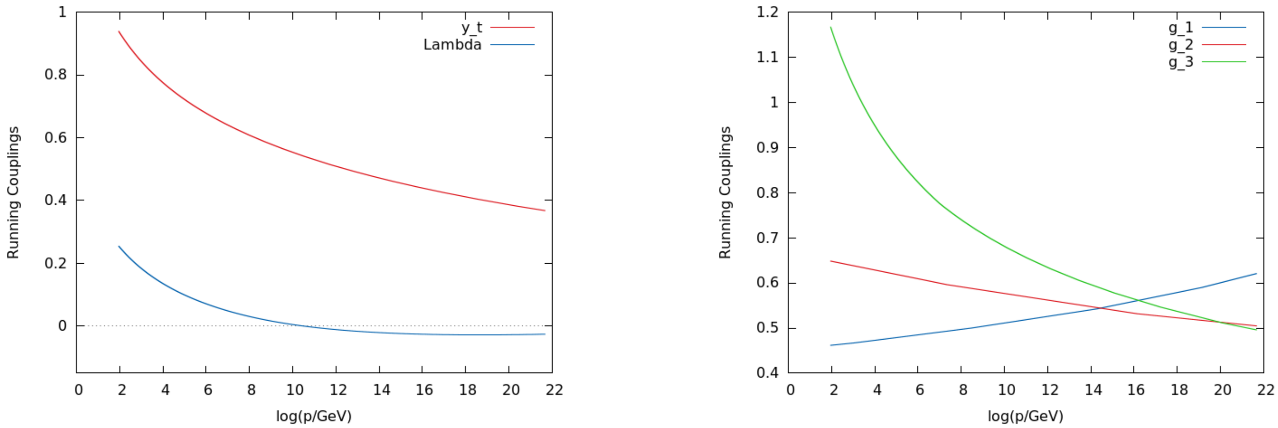

Figure 10: Modified SM, added one generation of the pair of fermions at mass $10^7 GeV$

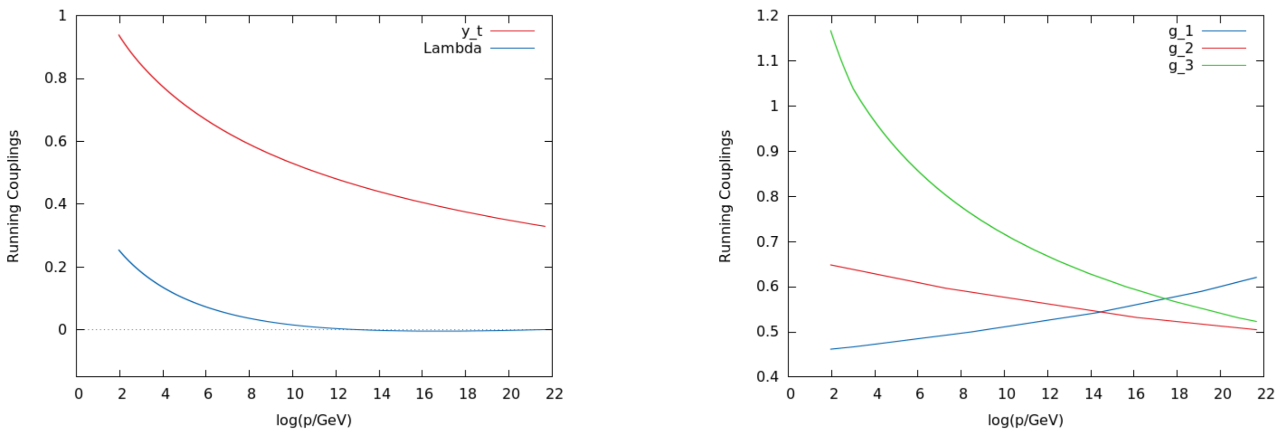

Figure 11: Modified SM, added two generations of the pair of fermions at mass $10^3 GeV$

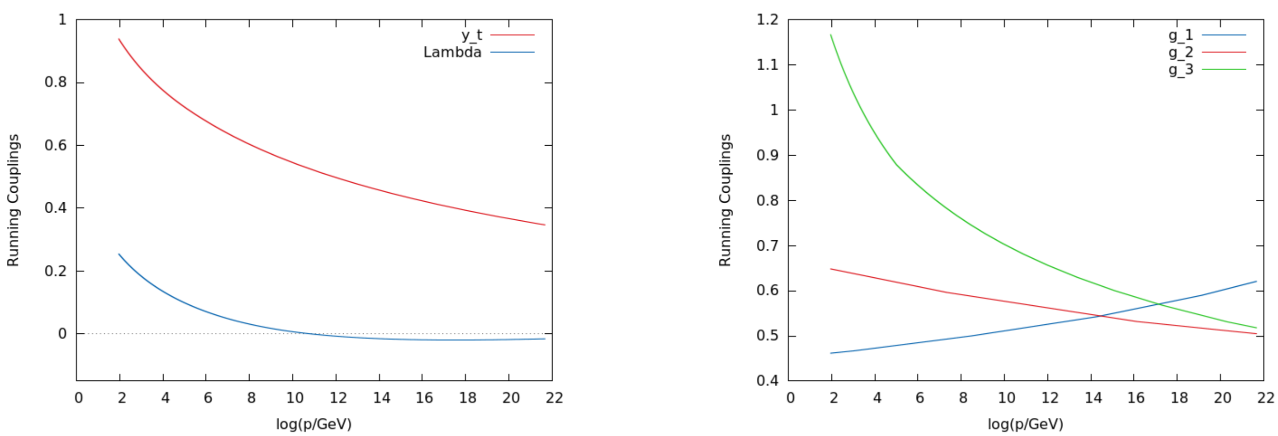

Figure 12: Modified SM, added two generations of the pair of fermions at mass $10^5 GeV$

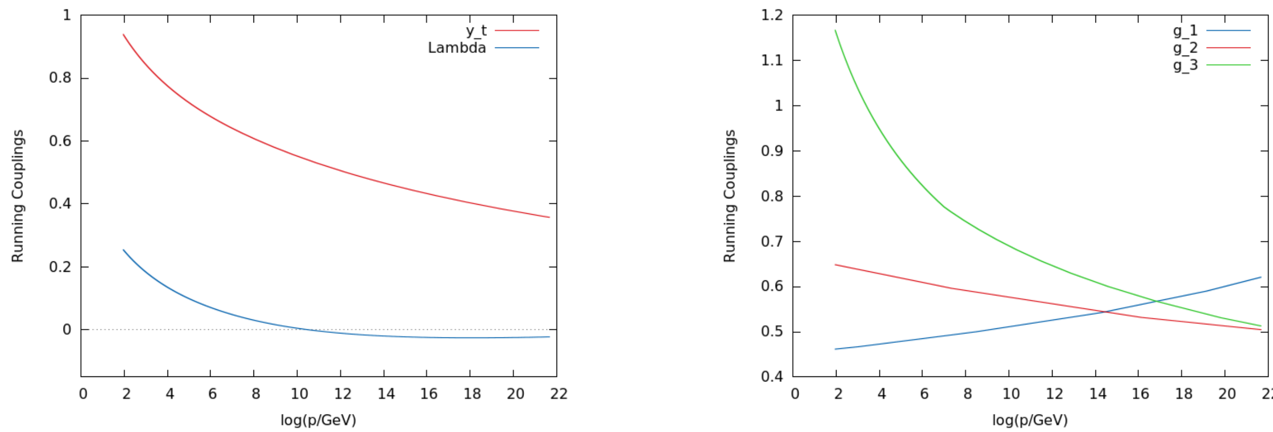

Figure 13: Modified SM, added two generations of the pair of fermions at mass $10^7 GeV$

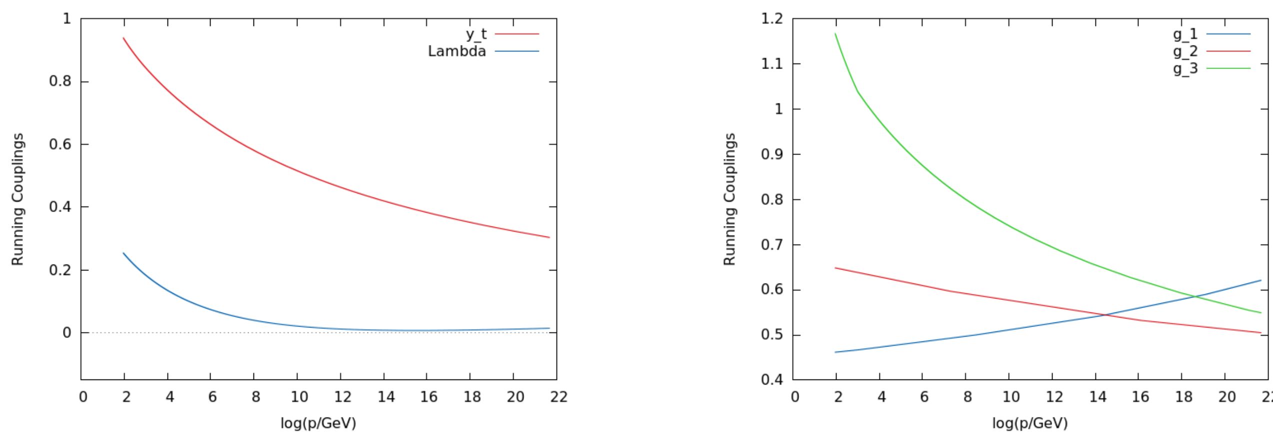

Figure 14: Modified SM, added three generations of the pair of fermions at mass $10^3 GeV$

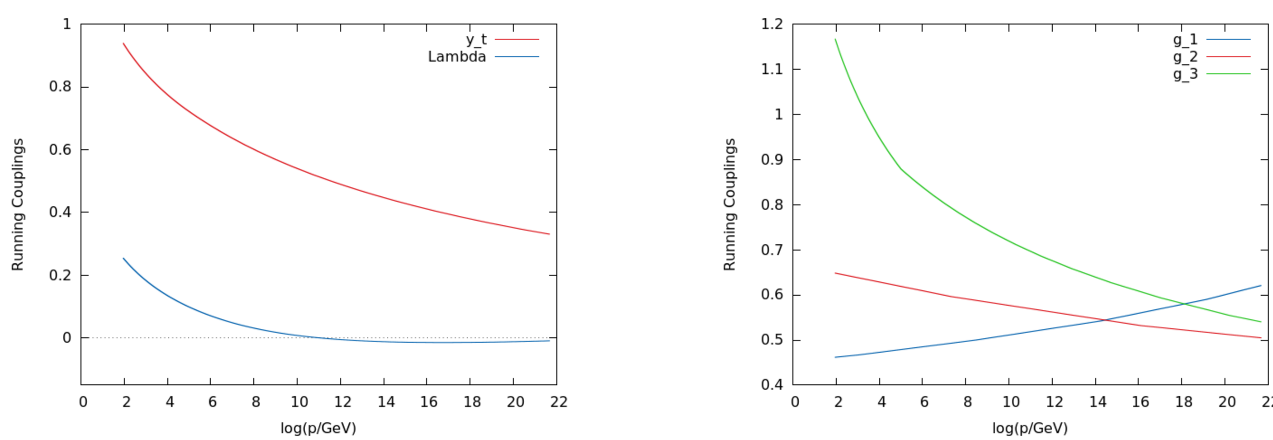

Figure 15: Modified SM, added three generations of the pair of fermions at mass $10^5 GeV$

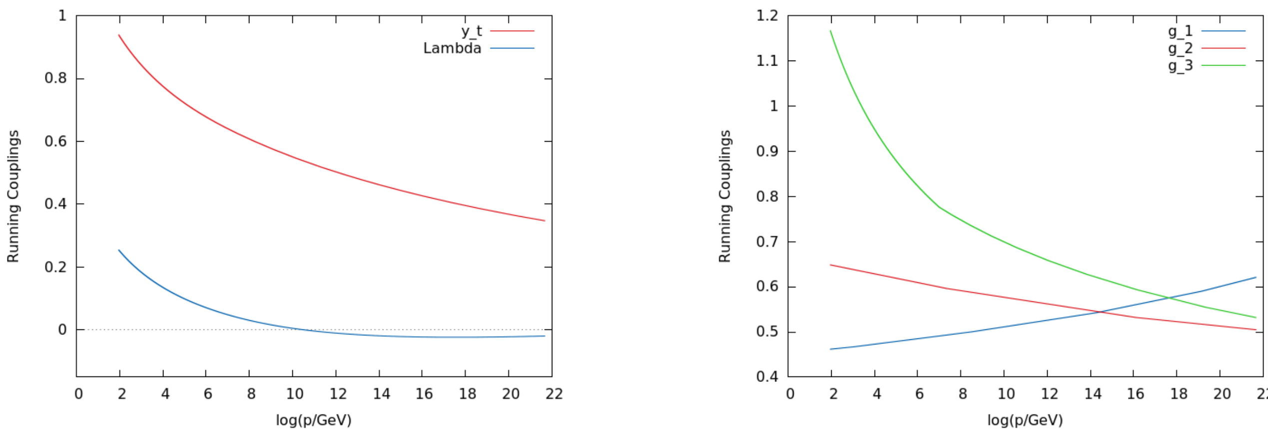

Figure 16: Modified SM, added three generations of the pair of fermions at mass $10^7 GeV$

Declaration

I hereby declare, that I have written this thesis by myself and have not used any other than the mentioned sources and auxiliary means.

Heidelberg, 04.07.2014,

Max Lautsch