UNIVERSITY OF CALIFORNIA
Los Angeles

Dequantization of the Dirac Equation: The Semiclassical Dirac Mechanics

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Physics

by

Alan R. Katz

1987

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to Christian Fronsdal for his help, advice, support and patience during these many years. I would also like to thank Moshe Flato for his help and for many illuminating discussions. Finally, I would like to thank Jon Postel and the U.S.C. Information Sciences Institute for their support.

VITA

December 19, 1955	Born, Middletown, Connecticut
1977	B.S., Physics University of Southern California Los Angeles, California
1977 - 1978	Tour Guide / Lecturer California Museum of Science and Industry Los Angeles, California
1978	M.S., Physics University of California, Los Angeles Los Angeles, California
1979 - 1983	Teaching Associate and Fellow Department of Physics University of California, Los Angeles Los Angeles, California
1983	Research Assistant Department of Physics University of California, Los Angeles Los Angeles, California
1983 - 1986	Instructor U.C.L.A. Extension, The Sciences Los Angeles, California
1984 - 1986	Instructor Department of Physics University of California, Los Angeles Los Angeles, California

1984	Award for outstanding service as an instructor Department of Physics University of California, Los Angeles Los Angeles, California
1978 - 1987	Research Assistant and Computer Scientist U.S.C. Information Sciences Institute Marina Del Rey, California

PUBLICATIONS

1. Format for Bitmap Files, RFC 797, USC Information Sciences Institute, (1981)

2. Decoding Facsimile Data from the Rapicom 450, RFC 798, USC Information Sciences Institute,(1981)

3. Proposal for the Body Item for a Multimedia Message (with J. Postel and G. Finn), Technical Report, USC Information Sciences Institute (1982)

4. An Experimental Internetwork Multimedia Mail System, Proceedings, IFIP 6.5 Working Conference on Computer Message Services, Nottingham, England (1984)

5. The DARPA Experimental Multimedia Mail System (with J. Postel, J. Reynolds, G. Finn, and A. DeSchon) IEEE Computer **18**, #10 (1985)

6. An Experimental Multimedia Mail System (with J. Postel, J. Reynolds, and G. Finn), Technical Report, USC Information Sciences Institute (1986)

7. The WKB Limit of the Dirac Equation and the g–2 Experiment, UCLA preprint UCLA/86/TEP/28, (1986)

8. Superspace Mechanics as the Classical Limit of Superfield Theory, Letters in Mathematical Physics **12**, 329 (1986)

ABSTRACT OF THE DISSERTATION

Dequantization of the Dirac Equation: The Semiclassical Dirac Mechanics

by

Alan R. Katz

Doctor of Philosophy in Physics

University of California, Los Angeles, 1987

Professor Christian Fronsdal, Chair

A relativistic Hamiltonian mechanics for a Dirac particle is derived as the semi-classical limit of the Dirac equation. The theory bears much resemblance to ordinary classical mechanics, except that some of the phase space variables are four by four matrices. This is necessary because of the spin degrees of freedom of the particle. Constraints in the theory connect the four by four matrices with observables.

In finding the semi-classical limit of the Dirac equation, we first find it useful to apply a WKB type of approximation to a scalar Superfield theory. By eliminating second class constraints, we obtain the Brink and Schwarz formulation of Casalbuoni's superspace Pseudomechanics. The spin 1/2 sector is then examined to find the corresponding WKB limit of the Dirac equation, a semi-classical mechanics.

We next reformulate this Dirac mechanics in terms of *-products utilizing phase space methods, guided in interpretation by what was obtained via the

WKB method. With the formalism in hand, we consider a Dirac particle in a homogeneous electromagnetic field. We are able to demonstrate that the g factor is equal to two, obtain the usual equation of motion for the position and momentum, and are able to DERIVE directly from QED the relativistic spin precession equation of Bargmann, Michel, and Telegdi; an equation originally given as just a simple relativization of the expression for non–relativistic precession. This establishes a vital link between QED theory and what is actually observed in the g – 2 experiments.

With this method, we can find a spin precession equation for inhomogeneous fields which contains quantum corrections to the usual equation and we present the general method. Some of these corrections may be easily deduced from gauge invariance, but only if everything is expressed in terms of *–products, not ordinary products. The equation of motion for any observable is always given as an explicit series in Planck's constant, thus allowing a classical limit to be easily taken. At all times we maintain a fully relativistic theory.

I. The Dirac Equation, g-2 Experiments, and the Semiclassical Limit

A. Introduction

This work deals with the *dequantization* of the Dirac equation. We would like to develop a (semi)classical, but fully relativistic mechanics of spin 1/2 particles as an $\hbar \to 0$ limit of Quantum Electrodynamics (QED). Observables and their equations of motion should be given in such a theory as an explicit (and possibly infinite) series in $\hbar$. Thus, quantum corrections to the (semi)classical mechanics would be clearly shown.

Much of today's physics deals with the subject of *quantization*, of somehow obtaining a quantum theory from a classical one. However, as Dirac has said,

> ...when we've got a given classical theory, in general there is not a unique quantum theory corresponding to it. There is no well-defined unique process for passing from classical theory to quantum theory. That means that when we set up a quantum theory we have to set it up to stand on its own feet, independent of the classical theory. *The only value of the classical theory is to provide us with hints for getting a quantum theory; the quantum theory is then something that has to stand*

> *in its own right.* If we were sufficiently clever to be able to think of a good quantum theory straight away, we could manage without classical theory at all. † (Emphasis mine)

In other words, what is called quantization is the use of a classical theory *as a guide* in guessing a quantum theory, which must then stand on its own. Presumably, the classical theory may then be looked at as the classical limit of the quantum one.

Here, we take the following point of view: the quantum theory is given and is *correct* and the classical theory is an approximation of the quantum one, which may be *derived* as the $\hbar \rightarrow 0$ limit. Starting from a relativistic quantum theory, we should be able to *derive* a fully relativistic classical theory. We should also be able to display explicitly the quantum corrections to this classical theory. This is what we mean by dequantization.

Semiclassical approximation methods abound in the literature and include such topics as geometric optics, the WKB method, *-products and phase space methods, limits of large N (number of particles), and coherent state methods. Many of these are useful in describing large numbers of bosons, such as photons in the field of quantum optics. The WKB method, as well as phase space methods and coherent states have been used to illustrate the classical limit of ordinary (and of

† Dirac, P.A.M., **Lectures on Quantum Field Theory**, Belfer Graduate School, Yeshiva University (1965), p 42-43

relativistic) quantum mechanics by dequantizing the Schrödinger (or Klein-Gordon) equation.

There are many difficulties when one tries to apply these methods to particles with spin, such as the Dirac particle. It is not clear how the spin degrees of freedom should be represented classically. Indeed, it has often been said that there is no classical analogue of spin. What do we mean by a classical theory of a Dirac particle? How will it differ from a classical mechanics of a particle without spin? What types of calculations would one be able to make in such a theory, and how would it be related to what is observed?

First of all, we should make clear that what is meant by such a (semi)classical theory is to *take the limit* $\hbar \to 0$ *but still keep the basic physical concepts, such as a full description of the spin.* That is why say such a limit may strictly be only semi-classical. We expect that we should obtain a mechanics which gives the usual equations of motion in the cases where spin should not make a difference, but that also allows extra degrees of freedom which adequately represents the spin. We should be able to derive the equation of motion of the spin (i.e. spin precession), in the same way we obtain the ordinary equations of particle motion.

Observation of spin precession and measurement of the g factor of the electron and other spin 1/2 particles has been of extreme importance in the development and verification of QED, for example. The $g - 2$ anomaly predicted by QED theory has been observed with a high degree of accuracy and agrees with theory (see, for example **[Rich72]**). There are two classes of experiments, those which measure the precession of the particle, and those where a resonance is observed. In both of these, however, there is not a direct connection between what is calculated and what is actually observed!

The analysis of the resonance experiments are done non-relativistically, using the exact eigenvalues of the Pauli Hamiltonian. This is valid, but does not really test the relativistic content of QED and Quantum Field Theory. The analysis of the precession experiments, on the other hand, requires a connection between the observed precessional frequency and the g factor which comes from the equation of Bargmann, Michel, and Telegdi **[Bargmann59]** or its equivalent. However, this equation *does not come* from QED; it is simply a relativization of the non-relativistic spin precession equation. In general, there may be many such precession equations with the same non-relativistic limit.

It is therefore possible that there is a hole in the verification of QED. The theory predicts a number for $g - 2$, and that number is observed, but the connection between the two does not come from the theory. We would be much more confident if the spin precession expression actually came from QED.

In this work, we are able to *derive* the Bargmann, Michel, Telegdi equation as a consequence of QED. For a spin 1/2 particle in a homogeneous electromagnetic field, the mechanics we derive as the (semi)classical limit of the Dirac equation predicts that equation as the equation of motion for the spin variables. **[Bargmann59]** (and some others) also gives an equation for the general case of inhomogeneous electromagnetic fields. We obtain quantum corrections to this equation proportional to the inhomogeneity, which could be made quite large. In the experiments, it is generally assumed that the field is essentially homogeneous. It would be interesting to design an experiment that makes use of the inhomogeneous corrections. (Homogeneous means here that $F_{\mu\nu,\rho} = 0$).

The formalism derived here as many advantages. First of all, it is fully relativistic. No non-relativistic approximations are used. For example, the usual way to demonstrate that the Dirac equation describes a particle with $g = 2$ is to break up the four component wave function into large and small components, make a non-relativistic approximation, and show one obtains a Hamiltonian with $g = 2$. Here, we make no such approximations, and can show that $g = 2$ by calculating the equation of motion of the spin. We obtain a spin precession equation, with the proper g value. If we consider a modified effective Dirac equation, with a term containing radiative and other corrections calculated from QED, we obtain in a similar way the $g - 2$ terms.

Our method lends itself equally well to a particle in an inhomogeneous electromagnetic field where we find there are quantum corrections to the spin precession equation. In the future, we should also be able to apply the method to a spin 1/2 particle in a gravitational field and calculate a spin precession equation, with quantum corrections. In principle, one could look at other effects, perhaps combining a gravitational and electromagnetic field.

We thus have *derived* a *classical*, fully relativistic theory of spin 1/2 particles from the theory of QED. In doing so, we obtain a mechanics which makes close contact with classical theory (and with our intuition), but fully characterizes the spin 1/2 nature of the particles.

B. Organization

This work is organized as follows. In the next chapter, we make our first attempt at dequantizing the Dirac equation via a WKB type of approximation. We will outline how this method works for a scalar particle, and the difficulties involved in trying to apply it to a Dirac particle. In order to overcome these difficulties, we find it useful to apply the method to a scalar *Superfield* theory, which includes the Dirac field as well as an ordinary scalar. We find we can follow in close analogy to the ordinary scalar particle, obtaining the Brink and Schwarz **[Brink81]** formulation of Casalbuoni's superspace mechanics **[Casalbuoni76]** as the semi-classical limit. Looking at the spin 1/2 sector of the theory, we then discover how to apply the WKB method to the Dirac equation, obtaining a classical limit for it. It is interesting to note here, that we therefore find the concept of supersymmetry of immense value *whether or not there really exists any such symmetry in nature.* We can use it as a tool, and at the end ~~through~~ throw away the supersymmetry, keeping only the spin 1/2 content.

The superspace method provides the keys to finding and interpreting the semiclassical Dirac mechanics, but it becomes clearer and more useful to reformulate it in terms of *-products, using phase space methods. In Chapter III, we review these methods and also explore how gauge transformations act under the *-product. In Chapter IV, we formulate the the Dirac mechanics in that language for a particle in a homogeneous electromagnetic field.

The Dirac mechanics so derived is a Hamiltonian formulation of the relativistic mechanics for a point particle, but with extra four by four matrix phase space variables (which are actually the Dirac gamma matrices) which represent the spin degrees of freedom. Constraints, in the sense of Dirac **[Dirac64]** relate the matrix variables with the usual 4-velocity and momentum. All observables correspond to matrices or numbers sandwiched between 4-spinors. By using a Heisenberg-like formulation, we can put all of the (proper) time dependence in the matrices and regard the 4-spinors as constant.

Armed with this formalism, we are able to find the usual equations of motion for the particle, find that $g = 2$, and are able derive the spin precession equation of Bargmann, Michel and Telegdi **[Bargmann59]** for homogeneous fields (i.e. $F_{\mu\nu,\rho} = 0$). In Chapter V, we then formulate the mechanics in general, for inhomogeneous fields, and calculate quantum corrections proportional to the inhomogeneity. These corrections could in principle be made quite large. We can easily see what some of these corrections must be from gauge invariance. Finally, in Chapter VI, we list conclusions and areas for future work.

Appendices A-F contain supplementary information and calculations which it was not appropriate to include in the body of the work.

C. Notation

We generally follow the notation of **[Bjorken64]** and **[Itzykson80]**, except in Chapter II, where we generally follow the notation of **[Wess83]**. This is because the work of Chapter II is most clearly explained in Weyl 2-spinor language. After that chapter, when we are through with superspace, it is clearer to go to a 4-spinor language.

Our metric is $(+---)$ except in Chapter II where it is $(-+++)$. Operators are indicated by hats, as in $\hat{A}$ or $\hat{x}$. The Einstein summation convention is used throughout, with repeated indices summed over. Generally Greek indices (μ , ν , λ , $\cdots$) take the value $0, 1, 2, 3$ while

Latin indices (i , j , k) take the value 1 , 2 , 3. Further conventions and notation that are used only in Chapter II are given in Appendix A.

Commutator brackets are written as,

$$\left[\hat{A} , \hat{B} \right]_{-} \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

The anticommutator is written as,

$$\left[\hat{A} , \hat{B} \right]_{+} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$$

Moyal brackets, which will be defined in Chapter III, and used thereafter, are written as,

$$\left[A , B \right]_{\mathrm{M}}$$

and the classical Poisson brackets, also defined in Chapter III are,

$$\left[A , B \right]_{\mathrm{pb}} \equiv \lim_{\hbar \to 0} \left[A , B \right]_{\mathrm{M}}$$

The Dirac Gamma matrices are γ_μ, with,

$$\left[\gamma_\mu , \gamma_\nu \right]_{+} = 2g_{\mu\nu} \quad , \quad \frac{i}{2} \left[\gamma_\mu , \gamma_\nu \right]_{-} \equiv \Sigma_{\mu\nu}$$

General spinor state vectors will be written as $<|$ and $|>$. The electromagnetic field tensor is $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$, with A_μ the 4-vector potential and commas denoting differentiation with respect to x_ν.

II. The Superfield WKB Method and Superspace Mechanics

A. Introduction

We now begin the somewhat tortuous route in search of a semi-classical limit of the Dirac equation. We would like to obtain a relativistic mechanics from a one-particle Dirac field theory. The usual way to do this type of derivation is via the WKB approximation, which is an approximation in terms of some characteristic length parameter. This is not (necessarily) an explicit dequantization, as $\hbar$ does not yet appear. However, if we apply WKB to a quantum wave equation (such as a Schrödinger or Klein-Gordon equation), and identify the characteristic length as $\hbar$, it is. By letting $\hbar \to 0$, one may obtain a classical limit. It should be emphasized that this requires the identification of the characteristic length with $\hbar$.

The WKB method has been used extensively for bosons, such as photons; for example in obtaining the geometrical optics limit. (see **[Born75]**). However, many difficulties arise when the same program is applyed to fermions (such as electrons). In particular, it is not clear how spin is represented in the final theory.

Because of difficulties in applying WKB to the Dirac equation directly, it was thought that perhaps it would be easier to apply it to a scalar superfield. Since the superfield is a scalar, the WKB method should be very similar to that of the ordinary scalar field. Since the scalar superfield equation contains the Dirac equation, it should then become clearer what the WKB method for the Dirac equation should be. This is what we will attempt in this chapter.

The question them becomes: what should the WKB limit of the superfield theory look like? In the usual scalar case, the classical limit is interpreted as a Hamilton-Jacobi equation, with derivatives of the Hamilton-Jacobi function corresponding to momenta. In the superfield case, there are derivatives with respect to anticommuting variables in the field equation. Thus, one may expect to end up with anticommuting momenta, which may possibly have something to do with the spin.

Mechanics with anticommuting momentum variables have been studied by many others, although not from the point of view of a WKB limit of a field theory. Casalbuoni **[Casalbuoni76]** developed a classical mechanics of Bose-Fermi systems by extending the phase space with the addition of anticommuting Grassman degrees of freedom. The phase space becomes a superspace and the anticommuting variables are shown to be connected with the spin. The Lagrangian used is invariant under the usual supersymmetry transformations. It is claimed that this mechanics may be regarded as the classical limit of a general quantum field theory with Bose and Fermi operators.

One problem with Casalbuoni's approach (from our point of view) is the presence of second class constraints, in the language of Dirac **[Dirac64]**. These constraints require that the Poisson Brackets must be modified in the Hamiltonian formulation of the theory and as will be shown, this obscures the relationship between this theory and the field theory. Also, with the new brackets, it turns out that:

$$\left[x_\mu , x_\nu \right]_{pb} \neq 0$$

In order to eliminate the constraints, Brink and Schwarz **[Brink81]** modify the Lagrangian by the addition of a term proportional to a characteristic length parameter. When this parameter goes to zero, the theory reduces to Casalbuoni's. Keeping the parameter non-zero, the second class constraints are eliminated and the usual Poisson Brackets and equations of motion may be used with the Hamiltonian. Also, the usual relationship,

$$\left[x_\mu , x_\nu \right]_{pb} = 0$$

is recovered. The reason we will want a Hamiltonian formulation is that we expect the WKB limit of field theory to correspond to a Hamilton-Jacobi equation, from which we can immediately see the corresponding Hamiltonian. Having second class constraints going along with the Hamiltonian obscures the physics.

Ironically, Brink and Schwarz try to avoid introducing a characteristic length in eliminating the second class constraints, and quote another theory with non-commuting x's developed to introduce a fundamental length in order to remove divergences in QED **[Snyder47]**. As will be shown in this chapter, this characteristic length can be interpreted as the same one used in the WKB reduction of a superfield theory and is in fact, $\hbar$ (actually, $\hbar/mc$).

The rest of this chapter is organized as follows: In section B, we apply the WKB method to an ordinary scalar field theory in order to illustrate features we may expect to appear in the superfield case. Next, in section C, we summarize the mechanics of Casalbuoni and Brink and Schwarz. We then show that their mechanics can be derived from superfield theory using a WKB approximation in section D. Finally, in section E, we use the results to apply WKB directly to the Dirac equation, without using superspace. The resulting limit will provide us with a formulation of the semi-classical limit of the Dirac equation, and will provide us with the essential key in interpretation when we rederive this mechanics in a different and clearer way in subsequent chapters. Much of this work appeared in **[Katz86A]**.

The notation used is generally that of reference **[Wess83]** and is summarized in Appendix A. The mathematics of anticommuting Grassman numbers may be found in **[Berezin66]**.

B. WKB for an Ordinary Scalar Field

To illustrate the WKB method of obtaining the classical limit of a field theory, we shall consider an ordinary free (complex) scalar field, φ. The Klein-Gordon field equation for a particle of mass m is:

$$\hbar^2 (\Box + m^2)\varphi = 0 \tag{2.1}$$

Let,

$$\varphi(x) = R(x) e^{\frac{i}{\hbar}S(x)} \tag{2.2}$$

where R and S are real scalar fields. This completely determines R and S for any complex field, φ. Inserting this into the field equation, one obtains for R and S :

$$\left[\hbar^2\Box R + i\hbar(2\partial^\mu R\partial_\mu S + R\Box S) - R(\partial^\mu S\partial_\mu S - m^2)\right] e^{\frac{i}{\hbar}S(x)} = 0 \tag{2.3}$$

The real and imaginary parts of the left hand side must separately vanish. The imaginary term has a factor of $\hbar$ and is just the equation of current conservation:

$$J_\mu{}^{,\mu} = 0 \quad , \quad with \quad J_\mu = i\hbar\left[\varphi^*\partial_\mu\varphi - (\partial_\mu\varphi^*)\varphi\right] \tag{2.4}$$

expressed in terms of R and S . In order to find the classical limit of the theory, we let $\hbar^2 \to 0$. The remaining real term, independent of $\hbar$ is:

$$R(\partial^\mu S\partial_\mu S - m^2) e^{\frac{i}{\hbar}S(x)} = 0 \tag{2.5}$$

or,

$$R(\partial^{\mu}S\partial_{\mu}S - m^2) = 0 \tag{2.6}$$

This last equation may be interpreted as the relativistic Hamilton-Jacobi equation for a free particle,

$$H(x_{\mu},\partial_{\mu}S) = \frac{\partial H}{\partial \tau} \equiv 0 \tag{2.7}$$

Here, S is the Hamilton-Jacobi function and H is the free Hamiltonian which is a function of x_{μ} and p_{μ} (the momentum) and is assumed to be independent of τ , the proper time. (It should be noted that if this same method is applied to the Schrödinger equation, one obtains the usual non-relativistic Hamilton-Jacobi equation). Making the connection between the derivative $\partial_{\mu}S$ and the momentum, p_{μ} , we find that the corresponding Hamiltonian is simply,

$$H(x,p) = R(p^{\mu}p_{\mu} - m^2) \tag{2.8}$$

or dividing by $2m$, we obtain the standard form,

$$H(x,p) = R\left[\frac{p^{\mu}p_{\mu}}{2m} - \frac{m}{2}\right] \tag{2.9}$$

Thus, the mechanics which is the classical limit of the original field theory is described by this Hamiltonian, using the usual equations of motion (dots indicate derivative with respect to τ, the proper time),

$$\dot{x}_{\mu} = \left[x_{\mu}, H\right]_{\text{pb}} , \quad \dot{p}_{\mu} = \left[p_{\mu}, H\right]_{\text{pb}} \tag{2.10}$$

where

$$\left[A , B\right]_{\text{pb}} \equiv \left[\frac{\partial A}{\partial x^{\mu}}\frac{\partial B}{\partial p_{\mu}} - \frac{\partial B}{\partial x^{\mu}}\frac{\partial A}{\partial p_{\mu}}\right] \tag{2.11}$$

are the usual Poisson Brackets. Varying R gives the first class constraint, $p^2 - m^2 = 0$.

It should be remembered that here, H is not the energy of the system (which is in fact the 0th component of the momentum), but is only something which plays the role of a a Hamiltonian in a Hamiltonian formulation of relativistic mechanics. I.e., its relation to the mechanical Lagrangian (assuming no second class constraints in the sense of Dirac **[Dirac64]**) is:

$$H(x,p) = \dot{x}^\mu p_\mu - L(x,\dot{x}) \tag{2.12}$$

In summary, the WKB approximation consists of first, writing down the field equation. Next decompose the field into real and imaginary parts by making approximation (2.2). Substituting into the field equation (2.1), and letting the highest power of $\hbar \to 0$, we find that the imaginary part of what is left is a statement of current conservation and that the real part may be interpreted as a Hamilton-Jacobi equation. Making the correspondence of $\partial_\mu S$ and p_μ, we then interpret the left hand side as the Hamiltonian. In the language of Dirac, we may regard this Hamiltonian as weakly equal to zero; i.e. we only care about the functional form of $H(x,p)$, not what its actual value is. (To remind us of this, we will follow the notation of Dirac in **[Dirac64]** and write $H(x,p) \approx 0$). We may then, if we wish, find the Lagrangian in the usual way.

C. Mechanics in Superspace

In this section, we briefly review the pseudomechanics of Casalbuoni **[Casalbuoni76]** and others which we will in section D of this chapter derive from a superfield theory. The mechanical Lagrangian is explicitly constructed to be invariant under ordinary supersymmetry transformations (with the speed of light 1, and the coordinates of superspace $z = (x,\theta)$):

$$\theta_\alpha \to \theta_\alpha + \varepsilon_\alpha \ , \qquad x_\mu \to x_\mu + \frac{i}{2m}\left[\theta\sigma_\mu\bar{\varepsilon} - \varepsilon\sigma_\mu\bar{\theta}\right] \tag{2.13}$$

The Lagrangian is:

$$L = m\left[\left(\dot{x}^\mu - \frac{1}{m}V^\mu\right)\left(\dot{x}_\mu - \frac{1}{m}V_\mu\right)\right]^{1/2} \tag{2.14}$$

where,

$$V_\mu \equiv i\left[\dot{\theta}\sigma_\mu\bar{\theta} - \theta\sigma_\mu\dot{\bar{\theta}}\right] \tag{2.15}$$

and dots indicate derivatives with respect to τ, the proper time.

The equations of motion are:

$$\dot{P}_\mu = 0 \to \ddot{x}_\mu = 0 \ , \quad \dot{\theta}_\alpha = 0 \ , \quad \dot{\bar{\theta}}_{\dot{\alpha}} = 0 \tag{2.16}$$

where the momentum P_μ will be defined below. The (four) spin is identified (in **[Casalbuoni76]**) as:

$$\Sigma_\mu = \theta\sigma_\mu\bar{\theta} \tag{2.17}$$

(for this chapter only, we label the spin as Σ_μ . In later chapters we will call it S_μ , but here we will be using S for something else). The equations of motion therefore imply that,

$$\dot{\Sigma}_\mu = 0 \tag{2.18}$$

i.e. the spin vector is constant. Casalbuoni shows that this theory is a sensible relativization of a mechanics with only three anticommuting momenta, which describes nonrelativistic spin. Presumably, if there were interactions in the theory, the equations for θ_α would give the correct spin precession equation for Σ_μ (Casalbuoni develops this for the nonrelativistic case only). One would expect that for an external electromagnetic interaction, one would obtain the relativistic spin precession equation of Bargmann, Michel, and Telegdi **[Bargmann59]** in the place of (2.18). We will do this for the spin 1/2 sector only when we reformulate the classical limit of the Dirac equation in Chapter IV.

When expressing this theory in Hamiltonian form, one finds that the phase space is reduced because of the presence of second class constraints. The generalized momenta are:

$$P_\mu = \frac{m^2}{L}\left[\dot{x}_\mu - \frac{1}{m}V_\mu\right]$$

$$\pi_\alpha = -i\ \sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P^\mu \ , \quad \bar{\pi}_{\dot{\alpha}} = -i\ \theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}P^\mu \tag{2.19}$$

where P_μ is conjugate to x_μ and π_α is conjugate to θ_α. The last two momenta are second class constraint equations,

$$D_\alpha \equiv \pi_\alpha + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}P^\mu = 0 \ , \quad \bar{D}_\alpha \equiv \bar{\pi}_\alpha + i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}P^\mu = 0 \tag{2.20}$$

in the sense of **[Dirac64]**. The Hamiltonian becomes,

$$H = \dot{x}^\mu P_\mu + \dot{\theta}^\alpha \pi_\alpha + \dot{\bar{\theta}}^{\dot{\alpha}} \bar{\pi}_{\dot{\alpha}} - L \approx 0 \tag{2.21}$$

H is weakly zero because of constraints. The first class constraint,

$$P^2 - m^2 = 0 \tag{2.22}$$

is handled using Dirac's method with a Lagrange multiplier, λ (which remains undetermined), so that the Hamiltonian is:

$$\lambda \left[P^2 - m^2 \right] \tag{2.23}$$

The θ degrees of freedom do not appear in H, but the Poisson Brackets have to be modified because of the second class constraints, D, so that,

$$\dot{x}_\mu = \left[x_\mu , H \right]^*_{\mathrm{pb}} , \quad \dot{P}_\mu = \left[P_\mu , H \right]^*_{\mathrm{pb}} \tag{2.24}$$

gives the same equations of motion as (2.16). The modified brackets found by applying the Dirac program are:

$$\left[A , B \right]^*_{\mathrm{pb}} = \left[A , B \right]_{\mathrm{pb}}$$

$$- \left[A , D^\alpha \right]_{\mathrm{pb}} \left[D_\alpha , B \right]_{\mathrm{pb}} - \left[A , \bar{D}_{\dot{\alpha}} \right]_{\mathrm{pb}} \left[\bar{D}^{\dot{\alpha}} , B \right]_{\mathrm{pb}} \tag{2.25}$$

(This definition guarantees that the constraints will remain satisfied at all times). Brink and Schwarz **[Brink81]**, eliminate the second class constraints by modifying the Lagrangian to be:

$$L = \frac{m}{2B} \left[\left[\dot{x}^\mu - \frac{1}{m} V^\mu \right] \left[\dot{x}_\mu - \frac{1}{m} V_\mu \right] \right]$$

$$+ \frac{l}{2B} \left[\dot{\theta}\dot{\theta} + \dot{\bar{\theta}}\dot{\bar{\theta}} \right] + \frac{Bm}{2} \tag{2.26}$$

Here, l is a length parameter which will later be put equal to zero, V_μ

is as before, and B is an external field to be varied separately. The equation of motion for B is:

$$\left[P^\mu P_\mu - m^2\right] + \frac{l}{B^2}\left[\dot{\theta}\dot{\theta} + \dot{\bar{\theta}}\dot{\bar{\theta}}\right] = 0 \tag{2.27}$$

where,

$$P_\mu = \frac{m}{B}\left[\dot{x}_\mu - \frac{1}{m}V_\mu\right] \tag{2.28}$$

Equation (2.27) may be regarded as a mass-shell condition, when l is put equal to zero. If we put $l = 0$ and solve the B equation of motion for B, we find that,

$$B = \frac{1}{m}\left[\left[\dot{x}^\mu - \frac{1}{m}V^\mu\right]\left[\dot{x}_\mu - \frac{1}{m}V_\mu\right]\right]^{1/2} \tag{2.29}$$

If we substitute this back into the Lagrangian (2.26), (with $l = 0$), we get precisely Casalbuoni's Lagrangian (2.14). By leaving l non-zero we avoid all second class constraints.

The equations of motion are now:

$$\dot{P}_\mu = 0 \;,\quad 2i\dot{\theta}^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} P_\mu = \frac{l}{B}\ddot{\bar{\theta}}_{\dot{\alpha}} \;,\quad 2i\sigma^\mu{}_{\alpha\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}} P_\mu = \frac{l}{B}\ddot{\theta}_\alpha \tag{2.30}$$

Also, these equations imply,

$$\ddot{x}_\mu = 0 \;,\quad \ddot{\theta}_\alpha = 0 \;,\quad \ddot{\bar{\theta}}_{\dot{\alpha}} = 0 \;,\quad and, \quad \dot{V}_\mu = 0 \tag{2.31}$$

If l is put to zero, these equations of motion become identical with the earlier equations (2.16). However, there are now no second class constraints. The momenta are now equation (2.28) and:

$$\pi_\alpha = -i\,\sigma^\mu{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}P^\mu + \frac{l}{B}\dot\theta_\alpha\ , \quad \bar\pi_{\dot\alpha} = -i\,\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}P^\mu - \frac{l}{B}\dot{\bar\theta}_{\dot\alpha} \qquad (2.32)$$

and become the same as in equations (2.19), if l is equal to zero. What were the constraints, equation (2.20), are now:

$$D_\alpha \equiv \pi_\alpha + i\sigma^\mu{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}P^\mu = \frac{l}{B}\,\dot\theta_\alpha$$

$$\bar D_\alpha \equiv -\bar\pi_\alpha - i\theta^\alpha\sigma^\mu{}_{\alpha\dot\alpha}P^\mu = \frac{l}{B}\,\dot{\bar\theta}_{\dot\alpha} \qquad (2.33)$$

and the equations of motion imply that,

$$\dot D_\alpha = 0\ , \quad \dot{\bar D}_{\dot\alpha} = 0 \qquad (2.34)$$

If we now calculate a Hamiltonian for the Brink and Schwarz Lagrangian, equation (2.26) (which is not done in their paper), we obtain,

$$H = \frac{B}{2m}\left[P^2 - m^2\right] + \frac{B}{2l}\left[DD + \bar D\bar D\right] \qquad (2.35)$$

If we substitute for B from equation (2.29) into the Lagrangian (2.26), but do not set $l = 0$, the Lagrangian is,

$$L = \frac{m}{2}\left[\left(\dot x^\mu - \frac{1}{m}V^\mu\right)\left(\dot x_\mu - \frac{1}{m}V_\mu\right)\right]^{1/2} + \frac{l}{2}\left[\dot\theta\dot\theta + \dot{\bar\theta}\dot{\bar\theta}\right] + \frac{m}{2} \qquad (2.36)$$

and the Hamiltonian becomes,

$$H = \frac{1}{2l}\left[DD + \bar D\bar D\right] + \frac{m}{2} \qquad (2.37)$$

Since there is no P^2 term in H, the relationship between $\dot x_\mu$ and P_μ will be different than before,

$$\dot{x}_\mu - \frac{1}{m} V_\mu = 0 \tag{2.38}$$

but the equations of motion (2.31) for x_μ and θ_α remain the same.

This theory reduces to Casalbuoni's as l goes to zero, but there are no second class constraints, so the ordinary Poisson Brackets may be used. Also, θ and π explicitly appear in the Hamiltonian. We will show that this Hamiltonian is exactly what is obtained from a WKB approximation applied to a scalar superfield theory, if l is replaced by $\hbar$ (to lowest order in $\hbar$).

The second class constraints provided a barrier to understanding the meaning of superspace mechanics. If we start from Lagrangian (2.14), and want a Hamiltonian formulation of the theory, we find we must modify the Poisson Brackets and are left with a very strange result that,

$$\left[x_\mu\, ,\, x_\nu\right]_{\text{pb}} \neq 0 \tag{2.39}$$

On the other hand, when we apply WKB to the superfield, the answer we will get does not look much like the pseudomechanics we expect. By introducing a characteristic length parameter, which we will put equal to zero at the very end of the calculations, we will see that these are in fact the same theory. We can formulate this theory without second class constraints and find from the superfield WKB method that we must identify the characteristic length, l , with $\hbar$.

D. WKB for a Superfield

We start with the standard superfield equations which contains the Dirac equation (with the correct factor of $\hbar$ included):

$$\frac{\hbar}{4}\hat{\mathbf{D}}\hat{\mathbf{D}}\Phi_1 = m\Phi_2^+ \tag{2.40a}$$

$$\frac{\hbar}{4}\overline{\hat{\mathbf{D}}}\overline{\hat{\mathbf{D}}}\Phi_2^+ = m\Phi_1 \tag{2.40b}$$

The operators $\hat{\mathbf{D}}_\alpha$ (which should not be confused with the constraints, D_α of equations (2.20) and (2.33)) are defined to be:

$$\hat{\mathbf{D}}_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \ , \quad \overline{\hat{\mathbf{D}}}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \tag{2.41}$$

and $\Phi_{1,2}$ ($\Phi^+{}_{1,2}$) are chiral (anti-chiral) superfields satisfying the constraints:

$$\overline{\hat{\mathbf{D}}}\Phi_{1,2} = 0 \ , \quad \hat{\mathbf{D}}\Phi_{1,2}{}^+ = 0 \tag{2.42}$$

In terms of component fields,

$$\Phi_{1,2} = A_{1,2} + \theta^\alpha\psi_{\alpha\,1,2} + \theta\theta F_{1,2} + i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu A_{1,2}$$

$$+ \theta\theta\partial_\mu\psi^\alpha{}_{1,2}\sigma^\mu{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + \theta\theta\bar{\theta}\bar{\theta}\Box A_{1,2} \tag{2.43}$$

$A_{1,2}$ are two ordinary scalar fields, $\psi_{1,2}$ are Weyl 2-spinors, together they make up one Dirac 4-spinor, and $F_{1,2}$ are auxiliary fields.

It should be noted here that the constraints (2.42) actually follow as a consequence of the field equations (2.40). Rather than work directly with (2.40), it is more advantageous for the WKB program to define a new superfield:

$$\Phi = \Phi_1 + \Phi_2{}^+ \quad , \quad \Phi^* = \Phi_1{}^+ + \Phi_2 \tag{2.44}$$

and to work with the single field equation which Φ satisfies:

$$\frac{\hbar}{4}\left[\hat{\mathbf{D}}^2 + \overline{\hat{\mathbf{D}}}^2 \right] \Phi = m\Phi \tag{2.45}$$

This single equation is of a form very similar to that of the Klein-Gordon equation. In fact, in analogy with the Klein-Gordon equation, one can define a (super) current to be (in two component language),

$$J_\alpha = \frac{1}{4}\left[\Phi^* \hat{\mathbf{D}}_\alpha \Phi - \left[\hat{\mathbf{D}}_\alpha \Phi^* \right] \Phi \right] \tag{2.46a}$$

$$J_{\dot{\alpha}} = \frac{1}{4}\left[\Phi^* \overline{\hat{\mathbf{D}}}_{\dot{\alpha}} \Phi - \left[\overline{\hat{\mathbf{D}}}_{\dot{\alpha}} \Phi^* \right] \Phi \right] \tag{2.46b}$$

such that the following current conservation equation is satisfied:

$$\hat{\mathbf{D}}^\alpha J_\alpha + \overline{\hat{\mathbf{D}}}_{\dot{\alpha}} J^{\dot{\alpha}} = 0 \tag{2.47}$$

(Note, this is a different type of supercurrent than usual, for example that described in **[Ferrara75]**). We now let,

$$\Phi = \mathbf{R} e^{\frac{i}{\hbar}\mathbf{S}} \tag{2.48}$$

as before, with $\mathbf{R}$ and $\mathbf{S}$ Real (or Vector) superfields. That we can always do this for any superfield is shown in Appendix B. Substituting into the field equation (2.45), we find,

$$\left\{ \frac{\hbar}{4}\left[\hat{\mathbf{D}}^2\mathbf{R} + \bar{\hat{\mathbf{D}}}^2\mathbf{R} \right] + \frac{1}{4}\left[2i\left[\hat{\mathbf{D}}\mathbf{R}\hat{\mathbf{D}}\mathbf{S} + \bar{\hat{\mathbf{D}}}\mathbf{R}\bar{\hat{\mathbf{D}}}\mathbf{S} \right] + i\mathbf{R}\left[\hat{\mathbf{D}}^2\mathbf{S} + \bar{\hat{\mathbf{D}}}^2\mathbf{S} \right] \right] \right.$$

$$\left. - \frac{1}{4\hbar}\mathbf{R}\left[\hat{\mathbf{D}}\mathbf{S}\hat{\mathbf{D}}\mathbf{S} + \bar{\hat{\mathbf{D}}}\mathbf{S}\bar{\hat{\mathbf{D}}}\mathbf{S} \right] - m\mathbf{R} \right\} e^{\frac{i}{\hbar}\mathbf{S}} = 0 \qquad (2.49)$$

We find that the imaginary, $\hbar^0$ term is simply the current conservation equation (2.46) in terms of $\mathbf{R}$ and $\mathbf{S}$, just as was true in the Klein-Gordon case. Again, if we now neglect the highest order in $\hbar$ of the real part, we are left with:

$$\frac{1}{4\hbar}\mathbf{R}\left[\hat{\mathbf{D}}\mathbf{S}\hat{\mathbf{D}}\mathbf{S} + \bar{\hat{\mathbf{D}}}\mathbf{S}\bar{\hat{\mathbf{D}}}\mathbf{S} \right] - m\mathbf{R} = 0 \qquad (2.50)$$

If we interpret this as a Hamilton-Jacobi equation in superspace, we may interpret $\mathbf{S}$ as the Hamilton-Jacobi function and $\mathbf{R}$ as an external field. Since we are working in superspace, and there are derivatives with respect to θ_α as well as with respect to x_μ, for the momenta we should make the identifications:

$$P_\mu = \frac{\partial \mathbf{S}}{\partial x^\mu} \ , \quad \pi_\alpha = \frac{\partial \mathbf{S}}{\partial \theta^\alpha} \ , \quad \bar{\pi}_{\dot{\alpha}} = \frac{\partial \mathbf{S}}{\partial \bar{\theta}^{\dot{\alpha}}} \qquad (2.51)$$

where π_α is conjugate to θ_α and P_μ is conjugate to x_μ. The previously defined D's from equations (2.20) and (2.33) are simply,

$$D_\alpha = \hat{\mathbf{D}}_\alpha \mathbf{S} \ , \quad \bar{D}_{\dot{\alpha}} = \bar{\hat{\mathbf{D}}}_{\dot{\alpha}} \mathbf{S} \qquad (2.52)$$

Thus the Hamiltonian to be associated with (2.50), and thus the WKB limit of the superfield theory is,

$$H = \frac{1}{4\hbar}\left[DD + \bar{D}\bar{D} \right] - m \qquad (2.53)$$

which is identical (within a constant) to equation (2.37) of section C, if we identify l as $\hbar$. We therefore find that the Pseudomechanics of **[Casalbuoni76]** and **[Brink81]** is in fact the WKB limit of this superfield theory (at least as far as the equations of motion and therefore the physics are concerned).

E. WKB for the Dirac Equation Directly

In this last section, we will look at the spin 1/2 sector of the WKB formula (2.48) to discover what the WKB limit of the Dirac equation is, without any reference to superspace.

In attempting to apply the WKB method to the Dirac equation directly, without any knowledge of superfield WKB, we find there are four main difficulties. First of all, since the equation is a spinor equation, we find that the resulting Hamilton-Jacobi equation, and therefore the Hamiltonian, obtained is a spinor. Second, it is not clear how spin should be represented in the final theory. Third, we may wonder if there should be momenta additional to the ordinary P_μ which represent the spin degrees of freedom. If so, what are they? Finally, when we make the approximation, we want to let:

$$\psi = R e^{\frac{i}{\hbar}S} \qquad (2.54)$$

Here, R must be a spinor. In order for WKB to work, we do not want it to be complex. Do we therefore want to say that R is Majorana?

The superfield approach gets around these problems. First, the spinor term is multiplied by a spinor degree of freedom, θ_α. Second, we have seen how spin is represented in terms of the extra anticommuting momenta, π_α. These additional momenta follow in exact analogy with the ordinary Klein-Gordon case. Finally, in the WKB approximation, **R** is a real superfield so we don't have the difficulty of H being a spinor. Let us see what the superfield approach tells us about the Dirac case.

From equation (2.43), we know that the two Weyl 2-spinors in the theory are:

$$\Phi_\theta = \Phi_{1\,\theta} = \psi \ , \quad \Phi_{\bar\theta} = \Phi^+{}_{2\,\bar\theta} = \bar\psi \tag{2.55}$$

(Here, we are using the representation (A.4) for the Dirac matrices from Appendix A. Also, we use the notation given in Appendix B where, for example, the θ component of Φ is Φ_θ). From (A.5) of Appendix A, we know that:

$$\Psi_a = \begin{bmatrix} \psi_{1\,\alpha} \\ \bar\psi_2{}^{\dot\alpha} \end{bmatrix} = \begin{bmatrix} \Phi_{\theta\,\alpha} \\ \Phi_{\bar\theta}{}^{\dot\alpha} \end{bmatrix} \tag{2.56}$$

From (B.10) in Appendix B, we know (we put in an $\hbar$ in the exponential),

$$\Phi_\theta = \left[i\mathbf{R}_0\mathbf{S}_\theta + \mathbf{R}_\theta \right] e^{\frac{i}{\hbar}\mathbf{S}_0} \tag{2.57}$$

and using the method of Appendix B, we can also find that:

$$\Phi_{\bar\theta} = \left[i\mathbf{R}_0\mathbf{S}_{\bar\theta} + \mathbf{R}_{\bar\theta} \right] e^{\frac{i}{\hbar}\mathbf{S}_0} \tag{2.58}$$

In four component language, then, if we let:

$$G = \mathbf{S}_0 \ , \quad \xi_a = \begin{bmatrix} \mathbf{S}_\theta \\ \mathbf{S}_{\bar\theta} \end{bmatrix} \ , \quad \chi_a = \begin{bmatrix} \mathbf{R}_\theta \\ \mathbf{R}_{\bar\theta} \end{bmatrix} \ , \tag{2.59}$$

we see that ξ and χ are Majorana spinors (as described in Appendix A) and we find that the WKB approximation for the Dirac field is:

$$\psi_a = \left[i\mathbf{R}_0\xi_a + \chi_a \right] e^{\frac{i}{\hbar}G} \tag{2.60}$$

G is the standard Hamilton-Jacobi function, and the momentum P_μ will be $\partial_\mu G$. Let us set the scalar $\mathbf{R}_0$ to be a constant 1, then we get

$$\psi_a = \left[i\xi_a + \chi_a \right] e^{\frac{i}{\hbar}G} \tag{2.61}$$

We now substitute into the Dirac equation:

$$i\hbar\gamma^\mu{}_{ab}\partial_\mu\psi_b - m\psi_a = 0 \tag{2.62}$$

and get

$$\left[i\hbar\gamma^\mu\partial_\mu\left[\chi + i\xi\right] + \left[\chi + i\xi\right]\gamma^\mu\partial_\mu G - m\left[\chi + i\xi\right]\right] e^{\frac{i}{\hbar}G} = 0 \tag{2.63}$$

Neglecting the highest order in $\hbar$, we identify this as a Hamilton-Jacobi equation and making the identification of P_μ with $\partial_\mu G$, following the previous procedure, we find that the Hamiltonian would be:

$$H_a = \left[\gamma^\mu P_\mu - m \right]_{ab} \left[\chi + i\xi \right]_b \approx 0 \qquad (2.64)$$

The problem with this Hamiltonian is that it is a spinor. The $\left[\chi + i\xi\right]$ is analogous to the R from section B. There, since R was only a Lagrange multiplier which multiplied everything, we could omit it from the Hamiltonian, or include it if we wished and the Hamiltonian remained a scalar. Here again, $\left[\chi + i\xi\right]$ is a Lagrange multiplier. We may therefore omit it if we wish, and regard the Hamiltonian as the bi-spinor:

$$H_{ab} = \left[\gamma^\mu P_\mu - m \right]_{ab} \approx 0 \qquad (2.65)$$

We will now have to provide an interpretation of a bi-spinor Hamiltonian, which we will do in Chapter IV. In the superfield case, the Hamiltonian was a scalar, not a bi-spinor, because this term was sandwiched between spinors which were Grassman phase space variables.

We obtain a bi-spinor Hamiltonian because, although we have dequantized completely the x_μ and p_μ degrees of freedom, the quantum nature of the spin degrees of freedom remains in the form of the operator (or matrix) character of H. Thus, we may consider (2.65) as the (relativistic) Hamiltonian for a Hamiltonian formulation of the mechanics if we also consider any phase space functions to possibly be matrices as well and always take expectation values with respect to some kind of spinor, such as $\left[\chi + i\xi\right]$ to be the actual observables.

In Chapter IV, we shall provide an interpretation to this and explore it in more detail, after we use what was found here to reformulate the classical limit in a clearer language (which will be introduced in the next chapter). We will then add an electromagnetic interaction, and obtain the spin precession equation of Bargmann, Michel, and Telegdi, with the correct g factor.

In retrospect, the Hamiltonian given in (2.65) seems to be what one would expect, it is just the Dirac equation. In fact, if one used the method of Fronsdal **[Fronsdal71]** instead of WKB, one would obtain the same thing. However, the superfield approach confirms that this is in fact the correct WKB limit and also helps us with the interpretation of the theory. By looking at what we did in the superfield case, we discover the solution to the four difficulties mentioned at the beginning of this section. We see that the superfield language is useful for talking about spinor WKB, even if there exists no such symmetry in nature. At the end we only use the spin 1/2 content and throw away the supersymmetry.

We now will introduce phase space methods and the concept of the *-product. In Chapter IV, we will finally obtain the semi-classical Dirac mechanics which is the $\hbar \to 0$ limit of the Dirac equation.

III. Dequantization and *-products

A. Introduction

We must now provide an interpretation of the Dirac Hamiltonian found via the WKB method in the previous chapter. Before doing this, however, we will review and develop further some phase space methods that will be needed. The Dirac Mechanics will then be formulated in the next chapter.

Phase space techniques in quantum mechanics were first developed by Weyl **[Weyl31]** and Wigner **[Wigner32]**. Later, Moyal **[Moyal49]** extended the method and introduced what is now called the Moyal bracket for functions on the phase space. These methods provide a probabilistic interpretation of quantum mechanics in terms of c-number phase space variables.

The basic idea is that quantum-mechanical, non-commuting operators are mapped into classical, commuting c-number functions. The non-commutivity of the operators are represented by the fact that products of operators are mapped into what is referred to as the *-product of the c-number functions, which in general is not equal to the ordinary product. In other words, $\hat{A} \to A$, $\hat{B} \to B$, but $\hat{A}\hat{B} \to A{*}B \neq AB$ and $A{*}B$ is generally not equal to $B{*}A$.

Any quantum-mechanical problem can be represented in this language, using ordinary commuting phase space functions, as long as the *-product is used instead of the ordinary product. This method has proven particularly useful in the field of quantum statistical mechanics and quantum optics (see **[Klauder60]**, **[Sudarshan63]**, **[Klauder68]**, **[Klauder85]** and the references provided therein). De Groot and Suttorp in **[deGroot72]** use these methods to formulate quantum-mechanical electrodynamics and the thermodynamics of material media.

The classical mapping is related to choosing a particular ordering of the quantum-mechanical operators. Thus, there are many mappings one could choose. Weyl's correspondence in **[Weyl31]** is defined by choosing the so called Weyl symmetrization ordering for the operators. Agarwal and Wolf in a series of three papers **[Agarwal70]** study general mappings for any chosen ordering. They show how the various orderings chosen in past work are related. Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer, in **[Bayen77]**, **[Bayen78]**, and **[Fronsdal78]**, further study the mathematical properties of these mappings and of the Moyal bracket formalism. In particular, they are able to calculate the spectrum of the harmonic oscillator and the hydrogen atom in this language.

For our purposes, it will be most convenient to choose the so-called normal ordering, which leads to a mapping obtained by taking coherent state expectation values of the operators. Coherent states are ***the most classical*** states; their properties are summarized in Appendix C and in **[Klauder85]**. The study in **[Agarwal70]** shows that the we are free to choose this ordering.

B. The Normal Order Map and *-Product

Let $\hat{a}$ and $\hat{a}^+$ be any canonical annihilation and creation operators which satisfy the basic commutation relation $\left[\hat{a}\,,\hat{a}^+\right]_- = 1$. (For example, the raising and lowering operators for a one dimensional harmonic oscillator). For now, we only consider one of each, but the method can easily be generalized to arbitrary dimension. Let z be a complex c-number. The coherent state $|z>$, whose definition and properties are given in Appendix C may be represented as:

$$|z> \equiv e^{\left(z\hat{a}^+ - \bar{z}\hat{a}\right)}|0> \tag{3.1}$$

where $|0>$ is the lowest (ground) state. It can be shown (see Appendix C) that $\hat{a}|z> = z|z>$ (also that $<z||z> = 1$) so that $<z|\hat{a}|z> = z$ and $<z|\hat{a}^+|z> = \bar{z}$.

Our mapping will be, for any function $\hat{f}(\hat{a},\hat{a}^+)$,

$$\hat{f}(\hat{a},\hat{a}^+) \rightarrow f(z,\bar{z}) \equiv <z|\hat{f}(\hat{a},\hat{a}^+)|z> \tag{3.2}$$

This is the normal ordering map, because it maps the normal ordered product $\hat{a}^n\,\hat{a}^{+m} \rightarrow z^n\,\bar{z}^m$. In order to find the *-product we must map the product of two arbitrary operator functions. If $\hat{f} \to f$ and $\hat{g} \to g$ then, it is shown in **[Agarwal70]** that,

$$\hat{f}(\hat{a}, \hat{a}^{+})\hat{g}(\hat{a}, \hat{a}^{+}) \to e^{\frac{\partial}{\partial z_1}\frac{\partial}{\partial \bar{z}_2}} f(z_1, \bar{z}_1)\, g(z_2, \bar{z}_2)\Bigg|_{\substack{z_1 = z_2 = z \\ \bar{z}_1 = \bar{z}_2 = \bar{z}}}$$

$$= fg + \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} + \frac{1}{2!}\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 g}{\partial \bar{z}^2} + \cdots = f(z, \bar{z}) * g(z, \bar{z}) \qquad (3.3)$$

We may express this in terms of real quantities, if we let $\hat{a} = 1/\sqrt{2\hbar}\,(\hat{q} + i\hat{p})$, $\hat{a}^{+} = 1/\sqrt{2\hbar}\,(\hat{q} - i\hat{p})$, and $z = 1/\sqrt{2\hbar}\,(q + ip)$. Then, $\left[\hat{q}, \hat{p}\right]_{-} = i\hbar$. Also, $\hat{q} \to q$ and $\hat{p} \to p$ and q and p are real. The *-product for functions of p and q is found to be:

$$\hat{f}(\hat{q}, \hat{p})\, \hat{g}(\hat{q}, \hat{p}) \to f(q,p) * g(q,p) =$$

$$e^{\frac{\hbar}{2}\left[\frac{\partial}{\partial p_1}\frac{\partial}{\partial p_2} - i\frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2} + i\frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2} + \frac{\partial}{\partial q_1}\frac{\partial}{\partial q_2}\right]} f(q_1, p_1) g(q_2, p_2)\Bigg|_{\substack{p_1 = p_2 = p \\ q_1 = q_2 = q}} \qquad (3.4)$$

In general, the *-product of two functions is a power series in $\hbar$. However, if either of the two functions is at most bi-linear in p and q, the series will terminate with the $\hbar^1$ term. For most of this work (except Chapter V), and for many cases of interest, this will be the case. Even if it is not, we may only be interested in terms up to order $\hbar^1$. In principle, however, we can keep all terms for an exact quantum-mechanical description. Terminating the series (if it does not terminate itself) will involve a classical approximation, neglecting higher orders of $\hbar$.

If the series does terminate, or if we neglect higher orders, we have that:

$$f * g = fg + \frac{\hbar}{2}\left[\frac{\partial f}{\partial p}\frac{\partial g}{\partial p} - i\,\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + i\,\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial g}{\partial q}\right] \tag{3.5}$$

We may generalize to more than one dimension by simply having more p's and q's (which, for the many-dimensional case, we will be calling x's). For the following chapters, we will be interested in relativistic mechanics, in four dimensions. Thus, we will have four p_λ 's and four x_λ 's ($\lambda = 0,1,2,3$). Since all the $\hat{p}_\mu$'s and $\hat{x}_\nu$'s commute except when $\mu = \nu$ ($\left[\hat{x}_\mu , \hat{p}_\nu\right]_- = g_{\mu\nu}$), we can immediately generalize (3.5) to (f and g are now arbitrary functions of p_λ and x_λ):

$$f * g = fg + \frac{\hbar}{2}\left[\frac{\partial f}{\partial p^\lambda}\frac{\partial g}{\partial p_\lambda} - i\,\frac{\partial f}{\partial p^\lambda}\frac{\partial g}{\partial x_\lambda} + i\,\frac{\partial f}{\partial x^\lambda}\frac{\partial g}{\partial p_\lambda} + \frac{\partial f}{\partial x^\lambda}\frac{\partial g}{\partial x_\lambda}\right] + \cdots \tag{3.6}$$

We next show how the *-product formalism is used with the dynamical equations of motion.

C. Dynamics and Equations of Motion

The (relativistic) quantum-mechanical Heisenberg equations of motion for any observable $\hat{A}$, given a quantum Hamiltonian $\hat{H}$ is:

$$i\hbar\frac{d\hat{A}}{d\tau} \equiv \dot{\hat{A}} = \left[\hat{A} , \hat{H}\right]_- \tag{3.7}$$

In order to make connection with classical mechanics, we may define the Moyal bracket between any two functions $A(p,x)$ and $B(p,x)$ as follows:

$$\left[A , B \right]_{\mathbf{M}} = \frac{1}{i\hbar} \langle z| \left[\hat{A} , \hat{B} \right]_{-} |z\rangle \tag{3.8}$$

where, $\hat{A} \to A$ and $\hat{B} \to B$. Then, in terms of the classical quantities, (3.7) becomes,

$$\dot{A} = \left[A , H \right]_{\mathbf{M}} \tag{3.9}$$

We see that the Moyal bracket can also be expressed as:

$$\left[A , B \right]_{\mathbf{M}} = \frac{1}{i\hbar} \left[A{*}B - B{*}A \right] \tag{3.10}$$

We may now use equations (3.4) or (3.6) to find the Moyal bracket in terms of p and x. We find it is:

$$\left[A , B \right]_{\mathbf{M}} = \frac{1}{i\hbar} \left[A{*}B - B{*}A \right] = \left[\frac{\partial A}{\partial x^{\lambda}} \frac{\partial B}{\partial p_{\lambda}} - \frac{\partial A}{\partial p^{\lambda}} \frac{\partial B}{\partial x_{\lambda}} \right]$$

$$- \frac{\hbar}{2} \left[\frac{\partial^2 A}{\partial p^2} \frac{\partial^2 B}{\partial p_{\lambda} \partial x^{\lambda}} - \frac{\partial^2 A}{\partial p_{\lambda} \partial x^{\lambda}} \frac{\partial^2 B}{\partial p^2} - \frac{\partial^2 A}{\partial x^2} \frac{\partial^2 B}{\partial p_{\lambda} \partial x^{\lambda}} + \frac{\partial^2 A}{\partial p_{\lambda} \partial x^{\lambda}} \frac{\partial^2 B}{\partial x^2} \right] + \cdots \tag{3.11}$$

If we let $\hbar \to 0$, we obtain the usual Poisson bracket (and equations of motion) of classical mechanics:

$$\left[A , B \right]_{\mathrm{pb}} = \lim_{\hbar \to 0} \left[A , B \right]_{\mathbf{M}} = \left[\frac{\partial A}{\partial x^{\lambda}} \frac{\partial B}{\partial p_{\lambda}} - \frac{\partial A}{\partial p^{\lambda}} \frac{\partial B}{\partial x_{\lambda}} \right] \tag{3.12}$$

Again we remember that this may be an exact expression for the Moyal bracket, without any classical approximations, if *either* A *or* B are at most bi-linear in x_{λ} and p_{λ}.

Finally, a very important point must be emphasized. As Bayen, et al point out in **[Bayen77]**, the usual rule for the time derivative of the product of two quantities does not (in general) hold. With,

$$\dot{A} = \left[A , H \right]_{\mathbf{M}} ,$$

$$\frac{d}{d\tau}\left[A B \right] \neq \dot{A}B + A\dot{B} \qquad (3.13)$$

However, it is true that,

$$\frac{d}{d\tau}\left[A{*}B \right] = \dot{A}{*}B + A{*}\dot{B} \qquad (3.14)$$

so that if we always express everything in terms of *-products, the usual time differentiation rule does hold. In fact, this point is an expression of the uncertainty principle and allows for a statistical interpretation of the mechanics (this point is clearly elucidated in the second paper of **[Bayen78]**).

D. Example: A Scalar Particle

As an example, we may consider a relativistic scalar particle of mass m and charge e in an external electromagnetic field. This example is given in **[Fronsdal71]**. In that paper, it is shown that the (relativistic) Hamiltonian (called L in that paper) is (if $\Pi_\mu = \left[p - eA \right]_\mu$, A_μ is the electromagnetic field potential, and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$) :

$$H = \frac{1}{2m}\left[\Pi^2 - m^2 \right] \qquad (3.15)$$

If we use the equations of motion (3.9) with the Moyal bracket (3.11) we obtain the usual relativistic equations of motion with quantum corrections. Calculating to order $\hbar$, we have,

$$\dot{x}_\mu = \frac{1}{m}\,\Pi_\mu$$

$$\ddot{x}_\mu = \frac{e}{m}\,\dot{x}^\nu F_{\nu\mu} + \frac{e^2\hbar}{2m}\,A_{\mu,\nu}{}^{,\nu}\,A_\lambda{}^{,\lambda} \;+\; \cdots \tag{3.16}$$

Thus, we see how the *-product language allows us to easily see the classical limit of quantum mechanics as well as quantum corrections to the equations of motion as an explicit power series in $\hbar$.

E. Gauge Invariance and the *-product

At first glance, it appears that the order $\hbar$ term in (3.16) is not gauge invariant, although the Hamiltonian (3.15) is. To see that (3.16) is in fact gauge invariant, we must examine how gauge transformations act with respect to the *-product.

We will first write down the standard (U(1)) gauge transformation for the wave function, Φ and the electromagnetic potential, $\hat{A}_\mu$:

$$\Phi \rightarrow e^{\frac{ie\alpha(x)}{\hbar}}\,\Phi = \hat{U}_G\Phi$$

$$\hat{A}_\mu \rightarrow \hat{A}_\mu + \hat{\alpha}_{,\mu} \tag{3.17}$$

Here, $\hat{\alpha}$ is a function of $\hat{x}_\mu$. Observables are bi-linear in Φ, so we have that the gauge transformation, for any quantum mechanical operator, $\hat{B}$, is,

$$\Phi^* \hat{B} \Phi \rightarrow \Phi^* \hat{U}_G^{-1} \hat{B}' \hat{U}_G \Phi$$

$$where \ , \quad \hat{B}' \equiv \hat{B}\Big|_{\hat{A}_\mu \rightarrow \hat{A}_\mu + \hat{\alpha}_{,\mu}} \tag{3.18}$$

The prime means to replace $\hat{A}_\mu$ by $\hat{A}_\mu + \hat{\alpha}_{,\mu}$, the usual gauge substitution. We may therefore regard a gauge transformation on the operator $\hat{B}$ to be:

$$\hat{B} \rightarrow \hat{U}_G^{-1} \hat{B}' \hat{U}_G = \hat{U}_G^{-1} \left[\hat{U}_G \hat{B}' + \left[\hat{B}' \, , \, \hat{U}_G \right]_- \right] \tag{3.19}$$

Generally, the commutator of $\hat{B}'$ with $\hat{U}_G$ will be proportional to $\hat{U}_G$ (or will be zero). If so, we may define $\hat{B}''$ by,

$$\left[\hat{B}' \, , \, \hat{U}_G \right]_- \equiv \hat{U}_G \hat{B}'' \tag{3.20}$$

Then, the gauge transformation on $\hat{B}$, may be written as,

$$\hat{B} \rightarrow \hat{B}' + \hat{B}'' \tag{3.21}$$

We now are ready to express this in the *-product language. Let us first consider, the electromagnetic potential, $\hat{A}_\mu$:

$$\hat{A}_\mu' = \hat{A}_\mu + \hat{\alpha}_{,\mu}$$

$$\hat{A}_\mu'' = 0$$

Thus,

$$\hat{A}_\mu \rightarrow \hat{A}_\mu + \hat{\alpha}_{,\mu} \tag{3.22}$$

which is the usual gauge substitution. Now, let us look at $\hat{p}_\mu$:

$$\hat{p}_\mu{}' = \hat{p}_\mu$$

$$\hat{p}_\mu{}'' = e\hat{\alpha}_{,\mu}$$

Thus,

$$\hat{p}_\mu \rightarrow \hat{p}_\mu + e\hat{\alpha}_{,\mu} \tag{3.23}$$

We see that $\hat{\Pi}_\mu = \hat{p}_\mu - e\hat{A}_\mu$ is gauge invariant, as expected.

We must, however, be very careful when we perform the normal order map. We may NOT simply make substitutions (3.22) and (3.23) in the classical quantities to obtain the gauge transformation. As an example, consider what happens to the product of two p_μ's. We have,

$$p_\mu * p_\nu = p_\mu p_\nu + \frac{\hbar}{2} g_{\mu\nu} \tag{3.24}$$

Under a gauge transformation, we obtain.

$$p_\mu * p_\nu \rightarrow \left[p_\mu + e\alpha_{,\mu}\right] * \left[p_\nu + e\alpha_{,\nu}\right]$$

$$= \left[p_\mu + e\alpha_{,\mu}\right]\left[p_\nu + e\alpha_{,\nu}\right] + \frac{\hbar}{2}\left[g_{\mu\nu} + e^2 \alpha_{,\mu\lambda}\alpha_{,\nu}{}^{,\lambda}\right] \tag{3.25}$$

Notice, that if we had just made the substitutions (3.22) and (3.23) in (3.24), we would have omitted the term:

$$\frac{e^2\hbar}{2}\alpha_{,\mu\lambda}\alpha_{,\nu}{}^{,\lambda} \tag{3.26}$$

Also, we notice that the gauge transformation mixes orders of $\hbar$. Thus, in general, we can only say something is gauge invariant to a certain order in $\hbar$. In a similar manner, one may show that the terms shown on the right hand side of (3.16) are gauge invariant to order $\hbar$. (Of course, the full expression is gauge invariant to all orders of $\hbar$).

As a final example, we will consider the *-product $\Pi_\mu * \Pi_\nu$ as this will appear again in later chapters. We have that Π_μ is gauge invariant, as is $\Pi_\mu * \Pi_\nu$. We have that to order $\hbar$,

$$\Pi_\mu * \Pi_\nu = \Pi_\mu \Pi_\nu + \frac{\hbar}{2}\Big[g_{\mu\nu} + e^2 A_{\mu,\lambda} A_\nu{}^{,\lambda} + ieF_{\nu\mu} \Big] + \cdots \tag{3.27}$$

Under a gauge transformation,

$$\Pi_\mu \Pi_\nu \rightarrow \Pi_\mu \Pi_\nu - \frac{e^2\hbar}{2}\Big[\alpha_{,\mu\lambda} A_\nu{}^{,\lambda} + A_{\mu,\lambda} \alpha_{,\nu}{}^{,\lambda} + \alpha_{,\mu\lambda} \alpha_{,\nu}{}^{,\lambda} \Big] \tag{3.28}$$

$$A_{\mu,\lambda} A_\nu{}^{,\lambda} \rightarrow A_{\mu,\lambda} A_\nu{}^{,\lambda} + \Big[\alpha_{,\mu\lambda} A_\nu{}^{,\lambda} + A_{\mu,\lambda} \alpha_{,\nu}{}^{,\lambda} + \alpha_{,\mu\lambda} \alpha_{,\nu}{}^{,\lambda} \Big] \tag{3.29}$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \tag{3.30}$$

$$g_{\mu\nu} \rightarrow g_{\mu\nu} \tag{3.31}$$

Thus, the *-product is gauge invariant, and the terms calculated to order $\hbar$ are gauge invariant to this order. However, the ordinary product $\Pi_\mu \Pi_\nu$ is NOT gauge invariant.

In general, *-products of Π_μ are gauge invariant, but not ordinary products. Thus, we have another reason for expressing everything in terms of *-products. Only in this way can we easily see the gauge invariance of an expression. Also, we see that a gauge transformation corresponds to making the substitutions (3.22) and (3.23) *in the *-product,* but not in the ordinary product. We will use this to great advantage in Chapter V in finding quantum corrections to the spin precession equation for inhomogeneous fields.

A similar situation occurs when we have constraints in the theory (as will be the case in the next chapter). If the constraint is expressed as,

$$D(x,p) = 0$$

then it is true that for any A , $A * D = 0$, but not in general that $AD = 0$.

We are now ready to interpret the WKB limit of the Dirac equation found in Chapter II in this language. The one big difference we will encounter in the Dirac mechanics will be that the A and B in the above equations will be four by four matrices. This means that the expression for the *-product and for the Moyal or Poisson bracket will be more complicated. In particular, we will have terms such as,

$$\frac{\partial A}{\partial p}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial p}\frac{\partial A}{\partial p}$$

In the case where A and B are ordinary numbers this is zero, but if they are matrices, it may not be. Otherwise, the formalism is very similar and the gauge invariance discussion above will still remain valid.

IV. The Dirac Semiclassical Mechanics:

A Particle in a Homogeneous Electromagnetic Field

We now proceed (finally) to interpret the mechanical Hamiltonian found as the WKB limit of the Dirac equation in Chapter II. We will find that it will be advantageous to redo the dequantization via the *-product method developed last chapter. However, the work of Chapter II provides the essential clues for interpreting the resulting mechanics.

In developing the formalism, we will first consider a free particle, but will quickly move (in section C) to study the interaction with a homogeneous electromagnetic field. We shall then be able to derive the correct relativistic equations of motion and the Bargmann, Michel, Telegdi relativistic spin precession equation of **[Bargmann59]** (for a homogeneous field). In Chapter V, we will develop a general formalism with an inhomogeneous field and obtain quantum corrections to this equation.

A. The Dirac Semi-Classical Mechanics

In this, section, we will consider a free Dirac particle of mass m. As found in Chapter II, the Hamiltonian obtained as the WKB limit of the Dirac Equation is:

$$H = \left[\gamma^{\mu} p_{\mu} - m \right] \tag{4.1}$$

This comes as no surprise, and one might have guessed it without going through the derivation in Chapter II, as it would be the usual Dirac Hamiltonian operator if p_μ was an operator. If we had applied the method of Fronsdal **[Fronsdal71]** we would have obtained this also.

We first note that the Hamiltonian is a bi-spinor, not a scalar. This is because, although we have dequantized completely the x_μ and p_μ degrees of freedom, the quantum nature of the spin degrees of freedom remains in the form of the matrix character of H. How should we proceed with such a Hamiltonian?

The key is to remember that (4.1) actually appears in the superspace WKB expression multiplied by spinors on either side. Each term in the WKB limit is actually a scalar. We therefore recognize that the Hamiltonian, and in fact all bi-spinors of the theory, are eventually to be sandwiched between spinors which we will write as:

$$<| H |> = <| \left[\gamma^{\mu} p_{\mu} - m \right] |> \tag{4.2}$$

Here, $<|$ and $|>$ are some kind of 4-spinors. (4.2) is what actually appears as the Dirac term in the superfield Hamiltonian.

Now, we have a four by four matrix Hamiltonian. If it is in fact a Hamiltonian, it should generate (proper) time translations. Thus, any observable of the theory will (in general) also be a four by four matrix. Suppose we have some observable A, and that it evolves in (proper) time according to:

$$\dot{A} = \left[A , H \right]_{\mathbf{M}} \tag{4.3}$$

with the Moyal bracket $\left[A , B \right]_{\mathbf{M}}$ yet to be determined. We now know the (four by four matrix) A at any point in time, but what is actually observed must be $<| A |>$ since spinors and bi-spinors cannot be directly observed.

We now almost have an interpretation, except for one problem. What is the time evolution of the $<|$ and $|>$? If we could have all of the dynamics of the theory in an equation like (4.3), we would have a formalism very similar to classical mechanics and would not have to worry about external spinor fields. Since what is actually observed is always the combination $<| A |>$, not A alone, we may proceed as is commonly done in ordinary quantum mechanics.

There, one is concerned with observables $<| \hat{A} |>$ also, but $<|$ and $|>$ are in general infinite-dimensional vectors in a Hilbert space, and $\hat{A}$ is a quantum-mechanical operator. What is actually observed is $<| \hat{A} |>$. One may take either a Schrödinger point of view where the states vary in time but the operators are fixed; or a Heisenberg point of view where the operators vary in time, but the states are fixed. (One could also take an intermediate view where both vary, as in the interaction picture). Both pictures are equivalent.

We can therefore keep a mechanical interpretation, with all of the dynamics in equation (4.3) if we take a Heisenberg-like picture. We say that whatever the spinor fields are, they do not change with time, but that the four by four observables (the A 's) do. We define all of the (proper) time dependence to be in the matrices, so that

$$\frac{d}{d\tau} \langle | A | \rangle = \langle | \dot{A} | \rangle \tag{4.4}$$

Note, that in Chapter II, we were actually taking a Schrödinger point of view. There, the four by four matrices were by definition fixed, but the anticommuting phase space variables θ (spinors) varied with time and we found how they varied.

The Dirac WKB limit found in Chapter II has taught us two things. First, that the proper time Hamiltonian in the WKB limit is just the ordinary Dirac operator, $\gamma^\mu p_\mu - m$. Second, that the interpretation of this is that the Hamiltonian and all other phase space functions are in general four by four matrices, but these matrices are to be sandwiched between spinors. For any phase space function A , we can only observe $\langle | A | \rangle$.

With this knowledge, we can proceed to formulate the semi-classical Dirac mechanics. Rather than worry about the spinor fields and their time evolution, we shall use the Heisenberg-like picture described above. It turns out that the clearest way to proceed is to redo the dequantization in the language of *-products. This will make clearer contact with ordinary mechanics and will provide a necessary connection between the phase space variables of the theory. (The previous work of **[Katz86B]** formulated the Dirac mechanics without the *-product language. Here, we will be able to be much clearer.)

In retrospect, perhaps, it may not have been necessary to go through the work of Chapter II to develop the Dirac mechanics (though it certainly was important for the superspace mechanics). However, despite much effort in the past to interpret a four by four phase space, no progress was made until the superspace method pointed the way. We now proceed to the Dirac mechanics for a free particle.

B. Dirac Mechanics and *-Dequantization

Here, instead of using the WKB method, we will redo the dequantization of the Dirac equation in the language of *-products developed in the last chapter. We start again with the Dirac equation for a free particle:

$$\left[\gamma^{\mu} \hat{p}_{\mu} - m \right] \Psi = 0 \tag{4.5}$$

Here, a hat (as in $\hat{p}$) denotes an ordinary quantum mechanical operator which acts in Hilbert space. No hat means we have an ordinary c-number (such as x_{μ} or p_{μ}) or a four by four matrix (such as γ_{μ}) From now on, what we mean by $|>$ or $<|$ are the Dirac one-particle states. In other words, in the coordinate representation with $|x>$ a position eigenket,

$$\Psi(x) = \langle x| \, |\rangle \ , \quad \overline{\Psi}(x) = \langle| \, |x\rangle \tag{4.6}$$

We now proceed in a way similar to that of Fronsdal **[Fronsdal71]** and guided by what we have said in the previous section and knowing what the WKB limit of the Dirac equation is, we say that the (relativistic) Hamiltonian for a Dirac particle is,

$$\hat{H} = \left[\gamma^{\mu} \hat{p}_{\mu} - m \right] \tag{4.7}$$

and interpret (4.5) as a constraint equation:

$$\hat{H} \ |\rangle \approx 0 \tag{4.8}$$

where " ≈ 0 " means weakly equal to zero, in the sense of Dirac (**[Dirac64]**). We recall that this means we only set it equal to zero at the end of all calculations. As usual when we have constraints, we have to follow Dirac's program and to check for secondary constraints.

We take the Heisenberg-like point of view discussed in the last section, so that the $|\rangle$'s are constant, but the γ_{μ} 's vary in time. Our phase space variables are therefore x_{μ} , p_{μ} , and γ_{μ} (note that γ_{μ} is a four by four c-number matrix, not a quantum-mechanical operator in the usual sense).

We will now use the *-product method of the previous chapter to formulate the mechanics in terms of the (semi) *classical* phase space variables x_{μ} , p_{μ} and γ_{μ}. We will be using the normal-order map and *-product to define a Moyal bracket, but we must remember that our quantities are matrices, so that the order may matter.

Since p_μ and x_μ are proportional to the unit matrix, we may easily generalize the normal order mapping of the previous chapter by multiplying everything by the four by four unit matrix. Then, via the normal order map, the four by four quantum operator $\hat{A}$ maps into the four by four c-number A . Thus (with $\hat{A} \to A$, $\hat{B} \to B$), from (3.6), the *-product is given by:

$$A * B = AB + \frac{\hbar}{2}\left[\frac{\partial A}{\partial p^\lambda}\frac{\partial B}{\partial p_\lambda} - i\,\frac{\partial A}{\partial p^\lambda}\frac{\partial B}{\partial x_\lambda} + i\,\frac{\partial A}{\partial x^\lambda}\frac{\partial B}{\partial p_\lambda} + \frac{\partial A}{\partial x^\lambda}\frac{\partial B}{\partial x_\lambda}\right] + \cdots \quad (4.9)$$

So far, all is the same as the usual case outlined in Chapter III. However, we must now remember that A and B are matrices. Notice that the *-product of γ_μ with anything is just the ordinary product. The equation of motion for any function A of the phase space variables is given by (3.9), so

$$\frac{dA}{d\tau} \equiv \dot{A} = \left[A\,,H\right]_M \equiv \frac{1}{i\hbar}\left[A * H - H * A\right] \quad (4.10)$$

Here, $H = \gamma^\mu p_\mu - m$. H is linear in p_μ so the *-product terminates with the order $\hbar^1$ term. We have,

$$\frac{\partial H}{\partial p_\lambda} = \gamma^\lambda\ , \quad \frac{\partial H}{\partial x_\lambda} = 0 \quad (4.11)$$

Hence,

$$A * H = AH + \frac{\hbar}{2}\left[\frac{\partial A}{\partial p^\lambda}\gamma^\lambda + i\frac{\partial A}{\partial x^\lambda}\gamma^\lambda\right] \tag{4.12}$$

and

$$H * A = HA + \frac{\hbar}{2}\left[\gamma^\lambda \frac{\partial A}{\partial p^\lambda} - i\gamma^\lambda \frac{\partial A}{\partial x^\lambda}\right] \tag{4.13}$$

Thus,

$$A * H - H * A = \left[A , H\right]_- + \frac{\hbar}{2}\left[\left[\frac{\partial A}{\partial p^\lambda} , \gamma^\lambda\right]_- + i\left[\frac{\partial A}{\partial x^\lambda} , \gamma^\lambda\right]_+\right] \tag{4.14}$$

where $\left[A , H\right]_- = AH - HA$ and $\left[A , H\right]_+ = AH + HA$ are the commutator and anticommutator with respect to the four by four matrices A and H (not with respect to the operators $\hat{A}$ and $\hat{H}$!). The Moyal bracket, therefore, of anything with this Hamiltonian is

$$\left[A , H\right]_M = \frac{1}{i\hbar}\left[A , H\right]_- + \frac{1}{2}\left[\left[\frac{\partial A}{\partial x^\lambda} , \gamma^\lambda\right]_+ - i\left[\frac{\partial A}{\partial p^\lambda} , \gamma^\lambda\right]_-\right] = \dot{A} \tag{4.15}$$

With this we can find the equations of motion for x_μ , p_μ , and γ_μ :

$$\dot{x}_\mu = \gamma_\mu$$

$$\dot{p}_\mu = 0$$

$$\dot{\gamma}_\mu = \frac{1}{i\hbar}(-2i)\Sigma_{\mu\nu}\, p^\nu \tag{4.16}$$

where, $\Sigma_{\mu\nu} \equiv \frac{i}{2}\left[\gamma_\mu , \gamma_\nu\right]_-$. We notice that the 4-velocity is just γ_μ , which has been known for a long time (see **[Bjorken64]** or **[Itzykson80]**, for example). The Dirac constraint equation (4.8) provides us with a connection between the momentum and the 4-velocity. As is shown in Appendix D, if we start with constraint (4.8) written as:

$$\gamma_\nu p^\nu \; |> \approx m \; |> \tag{4.17}$$

and also

$$<| \; \gamma_\nu p^\nu \approx <| \; m \tag{4.18}$$

and multiply (4.17) on the left by $<| \; \gamma_\mu$, we find that

$$<| \, p_\mu \, |> \; = m <| \; \gamma_\mu \, |> \; = m <| \; \dot{x}_\mu \, |> \tag{4.19}$$

(This is essentially equation (D.4)). Thus we have the usual relationship between p_μ and $\dot{x}_\mu$, but only for $<| \, p_\mu \, |>$ and $<| \, \dot{x}_\mu \, |>$, and only as a weak condition (we cannot further use the equations of motion).

Using (4.19) and remembering that the p_μ are diagonal matrices, we have that,

$$<| \, \dot{x}_\mu \, |> \; <| \, \dot{x}^\mu \, |> \; = \; <| \, \dot{x}_\mu \, |> \frac{p^\mu}{m}$$

$$= \frac{1}{m} <| \; \gamma_\mu \, p^\mu \, |> \; = \frac{1}{m} <| \; m \; |> \; = 1 \tag{4.20}$$

Thus, we check that $d\tau$ is the usual invariant, with,

$$d\tau^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \tag{4.21}$$

We must now check that the constraint (4.8) and the equations of motion (4.16) are consistent. We follow Dirac's program and remember that the constraint is a weak condition, to be applied only at the very end. If we use (4.15) to calculate $\dot{H}$, we find that $\dot{H} = 0$. Thus, the Hamiltonian is constant and the constraint (4.8) holds for all time so we are consistent there.

We also have (4.19). If we take its time derivative, we find that (4.19) generates what Dirac calls a secondary constraint:

$$<| \dot{p}_\mu |> = m <| \dot{\gamma}_\mu |> \tag{4.22}$$

or that,

$$\frac{1}{i\hbar}(-2i) <| \ \Sigma_{\mu\nu} p^\nu \ |> = 0 \tag{4.23}$$

For the ordinary scalar particle, this is as far as one could proceed. However, since we are dealing with a spin 1/2 particle, we should be able to calculate the equation of motion for the spin. We expect it to be constant in the free case, but to exhibit precession, for example, in an electromagnetic field.

Before we do this, we would like to make one other point. We will want to keep the formalism in terms of *-products as much as possible. This is because, as was discussed in the last section, the usual time derivation formula holds for the *-product of two functions (equation (3.14)) but not, in general for the ordinary product. Also, as was shown in the previous chapter, gauge transformations are simple for the *-product. Finally, we must be very careful with the constraints. If A is any function on phase space, the constraint $H \, |> \approx 0$ implies that

$$A * H \ |> \approx 0 \tag{4.24}$$

but NOT (necessarily) that $AH \, |> \approx 0$. Since $\gamma_\mu * p_\nu = \gamma_\mu p_\nu$ and $\gamma_\mu * x_\nu = \gamma_\mu x_\nu$, our previous work leading to (4.22) and in Appendix D is correct.

The spin, S_μ , is defined for the Dirac particle, to be:

$$S_\mu = -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\Sigma^{\nu\lambda}p^\rho \tag{4.25}$$

(See **[Bjorken64]** and **[Itzykson80]** , for example). $\varepsilon_{\mu\nu\lambda\rho}$ is the totally antisymmetric tensor, with $\varepsilon^{0123} = 1$, $\varepsilon_{0123} = -1$, and $\varepsilon_{\mu\nu\lambda\rho}\,\varepsilon^{\mu\nu\lambda\rho} = -24$.

We have,

$$S_\mu = -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\Sigma^{\nu\lambda} * p^\rho \tag{4.26}$$

so,

$$\dot{S}_\mu = -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\dot{\Sigma}^{\nu\lambda} * p^\rho + \Sigma^{\nu\lambda} * \dot{p}^\rho \tag{4.27}$$

Now, $\dot{p}_\mu = 0$ and,

$$\dot{\Sigma}^{\nu\lambda} = \left[\Sigma^{\nu\lambda}\,,\gamma^\alpha\right]_- p_\alpha = 2i\left[\gamma^\nu p^\lambda - \gamma^\lambda p^\nu\right] \tag{4.28}$$

so,

$$\dot{S}_\mu = -2i\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\left[\gamma^\nu p^\lambda - \gamma^\lambda p^\nu\right] * p^\rho = 0 \tag{4.29}$$

as expected. We now have a formalism which gives the correct equations of motion and spin for a free particle. We next consider interactions.

C. Dirac Particle in an External Homogeneous Field

We consider a particle in an external homogeneous electromagnetic field, with 4-potential A_μ and $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ (the comma denotes differentiation with respect to x_μ). The Hamiltonian is generalized from the free case, using minimal coupling, with p_μ replaced by $\Pi_\mu = p_\mu - eA_\mu$ (e is the charge of the particle) and is therefore gauge invariant. We have,

$$H = \gamma^\mu \left[p_\mu - eA_\mu\right] - m = \gamma^\mu \Pi_\mu - m \tag{4.30}$$

The Dirac constraint is modified from (4.17) to be

$$\gamma_\nu \Pi^\nu |> \approx m |> \tag{4.31}$$

As shown in Appendix D (equation (D.4)), instead of (4.19), we have,

$$<| \Pi_\mu |> = m <| \gamma_\mu |> \tag{4.32}$$

which will give us the usual relation for a particle in an external electromagnetic field (as we will see when we calculate $\dot{x}_\mu$).

With the Hamiltonian (4.30), we have that

$$\frac{\partial H}{\partial p^\lambda} = \gamma_\lambda \quad , \quad \frac{\partial H}{\partial x^\lambda} = -e\gamma^\nu A_{\nu,\lambda} \tag{4.33}$$

If the field is homogeneous, so that $F_{\mu\nu,\lambda} = 0$, the expression for the *-product again terminates with the $\hbar^1$ term (H is linear in p and x). For any function on phase space, B ,

$$B * H = BH + \frac{\hbar}{2}\left[\left[\frac{\partial B}{\partial p^\lambda}\gamma^\lambda - eB_{,\lambda}\gamma^\nu A_\nu{}^{,\lambda}\right]\right.$$

$$\left. + i\left[B_{,\lambda}\gamma^\lambda + e\frac{\partial B}{\partial p^\lambda}\gamma^\nu A_\nu{}^{,\lambda}\right]\right]$$

$$H * B = HB + \frac{\hbar}{2}\left[\left[\gamma^\lambda\frac{\partial B}{\partial p^\lambda} - e\gamma^\nu A_\nu{}^{,\lambda}B_{,\lambda}\right]\right.$$

$$\left. + i\left[-\gamma^\lambda B_{,\lambda} - e\gamma^\nu A_\nu{}^{,\lambda}\frac{\partial B}{\partial p^\lambda}\right]\right] \tag{4.34}$$

Thus, the Moyal bracket and general equation of motion for B for this Hamiltonian is given by,

$$\dot{B} = \left[B\,,H\right]_M = \frac{1}{i\hbar}\left[B * H - H * B\right] =$$

$$\frac{1}{i\hbar}\left[B\,,H\right]_- - \frac{i}{2}\left[\left[\frac{\partial B}{\partial p^\lambda}\,,\gamma^\lambda\right]_- - e\left[B_{,\lambda}\,,\gamma^\nu A_\nu{}^{,\lambda}\right]_-\right]$$

$$+ \frac{1}{2}\left[\left[B_{,\lambda}\,,\gamma^\lambda\right]_+ + e\left[\frac{\partial B}{\partial p^\lambda}\,,\gamma^\nu A_\nu{}^{,\lambda}\right]_+\right] \tag{4.35}$$

From this, we can find the equations of motion for the phase space variables. We get,

$$\dot{x}_\mu = \gamma_\mu$$

$$\dot{p}_\mu = e\gamma^\nu A_{\nu,\mu}$$

$$\dot{\gamma}_\mu = \frac{1}{i\hbar}(-2i)\Sigma_{\mu\nu}\,\Pi^\nu \tag{4.36}$$

so,

$$\dot{\Pi}_\mu = e\gamma^\nu F_{\nu\mu} = e\dot{x}^\nu F_{\nu\mu} \tag{4.37}$$

Note, that again, the 4-velocity is γ_μ , just as in the free case. Thus, (4.32) provides the usual relationship between the 4-velocity, $\dot{x}_\mu$, and Π_μ . Also, we have

$$<|\ \dot{x}_\mu\ |>\ <|\ \dot{x}^\mu\ |> = 1 \tag{4.38}$$

as before (equation (4.20)), so (4.21) is still true. One may check that $\dot{H} = 0$ still, but arising from (4.32), we have the secondary constraint,

$$\frac{1}{i\hbar}(-2i)<|\ \Sigma_{\mu\nu}\,\Pi^\nu\ |> = \frac{e}{m}<|\ \dot{x}^\nu\,F_{\nu\mu}\ |> \tag{4.39}$$

which makes the equations of motion (4.36) and the phase space relation (4.32) consistent (note that the term on the left seems to be of order $\hbar^{-1}$, while the term on the right is of order 1). In appendix D, we have derived other phase space relations which will be used in the rest of this chapter.

We therefore have the usual equation of motion for x_μ,

$$<|\ \ddot{x}_\mu\ |> = \frac{e}{m}<|\ \dot{x}^\nu\,F_{\nu\mu}\ |> \tag{4.40}$$

which is the same as for a scalar particle, as expected. (This is given in **[Bargmann59]**). We expect the equation of motion for the spin, however, to be different from the scalar case.

We first need a definition of the spin. In QED (which we are trying to express in this semi-classical language), one has only to define the spin for the asymptotically free *in* and *out* states (in the sense of the LSZ language). Here, we therefore need an expression which can interpolate between the *in* and *out* states. We can do this by defining the spin in a gauge invariant way which reduces to the expression for the free case, (4.25) as $A_\mu \rightarrow 0$. Minimal coupling, substituting Π_μ for p_μ, accomplishes this. We therefore define the spin as:

$$S_\mu \equiv -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\,\Sigma^{\nu\lambda}\,\Pi^\rho \tag{4.41}$$

Note, that with the *-product as given in (4.9), this is also,

$$S_\mu \equiv -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\,\Sigma^{\nu\lambda} * \Pi^\rho \tag{4.42}$$

and that from the discussion in the previous chapter, we know that this is gauge invariant. Taking the time derivative, we have,

$$\dot{S}_\mu \equiv -\frac{\hbar}{4}\varepsilon_{\mu\nu\lambda\rho}\left[\,\dot{\Sigma}^{\nu\lambda} * \Pi^\rho + \Sigma^{\nu\lambda} * \dot{\Pi}^\rho\,\right] \tag{4.43}$$

We remember to express things in terms of the *-product before taking the time derivative, so that the product differentiation rule will hold. The *-product may be replaced by the ordinary product in (4.42) but *NOT* in (4.43).

From (4.37), $\dot{\Pi}_\mu = e\gamma^\nu F_{\nu\mu}$ and,

$$\dot{\Sigma}^{\nu\lambda} = \left[\Sigma^{\nu\lambda} , \gamma^{\alpha} \right]_{-} \Pi_{\alpha} = 2i \left[\gamma^{\nu}\Pi^{\lambda} - \gamma^{\lambda}\Pi^{\nu} \right] \tag{4.44}$$

Now, $\varepsilon_{\mu\nu\lambda\rho}\, \Pi^{\lambda}\Pi^{\rho}$ would be zero, but $\varepsilon_{\mu\nu\lambda\rho}\, \Pi^{\lambda} * \Pi^{\rho} \neq 0$. The *-product between two Π 's is found to be,

$$\Pi_{\mu} * \Pi_{\nu} = \Pi_{\mu}\, \Pi_{\nu} + \frac{\hbar}{2} \left[g_{\mu\nu} + e^{2} A_{\mu,\lambda} A_{\nu,}{}^{\lambda} + ieF_{\nu\mu} \right] \tag{4.45}$$

Notice that it contains a term antisymmetric in μ and ν (the $F_{\nu\mu}$ term), which survives multiplication by the $\varepsilon_{\mu\nu\lambda\rho}$. We showed in Chapter III that this expression is gauge invariant.

For clarity (especially when we get to the next section), we will break up (4.43) into two parts, I_{μ} and II_{μ} .

$$I_{\mu} \equiv -\frac{\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \Sigma^{\nu\lambda} * \dot{\Pi}^{\rho} = -\frac{e\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \Sigma^{\nu\lambda}\, \gamma_{\alpha}\, F^{\alpha\rho} \tag{4.46}$$

and,

$$II_{\mu} \equiv -\frac{\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \dot{\Sigma}^{\nu\lambda} * \Pi^{\rho} = -\frac{2i\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho} \left[\gamma^{\nu}\, \Pi^{\lambda} - \gamma^{\lambda}\, \Pi^{\nu} \right] * \Pi^{\rho}$$

$$= -\frac{ei\hbar}{2}\, \varepsilon_{\mu\nu\lambda\rho} \left[2\gamma^{\nu} \right] \Pi^{\lambda} * \Pi^{\rho} = \frac{ei\hbar}{2}\, \varepsilon_{\mu\nu\lambda\rho}\, \gamma^{\nu} F^{\lambda\rho} \tag{4.47}$$

(Only the antisymmetric part of the star product survives the $\varepsilon_{\mu\nu\lambda\rho}$ multiplication). The Bargmann, Michel, Telegdi equation of **[Bargmann59]** for a particle in a homogeneous electromagnetic field is,

$$\dot{S}_{\mu} = \frac{e}{m} \left[\left(\frac{g}{2} \right) F_{\mu\nu}\, S^{\nu} + \left(\frac{g}{2} - 1 \right) S_{\alpha}\, F^{\alpha\beta} \dot{x}_{\beta}\, \dot{x}_{\mu} \right] \tag{4.48}$$

If $g=2$,

$$\dot{S}_\mu = \frac{e}{m} F_{\mu\nu} S^\nu \qquad (4.49)$$

In Appendix E, we show that $<| I_\mu + II_\mu |>$ is in fact (weakly) equal to this (when sandwiched between $<|$ and $|>$):

$$<| \dot{S}_\mu |> = <| I_\mu + II_\mu |> = \frac{e}{m} <| F_{\mu\nu} S^\nu |> \qquad (4.50)$$

In order to see this, it is convenient to express everything in terms of the dual of the electromagnetic field tensor,

$$F^*_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \qquad (4.51)$$

Also,

$$F_{\mu\nu} = -\frac{1}{4} \varepsilon_{\mu\nu\lambda\rho} F^{*\lambda\rho} \qquad (4.52)$$

We find in Appendix E, that both sides of (4.50), when expressed in terms of $F^*_{\mu\nu}$ are equal to,

$$<| I_\mu + II_\mu |> = \frac{i\hbar m}{4} \left[<| \gamma^\gamma g_\mu{}^\delta - \frac{1}{2} \gamma^\gamma \gamma^\delta \gamma_\mu |> \right] F^*{}_{\gamma\delta} \qquad (4.53)$$

We have therefore derived the Bargmann, Michel, Telegdi equation, with $g = 2$ and we see that this equation therefore follows naturally from the Dirac mechanics. Our formalism is very similar to classical mechanics, and we have made no non-relativistic approximations.

The usual way to show the Dirac equation implies that $g=2$ is to separate out small and large components from the four component $|>$ and to make a non-relativistic approximation (see **[Bjorken64]** and **[Itzykson80]**, for example). Here, we have made no such approximation. Also, the usual method does not produce a spin precession equation, which has to be put in by hand. With the method described in this chapter, we obtain the spin equation hand in hand with obtaining the usual equations of motion.

Finally, we consider the $(g-2)$ term. The Dirac equation predicts that $g=2$, but because of radiative corrections which may be calculated in QED, the effective Dirac equation is modified, and produces a g different from 2. If we apply our method to such a modified Dirac Hamiltonian, we should obtain a spin precession equation with $g \neq 2$. We will do this in the next section.

D. The Modified Dirac Equation and the (g - 2) Terms

A modified Dirac Hamiltonian which contains radiative and other corrections calculated from QED may be written as:

$$H = \gamma^\nu \Pi_\nu - m - \frac{ie}{4m} \Delta \Sigma^{\alpha\beta} F_{\alpha\beta} \tag{4.54}$$

This is given, for example, in **[Schwinger70]** (see also [Bjorken64] and **[Itzykson80]**). Δ is the g factor anomaly, $(g/2 - 1)$. For one loop corrections, it would be given as $\alpha/2\pi$, where α is the dimensionless coupling constant $\approx 1/137$. We will consider only a homogeneous field here (as is done in **[Bargmann59]** and also as is usually assumed in the experiments). In the next chapter, we will write down a general formalism for any external field.

If $F_{\mu\nu}$ is homogeneous, so that $F_{\mu\nu,\sigma} = 0$, then the equations of motion (4.36) and (4.37) remain unchanged. In fact, the Moyal bracket given in (4.35) is still valid. We only get an extra contribution to $\dot{\Sigma}_{\mu\nu}$ of

$$- \frac{ie\Delta}{4m} \left[\Sigma_{\mu\nu} , \Sigma_{\alpha\beta} \right]_{-} F^{\alpha\beta} \tag{4.55}$$

Thus, the equation of motion for x_μ , equation (4.40) is the same, but there is an extra contribution to $\dot{S}_\mu$, namely

$$III_\mu = \frac{i\hbar e\Delta}{16m} \varepsilon_{\mu\nu\lambda\rho} \left\{ \Sigma^{\nu\lambda} , \Sigma^{\alpha\beta} \right\}_{-} F_{\alpha\beta} * \Pi^\rho \tag{4.56}$$

Evaluating the commutator and rearranging, we find,

$$III_\mu = \frac{e\hbar\Delta}{2m} \varepsilon_{\mu\nu\lambda\rho} \Sigma^{\nu\alpha} F_\alpha{}^\lambda \Pi^\rho \tag{4.57}$$

(note that $F_{\mu\nu} * \Pi_\rho = F_{\mu\nu} \Pi_\rho$) and it is shown in Appendix E, (E.23) that,

$$<| III_\mu |> =$$

$$\frac{e}{m} \Delta \left[<| F_{\mu\nu} S^\nu |> + <| S_\alpha F^{\alpha\beta} |> <| \gamma_\beta |> <| \gamma_\mu |> \right] \tag{4.58}$$

We therefore have that,

$$<| I_\mu + II_\mu + III_\mu |> =$$

$$\frac{e}{m} \left[<| F_{\mu\nu} S^\nu |> + \Delta \left[<| S_\alpha |> F^{\alpha\beta} \dot{x}_\beta \dot{x}_\mu \right] \right] \tag{4.59}$$

This is the Bargmann, Michel Telegdi equation (4.48) in terms of Δ instead of g, if $\Delta = (g/2 - 1)$. Therefore, for homogeneous fields, we do indeed find the correct g factor anomaly.

If the field had not been homogeneous, we would have had a more complicated equation on our hands, but would have been able to derive a similar expression. We consider this in the next chapter where we generalize to inhomogeneous fields.

V. The Dirac Mechanics:

Inhomogeneous Field

We will examine, in a general way, what corrections there are (if any) to the equations of motion of the previous chapter if we allow the field to be inhomogeneous (i.e. $F_{\mu\nu,\rho} \neq 0$). In the actual existing precession experiments for measuring $g - 2$, the field is assumed to be homogeneous (or at least that $F_{\mu\nu,\rho}$ is very small). The resonance experiments, remember, are analyzed non-relativistically, but do have an oscillating (inhomogeneous) field.

If we can calculate quantum corrections to the Bargmann, Michel, Telegdi equation (4.48), new precession experiments for measuring $g - 2$ which utilize an inhomogeneous field may suggest themselves. Also, there is some interest in the behavior of an electron gas in an inhomogeneous field in the study of neutron stars. Currently, such behavior is evaluated numerically in certain limits (see **[Achuthan82]**).

Bargmann, et al in **[Bargmann59]** and others have postulated a general spin precession, from which they obtain (4.48). It is:

$$\dot{S}_\mu = \left[\frac{ge}{2m} \right] \left[F_{\mu\nu} S^\nu + S_\alpha F^{\alpha\beta} \dot{x}_\beta \dot{x}_\mu \right] - \left[\ddot{x}^\nu S_\nu \dot{x}_\mu \right] \tag{5.1}$$

To the author's knowledge, this equation has not actually been tested experimentally, except in the homogeneous case. It is claimed that (5.1) is true in general for any interaction. However, with the formalism developed here, we see that the equation of motion for x_μ is obtained from the Dirac Hamiltonian, as is the equation of motion for the spin, S_μ. Thus, in principle, if the equation of motion for x_μ is different from that for the homogeneous case, the first term in equation (5.1) may also be. The only way to know what the spin precession equation is for a general interaction is to actually calculate it.

Also, knowing what we now know about gauge transformations in this mechanics (Chapter III, Section E), we immediately see that (5.1) *is not gauge invariant* except if the field is homogeneous! It is only gauge invariant to the lowest order in $\hbar$. If we replaced the ordinary products in (5.1) with *-products, it *would* be gauge invariant. Most of the quantum corrections we will obtain via calculation may also be immediately obtained by making this replacement.

Thus, gauge invariance alone will give us most of the corrections. However, we shall proceed to actually calculate them, at least to the next order in $\hbar$. If we can, we will make statements about quantities to all orders. We shall not attempt to interpret these extra terms or guess how they may be tested experimentally here. However, we should keep in mind that Δ or $(g/2-1)$ is of order $\hbar$, so that terms with an $\hbar$ AND a Δ , or an $\hbar$ AND higher powers of the field are probably too small to be detected. However, terms involving the inhomogeneity, $F_{\mu\nu,\lambda}$ may be made quite large. It may be possible to design an experiment where this term is made large enough to overcome being multiplied by $\hbar$ or Δ.

The first important fact we will show is that our expression for the spin precession for the unmodified Dirac equation (for $g = 2$), equation in terms of the γ_μ's, (4.53) is correct as it stands to all orders in $\hbar$, even for inhomogeneous fields. However, the expression for the precession in terms of the spin, the Bargmann, Michel, Telegdi equation (4.49), *DOES* have to be modified.

To see this, we look again at the calculation of I_μ and II_μ from the previous chapter. We had that

$$I_\mu = -\frac{\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \Sigma^{\nu\lambda} * \dot{\Pi}^\rho =$$

$$- \frac{e\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \Sigma^{\nu\lambda}\, \gamma_\alpha \left[\left(\frac{1}{i\hbar} \right) \left(\Pi^\rho * \Pi^\alpha - \Pi^\alpha * \Pi^\rho \right) \right] \tag{5.2}$$

and that

$$II_\mu = -\frac{\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \dot{\Sigma}^{\nu\lambda} * \Pi^\rho =$$

$$- \frac{2i\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho} \left(\gamma^\nu\, \Pi^\lambda - \gamma^\lambda\, \Pi^\nu \right) * \Pi^\rho \tag{5.3}$$

(Remember that the *-product of γ_μ with anything is the same as the ordinary product). Notice, that in both cases, any higher order quantum correction to what was found in the previous chapter can only come from terms like $\Pi_\mu * \Pi_\nu$. In fact, in both cases, only higher order terms which are antisymmetric in μ and ν will contribute. The order $\hbar^2$ correction to the *-product may be calculated from a generalization of (3.4) to be,

$$A * B = AB + \frac{\hbar}{2}\left[\frac{\partial A}{\partial x^\lambda}\frac{\partial B}{\partial x_\lambda} + \cdots\right] + \frac{\hbar^2}{8}\left[\frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial x^2} + \quad \cdots\right] + \cdots \tag{5.4}$$

The order $\hbar^2$ term contains second derivatives with respect to x_μ and p_μ. Here, we have that

$$\Pi_{\mu,\lambda} = -eA_{\mu\ \lambda} \ , \quad \frac{\partial \Pi_\mu}{\partial p^\lambda} = g_{\mu\lambda} \tag{5.5}$$

and that,

$$\frac{\partial^2 \Pi_\mu}{\partial x^2} = -eA_{\mu,\lambda}{}^{,\lambda} \ , \quad \frac{\partial^2 \Pi_\mu}{\partial p^2} = 0 \ , \quad \frac{\partial^2 \Pi_\mu}{\partial x^\lambda p_\lambda} = 0 \tag{5.6}$$

and so the only order $\hbar^2$ correction to $\Pi_\mu * \Pi_\nu$ is,

$$\hbar^2 \frac{e^2}{8} A_{\mu,\lambda}{}^{\lambda} A_{\nu,\sigma}{}^{\sigma} \tag{5.7}$$

The only higher order corrections will involve higher order derivatives of A_μ and will be symmetric in μ and ν , as in (5.7). Thus, they do not contribute to either the calculation of I_μ or II_μ. However, it will turn out that the connection between $<| I_\mu + II_\mu |>$ and $F_{\mu\nu}S^\nu$ does have quantum corrections, which are both higher order in $\hbar$ AND involve derivatives of the field.

Before considering this connection, let us first look at III_μ, which is given as,

$$III_\mu = \frac{e\hbar\Delta}{2m} \varepsilon_{\mu\nu\lambda\rho} \Sigma^{\nu\alpha} F_\alpha{}^\lambda * \Pi^\rho \tag{5.8}$$

Here, we may obtain higher order corrections from the *-product, $F_{\alpha\beta} * \Pi_\rho$. Using the definition of the *-product, we find that the correction to the next order in $\hbar$ is (remember that there is already one order of $\hbar$ in the definition of the spin),

$$\frac{ie\hbar^2\Delta}{4m} \varepsilon_{\mu\nu\lambda\rho} <| \Sigma^{\nu\alpha} \left[F_\alpha{}^{\lambda,\rho} + ieA^{\rho,\sigma} F_\alpha{}^\lambda{}_{,\sigma} \right] |> + \cdots \tag{5.9}$$

In Appendix F, we show that,

$$<| \Sigma^{\mu\nu} |> = \frac{2}{m\hbar} \varepsilon^{\mu\nu\lambda\rho} <| S_\lambda |><| \Pi_\rho |> \tag{5.10}$$

and that, therefore, the correction to the next order for $<| III_\mu |>$ may be expressed as,

$$\frac{i\hbar e\Delta}{2m^2} \begin{vmatrix} \mu & \lambda & \rho \\ \alpha & \beta & \gamma \end{vmatrix} <| S_\beta |><| \Pi_\gamma |> <| F_\alpha{}^{\lambda,\rho} + ieA^{\rho,\sigma} F_\alpha{}^\lambda{}_{,\sigma} |> \tag{5.11}$$

Here, we use (E.8) of Appendix E and the notation defined there. $\begin{vmatrix} \mu & \lambda & \rho \\ \alpha & \beta & \gamma \end{vmatrix}$ is a determinant which is defined in (E.7) and makes (5.11) antisymmetric in μ , λ , ρ. We notice that this term is proportional to $\hbar\Delta$ and is therefore very small.

We also obtain another correction to III_μ due to the fact that the equation of motion for Π_μ is modified. We shall call this term IV_μ. This is the type of correction one would ordinarily expect to arise from the second term of (5.1), and we will see that it is almost that, but not quite.

The equation of motion for Π_μ, (4.37) is modified when the field is inhomogeneous, because of the Δ term in the Hamiltonian. We now have,

$$\dot{\Pi}_\mu = e\gamma^\nu F_{\nu\mu} + \frac{ie\Delta}{4m}\Sigma^{\alpha\beta}F_{\alpha\beta,\mu} \tag{5.12}$$

This gives rise to the extra term, IV_μ:

$$IV_\mu = \left[-\frac{\hbar}{4}\right]\left[\frac{ie\Delta}{4m}\right]\varepsilon_{\mu\nu\lambda\rho}\Sigma^{\nu\lambda}\Sigma_{\alpha\beta}F^{\alpha\beta,\rho} \tag{5.13}$$

In Appendix F, we also show (equation (F.2)) that,

$$\varepsilon_{\mu\nu\alpha\beta}<|\ \Sigma^{\mu\nu}\ |> =$$

$$-\frac{4}{m\hbar}\Big[<|\ S_\alpha\ |><|\ \Pi_\beta\ |> - <|\ S_\beta\ |><|\ \Pi_\alpha\ |>\Big] \tag{5.14}$$

using this, we find that $<|\ IV_\mu\ |>$ may be expressed as,

$$<|\ IV_\mu\ |> = \frac{ie\Delta}{8m^2}<|\ S^\mu\Pi^\rho - S^\rho\Pi^\mu\ |> <|\ \Sigma_{\alpha\beta}\ |>F^{\alpha\beta}{}_{,\rho} \tag{5.15}$$

If we just substituted (5.12) into the general Bargmann, Michel, Telegdi equation (5.1), we would obtain almost this expression, except that the $1/2<|\ (S_\rho\Pi_\mu - S_\mu\Pi_\rho)\ |>$ in (5.15) would be replaced by $S_\mu\Pi_\nu$. This is the result people usually use when they claim to write down the spin precession equation for inhomogeneous fields. Equation (5.15) is the correct expression.

Finally, we have that the expression for $<| I_\mu + II_\mu |>$ and for $<| III_\mu |>$ when expressed in terms of S_μ will have to be modified if the field in inhomogeneous. The identities of Appendix D are still valid, but when we multiply by $F_{\mu\nu}$, as we do to put things in terms of S_μ in Appendix E, we must multiply with the *-product, not the ordinary product. We will then obtain corrections of higher order in $\hbar$ proportional to the derivative of $F_{\mu\nu}$.

Thus, in addition to the corrections (5.11) and (5.15), we obtain the higher order terms from the *-product in the generalization of (4.59),

$$<| I_\mu + II_\mu + III_\mu |> =$$

$$\frac{e}{m}\left[<| F_{\mu\nu} * S^\nu |> + \Delta \left[<| S_\alpha |> * F^{\alpha\beta} * \dot{x}_\beta * \dot{x}_\mu \right] \right] \tag{5.16}$$

These terms are precisely those necessary to make the Bargmann, Michel, Telegdi equation gauge invariant. In fact, (5.16) IS the Bargmann, Michel, Telegdi equation, with ordinary products replaced by *-products. The second term in (5.16) is proportional to higher powers of $\hbar$ times Δ and are therefore very small. We shall not evaluate them further. However, the correction arising from the first term (to order $\hbar^1$) is,

$$\frac{\hbar^2}{8}\varepsilon^{\nu}{}_{\lambda\rho\sigma} <| \Sigma^{\lambda\rho} |> <| F_{\mu\nu}{}^{,\sigma} + ieA^{\rho,\sigma}F_{\mu\nu,\lambda} |> \tag{5.17}$$

Using (F.2) from Appendix F, we may express this as,

$$\frac{i\hbar}{2} <| S^{\nu}\dot{x}^{\sigma} - S^{\sigma}\dot{x}^{\nu} |><| F_{\mu\nu,\sigma} + ieA_{\sigma,\rho}F_{\mu\nu}{}^{,\rho} |> \tag{5.18}$$

This term will be present even with $\Delta \to 0$ and is therefore an order $\hbar$ correction to the spin precession equation obtained from the usual Dirac Hamiltonian with no radiative corrections.

In summary, we obtain the following corrections to $\dot{S}_\mu$ when the field is not homogeneous:

1. The expression for $<| I_\mu + II_\mu |>$ in terms of the gamma matrices is unchanged to all orders of $\hbar$.

2. The expression for $<| III_\mu |>$ has the correction given in (5.11). It is of order $\hbar\Delta$ and smaller and involves derivatives of the field.

3. There is also another term, $<| IV_\mu |>$ given in (5.15). This is of order Δ (no $\hbar$) and is *almost* what people usually write down for an inhomogeneous field.

4. The rest of the corrections may be obtained by replacing the ordinary product by the *-product in the identities relating products of gamma matrices and the spin. This is equivalent to making this replacement in the standard Bargmann, Michel, Telegdi equation which is also exactly what is necessary to make that equation gauge invariant to the next order in $\hbar$. Even for the unmodified Dirac Hamiltonian ($\Delta = 0$), we obtain an order $\hbar^1$ correction, given by (5.18).

Thus, the two corrections obtained which are biggest, (5.15) and (5.18) are of order $\hbar$ or Δ. (5.15) is a correction from what people usually obtain, and (5.18) is the first quantum correction to the Bargmann, Michel, Telegdi equation, with $g = 2$.

If the inhomogeneity in the field, $F_{\mu\nu,\lambda}$ is large enough to overcome the factor of $\hbar$ or Δ, (5.15) and (5.18) should be observable. It is hoped that such an experiment could be designed.

VI. Conclusions and Future Work

We now have a complete semi-classical formalism for describing the mechanics of spin 1/2 particles. Using the language of *-products and Moyal brackets we may in principle calculate exact equations of motion and of spin precession to all orders of $\hbar$. We are able to use the fully relativistic Dirac Hamiltonian at all times and make *no non-relativistic approximations* as is usually done.

The usual way to demonstrate that the Dirac equation describes a particle with $g = 2$ is to break up the four component wave function into large and small components, make a non-relativistic approximation, and show one obtains a Hamiltonian with $g = 2$. Here, we have made no such approximations, and have shown that $g = 2$ by calculating the equation of motion of the spin. We obtain a spin precession equation, with the proper g value. If we consider a modified effective Dirac equation, with a term containing radiative and other corrections calculated from QED, we obtain in a similar way the $g - 2$ terms.

We find that the usual Bargmann, Michel, Telegdi spin precession equation is correct for homogeneous fields, but that there are quantum corrections for the inhomogeneous case. Most of these corrections may be obtained by replacing the ordinary product with the *-product, thus keeping gauge invariance. The usual equation is gauge invariant only for the homogeneous case.

With the *-product language, we have determined how to apply gauge transformations to classical quantities. The rule is to make the usual substitutions,

$$A_\mu \to A_\mu + \alpha_{,\mu}(x)$$

$$p_\mu \to p_\mu + e\alpha_{,\mu}(x)$$

*in the *-product.* This may turn out to be useful in finding quantum corrections to other classical equations.

Finally, we conclude by noting areas for future work which have suggested themselves during the course of this investigation:

1. It should be possible to apply this method to an electron in an external gravitational field by starting with the general relativistic Dirac equation. The standard general relativistic spin precession equation should be obtained, but with quantum corrections. It is possible that the effect of the corrections could be observed, possibly by affecting the results of an electromagnetic precession experiment.

2. It would be interesting to devise an experiment which might be able to detect the correction terms found in the previous chapter. If the inhomogeneity of the field were made quite high, they may be detectable.

3. The next step in the Dequantization program would be to dequantize the second-quantized Dirac equation, where the states are not restricted to the one particle ones. We would thus be able to derive the modification to the ordinary Dirac equation (i.e. the Δ term) in equation (4.54) directly, for example. The Heisenberg formulation of quantum field theory of Dirac (**[Dirac65]**) may be useful here.

4. Using the superspace mechanics derived in Chapter II, it would be interesting to study the thermodynamics of a supersymmetric gas of these particles. We should be able to obtain a thermodynamic potential which can be either Fermi-Dirac or Bose-Einstein, depending on the initial state of the gas. To the author's knowledge, this has not been done.

References

[Achuthan82] Achuthan, P., Benjamin, S., and Venkatesan, K, Journal of Physics A15, 3607 (1982)

[Agarwal70] Agarwal, G. S. and Wolf, E., Physical Review D2, #10, (1970), P 2161, 2187, 2206

[Bargmann59] Bargmann, V., Michel, L, and Telegdi, V.; Physics Review Letters 2, #10, 435 (1959)

[Bayen77] Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer, Letters in Mathematical Physics 1, 521 (1977)

[Bayen78] Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer, Annals of Physics 111, # 1, 61 and 111 (1978)

[Berezin66] Berezin, F.A., **The Method of Second Quantization**, p 49, Academic Press (1966)

[Born75] Born and Wolf, **Principles of Optics**, 5th. edition, Pergamon (1975)

[Bjorken64] Bjorken and Drell, **Relativistic Quantum Mechanics**, McGraw Hill, 1964; **Relativistic Quantum Fields**, McGraw Hill, 1965

[Brink81] Brink, L. and Schwarz, J., Physics Letters 100B, #4, 310 (1981)

[Casalbuoni76] Casalbuoni, R. Nuovo Cimento, 33A, #3, 389 (1976)

[deGroot72] de Groot, S.R. and Suttorp, L.G., **Foundations of Electrodynamics**, North-Holland, 1972

[Dirac64] Dirac, P.A.M., **Lectures on Quantum Mechanics**, Belfer Graduate School, Yeshiva University (1964)

[Dirac65] Dirac, P.A.M., **Lectures on Quantum Field Theory**, Belfer Graduate School, Yeshiva University (1965)

[Ferrara75] Ferrara, S. and Zumino, B. Nuclear Physics B87, 207 (1975)

[Fronsdal71] Fronsdal, C., Physics Review D4, #6, 1689 (1971)

[Fronsdal78] Fronsdal, C., Reports on Mathematical Physics 15, 111 (1978)

[Itzykson80] Itzykson and Zuber, **Quantum Field Theory**, Mcgraw-Hill, 1980

[Katz86A] Katz, A., Letters in Mathematical Physics 12, 329 (1986)

[Katz86B] Katz, A. *WKB Limit of the Dirac Equation and the g-2 Experiment*, UCLA preprint (UCLA/86/TEP/28), July, 1986

[Klauder60] Klauder, J. Annals of Physics 11, 123 (1960)

[Klauder68] Klauder, J. and Sudarshan, E. **Fundamentals of Quantum Optics**, W. A. Benjamin (1968)

[Klauder85] Klauder, J and Skagerstam, B, **Coherent States**, World Scientific, 1985. See pages 1-114 for an overview of coherent states and their uses

[Moyal49] Moyal, J.E. Proc. Cambridge Phil. Soc. 45, 99 (1949)

[Rich72] Rich, A. and Wesley, J. Reviews of Modern Physics, Vol. 44, #2 (1972)

[Schwinger70] Schwinger, J., **Particles, Sources, and Fields**, Volume I, Addison Wesley (1970)

[Snyder47] Snyder, H.S. Physics Review 71, 38 (1947)

[Sudarshan63] Sudarshan, E. C. G. Physical Review Letters 10, #7, 277 (1963)

[Wess83] Wess and Bagger, **Supersymmetry and Supergravity**, Princeton University Press, 1983

[Weyl31] Weyl, H. **The Theory of Groups and Quantum Mechanics**, Dover, 1931

[Wigner32] Wigner, E.P. Physical Review 40, 749 (1932)

Appendix A: Chapter II Notation

The notation given here is relevant mainly to Chapter II and may not apply to the other chapters. Notation for the other chapters is given in Chapter I. In particular, our choice of metric is different for Chapter II. This is because we there mainly follow the notation of **[Wess83]** (also of **[Katz86A]**) and use a 2-spinor language (we use 4-spinors at the end of that chapter).

We use Greek indices from the middle of the alphabet $(\mu,\nu,\cdots)$ on Lorentz 4-vectors which take the value 0,1,2,3. Ordinary 3-vectors have Latin] indices $(i,j,\cdots)$ which take the values 1,2,3. Two component Weyl spinors have Greek indices from the beginning of the alphabet $(\alpha,\beta,\gamma,\cdots)$ which may be dotted or undotted and take the value 1 or 2 and four component Dirac spinors have indices $a,b,c,\cdots$ which take the values 1,2,3,4. The standard Einstein summation convention is used throughout for vector indices, with repeated indices summed over, and the following convention is used for Weyl spinor indices:

$$\Psi\Psi = \Psi^{\alpha}\Psi_{\alpha} \ , \quad \overline{\Psi}\overline{\Psi} = \overline{\Psi}_{\dot{\alpha}}\overline{\Psi}^{\dot{\alpha}} \tag{A.1}$$

The metric (for Chapter II only!) is $g_{\mu\nu} = (-1,1,1,1)$ and:

$$\overline{\sigma}^{\mu\,\dot{\beta}\alpha} = (1,\sigma^{i}) \ , \quad \sigma^{\mu}{}_{\alpha\dot{\beta}} = (-1,\overline{\sigma}^{i}) \tag{A.2}$$

where σ^i are the ordinary Pauli matrices. Spinor indices are raised or lowered by the antisymmetric tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$, with $\varepsilon_{21} = \varepsilon^{12} = 1$, $\varepsilon_{12} = \varepsilon^{21} = -1$ and,

$$\varepsilon_{\alpha\beta}\,\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma \tag{A.3}$$

(and similarly for $\varepsilon_{\dot\alpha\dot\beta}$). Four component spinors may be related to two component ones through the following representation of the Dirac gamma matrices:

$$\gamma^\mu{}_{ab} = \begin{bmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{bmatrix} \quad (a,b = 1,2,3,4) \tag{A.4}$$

In this basis (called the Weyl basis), Dirac spinors contain two Weyl spinors, while Majorana spinors contain only one:

$$\textit{Dirac}:\ \Psi_a = \begin{bmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{bmatrix}, \quad \textit{Majorana}:\ \Psi^{(M)}{}_a = \begin{bmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{bmatrix} \tag{A.5}$$

Superfields are written as bold Greek letters ($\boldsymbol{\Phi}$). The eight coordinates of superspace are:

$$z = (\, x^\mu, \theta_\alpha, \bar\theta^{\dot\alpha} \,) \tag{A.6}$$

Here, the x_μ's are the ordinary four coordinates in spacetime and the θ's are the anticommuting spinor elements of a Grassman algebra. Because of the anticommutivity of the θ's, any superfield may be expanded into a finite power series in θ.

Appendix B: Superfield WKB

In relation to the work of Chapter II (and of **[Katz86A]**), we show that the superfield Φ can always be expressed in the form of equation (2.48):

$$\Phi = \mathbf{R}\, e^{i\mathbf{S}} \qquad \text{(B.1)}$$

where $\mathbf{R}$ and $\mathbf{S}$ are **Real** superfields. Let us expand Φ , $\mathbf{R}$, and $\mathbf{S}$ as shown:

$$\Phi = \Phi_0 + \Phi_\theta\, \theta + \Phi_{\bar\theta}\, \bar\theta + \Phi_{\theta\theta}\, \theta\theta + \cdots \qquad \text{(B.2)}$$

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}_\theta\, \theta + \mathbf{R}_{\bar\theta}\, \bar\theta + \mathbf{R}_{\theta\theta}\, \theta\theta + \cdots$$

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{S}_\theta\, \theta + \mathbf{S}_{\bar\theta}\, \bar\theta + \mathbf{S}_{\theta\theta}\, \theta\theta + \cdots \qquad \text{(B.3)}$$

The exponential is defined by its power series:

$$e^{i\mathbf{S}} = 1 + i\,\mathbf{S} - \frac{\mathbf{S}^2}{2\,!} - \frac{i\,\mathbf{S}^3}{3\,!} + \cdots \qquad \text{(B.4)}$$

Therefore, the 0th component of $e^{i\mathbf{S}}$ is:

$$e^{i\mathbf{S}}\Big]_0 = 1 + i\,\mathbf{S}_0 - \frac{{\mathbf{S}_0}^2}{2\,!} - \frac{i\,{\mathbf{S}_0}^3}{3\,!} + \cdots = e^{i\mathbf{S}_0} \qquad \text{(B.5)}$$

Now,

$$\Phi_0 = \mathbf{R}_0 \left[e^{i\mathbf{S}}\right]_0$$

$$\Phi_\theta = \mathbf{R}_\theta \left[e^{i\mathbf{S}}\right]_0 + \mathbf{R}_0 \left[e^{i\mathbf{S}}\right]_\theta$$

$$\Phi_{\theta\theta} = \mathbf{R}_{\theta\theta} \left[e^{i\mathbf{S}}\right]_0 + \mathbf{R}_\theta \left[e^{i\mathbf{S}}\right]_\theta + \mathbf{R}_0 \left[e^{i\mathbf{S}}\right]_{\theta\theta} \qquad \text{(B.6)}$$

and so on. $\mathbf{R}_0$ and $\mathbf{S}_0$ are completely determined as the real part and imaginary phase of Φ_0 via:

$$\Phi_0 = \mathbf{R}_0\, e^{i\mathbf{S}_0} \tag{B.7}$$

Once $\mathbf{R}_0$ and $\mathbf{S}_0$ have been determined, $\mathbf{R}_\theta$ and $\mathbf{S}_\theta$ may be determined from Φ_0 , $\mathbf{R}_0$, and $\mathbf{S}_0$ since:

$$\left[\mathbf{S}^2\right]_\theta = 2\mathbf{S}_\theta\mathbf{S}_0 \ , \quad \left[\mathbf{S}^3\right]_\theta = 3\mathbf{S}_\theta\mathbf{S}_0^{\,2} \ , \quad \cdots \tag{B.8}$$

so,

$$\left[e^{i\mathbf{S}}\right]_\theta = i\mathbf{S}_\theta - \frac{2\mathbf{S}_\theta\mathbf{S}_0}{2!} - \frac{3i\mathbf{S}_\theta\mathbf{S}_0^{\,2}}{3!} + \cdots = i\mathbf{S}_\theta e^{i\mathbf{S}_\theta} \tag{B.9}$$

and,

$$\Phi_\theta = i\Phi_0\mathbf{S}_\theta + \mathbf{R}_\theta e^{i\mathbf{S}_0} \tag{B.10}$$

Similarly, once $\mathbf{R}_\theta$, $\mathbf{S}_\theta$, $\mathbf{R}_0$, and $\mathbf{S}_0$ are known, $\mathbf{R}_{\theta\theta}$ and $\mathbf{S}_{\theta\theta}$ may be determined from $\Phi_{\theta\theta}$ and so on for all of the other components of Φ . Thus, one can always represent a complex superfield as in (B.1) where $\mathbf{R}$ and $\mathbf{S}$ are real superfields.

It should also be noted that (B.10) shows the correct expansion for a Weyl spinor, which is what Φ_θ is. This, together with the analogous expression for $\Phi_{\bar{\theta}}$, thus shows what the WKB approximation for the Dirac spinor (which is two Weyl spinors) should be. This is used in Chapter II, Section E.

Appendix C: Coherent States

Here, we outline the definition of and some properties of coherent states. Derivations and proofs, as well as much more information can be found in **[Klauder68]** and **[Klauder85]** and references therein.

1. Definition

Coherent states can be defined in many ways. Here, we will deal with only one degree of freedom, the generalization to many is straightforward.

We consider a one dimensional harmonic oscillator, with states $|n\rangle$, eigenstates of the number operator $\hat{N} = \hat{a}^+\hat{a}$. $\hat{a}^+$ and $\hat{a}$ are the usual creation and annihilation operators which satisfy the commutation relationship $\left[\hat{a}, \hat{a}^+\right]_- = 1$. We may define the coherent states as:

$$|z\rangle = e^{\left[z\hat{a}^+ - \bar{z}\hat{a}\right]}|0\rangle = e^{-\frac{1}{2}z\bar{z}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \qquad \text{(C.1)}$$

for any complex number z. It is generally useful to consider the real and imaginary parts of z. We let $z = 1/\sqrt{2\hbar}(q + ip)$. The coherent states are then given by:

$$|z\rangle = |p,q\rangle = e^{-\frac{1}{4}\hbar(p^2+q^2)} \sum_{n=0}^{\infty} \frac{(q + ip)^n}{(2\hbar)^{\frac{n}{2}}(n!)^{\frac{1}{2}}} |n\rangle \qquad \text{(C.2)}$$

To generalize the definition to K degrees of freedom, we have K $\hat{a}$'s, K

$\hat{a}^+$'s, and K N_k 's ($k=1,..K$); with eigenvectors $|n_1,n_2,\cdots,n_k> = |\{n_k\}>$.
The many-dimensional coherent states are then defined as:

$$|\{n_k\}> = \sum_{\{n_k\}=0}^{\infty} \left[\prod_{i=1}^{K} e^{-\frac{1}{2}z_i z_i^*} \frac{z_i^{\,n_i}}{\sqrt{n_i!}} \, |\{n_k\}> \right] \tag{C.3}$$

2. Properties

We list here the basic properties of the coherent states. We will write them for one dimension, but they hold true for any number (even, essentially, for an infinite number of degrees of freedom).

Ground state for a Displaced or Driven Oscillator

If $|n=0>$ is the ground state for an simple harmonic oscillator, with vanishing mean position and momentum:

$$<0|\hat{p}|0> = 0 \ , \ <0|\hat{q}|0> = 0 \tag{C.4}$$

then the coherent state $|p,q>(= |z>)$ is the ground state for both the displaced oscillator ($\hat{p}$ and $\hat{q}$ displaced by p and q) and the forced harmonic oscillator (a forcing term of $\hat{p}p + \hat{q}q$ added to the free Hamiltonian).

Minimum Uncertainty States

The coherent states satisfy the minimum Heisenberg uncertainty states and are therefore the *most classical* states. If $\Delta\hat{A} = \hat{A} - <z|\hat{A}|z>$ for any operator $\hat{A}$, then

$$\sqrt{<z|\Delta q^2|z> \ <z|\Delta p^2|z>} = \Delta q \ \Delta p = \frac{\hbar}{2} \tag{C.5}$$

which *equalizes* the uncertainty relation.

Annihilation Operator Eigenstates

$$\hat{a}|z> = z|z> \tag{C.6}$$

so that $<z|\hat{a}|z> = z$ and $<z|\hat{a}^+|z> = \bar{z}$. This provides a natural mapping of normally ordered products of $\hat{a}$ and $\hat{a}^+$, since,

$$<z|\hat{a}^{+n}\ \hat{a}^m|z> = \bar{z}^n\ z^m \tag{C.7}$$

Resolution of the Identity

The *Resolution of the Identity* operator, I, is given by:

$$I = \frac{1}{\pi}\int d^2z\ |z><z| \tag{C.8}$$

($d^2z \equiv d(\text{Im } z)\ d(\text{Re } z)$). Thus, arbitrary matrix elements may be represented in terms of the coherent states (and arbitrary wave functions by coherent state wave functions), since therefore,

$$<\varphi|\psi> = \frac{1}{\pi}\int d^2z\ <\varphi|z><z|\psi> \tag{C.9}$$

Overcompleteness

Relation (C.8) also implies that:

$$|z'> = \frac{1}{\pi}\int d^2z\ |z><z|z'> \tag{C.10}$$

Thus, the coherent states form an overcomplete set.

Most applications of coherent states use them as a basis to represent quantum-mechanical wave functions. We do not use this property here, but mainly use the fact that these states naturally map normal ordered products of operators and therefore provide a nice way of mapping functions of $\hat{p}_k$ and $\hat{q}_k$.

The concept of coherent states can be generalized further, such as in the case of *spin coherent states* and *group coherent states*, but we are not concerned with these here (see **[Klauder85]**).

Appendix D: Phase Space Identities

Here, we derive a number of useful identities relating the phase space variables γ_μ and $\Pi_\mu = (p - eA)_\mu$. These identities also apply to the free case, if Π_μ is replaced by p_μ .

These relationships are only true for the expectation values, and are only weakly true (in the sense of Dirac's treatment of constraints). They may only be applied at the end of all calculations.

We begin with the basic constraint of the theory; the Dirac constraint:

$$\gamma_\nu \Pi^\nu |> \approx m |> \tag{D.1}$$

If we multiply on the left by one gamma matrix, γ_ν , and then by $<|$, we obtain:

$$<|\gamma_\mu \gamma_\nu \Pi^\nu|> = m<|\gamma_\mu|> \tag{D.2}$$

Anticommuting the γ 's, (since $(\gamma_\mu, \gamma_\nu)_+ = 2g_{\mu\nu}$) , we obtain:

$$<|(2g_{\mu\nu} - \gamma_\nu\gamma_\mu) \Pi^\mu|> = m<|\gamma_\mu|> \tag{D.3}$$

and, applying the constraint on the left (on $<|$), we find,

$$<|\Pi_\mu|> = m<|\gamma_\mu|> \tag{D.4}$$

which is the usual relationship between Π_μ and the 4-velocity (which is γ_μ).

If we follow the same method, but multiply the constraint (D.1) by $<|\ \gamma_\lambda \gamma_\mu$, then anticommute the γ 's to the left, we obtain,

$$<|\gamma_\lambda\, \Pi_\mu|> \; = \; <|\gamma_\mu\, \Pi_\lambda|> \qquad \text{(D.5)}$$

which we knew anyway since, because Π_μ is proportional to the unit matrix, and using (D.4)

$$<|\gamma_\lambda\, \Pi_\mu|> \; = \; <|\gamma_\lambda|>\Pi_\mu \; = \; \frac{1}{m}\, \Pi_\lambda \Pi_\mu \; = \; <|\gamma_\mu|>\Pi_\lambda$$

A very useful relation is obtained when multiplying (D.1) by three γ 's. Anticommuting them through, we obtain,

$$<|\gamma_\alpha \gamma_\beta \Pi_\gamma|> \; - \; <|\gamma_\alpha \gamma_\gamma \Pi_\beta|> \; + \; <|\gamma_\beta \gamma_\gamma \Pi_\alpha|> \; = \; m\; <|\gamma_\alpha \gamma_\beta \gamma_\gamma|> \qquad \text{(D.6)}$$

Often in expressions involving the spin, we have three γ 's multiplied by the totally antisymmetric $\varepsilon^{\mu\nu\lambda\rho}$. (D.5) then implies that

$$3\; \varepsilon^{\mu\nu\lambda\rho}\; <|\gamma_\nu \gamma_\lambda \Pi_\rho|> \; = \; m\; \varepsilon^{\mu\nu\lambda\rho}\; <|\gamma_\alpha \gamma_\beta \gamma_\gamma|> \qquad \text{(D.7)}$$

The factor of 3 is very important for obtaining the correct expression for the spin precession equation.

One other relation will prove useful. Again, since Π_μ is proportional to the unit matrix,

$$<|\gamma_\alpha|> \; <|\gamma^\alpha \gamma_\mu \gamma^\nu|> \; = \; \frac{1}{m}\, \Pi_\alpha <|\gamma^\alpha \gamma^\mu \gamma^\nu|>$$

$$= \frac{1}{m}\; <|\Pi_\alpha\, \gamma^\alpha \gamma^\mu \gamma^\nu|> \; = \; <|\gamma^\mu \gamma^\nu|> \qquad \text{(D.8)}$$

Appendix E: Calculations for Chapter IV

Here, we provide detailed calculation which lead to equation (4.53) and (4.58) in Chapter IV. We will be utilizing the phase space relationships derived in Appendix D.

We first wish to show that $<| I_\mu + II_\mu |>$ from Chapter IV is equal to,

$$\frac{e}{m} <| F_{\mu\nu} S^\nu |> \tag{E.1}$$

which is the **[Bargmann59]** result for $g = 2$. In order to show this, we must express everything in terms of the duel of the electromagnetic field tensor, $F_{\mu\nu}$,

$$F^*_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \tag{E.2}$$

with $\varepsilon_{\mu\nu\lambda\rho}$ the totally antisymmetric Levi-Civita tensor, and $\varepsilon_{\mu\nu\lambda\rho}\, \varepsilon^{\mu\nu\lambda\rho} = -24$. Since,

$$\varepsilon_{\mu\nu\lambda\rho}\, \varepsilon^{\mu\alpha\beta\gamma} = -\det({g_a}^b) \qquad \begin{array}{l} a = \nu,\lambda,\rho \\ b = \alpha,\beta,\gamma \end{array} \tag{E.3}$$

which is given, for example, in **[Itzykson80]** , page 692, we find that,

$$F_{\mu\nu} = -\frac{1}{4}\varepsilon_{\mu\nu\lambda\rho} F^{*\lambda\rho} \tag{E.4}$$

I_μ is given in (4.46) as,

$$I_\mu = -\frac{e\hbar}{4}\, \varepsilon_{\mu\nu\lambda\rho}\, \Sigma^{\nu\lambda}\, \gamma_\alpha F^{\alpha\rho} \tag{E.5}$$

We may write this as,

$$- \frac{ei\hbar}{4} \varepsilon_{\mu\nu\lambda\rho} \gamma^{\nu} \gamma^{\lambda} \gamma_{\alpha} F^{\alpha\rho} \tag{E.6}$$

because of the antisymmetry of $\Sigma^{\nu\lambda}$. Let us write the determinant in (E.3) as,

$$\begin{vmatrix} \nu & \lambda & \rho \\ \alpha & \beta & \gamma \end{vmatrix} \equiv g_{\nu}{}^{\alpha} g_{\lambda}{}^{\beta} g_{\rho}{}^{\gamma} + g_{\nu}{}^{\beta} g_{\lambda}{}^{\gamma} g_{\rho}{}^{\alpha} + g_{\nu}{}^{\gamma} g_{\lambda}{}^{\alpha} g_{\rho}{}^{\beta}$$

$$- g_{\nu}{}^{\gamma} g_{\lambda}{}^{\beta} g_{\rho}{}^{\alpha} - g_{\nu}{}^{\beta} g_{\lambda}{}^{\alpha} g_{\rho}{}^{\gamma} - g_{\nu}{}^{\alpha} g_{\lambda}{}^{\gamma} g_{\rho}{}^{\beta} \tag{E.7}$$

This will turn out to be a bit clearer notation. (Notice we will take cyclic combinations with a + sign, and anticyclic combinations with a - sign and then add). So, (E.3) is written

$$\varepsilon_{\mu\nu\lambda\rho}\, \varepsilon^{\mu\alpha\beta\gamma} = - \begin{vmatrix} \nu & \lambda & \rho \\ \alpha & \beta & \gamma \end{vmatrix} \tag{E.8}$$

Using (E.4) and (E.8), we have that,

$$<| I_{\mu} |> = - \frac{i\hbar e}{16} \begin{vmatrix} \mu & \nu & \lambda \\ \gamma & \delta & \alpha \end{vmatrix} <| \gamma^{\nu} \gamma^{\lambda} \gamma_{\alpha} |> F^{*}{}_{\gamma\delta} \tag{E.9}$$

Working through the determinant, we obtain,

$$<| I_{\mu} |> = \frac{i\hbar e}{4} \left[<| 3\gamma^{\gamma} g_{\mu}{}^{\delta} - \frac{1}{2} \gamma^{\gamma} \gamma^{\delta} \gamma_{\mu} |> \right] F^{*}{}_{\gamma\delta} \tag{E.10}$$

We now turn to II_{μ} . It is (from (4.47)),

$$II_{\mu} = \frac{ei\hbar}{2} \varepsilon_{\mu\nu\lambda\rho} \gamma^{\nu} F^{\lambda\rho} \tag{E.11}$$

by the definition of $F^{*}{}_{\mu\nu}$, (E.2), we see immediately that,

$$II_{\mu} = \frac{ei\hbar}{2} \gamma^{\nu} F^{*}{}_{\mu\nu} = \frac{i\hbar e}{4} \left[-2\gamma^{\gamma} g_{\mu}{}^{\delta} \right] F^{*}{}_{\gamma\delta} \tag{E.12}$$

Thus,

$$<| I_\mu + II_\mu |> = \frac{ei\hbar}{4}\left[<| \gamma^\gamma g_\mu{}^\delta - \frac{1}{2}\gamma^\gamma \gamma^\delta \gamma_\mu |> \right] F^*{}_{\gamma\delta} \tag{E.13}$$

Now we need to see if this is the same as (E.1). We have from the definition of the spin, (4.41) and the phase space relation (D.7) from Appendix D, that

$$<| S_\mu |> = -\frac{i\hbar m}{12}\varepsilon_{\mu\nu\lambda\rho} <| \gamma^\nu \gamma^\lambda \gamma^\rho |> \tag{E.14}$$

Thus,

$$<| F_{\mu\nu} |> <| S^\nu |> = \frac{i\hbar m}{48} <| \varepsilon^{\mu\nu\alpha\beta} F^*{}_{\alpha\beta} |> <| \varepsilon_{\nu\lambda\rho\sigma}\gamma^\lambda \gamma^\rho \gamma^\sigma |>$$

$$= -\frac{i\hbar m}{48}\begin{vmatrix} \lambda & \rho & \sigma \\ \alpha & \beta & \mu \end{vmatrix} <| \gamma^\lambda \gamma^\rho \gamma^\sigma F^*{}_{\alpha\beta} |> \tag{E.15}$$

and, evaluating the determinant and multiplying by $\frac{e}{m}$,

$$\frac{e}{m} <| F_{\mu\nu} S^\nu |> = \frac{ei\hbar}{4}\left[<| \gamma^\gamma g_\mu{}^\delta - \frac{1}{2}\gamma^\gamma \gamma^\delta \gamma_\mu |> \right] F^*{}_{\gamma\delta} \tag{E.16}$$

which is the same as $<| I_\mu + II_\mu |>$ in (E.13). Note that we use $<| F_{\mu\nu} |>$ and $<| S_\nu |>$, since that is what is actually observed.

We now verify the $g - 2$ term, III_μ. We have from (4.57) that,

$$III_\mu = \frac{e\hbar\Delta}{2m}\varepsilon_{\mu\nu\lambda\rho}\Sigma^{\nu\alpha} F_\alpha{}^\lambda \Pi^\rho \tag{E.17}$$

again, we write everything in terms of $F^*{}_{\mu\nu}$, using (E.4) and (E.8) and the definition of $\Sigma_{\mu\nu}$, we have,

$$III_\mu = \frac{ei\hbar\Delta}{4m} \varepsilon_{\mu\nu\lambda\rho} \left[\gamma^\nu , \gamma^\alpha \right]_- \left[-\frac{1}{4} \right] \varepsilon_\alpha{}^{\lambda\gamma\delta}\, \Pi^\rho F^*{}_{\gamma\delta}$$

$$= \frac{ei\hbar\Delta}{16m} \begin{vmatrix} \mu & \nu & \rho \\ \alpha & \gamma & \delta \end{vmatrix} \left[\gamma^\nu , \gamma^\alpha \right]_- \Pi^\rho F^*{}_{\gamma\delta} \tag{E.18}$$

Evaluating the determinant, we obtain,

$$\frac{ei\hbar\Delta}{4} \left[\gamma^\gamma g_\mu{}^\delta - \frac{1}{2}\gamma^\gamma \gamma^\delta \gamma_\mu + \frac{1}{2m} \gamma^\gamma \gamma^\delta \Pi_\mu \right] F^*{}_{\gamma\delta} \tag{E.19}$$

We notice that the first two terms are essentially $<| F_{\mu\nu} S^\nu |>$, from (E.16). The third term may be related to $S_\alpha F^{\alpha\beta} \dot{x}_\beta \dot{x}_\mu$ as follows:

$$<| S_\alpha |> <| F^{\alpha\beta} |> <| \gamma_\beta |> <| \gamma_\mu |>$$

$$= \frac{i\hbar}{48} \varepsilon_{\alpha\nu\lambda\rho} <| \gamma^\nu \gamma^\lambda \gamma^\rho |> \varepsilon^{\alpha\beta\gamma\delta} F^*{}_{\gamma\delta} <| \gamma_\beta |> <| \gamma_\mu |>$$

$$= \frac{i\hbar}{48} \begin{vmatrix} \nu & \lambda & \rho \\ \beta & \gamma & \delta \end{vmatrix} <| \gamma^\nu \gamma_\lambda \gamma^\rho |> <| \gamma_\beta |> <| \gamma_\mu |> F^*{}_{\gamma\delta} \tag{E.20}$$

We obtain,

$$\frac{i\hbar}{4} \Big[<| \gamma^\gamma |><| \gamma^\delta |><| \gamma_\mu |>$$

$$+ \frac{1}{2} <| \gamma_\beta |><| \gamma^\beta \gamma^\gamma \gamma^\delta |><| \gamma_\mu |> \Big] F^*{}_{\gamma\delta} \tag{E.21}$$

The first term is zero, by symmetry, and the second is,

$$\frac{i\hbar}{8} <| \gamma^\gamma \gamma^\delta |> \Pi_\mu F^*{}_{\gamma\delta} \tag{E.22}$$

where we have used phase space relation (D.4) and (D.8), from Appendix D. Thus, we have that,

$$< | III_\mu | > =$$

$$\frac{e}{m} \Delta \left[< | F_{\mu\nu} S^\nu | > + < | S_\alpha F^{\alpha\beta} | > < | \gamma_\beta | > < | \gamma_\mu | > \right] \tag{E.23}$$

and, therefore, that

$$< | I_\mu + II_\mu + III_\mu | > =$$

$$\frac{e}{m} \left[< | F_{\mu\nu} S^\nu | > + \Delta \left(< | S_\alpha | > F^{\alpha\beta} \dot{x}_\beta \dot{x}_\mu \right) \right] \tag{E.24}$$

Appendix F: Calculations for Chapter V

Here, we will show that,

$$<| \Sigma^{\mu\nu} |> = \frac{2}{m\hbar} \varepsilon^{\mu\nu\lambda\rho} <| S_\lambda |><| \Pi_\rho |> \tag{F.1}$$

and therefore, that,

$$\varepsilon_{\mu\nu\alpha\beta} <| \Sigma^{\mu\nu} |> =$$

$$-\frac{4}{m\hbar}\Big(<| S_\alpha |><| \Pi_\beta |> - <| S_\beta |><| \Pi_\alpha |> \Big) \tag{F.2}$$

We will be using the same notation as in Appendix D and E. We know from (E.14), that,

$$<| S_\mu |> = -\frac{i\hbar m}{12} \varepsilon_{\mu\nu\lambda\rho} <| \gamma^\nu \gamma^\lambda \gamma^\rho |> \tag{F.3}$$

Thus,

$$\varepsilon^{\mu\nu\lambda\rho} <| S_\lambda |><| \Pi_\rho |> = \frac{i\hbar}{12} \begin{bmatrix} \mu & \nu & \rho \\ \alpha & \beta & \gamma \end{bmatrix} <| \gamma^\alpha \gamma^\beta \gamma^\gamma |><| \Pi_\rho |> \tag{F.4}$$

We evaluate the determinant and, using the Dirac constraint, (D.1), find this equal to,

$$\frac{i\hbar m}{4} <| \Big[\gamma^\mu , \gamma^\nu \Big]_- |> \tag{F.5}$$

or, with the definition of $\Sigma_{\mu\nu}$, we obtain (F.1)

We now multiply by $\varepsilon_{\mu\nu\alpha\beta}$, and obtain,

$$\varepsilon_{\mu\nu\alpha\beta} <| \Sigma^{\mu\nu} |> = -\frac{2}{m\hbar} \begin{vmatrix} \nu & \alpha & \beta \\ \nu & \lambda & \rho \end{vmatrix} <| S_\lambda |><| \Pi_\rho |> \qquad \text{(F.6)}$$

Evaluating the determinant, we obtain (F.2)