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Bohm's potential, classical/quantum duality and repulsive gravity

Carlos Castro Perelman [a,b]

[a] Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta, GA, 30314, United States of America
[b] Ronin Institute, 127 Haddon Place, Montclair, NJ, 07043, United States of America

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ABSTRACT

We propose the notion of a classical/quantum *duality* in the gravitational case (it can be extended to other interactions). By this one means *exchanging* Bohm's quantum potential for the classical potential $V_Q \leftrightarrow V$ in the stationary quantum Hamilton–Jacobi equation (QHJE) so that $V_Q + V = -V_0$ (ground state energy). Despite that the corresponding Schrödinger equations, and their solutions differ, their associated quantum Hamilton–Jacobi equation, and *ground* state energy remains the *same*. This is how the classical/quantum duality is implemented. In this scenario Bohm's quantum potential (which coincides with the attractive Newtonian potential) is now correlated to a classical *repulsive* gravitational potential (plus a constant). These results suggest that there might be a *quantum* origin to the classical *repulsive* gravitational behavior (of the accelerated expansion) of the universe which is based on this notion of classical/quantum *duality*. We hope that the notion of classical/quantum duality raised in this work in connection to the QHJE may cast further light into the deep interplay between gravity and quantum mechanics.

David Bohm showed long ago [1] that the Schrödinger equation for the complex valued wave function $\Psi(\vec{r},t)$ is equivalent to the coupled pair of equations

$$-\frac{\partial S}{\partial t} = \frac{(\vec{p})^2}{2m} + V_Q + V = \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + V \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\frac{\nabla S}{m}) = 0 \tag{2}$$

The first equation is the Quantum Hamilton–Jacobi Equation (QHJE) involving an external potential $V(\vec{r})$, and including Bohm's quantum potential $V_Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$ (ρ is the probability density and S is the action). The second equation is the continuity equation. The substitution

$$\Psi(\vec{r},t) \equiv \sqrt{\rho(\vec{r},t)}e^{iS(\vec{r},t)/\hbar} \tag{3}$$

into eqs. (1)–(2) yields the Schrödinger equation

$$i\hbar\,(\frac{\partial\Psi(\vec{r},t)}{\partial t}) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t)) + V(\vec{r})\Psi(\vec{r},t) \tag{4}$$

Bohm's quantum potential $V_Q = -\frac{\hbar^2}{2m}(\nabla^2\sqrt{\rho}/\sqrt{\rho})$ was shown to be *proportional* to the difference of the Weyl and Riemann scalar spatial curvature produced by an ensemble density of paths associated with one, and only one particle, as shown in [2]. The constant of proportionality is $-\frac{\hbar^2}{2m}$. It can be generalized to the relativistic case. This geometrization process of quantum mechanics (not to be confused with geometric quantization) allowed to derive the Schrödinger, Klein–Gordon [2] and Dirac equations [3–5]. Most recently, a related geometrization of quantum mechanics was proposed [6] that describes the time evolution of particles as geodesic lines in a curved space, whose curvature is induced by the quantum potential. This formulation allows therefore the incorporation of all quantum effects into the geometry of space–time, as it is the case for gravitation in the general relativity.

The above result of Bohm can be generalized to *many* particles as well where the wavefunction depends on *all* of the particle coordinates (configuration space). In Bohmian mechanics the time evolution of a quantum system comprised of N particles is driven by the nonlocal quantum potential $V_Q(\vec{r}_1,\vec{r}_2,\cdots,\vec{r}_N;t)$ which is a function of the entire configuration space and time. In the case of two particles [9], it is convenient to express all physical quantities in terms of the center of mass coordinate $\vec{R}$, and the relative radial coordinate $\vec{r}$ given by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2},\ \ \vec{r} = \vec{r}_1 - \vec{r}_2 \tag{5}$$

E-mail address: perelmanc@hotmail.com.

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the total mass M and reduced mass m are

$$M = m_1 + m_2, \quad m = \frac{m_1 m_2}{m_1 + m_2} \tag{6}$$

When the temporal dependence of the action is of the form

$$S = S(\vec{r}, \vec{R}, t) = S(\vec{r}, \vec{R}) - E\, t \tag{7}$$

the *stationary* quantum Hamilton–Jacobi equation (QHJE) associated with the two particles can be decomposed in terms of the motion of the center of mass plus the motion relative to the center of mass as follows

$$E = \frac{(\nabla_r S)^2}{2m} + \frac{(\nabla_R S)^2}{2M} - \frac{\hbar^2}{2m}\frac{\nabla_r^2\sqrt{\rho}}{\sqrt{\rho}} - \frac{\hbar^2}{2M}\frac{\nabla_R^2\sqrt{\rho}}{\sqrt{\rho}} + V(\vec{r}) \tag{8}$$

$$\rho(\vec{r}, \vec{R}) = \sigma(\vec{r})\zeta(\vec{R}) = \sigma(r,\theta,\phi)\zeta(R,\theta_{cm},\phi_{cm}), \quad R = |\vec{R}|, \quad r = |\vec{r}| \tag{9}$$

$$S(\vec{r}, \vec{R}) = S_1(\vec{r}) + S_2(\vec{R}) \tag{10}$$

To simplify matters we shall freeze the angular and temporal dependence of the physical quantities and focus solely on the radial dependence only. Hence, the functional dependence of ρ and S simplifies and reduces to the form

$$\rho(\vec{r}, \vec{R}) = \sigma(r)\zeta(R), \quad S(\vec{r}, \vec{R}) = S_1(r) + S_2(R) \tag{11}$$

so that

$$\frac{\hbar^2}{2m\sqrt{\rho(r,R)}}\frac{1}{r^2}\partial_r(r^2\partial_r\sqrt{\rho(r,R)}) = \frac{\hbar^2}{2m\sqrt{\sigma(r)}}\frac{1}{r^2}\partial_r(r^2\partial_r\sqrt{\sigma(r)}) \tag{12}$$

$$\frac{\hbar^2}{2M\sqrt{\rho(r,R)}}\frac{1}{R^2}\partial_R(R^2\partial_R\sqrt{\rho(r,R)}) = \frac{\hbar^2}{2M\sqrt{\zeta(R)}}\frac{1}{R^2}\partial_R(R^2\partial_R\sqrt{\zeta(R)}) \tag{13}$$

The continuity equation in the stationary case is

$$\nabla_r(\rho\frac{\nabla_r S}{m}) + \nabla_R(\rho\frac{\nabla_R S}{M}) = 0 \tag{14}$$

Introducing a potential of the form $V = V(r)$, separating (decoupling) the motion of the center of mass from the motion relative to the center of mass, and inserting the expressions in eq. (12)–(13), leads to the following stationary QHJEs, and continuity equations

$$E - \mathcal{E} = \frac{(\nabla_r S_1(r))^2}{2m} - \frac{\hbar^2}{2m\sqrt{\sigma(r)}}\frac{1}{r^2}\partial_r(r^2\partial_r\sqrt{\sigma(r)}) + V(r) \tag{15}$$

$$\mathcal{E} = \frac{(\nabla_R S_2(R))^2}{2M} - \frac{\hbar^2}{2M\sqrt{\zeta(R)}}\frac{1}{R^2}\partial_R(R^2\partial_R\sqrt{\zeta(R)}) \tag{16}$$

$$\nabla_r(\,\sigma(r)\,\frac{\nabla_r S_1(r)}{m}\,) = 0, \quad \nabla_R(\,\zeta(R)\,\frac{\nabla_R S_2(R)}{M}) = 0 \tag{17}$$

where $\mathcal{E}$ is the energy associated with the center of mass motion.

A careful inspection reveals that the following solutions (below we shall explain the physical origin of these solutions)

$$\sigma(r) = \sigma_o e^{Ar}, \ (A<0), \quad \zeta(R) = \zeta_o\frac{e^{BR}}{R^2}, (B<0) \tag{18}$$

with σ_o, ζ_o constants, yield the following quantum Bohm potentials

$$-\frac{\hbar^2}{2m\sqrt{\sigma(r)}}\frac{1}{r^2}\partial_r(r^2\partial_r\sqrt{\sigma(r)}) = -\frac{\hbar^2}{2m}(\frac{A}{r} + \frac{A^2}{4}) \tag{19}$$

$$-\frac{\hbar^2}{2M\sqrt{\zeta(R)}}\frac{1}{R^2}\partial_R(R^2\partial_R\sqrt{\zeta(R)}) = -\frac{\hbar^2}{2m}(\frac{B^2}{4}) \tag{20}$$

$A<0, B<0$ must be *negative* due to the normalization condition

$$\int_0^\infty \sigma(r)4\pi r^2 dr = 1, \quad \int_0^\infty \zeta(R)4\pi R^2 dR = 1 \tag{21}$$

otherwise the integrals would diverge. Despite that $\zeta(R)$ diverges at $R=0$ it is normalizable.

Because $A<0$, the quantum Bohm potential in eq. (19) leads to a *repulsive* gravitational potential plus a constant (a zero-point energy)

$$-\frac{\hbar^2}{2m}(\frac{A}{r} + \frac{A^2}{4}) = \frac{Gm_1m_2}{r} - V_0, \quad V_0 = \frac{\hbar^2 A^2}{8m} \tag{22}$$

From eq. (22) one learns that

$$A = -\frac{2G}{\hbar^2}m_1m_2m = \ -\frac{2G}{\hbar^2}\frac{(m_1m_2)^2}{m_1+m_2} \Rightarrow$$
$$V_0 = \frac{1}{2\hbar^2}\,m(Gm_1m_2)^2 \tag{23}$$

The quantum Bohm potential is tantamount of a *repulsive* gravitational potential (plus a constant), and is cancelled out by the *attractive* Newtonian gravitational potential $V = V(r) = -(Gm_1m_2/r)$ stemming from the gravitational interaction of the two particles of masses m_1, m_2. In doing so, the stationary QHJE (15) becomes effectively a classical-like stationary Hamilton–Jacobi equation for a free particle with a *shifted* energy

$$E - \mathcal{E} + V_0 = \frac{(\nabla_r S_1(r))^2}{2m} \tag{24}$$

and from which one learns

$$p_r = \partial_r S_1(r) = \sqrt{2m(E - \mathcal{E} + V_0)} \Rightarrow \ S_1(r) = p_r r \tag{25}$$

we set the constant of integration to zero in the last terms of eq. (25).

At this stage it is very important to emphasize that there are key differences from our results and those of [9]. Firstly, Matone [9] set the potential $V(r)=0$, which is not the case here. One cannot *ignore* the gravitational potential generated by the presence of two masses m_1, m_2. Secondly, he proposed a quantum potential of the form

$$V_Q = -\frac{Gm_1m_2}{r} + \mathcal{O}(\hbar) \tag{26}$$

which is very different from our findings in eq. (22). Our quantum potential generates a *repulsive* gravitational interaction *plus* a constant proportional to $\hbar^{-2}$.

The quantum potential associated to the center of mass is

$$-\frac{\hbar^2}{2M\sqrt{\zeta(R)}}\frac{1}{R^2}\partial_R(R^2\partial_R\sqrt{\zeta(R)}) = \ -\frac{\hbar^2}{2M}(\frac{B^2}{4}) \tag{27}$$

such that

$$(\nabla_R S_2(R))^2 - \frac{\hbar^2 B^2}{4} = 2M\,\mathcal{E} \Rightarrow$$
$$\partial_R S_2(R) = P_R = \sqrt{\frac{\hbar^2 B^2}{4} + 2M\mathcal{E}} \Rightarrow \ S_2(R) = P_R R \tag{28}$$

we set the constant of integration in the last term of eq. (28) to zero.

To finalize one needs to study the continuity equations

$$\frac{1}{m}\ \nabla_r\cdot(\sigma(r)\nabla_r S_1(r)) = \frac{1}{m}\frac{1}{r^2}\partial_r(\ r^2\sigma(r)\partial_r S_1(r)\ ) = 0,$$
$$\frac{1}{M}\nabla_R\cdot(\zeta(R)\nabla_R S_2(R)) = \frac{1}{M}\frac{1}{R^2}\partial_R(\ R^2\zeta(R)\partial_R S_2(R)\ ) = 0 \qquad (29)$$

The solutions to the above continuity equations, when $\sigma(r), \zeta(R)$ are given by eqs. (18), will require then that the constant momenta p_r, P_R in eqs. (25), (28) should be trivially zero

$$\nabla_r S_1(r) = p_r = \sqrt{2m(E-\mathcal{E}+V_0)} = 0 \Rightarrow\ E-\mathcal{E}+V_0 = 0 \Rightarrow$$
$$E\ -\mathcal{E} = E' = -V_0 = -\frac{m}{2\hbar^2}(Gm_1m_2)^2 \qquad (30)$$

$$\nabla_R S_2(R)\ = P_R = \sqrt{\frac{\hbar^2B^2}{4}+2M\mathcal{E}} = 0\ \Rightarrow\ -\frac{\hbar^2B^2}{8M} = \mathcal{E} \qquad (31)$$

The ground state energy of the Hydrogen atom (ignoring relativistic corrections) is

$$E(n=1) = -\frac{me^4}{2\hbar^2} = -\frac{1}{2m}(m\frac{e^2}{\hbar c}c)^2 = -\frac{mc^2}{2}\ \alpha_e^2 \qquad (32)$$

we wrote the $E(n=1)$ in this form above to remind the reader that the mean electron velocity in the ground state is $c/137$. By analogy, one can define the "gravitational" fine structure α_G from the correspondence

$$\alpha_e = \frac{e^2}{\hbar c} \leftrightarrow \frac{Gm_1m_2}{\hbar c} = \alpha_G \qquad (33)$$

such that

$$E' = -V_0 = -\frac{mc^2}{2}(\frac{Gm_1m_2}{\hbar c})^2 = -\frac{mc^2}{2}\alpha_G^2 \qquad (34)$$

The value of $E' < 0$ is negative as expected for a (gravitationally) bound two-particle system. In the ground state the two particles are in static equilibrium configuration with respect to their center of mass.

The *stationary* Schrödinger equation in the spherically symmetric case for the two-particle system can be decomposed into the following two Schrödinger equations. The first one is

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r(r^2\partial_r\Psi(r)) + V_N(r)\Psi(r) = (E-\mathcal{E})\Psi(r) = E'\Psi(r) \quad (35)$$

where V_N is the attractive gravitational Newtonian potential $-(Gm_1m_2/r)$. As usual, m is the reduced mass, and $r = |\vec{r}_1 - \vec{r}_2|$ is the relative separation of the two particles. The other equation is

$$-\frac{\hbar^2}{2M}\frac{1}{R^2}\partial_R(R^2\partial_R\Psi(R)) = \mathcal{E}\Psi(R) \qquad (36)$$

where M is the total mass m_1+m_2, and R is the radial coordinate of the center of mass. One can verify by a simple inspection that $\Psi(r) = \sqrt{\sigma(r)} \sim exp(Ar/2), (A<0)$, and $\Psi(R) = \sqrt{\zeta(R)} \sim exp(BR/2)/R, (B<0)$, solve the above two Schrödinger equations, respectively.

Another physical solution for the center of mass motion occurs when $B=0 \Rightarrow P_R = \sqrt{2M\mathcal{E}}, \mathcal{E} > 0$. In this case one has $\zeta(R) \sim 1/R^2$, and the continuity equation $\frac{1}{MR^2}\partial_R(R^2\zeta(R)P_R) \sim \partial_R P_R = 0$ is obeyed without restricting the value of P_R to zero. One can verify again that given $S_2(R) = P_R R = \sqrt{2M\mathcal{E}}R$, the wavefunction $\Psi(R) \sim \frac{1}{R}exp(iP_R R/\hbar)$ (a spherical analog of a plane wave solution) obeys the Schrödinger equation (36). Since $\Psi(R)$ is not normalizable, as it is usual with the plane wave solutions one confines the particle to a "box" of finite size; i.e. one introduces an infrared cutoff.

The normalized wavefunction solutions to the Schrödinger equation corresponding to the Hydrogen atom (associated to a Coulomb potential) in spherical coordinates are

$$\Psi_{nlm}(r,\theta\phi) = \sqrt{(\frac{2}{na_o})^3\frac{(n-l-1)!}{2n(n+l)!}}e^{-\varrho/2}\ \varrho^l L^{2l+1}_{n-l-1}(\varrho)Y^l_m(\theta,\phi) \qquad (37)$$

where ϱ is defined by $\varrho = 2r/na_o$, and a_o is the Bohr radius $a_o = \hbar^2/me^2$. $L^{2l+1}_{n-l-1}(\varrho)$ is a generalized Laguerre polynomial of degree $n-l-1$, and $Y^l_m(\theta,\phi)$ are the spherical harmonics. In the gravitational case one just replaces $e^2 \to Gm_1m_2$ in all the expressions.

One may notice that the ground state wavefunction, $\Psi_{n=1,l=0,m=0} \sim exp(-\varrho/2)$, has the *same* functional form as $\sqrt{\sigma(r)} \sim exp(Ar/2)$, $A<0$ in eq. (18). The expression for the quantum potential $V_Q(\sqrt{\sigma(r)})$ based on $\sqrt{\sigma(r)} \sim exp(Ar/2), A<0$, coincided *exactly* with the *repulsive* gravitational potential (plus a constant). The continuity equations led to $p_r = P_R = 0$, and finally to the value of $E' = E - \mathcal{E}$ in eq. (34), and which has exactly the *same* expression as the ground state energy for the Hydrogen atom (32), after the correspondence $e^2 \leftrightarrow Gm_1m_2$ is made.

Does this correspondence occur for the other excited states $n = 2, 3, \ldots$?, and for other values of l, m besides $l = m = 0$ when one includes the rotational degrees of freedom? It should occur because the Schrödinger equation is *equivalent* to the coupled system of 2 differential equations (1)–(2). Namely, $\Psi_{nlm}(r,\theta,\phi) = \sqrt{\sigma_{nlm}(r,\theta,\phi)}cos(S_{nlm}(r,\theta,\phi)/\hbar)$ solves the Schrödinger equation if, and only if, $\sigma_{nlm}(r,\theta,\phi)$ and $S_{nlm}(r,\theta,\phi)$ solve the coupled system of 2 differential equations (1)–(2), and vice versa.

The excited states will no longer correspond to static mass configurations with respect to the center of mass, and the expression for $V_Q(\rho_{excited})$ will no longer be equal to the repulsive gravitational potential (up to a constant) cancelling the attractive Newtonian potential, but it will be a more complicated function. The momenta p_r will no longer be constant and there will be a nontrivial motion relative to the center of mass.

Can these results be generalized to other potentials $V(r)$ reflecting the interaction between two particles, besides the Newtonian one, or $V \sim 1/r$ potentials are special? Let us study the $3D$ spherically symmetric harmonic oscillator case, assuming the interaction between the particles is governed by a harmonic oscillator. The Gaussian ground state wavefunction $\Psi \sim \sqrt{\rho} \sim exp(-\lambda r^2)$ yields a quantum potential

$$V_Q = \frac{3\hbar^2\lambda}{m} - \frac{2\hbar^2\lambda^2}{m}r^2 \qquad (38)$$

which will cancel out the contribution of the $3D$ spherically symmetric harmonic oscillator potential $V_{osc} = \frac{1}{2}m\omega^2r^2$ (leaving a constant) when $(2\hbar^2\lambda^2)/m = \frac{1}{2}m\omega^2$. Solving for $\lambda = (m\omega/2\hbar)$ gives $V_Q + V_{osc} = \frac{3\hbar^2\lambda}{m} = (\frac{3\hbar^2}{m})(m\omega/2\hbar) = \frac{3}{2}\hbar\omega$ which is precisely the ground state energy of the $3D$ spherically symmetric harmonic oscillator. This procedure works for other potentials since setting $V_Q(\rho_{ground}) + V = E_0$, and $\nabla S = 0$ (zero momentum is a trivial solution to the continuity equation) into the QHJE leads always to $E = E_0$.

We propose next the notion of a classical/quantum *duality* in the gravitational case (it can be extended to other interactions).

By this one means *exchanging* $V_Q \leftrightarrow V$ in the stationary QHJE, so that $V_Q + V = -V_0$ (as before) and leading to

$$V_Q = -\frac{\hbar^2}{2m}\left(\frac{\nabla^2\sqrt{\sigma(r)}}{\sqrt{\sigma(r)}}\right) = V_N = -\frac{Gm_1m_2}{r},$$
$$V(r) = -V_N - V_0 = \frac{Gm_1m_2}{r} - V_0 \tag{39}$$

We should remark that over the years *different* notions of classical/quantum duality than ours have appeared in the literature, see [10] for some references. One must not confuse these different notions of classical/quantum duality.

The differential equation to be solved now is

$$r^2\frac{d^2\sqrt{\sigma(r)}}{dr^2} + 2\,r\frac{d\sqrt{\sigma(r)}}{dr} - Cr\sqrt{\sigma(r)} = 0,\quad C = \frac{2Gm_1m_2m}{\hbar^2} > 0 \tag{40}$$

The solutions must also be normalizable $\int_0^\infty \sigma(r)4\pi r^2 dr = 1$. They are given in terms of modified Bessel functions of the first I_1, and second kind K_1

$$\sqrt{\sigma(r)} = a_1\frac{I_1(2\sqrt{Cr})}{\sqrt{Cr}} + a_2\frac{K_1(2\sqrt{Cr})}{\sqrt{Cr}} \tag{41}$$

The normalization condition requires to discard the I_1 contribution because I_1 diverges at $r=\infty$, leaving the K_1 function which vanishes at $r=\infty$. In doing so, the normalization condition will fix the value of the a_2 coefficient in terms of $C>0$. Note that C has an explicit $\hbar$-dependence, as it should, in order to have $V_Q = V_N$. The right hand side has no $\hbar$ factor, so there must be a cancellation of the $\hbar$ factors in the left hand side.

The stationary Schrödinger equation in the spherically symmetric case involving the repulsive gravitational potential (plus a constant) is now given by

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\partial_r(r^2\partial_r\Psi(r)) + \left(\frac{Gm_1m_2}{r} - V_0\right)\Psi(r)$$
$$= (E-\mathcal{E})\Psi(r) = E'\Psi(r) \tag{42}$$

Eq. (42) should be compared with eq. (35). They both differ however the QHJE (15) remains the *same*. This is how the classical/quantum duality is implemented. Despite that the solutions to eq. (42) differ from the solutions to eq. (35), the *ground* state energy solution to eq. (42) is the *same* as before (34) $E' = E - \mathcal{E} = -V_0 = -\frac{mc^2}{2}\alpha_G^2$. One can verify this by checking that $\Psi = a_2\,\frac{K_1(2\sqrt{Cr})}{\sqrt{Cr}}$ solves the stationary Schrödinger equation (42) with the ground state energy $E' = E - \mathcal{E} = -V_0 = -\frac{mc^2}{2}\alpha_G^2$.

In this scenario Bohm's quantum potential associated to the *ground* state $V_Q(\sigma_{ground}(r)) = V_N$ coincides with the attractive Newtonian potential, and is now correlated with the classical *repulsive* gravitational potential (plus a constant) of eq. (39). And vice versa, under the exchange $V_Q \leftrightarrow V$. This 2-particle model can be generalized to the N-particle case. One would have to verify that the gravitational attraction (in the ground state configuration) is compensated by the repulsive contribution of the N-particle quantum potential corresponding to the ground state probability density.

The accelerated expansion of the Universe is generally assumed to be driven by a positive vacuum energy density (like a positive cosmological constant), or by some scalar field $\phi(t)$ (quintessence) whose potential $V(\phi(t))$ at late stages of the universe mimics the behavior of the cosmological constant. Here we are proposing a very different scenario in which there might be a *quantum* origin to the classical *repulsive* gravitational behavior of the universe based on this notion of classical/quantum *duality*. An account of the historical developments of gravity from Newton to the repulsive gravity of the vacuum energy can be found in [11].

A key remark is in order. The explicit presence of $\hbar^{-2}$ in the expression for V_0 (34) (and which is part of the potential $V(r)$ in eq. (39)) is not very common in the Quantum Mechanical problems that we are familiar with. However, as emphasized by Klauder in his monograph [8], the principal purpose of his Enhanced Quantization program is to describe and apply a new way to quantize classical systems, which in turn, lead to classical *enhanced* Hamiltonians that explicitly *contain* nonvanishing $\hbar$ terms. The Enhanced Quantization program of Klauder relies on the *coexistence* of the classical and quantum world, and consequently it involves the explicit presence of $\hbar$, and *without* taking the $\hbar\to 0$ limit. Therefore, having an $\hbar$-dependence on V_0 is not an alien property that should be dismissed and which would disqualify $V(r)$ as a "classical" potential.

To finalize we shall discuss another way in which repulsive gravity emerges. In [7] we proposed the *novel* equation[1] which we coined as the "Bohm–Poisson" (BP) equation (for *static* solutions $\rho = \rho(\vec{r})$)

$$\nabla^2 V_Q = 4\pi G m\rho \;\Rightarrow\; -\frac{\hbar^2}{2m}\nabla^2\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 4\pi G m\rho \tag{43}$$

the purpose of eq. (43) was to *replace* the well known (among the experts) nonlinear Newton–Schrödinger equation. The fundamental quantity is *no* longer the wave-function Ψ (complex-valued in general) but the real-valued probability density $\rho = \Psi^*\Psi$. The logic behind eq. (43) was based on the idea that the laws of physics should themselves determine the distribution of matter. This is going one step further from General Relativity where a given distribution of matter determines the geometry.

The BP equation (43) is invariant under the transformations $\rho \leftrightarrow -\rho$, and $G\to -G$. Thus solutions with $G<0$ are associated to *repulsive* gravity. If, in addition to the Bohm–Poisson (BP) equation one were to *add* the Schrödinger equation for the complex-valued wave-function $\Psi \equiv \sqrt{\rho}e^{iS/\hbar}$, one can obtain consistent solutions, which *avoids* having an overdetermined system of equations, when the external potential is itself a function of ρ. The functional form of the potential V *cannot* be arbitrary, but it is subjected to satisfy a system of equations. One equation is the BP equation (43). The second equation is the QHJE, and the third equation is the continuity equation. The latter two equations are equivalent to the Schrödinger equation. Therefore, one has a system of 3 equations for the 3 unknowns $\rho(r), S(r), V(r)$. The potential itself is determined from the equations instead of being put in by hand.

One can also propose another system of 3 equations (for the 3 unknowns $\rho(r), S(r), V(r)$) where the first equation is $V_Q(\rho) = V_N$. The second and third equations are the usual QHJE and continuity equation, respectively. In principle, one could have a family of many solutions consisting of the many triplets $\{\rho_1(r), S_1(r), V_1(r)\}, \{\rho_2(r), S_2(r), V_2(r)\}, \cdots \{\rho_N(r), S_N(r), V_N(r)\}$. There could even be an infinite number of solutions. This last set of 3 equations would correlate the classical potentials $V_1, V_2, \cdots$ with the quantum potential $V_Q = V_N$. Classical/Quantum duality selected one specific classical potential given by $V = -V_N - V_0$ in eq. (39).

Extensions of the Bohm–Poisson equation to the full *relativistic* regime were developed in [13]. In the case of a *real*-valued scalar field $\phi = \phi^*$, the relativistic field theory analog of the Bohm–Poisson equation in D dimensions was given by [13]

[1] To our knowledge eq. (43) has not appeared before.

$$(\frac{\hbar}{m})^2\Box\left(\frac{\Box\phi(\vec{r},t)}{\phi(\vec{r},t)}\right)=4\pi G g^{\mu\nu}T_{\mu\nu},$$
$$\Box\equiv\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,g^{\mu\nu}\partial_\nu),\ \ c=1 \quad (44)$$

where the trace of the stress energy tensor $T=g^{\mu\nu}T_{\mu\nu}$ associated with the scalar field appears in the right hand side. The stress energy tensor for the scalar field is defined in term of the matter terms S_m in the action by $T_{\mu\nu}=-\frac{2}{\sqrt{g}}\frac{\delta S_m(\phi,g_{\mu\nu})}{\delta g^{\mu\nu}}$.

In $4D$, given the Lorentzian signature $(-,+,+,+)$, the action in a curved background with a cosmological constant was chosen to be

$$S=\int d^4x\sqrt{-g}\left(\frac{(R-2\Lambda)}{16\pi G}-\frac{g^{\mu\nu}}{2}(\partial_\mu\phi)\,(\partial_\nu\phi)-V(\phi)\right) \quad (45)$$

and is associated with a canonical real scalar field ϕ with a potential $V(\phi)$. The FLRW metric is

$$ds^2=-(dt)^2+a^2(t)\left(\frac{(dr)^2}{1-kr^2}+r^2(d\Omega)^2\right),\ \ k=1,0;-1 \quad (46)$$

k is the spatial scalar curvature parameter with units of $(length)^{-2}$.

The equations of motion corresponding to the action (45), combined with the Relativistic Bohm–Poisson equation (44), leads to a family of solutions for $a(t),\phi(t)$ and $V(\phi)$. Once again, the potential $V(\phi)$ is *not* put in by hand but instead it is *derived* from the above system of equations. Two specific solutions for $a(t),\phi(t),V(\phi)$ were provided [13] encoding the repulsive nature of dark energy. One solution leads to an exact cancellation of the cosmological constant, but an expanding decelerating cosmos; while the other solution leads to an exponential accelerated cosmos consistent with a de Sitter phase, and whose extremely small cosmological constant is $\Lambda=\frac{3}{R_H^2}$, consistent with current observations. For further details we refer [13].

In passing we should mention that of the many articles surveyed in the literature pertaining the role of Bohm's quantum potential and cosmology, [14], [15], [16] we did not find any related to the novel Bohm–Poisson equation proposed in this work.[2] The authors [15], for instance, have shown that replacing classical geodesics with quantal (Bohmian) trajectories gives rise to a quantum corrected Raychaudhuri equation (QRE). They derived the second order Friedmann equations from the QRE, and showed that this also contains a couple of quantum correction terms, the first of which can be interpreted as cosmological constant (and gives a correct estimate of its observed value), while the second as a radiation term in the early universe, which gets rid of the big-bang singularity and predicts an infinite age of our universe. The model of "dark energy without dark energy" based on the sub-quantum potential associated with the CMB particles by [17] also differs from the work presented here.

A different quantum potential than Bohm's was proposed by [9] based on the Quantum Equivalence postulate of Quantum Mechanics under D-dimensional Mobius transformations. In one-dimension, their quantum potential Q was given in terms of the Schwarzian derivative of the action with respect to x by $Q=\frac{\hbar^2}{4m}\{S,x\}$. The Schwarzian derivative is defined by $\{S,x\}=(S'''/S')-\frac{3}{2}(S''/S')^2$. The Schwarzian derivative is Mobius invariant $\{\gamma(S),x\}=\{S,x\}$, where the Mobius transformation is defined as $\gamma(S)=\frac{aS+b}{cS+d}, ad-bc=1$. In one-dimension the continuity equation in the stationary case is $\frac{d}{dx}[(\rho(x)/m)(dS/dx)]=0\Rightarrow\rho(dS/dx)=constant$. Inserting $\sqrt{\rho}\sim(dS/dx)^{-\frac{1}{2}}$ into $Q=\frac{\hbar^2}{4m}\{S,x\}$ yields the expression for Bohm's quantum potential after some straightforward algebra [9].

Schwarzian Quantum Mechanics has recently been a very active topic of research in connection to the Sachdev–Ye–Kitaev (SYK) model [18]. Another very relevant topics of current research related to the emergence of gravity are holographic quantum complexity, entanglement entropy, information geometry, quantum computation and information theory, black holes, Cayley graphs, $\cdots$, see [19] and the references therein. The close relation between gravity and quantum mechanics has been analyzed by Susskind [20]. Our main goal, if possible, is to geometrize quantum mechanics. The emergence of quantum mechanics from the fractal geometry of spacetime has been advanced long ago by Nottale [12]. We hope that the notion of Classical/Quantum Duality raised in this work in connection to the QHJE may cast further light into the deep interplay between gravity and quantum mechanics.

[2] A Google Scholar search provided the response "Bohm–Poisson equation and cosmological constant did not match any articles".

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