SPIN TUNE DECOHERENCE IN MULTIPOLE FIELDS

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Abstract

This article analyses possible limitations in the method to search for the electric dipole moment (EDM) using polarized particles in a storage ring [1]. It is well known that for discovery of the electric dipole moment one needs to create such conditions where the particle's spin oscillations can be caused only by the EDM. There is some number of possible methods for EDM search using a storage ring. For instance at present we know the resonant spin build-up method in a magnetostatic ring and "frozen" spin method in a pure or partly electrostatic ring with magic energy. Both methods have common limitations caused by spin decoherence. In a previous article [2] we considered the main reasons leading to incoherent spread of spin tune taking into account only second-order nonlinearity. In this article, we have extended this method to the case for the higher orders, and the reasons of the spin decoherence are classified independently on method.

SPIN DECOHERENCE ORIGIN

As we know from the T-BMT and motion equations the spin tune is $\nu_s = \gamma G$. If the equilibrium energy (energy averaged over one turn) has a spread $\langle \Delta\gamma_{eq} \rangle$ dependent on the particle parameters the spin tune spread for N_t turns has incoherent spread:

$$2\pi \langle \Delta\nu_s \rangle_{N_t} = 2\pi G \langle \Delta\gamma_{eq} \rangle N_t \tag{1}$$

Thus the source of incoherent spread of spin tune is an incoherent energy spread. Then we have to ask what determines the incoherent energy spread. Obviously, this answer may be obtained from the motion equations. In paper [2] an analysis of longitudinal motion together with transverse motion has been done using the basic "synchrotronous acceleration principle". For small amplitude of synchrotron oscillation we can write the equation for momentum deviation $\delta = \frac{p - p_s}{p_s}$ from synchronous level:

$$\frac{d^2\delta}{dt^2} + \frac{eV_{rf}\omega_{rf}^2}{2\pi h \beta^2 E}\left(\alpha_0 - \frac{1}{\gamma^2}\right)\cdot\delta = \frac{eV_{rf}\omega_{rf}^2}{2\pi h \beta^2 E}\cdot\left[-\left(\alpha_1 + \frac{3}{2}\frac{\beta^2}{\gamma^2} - \frac{\alpha_0}{\gamma^2} + \frac{1}{\gamma^4}\right)\cdot\delta^2 - \left(\frac{\Delta L}{L}\right)_\beta\right], \tag{2}$$

where E is full energy, β is relative velocity, eV_{rf} is energy gain per turn with V_{rf} voltage gap, $\omega_{rf} = 2\pi h f_{rev}$ is angular frequency of RF field, h is a harmonic number, $f_{rev} = 1/T_{rev}$ is revolution frequency, the momentum compaction factor $\alpha = \alpha_0 + \alpha_1 \frac{\Delta p}{p_s}$ introduced through sleep-factor $\eta = \alpha - \frac{1}{\gamma^2}$ and the orbit lengthening due to the betatron motion $\left(\frac{\Delta L}{L}\right)_{\hat\beta}$.

Solving (2) we can define an influence of the betatron oscillation, the square term of momentum compaction factor α_1 and the slip factor η onto the equilibrium level energy shift $\Delta\delta_{eq}$:

$$\Delta\delta_{eq} = \frac{\gamma_s^2}{\gamma_s^2\alpha_0 - 1}\left[\frac{\delta_m^2}{2}\left(\alpha_1 + \frac{3}{2}\frac{\beta^2}{\gamma^2} - \frac{\alpha_0}{\gamma_s^2} + \frac{1}{\gamma_s^4}\right) + \left(\frac{\Delta L}{L}\right)_{\hat\beta}\right] \tag{3}$$

Thus due to the asymmetric shape of the separatrix in the momentum and lengthening orbit the equilibrium momentum has incoherent character.

ORBIT LENTHENING WITH MULTIPOLE FIELD

If with the contribution of asymmetry separatrix everything is clear, and it is quite easy to be taken into account in determining of the spin tune spread then the contribution of the betatron motion depends on the order of the considered multipole.

In absence of multipole in paper [2] has been shown that the orbit lengthening just due to the linear betatron oscillation is:

$$\left(\frac{\Delta L}{L}\right)_{\hat\beta} = \frac{\pi}{2L}\left[\varepsilon_x\nu_x + \varepsilon_y\nu_y\right], \tag{4}$$

where $\varepsilon_{x,y}$ and $\nu_{x,y}$ are emittance and tune in horizontal and vertical planes correspondingly and $\varepsilon_{x,y}$ for each particle has own value $\left\langle \frac{\varepsilon_{x,y}}{\beta_{x,y}} \right\rangle = 2\left\langle \begin{matrix} x_\beta^2 \\ y_\beta^2 \end{matrix} \right\rangle$.

Sextupoles

First let us suppose that we have quadrupoles and sextupoles in the ring:

$$x'' + \frac{K(s)}{1+\delta}\cdot x + \frac{1}{2}\cdot\frac{S(s)}{1+\delta}\cdot\left(x^2 - y^2\right) = \frac{1}{\rho(s)}\cdot\frac{\delta}{1+\delta}, \tag{5}$$
$$y'' - \frac{K(s)}{1+\delta}\cdot y - \frac{S(s)}{1+\delta}\cdot xy = 0$$

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where $x''=\frac{d^2x}{ds^2}$, $y''=\frac{d^2y}{ds^2}$, $K(s)=\frac{e}{p_s}\frac{\partial B_y(s)}{\partial x}$ and $S(s)=\frac{e}{p}\frac{\partial B_y^2(s)}{\partial x^2}$ is quadrupole and sextupole components in the ring correspondingly. Since we should exclude the third integer resonance we can use just the average value of $\overline{S}=\frac{1}{2\pi R}\sum_i S_i l_i$. Following Courant-Snyder formalism [3] we introduce the new variables:

$$\eta_x = x/\sqrt{\beta_x};\quad \eta_y = y/\sqrt{\beta_y};\qquad \beta_{x,y}\nu_{0x,y}d\phi = ds \quad (6)$$

The new independent variable ϕ is periodic with 2π and corresponds to the period of circumference $C=2\pi\overline{R}$ by s. Taking $\delta \ll 1$ we get the equations system:

$$\eta_x'' + \nu_{0x}^2(1-\delta)\eta_x = -\frac{1}{2}S\nu_{0x}^2(1-\delta)\beta_x^{3/2}\left(\beta_x\eta_x^2-\beta_y\eta_y^2\right)+\nu_{0x}^2(1-\delta)\beta_x^{3/2}\frac{\delta}{\rho}$$
$$\eta_y'' + \nu_{0y}^2(1-\delta)\eta_y = S\nu_{0y}^2(1-\delta)\beta_y^{3/2}\beta_x^{1/2}\eta_x\beta_y^{1/2}\eta_y \quad (7)$$

To simplify further discussion we will pass to a simpler representation of equations (7) in coordinate $d\vartheta_{x,y}=\omega_{x,y}\cdot d\phi$ with the new designations of the coefficients:

$$\varpi_{0x}^2=\nu_{0x}^2(1-\delta);\ \varpi_{0y}^2=\nu_{0y}^2(1-\delta);\ A=\frac{1}{2}\varpi_{0x}^2S\beta_x^{3/2};$$
$$C_\delta=\frac{\varpi_{0x}^2}{\rho}\beta_x^{3/2};\ B=\varpi_{0y}^2S\beta_y^{3/2} \quad (8)$$

We retain only the average value of coefficients in the right side of equations (7), because their typical frequencies differ from the betatron tunes. Then the equation system (7) is written as:

$$\omega_x^2\eta_x''+\varpi_{0x}^2\eta_x=-A\left(\beta_x\eta_x^2-\beta_y\eta_y^2\right)+C_\delta\cdot\delta$$
$$\omega_y^2\eta_y''+\varpi_{0y}^2\eta_y=B\beta_x^{1/2}\beta_y^{1/2}\eta_x\eta_y \quad (9)$$

Obviously the sextupole term gives small contribution in the solution, and it can be considered as small perturbation of motion.

Following Landau method [4] we seek a solution in:

$$\eta_{x,y}=\eta_{0x,y}+\eta_{1x,y}+\eta_{2x,y}+\ldots$$
$$\omega_{x,y}=\omega_{0x,y}+\omega_{1x,y}+\omega_{2x,y}+\ldots \quad (10)$$

Omitting the intermediate steps, we obtain the solution for first approach:

$$\eta_{0x}=\sqrt{\varepsilon_x}\cos\vartheta_x+\frac{C_\delta\cdot\delta}{\varpi_{0x}^2}$$
$$\omega_{0x}=\varpi_{0x} \quad (11)$$
$$\eta_{0y}=\sqrt{\varepsilon_y}\cos\vartheta_y$$
$$\omega_{0y}=\varpi_{0y}$$

and for the next approach:

$$\eta_{1x}=-\frac{A}{\varpi_{0x}^2}\left(\frac{\beta_x\varepsilon_x}{2}-\frac{\beta_y\varepsilon_y}{2}\right)-\frac{A\beta_x}{\varpi_{0x}^2}\left(\frac{C_\delta\delta}{\varpi_{0x}^2}\right)^2+\frac{A}{\varpi_{0x}^2}\left(\frac{\beta_x\varepsilon_x}{6}\cos 2\vartheta_x+\frac{\beta_y\varepsilon_y}{2(1-4\varpi_{0y}^2/\varpi_{0x}^2)}\cos 2\vartheta_y\right)$$
$$\eta_{1y}=\frac{B}{\varpi_{0y}^2}\sqrt{\varepsilon_x\beta_x}\sqrt{\varepsilon_y\beta_y}\cdot\left\{\frac{\cos(\vartheta_y-\vartheta_x)}{2[1-(\varpi_{0y}-\varpi_{0x})^2/\varpi_{0y}^2]}+\frac{\cos(\vartheta_y+\vartheta_x)}{2[1-(\varpi_{0y}+\varpi_{0x})^2/\varpi_{0y}^2]}\right\} \quad (12)$$

Substituting (11, 12) in $x=\beta_x^{1/2}(\eta_{0x}+\eta_{1x})$ and $y=\beta_y^{1/2}(\eta_{0y}+\eta_{1y})$, we have:

$$x=\frac{\beta_x^2}{\rho}\delta-\frac{1}{L}\sum_i S_i\beta_{xi}^2\left(\frac{\beta_x\varepsilon_x}{2}-\frac{\beta_y\varepsilon_y}{2}\right)-\delta^2\frac{1}{L}\sum_i\frac{S_i\beta_{xi}^6}{\rho^2}+\frac{1}{L}\sum_i S_i\beta_{xi}^2\left(\frac{\beta_x\varepsilon_x}{6}\cos 2\vartheta_x+\frac{\beta_y\varepsilon_y}{2(1-4\varpi_{0y}^2/\varpi_{0x}^2)}\cos 2\vartheta_y\right)+\sqrt{\varepsilon_x\beta_{xm}}\cos\vartheta_x$$
$$y=\sqrt{\varepsilon_x\beta_x}\sqrt{\varepsilon_y\beta_y}\frac{1}{L}\sum_i S_i\beta_{yi}^2\left\{\frac{\cos(\vartheta_y-\vartheta_x)}{1-(\varpi_{0y}-\varpi_{0x})^2/\varpi_{0y}^2}+\frac{\cos(\vartheta_y+\vartheta_x)}{1-(\varpi_{0y}+\varpi_{0x})^2/\varpi_{0y}^2}\right\}+\sqrt{\varepsilon_y\beta_y}\cos\vartheta_y \quad (13)$$

and:

$$x'=-\frac{1}{L}\sum_i S_i\beta_{xi}^2\left(\frac{\varepsilon_x}{3}\sin 2\vartheta_x+\frac{\varepsilon_y}{(1-4\varpi_{0y}^2/\varpi_{0x}^2)}\sin 2\vartheta_y\right)-\sqrt{\frac{\varepsilon_x}{\beta_x}}\sin\vartheta_x$$
$$y'=-\frac{1}{L}\sum_i S_i\beta_{yi}^2\sqrt{\frac{\varepsilon_x}{\beta_x}}\sqrt{\frac{\varepsilon_y}{\beta_y}}\left\{\frac{(\beta_x-\beta_y)\cdot\sin(\vartheta_y-\vartheta_x)}{1-(\varpi_{0y}-\varpi_{0x})^2/\varpi_{0y}^2}+\frac{(\beta_x+\beta_y)\cdot\sin(\vartheta_y+\vartheta_x)}{1-(\varpi_{0y}+\varpi_{0x})^2/\varpi_{0y}^2}\right\}-\sqrt{\frac{\varepsilon_y}{\beta_y}}\sin\vartheta_y \quad (14)$$

The same can be done for the tune shift:

$$\omega_{1x}=\frac{\delta}{4\pi}\int_0^{2\pi}\beta_x SD_x d\vartheta=\delta\frac{1}{4\pi C}\sum_i S_i l_{si}D_{xi}\beta_{xi}$$
$$\omega_{1y}=-\frac{\delta}{4\pi}\int_0^{2\pi}\beta_y SD_x d\vartheta=\delta\frac{1}{4\pi C}\sum_i S_i l_{si}D_{xi}\beta_{yi} \quad (15)$$

Finally we should substitute (13, 14) with $D_x \approx \beta_x^2 / \rho$ in the common expression for the orbit lengthening:

$$\left(\frac{\Delta L}{L}\right)_{\hat{\beta}} = \frac{1}{L}\oint\left(\frac{x}{\rho} + \frac{x'^2 + y'^2}{2}\right)ds \tag{16}$$

As we can see it consists of two terms. The first is:

$$\left(\frac{\Delta L}{L}\right)_{\frac{x}{\rho}} = \frac{1}{L}\oint\frac{x}{\rho}ds = \frac{\delta}{L}\int\frac{D_x}{\rho}ds - \frac{\varepsilon_x}{2L}\sum_i S_i l_{si} D_{xi}\beta_{xi} + \tag{17}$$

$$\frac{\varepsilon_y}{2L}\sum_i S_i l_{si} D_{xi}\beta_{yi} - \frac{\delta^2}{L}\sum_i S_i l_{si} D_{xi}^3$$

and the second one for the case $\beta_x = \beta_y$ and $\varpi_{0x} = \varpi_{0y}$

$$\left(\frac{\Delta L}{L}\right)_{x',y'} = \frac{1}{L}\oint\frac{x'^2 + y'^2}{2}ds = \frac{5}{4}\left(\frac{\varepsilon_x \frac{1}{L}\sum_i S_i\beta_{xi}^2}{3}\right)^2 + \tag{18}$$

$$\frac{\varepsilon_x}{4\beta_x} + \frac{\varepsilon_y}{4\beta_y}$$

In the second term both quadrupoles and sextupoles introduce the orbit lengthening, but the sextupoles have the contribution several orders of magnitude less because of the factor $(\bar{S}\varepsilon_x)^{1/2}$ and we can take that finally:

$$\left(\frac{\Delta L}{L}\right)_{x',y'} = \frac{\pi}{2L}\left[\varepsilon_x\nu_x + \varepsilon_y\nu_y\right] \tag{19}$$

Obviously in order to compensate the orbit lengthening we have to fulfil conditions:

$$-\frac{\varepsilon_x}{2L}\sum_i S_i l_{si} D_{xi}\beta_{xi} = \frac{\pi}{2L}\varepsilon_x\nu_x$$

$$\frac{\varepsilon_y}{2L}\sum_i S_i l_{si} D_{xi}\beta_{yi} = \frac{\pi}{2L}\varepsilon_y\nu_y \tag{20}$$

$$-\frac{\delta^2}{L}\sum_i S_i l_{si} D_{xi}^3 = \alpha_2\delta^2$$

and for the chromaticity compensation:

$$\nu_{0x}\delta\frac{1}{L}\sum_i S_i l_{si} D_{xi}\beta_{xi} = -\delta\nu_{0x}^2$$

$$-\nu_{0y}\delta\frac{1}{L}\sum_i S_i l_{si} D_{xi}\beta_{yi} = -\delta\nu_{0y}^2 \tag{21}$$

Multipoles

For the higher order multipoles up to M_n we have:

$$x'' + \frac{K(s)}{1+\delta}\cdot x + \frac{1}{2}\cdot\frac{S(s)}{1+\delta}\cdot\left(x^2 - y^2\right) +$$

$$\frac{1}{6}\cdot\frac{O(s)}{1+\delta}\cdot\left(x^3 - 3xy^2\right) + \ldots + M_{nx} = \frac{1}{\rho(s)}\cdot\frac{\delta}{1+\delta}$$

$$y'' - \frac{K(s)}{1+\delta}\cdot y - \frac{S(s)}{1+\delta}\cdot xy -$$

$$\frac{1}{6}\cdot\frac{O(s)}{1+\delta}\cdot\left(3x^2y - y^3\right) + \ldots + M_{ny} = 0 \tag{22}$$

Following once again Courant-Snyder formalism with the new variable for the horizontal plane only in order simplify the final expression we can reduce the equation to the general form:

$$\omega_x^2\eta_x'' + \varpi_{0x}^2\eta_x = \alpha_0 + \alpha_1\eta_x + \alpha_2\eta_x^2 + \alpha_3\eta_3^3 + \alpha_4\eta_4^4 + \ldots, \tag{23}$$

with coefficients:

$\alpha_0 \sim D_x\cdot\delta;\ \alpha_1 \sim \delta\varpi_0^2;\ \alpha_2 \sim S;\ \alpha_3 \sim O;\ \alpha_4 \sim D$

where δ is momentum spread, D_x is dispersion, S, O, D are sextupole, octupole and decapole terms correspondingly. It is no reason to repeat all routine calculations which have been done for the sextupole in previous section, and we will write the final solution for the tune shift affected on chromaticity and change of average position affected on the orbit lengthening:

$$\frac{\delta\varpi}{\varpi} = \frac{1}{2}\left\{\underbrace{\frac{\alpha_1}{\varpi_0^2}}_{\text{quadrupole}} + \underbrace{2\frac{\alpha_0\alpha_2}{\varpi_0^4}}_{\text{sextupole}} + \underbrace{3\frac{\alpha_0^2\alpha_3}{\varpi_0^6} + \frac{\alpha_3}{\varpi_0^2}\frac{3a^3}{4}}_{\text{octupole}} + \ldots\right.$$

$$\left.\underbrace{4\frac{\alpha_0^3\alpha_3}{\varpi_0^8}}_{\text{decapole}} + \ldots + \underbrace{n\frac{\alpha_0^{n-1}\alpha_n}{\varpi_0^{2n}}}_{\text{n-pole}}\right\} \tag{24}$$

$$\delta x_{av} = \underbrace{\frac{\alpha_0\alpha_1}{\varpi_0^4}}_{\text{quadrupole}} + \underbrace{\frac{1}{2}a^2\frac{\alpha_2}{\varpi_0^2} + \frac{\alpha_0^2\alpha_2}{\varpi_0^6}}_{\text{sextupole}} + \underbrace{\frac{3}{2}a^3\frac{\alpha_0\alpha_3}{\varpi_0^4} + \frac{\alpha_0^3\alpha_3}{\varpi_0^8}}_{\text{octupole}} +$$

$$\underbrace{\frac{3}{8}a^4\frac{\alpha_4}{\varpi_0^2} + 3a^2\frac{\alpha_0^2\alpha_4}{\varpi_0^4} + \frac{\alpha_0^4\alpha_4}{\varpi_0^{10}}}_{\text{decapole}} + \ldots, \tag{25}$$

where $a = (\varepsilon_x\beta_x)^{1/2}$ is amplitude of radial oscillation.

CONCLUSION

Thus, we can see that any multipole changes the betatron tune and simultaneously it shifts the average position of the orbit, extending or shortening it. By the latter we can change the average level of energy, and hence the spin tune.

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