Phase Diagram and Baryon-Baryon Scattering in the Gross-Neveu Model

Gerald V. Dunne[1]*

[1] *Deptartment of Physics, University of Connecticut, Storrs, CT 06269, USA*

One of the most important outstanding problems in modern particle and nuclear physics is to describe the finite density and finite temperature behavior of strongly interacting systems involving fermions. The primary motivation is to understand, as a function of temperature and density, quantum chromodynamics (QCD), the theory of the strong force that binds quarks and gluons. The Gross-Neveu models are 1+1 dimensional quantum field theories that capture much of the important physics, especially that associated with chiral symmetry breaking, and have somewhat magical integrability properties which permit exact analytic solutions to a number of important physical questions. The first is a complete description of the large N_f Gross-Neveu density-temperature phase diagram, showing the presence of crystalline condensate regions, and the second concerns the exact analytic description of baryon-baryon interactions. In this talk I review these results and discuss lessons for studying QCD and nuclear physics.

1. Introduction

This work is motivated by an attempt to understand temperature/density phase diagrams of interacting fermion systems [1]. Gross-Neveu (GN) models [2, 3, 4] are remarkable $(1+1)$-dimensional interacting fermionic models that share some important features with quantum chromodynamics (QCD): they are renormalizable, asymptotically free, exhibit dynamical chiral symmetry breaking, and have a limit of large number N_f of flavors that behaves like the 't Hooft large N_c limit of QCD. The GN models are well studied, but some features have come to light only relatively recently. For example, at finite temperature and density, in the $N_f \to \infty$ limit, the phase diagram shows regions in which the system prefers to form a spatially inhomogeneous crystalline condensate. This is true both of the Gross-Neveu model, GN_2, which has a *discrete* chiral symmetry, and of the NJL_2 model, which has a *continuous* chiral symmetry:

$$\mathcal{L}_{\mathrm{GN}_2} = \bar{\psi}\, i\,\partial\!\!\!/\,\psi + \frac{g^2}{2}\left(\bar{\psi}\psi\right)^2 \qquad (1)$$

$$\mathcal{L}_{\mathrm{NJL}_2} = \bar{\psi}\, i\,\partial\!\!\!/\,\psi + \frac{g^2}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i\gamma^5\psi\right)^2\right] \qquad (2)$$

However, the phase diagrams are very different [5, 6]. The inhomogeneous phase of the GN_2 model has been verified on the lattice [7], and similar 1d inhomogeneous condensates have been found in related models, also in higher-dimensions [8, 9].

The reason that the large N_f phase diagrams can be described analytically can be seen in two complementary ways. One approach to the phase diagram is via Hartree-Fock, in which one solves the single-particle Dirac equation self-consistently, subject to the constraint that the resulting expectation value of the condensate matches the trial form. This is a seemingly intractable problem for a non-uniform condensate, but the underlying integrability of the GN systems [10] enables an exact analytic solution of the relevant Dirac equations and the associated filling of the Dirac sea [6, 11]. For example, in the GN_2 system, the resulting solution actually solves the Hartree-Fock equation mode-by-mode, which in turn means that one is actually solving a nonlinear Dirac equation. This also explains why it is possible to find time-dependent solutions that can describe baryon-baryon scattering in the GN_2 model, and in this case the condensates are associated with the Sinh-Gordon equation, in light-cone coordinates. A second approach to the GN phase diagram is via the gap equation. This requires solving a non-

*Electronic address: gerald.dunne@uconn.edu

linear functional differential equation for the condensate, a formidable task for an inhomogeneous condensate. But for the GN systems, there is a deep integrability structure underlying the gap equation, which permits its closed-form solution, to all orders. For example, the thermodynamic Ginzburg-Landau expansions of the GN_2 and NJL_2 models are expansions in functionals of the associated condensate, and these functionals are precisely [6, 12] the conserved quantities of the modified Korteweg-de Vries (mKdV) and Ablowitz-Kaup-Newell-Segur (AKNS) hierarchies, respectively; this permits the gap equation to be solved and resummed to all orders, as described below.

2. Gross-Neveu Phase Diagrams

By a Hubbard-Stratonovich transformation, the four-fermion interaction terms can be expressed in terms of scalar and pseudo-scalar bosonic condensate fields, σ and π (respectively), which are conveniently expressed in terms of a complex condensate field: $\Delta = \sigma - i\pi$. For GN_2 we only have σ, and so the condensate field Δ is real. The general NJL_2 system can be described equivalently by the effective Lagrangian:

$$\mathcal{L} = \bar{\psi}\left[i\partial\!\!\!/ - \frac{1}{2}(1-\gamma^5)\Delta - \frac{1}{2}(1+\gamma^5)\Delta^*\right]\psi - \frac{1}{2g^2}|\Delta|^2, \qquad (3)$$

which is now quadratic in the fermion fields. The corresponding single particle fermionic Hamiltoninan is (with Dirac matrices: $\gamma_0 = \sigma_1$, $\gamma_1 = -i\sigma_2$, $\gamma^5 = \sigma_3$):

$$H = \begin{pmatrix} -i\frac{d}{dx} & \Delta(x) \\ \Delta^*(x) & i\frac{d}{dx} \end{pmatrix} \qquad (4)$$

This Hamiltonian is known as the Bogoliubov-de Gennes (BdG) Hamiltonian.

In the Hartree-Fock approach we solve the single particle Dirac equation, $H\psi = E\psi$, subject to the consistency condition relating the condensate field $\Delta(x)$ to the expectation values of the scalar and pseudoscalar fermionic bilinears:

$$\langle\bar{\psi}\psi\rangle(x) - i\langle\bar{\psi}i\gamma^5\psi\rangle(x) = -\Delta(x)/g^2 \qquad (5)$$

This is clearly a highly nontrivial problem, compounded by the fact that the expectation values include the Fermi factors appropriate for finite temperature and density. The gap equation approach involves integrating out the fermion fields in (3) to obtain an effective action (per fermion flavor) for the condensate field:

$$S_{\rm eff}[\Delta] = -\frac{1}{2g^2 N_f}\int d^2x|\Delta|^2 \qquad (6)$$
$$-i\ln\det\left[i\partial\!\!\!/ - \frac{1}{2}(1-\gamma^5)\Delta - \frac{1}{2}(1+\gamma^5)\Delta^*\right]$$

The gap equation for the condensate field identifies stationary points of $S_{\rm eff}[\Delta]$:

$$\frac{\delta S_{\rm eff}}{\delta\Delta^*(x)} = 0 \qquad (7)$$

This is a complicated nonlinear functional differential equation, which moreover should be solved at finite temperature and density.

Despite the apparent complexity of each of these two problems, each can be solved analytically. One finds explicit static but spatially inhomogeneous condensates:

$$\sigma(x) = m\,k\,{\rm sn}(m\,x|\,k^2) \quad , \quad GN_2 \qquad (8)$$
$$\Delta(x) = m\,e^{2iqx} \quad , \quad NJL_2 \qquad (9)$$

where sn is the usual Jacobi elliptic function, with elliptic parameter k^2. Note that each solution in (8, 9) depends on two parameters, whose dependence on temperature T and chemical potential μ remains to be determined, by thermodynamic minimization of the grand potential. As (T, μ) change, these parameters change, mapping out different condensates in different regions of the phase diagram. These solutions are most readily obtained [11] by consideration of the resolvent $R(x; E)$, the coincident-point limit of the Green's function $G(x, y; E)$:

$$R(x;E) \equiv \langle x|\frac{1}{H-E}|x\rangle \quad . \qquad (10)$$

The local density of states for fermions in the presence of the condensate follows from the resolvent

$$r(x;E) = \frac{1}{\pi}{\rm Im\,tr}_D\left(R(x;E+i\epsilon)\right) \qquad (11)$$

Given the fermioni density of states, $r(E) = \int dx\, r(x;E)$, all relevant thermodynamic quantities, at finite temperature and chemical potential, can be derived from the grand canonical potential

$$\Psi[\Delta(x);T,\mu] = \frac{1}{2N_f g^2}\frac{1}{L}\int_0^L dx|\Delta(x)|^2 - \frac{1}{\beta}\int_{-\infty}^{\infty} dE\, r(E)\ln\left(1+e^{-\beta(E-\mu)}\right) \quad (12)$$

Since we know $r(E)$ exactly, we can analyze

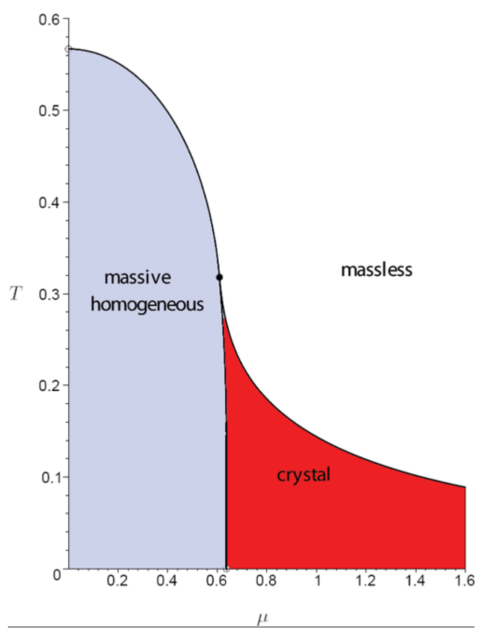

FIG. 1: Phase diagram for the GN_2 model, showing the crystalline phase in which the discrete chiral symmetry is broken by a periodic crystalline scalar condensate.

the thermodynamic properties of this model precisely. Minimizing the grand potential with respect to the parameters m, ν, and q appearing in the solutions (8, 9) determines these parameters as functions of temperature T and chemical potential μ [6]. The resulting phase diagram for the GN_2 and NJL_2 models are shown in Fig 1 and Fig 2, respectively. Note that the phase diagrams are quite different. The GN_2 model, has a tricritical point, and a

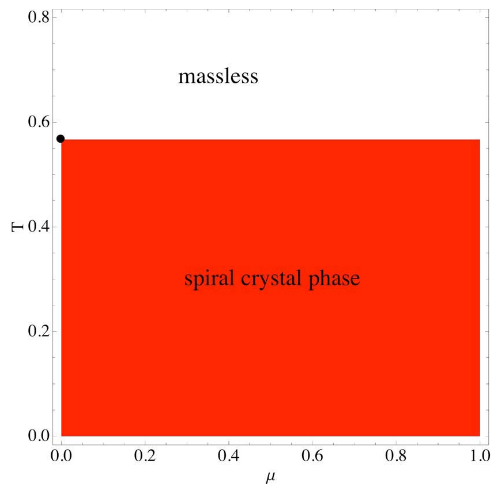

FIG. 2: Phase diagram for the NJL_2 model, showing the chiral spiral phase in which the continuous chiral symmetry is broken by a spiral complex scalar-pseudoscalar condensate $\Delta(x) = A\, e^{2i\mu\, x}$.

crystalline phase at high density and low temperature. Close to the upper boundary, the crystal is well approximated by a weak periodic LOFF form, $\sigma(x) \approx A\sin(m\,x)$, while near the left boundary, close to the critical chemical potential $\mu_c = 1/\pi$ [which is the mass of the lowest baryon], the crystal looks like a kink-antikink lattice of baryons. The form of the condensate is actually a direct consequence of the discrete chiral symmetry, which implies that the single-particle spectrum of the BdG hamiltonian is necessarily symmetric about 0; in turn this means that if a gap opens it must be centered at 0, or there must be two gaps, equally spaced about 0. The opening of such gaps is a physical consequence of the Peierls instability, here in a relativistic system, which states that a one dimensional system can lower its energy by forming a periodic structure, which opens a gap in the spectrum at the Fermi energy. In 1 dimension this is actually enough information to deduce the form of the GN_2 condensate, since it is the simplest of the so-called "finite-gap" potentials. On the other hand, the phase diagram for the NJL_2 model is very different.

Thermodynamics determines the parameter q to be equal to μ, and the amplitude is only a function of T. The crystal phase below T_c is called the "chiral spiral", $\Delta(x) = A(T)\, e^{2i\mu\, x}$. In the NJL_2 model the continuous chiral symmetry breaks the energy-reflection symmetry of the associated BdG hamiltonian, and the Peierls instability now opens just a single gap, but the phase of the complex condensate shifts the spectrum so that the chemical potential μ lies in the gap, thereby minimizing the grand potential. This phase dependence is also directly related to the axial anomaly in $(1+1)$-dimensions [6].

3. Ginzburg-Landau expansion and Integrability of GN Models

The Ginzburg-Landau expansion of the renormalized grand potential (12) is in terms of functionals $c_n[\Delta]$ of the condensate $\Delta(x)$:

$$\Psi_{\rm GL} = \sum_{n=0}^{\infty} \alpha_n(T,\mu)\, c_n[\Delta] \qquad (13)$$

where the coefficients $\alpha_n(T,\mu)$ are simple known functions of T and μ [6]. The first few functionals are familiar: c_0 is a constant,

$$\begin{aligned} c_2[\Delta] &= \int dx\, |\Delta|^2 \\ c_3[\Delta] &= \int dx\, {\rm Im}(\Delta\, \Delta'^*) \\ c_4[\Delta] &= \int dx\, (|\Delta|^4 + |\Delta'|^2) \\ c_5[\Delta] &= \int dx\, {\rm Im}\left((\Delta'' - 3|\Delta|^2\Delta)\Delta'^*\right) \end{aligned} \qquad (14)$$

For the GN_2 model, the condensate is real, so all the odd-index c_n vanish. Given this expansion, the gap equation now reads:

$$\sum_{n=3}^{\infty} \alpha_n(T,\mu)\, \frac{\delta c_n[\Delta]}{\delta \Delta^*(x)} = -\alpha_2(T,\mu)\, \Delta(x) \qquad (15)$$

As n increases, the $c_n[\Delta]$ rapidly become more and more complicated, involving higher powers of Δ and higher derivatives, which makes the variation $\delta c_n[\Delta]/\delta\Delta^*(x)$ also very complicated. But, remarkably, the $c_n[\Delta]$ are precisely the conserved quantities of the Ablowitz-Kaup-Newell-Segur (AKNS) integrable hierarchy, which for a real condensate reduces to the modified-Korteweg-de Vries (mKdV) integrable hierarchy [6, 12]. These have special properties: for e.g., for mKdV, when evaluated on a solution of the lowest non-trivial equation of the hierarchy, the nonlinear Schrödinger equation, the variation of each and every $c_n[\sigma]$ is proportional to $\sigma(x)$:

$$\frac{\delta c_n[\Delta]}{\delta \sigma(x)} = d_n\, \sigma(x) \quad , \quad \forall\, n \qquad (16)$$

for some constants d_n. This is astonishing, and has the immediate consequence that we can solve the gap equation (15) simply by determining the parameters of the solution as implicit functions of T and μ. A similar result holds for the AKNS hierarchy and the Ginzburg-Landau expansion of the NJL_2 model [6]. Thus we have an all-orders solution to the Ginzburg-Landau equations, due to the integrable hierarchies underlying the thermodynamic Ginzburg-Landau expansion.

There is analogous manifestation of this integrability in the Hartree-Fock approach to the GN models. To illustrate, consider the GN_2 model, where we seek to solve

$$(i\,\partial\!\!\!/ - \sigma(x))\,\psi \;=\; 0 \qquad (17)$$

subject to the constraint

$$\langle \bar\psi\, \psi\rangle(x) \equiv \sum_{\text{states}\, p} \bar\psi_p(x)\, \psi_p(x) = -\frac{1}{g^2}\, \sigma(x) \qquad (18)$$

The simplest way to solve the second equation would be to find single-particle spinor solutions ψ_p such that $\bar\psi_p(x)\psi_p(x) \propto \sigma(x)$, *for every mode p separately.* (This is the analogue of the mKdV property in (16)). Then the Dirac equation (17) would become a *nonlinear* Dirac equation:

$$\left(i\,\partial\!\!\!/ - l_p\, \bar\psi_p(x)\, \psi_p(x)\right) \psi_p(x) \;=\; 0 \qquad (19)$$

for some constant l_p, and we obtain solutions to the Hartree-Fock system of equations.

The spinors are 2-component, and we can rewrite this nonlinear Dirac equation in terms of the bilinear $S(x) = \bar{\psi}(x)\psi(x)$ [10]. Then $S(x)$ must satisfy $SS'' - (S')^2 - S^4 = -1$, which is equivalent to the nonlinear Schrödinger equation: $S'' - 2S^3 + S = 0$, the lowest member of the mKdV hierarchy. We can also write the equation in terms of another field $\theta(x)$ defined by $S(x) \equiv e^{\theta(x)/2}$, resulting in the 1 dimensional Sinh-Gordon equation

$$\theta'' - 4\sinh\theta = 0 \tag{20}$$

also an integrable nonlinear equation.

4. Baryon Scattering in Gross-Neveu Models

Perhaps even more surprising than the integrability properties of the static condensate solutions outlined in the previous section is the fact that one can also find nontrivial *space-time dependent* solutions to the Hartree-Fock problem [14, 15, 16]. Semiclassical methods are not restricted to static solutions. Since we are dealing with a relativistic field theory, we can boost any static solution to an arbitrary Lorentz frame, turning a static Hartree-Fock (HF) solution into a solution of time-dependent Hartree-Fock (TDHF):,

$$\begin{aligned} \left(i\gamma_5\partial_x + \gamma^0 S(x,t)\right)\psi_\alpha &= i\partial_t\psi_\alpha \qquad (21) \\ S(x,t) &= -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta\psi_\beta \end{aligned}$$

It was anticipated already in Witten's seminal paper on baryons in the $1/N$ expansion in QCD [13], that baryon-baryon scattering might be solved with the help of the TDHF approach at large N. This scenario is realized explicitly for the GN model [14, 15, 16].

Dashen, Hasslacher and Neveu (DHN) found multifermion bound states (baryons) in the GN_2 gap equation [3]. The *static* DHN baryon scalar potential is

$$\begin{aligned} S_{\rm DHN}(x) &= 1 + y\,\tanh(y\,x - \frac{1}{2}\text{arctanh}(y)) \\ &\quad -y\,\tanh(y\,x + \frac{1}{2}\text{arctanh}(y)) \qquad (22) \end{aligned}$$

where the $y \in [0, 1]$ is a real parameter. DHN baryons have a valence bound state which can be filled with up to N fermions. In the large N limit, the filling fraction $\nu = n/N$ becomes a continuous parameter, so that there is in fact a whole one-parameter family of baryons. For self-consistency in the gap equation, the filling fraction ν is related to y by $y = \sin(\pi\nu/2)$. The DHN baryons span the region from a weakly bound, non-relativistic state at small filling to the ultrarelativistic limit of a decoupled kink and antikink at complete filling. At large filling fraction, the DHN baryon looks like a bound kink-antikink molecule, and it is stable not because of topology but because of a balance between the kink-antikink interaction and the effect of the fermions bound to the kink and antikink. This leads to a direct relation between the baryon size (the distance between the kink and antikink) and the fermion filling fraction, giving a beautiful example of dynamical stability as well as of the Jackiw-Rebbi mechanism of fermion modes bound to localized defects such as kinks [17]. A particularly interesting special case is the baryon with non-trivial topology, the kink, which is attributed to Callan, Coleman, Gross and Zee (CCGZ). Later, Feinberg established the complete set of static solutions to the large N gap equation [4], combining inverse scattering theory with resolvent techniques. The general solution consists of marginally bound multibaryon configurations whose energy does not depend on the distance between the constituents. A common feature of all static solutions is the fact that the self-consistent scalar potentials are reflectionless, generalizing the Kay-Moses potentials of the Schrödinger equation to the Dirac equation.

In addition to boosted static solutions, some truly dynamical solutions of Eqs. (22) are also known. They are harder to find than static HF solutions since inverse scattering theory is not as well developed for TDHF. The efforts to find non-trivial, time-dependent mean field solutions were also pioneered by DHN who already presented a breather solution, a periodically oscillating (in time) multifermion bound state [3]. DHN also pointed out the possibility to relate the breather to the kink-antikink scattering problem by analytic continuation.

This suggestion was taken up and elaborated in several recent works. Kink-antikink scattering was addressed in [14]. Apart from a first glimpse of the scattering problem of composite, relativistic objects, the solution also gave several new insights into the mathematical structure of the theory. A special feature of kink dynamics is the fact that the (valence) fermions do not react back on the solitons that are carrying them. This decoupling made it possible to formulate kink dynamics in the language of a well-studied, classical soliton theory, the sinh-Gordon model [18], and to use the known n-soliton solution of this model to generalize kink-antikink scattering to the case of an arbitrary number of kink-like baryons [15]. From the point of view of TDHF theory, the most striking feature of all kink-antikink solutions is the fact that the scalar density of each single-fermion level is proportional to the full self-consistent potential $S(x,t)$. Thus, TDHF reduces again to a nonlinear Dirac equation, and when expressed in terms of the bilinear $S(x,t) \equiv e^{\theta(x,t)/2}$, we obtain the $(1+1)$-dim Sinh-Gordon equation: $\partial_\mu^2\theta + \sinh\theta = 0$ [10].

The simplest nontrivial time-dependent solution is the boosted kink

$$S(x,t) = \tanh(2(x-vt)/\sqrt{1-v^2}) \qquad (23)$$

from which one can construct a real kink-antikink scattering solution:

$$S(x,t) = \frac{v\,\cosh\left(\frac{2x}{\sqrt{1-v^2}}\right) - \cosh\left(\frac{2vt}{\sqrt{1-v^2}}\right)}{v\,\cosh\left(\frac{2x}{\sqrt{1-v^2}}\right) + \cosh\left(\frac{2vt}{\sqrt{1-v^2}}\right)} \qquad (24)$$

Using the relation to the Sinh-Gordon equation, Bäcklund transformations generate more complicated solutions describing multi-kink scattering processes [14, 15]. These time-dependent $S(x,t)$ satisfy the TDHF problem, and the corresponding solutions ψ_p to (17) can be constructed analytically, and satisfy the nonlinear Dirac equation for each mode p.

As nice as kinks are mathematically, they form only one extreme endpoint of the DHN baryon family. To complete the picture, we need to understand the scattering of general DHN baryons, without the restriction to kinks and antikinks. The scattering of two arbitrary DHN baryons allows one to probe the degree to which the internal bound state structure is relativistic, all the way from the non-relativistic limit to the ultrarelativistic one. Furthermore, by choosing the velocity of the baryons, one can cover the range from non-relativistic to relativistic scattering in the external kinematics as well. This general problem is not easy, because the valence fermions are now expected to react back. However, one can solve this problem by a method based upon an ansatz [16]. The solution of the Dirac equation and the requirement of self-consistency are sufficient to determine the unknown parameters of the ansatz and to establish an exact baryon-baryon scattering solution in the large N limit of the GN model. Perhaps more important than the specific solution found in [16] is the fact that the ansatz can be generalized in a natural way to a whole class of more complicated scattering problems, also involving multibaryon bound states and breathers, in addition to DHN baryons.

Consider the scattering of two DHN baryons with different baryon numbers (parameters y_1, y_2). To keep the number of parameters as small as possible, we work in the center-of-velocity frame where the baryon velocities are $\pm v$. Since the calculation is fully covariant, we can transform the results into any other Lorentz frame. In analogy to the one baryon problem, we parametrize the scalar potential as a rational function of exponentials. The spinors are also taken to be exponentials times functions similar to S, where we always insist on keeping the same denominators. In the single baryon case, the asymptotic information used to reduce the number of parameters came from the vacuum. We can similarly exploit the asymptotic information from the incoming and outgoing baryons. This recursive way of proceeding greatly reduces the number of parameters which then have to be determined algebraically via the Dirac equation.

The ansatz for the scalar potential is

$$S = \frac{\mathcal{N}}{\mathcal{D}} \qquad (25)$$

where

$$\begin{aligned}\mathcal{N} &= 1 + a_1 U_1 + a_2 U_2 + a_{11} U_1^2 + a_{12} U_1 U_2 \\ &\quad + a_{22} U_2^2 + a_{112} U_1^2 U_2 + a_{122} U_1 U_2^2 \\ &\quad + a_{1122} U_1^2 U_2^2 \end{aligned} \tag{26}$$

with a similar expansion for $\mathcal{D}$, with different coefficients. Here (with $\gamma = 1/\sqrt{1-v^2}$):

$$\begin{aligned} U_1 &= \exp\{2y_1\gamma(x - vt)\} \\ U_2 &= \exp\{2y_2\gamma(x + vt)\} \end{aligned} \tag{27}$$

The U_1, U_2 dependence is motivated by the product of the two baryon potentials, which we must recover when the scatterers are well separated. In this sense, the ansatz is the minimal one having a chance of describing baryon-baryon scattering. Almost all of the 16 real parameters in S are in fact determined by the asymptotic in- and out-states. The remaining parameters can be determined explicitly [16]. An example of the scattering of two DHN baryons is shown in Fig. 3. A novel feature of these scattering events is that the fermions that bind each individual DHN baryon together play a nontrivial role in the scattering process [16].

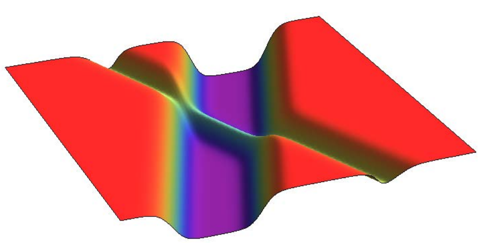

FIG. 3: Scattering of a small ($y_1 = 0.8$) and a large ($1 - y_2 = 10^{-7}$) DHN baryon with velocities ± 0.4. The self-consistent scalar potential $S(x,t)$ is shown for a range of (x,t) values in the vicinity of the collision.

These explicit TDHF solutions have a surprising geometrical interpretation, in terms of embedding 2d surfaces into 3d spaces [19]. This embedding problem can be formulated naturally in terms of 2-component spinors, with the following correspondences:

$$\begin{aligned} \text{Dirac equation} &: (i\partial\!\!\!/ - S)\psi = 0 \\ \text{mean curvature} &: H = S/(\bar\psi\psi) \\ \text{induced metric factor} &: f = \bar\psi\psi \\ \text{Hopf differentials} &: \\ &\begin{cases} Q^{(+)} = -i(\psi_1^*\psi_{1,+} - \psi_{1,+}^*\psi_1) \\ Q^{(-)} = i(\psi_2^*\psi_{2,-} - \psi_{2,-}^*\psi_2) \end{cases} \end{aligned} \tag{28}$$

Given a solution ψ of the Dirac equation with potential S, if one defines the mean curvature H and induced metric factor f ($ds^2 = f^2(x_+, x_-)dx_+dx_-$), as in the second and third equations, then in fact this H and f define the embedding of a 2d surface into 3d Minkowski space. The condition of constant mean curvature says that S is proportional to the bilinear $\bar\psi\psi$, so we have the nonlinear Dirac equation satisfied by the solutions of the TDHF equations. This is a reflection of the underlying integrability, and explains geometrically why the Sinh-Gordon equation plays such an important role in constructing solutions. The relation to minimal ($H = 0$) surfaces in AdS_3, and hence string worldsheets, follows as the constant mean curvature of the embedded surface is traded for a constant curvature of the 3d space [14, 16].

5. Conclusions

The phase diagrams of the large N_f Gross-Neveu models, GN_2 and NJL_2, can be solved analytically, revealing a rich structure of crystalline condensates. This solution is possible due to a deep integrability structure underlying the GN models, and manifests itself both in the Hartree-Fock and gap equation approaches. Furthermore, explicit space-time dependent solutions of the time-dependent Hartree-Fock problem can be constructed, realizing the nuclear physics picture of the scattering of self-consistently bound baryons. These solutions provide the correct starting point for studies going beyond the leading large N_f limit, and suggest certain features that could be applied to higher-dimensional systems such as 4 dimensional NJL models.

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References

[1] K. Rajagopal and F. Wilczek, "The condensed matter physics of QCD," in *At the Frontier of Particle Physic; Handbook of QCD*, M. Shifman, ed., (World Scientific, 2001); arXiv:hep-ph/0011333; P. de Forcrand and O. Philipsen, "Towards the QCD phase diagram," PoS LAT **2006**, 130 (2006) [arXiv:hep-lat/0611027].

[2] D. J. Gross and A. Neveu, "Dynamical Symmetry Breaking In Asymptotically Free Field Theories," Phys. Rev. D **10**, 3235 (1974).

[3] R. F. Dashen, B. Hasslacher and A. Neveu, "Semiclassical Bound States In An Asymptotically Free Theory," Phys. Rev. D **12**, 2443 (1975).

[4] J. Feinberg, "All about the static fermion bags in the Gross-Neveu model," Annals Phys. **309**, 166 (2004) [arXiv:hep-th/0305240].

[5] M. Thies, "Analytical solution of the Gross-Neveu model at finite density," Phys. Rev. D **69**, 067703 (2004) [arXiv:hep-th/0308164]; "From relativistic quantum fields to condensed matter and back again: Updating the Gross-Neveu phase diagram," J. Phys. A **39**, 12707 (2006) [arXiv:hep-th/0601049].

[6] G. Basar, G. V. Dunne and M. Thies, "Inhomogeneous Condensates in the Thermodynamics of the Chiral NJL(2) model," Phys. Rev. D **79**, 105012 (2009) [arXiv:0903.1868 [hep-th]].

[7] P. de Forcrand and U. Wenger, "New baryon matter in the lattice Gross-Neveu model," PoS **LAT2006**, 152 (2006) [arXiv:hep-lat/0610117].

[8] D. Nickel and M. Buballa, "Solitonic ground states in (color-) superconductivity," Phys. Rev. D **79**, 054009 (2009) [arXiv:0811.2400 [hep-ph]].

[9] T. Kojo, Y. Hidaka, L. McLerran and R. D. Pisarski, "Quarkyonic Chiral Spirals," Nucl. Phys. A **843**, 37 (2010) [arXiv:0912.3800 [hep-ph]].

[10] A. Neveu and N. Papanicolaou, "Integrability Of The Classical Scalar And Symmetric Scalar-Pseudoscalar Contact Fermi Interactions In Two-Dimensions," Commun. Math. Phys. **58**, 31 (1978).

[11] G. Basar and G. V. Dunne, "Self-consistent crystalline condensate in chiral Gross-Neveu and Bogoliubov-de Gennes systems," Phys. Rev. Lett. **100**, 200404 (2008), [arXiv:0803.1501 [hep-th]]; "A Twisted Kink Crystal in the Chiral Gross-Neveu model," Phys. Rev. D **78**, 065022 (2008), [arXiv:0806.2659 [hep-th]].

[12] F. Correa, G. V. Dunne and M. S. Plyushchay, "The Bogoliubov/de Gennes system, the AKNS hierarchy, and nonlinear quantum mechanical supersymmetry," Annals Phys. **324**, 2522 (2009) [arXiv:0904.2768 [hep-th]].

[13] E. Witten, "Baryons in the 1/n Expansion," Nucl. Phys. B **160**, 57 (1979).

[14] A. Klotzek and M. Thies, "Kink dynamics, sinh-Gordon solitons and strings in AdS_3 from the Gross-Neveu model," J. Phys. A **43**, 375401 (2010) [arXiv:1006.0324 [hep-th]].

[15] C. Fitzner and M. Thies, "Exact solution of N baryon problem in the Gross-Neveu model," Phys. Rev. D **83**, 085001 (2011) [arXiv:1010.5322 [hep-th]].

[16] G. V. Dunne, C. Fitzner and M. Thies, "Baryon-baryon scattering in the Gross-Neveu model: the large N solution," Phys. Rev. D **84**, 105014 (2011) [arXiv:1108.5888 [hep-th]].

[17] R. Jackiw and C. Rebbi, "Solitons with Fermion Number 1/2," Phys. Rev. D **13**, 3398 (1976).

[18] A. Jevicki and K. Jin, "Solitons and AdS String Solutions," Int. J. Mod. Phys. A **23**, 2289 (2008) [arXiv:0804.0412 [hep-th]].

[19] G. Basar and G. V. Dunne, "Gross-Neveu Models, Nonlinear Dirac Equations, Surfaces and Strings," JHEP **1101**, 127 (2011) [arXiv:1011.3835 [hep-th]].