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Sigma-model limit of Yang–Mills instantons in higher dimensions

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Abstract

We consider the Hermitian Yang–Mills (instanton) equations for connections on vector bundles over a $2n$-dimensional Kähler manifold X which is a product $Y \times Z$ of p- and q-dimensional Riemannian manifold Y and Z with $p+q=2n$. We show that in the adiabatic limit, when the metric in the Z direction is scaled down, the gauge instanton equations on $Y \times Z$ become sigma-model instanton equations for maps from Y to the moduli space $\mathcal{M}$ (target space) of gauge instantons on Z if $q \geq 4$. For $q < 4$ we get maps from Y to the moduli space $\mathcal{M}$ of flat connections on Z. Thus, the Yang–Mills instantons on $Y \times Z$ converge to sigma-model instantons on Y while Z shrinks to a point. Put differently, for small volume of Z, sigma-model instantons on Y with target space $\mathcal{M}$ approximate Yang–Mills instantons on $Y \times Z$.

1. Introduction and summary

The Yang–Mills equations in two, three and four dimensions were intensively studied both in physics and mathematics. In mathematics, this study (e.g. projectively flat unitary connections and stable bundles in $d = 2$ [1], the Chern–Simons model and knot theory in $d = 3$, instantons and Donaldson invariants [2] in $d = 4$ dimensions) has yielded a lot of new results in differential

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and algebraic geometry. There are also various interrelations between gauge theories in two, three and four dimensions. In particular, Chern–Simons theory in $d = 3$ dimensions reduces to the theory of flat connections in $d = 2$ (see e.g. [3,4]). On the other hand, the gradient flow equations for Chern–Simons theory on a $d = 3$ manifold Y are the first-order anti-self-duality equations on $Y \times \mathbb{R}$, which play a crucial role in $d = 4$ gauge theory.

The program of extending familiar constructions in gauge theory, associated to problems in low-dimensional topology, to higher dimensions was proposed by Donaldson and Thomas in the seminal paper [5] (see also [6]) and developed in [7–14] among others. An important role in this investigation is played by first-order gauge-field equations which are a generalization of the anti-self-duality equations in $d = 4$ to higher-dimensional manifolds with special holonomy (or, more generally, with G-structure [15,16]). Such equations were first introduced in [17] and further considered in [18–22] (see also the references therein).

Instanton equations on a d-dimensional Riemannian manifold X can be introduced as follows [17,5,10]. Suppose there exist a 4-form Q on X. Then there exists a $(d-4)$-form $\Sigma := *Q$, where $*$ is the Hodge operator on X. Let $\mathcal{A}$ be a connection on a bundle E over X with curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. The generalized anti-self-duality (instanton) equation on the gauge field then is [10]

$$*\mathcal{F} + \Sigma \wedge \mathcal{F} = 0 . \tag{1.1}$$

For $d > 4$ these equations can be defined on manifolds X with *special holonomy*, i.e. such that the holonomy group G of the Levi-Civita connection on the tangent bundle TX is a subgroup in $\mathrm{SO}(d)$. Solutions of (1.1) satisfy the Yang–Mills equation

$$\mathrm{d} * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - (-1)^d * \mathcal{F} \wedge \mathcal{A} = 0 . \tag{1.2}$$

The instanton equation (1.1) is also well defined on manifolds X with non-integrable *G-structures*, i.e. when $\mathrm{d}\Sigma \neq 0$. In this case (1.1) implies the Yang–Mills equation with (3-form) torsion $T := *\mathrm{d}\Sigma$, as is discussed e.g. in [23–27].

Manifolds X with a $(d-4)$-form Σ which admits the instanton equation (1.1) are usually *calibrated* manifolds with *calibrated submanifolds*. Recall that a calibrated manifold is a Riemannian manifold (X, g) equipped with a closed p-form φ such that for any oriented p-dimensional subspace ζ of $T_x X$, $\varphi|_\zeta \leq vol_\zeta$ for any $x \in X$, where vol_ζ is the volume of ζ with respect to the metric g [28]. A p-dimensional submanifold Y of X is said to be a calibrated submanifold with respect to φ (φ-calibrated) if $\varphi|_Y = vol_Y$ [28]. In particular, suitably normalized powers of the Kähler form on a Kähler manifold are calibrations, and the calibrated submanifolds are complex submanifolds. On a G_2-manifold one has a 3-form which defines a calibration, and on a Spin(7)-manifold the defining 4-form (the Cayley form) is a calibration as well [5,6].

It is not easy to construct solutions of (1.1) for $d > 4$ and to describe their moduli space.[1] It was shown by Donaldson, Thomas, Tian [5,10] and others that the *adiabatic limit* method provides a useful and powerful tool. The adiabatic limit refers to the geometric process of shrinking a metric in some directions while leaving it fixed in the others.[2] It is assumed that on X there is

1 Some explicit solutions for particular manifolds X were constructed e.g. in [21,23,25,14,27].

2 In lower dimensions, the adiabatic limit was successfully used for a description of solutions to the d=2+1 Ginzburg–Landau equations and to the d=4 Seiberg–Witten monopole equations (see e.g. reviews [29,30] and the references therein).

a family Σ_ε of $(d-4)$-forms with a real parameter ε such that $\Sigma_0 = \lim_{\varepsilon\to 0} \Sigma_\varepsilon$ defines a calibrated submanifold Y of X. Then one can define a normal bundle $N(Y)$ of Y with a projection

$$\pi : N(Y) \to Y . \tag{1.3}$$

The metric on X induces on $N(Y)$ a Riemannian metric

$$g_\varepsilon = \pi^* g_Y + \varepsilon^2 g_Z , \tag{1.4}$$

where $Z \cong \mathbb{R}^4$ is a typical fibre. In fact, the fibres are calibrated by a 4-form Q_ε dual to Σ_ε. The metric (1.4) extends to a tubular neighborhood of Y in X, and (1.1) may be considered on this subset of X. Anyway, it was shown [5,10,6] that solutions of the instanton equation (1.1) defined by the form Σ_ε on (X, g_ε) in the adiabatic limit $\varepsilon \to 0$ converge to sigma-model instantons describing a map from the $(d-4)$-dimensional submanifold Y into the hyper-Kähler moduli space of framed Yang–Mills instantons on fibres $\mathbb{R}^4$ of the normal bundle $N(Y)$.

The submanifold $Y \hookrightarrow X$ is calibrated by the $(d-4)$-form Σ defining the instanton equation (1.1). However, on X there may exist other p-forms φ and associated φ-calibrated submanifolds Y of dimension $p \neq d-4$. In such a case one can define a different normal bundle (1.3) with fibres $\mathbb{R}^{d-p}$ and deform the metric as in (1.4). However, this task is quite difficult technically and will be postponed for a future work. As a more simple task, one may take a direct product manifold $X = Y \times Z$ with $\dim_\mathbb{R} Y = p$ and $\dim_\mathbb{R} Z = q = d-p$ with a p-form $\varphi = vol_Y$, or consider non-flat manifolds Z and a $(d-4)$-form Σ defining (1.1). In string theory $\dim_\mathbb{R} X = 10$, and calibrated submanifolds Y are identified with worldvolumes of p-branes where p varies from zero to ten.

In this short paper we explore the direct product case $X = Y \times Z$ with $\dim_\mathbb{R} Y = p \neq d-4$ for Kähler manifolds X and the adiabatic limit of the Hermitian Yang–Mills equations on bundles over X. We will show that for even p (and hence even q) the adiabatic limit of (1.1) yields sigma-model instanton equations describing holomorphic maps from Y into the moduli space of Hermitian Yang–Mills instantons on Z. For odd p and q the consideration is more involved, and we describe only the case $p{=}q{=}3$ in which we obtain maps from Y into the moduli space of flat connections on Z. For the purpose of this paper, this special case sufficiently illustrates the main features of the odd-dimensional cases.

2. Moduli space of instantons in $d \geq 4$

Bundles. Let X be an oriented smooth manifold of dimension d, G a semisimple compact Lie group, $\mathfrak{g}$ its Lie algebra, P a principal G-bundle over X, $\mathcal{A}$ a connection 1-form on P and $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ its curvature. We consider also the bundle of groups $\mathrm{Int}P = P \times_G G$ (G acts on itself by internal automorphisms: $h \mapsto ghg^{-1}$, $h, g \in G$) associated with P, the bundle of Lie algebras $\mathrm{Ad}P = P \times_G \mathfrak{g}$ and a complex vector bundle $E = P \times_G V$, where V is the space of some irreducible representation of G. All these associated bundles inherit their connection $\mathcal{A}$ from P.

Gauge transformations. We denote by $\mathbb{A}'$ the space of connections on P and by $\mathcal{G}'$ the infinite-dimensional group of gauge transformations (automorphisms of P which induce the identity transformation of X),

$$\mathcal{A} \mapsto \mathcal{A}^g = g^{-1}\mathcal{A}g + g^{-1}\mathrm{d}g , \tag{2.1}$$

which can be identified with the space of global sections of the bundle $\mathrm{Int}P$. Correspondingly, the infinitesimal action of $\mathcal{G}'$ is defined by global sections χ of the bundle $\mathrm{Ad}P$,

$$\mathcal{A} \mapsto \delta_\chi \mathcal{A} = \mathrm{d}\chi + [\mathcal{A}, \chi] =: D_{\mathcal{A}}\chi \tag{2.2}$$

with $\chi \in \mathrm{Lie}\mathcal{G}' = \Gamma(X, \mathrm{Ad}P)$.

Moduli space of connections. We restrict ourselves to the subspace $\mathbb{A} \subset \mathbb{A}'$ of irreducible connections and to the subgroup $\mathcal{G} = \mathcal{G}'/Z(\mathcal{G}')$ of $\mathcal{G}'$ which acts freely on $\mathbb{A}$. Then the *moduli space* of irreducible connections on P (and on E) is defined as the quotient $\mathbb{A}/\mathcal{G}$. We do not distinguish connections related by a gauge transformation. Classes of gauge equivalent connections are points $[\mathcal{A}]$ in $\mathbb{A}/\mathcal{G}$.

Metric on $\mathbb{A}/\mathcal{G}$. Since $\mathbb{A}$ is an affine space, for each $\mathcal{A} \in \mathbb{A}$ we have a canonical identification between the tangent space $T_{\mathcal{A}}\mathbb{A}$ and the space $\Lambda^1(X, \mathrm{Ad}P)$ of 1-forms on X with values in the vector bundle $\mathrm{Ad}P$. We consider $\mathfrak{g}$ as a matrix Lie algebra, with the metric defined by the trace. The metrics on X and on the Lie algebra $\mathfrak{g}$ induce an inner product on $\Lambda^1(X, \mathrm{Ad}P)$,

$$\langle \xi_1, \xi_2 \rangle = \int_X \mathrm{tr}\,(\xi_1 \wedge *\xi_2) \qquad \text{for} \qquad \xi_1, \xi_2 \in \Lambda^1(X, \mathrm{Ad}P)\ . \tag{2.3}$$

This inner product is transferred to $T_{\mathcal{A}}\mathbb{A}$ by the canonical identification. It is invariant under the $\mathcal{G}$-action on $\mathbb{A}$, whence we get a metric (2.3) on the moduli space $\mathbb{A}/\mathcal{G}$.

Instantons. Suppose there exists a $(d-4)$-form Σ on X which allows us to introduce the instanton equation

$$*\mathcal{F} + \Sigma \wedge \mathcal{F} = 0 \tag{2.4}$$

discussed in Section 1. We denote by $\mathcal{N} \subset \mathbb{A}$ the space of irreducible connections subject to (2.4) on the bundle $E \to X$. This space $\mathcal{N}$ of instanton solutions on X is a subspace of the affine space $\mathbb{A}$, and we define the moduli space $\mathcal{M}$ of instantons as the quotient space

$$\mathcal{M} = \mathcal{N}/\mathcal{G} \tag{2.5}$$

together with a projection

$$\pi : \mathcal{N} \overset{\mathcal{G}}{\to} \mathcal{M}\ . \tag{2.6}$$

According to the bundle structure (2.6), at any point $\mathcal{A} \in \mathcal{N}$, the tangent bundle $T_{\mathcal{A}}\mathcal{N} \to \mathcal{N}$ splits into the direct sum

$$T_{\mathcal{A}}\mathcal{N} = \pi^* T_{[\mathcal{A}]}\mathcal{M} \oplus T_{\mathcal{A}}\mathcal{G}\ . \tag{2.7}$$

In other words,

$$T_{\mathcal{A}}\mathcal{N} \ni \tilde{\xi} = \xi + D_{\mathcal{A}}\chi \qquad \text{with} \qquad \xi \in \pi^* T_{[\mathcal{A}]}\mathcal{M} \quad \text{and} \quad D_{\mathcal{A}}\chi \in T_{\mathcal{A}}\mathcal{G}\ , \tag{2.8}$$

where $\tilde{\xi}, \xi \in \Lambda^1(X, \mathrm{Ad}P)$ and $\chi \in \Lambda^0(X, \mathrm{Ad}P) = \Gamma(X, \mathrm{Ad}P)$. The choice of ξ corresponds to a local fixing of a gauge.

Metric on $\mathcal{M}$. Denote by ξ_α a local basis of vector fields on $\mathcal{M}$ (sections of the tangent bundle $T\mathcal{M}$) with $\alpha = 1, \dots, \dim_{\mathbb{R}}\mathcal{M}$. Restricting the metric (2.3) on $\mathbb{A}/\mathcal{G}$ to the subspace $\mathcal{M}$ provides a metric $\mathbb{G} = (G_{\alpha\beta})$ on the instanton moduli space,

$$G_{\alpha\beta} = \int_X \mathrm{tr}\,(\xi_\alpha \wedge *\xi_\beta)\ . \tag{2.9}$$

Kähler forms on $\mathcal{M}$. If X is Kähler with a complex structure J and a Kähler form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, then the Kähler 2-form $\Omega = (\Omega_{\alpha\beta})$ on $\mathcal{M}$ is given by

$$\Omega_{\alpha\beta} = -\int_X \mathrm{tr}\,(J\xi_\alpha \wedge *\xi_\beta)\ . \tag{2.10}$$

It is well known that the moduli space of framed instantons[3] on a hyper-Kähler 4-manifold X (with three integrable almost complex structures J^i) is hyper-Kähler, with three Kähler forms

$$\Omega^i_{\alpha\beta} = -\int_X \mathrm{tr}\,(J^i\xi_\alpha \wedge *\xi_\beta)\ . \tag{2.11}$$

3. Hermitian Yang–Mills equations

Instanton equations. On any Kähler manifold X of dimension $d = 2n$ there exists an integrable almost complex structure $J \in \mathrm{End}(TX)$, $J^2 = -\mathrm{Id}$, and a Kähler $(1,1)$-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ compatible with J. The natural 4-form

$$Q = \tfrac{1}{2}\omega\wedge\omega \tag{3.1}$$

and its dual $\Sigma = *Q$ allow one to formulate the instanton equation (2.4) for a connection $\mathcal{A}$ on a complex vector bundle E over X associated to the principal bundle $P(X,G)$. The fibres $\mathbb{C}^N$ of E support an irreducible G-representation. For simplicity, we have in mind the fundamental representation of $\mathrm{SU}(N)$. One can endow the bundle E with a Hermitian metric and choose $\mathcal{A}$ to be compatible with the Hermitian structure on E.

The instanton equations in the form (2.4) with $\Sigma = \frac{1}{2} * (\omega\wedge\omega)$ may then be rewritten as the following pair of equations,

$$\mathcal{F}^{0,2} = -(\mathcal{F}^{2,0})^\dagger = 0 \tag{3.2}$$

and

$$\omega^{n-1}\wedge\mathcal{F} = 0 \qquad \Leftrightarrow \qquad \omega\,\lrcorner\,\mathcal{F} = \omega^{\hat\mu\hat\nu}\mathcal{F}_{\hat\mu\hat\nu} = 0\ , \tag{3.3}$$

where $\hat\mu, \hat\nu, \ldots = 1, \ldots, 2n$, and the notation $\omega\lrcorner$ exploits the underlying Riemannian metric of X for raising indices of ω. Eqs. (3.2)–(3.3) were introduced by Donaldson, Uhlenbeck and Yau [19] and are called the Hermitian Yang–Mills (HYM) equations.[4] The HYM equations have the following algebro-geometric interpretation. Eq. (3.2) implies that the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ is of type $(1,1)$ with respect to J, whence the connection $\mathcal{A}$ defines a *holomorphic structure* on E. Eq. (3.3) means that $E \to X$ is a *polystable* vector bundle. The moduli space $\mathcal{M}_X$ of HYM connections on E, the metric $\mathbb{G} = (G_{\alpha\beta})$ and the Kähler form $\Omega = (\Omega_{\alpha\beta})$ on $\mathcal{M}_X$ are introduced as described in Section 2 after specializing X to be Kähler.

Direct product of Kähler manifolds. The subject of this paper is the adiabatic limit of the HYM equations (3.2)–(3.3) on a direct product

[3] Framed instantons are instantons modulo gauge transformations which approach the identity at a fixed point.

[4] Instead of (3.3) one sometimes finds $\omega\lrcorner\mathcal{F} = \mathrm{i}\lambda\,\mathrm{Id}_E$ with $\lambda\in\mathbb{R}$. We take $\lambda = 0$, i.e. assume $c_1(E) = 0$, since one may always pass from a rank-N bundle of non-zero degree to one of zero degree by considering $\tilde{\mathcal{F}} = \mathcal{F} - \frac{1}{N}(\mathrm{tr}\mathcal{F})\mathbf{1}_N$.

$$X \ = \ Y \times Z \tag{3.4}$$

of Kähler manifolds Y and Z. The dimensions p and q of Y and Z are even, and $p+q=2n$. Let $\{e^a\}$ with $a=1,\ldots,p$ and $\{e^\mu\}$ with $\mu=p+1,\ldots,2n$ be local frames for the cotangent bundles T^*Y and T^*Z, respectively. Then $\{e^{\hat\mu}\}=\{e^a,e^\mu\}$ with $\hat\mu=1,\ldots,2n$ will be a local frame for the cotangent bundle $T^*X=T^*Y\oplus T^*Z$. We introduce on $Y\times Z$ the metric

$$g \ = \ g_Y+g_Z \ = \ \delta_{ab}\, e^a\otimes e^b+\delta_{\mu\nu}\, e^\mu\otimes e^\nu \ = \ \delta_{\hat\mu\hat\nu}\, e^{\hat\mu}\otimes e^{\hat\nu} \tag{3.5}$$

and an integrable almost complex structure

$$J \ = \ J_Y\oplus J_Z\in \mathrm{End}(TY)\oplus\mathrm{End}(TZ)\ , \qquad J_Y^2=-\mathrm{Id}_Y \quad \text{and} \quad J_Z^2=-\mathrm{Id}_Z\ , \tag{3.6}$$

whose components are defined by $J_Y e^a = J^a_b e^b$ and $J_Z e^\mu = J^\mu_\nu e^\nu$. Likewise, the Kähler form $\omega(\cdot,\cdot)=g(J\,\cdot,\cdot)$ on $Y\times Z$ decomposes as

$$\omega=\omega_Y+\omega_Z \tag{3.7}$$

with components $\omega_Y=(\omega_{ab})$ and $\omega_Z=(\omega_{\mu\nu})$.

Splitting of the HYM equations. We introduce on $X=Y\times Z$ local coordinates $\{y^a\}$ and $\{z^\mu\}$ and choose $e^a=\mathrm{d}y^a$, $e^\mu=\mathrm{d}z^\mu$. Any connection on the bundle $E\to X$ is decomposed as

$$\mathcal{A} \ = \ \mathcal{A}_Y+\mathcal{A}_Z \ = \ \mathcal{A}_a\mathrm{d}y^a+\mathcal{A}_\mu\mathrm{d}z^\mu\ , \tag{3.8}$$

where the components $\mathcal{A}_a$ and $\mathcal{A}_\mu$ depend on $(y,z)\in Y\times Z$. The curvature $\mathcal{F}$ of $\mathcal{A}$ has components $\mathcal{F}_{ab}$ along Y, $\mathcal{F}_{\mu\nu}$ along Z, and $\mathcal{F}_{a\mu}$ which we call "mixed".

Note that the holomorphicity conditions (3.2) may be expressed through the projector

$$\bar P=\tfrac{1}{2}\,(\mathrm{Id}+\mathrm{i}J)\ , \qquad \bar P^2=\bar P \tag{3.9}$$

onto the $(0,1)$-part of the complexification of the cotangent bundle $T^*X=T^*Y\oplus T^*Z$ as

$$\bar P\bar P\mathcal{F} \ = \ 0\ , \tag{3.10}$$

which in components reads

$$\big(\delta^{\hat\sigma}_{\hat\mu}+\mathrm{i}J^{\hat\sigma}_{\hat\mu}\big)\big(\delta^{\hat\lambda}_{\hat\nu}+\mathrm{i}J^{\hat\lambda}_{\hat\nu}\big)\mathcal{F}_{\hat\sigma\hat\lambda} \ = \ 0\ . \tag{3.11}$$

From (3.6) it follows that these equations split into three parts:

$$\big(\delta^c_a+\mathrm{i}J^c_a\big)\big(\delta^d_b+\mathrm{i}J^d_b\big)\mathcal{F}_{cd} \ = \ 0 \qquad \Leftrightarrow \qquad \mathcal{F}^{0,2}_Y=0\ , \tag{3.12}$$

$$\big(\delta^\sigma_\mu+\mathrm{i}J^\sigma_\mu\big)\big(\delta^\lambda_\nu+\mathrm{i}J^\lambda_\nu\big)\mathcal{F}_{\sigma\lambda} \ = \ 0 \qquad \Leftrightarrow \qquad \mathcal{F}^{0,2}_Z=0\ , \tag{3.13}$$

and

$$\mathcal{F}_{a\nu}J^\nu_\mu+J^c_a\mathcal{F}_{c\mu} \ = \ 0 \qquad \Leftrightarrow \qquad \mathcal{F}_{a\mu}-J^c_aJ^\nu_\mu\mathcal{F}_{c\nu} \ = \ 0\ . \tag{3.14}$$

Finally, with the help of (3.7) the stability equation (3.3) takes the form

$$\omega_Y\,\lrcorner\,\mathcal{F}_Y+\omega_Z\,\lrcorner\,\mathcal{F}_Z \ = \ \omega^{ab}\mathcal{F}_{ab}+\omega^{\mu\nu}\mathcal{F}_{\mu\nu} \ = \ 0\ . \tag{3.15}$$

4. Adiabatic limit of the HYM equations for even p and q

Moduli space $\mathcal{M}_Z$. In order to investigate the adiabatic limit of (3.12)–(3.15), we introduce on $X = Y \times Z$ the deformed metric and Kähler form

$$g_\varepsilon = g_Y + \varepsilon^2 g_Z \qquad \text{and} \qquad \omega_\varepsilon = \omega_Y + \varepsilon^2 \omega_Z , \tag{4.1}$$

while the complex structure $J = J_Y \oplus J_Z$ does not depend on ε according to (3.6). Since J_Y and J_Z are untouched, (3.12)–(3.14) keep their form in the adiabatic limit $\varepsilon \to 0$. In particular, (3.12) implies that $\mathcal{F}_Y^{0,2} = 0$, i.e. the bundle $E \to Y \times Z$ is holomorphic along Y for any $z \in Z$.[5] On the other hand, (3.15) for $\varepsilon \to 0$ becomes

$$\omega_Z \lrcorner \mathcal{F}_Z = \omega^{\mu\nu} \mathcal{F}_{\mu\nu} = 0 , \tag{4.2}$$

which together with (3.13) means that $\mathcal{A}_Z$ is a HYM connection (framed instanton) on Z for any given $y \in Y$. We denote the moduli space of such connections by

$$\mathcal{M}_Z = \mathcal{N}_Z / \mathcal{G}_Z , \tag{4.3}$$

where $\mathcal{N}_Z$ is the space of all instanton solutions on Z for a fixed $y \in Y$, and $\mathcal{G}_Z$ consists of the elements of $\mathcal{G}$ with the same fixed value of y. We here suppress the y dependence in our notation. The moduli space $\mathcal{M}_Z$ is a Kähler manifold on which we introduce the metric $\mathbb{G}$ and Kähler form Ω with components

$$G_{\alpha\beta} = \int_Z \mathrm{tr}\,(\xi_\alpha \wedge *_Z \xi_\beta) \qquad \text{and} \qquad \Omega_{\alpha\beta} = -\int_Z \mathrm{tr}\,(J_Z \xi_\alpha \wedge *_Z \xi_\beta) \tag{4.4}$$

similar to (2.9) and (2.10) but now with $\xi_\alpha \in \Lambda^1(Z, \mathrm{Ad}P)$ and the Hodge operator $*_Z$ defined on Z. Note that for $\dim_{\mathbb{R}} Z = 2$ the HYM equations (3.13) and (4.2) enforce $\mathcal{F}_Z = 0$, i.e. $\mathcal{M}_Z$ becomes the moduli space of flat connections on bundles $E(y)$ over a two-dimensional Riemannian manifold Z.

A map into $\mathcal{M}_Z$. The bundle $E(y)$ is a HYM vector bundle over Z for any $y \in Y$. Letting the point y vary, the connection $\mathcal{A}_Z = \mathcal{A}_\mu(y,z)\mathrm{d}z^\mu$ on $E(y)$ defines a map

$$\phi : Y \to \mathcal{M}_Z \qquad \text{with} \qquad \phi(y) = \{\phi^\alpha(y)\} , \tag{4.5}$$

where ϕ^α with $\alpha = 1, \dots, \dim_{\mathbb{R}} \mathcal{M}_Z$ are local coordinates on $\mathcal{M}_Z$. This map is constrained by our remaining set of equations, namely (3.14) for the mixed field-strength components

$$\mathcal{F}_{a\mu} = \partial_a \mathcal{A}_\mu - \partial_\mu \mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_\mu] = \partial_a \mathcal{A}_\mu - D_\mu \mathcal{A}_a . \tag{4.6}$$

Similarly to (2.7) and (2.8), $\partial_a \mathcal{A}_\mu$ decomposes into two parts,

$$T_{\mathcal{A}_Z}\mathcal{N}_Z = \pi^* T_{[\mathcal{A}_Z]}\mathcal{M}_Z \oplus T_{\mathcal{A}_Z}\mathcal{G}_Z \qquad \Leftrightarrow \qquad \partial_a \mathcal{A}_\mu = (\partial_a \phi^\alpha)\xi_{\alpha\mu} + D_\mu \epsilon_a , \tag{4.7}$$

where $\{\xi_\alpha = \xi_{\alpha\mu}\mathrm{d}z^\mu\}$ is a local basis of vector fields on $\mathcal{M}_Z$. Here, ϵ_a are $\mathfrak{g}$-valued gauge parameters which are determined by the gauge-fixing equations

$$(\partial_a \phi^\alpha)\, g^{\mu\nu} D_\mu \xi_{\alpha\nu} = 0 \qquad \Rightarrow \qquad g^{\mu\nu} D_\mu D_\nu \epsilon_a = g^{\mu\nu} D_\mu \partial_a \mathcal{A}_\nu . \tag{4.8}$$

5 We can always choose a gauge such that $\mathcal{A}_Y^{0,1} = 0$ and locally $\mathcal{A}_Y^{1,0} = h^{-1}\partial_Y h$ for a G-valued function $h(y,z)$.

Substituting (4.7) into (4.6), the mixed field-strength components simplify to

$$\mathcal{F}_{a\mu} = (\partial_a \phi^\alpha)\,\xi_{\alpha\mu} - D_\mu(\mathcal{A}_a - \epsilon_a)\,. \tag{4.9}$$

Inserting this expression into our remaining equations (3.14), we obtain

$$(\partial_a \phi^\alpha)\,\xi_{\alpha\mu} - J_a^c J_\mu^\sigma (\partial_c \phi^\alpha)\,\xi_{\alpha\sigma} = D_\mu(\mathcal{A}_a - \epsilon_a) - J_a^c J_\mu^\sigma D_\sigma(\mathcal{A}_c - \epsilon_c) \tag{4.10}$$

as a condition on the map ϕ.

Sigma-model instantons. In order to better interpret the above equations, we multiply both sides with $\mathrm{d}z^\mu \wedge *_Z \xi_\beta$, take the trace over $\mathfrak{g}$, integrate over Z and recognize the integrals in (4.4). The integral of the right-hand side of (4.10) vanishes due to (4.7)–(4.8) (orthogonality of $\xi_\alpha \in T\mathcal{M}_Z$ and $D\chi \in T\mathcal{G}_Z$), and we end up with

$$(\partial_a \phi^\alpha) G_{\alpha\beta} + J_a^c\,(\partial_c \phi^\alpha)\Omega_{\alpha\beta} = 0\,. \tag{4.11}$$

Inverting the moduli-space metric G and introducing the almost complex structure $\mathcal{J}$ on $\mathcal{M}_Z$ via its components

$$\mathcal{J}_\beta^\alpha := \Omega_{\beta\gamma} G^{\gamma\alpha}\,, \tag{4.12}$$

we rewrite (4.11) as

$$\partial_a \phi^\alpha = -J_a^c\,(\partial_c \phi^\beta)\mathcal{J}_\beta^\alpha \qquad \Leftrightarrow \qquad \mathrm{d}\phi = -\mathcal{J} \circ \mathrm{d}\phi \circ J\,. \tag{4.13}$$

Using $J_c^a J_b^c = -\delta_b^a$ and $\mathcal{J}_\gamma^\alpha \mathcal{J}_\beta^\gamma = -\delta_\beta^\alpha$, alternative versions are

$$(\partial_a \phi^\beta)\mathcal{J}_\beta^\alpha - J_a^b\,(\partial_b \phi^\alpha) = 0 \qquad \Leftrightarrow \qquad \mathcal{J} \circ \mathrm{d}\phi = \mathrm{d}\phi \circ J \tag{4.14}$$

and

$$(\delta_a^b + \mathrm{i}J_a^b)\,(\partial_b \phi^\beta)(\delta_\beta^\alpha - \mathrm{i}\mathcal{J}_\beta^\alpha) = 0 \qquad \Leftrightarrow \qquad \mathcal{P} \circ \mathrm{d}\phi \circ \bar{P} = 0\,, \tag{4.15}$$

with the obvious definition for $\mathcal{P}$.

These equations mean that $\phi^1 + \mathrm{i}\phi^2,\ \phi^3 + \mathrm{i}\phi^4,\ \ldots$ are holomorphic functions of complex coordinates on Y, i.e. ϕ is a holomorphic map. It is clear that our equations (4.15) are BPS-type (instanton) first-order equations for the sigma model on Y with target space $\mathcal{M}_Z$, whose field equations define harmonic maps from Y into $\mathcal{M}_Z$. For $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} Z = 2$ these equations have appeared in [31] as the adiabatic limit of the HYM equations on the product of two Riemann surfaces.[6] Our (4.15) generalize [31] to the case $\dim_{\mathbb{R}} Y > 2$ and $\dim_{\mathbb{R}} Z \geq 2$. From the implicit function theorem it follows that near every solution ϕ of (4.15) there exists a solution $\mathcal{A}_\varepsilon$ of the HYM equations (3.2)–(3.3) for ε sufficiently small. In other words, solutions of (4.15) approximate solutions of the HYM equations on X.

5. Adiabatic limit of gauge instantons for $p = q = 3$

If the Kähler manifold X is a direct product of two *odd*-dimensional manifolds Y and Z, i.e. if $p = \dim_{\mathbb{R}} Y$ and $q = \dim_{\mathbb{R}} Z$ are both odd, then we may need to impose conditions on the geometry of Y and Z for $X = Y \times Z$ to be Kähler. However, we are not aware of these demands

[6] See also [32] where this limit was discussed in the framework of topological Yang–Mills theories.

outside of special cases, such as products of tori. Therefore, we restrict ourselves to tori Y and Z with $p = q = 3$ since already this case illustrates essential differences from the case of even p and q. More general situations demand more effort and will be considered elsewhere.

Deformed structures. We consider the Calabi–Yau space

$$X \;=\; Y \times Z \;=\; T^3 \times T^3_r \;, \tag{5.1}$$

where T^3 is a 3-torus and T^3_r is another 3-torus, with r marked points (punctures). We endow X with the deformed metric

$$\begin{aligned} g_\varepsilon &= g_{T^3} + \varepsilon^2 g_{T^3_r} \\ &= e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + \varepsilon^2 (e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6) \end{aligned} \tag{5.2}$$

and choose the basis of $(1, 0)$-forms as

$$\theta^1 = e^1 + \mathrm{i}\varepsilon e^4 \;, \quad \theta^2 = e^2 + \mathrm{i}\varepsilon e^5 \quad \text{and} \quad \theta^3 = e^3 + \mathrm{i}\varepsilon e^6 \tag{5.3}$$

with a real deformation parameter ε.

The combined torus $T^3 \times T^3_r$ supports an integrable almost complex structure J satisfying $J\theta^j = \mathrm{i}\theta^j$ for $j = 1, 2, 3$, which determines its components,

$$J e^{\hat\mu} = J^{\hat\mu}_{\hat\nu} e^{\hat\nu} : \quad J^1_4 = J^2_5 = J^3_6 = -\varepsilon \quad \text{and} \quad J^4_1 = J^5_2 = J^6_3 = \varepsilon^{-1} \;. \tag{5.4}$$

For the Kähler form $\omega(\cdot,\cdot) = g(J\,\cdot\,,\cdot)$ the components are

$$\omega_{14} = \omega_{25} = \omega_{36} = \varepsilon \qquad \text{and} \qquad \omega_{41} = \omega_{52} = \omega_{63} = -\varepsilon \;. \tag{5.5}$$

Adiabatic limit for instantons. The HYM equations (3.2) and (3.3) on $T^3 \times T^3_r$ with J and ω given by (5.4) and (5.5) read

$$\begin{aligned} \mathcal{F}_{ab} + \mathrm{i}\mathcal{F}_{a\mu} J^\mu_b + \mathrm{i}J^\mu_a \mathcal{F}_{\mu b} - J^\mu_a J^\nu_b \mathcal{F}_{\mu\nu} &= 0 \;, \\ \mathcal{F}_{\mu\nu} + \mathrm{i}\mathcal{F}_{\mu b} J^b_\nu + \mathrm{i}J^b_\mu \mathcal{F}_{b\nu} - J^a_\mu J^b_\nu \mathcal{F}_{ab} &= 0 \;, \\ \mathcal{F}_{a\mu} + \mathrm{i}\mathcal{F}_{ab} J^b_\mu + \mathrm{i}J^\nu_a \mathcal{F}_{\nu\mu} - J^\nu_a J^b_\mu \mathcal{F}_{\nu b} &= 0 \;, \end{aligned} \tag{5.6}$$

with $a, b = 1, 2, 3$ and $\mu, \nu = 4, 5, 6$, as well as

$$\mathcal{F}_{14} + \mathcal{F}_{25} + \mathcal{F}_{36} \;=\; 0 \;. \tag{5.7}$$

In the adiabatic limit $\varepsilon \to 0$ the first two lines of (5.6) reduce to

$$\mathcal{F}_{45} \;=\; \mathcal{F}_{46} \;=\; \mathcal{F}_{56} \;=\; 0 \tag{5.8}$$

while the mixed-component part of (5.6) together with (5.7) produces

$$\begin{aligned} &\mathcal{F}_{16} - \mathcal{F}_{34} \;=\; 0 \;, \quad \mathcal{F}_{35} - \mathcal{F}_{26} \;=\; 0 \;, \quad \mathcal{F}_{24} - \mathcal{F}_{15} \;=\; 0 \qquad \text{and} \\ &\mathcal{F}_{14} + \mathcal{F}_{25} + \mathcal{F}_{36} \;=\; 0 \;. \end{aligned} \tag{5.9}$$

Recall that

$$\mathcal{A} \;=\; \mathcal{A}_Y + \mathcal{A}_Z \;=\; \mathcal{A}_a(y, z)\mathrm{d}y^a + \mathcal{A}_\mu(y, z)\mathrm{d}z^\mu \tag{5.10}$$

is a connection on a vector bundle E over $X = T^3 \times T^3_r$. From (5.8) we learn that $\mathcal{A}_Z$ is a flat connection on $Z = T^3_r$ for any $y \in Y = T^3$. We denote by $\mathcal{N}_Z$ the space of solutions to (5.8) and by $\mathcal{M}_Z$ the moduli space of all such connections. From (5.9) we see that in the adiabatic limit there are no restrictions on $\mathcal{A}_Y$, since the components $\mathcal{A}_a$ and $\mathcal{F}_{ab}$ no longer appear.

Sigma-model equations. For the mixed components $\mathcal{F}_{a\mu}$ of the field strength we have

$$\mathcal{F}_{a\mu} = \partial_a \mathcal{A}_\mu - D_\mu \mathcal{A}_a = (\partial_a \phi^\alpha)\xi_{\alpha\mu} - D_\mu(\mathcal{A}_a - \epsilon_a) \tag{5.11}$$

where, as in Section 4, we used for $\partial_a \mathcal{A}_\mu$ the decomposition formula (4.7) and introduced the map

$$\phi:\ T^3 \to \mathcal{M}_{T^3_r}\ . \tag{5.12}$$

Let us, for a short while, relax the gauge fixing (4.8) and allow $\phi(y)$ to take values in the full solution space $\mathcal{N}_{T^3_r}$. Correspondingly $\xi_\alpha = \xi_{\alpha\mu}\mathrm{d}z^\mu$ will be momentarily a basis of all vector fields on $\mathcal{N}_{T^3_r}$, and ϵ_a are undetermined.

Substituting (5.11) into (5.9), we obtain the equations

$$\begin{aligned}
(\partial_1\phi^\alpha)\,\xi_{\alpha 6} - (\partial_3\phi^\alpha)\,\xi_{\alpha 4} &= D_6(\mathcal{A}_1 - \epsilon_1) - D_4(\mathcal{A}_3 - \epsilon_3)\ ,\\
(\partial_3\phi^\alpha)\,\xi_{\alpha 5} - (\partial_2\phi^\alpha)\,\xi_{\alpha 6} &= D_5(\mathcal{A}_3 - \epsilon_3) - D_6(\mathcal{A}_2 - \epsilon_2)\ ,\\
(\partial_2\phi^\alpha)\,\xi_{\alpha 4} - (\partial_1\phi^\alpha)\,\xi_{\alpha 5} &= D_4(\mathcal{A}_2 - \epsilon_2) - D_5(\mathcal{A}_1 - \epsilon_1)
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&(\partial_1\phi^\alpha)\,\xi_{\alpha 4} + (\partial_2\phi^\alpha)\,\xi_{\alpha 5} + (\partial_3\phi^\alpha)\,\xi_{\alpha 6}\\
&\quad = D_4(\mathcal{A}_1 - \epsilon_1) + D_5(\mathcal{A}_2 - \epsilon_2) + D_6(\mathcal{A}_3 - \epsilon_3)\ .
\end{aligned} \tag{5.14}$$

Multiplying both sides with $\xi_{\beta\mu}$ for $\mu = 4, 5, 6$ and integrating $\mathrm{tr}\,(\xi_{\alpha\mu}\xi_{\beta\nu})$ over T^3_r, the above four equations yield the $3\dim_{\mathbb{R}}\mathcal{N}_{T^3_r}$ relations

$$\partial_a\phi^\alpha + \pi_a{}^b{}_c\,(\partial_b\phi^\beta)\,\Pi^c{}^\alpha_\beta = \mathrm{j}^\alpha_a\ , \tag{5.15}$$

where

$$\pi_a{}^b{}_c := \varepsilon^b_{ac} \qquad \text{and} \qquad \Pi^a{}^\alpha_\beta := \Pi^a_{\beta\gamma}G^{\gamma\alpha} \tag{5.16}$$

with

$$G_{\alpha\beta} = \int_{T^3_r} \mathrm{d}^3z\ \delta^{\mu\nu}\,\mathrm{tr}\,(\xi_{\alpha\mu}\xi_{\beta\nu}) \qquad \text{and} \qquad \Pi^a_{\alpha\beta} = \int_{T^3_r} \mathrm{d}^3z\ \varepsilon^{a+3\,\mu\nu}\,\mathrm{tr}\,(\xi_{\alpha\mu}\xi_{\beta\nu})\ . \tag{5.17}$$

The right-hand side of (5.15) is given by

$$\mathrm{j}^\alpha_a = G^{\alpha\beta}\int_{T^3_r} \mathrm{d}^3z\ \mathrm{tr}\left\{\delta^b_a\,\delta^{\mu\nu} + \varepsilon^b_{ac}\,\varepsilon^{c+3\,\mu\,\nu}\right\} D_\mu(\mathcal{A}_b - \epsilon_b)\,\xi_{\beta\nu}\ . \tag{5.18}$$

The $(1,1)$ tensors $\pi_a = (\varepsilon^b_{ac})$, $a = 1, 2, 3$, on T^3 and the $(1,1)$ tensors $\Pi_a = (\delta_{ab}\Pi^b{}^\alpha_\beta)$ on $\mathcal{N}_{T^3_r}$ satisfy the identities

$$\pi^3_a + \pi_a = 0 \qquad \text{and} \qquad \Pi^3_a + \Pi_a = 0\ , \tag{5.19}$$

i.e. they define three so-called f-structures [33] correspondingly on T^3 and on $\mathcal{N}_{T^3_r}$. To clarify their meaning we observe that (5.19) defines orthogonal projectors

$$P_a := -\pi_a^2 \qquad \text{and} \qquad P_a^\perp := \mathbb{1}_3 + \pi_a^2 \tag{5.20}$$

of rank two and rank one on T^3 and similarly orthogonal projectors

$$\mathcal{P}_a := -\Pi_a^2 \qquad \text{and} \qquad \mathcal{P}_a^\perp := \mathrm{Id} + \Pi_a^2 \tag{5.21}$$

on $\mathcal{N}_{T_r^3}$, where Id is the identity tensor. The tangent bundle $T(T^3)$ splits into eigenspaces of P_a,

$$T(T^3) = T(T_a^2 \times S_a^1) = T(T_a^2) \oplus T(S_a^1) = L_a \oplus N_a \qquad \text{for} \quad a = 1, 2, 3\,, \tag{5.22}$$

which defines on T^3 two distributions L_a and N_a of rank two and one, respectively, and decomposes the 3-torus in three different ways. Analogously, the projector $\mathcal{P}_a$ yields a splitting

$$T(\mathcal{N}_{T_r^3}) = \mathcal{L}_a \oplus \mathcal{N}_a \tag{5.23}$$

which is in fact induced by the factorization of T_r^3 into a two-dimensional torus and a circle.

We now come back to the question of gauge fixing. Recalling that $\mathcal{A}_Z$ is flat on T_r^3, we gauge away one component, say

$$\mathcal{A}_6 = 0 \qquad \Rightarrow \qquad \xi_{\alpha 6} = \delta_\alpha \mathcal{A}_6 = 0\,, \tag{5.24}$$

from which it follows in (5.17) that

$$\Pi_{\alpha\beta}^1 = \Pi_{\alpha\beta}^2 = 0 \tag{5.25}$$

and only $\Pi_{\alpha\beta}^3$ is non-vanishing. With (5.24) our moduli space $\mathcal{M}_{T_r^3}$ is reduced to the moduli space $\mathcal{M}_{T_r^2}$ of flat connections on the torus T_r^2.[7] Furthermore, j_a^α defined by (5.18) must be zero since ξ_α with the gauge-fixing condition (5.24) are tangent to the moduli space $\mathcal{M}_{T_r^2}$ of flat connections on T_r^2 and therefore orthogonal to $D_\mu(\mathcal{A}_b - \epsilon_b)$ in (5.18) tangent to the gauge orbits. Thus, after fixing the gauge $\mathcal{A}_6 = 0$ the sigma-model instanton equations (5.15) reduce to

$$(\partial_1 + \mathrm{i}\partial_2)\phi^\beta(\delta_\beta^\alpha - \mathrm{i}\mathcal{J}_\beta^\alpha) = 0 \qquad \text{and} \qquad \partial_3\phi^\alpha = 0\,, \tag{5.26}$$

where $\partial_a := \partial/\partial y^a$ and $\mathcal{J}_\beta^\alpha := \Pi^{3\alpha}{}_\beta$ is a complex structure on the Kähler moduli space $\mathcal{M}_{T_r^2}$ of flat connections on T_r^2. Hence, for $p = q = 3$ we obtain the degenerate case of holomorphic maps

$$\phi :\ T^2 \to \mathcal{M}_{T_r^2} \tag{5.27}$$

from T^2 into the moduli space $\mathcal{M}_{T_r^2}$. This is degenerate in the sense that the HYM connection on $T^3 \times T_r^3$ in the adiabatic limit for (5.2) is implicitly reduced to a HYM connection on $T^2 \times T_r^2$.

Remark. The above degeneracy is not generic but relates only to the case of $q = 3$. As a counterexample, let us consider $q = 4$, for instance the G_2-instanton equations (for a definition see e.g. [5,6,12,14]) on the 7-manifold

$$X = Y \times Z = T^3 \times Z \qquad \text{with} \quad Z = T^4\,, \quad K3 \quad \text{or} \quad \mathbb{R}^4\,. \tag{5.28}$$

In the adiabatic limit of $\varepsilon \to 0$ with the deformed metric $g_\varepsilon = g_Y + \varepsilon^2 g_Z$ the G_2-instanton equations become

[7] For simplicity we locate all punctures on the two-dimensional torus.

$$\partial_a \phi^\alpha + \varepsilon^b_{ac}\,(\partial_b \phi^\beta)\,\mathcal{J}^{c\,\alpha}_{\ \ \beta} \,=\, 0\,. \tag{5.29}$$

This looks similar to (5.15) with $\mathrm{j}^\alpha_a = 0$ and features three complex structures $\mathcal{J}^c = (\mathcal{J}^{c\,\alpha}_{\ \ \beta})$ (instead of f-structures Π^c) on the hyper-Kähler moduli space $\mathcal{M}_Z$ of framed Yang–Mills instantons on the hyper-Kähler 4-manifold Z. These equations were discussed e.g. in [6,13] in the form of Fueter equations. In the above case (5.28) they define maps $\phi : T^3 \to \mathcal{M}_Z$ which are sigma-model instantons minimizing the standard sigma-model energy functional.

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