

STABILITY OF TWO-DIMENSIONAL NON-LINEAR OSCILLATIONS

with periodical hamiltonian; applied to the

NON-LINEAR BETATRON OSCILLATIONS OF AN ALTERNATING-GRADIENT SYNCHROTRON.

Summary.

We consider a mechanical system of two degrees of freedom, which can be described by a system of canonical differential equations. The hamiltonian is supposed to be explicitly time-dependent with period 2π . The aim is to bring this system by a sequence of canonical and periodical transformations into a form, where the new hamiltonian is constant and as simple as possible. After the first of these transformations one gets constant coefficients in the quadratic part of the hamiltonian and these are to be interpreted as the frequencies of the corresponding linearized system. These two constants, ω_1 and ω_2 , play a dominating part in this theory. If n_1 , n_2 and p are integers (positive, negative and zero included), then it turns out that in principle resonancelike behaviour can occur on the lines $n_1\omega_1 + n_2\omega_2 = p$. Discussion of the final form of the hamiltonian leads to the following results

- a) If n_1 and n_2 have different signs: no resonance
- b) If n_1 and n_2 have equal signs (including the case that one of them is zero) the stability behaviour depends on more specifications:
 - a) Instability (dangerous) for $|n_1 + n_2| = 2$
 - β) Instability (not necessarily dangerous) for $|n_1 + n_2| = 3$
 - γ) Whether instability or stability depends on some numerical constants for $|n_1 + n_2| = 4$
 - δ) Stability (in almost all cases) for $|n_1 + n_2| \geq 5$

The corresponding resonance-lines are given in fig. 9, 10, 11, 12 (p. 60-63).

These results are due to the construction of two constants of the motion (invariants) one of which is the new hamiltonian itself, the other one being simply a quadratic form in the amplitudes of the transformed coordinates. Whereas the latter one is exactly constant, the hamiltonian can in practice be calculated only up to some order N and is constant then up to terms of the order $N + 1$ in the new coordinates.

From these two invariants one can, for each explicit example, decide which of the possibilities in b β) and b γ) will hold, and one can calculate the amplitude ranges once the initial amplitudes are given. It is difficult however to estimate how accurate the numerical values for amplitude ranges will be. Comparisons between numerical solutions and the predictions of the discussion of an invariant in the one-dimensional case came out quite satisfactory - provided not too large non-linearities (or amplitudes) were considered. One should at least expect the calculated amplitude ranges to be a good qualitative estimate, which becomes quantitatively accurate for very small non-linearities (or amplitudes). This condition seems to be fulfilled quite good in the CERN proton synchrotron.

The above qualitative statements, however, concern the behaviour of solutions neighbouring the identical-zero-solution (and that means for vanishing amplitudes) so that they remain valid also for large non-linear terms-coefficients.

Contents

A) Introduction	4
B) The equations of motion	5
C) The zeroth transformation	6
D) The first transformation	6
E) Significance of the first transformation	10
F) The complete first transformation	13
G) The second transformation	17
H) Formal solution of the equations for s	20
J) Discussion of the principally possible instabilities	27
K) On the definition of "stability"	30
L) Classification of resonance-lines	32
M) The new hamiltonian $g(y_1 y_2 y_3 y_4 t)$ at resonance	37
N) The third transformation into the normal form	40
O) Discussion of the hamiltonian $P(\xi_1 \bar{\xi}_1 \xi_2 \bar{\xi}_2)$	42
P) The two invariant functions	42
Q) Discussion of the two invariants	45
1) The quadratic invariant	46
2) The $(N+1)^{th}$ -order invariants	48
R) More explicit form of the $(N+1)^{th}$ -order invariants	51
S) Discussion of the $(N+1)^{th}$ -order invariants	54
1) Stability at the origin	54
2) Stability near the origin	57
T) General results	58
U) Graphical representation of resonance-lines	58
V) Concluding remarks	64
W) Literature [in text quoted by the first characters of the author's name; for instance (Be)]	67
X) Appendix 1 (the first transformation)	68
Y) Appendix 2 (proof of existence of real periodic solutions for s).	73

A. Introduction.

The equations of motion in an alternating gradient synchrotron are of the Hill-type containing non-linear terms too. In a slightly idealized case they can be written in hamiltonian form where the hamiltonian contains terms higher than quadratic and has time-dependent periodical coefficients. So we wish to treat two-dimensional non-linear oscillations obeying Hamilton's equations with periodical coefficients but we shall have a look at the synchrotron every now and then.

Systems with one degree of freedom and constant coefficients have been attacked most successfully by using the phase-space and discussing the phase-trajectories. They are given by a function $f(p, q) = \text{const}$ which means a function, which is invariant under the transformation $t \rightarrow t + \delta$. If one has no constant coefficients, it is very difficult to find such functions and generally they are constant only approximately. Nevertheless, if one gets such functions they will prove to be useful insofar as one can discuss stability and amplitude-ranges. This has been done in the one-dimensional case by several authors, for instance by J. Moser (Mo) ^(*), A. Schoch (Scho) and the present writer (not published). The two methods of Moser and Schoch lead to the same invariant functions, although it is not easy to see how. A third method stems from P. Sturrock (Stu 1) which also leads to the same functions. Finally the present writer attacked the problem in a fourth way and arrived again there. This and the comparison of the results with some numerical integrations gave us much confidence into the usefulness of these methods.

The generalization into the case of two degrees of freedom is difficult in two respects: the phase-space becomes 4-dimensional and loses its evidence. Furthermore the phase-trajectory would be determined not by one but by three independent invariant functions and the methods used in the one-dimensional case will be helpful to find only one of these.

Yet it is possible to overcome somewhat these difficulties by restricting oneself on amplitude ranges instead of asking for the whole orbit. Thus the number of variables of interest is reduced to two and having two invariant functions would allow to fix the values of them both. These two functions are constructed in this paper by several transformations leading to a hamiltonian with constant coefficients in a simple normal form. This is the first invariant function and because it is not only constant but also generates the equations of motion, it helps to get the second invariant too, both being expressed as functions of the amplitudes.

^(*) See Literature p.67

B. The equations of motion

The general equations of motion in an A.G. Synchrotron can be written

$$\ddot{x}_1 + n_{11}(t)x_1 + n_{12}(t)x_2 = -\frac{\partial \Delta K}{\partial x_1} + f_1(t) \quad (a) \quad (*)$$

$$\ddot{x}_2 + n_{21}(t)x_1 + n_{22}(t)x_2 = -\frac{\partial \Delta K}{\partial x_2} + f_2(t)$$

where x_1 and x_2 mean the deviations in radial and horizontal direction respectively from the equilibrium orbit of an ideal machine. The crossterms n_{12} and n_{21} refer to twists, the functions $f_1(t)$ and $f_2(t)$ to misalignments and the derivatives of $\Delta K(x_1, x_2, t)$ to non-linear terms. The independent variable t does not mean the time but the arclength or the azimuthal angle or something else with the only condition, that all functions of t occurring in the equations are periodic with period 2π . Nevertheless, for convenience's sake we shall simply speak of "time", keeping in mind that it has not necessarily the physical dimension of time. The period 2π can be achieved by normalizing and this period then corresponds either to the structure period in an ideal machine or to the full revolution in a real machine with construction errors.

We now start simplifying the above equations by several transformations till we end up with a form where we can discuss the possible amplitude variations. This form will not be expressed in the original coordinates and therefore the discussion may seem to be of little use. We will see, however, that if we take care to keep the transformations in limits, this discussion leads at least to estimates of orders of magnitudes and applies the more directly to the original coordinates the smaller the nonlinear parts become.

(*) Formulas are quoted like this: (5a) means page 5, formula (a).

C. The Zeroth transformation

The zeroth transformation serves to eliminate the functions $f_1(t)$ and $f_2(t)$ and consists simply in introducing a new reference orbit φ_1 and φ_2 which is defined as the closed orbit (the periodic solution) of the simplified system

$$\ddot{x}_1 + n_{11}x_1 + n_{12}x_2 = f_1(t)$$

$$\ddot{x}_2 + n_{21}x_1 + n_{22}x_2 = f_2(t)$$

If we introduce $x_1 - \varphi_1 = z_1$; $x_2 - \varphi_2 = z_2$ we find

$$\ddot{z}_1 + n_{11}z_1 + n_{12}z_2 = - \frac{\partial \Delta F(z_1, z_2, t)}{\partial z_1}$$

$$\ddot{z}_2 + n_{21}z_1 + n_{22}z_2 = - \frac{\partial \Delta F}{\partial z_2}$$

Now, to save symbols, we replace $z_1 \rightarrow x_1$ and $\Delta F \rightarrow \Delta K$ so arriving at (5a) but without the terms f_1 and f_2 . Our coordinates now describe the deviations from the closed orbit $\varphi_1(t)$ $\varphi_2(t)$. of course we suppose that this closed orbit exists and has a small amplitude. This is a question which can be considered as already settled by the linear theory. In the following we shall forget about this transformation (which therefore is called the "zeroth").

D. The first transformation

The first transformation also falls into the scope of the linear theory and serves to transform the time dependence of the linear coefficients away and to decouple the linear part. If one considers an ideal machine, the n_{12} and n_{21} are not present; in a real machine, however, they are present, but they may be small. We shall consider the general case and not use its

actual smallness, which of course is very convenient for the treatment of explicit examples. The equations

$$\ddot{x}_1 + n_{11}x_1 + n_{12}x_2 = -\Delta K_{x1}$$

can be written

$$\ddot{x}_2 + n_{21}x_1 + n_{22}x_2 = -\Delta K_{x2}$$

$$\begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \equiv \xi$$

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 & &= K_{\xi 2} \\ \dot{\xi}_2 &= -n_{11}\xi_1 - n_{12}\xi_3 - \Delta K_{\xi 1} & &= -K_{\xi 1} \\ \dot{\xi}_3 &= \xi_4 & &= K_{\xi 4} \\ \dot{\xi}_4 &= -n_{21}\xi_1 - n_{22}\xi_3 - \Delta K_{\xi 3} & &= -K_{\xi 3} \end{aligned}$$

(a) and $K = K_0(\xi_1, \xi_2, \xi_3, \xi_4, t) + \Delta K(\xi_1, \xi_3, t)$. Here

(b) $K_0 = \frac{1}{2} (n_{11}\xi_1^2 + \xi_2^2 + 2n_{12}\xi_1\xi_3 + n_{22}\xi_3^2 + \xi_4^2)$ is the hamiltonian of the linear problem. (Note that this implies $n_{12} = n_{21}$ which in fact is fulfilled in our machine.) We now try to transform to new variables $\eta_1 \dots \eta_4$ so that the hamiltonian becomes

(c) $h(\eta_1, \eta_2, \eta_3, \eta_4, t) = \frac{\omega_1}{2} (\eta_1^2 + \eta_2^2) + \frac{\omega_2}{2} (\eta_3^2 + \eta_4^2) + \Delta h(\eta_1, \eta_2, \eta_3, \eta_4, t)$.

That is, the linear part of the equations for η has constant coefficients ω_1 and ω_2 and is decoupled. The equations

$$(a) \quad \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \end{pmatrix} = \begin{pmatrix} h\eta_2 \\ -h\eta_1 \\ h\eta_4 \\ -h\eta_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} + \begin{pmatrix} \Delta h\eta_2 \\ -\Delta h\eta_1 \\ \Delta h\eta_4 \\ -\Delta h\eta_3 \end{pmatrix}$$

whereas the old ones may be written

$$(b) \quad \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \begin{pmatrix} K_{\xi 2} \\ -K_{\xi 1} \\ K_{\xi 4} \\ -K_{\xi 3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -n_{11} & 0 & -n_{12} & 0 \\ 0 & 0 & 0 & 1 \\ -n_{12} & 0 & -n_{22} & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} + \begin{pmatrix} \Delta K_{\xi 2} \\ -\Delta K_{\xi 1} \\ \Delta K_{\xi 4} \\ -\Delta K_{\xi 3} \end{pmatrix}$$

At first we do not consider the higher terms but only try to transform the linear problem, which is in matrix notation

$$(c) \quad \begin{aligned} \dot{\eta} &= \Omega \eta \\ \dot{\xi} &= N \xi \end{aligned} \quad \text{where } \Omega \text{ and } N \text{ are the matrices in (8a,b)}$$

The transformation is done by a matrix A_t :

$$(d) \quad A_t \xi(t) = \eta(t), \quad \text{where} \quad A_{t+2\pi} = A_t \quad \text{and} \quad \det A_t \equiv 1$$

In the appendix 1 (page 68) is derived, that this matrix has the following form

$$(e) \quad A_t = U_t R X_t^{-1} T_t^{-1} = R D_t X_t^{-1} = R F_t^{-1}$$

The symbols U_t , R , D_t , T_t , X_t , F_t have the following meaning:

$$U_t = \begin{pmatrix} \cos \omega_1 t & \sin \omega_1 t & 0 & 0 \\ -\sin \omega_1 t & \cos \omega_1 t & 0 & 0 \\ 0 & 0 & \cos \omega_2 t & \sin \omega_2 t \\ 0 & 0 & -\sin \omega_2 t & \cos \omega_2 t \end{pmatrix} \quad \text{and}$$

$$R = \rho \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad (\text{with } \rho \text{ arbitrary}) \text{ has the property that}$$

$$R^{-1} U_t R = D_t = \begin{pmatrix} e^{i\omega_1 t} & & & 0 \\ & e^{-i\omega_1 t} & & \\ & & e^{i\omega_2 t} & \\ 0 & & & e^{-i\omega_2 t} \end{pmatrix}$$

T_t is the transfermatrix which solves the equation $\dot{\xi} = N\xi$ by

(a) $\xi(t) = T_t \xi(0) = T_t \xi_0$

X_t is the matrix, the column vectors of which are the four Floquet-solutions $\xi^{(k)}(t)$ which are defined by the property, that [because of $T_{t+2\pi} = T_t T_{2\pi}$ (see appendix p. 6)]

$$\xi^{(k)}(t+2\pi) = \lambda_k \xi^{(k)}(t) = T_t T_{2\pi} \xi_0^{(k)} = T_t \lambda_k \xi_0^{(k)} \quad \text{so that}$$

$$T_{2\pi} \xi_0^{(k)} = \lambda_k \xi_0^{(k)} \quad \text{or} \quad T_{2\pi} X_0 = X_0 \Lambda; \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \equiv D_{2\pi}$$

(The identity of Λ and $D_{2\pi}$ is proven in the appendix p. 7).

Therefore the initial vectors $\xi_0^{(k)}$, which lead to the Floquet-solutions, are the eigen vectors of $T_{2\pi}$. Then $X_t = T_t X_0$. Apart from a factor which may be chosen to be one, the Floquet-solutions are pairwise complex conjugated [because the eigenvalues of $T_{2\pi}$, which is real, are also] : $\xi^{(2)} = \bar{\xi}^{(1)}$; $\xi^{(4)} = \bar{\xi}^{(3)}$. We shall speak of X_t as "the" Floquet-solution because X_t consists of the four independent $\xi^{(k)}$ as columnvectors.

Finally F_t is the periodic part of the Floquet-solution, which can namely be decomposed into factors in the following way (see p.72)

$$X_t = F_t D_t \quad \text{where } F_{t+2\pi} = F_t \text{ and } D_t \text{ is defined above.}$$

Because D_t is periodic if and only if $\frac{\omega_1}{\omega_2} = \frac{p}{q}$, p, q integers, we shall call D_t the non-periodic part of the Floquet-solution (inspite of that it is practically always periodic).

The unity determinant of A_t is achieved by normalizing the columnvectors of F_0 and then adjusting the numerical value of ρ in R .

E. Significance of the first transformation

We now consider the significance of our transformation and try to see what the new coordinates η mean. Because of the important properties of the Floquet-solutions it is most natural to express any solution of the linear equations as a linear combination of Floquet vectors:

$$\xi(t) = a_1 \xi^{(1)}(t) + a_2 \xi^{(2)}(t) + a_3 \xi^{(3)}(t) + a_4 \xi^{(4)}(t) \equiv X_t a$$

$$\text{where } a \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = X_0^{-1} \xi_0$$

Then with (8e)

$$\eta = A_t \xi = R D_t X_t^{-1} \cdot X_t a = R D_t a = R F_t^{-1} \cdot \xi$$

That means, regarding the two last forms, η is (apart from the constant R) the "non-periodic part" of the solution ξ and A_t is (apart from R) the reciprocal of the "periodic part" of the Floquet-solution.

Finally we remark that, if ξ has real components, the same holds for η . Because the $\xi^{(k)}$ are pairwise complex conjugated, the components of a must be too: $a_2 = \bar{a}_1$; $a_4 = \bar{a}_3$.

Thus

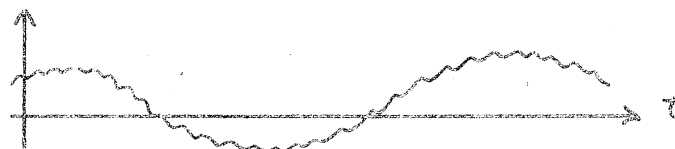
$$D_t a = \begin{pmatrix} a_1 e^{i\omega_1 t} \\ \bar{a}_1 e^{-i\omega_1 t} \\ a_3 e^{i\omega_2 t} \\ \bar{a}_3 e^{-i\omega_2 t} \end{pmatrix}.$$

Since the column-vectors of R are again pairwise complex conjugated, $\eta = R D_t a$ has real components.

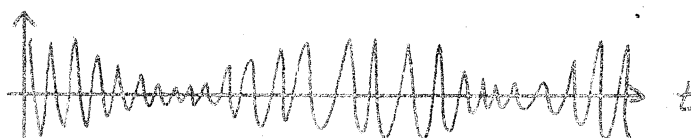
In the following we shall only consider the motion of η and its stability. If quantitative statements on the particles' amplitudes are wanted, one has to transform the results for η back into

$\xi = A_t^{-1} \eta = F_t R^{-1} \eta$. This, however, is not always necessary, because sometimes η already provides enough information.

This is the case if F_t , the periodic part of the Floquet-solution, behaves reasonably. This is to be investigated in each explicit case separately. No general answer is here possible, but ^{one} can express "reasonable behaviour" of F_t in words: The back-transformation is not necessary if the linear motion has the following character



$$\omega_{1,2} \ll 1$$



$$\omega_{1,2} \gg 1$$

which are limiting cases corresponding to $\omega_{1,2} \ll 1$ or $\omega_{1,2} \gg 1$ respectively.

For the AG-Synchrotron the first is true if one considers an ideal machine with many structure periods per revolution and low Q -values because then $\omega_{1,2}$ are the numbers of betatron oscillations per structure period $\omega_{1,2} = Q_{1,2}/M \ll 1$. The other picture is true if one considers a real machine, where the structure period is necessarily 2π . Then $\omega_{1,2} = Q_{1,2} \gg 1$. The ideal machine with superperiod leads to an intermediate picture because $\omega_{1,2}$ are then the numbers of betatron oscillations per superperiod and have the order 1.

Of course we presuppose for all the following, that the linear system is entirely stable. By the way, instability of the linear system would express itself not in the periodic part F_t but in D_t , where ω_1 and ω_2 become then complex, so that some elements of D_t grow exponentially. As is well known from linear theory, the real parts of $\omega_{1,2}$ obey then the relation

$$\operatorname{re} [n_1 \omega_1 + n_2 \omega_2] = p \text{ (integer)} \quad \text{where} \quad n_1 + n_2 = 2$$

$$n_1, n_2 \geq 0$$

We therefore assume for the following

$$n_1 \omega_1 + n_2 \omega_2 \neq p \text{ (integer)} \quad \text{for} \quad |n_1| + |n_2| \leq 2$$

So far we have considered only the linear part of the equations of motion and only in this restricted frame η has the meaning stated above. If instead the whole set of equations is considered, η cannot be interpreted longer as "the non-periodic part" of a solution which is combined of Floquet solutions. Instead we must now read all this the other way round: We have constructed a transformation A_t by which we can simplify the equations of motion. The new variables η follow these simplified equations. Suppose them to be solved, then the actual motion of particles is described by $\xi = A_t^{-1} \eta$. Neither ξ nor η have then, in principle, anything to do with Floquet solutions. In fact, the latter only served to construct a suitable transformation and to illustrate its significance in saying what ξ and η would be if the equations were linear.

F. The complete first transformation

Our transformation A_t transforms the linear equations with periodic coefficients into such ones with constant coefficients. But we wish to transform the whole set of equations including the non-linear terms in such a way, that the transformed equations are again canonical (i.e. follow from a hamiltonian). For the linear part this is achieved by suitable construction of A_t . But as it is well known, it is not generally true, that if the equations for ξ follow from $K(\xi, t)$, the equations for $\eta = A_t \xi$ follow from $K(A_t^{-1} \eta, t)$.

Therefore we must ask, how we get the new hamiltonian, of which we know as yet only the quadratic terms:

$$h_0(\eta) = \frac{\omega_1}{2} (\eta_1^2 + \eta_2^2) + \frac{\omega_2}{2} (\eta_3^2 + \eta_4^2)$$

This we achieve with the formalism of canonical transformations, which is used throughout the paper:

If there is a set of canonical variables and equations

$$\begin{aligned} \dot{y}_1 &= K_{y2} \\ \dot{y}_2 &= -K_{y1} \\ \dot{y}_3 &= K_{y4} \\ \dot{y}_4 &= -K_{y3} \end{aligned} \quad \text{with a Hamiltonian } K(y_1 y_2 y_3 y_4 t)$$

and we wish to go over to a new set of canonical variables obeying the equations

$$\begin{aligned} \dot{u}_1 &= G_{u2} \\ \dot{u}_2 &= -G_{u1} \\ \dot{u}_3 &= G_{u4} \\ \dot{u}_4 &= -G_{u3} \end{aligned} \quad \text{with a new Hamiltonian } G(u_1 u_2 u_3 u_4 t),$$

(a)

This transformation can be generated by a function $S(y_1 u_2 y_3 u_4 t)$ where

$$(a) \quad \begin{aligned} u_1 &= S_{u_2} \\ y_2 &= S_{y_1} \\ u_3 &= S_{u_4} \\ y_4 &= S_{y_3} \end{aligned} \quad \text{defines implicitly the transformation } y \longleftrightarrow u.$$

$$\text{Then } G(u_1 u_2 u_3 u_4 t) = K(y_1 y_2 y_3 y_4 t) + \frac{\partial S}{\partial t},$$

with the y 's expressed as functions of the u 's, is the new hamiltonian.

Assume the four equations (a) solved for y :

$$\begin{aligned} y_1 &= y_1(u_1 u_2 u_3 u_4 t) \\ y_2 &= y_2(u_1 u_2 u_3 u_4 t) \\ y_3 &= y_3(u_1 u_2 u_3 u_4 t) \\ y_4 &= y_4(u_1 u_2 u_3 u_4 t) \end{aligned} \quad \text{then it follows that the Jacobian } \frac{\partial(y_1 y_2 y_3 y_4)}{\partial(u_1 u_2 u_3 u_4)} \equiv 1$$

This property of the Jacobian is necessary and sufficient for the transformation being canonical.

We now treat in this manner the transformation of the hamiltonian.

We have the coordinate transformation $\eta = A\xi$ with $\det A \equiv 1$, so the transformation is canonical. Hence

$$h(\eta_1 \eta_2 \eta_3 \eta_4 t) = K(\xi_1 \xi_2 \xi_3 \xi_4 t) + \frac{\partial S}{\partial t}.$$

If we split the hamiltonians in the quadratic part and the higher terms, we have

$$h_0 + \Delta h = \frac{\omega_1}{2}(\eta_1^2 + \eta_2^2) + \frac{\omega_2}{2}(\eta_3^2 + \eta_4^2) + \Delta h(\eta) =$$

$$K_0(A^{-1}\eta) + \frac{\partial S}{\partial t} + \Delta K(A^{-1}\eta).$$

Here we have inserted $\xi = A^{-1}\eta$ and so the old variables expressed by the new ones. Because A was a linear transformation, S must be a quadratic function and $\frac{\partial S}{\partial t}$ does not contribute to $\Delta h(\eta)$ which therefore is simply equal to $\Delta K(A^{-1}\eta)$. Thus without knowing $S(\xi_1, \xi_2, \xi_3, \xi_4, t)$ explicitly, we can state that

$$(a) \quad h(\eta_1, \eta_2, \eta_3, \eta_4, t) = \frac{\omega_1}{2} (\eta_1^2 + \eta_2^2) + \frac{\omega_2}{2} (\eta_3^2 + \eta_4^2) + \Delta K(A_t^{-1}\eta, t)$$

is the new hamiltonian.

$h(\eta, t)$ has again period 2π because $K(\xi, t)$ had, and A_t has too.

This leads to a statement, which in several respects is useful.

If the equations of motion in an AG-Synchrotron can be written in canonical form, then it is always possible to replace them exactly by another set of equations which have constant coefficients in the linear part and of which all coefficients have period 2π (for an ideal machine the structureperiod is assumed to be normalized to be 2π). The connection between the original coordinates and the new ones involves only the solution of the linear problem. So the equations

$$\ddot{x}_1 + n_{11}x_1 + n_{12}x_2 = F_1(x_1, x_2, t)$$

are equivalent to the other set

$$\ddot{x}_2 + n_{21}x_1 + n_{22}x_2 = F_2(x_1, x_2, t) \quad \text{with constant frequencies } \omega_1 \text{ and } \omega_2$$

$$\ddot{\eta}_1 + \omega_1^2 \eta_1 = \phi_1(\eta_1, \eta_2, \dot{\eta}_1, \dot{\eta}_2, t)$$

$$\ddot{\eta}_2 + \omega_2^2 \eta_2 = \phi_2(\eta_1, \eta_2, \dot{\eta}_1, \dot{\eta}_2, t)$$

This means, that the effect of non-linear terms in these new coordinates is described by the deviations of these coordinates from sinusoidal motion, the transformation back to the original particles coordinates being unaffected by non-linearities. We shall assume that this linear transformation back to the old coordinates behaves quite normal i.e. the linear problem has a solution which does not lead to large amplitudes. If this is the case,

we may investigate the new coordinates under the following aspects :
 under which conditions for ω_1 , ω_2 and the non-linear terms is it possible to excite large oscillations even if the initial values are small? We shall see, that this depends first of all on some rationality properties of ω_1 and ω_2 and secondly on the grade of non-linear terms but not so essentially on the detailed structure of the right-hand side in the above equations. This comes, roughly speaking, from the fact, that the left-hand side of the equations looks to the right-hand side selectively, it fears only those Fourier components which are in or near resonance with its own frequencies and does not worry about all the other terms. So, for instance it is, apart from a numerical factor, almost irrelevant whether the coefficients of the right-hand side, which have period 2π , are pure sin and cos functions or any other periodic function including periodic δ -functions. This gives some confidence into results coming from analog models which work with a two-dimensional linear oscillator of constant frequencies and externally introduced non-linear terms, the coefficients of which are sinusoidal or rectangular shaped functions. It is to believe that these models, one of which is beginning to work in the CERN-PS Group, will show all essential features of the behaviour of the particles. The existence of the linear transformation A_t derived above justifies therefore the use of simplified models and also the application of perturbation methods on the equations with constant linear coefficients (which is relatively easy) instead of the original ones.

Always keeping in mind that the transformation back to the original coordinates $A_t^{-1} = F_t R^{-1}$ must be considered separately and carefully, we deal from now on only with the coordinate set $\eta(t)$ and the hamiltonian $h(\eta, t)$, and we shall refer to these two as to the original coordinates and the original hamiltonian.

G. The second transformation

In order to investigate the behaviour of $\eta(t)$ we shall make a further canonical transformation which serves to simplify the hamiltonian by removing its explicit time dependence and decoupling the two directions as far as possible. For all practical cases it must be pre-supposed that the non-linearities are small (in the same sense as for a perturbation treatment) and that therefore the transformation has somewhat the character of an infinitesimal one. This has one considerable advantage: Suppose we have transformed into a form, where the behaviour of the new coordinates is easily discussed, then it is not necessary to transform back to η , since this does not change very much the results, because the new coordinates are then equal to η plus small higher terms. Therefore we need not worry about the significance of the new coordinates and can, without great error, interpret them as if they were equal to η . In practical cases it is easily decided whether one can do so or whether one must transform back: One solves the transformation formulae (18c) by iteration up to second order terms and sees whether they are small or not. If not, then either one has chosen the transformation unsuitable (see later, especially p.24/25) or the initial amplitudes are already too large. In the first case one changes the transformation and in the second one, it is perhaps sufficient to transform back, using the explicit transformation as found by iteration up to quadratic terms. If not, then the whole method will practically become too complicated to be useful for such large initial amplitudes and one must restrict oneself to smaller ones. Some of the general results, however, do not so much depend on the assumption of small non-linearities and remain valid even if it would be no more justified to interpret the new coordinates as equal to η .

We try to determine the generating function S of this transformation in such a way, that as many coefficients of the new hamiltonian g as possible become zero or at least constant. But we have to demand that S is periodic, so that the transformation between η and y is also periodic and we are sure, that if we find y staying in limits, η cannot run away after some (strictly speaking: short) time. But even if S is periodic, it could happen, that y remains small, but within each period of the transformation η assumes very large values. This can be excluded if we demand also that the coefficients (of period 2π) of S have small amplitudes.

It will turn out that by these conditions

- a) simplest possible form of g
- b) S periodic (strictly speaking: nearly periodic)
- c) small coefficients in S

the generating function S as well as the new hamiltonian g are almost uniquely fixed. But it will also turn out, that strictly speaking, the condition c) is impossible to fulfil because of the same difficulty which, in perturbation theory, is very well known under the name "small denominators". Fortunately, this does not affect the practical validity of the following treatment provided, that the initial values of the variables are not too large. (for this, see also discussion on p.24/25) This has been investigated from the mathematical point of view for the one dimensional case by J.Moser (Mo).

We start now with the following equations:

$$\begin{aligned}
 \dot{\eta}_1 &= h\eta_2 \\
 \dot{\eta}_2 &= -h\eta_1 \\
 \dot{\eta}_3 &= h\eta_4 \\
 \dot{\eta}_4 &= -h\eta_3
 \end{aligned}
 \quad
 h(\eta_1\eta_2\eta_3\eta_4t) = \frac{\omega_1}{2}(\eta_1^2 + \eta_2^2) + \frac{\omega_2}{2}(\eta_3^2 + \eta_4^2) + \Delta h(\eta_1\eta_2\eta_3\eta_4t)$$

(a)

Δh contains terms higher than quadratic and may be ordered in a sequence of homogeneous parts of order n

$$\Delta h = h^{(3)} + h^{(4)} + \dots + h^{(n)} + \dots$$

We transform to a new set of variables, which we call y_i by means of the generating function (see p. 13)

$$s(\eta_1\eta_2\eta_3\eta_4t) = \eta_1\eta_2 + \eta_3\eta_4 + s^{(3)} + s^{(4)} + \dots$$

(b)

where $s^{(n)}$ means a homogeneous polynomial of degree n .

Then

$$\begin{aligned}
 y_1 &= s_{y2} = \eta_1 + \dots + s_{y2}^{(n)} + \dots \\
 \eta_2 &= s_{\eta1} = y_2 + \dots + s_{\eta1}^{(n)} + \dots \\
 y_3 &= s_{y4} = \eta_3 + \dots + s_{y4}^{(n)} + \dots \\
 \eta_4 &= s_{\eta3} = y_4 + \dots + s_{\eta3}^{(n)} + \dots
 \end{aligned}$$

(c)

i.e. the transformation becomes the identical transformation if non-linear terms are neglected. This is because we do not wish of course to transform more than necessary: The quadratic part of the hamiltonian shall remain unchanged.

The new hamiltonian $g(y_1 y_2 y_3 y_4, t)$ becomes

$$(a) \quad g(y_1 y_2 y_3 y_4, t) = h(\eta_1 \eta_2 \eta_3 \eta_4, t) + \frac{\partial}{\partial t} s(\eta_1 \eta_2 \eta_3 \eta_4, t)$$

and we wish to make g as simple as possible. Assume also g split into homogeneous parts:

$$(b) \quad g = \frac{\omega_1}{2} (y_1^2 + y_2^2) + \frac{\omega_2}{2} (y_3^2 + y_4^2) + g^{(3)} + \dots + g^{(n)} + \dots$$

We have to equate parts of equal degree on both sides of (a). For this, it is still necessary to express the old variables η as functions of the new ones or rather, because this is impossible, to write both sides of the equation in those variables which occur explicitly in s , namely $\eta_1 \eta_2 \eta_3 \eta_4$ and to use (18c) to replace $y_1 \eta_2 y_3 \eta_4$ wherever they occur, by derivatives of s . This yields

$$\frac{\omega_1}{2} (s_{y_2}^2 + y_2^2) + \frac{\omega_2}{2} (s_{y_4}^2 + y_4^2) + g^{(3)}(s_{y_2}, y_2, s_{y_4}, y_4, t) + g^{(4)} + \dots =$$

$$\frac{\omega_1}{2} (\eta_1^2 + s_{\eta_1}^2) + \frac{\omega_2}{2} (\eta_3^2 + s_{\eta_3}^2) + h^{(3)}(\eta_1, s_{\eta_1}, \eta_3, s_{\eta_3}, t) + h^{(4)} + \dots + \frac{\partial s^{(3)}}{\partial t} + \frac{\partial s^{(4)}}{\partial t} + \dots$$

Assume the expansion (18c) for s inserted here. Equating homogeneous parts of degree n on both sides leads to

$$\frac{\omega_1}{2} (2\eta_1 s_{y_2}^{(n)}) + \frac{\omega_2}{2} (2\eta_3 s_{y_4}^{(n)}) + \dots + g^{(n)}(\eta_1 \eta_2 \eta_3 \eta_4, t) =$$

$$\frac{\omega_1}{2} (2y_2 s_{\eta_1}^{(n)}) + \frac{\omega_2}{2} (2y_4 s_{\eta_3}^{(n)}) + \dots + h^{(n)}(\eta_1 \eta_2 \eta_3 \eta_4, t) + \frac{\partial s^{(n)}}{\partial t}.$$

Here the dots indicate terms of order n containing only derivatives of $s^{(n-1)}$, $s^{(n-2)}$... multiplied with powers of η_1, y_2, η_3, y_4 . Suppose the $s^{(n-1)}$, $s^{(n-2)}$... $s^{(3)}$ and $g^{(3)}$... $g^{(n-1)}$ already known, then we have, because $h^{(n)}(\eta_1 y_2 \eta_3 y_4, t)$ is given,

$$(a) \quad \frac{\partial s^{(n)}}{\partial t} + \omega_1 \left(y_2 s_{\eta_1}^{(n)} - \eta_1 s_{y_2}^{(n)} \right) + \omega_2 \left(y_4 s_{\eta_3}^{(n)} - \eta_3 s_{y_4}^{(n)} \right) - g^{(n)}(\eta_1 y_2 \eta_3 y_4, t) = f^{(n)}(\eta_1 y_2 \eta_3 y_4, t)$$

Here $f^{(n)}$ is a known function and (a) therefore gives a first order partial differential equation for $s^{(n)}$.

Whenever possible we try to set the coefficients of $g^{(n)}$ zero, but sometimes this is not possible because parts of $g^{(n)}$ may serve to remove a resonance between the inhomogeneous part and the rest of the left-hand side of the equations. This will become clear if we now proceed to the

H. Formal solution of the equations for s

First of all we remark, that it is irrelevant, how the variables are called in these equations. That is, we may write

$$s^{(n)} = \sum_{jklm} \sigma_{jklm}^{(n)} \eta_1^j y_2^k \eta_3^l y_4^m ; \quad g^{(n)} = \sum_{jklm} \gamma_{jklm}^{(n)} \eta_1^j y_2^k \eta_3^l y_4^m$$

$$f^{(n)} = \sum_{jklm} \varphi_{jklm}^{(n)} \eta_1^j y_2^k \eta_3^l y_4^m$$

where the coefficients are functions of time. This may be inserted into (a) and the coefficients equated. We get therefore the same coefficients independently of how we call the variables and we may therefore to solve this system replace $\eta_1 \rightarrow y_1$ and $\eta_3 \rightarrow y_4$.

This gives

$$(a) \quad \frac{\partial s^{(n)}}{\partial t} + \omega_1 \left[y_2 s_{y1}^{(n)} - y_1 s_{y2}^{(n)} \right] + \omega_2 \left[y_4 s_{y3}^{(n)} - y_3 s_{y4}^{(n)} \right] - g^{(n)}(y_1 y_2 y_3 y_4 t) = f^{(n)}(y_1 y_2 y_3 y_4 t)$$

It is now convenient to introduce complex notation

$$\begin{aligned} y_1 + iy_2 &= z_1 & y_3 + iy_4 &= z_2 \\ y_1 - iy_2 &= \bar{z}_1 & y_3 - iy_4 &= \bar{z}_2 \end{aligned}$$

This transformation is not strictly canonical because its determinant is not unity. But we can achieve, that the z 's also obey hamiltons equations. For the y 's we have

$$\dot{y}_1 = g_{y2}$$

$$\dot{y}_2 = -g_{y1}$$

$$\dot{y}_3 = g_{y4}$$

$$\dot{y}_4 = -g_{y3}$$

and for the z 's we assume a new hamiltonian $G(z_1 \bar{z}_1 z_2 \bar{z}_2, t)$ so that

$$\dot{z}_1 = G_{z1} = \dot{y}_1 + i\dot{y}_2 = g_{y2} - ig_{y1} = i(g_{z1} - g_{\bar{z}1}) - i(g_{z1} + g_{\bar{z}1}) = -2ig_{\bar{z}1}$$

because

$$\frac{\partial}{\partial y_1} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_1}; \quad \frac{\partial}{\partial y_2} = i\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial \bar{z}_1}\right) \text{ and similar for } y_{3,4} \text{ and } z_2.$$

Therefore

$$\dot{z}_1 = G_{z1}$$

$$\dot{z}_2 = G_{z2}$$

$$\text{with } G(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \equiv -2ig(y_1 y_2 y_3 y_4 t)$$

with y_1 expressed by the z 's.

Furthermore

$$y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2} = i \left(\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - z_1 \frac{\partial}{\partial z_1} \right) \quad \text{similar for } y_{3,4} \text{ and } z_2.$$

We get then, if we denote

$$(a) \quad \begin{cases} s(y_1 y_2 y_3 y_4 t) \equiv S(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \\ g(y_1 y_2 y_3 y_4 t) \equiv \frac{1}{2} G(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \\ h(y_1 y_2 y_3 y_4 t) \equiv \frac{1}{2} H(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \\ f^{(n)}(y_1 y_2 y_3 y_4 t) \equiv F^{(n)}(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \end{cases}$$

the equation (21a) in complex notation:

$$(b) \quad \frac{\partial S^{(n)}}{\partial t} + i\omega_1 \left[\bar{z}_1 S_{z_1} - z_1 S_{\bar{z}_1} \right] + i\omega_2 \left[\bar{z}_2 S_{z_2} - z_2 S_{\bar{z}_2} \right] - \frac{1}{2} G(z_1 \bar{z}_1 z_2 \bar{z}_2 t) = F^{(n)}(z_1 \bar{z}_1 z_2 \bar{z}_2 t)$$

By this notation we do not lose any clarity because the quantities we are most interested in, namely the amplitudes of the motion, are given by $y_1^2 + y_2^2 = z_1 \bar{z}_1$ and $y_3^2 + y_4^2 = z_2 \bar{z}_2$.

Writing

$$(c) \quad \left. \begin{aligned} S^{(n)} &= \sum s_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \\ G^{(n)} &= \sum g_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \\ H^{(n)} &= \sum h_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \\ F^{(n)} &= \sum f_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \end{aligned} \right\} \quad j + k + l + m = n$$

We get for the n^{th} order coefficients

$$(a) \quad s_{jklm}^{(n)} + i [\omega_1(k-j) + \omega_2(m-l)] s_{jklm}^{(n)} = f_{jklm}^{(n)} + \frac{i}{2} g_{jklm}^{(n)}$$

Here the $f_{jklm}^{(n)}$ are given functions of t with period 2π , whereas the coefficients $g_{jklm}^{(n)}$ are still at our disposal and wherever we can, we shall set them zero or constant. The solution of (a), if we abbreviate

(b) $\omega_1(k-j) + \omega_2(m-l) = \Omega$, can be written either by expanding $s_{jklm}^{(n)}$ into a Fourier-series, thereby exhibiting immediately the demanded periodicity, or by means of Greens formula, where the periodicity condition is fulfilled by choosing a suitable initial value. The solution is in Fourier representation (omitting here the indices $jklm$ and (n))

$$(c) \quad s = \sum_{\nu=-\infty}^{+\infty} s_{\nu} e^{i\nu t} = -i \sum_{\nu=-\infty}^{+\infty} \frac{f_{\nu} + \frac{i}{2} g_{\nu}}{\nu + \Omega} e^{i\nu t}$$

where f_{ν} and g_{ν} are the Fourier-coefficients of f and g .

One sees immediately, that one can set all $g_{\nu} = 0$ provided, that Ω is not an integer p . Hence $s_{jklm}^{(n)} \equiv 0$ is possible. If, however, $\Omega = p$, then the denominator with $\nu = -p$ will vanish and for all $g_{\nu} = 0$ we would have one infinite term and no periodic solution. But we can achieve periodicity by choosing in this case

$$(d) \quad g_{-p} = -\frac{2}{i} f_{-p} \quad \text{and giving } s_{-p} \text{ any desired value.}$$

All the other Fourier components of g we may again set zero as the simplest possibility. Therefore in the case of resonance: $n_1 \omega_1 + n_2 \omega_2 = p$

$$(e) \quad s_{jklm}^{(n)}(t) = -e^{-ipt} \cdot \frac{2}{i} f_{jklm, -p}^{(n)} = e^{-ipt} \cdot \left[\frac{1}{\pi} \int_0^{2\pi} f_{jklm}^{(n)}(\tau) e^{ip\tau} d\tau \right]$$

In the integral representation the periodic solution is

$$(a) \quad s(t) = \frac{e^{-i\Omega t}}{e^{2\pi i\Omega} - 1} \int_t^{t+2\pi} \left[f + \frac{1}{2} g \right] e^{i\Omega \tau} d\tau, \quad \text{which is uniquely determined by this integral if } \Omega \neq p.$$

If $\Omega = p$ we obtain again zero in the denominator and must compensate this by demanding

$$\int_t^{t+2\pi} \left[f(\tau) + \frac{1}{2} g(\tau) \right] e^{ip\tau} d\tau = 0 \quad \text{which is identical to (23d)}$$

Now we shall remove the ambiguity arising from the fact that in this case $\frac{0}{0}$ is involved. If namely $\Omega = p + \epsilon$ with small ϵ , we demand, that the solution behaves continuously for $\epsilon \rightarrow 0$. By expanding (a) with respect to ϵ we get for $\epsilon \rightarrow 0$.

(b)

$$s_{jklm}^{(n)}(t) = \frac{e^{-ipt}}{2\pi} \int_t^{t+2\pi} \left[f_{jklm}^{(n)}(\tau) + \frac{1}{2} g_{jklm}^{(n)}(\tau) \right] e^{ip\tau} d\tau$$

with $g_{jklm}^{(n)}(\tau)$ given by (23e). This is for $\Omega = p$

Of course, this corresponds to setting $s_{jklm, -p}^{(n)} = 0$ in the Fourier solution (as seen by the argument of continuity).

For $\Omega \neq p$ (integer) we can set $g_{jklm}^{(n)}(t) \equiv 0$ and get

(c)

$$s_{jklm}^{(n)}(t) = \frac{e^{-i\Omega t}}{e^{2\pi i\Omega} - 1} \int_t^{t+2\pi} f_{jklm}^{(n)}(\tau) e^{i\Omega \tau} d\tau; \quad \Omega \neq p \text{ (integer)}$$

The above argument of continuity for $\Omega \rightarrow p$ has a practical reason. In a physical system the frequencies ω_1 and ω_2 are always determined to a finite accuracy and therefore numbers j, k, l, m always exist to make Ω integer within the physical accuracy. We shall be interested, especially in those cases, where this can be achieved with low values of $n_1 = (k-j)$ and $n_2 = (m-l)$. If ω_1 and ω_2 change a little from such values, then one generally

needs large n_1 and n_2 to make Ω again integer (one-dimensional example: $1/3 + \epsilon$ is a ratio of large numbers, if ϵ is small). To do so, would mean to compute for the new values of $\omega_{1,2}$ the transformation again with entirely different results. If namely for the case of $\Omega = p$ with low $n_1 = k-j$ and $n_2 = m-l$ (and low j, k, l, m), the corresponding $g_{jklm}^{(n)}(t)$ with low n , had to be adjusted to remove resonance, this is no longer necessary if ω_1 and ω_2 have slightly changed because there is then no resonance at all for $s_{jklm}^{(n)}$. So we may set $g_{jklm}^{(n)} = 0$. The resonance occurs now for another set $j' k' l' m'$ in a very high order - in such a high order perhaps, that we need not consider it. We may then resign to demand periodicity of S in this high order and solve the equation for $s_{j'k'l'm'}^{(n')}$ with initial condition $s(0) = 0$. In this high order it will build up very slowly and is to be neglected even for considerable large times (see p.31). But we have then the curious behaviour that a small change of ω_1 and ω_2 would bring to vanish g -coefficients discontinuously and bring to change s -coefficients also discontinuously: At the low-order resonance (j, k, l, m small numbers) we have enforced the s -coefficient to be periodic with finite amplitude (by adjusting the g -coefficient according to 23e) but as mentioned, we were still free to choose any desired mean value. Very near to the resonance (i.e. with very slightly changed ω_1 and ω_2) we do not need more to adjust the g -coefficient and can set it equal to zero. This, however, would have the consequence of very large amplitudes for the s -coefficient (of order $\frac{1}{\Omega - p}$). It seems therefore reasonable to investigate small environs of a "resonance line" (in the ω_1, ω_2 -plane) $n_1 \omega_1 + n_2 \omega_2 = \Omega = p$ by keeping the resonance destructing g -coefficient in the equation and giving it the same value as for the exact resonance. Then the s -amplitudes behave smoothly within a small neighbourhood of the resonance line. So this is the reason for the above argument. Of course, the neighbourhood of the resonance line, in which this is allowed, must be taken small. So small, that the higher resonance lines nearby (there are of course infinitely many!) are all of considerably higher order than the one that has been considered.

There are cases, however, where the smoothness argument does not apply. Whatever the values of ω_1 and ω_2 are, $\omega_1(k-j) + \omega_2(m-l) = \Omega = p = 0$ (which is, in the above sense, an integer) occurs if $k = j$ and $m = l$.

Then by (23e) $g_{jjll}^{(n)}$ is simply a constant. Here again $s_{jjll}^{(n)}(t)$ is a periodic function with undetermined mean value (or initial value) and there is at first no reason for giving it any special value. In this way, from each equation with $k=j, m=l$ we obtain a new free constant and these constants of course enter into the right-hand sides $f_{j'k'l'm'}^{(n')}$ of the equations of higher order and accumulate there. In principle it would probably be possible to leave them undetermined when they just occur and to use them in higher order equations, where again $j'=k'$ and $m'=l'$ to bring the mean value there (zeroth Fouriercoefficient) of $f_{j'j'l'l'}^{(n')}$ to zero. There it would not be necessary then to adjust $g_{j'j'l'l'}^{(n')}$ which could be set simply equal to zero.

In the present general exposé it is hardly possible to see what would happen if one keeps these new constants at first undetermined and uses them where they may be useful. This would only be seen in explicit examples because the functions $f_{jklm}^{(n)}(t)$ become, with increasing order n , rapidly very complicated and to pursuit undetermined constants into higher orders, makes them still more complicated. However, our aim to keep the coefficients s of the generating function S as small as possible, suggests to put the mean value of all coefficients $s_{jjll}^{(n)}$ equal to zero and to use the g -coefficients to remove all resonances for $\Omega = p = 0$. This also has the economic reason of keeping the involved work in explicit cases low. But there might arise examples, where the whole question is to be considered anew.

Finally we mention that the solution of the whole set of equations leads to a real function S , which is necessary to assure the new hamiltonian $g(y_1 y_2 y_3 y_4 t) = \frac{1}{2} G(z_1 \bar{z}_1 z_2 \bar{z}_2 t)$ and the variables $y_1 \dots y_4$ being real too. The proof is given in the appendix 2 (p. 73) This has the consequence that

$$\begin{aligned} \bar{s}_{jklm}^{(n)} &= s_{kjml}^{(n)} \\ (a) \quad \bar{f}_{jklm}^{(n)} &= f_{kjml}^{(n)} \\ \bar{g}_{jklm}^{(n)} &= -g_{kjml}^{(n)} \end{aligned}$$

J. Discussion of the principally possible instabilities

Only to exhibit the kinds of arguments involved, let us at first assume the academic case, that the values of ω_1 and ω_2 are such that there are no integers n_1, n_2, p at all for which $n_1\omega_1 + n_2\omega_2 = p \neq 0$ is fulfilled. ($\omega_1 = \pi; \omega_2 = \lg 2$ for instance). Then the only resonances left are those with $n_1 = n_2 = p = 0$ and (23e) shows that all g -coefficients $g_{jjll}^{(n)}$ are constant and all the other ones $[k \neq j \text{ and } m \neq l]$ may be set zero. [In the following we replace g_{jjkk} by g_{jk}]. Thus the new hamiltonian is independent of t :

(a)

$$G(z_1 \bar{z}_1 z_2 \bar{z}_2) = -i\omega_1 z_1 \bar{z}_1 - i\omega_2 z_2 \bar{z}_2 + g_{20}(z_1 \bar{z}_1)^2 + g_{02}(z_2 \bar{z}_2)^2 + g_{11}(z_1 \bar{z}_1)(z_2 \bar{z}_2) + \dots = G(r_1^2, r_2^2)$$

It is a function of

$$r_1^2 = z_1 \bar{z}_1 = y_1^2 + y_2^2 \quad \text{and} \\ r_2^2 = z_2 \bar{z}_2 = y_3^2 + y_4^2 \quad \text{only}$$

First of all we see that in this case $G(z_1 \bar{z}_1 z_2 \bar{z}_2)$ is a constant of the motion or an invariant under the (canonical) transformation $t \rightarrow t + \delta$

This is because

(b)

$\dot{z}_1 = G_{z1}$ and $\dot{z}_2 = G_{z2}$. The equations for $\bar{z}_1$ and $\bar{z}_2$ follow by complex conjugating:
 $\dot{\bar{z}}_1 = -G_{z1}$ and $\dot{\bar{z}}_2 = G_{z2}$, so that complex conjugated variables are at the same time canonically conjugated (as it must be)

Then for any function $\phi(z_1, \bar{z}_1, z_2, \bar{z}_2, t)$

(a)
$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \dot{z}_1 \phi_{z_1} + \dot{\bar{z}}_1 \phi_{\bar{z}_1} + \dot{z}_2 \phi_{z_2} + \dot{\bar{z}}_2 \phi_{\bar{z}_2} \quad \text{or}$$

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + G_{z_1} \phi_{z_1} - G_{z_1} \phi_{\bar{z}_1} + G_{z_2} \phi_{z_2} - G_{z_2} \phi_{\bar{z}_2}$$

This is zero for $\phi = G$ because $\frac{\partial G}{\partial t} = 0$.

But even more:

With $\phi = z_1 \bar{z}_1$

$$\frac{d\phi}{dt} = (r_1^2)' = \bar{z}_1 G_{z_1} - z_1 G_{\bar{z}_1} = 0 \quad \text{and the same holds good for } z_2 \bar{z}_2.$$

This one immediately sees by inserting the general term $G_{j1}^{(n)} (z_1 \bar{z}_1)^j (z_2 \bar{z}_2)^1$ into this formula. So we have:

(b)

If ω_1 and ω_2 are such, that no integers n_1, n_2 (positive and negative) exist so that $n_1 \omega_1 + n_2 \omega_2 = p \neq 0$, then (27a) is the transformed hamiltonian and

$$r_1^2 = z_1 \bar{z}_1 = y_1^2 + y_2^2$$

are constant. The motion is stable.

$$r_2^2 = z_2 \bar{z}_2 = y_3^2 + y_4^2$$

(for the definition of "stability" see 30-32)

For small amplitudes, there is no need to transform back to the η coordinates because $\eta = y + \text{small terms}$. But (28b) is still true for large initial amplitudes; then of course one must transform back because the transformation may introduce large oscillations into the η - coordinates (see p. 17/18).

In this case one can also give the explicit solution:

(c)

$$\dot{z}_\rho = G_{z_\rho} = \frac{\partial G}{\partial (z_\rho \bar{z}_\rho)} \cdot z_\rho \quad ; \quad z_\rho(t) = z_{\rho 0} \cdot \exp. [-i\varphi_\rho t] \quad \rho = 1, 2$$

Here φ_1 and φ_2 are constant "effective frequencies"

(a)

$$\varphi_1 = i \frac{\partial G}{\partial (z_1 \bar{z}_1)} = \omega_1 + i \left[2g_{20} r_{10}^2 + g_{11} r_{20}^2 + \dots + l g_{1k} r_{10}^{2l-2} r_{20}^{2k} + \dots \right]$$

$$\varphi_2 = i \frac{\partial G}{\partial (z_2 \bar{z}_2)} = \omega_2 + i \left[2g_{02} r_{20}^2 + g_{11} r_{10}^2 + \dots + k g_{1k} r_{10}^{2l} r_{20}^{2k-2} + \dots \right]$$

because $z_p \bar{z}_p = r_p = r_{p0} = \text{const.}$ The g_{lk} always are purely imaginary numbers (see 26a) and $\varphi_{1,2}$ are therefore real. The "frequencyshifts" $\varphi_1 - \omega_1$ and $\varphi_2 - \omega_2$ are the most characteristic features of all non linear oscillations. They will become very important for the suppression of resonances. (p. 56).

Now we abandon the assumption that $n_1 \omega_1 + n_2 \omega_2 = p \neq 0$ is impossible and consider the more realistic case that this is fulfilled somewhere, say for $n_1 = k-j$ and $n_2 = m-l$. This determines at once a one-dimensional manifold of ω -values for which it is also fulfilled, namely the

(b)

$$\text{"resonance line"} \quad n_1 \omega_1 + n_2 \omega_2 = p.$$

We shall consider a fixed resonance line, i.e. given values

(c)

$$\begin{aligned} n_1 &= k-j \\ n_2 &= m-l \end{aligned} \quad \text{and } p = \omega_1(k-j) + \omega_2(m-l) = n_1 \omega_1 + n_2 \omega_2$$

Obviously there are infinitely many different values of k, j, m, l leading to the same n_1, n_2, p and for each such combination we must compensate the resonance by

(d)

$$g_{jklm}^{(n)}(t) = e^{-ipt} \cdot \gamma_{jklm}^{(n)} \quad \text{where } \gamma_{jklm}^{(n)} = \left[\frac{1}{\pi} \int_0^{2\pi} f_{jklm}^{(n)}(\tau) e^{ipt} d\tau \right] = \text{const.}$$

In this case the transformation $\eta \rightarrow y$ remains periodic with small coefficients but the new hamiltonian contains t explicitly

$$g(z_1 \bar{z}_1 z_2 \bar{z}_2 t) = -i\omega_1 z_1 \bar{z}_1 - i\omega_2 z_2 \bar{z}_2 + \dots + e^{-ipt} \gamma_{jklm}^{j-k, l-m} z_1^{j-k} z_2^{l-m} + \dots$$

and the arguments used for the statement (28b) break down. Whether there is stability or not must be decided more or less explicitly and for the moment we must assume the possibility of instable motions. Therefore on any resonance line principally instability can occur. This leads at

first sight to a hopeless situation because the $\omega_1 - \omega_2$ plane is densely covered with such lines and one can fear that there is no stable motion at all - in contradiction to experience,

K. On the definition of "stability"

Before going ahead, we must define more precisely the term stability but without doing it in a very rigorous mathematical way.

Stability in the synchrotron means practically amplitudes of less than a few centimeters. Infinity is here the same as the aperture of the vacuum chamber. Even if in a nonlinear system no resonance can lead to infinite amplitudes because of frequency shifts (29a), this stabilizing effect of nonlinearities can be practically worth less. Stability in the sense of bounded amplitudes is not a useful concept because it is too weak. In this sense nearly all nonlinear systems will be stable. There is the definition of Liapunoff according to which a solution is stable with respect to a given one (here $y \equiv 0$) if it remains always in its neighbourhood. That is here: Small oscillations remain small. This definition is in some respect too strong because we may wish to call a motion stable even if it runs away from zero but has a bounded amplitude within the allowed aperture of the vacuum chamber. So our definition has to keep the middle and we shall call our system stable if

- 1) it is stable in Liapunoff's sense around $y \equiv 0$ but also if
- 2) it is not stable around $y \equiv 0$ but has small bounded amplitudes. We shall distinguish these two cases as "strong" and "weak" stability respectively.

But there is another point. All that we can say concerns terms up to a finite order and there always remains an uncertainty of how the situation of the system will be after very long times. Suppose the case considered above, where $n_1 \omega_1 + n_2 \omega_2 \neq p$ for all $n_1, n_2 \neq 0$. The new hamiltonian shows a stable motion. But in the transformation occur coefficients, which indeed are periodic, but which have very large amplitudes of order $\frac{1}{\Omega - p}$ where $n_1 \omega_1 + n_2 \omega_2 = \Omega$ (see (24c)) and Ω approaches more and more an integer for suitable (high) n_1 and n_2 . On the other hand, the functions $f_{jklm}^{(n)}$ become smaller and smaller for high orders if the coefficients of the original hamiltonian decrease not too slowly. It is difficult to

oversee this and to decide whether the $f_{jklm}^{(n)}$ decrease faster than the $\frac{1}{\Omega - p}$. A rough estimate, however, may arise from the following consideration: from ordinary perturbation theory, using the representation of the solution by a series of trigonometric functions, it is a well known fact that one can proceed in two ways: in one way one solves the equations iteratively beginning with the linear part, inserts the solution into the non-linear terms and integrates the new inhomogeneous system and so on. Then one gets the "secular terms" which increase with t , $t^2 \dots$. The other one is, to begin with undetermined effective frequencies which are then determined in such a way that the secular terms vanish. It can be shown that both methods are equivalent for not too long times, because the secular terms correspond exactly to a Taylor expansion of the trigonometric functions with changed frequencies around the original ones:

$$\begin{aligned} \sin [(\omega + \Delta\omega)t] &= \sin \omega t \cdot \cos \Delta\omega t + \cos \omega t \cdot \sin \Delta\omega t \\ &= \sin \omega t \cdot \left[1 - \frac{(\Delta\omega t)^2}{2!} + \dots \right] + \cos \omega t \cdot \left[\Delta\omega t - \frac{(\Delta\omega t)^3}{3!} + \dots \right] \end{aligned}$$

So one can also renounce here on strict periodicity of S in high orders and imagine all terms near to resonance to be integrated with the initial condition $s_{jklm}^{(n)} = 0$ giving

$$s_{jklm}^{(n)} = e^{-i\Omega t} \int_0^t f_{jklm}^{(n)} e^{i\Omega \tau} d\tau \quad \text{Set } \Omega = p + \epsilon$$

then $f_{jklm}^{(n)} = \dots + f_{jklm, -p}^{(n)} e^{-ip\tau} + \dots$ and the

leading term of $s_{jklm}^{(n)}$ becomes

$$s_{jklm}^{(n)}(t) \approx f_{jklm, -p}^{(n)} e^{-i\Omega t} \frac{e^{i\epsilon t} - 1}{i\epsilon} \approx t \cdot f_{jklm, -p}^{(n)} e^{-i\Omega t}$$

If, in the original hamiltonian the coefficients decrease fast enough with increasing order (or if the amplitudes are sufficiently small) $f_{jklm, -p}^{(n)}$ will be a very small number and $s_{jklm}^{(n)}$ will remain small for long time. Of course, there is no increase proportional to time for

very long time intervals but only at the beginning. Later on it increases more slowly and finally decreases again, reaching a maximum amplitude $f_{jklm,-p}^{(n)}/(4-p)$ as the periodic solution. But by renouncement of strict periodicity of S in high orders we know that for $t = 0$ the coefficients $s_{jklm}^{(n)}$ of high order start with zero amplitude and therefore no back-transformation is necessary - for a long, but not infinite, time, one can forget at all that there are high coefficients with small denominators. (A more rigorous mathematical estimate of the influence of higher terms is given by Moser (Mo) for the one-dimensional case.) Our statements about stability are always the result of considerations on terms of finite (and in fact low) order. We shall indicate the effects of higher terms in the following by speaking of "quasi-stability" rather than of stability. So we adopt from now on the terms "strong quasi-stability" and "weak quasi-stability" in the just developed sense.

L. Classification of resonance-lines

Going back to p.19 we see that in (20a) the inhomogeneous part $f^{(n)}(\eta_1 y_2 \eta_3 y_4, t)$ consists of $-h^{(n)}(\eta_1 y_2 \eta_3 y_4, t)$ plus terms which come from $h^{(n-1)}(\eta_1 s_{\eta_1} \eta_3 s_{\eta_3}, t)$, $h^{(n-2)}(\eta_1 s_{\eta_1} \eta_3 s_{\eta_3}, t)$, $h^{(3)}(\eta_1 s_{\eta_1} \eta_3 s_{\eta_3}, t)$ and $g^{(n-1)}(s_{y_2} y_2 s_{y_4} y_4, t)$, $g^{(n-2)}(s_{y_2} y_2 s_{y_4} y_4, t)$, $g^{(3)}(s_{y_2} y_2 s_{y_4} y_4, t)$ where in these functions the derivatives of $s = s^{(2)} + s^{(3)} + s^{(4)} + \dots + s^{(n-1)}$ are inserted and then all the terms of order n are collected. So in the most general case $f^{(n)}(\eta_1 y_2 \eta_3 y_4, t) = -h^{(n)}(\eta_1 y_2 \eta_3 y_4, t) + p^{(n)}(\eta_1 y_2 \eta_3 y_4, t)$ where $p^{(n)}$ is a polynomial of n^{th} degree combined with the derivatives of $s^{(n-1)}$, $s^{(n-2)}$, ..., $s^{(2)}$. In the form (23a) where equal powers have been equated, we have therefore $f_{jklm}^{(n)} = -\frac{1}{2} h_{jklm}^{(n)} + p_{jklm}^{(n)}$ where $p_{jklm}^{(n)}$ is a more or less complicated sum of products of coefficients $s_{\alpha\beta\gamma\delta}^{(n-1)}$, $s_{\alpha'\beta'\gamma'\delta'}^{(n-2)}$, $s_{\lambda\mu\nu\rho}^{(n-1)}$, $s_{\lambda'\mu'\nu'\rho'}^{(n-2)}$, and $h_{\rho\sigma\tau\epsilon}^{(n-1)}$, $h_{\rho'\sigma'\tau'\epsilon'}^{(n-2)}$, combined in such a way that in any product the sum of upper indices is always equal to n .

The equation for $s_{jklm}^{(n)}$ is then

$$s_{jklm}^{(n)} + i \left[\omega_1(k-j) + \omega_2(m-l) \right] s_{jklm}^{(n)} = -\frac{1}{2} h_{jklm}^{(n)} + \frac{1}{2} g_{jklm}^{(n)} + p_{jklm}^{(n)}$$

and therefore the solution $s_{jklm}^{(n)}$ contains one additive part the magnitude of which is proportional to the absolute value of $h_{jklm}^{(n)}$ and another additive part proportional to the absolute value of $p_{jklm}^{(n)}$, which in turn contains lower h-coefficients, g-coefficients and s-coefficients mixed.

The latter ones are again partly proportional to a h-coefficient and contain partly mixed, still lower, coefficients. So one can reduce by recursion in this n^{th} order equation, the right-hand side to $-h_{jklm}^{(n)} + \frac{1}{2} g_{jklm}^{(n)} + (\text{combination of lower h-coefficients})$.

That means:

- (a) Only $h_{jklm}^{(n)}$ enters linearly into $s_{jklm}^{(n)}$. All lower h-coefficients are contained in $s_{jklm}^{(n)}$ only in a mixed form and the higher ones do not appear at all in $s_{jklm}^{(n)}$.
- From this follows: if in the hamiltonian $h(\eta_1, \eta_2, \eta_3, \eta_4, t)$ a term of order n is present, then it will show its strongest effect in the equation for $s_{jklm}^{(n)}$, smaller effects in higher order s-coefficients and no effect in lower ones.

We now consider the equation for a fixed order $n=N$.

(b)
$$\begin{cases} s_{jklm}^{(N)} + i \left[\omega_1(k-j) + \omega_2(m-l) \right] s_{jklm}^{(N)} = -\frac{1}{2} h_{jklm}^{(N)} + \frac{1}{2} g_{jklm}^{(N)} + p_{jklm}^{(N)} \\ j+k+l+m = N ; j,k,l,m \geq 0 \end{cases}$$

As already pointed out, one has to adjust a $g_{jklm}^{(N)}(t)$ if $\omega_1(k-j) + \omega_2(m-l) = \Omega = p(\text{integer})$ and this means in principle that an instability can occur (see p. 29) Therefore we have called the line (in $\omega_1\omega_2$ -plane)

$$\omega_1(k-j) + \omega_2(m-l) = p \quad \text{a resonance-line}$$

We can classify these lines by the following

Let be $k-j = n_1$ and j, k, l, m all ≥ 0 .
 $m-l = n_2$

(a)

Then $n_1 \omega_1 + n_2 \omega_2 = p$ (integer) is called a resonance-line of the order M , if $|n_1| + |n_2| = M$.

Furthermore, if both n_1 and n_2 are different from zero, we speak of a sum-resonance-line if n_1 and n_2 have equal signs; of a difference-resonance-line if n_1 and n_2 have different signs.

From the definition (a) follows (proof below)

(b)

- a) Apart from a parallel shifting there are always just $2M$ lines of the order M present
- b) In the N^{th} order equation (33b) these and only these orders of resonance-lines can occur: $N, N-2, N-4, \dots$
- γ) If ω_1 and ω_2 lie on a definite resonance-line $n_1 \omega_1 + n_2 \omega_2 = p$ of the order $|n_1| + |n_2| = N$, then in the set of the N^{th} order equations there will be just two, for which resonance occurs. These are for $s_{j'k'l'm'}^{(N)}$ and $s_{k'j'l'm'}^{(N)}$ where the four indices $j'k'l'm'$ are uniquely determined by n_1 and n_2 alone and two of them - one of the first pair (j', k') and one of the second pair (l', m') - will be zero, hence $j+k = |n_1|$ $l+m = |n_2|$.

All this is nearly trivial:

α) $|n_1| + |n_2| = M$ implies
 $n_1 = \pm (M - a)$
 $n_2 = \pm a$ with $a = 0, 1, 2, \dots, M$

For $a = 0$ or M there are two possibilities respectively, for

$a = 1, 2, \dots, M-1$ four possibilities at a time. So together

$2 \times 2 + 4 \times (M-1) = 4M$. But now each line occurs twice

(as $n_1 \omega_1 + n_2 \omega_2 = p$ and as $-n_1 \omega_1 - n_2 \omega_2 = -p$) and the number of lines is $2M$.

8) Assume the possibility of lineorders $N-L$, where at first L may be any integer. Then, in the N^{th} order equation, the possible sign combinations of n_1 and n_2 (left-hand column) give the four possible forms of the equation $|n_1| + |n_2| = N-L$ (second column from the left)

n_1	n_2	$ n_1 + n_2 = N-L$		$2k + 2m = 2N-L$	$2j + 2l = L$
$+$	$+$	$-j+k-l+m = N-L$	adding and	$2j + 2m = 2N-L$	
$-$	$+$	$j-k-l+m = N-L$	subtracting	$2k + 2l = 2N-L$	and $2k + 2l = L$
$+$	$-$	$-j+k+l-m = N-L$	$j+k+l+m = N$	$2j + 2l = 2N-L$	$2j + 2m = L$
$-$	$-$	$j-k+l-m = N-L$	gives	$2j + 2l = 2N-L$	$2k + 2m = L$

At the extreme right-hand side one sees that L is positive because j, k, l, m are and is an even number; $L = 0, 2, 4, \dots$. This proves 8).

9) As an example take the second case $n_1 \leq 0; n_2 \geq 0; |n_1| + |n_2| = N$.

The resonance-line is of the order N ; L must be zero. Again, because j, k, l, m are positive, $k = l = 0$ and $j + m = N = |n_1| + |n_2|$. This determines $j = |n_1|$ and $m = |n_2|$. It is similar for any of the other cases. One also sees that the multiplication of both n_1 and n_2 with -1 simply causes a change of j with k and l with m ; but the resonance-line remains the same by multiplication with -1 .

Finally we combine (33a) and (34b) to obtain the following statement:

If in the original hamiltonian, terms of the order $n \gg 3$ are present, any one of these can in principle excite resonance on lines of any order, but a term of order N will generally have the strongest effect for the lines of the order $N, N-2, N-4, \dots$, and smaller effect on all the other lines.

It will be shown, however, in the following, that in nearly all cases the lines of the order $N \gg 5$ ($|n_1| + |n_2| \gg 5$) are not excited at all (i.e. they are strongly quasi-stable) and that for the lower ones their stability behaviour can be discussed explicitly if the original hamiltonian is given explicitly.

(a)

The foregoing is quite generally true if the coefficients of the original hamiltonian decrease with increasing order. If they strongly decrease and all have absolute values < 1 , one can even say more: a coefficient of the order N in the hamiltonian will cause according to (33a) and (34b) resonances of the orders $N, N-2, N-4, \dots$ by a force directly proportional to its absolute value, higher ones only by a force proportional to the absolute value of products of coefficients in which the mentioned N^{th} order hamiltonian coefficient enters together with other ones. These products have, for strongly decreasing coefficients of the hamiltonian, considerably smaller absolute values than the N^{th} order coefficient itself and may therefore be neglected. Then one would say that the N^{th} order term of the hamiltonian causes only the resonances $N, N-2, N-4, \dots$. But now, generally, in the hamiltonian also terms of the orders $N-2, N-4, \dots$ will be present and according to our presupposition they will increase in decreasing order. That means, the resonance-line of the order $N-6$, for instance, may be excited by the terms of the orders $N, N-2, N-4$ and $N-6$ of the original hamiltonian. But because the coefficient of the order $N-6$ in the hamiltonian is much larger than the higher ones, its effect will cover that of all the latter, which only give small correcting contributions. In this sense one may come to the qualitative statement:

(a)

If in the original hamiltonian $h(\eta_1, \eta_2, \eta_3, \eta_4, t)$ the coefficients of terms of the order $n \geq 3$ have absolute values, smaller than one, and decrease strongly with increasing n (all this attainable by a simple scale transformation) then the term of the order N in the hamiltonian can excite resonances just of the order N (and practically only this one) proportionally to the absolute value of the N^{th} order coefficients in the hamiltonian.

The above mentioned scale-transformation of course cannot change the physical properties of the system - it merely liberates the system of differential equations for the $s_{jklm}^{(n)}$ from some irregular order-of-magnitude-behaviour with respect to the order n and shifts the difficulty to the question, for which range of amplitudes of the coordinates (36a) remains true.

M. The new hamiltonian $g(y_1 y_2 y_3 y_4 t)$ at resonance

We have already considered the form of the new hamiltonian

$g(y_1 y_2 y_3 y_4 t) \equiv \frac{1}{2} G(z_1 \bar{z}_1 z_2 \bar{z}_2 t)$ for the case that ω_1 and ω_2 do not lie on any resonance-line. Let us now assume that they do:

(a) $n_1 \omega_1 + n_2 \omega_2 = p$ (integer) with $|n_1| + |n_2| = N$

but that they do not lie at the same time on any other line.

(If the case, that ω_1 and ω_2 lie on no resonance-line at all, is considered as unrealistic of the order ∞^2 , then the present assumption is still unrealistic of the order ∞ because the line $n_1 \omega_1 + n_2 \omega_2 = p$ will be crossed by infinitely many other lines. We rather mean that ω_1 and ω_2 lie on the line $n_1 \omega_1 + n_2 \omega_2 = p$, not in the immediate neighbourhood of a point, where another line crosses which is of an order comparable with or even lower than N).

According to (34b β), the lowest order equations, in which (a) causes a resonance, are of the order N :

(b) $s_{jklm}^{(N)} + i [\omega_1(k-j) + \omega_2(m-l)] s_{jklm}^{(N)} = f_{jklm}^{(N)}(t) + \frac{1}{2} g_{jklm}^{(N)}(t); j+k+l+m = N$

and from (34b γ) just two of these equations lead to resonance, because by n_1 and n_2 a set of indices, say j', k', l', m' is uniquely determined and by multiplying (a) with -1 the line remains the same but j' changes with k' and m' with l' . According to (23e) one has now two $g_{jklm}^{(N)}(t)$ to adjust to remove resonance:

$g_{j'k'l'm'}^{(N)}(t) = e^{-ipt} \left[\frac{1}{\pi} \int_0^{2\pi} f_{j'k'l'm'}^{(N)}(\tau) e^{ip\tau} d\tau \right] = e^{-ipt} \gamma_{j'k'l'm'}$ and

(c) $g_{k'j'l'm'}^{(N)}(t) = e^{ipt} \left[\frac{1}{\pi} \int_0^{2\pi} f_{k'j'l'm'}^{(N)}(\tau) e^{-ip\tau} d\tau \right] = e^{ipt} \gamma_{k'j'l'm'}$

for $n_1 \omega_1 + n_2 \omega_2 = p$ and $-n_1 \omega_1 - n_2 \omega_2 = -p$ respectively, where the γ 's are constant. As it has to be (see Appendix 2, p. 73): $\bar{g}_{j'k'l'm'}^{(N)} = -g_{k'j'l'm'}^{(N)}$. We now know already how the new hamiltonian looks up to terms of the order $N-1$ from the

case without resonance. To this form (27a), which remains unchanged up to the order $N-1$ one has now to add the terms coming from

$$e^{-ipt} \gamma_{j'k'l'm'} z_1^{j'-k'} \bar{z}_1^{l'-m'} z_2^{j'-k'} \bar{z}_2^{l'-m'} + e^{ipt} \gamma_{k'j'm'l'} z_1^{k'-j'} \bar{z}_1^{m'-l'} z_2^{k'-j'} \bar{z}_2^{m'-l'}$$

and up to these terms the new hamiltonian becomes now (we replace $j'k'l'm'$ by $j k l m$)

$$\begin{aligned} G(z_1 \bar{z}_1 z_2 \bar{z}_2 t) = & -i\omega_1 z_1 \bar{z}_1 - i\omega_2 z_2 \bar{z}_2 + g_{20}^{(4)} (z_1 \bar{z}_1)^2 + g_{11}^{(4)} (z_1 \bar{z}_1) (z_2 \bar{z}_2) \\ & + g_{02}^{(4)} (z_2 \bar{z}_2)^2 + \dots + g_{\frac{\bar{N}}{2}0}^{(N)} (z_1 \bar{z}_1)^{\frac{\bar{N}}{2}} + g_{\frac{\bar{N}}{2}-1,1}^{(N)} (z_1 \bar{z}_1)^{\frac{\bar{N}}{2}-1} (z_2 \bar{z}_2) \\ & + \dots + g_{0\frac{\bar{N}}{2}}^{(N)} (z_2 \bar{z}_2)^{\frac{\bar{N}}{2}} + e^{-ipt} \gamma_{jklm} z_1^{j-k} \bar{z}_1^{l-m} z_2^{j-k} \bar{z}_2^{l-m} \\ & + e^{ipt} \gamma_{kjml} z_1^{k-j} \bar{z}_1^{m-l} z_2^{k-j} \bar{z}_2^{m-l} + \dots \end{aligned}$$

$$\left(\frac{\bar{N}}{2} \text{ is the smallest integer } \geq \frac{N}{2} \right)$$

It is now easy to see what will be the general form of the higher terms:

- 1) The resonance-line remains the same, if it is multiplied by an arbitrary factor. However, only integer factors a will let an_1 , an_2 and ap integer also. The same resonance-line comes therefore into action again in all those higher equations, where $j'' = aj$; $k'' = ak$; $l'' = al$ and $m'' = am$ with a any integer. But then $p'' = ap$ and that means simply that we shall have higher terms from this argument of the form

$$\gamma_{j''k''l''m''} \left[e^{-ipt} z_1^{j-k} \bar{z}_1^{l-m} z_2^{j-k} \bar{z}_2^{l-m} \right]^a \text{ and the corresponding complex conjugated ones.}$$

2) Because $n_1 = k-j$ and $n_2 = m-l$, one can change the values of j, k, l, m without changing n_1 and n_2 in the following way:

$$\begin{aligned} k &\rightarrow k + \alpha & m &\rightarrow m + \beta \\ j &\rightarrow j + \alpha & l &\rightarrow m + \beta \end{aligned} \quad \text{with } \alpha, \beta \text{ positive integers}$$

$$N \rightarrow N + 2\alpha + 2\beta$$

(Because j, k, l, m are positive by definition and two of them are zero in the N^{th} order equation, α and β must be positive.)

From this argument we see that any value of α and β leads again to a resonance and demands another g -coefficient. But p is not changed and we shall therefore have terms of the form

$$\gamma_{j''k''l''m''} \left[e^{-ipt} z_1^{j+\alpha} \bar{z}_1^{k+\alpha} z_2^{l+\beta} \bar{z}_2^{m+\beta} \right] = \gamma_{j''k''l''m''} \left[e^{-ipt} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \right] (z_1 \bar{z}_1)^\alpha (z_2 \bar{z}_2)^\beta$$

and the complex conjugated.

From 1) and 2) together follows that all higher terms containing the time explicitly must be of the form

$$\left[e^{-ipt} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \right]^a \cdot (z_1 \bar{z}_1)^\alpha \cdot (z_2 \bar{z}_2)^\beta \quad (\text{and complex conjugated})$$

multiplied with constant coefficients. Here $a = 1, 2, 3, \dots$ $\alpha, \beta = 0, 1, 2, \dots$ (all independently).

Finally there are still higher terms not containing the time. They are the same as in the non-resonance case. If we denote

$$e^{-ipt} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m = \sigma(t)$$

$$(a) \quad z_1 \bar{z}_1 = r_1^2$$

$$z_2 \bar{z}_2 = r_2^2$$

the total hamiltonian will be a power series with constant coefficients in these three variables : r_1^2 ; r_2^2 and $\sigma(t)$.

(a)

$$G(z_1 z_1 z_2 z_2 t) = F[\sigma(t), r_1^2, r_2^2]$$

This is only formal - nothing is said on convergence.

Finally, from (34b, γ), $\sigma(t)$ must have one of the following forms:

(b)

$$\sigma(t) = e^{\pm i p t} \begin{cases} \begin{matrix} |n_1| & |n_2| \\ z_1 & z_2 \\ |n_1| & |n_2| \\ \bar{z}_1 & z_2 \\ |n_1| & |n_2| \\ z_1 & \bar{z}_2 \\ |n_1| & |n_2| \\ \bar{z}_1 & \bar{z}_2 \end{matrix} & \begin{matrix} n_1 & n_2 \\ \hline - & - \\ + & - \\ - & + \\ + & + \end{matrix} \end{cases}$$

N. The third transformation into the normal-form

We can now get rid of the whole time dependence of the hamiltonian by a simple rotation of the coordinate system. This third transformation can be performed not in a general form but is specified by the resonance-line on which ω_1 and ω_2 lie. The generating function for $n_1 \omega_1 + n_2 \omega_2 = p$ is given by

(c)

$$W(z_1 \bar{z}_1 z_2 \bar{z}_2 t) = (z_1 \bar{z}_1 + z_2 \bar{z}_2) \cdot \exp \left[i \frac{p}{n_1 + n_2} t \right] \quad \text{so that}$$

$$\xi_1 = W_{\xi 1} \quad z_1 = \xi_1 \exp \left[-i \frac{p}{n_1 + n_2} t \right]$$

$$\xi_2 = W_{\xi 2} \quad \bar{z}_1 = \bar{\xi}_1 \exp \left[i \frac{p}{n_1 + n_2} t \right]$$

$$z_1 \bar{z}_1 = \xi_1 \bar{\xi}_1$$

(d)

or

obviously

$$\bar{z}_1 = W_{z 1} \quad z_2 = \xi_2 \exp \left[-i \frac{p}{n_1 + n_2} t \right]$$

$$z_2 \bar{z}_2 = \xi_2 \bar{\xi}_2$$

$$\bar{z}_2 = W_{z 2} \quad \bar{z}_2 = \bar{\xi}_2 \exp \left[i \frac{p}{n_1 + n_2} t \right]$$

Then a term $\sigma(t)$ goes over into

$$\exp[-ipt] z_1^{j-k} z_2^{l-m} = \exp[-ipt] \exp[i(-j+k-l+m) \cdot \frac{p}{n_1+n_2} t] z_1^{j-k} z_2^{l-m}.$$

Since $-j+k-l+m = n_1 + n_2$, the two exponential functions are cancelled.

The new hamiltonian becomes now

$$\Gamma(\xi_1 \bar{\xi}_1 \xi_2 \bar{\xi}_2) = G(z_1 \bar{z}_1 z_2 \bar{z}_2 t) + \frac{\partial W}{\partial t}. \quad \text{In } G \text{ the } z\text{'s are expressed by the } \xi\text{'s very simply by replacing } z \rightarrow \xi \text{ omitting the exponentials. } \frac{\partial W}{\partial t} \text{ expressed in } \xi \text{ is}$$

$$\frac{\partial W}{\partial t} = i \frac{p}{n_1 + n_2} (\xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2) \text{ so that the final normal-form becomes}$$

(a)

$$\begin{aligned} \Gamma(\xi_1 \bar{\xi}_1 \xi_2 \bar{\xi}_2) = & -i \left[\left(\omega_1 - \frac{p}{n_1 + n_2} \right) \xi_1 \bar{\xi}_1 + \left(\omega_2 - \frac{p}{n_1 + n_2} \right) \xi_2 \bar{\xi}_2 \right] \\ & + \sum_{\gamma=2}^{\frac{N}{2}} \sum_{\alpha+\beta=\gamma} \xi_{\alpha\beta}^{(2\gamma)} (\xi_1 \bar{\xi}_1)^\alpha (\xi_2 \bar{\xi}_2)^\beta \\ & + (\gamma_{jklm}^{(N)} \xi_1^j \bar{\xi}_1^k \xi_2^l \bar{\xi}_2^m + \text{c.c.}) + \dots = \psi(x_1^2, x_2^2, \zeta) \end{aligned}$$

$\frac{N}{2}$ smallest integer $\leq \frac{N}{2}$

with $\zeta = \xi_1^j \bar{\xi}_1^k \xi_2^l \bar{\xi}_2^m$ (where either j or k and either l or m is zero)

Because $\frac{\partial W}{\partial t}$ is purely imaginary, Γ is too.

0. Discussion of the hamiltonian $\Gamma(\xi_1 \bar{\xi}_1 \xi_2 \bar{\xi}_2)$

The hamiltonian Γ has two important properties

- 1) it gives the equations of motion for the ξ -coordinates

$$\dot{\xi}_1 = \Gamma_{\bar{\xi}_1} ; \quad \dot{\xi}_2 = \Gamma_{\bar{\xi}_2} \quad (\text{and complex conjugated})$$

- 2) It does not contain the time explicitly and is therefore a constant of the motion or an invariant function under a transformation $t \rightarrow t + \delta$ (see 28). This remains valid, of course, also if one expresses everything again in the z 's.

One can now try to solve the equations of motion (which indeed do not contain time-dependent coefficients but are still coupled by non-linear terms). In special cases this may be of interest and can be done by numerical methods.

This requests special initial conditions and will never lead to general statements.

In the case without resonance (p. 27) we came to a quite general statement (28b) about stability, which was based on the fact that r_1^2 and r_2^2 were invariants under time change. It is now our aim to find a similar set of invariant functions of r_1^2 and r_2^2 .

P. The two invariant functions

As in the non-resonance case we look for $\frac{d}{dt} r_1^2$ and $\frac{d}{dt} r_2^2$:

$$(r_1^2)^\circ = \dot{\xi}_1 \bar{\xi}_1 + \xi_1 \dot{\bar{\xi}}_1 = \xi_1 \Gamma_{\bar{\xi}_1} - \bar{\xi}_1 \Gamma_{\xi_1} ; \quad (r_2^2)^\circ = \xi_2 \Gamma_{\bar{\xi}_2} - \bar{\xi}_2 \Gamma_{\xi_2}$$

Now all terms in Γ are of one of the three forms (see p.39)

$$\varepsilon_{\alpha\beta} (\xi_1 \bar{\xi}_1)^\alpha (\xi_2 \bar{\xi}_2)^\beta \quad \text{or}$$

$$\gamma_{j'k'l'm'} \xi_1^{j'} \xi_1^{k'} \xi_2^{l'} \xi_2^{m'} \quad \text{with}$$

$$\begin{aligned} j' &= j + \lambda & l' &= l + \mu & \lambda &= 0, 1, 2, \dots \\ k' &= k + \lambda & m' &= m + \mu & \mu &= 0, 1, 2, \dots \end{aligned} \quad \text{and} \quad \text{independently}$$

or

$$\gamma_{j''k''l''m''} \xi_1^{j''} \xi_1^{k''} \xi_2^{l''} \xi_2^{m''} \quad \text{with}$$

$$\begin{aligned} j'' &= aj & l'' &= al & a &= 2, 3, 4, \dots \\ k'' &= ak & m'' &= am & & \text{(the same for all forms)} \end{aligned}$$

Applying the operators

$$D_1 = \bar{\xi}_1 \frac{\partial}{\partial \xi_1} - \xi_1 \frac{\partial}{\partial \bar{\xi}_1}$$

$$(r_1^2)^\circ = D_1 \Gamma$$

we have

and

$$D_2 = \bar{\xi}_2 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \bar{\xi}_2}$$

$$(r_2^2)^\circ = D_2 \Gamma$$

for the three types of terms we find

$$D_{1,2} \varepsilon_{\alpha\beta} (\xi_1 \bar{\xi}_1)^\alpha (\xi_2 \bar{\xi}_2)^\beta = 0$$

$$D_1 \gamma_{j'k'l'm'} \xi_1^{j'} \xi_1^{k'} \xi_2^{l'} \xi_2^{m'} = (k-j) \gamma_{j'k'l'm'} \xi_1^{j'} \xi_1^{k'} \xi_2^{l'} \xi_2^{m'}$$

$$D_1 \gamma_{j''k''l''m''} \xi_1^{j''} \xi_1^{k''} \xi_2^{l''} \xi_2^{m''} = (k-j) a \gamma_{j''k''l''m''} \xi_1^{j''} \xi_1^{k''} \xi_2^{l''} \xi_2^{m''}$$

Therefore (symbolically)

$$D_1 \Gamma = (k-j) \left[\sum \gamma_{j'k'l'm'} \xi_1^{j'} \xi_1^{k'} \xi_2^{l'} \xi_2^{m'} + \sum_a a \gamma_{j''k''l''m''} \xi_1^{j''} \xi_1^{k''} \xi_2^{l''} \xi_2^{m''} \right]$$

and similar

$$D_2 \Gamma = (m-l) \left[\sum \gamma_{j'k'l'm'} \xi_1^{j'} \xi_1^{k'} \xi_2^{l'} \xi_2^{m'} + \sum_a a \gamma_{j''k''l''m''} \xi_1^{j''} \xi_1^{k''} \xi_2^{l''} \xi_2^{m''} \right]$$

Because each time a is the same number for all four indices and because the λ and μ which can be different for the 1- and the 2- coordinate, fall out, the two square brackets contain exactly the same sums (including the terms where j is changed with k and l with m for complex conjugation. In the formula these terms are only omitted for the sake of convenience). This gives

$$\frac{dr_1^2}{dt} = \frac{D_1 f}{D_2 f} = \frac{dr_1^2}{dr_2^2} = \frac{k-j}{m-l} = \frac{n_1}{n_2} \text{ and therefore by integration}$$

(a)
$$r_1^2 - \frac{n_1}{n_2} r_2^2 = r_{10}^2 - \frac{n_1}{n_2} r_{20}^2 = A = \text{constant or}$$

$$r_2^2 - \frac{n_2}{n_1} r_1^2 = r_{20}^2 - \frac{n_2}{n_1} r_{10}^2 = B = \text{constant. (exactly!)}$$

$$B = -\frac{n_2}{n_1} A. \quad \text{For vanishing } n_1 \text{ or } n_2 \text{ one form remains valid.}$$

r_{10} and r_{20} denote the initial values.

As far as our presupposition is fulfilled, that ω_1 and ω_2 lie on one resonance-line, but not on any other one (crossing it), we have here an exact invariant. If there is a crossing with another line of the order $N' \gg N$, it will remain still approximately invariant - at least up to the order N' . In all the following we will not worry about that and idealize the situation by assuming our presupposition to be valid. The behaviour near crossing points will perhaps be treated in a later report.

Apart from (44a) we still have another invariant function, that is the hamiltonian itself. In order to write it also in r_1 and r_2 (as far as it is considered as an invariant rather than as the hamiltonian) we introduce

(c)
$$\xi_1 = r_1 e^{i\varphi_1}$$

$$\xi_2 = r_2 e^{i\varphi_2}$$

$$v_{jklm} = \gamma e^{i\delta}$$

We need to express only this one γ_{jklm} because firstly the stability behaviour can be discussed from terms up to the order of the considered resonance alone and it secondly cannot be discussed, if higher terms are included (i.e. if the amplitudes become so large, that higher terms become important). This will become clear in what follows. With this we get for (41a) up to N^{th} -order terms:

(a)

$$\begin{aligned} i \cdot P(\xi_1 \bar{\xi}_1 \xi_2 \bar{\xi}_2) \equiv \phi(r_1 r_2 \varphi_1 \varphi_2) &= (\omega_1 - \frac{p}{n_1 + n_2}) r_1^2 + (\omega_2 - \frac{p}{n_1 + n_2}) r_2^2 + \\ &+ \sum_{\lambda=2}^{N/2} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} r_1^{2\alpha} r_2^{2\beta} + \\ &+ 2\gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) r_1^{|n_1|} r_2^{|n_2|} + \dots \end{aligned}$$

[$i g_{\alpha\beta}$ is a real quantity; $j+k=|n_1|$ and $l+m=|n_2|$ according to (34by)]

Whereas (44a) is exactly invariant and very simply constructed, neither one is true for (45a), if the dotted higher terms are neglected. It is indeed very complicated, because in any explicit case one will have long calculations to do before getting the coefficients $g_{\alpha\beta}$ and γ_{jklm} .

But at least we possess now two invariants and will use them in the following to discuss stability.

Q. Discussion of the two invariants

We are left with the two invariants

(b)

$$r_1^2 - \frac{n_1}{n_2} r_2^2 = r_{10}^2 - \frac{n_1}{n_2} r_{20}^2 = A \quad \text{or} \quad r_2^2 - \frac{n_2}{n_1} r_1^2 = r_{20}^2 - \frac{n_2}{n_1} r_{10}^2 = B \quad \text{and}$$

$$\begin{aligned} (\omega_1 - \frac{p}{n_1 + n_2}) r_1^2 + (\omega_2 - \frac{p}{n_1 + n_2}) r_2^2 &+ \sum_{\lambda=2}^{\frac{N}{2}} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} r_1^{2\alpha} r_2^{2\beta} + \\ &+ 2\gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) r_1^{|n_1|} r_2^{|n_2|} = C \end{aligned}$$

$$\frac{N}{2} \text{ smallest integer} \geq \frac{N}{2}$$

Here A and B can be calculated from the initial amplitudes alone whereas in C also the initial phases φ_{10} and φ_{20} enter.

In the following we shall refer to these invariants as to the quadratic invariant and the $(N+1)^{\text{th}}$ -order invariant respectively.

The latter notation indicates that this form is indeed invariant up to the order $N+1$. This is because we have in the double-sum terms up to

$$2. \frac{\bar{N}}{2} = \begin{cases} N & \text{for } N \text{ even} \\ N+1 & \text{for } N \text{ odd} \end{cases}$$

and because the next higher terms, which we have neglected, are of the order $N+2$. (see p. 38/39).

1) First we consider the quadratic invariant.

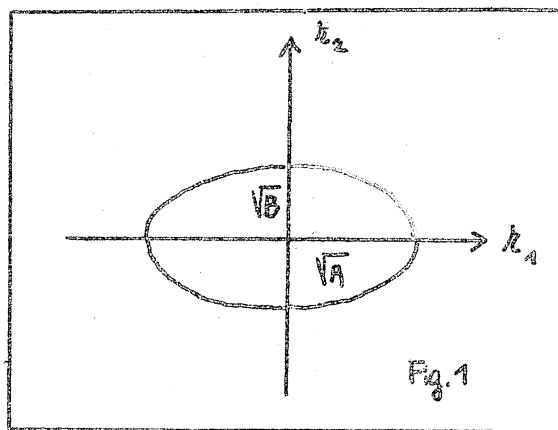
If $\frac{n_1}{n_2}$ is negative, then A(or B) must be positive and

$r_1^2 + \left| \frac{n_1}{n_2} \right| r_2^2 = A$ is an ellipse with the half axes:

$$a_1 = \sqrt{A} = \sqrt{\left| \frac{n_1}{n_2} \right|} B$$

$$a_2 = \sqrt{\left| \frac{n_2}{n_1} \right|} A = \sqrt{B}$$

(a)



(of course, only positive values of r_1 and r_2 are of sense)

If $\frac{n_1}{n_2}$ is positive, then A(or B) can be positive or negative (or zero)

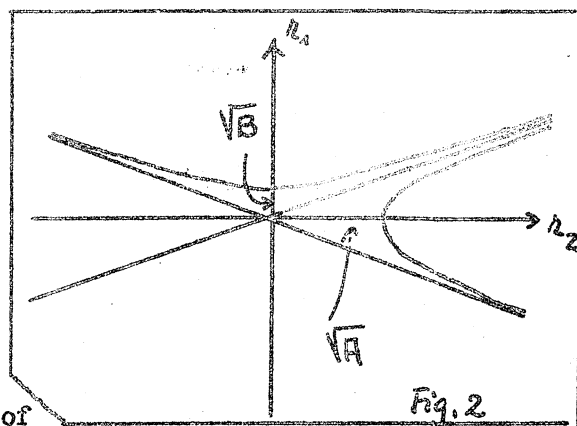
and A and B have different signs. The curves are now hyperbolas with slope of the asymptote equal to $\frac{n_2}{n_1}$ and the half axes

(a)

$$a_1 = \sqrt{A} = \sqrt{-\frac{n_1}{n_2} B} \quad \text{for } \begin{matrix} A > 0 \\ B < 0 \end{matrix}$$

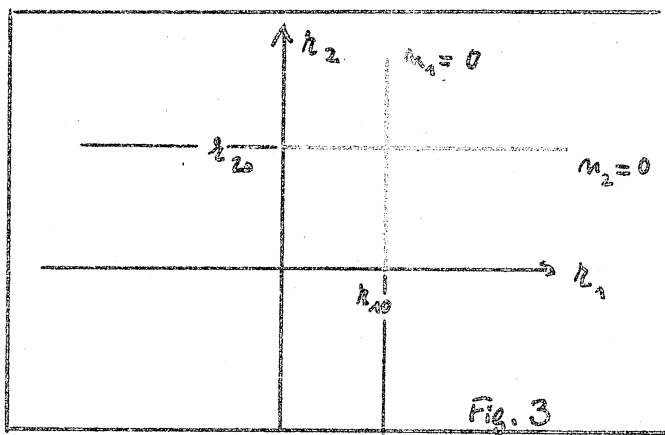
$$a_2 = \sqrt{-\frac{n_2}{n_1} A} = \sqrt{B} \quad \text{for } \begin{matrix} A < 0 \\ B > 0 \end{matrix}$$

$$a_1 = a_2 = 0 \quad \text{for } A = B = 0 \quad (\text{pair of straight lines})$$



It is immediately clear, that the elliptic case means strong quasi-stability (p.32): neither r_1 nor r_2 can reach values greater than the larger of the two half axes, which are given by the initial amplitudes and which vanish for zero initial values. In fact, the discussion of the $(N+1)^{\text{th}}$ -order invariant may restrict the range even more on a small piece of the first quadrant of the ellipse. (see fig. 4d)

The hyperbolic case does not allow any conclusion - it must be discussed with the help of the $(N+1)^{\text{th}}$ -order invariant. Finally there remain the cases that one of the n_1 or n_2 is zero. Here also the $(N+1)^{\text{th}}$ -order invariant is to be discussed, because the picture is as follows:



For the following the elliptic case drops out by the statement:

(b)

If ω_1 and ω_2 lie on a difference-resonance-line (n_1 and n_2 of different signs) then one has strong quasi-stability and the maximum amplitudes are given by the initial values through (46a).

In other words: the resonance-lines with positive slope are never excited, they are "suppressed resonance-lines". This is the non-linear analogon to the fact, that the lines $\omega_1 - \omega_2 = p$ are suppressed in the linear theory also. [See for instance G.Lüders (Lü)]

2) The $(N+1)^{\text{th}}$ -order invariants.

In order to cover also the cases where n_1 or n_2 is equal to zero, we use the quadratic invariant to eliminate r_1 or r_2 respectively from the $(N+1)^{\text{th}}$ -order invariant. In this way we get two forms:

a) Inserting $r_2^2 = B + \frac{n_2}{n_1} r_1^2$ into the $(N+1)^{\text{th}}$ order invariant:

(a)

$$r_1^2 (n_1 \omega_1 + n_2 \omega_2 - p) + n_1 \sum_{\lambda=2}^{\frac{N}{2}} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} r_1^{2\alpha} \left(B + \frac{n_2}{n_1} r_1^2 \right)^\beta +$$

$$+ 2n_1 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) r_1^{|n_1|} \left(B + \frac{n_2}{n_1} r_1^2 \right)^{\frac{|n_2|}{2}} = C_1$$

$\frac{N}{2}$ smallest integer $\geq \frac{N}{2}$

$n_1 \neq 0$ necessary

or

β) Inserting $r_1^2 = A + \frac{n_1}{n_2} r_2^2$ we get the alternative form

(b)

$$r_2^2 (n_1 \omega_1 + n_2 \omega_2 - p) + n_2 \sum_{\lambda=2}^{\frac{N}{2}} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} \left(A + \frac{n_1}{n_2} r_2^2 \right)^\alpha r_2^{2\beta} +$$

$$+ 2n_2 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) \left(A + \frac{n_1}{n_2} r_2^2 \right)^{\frac{|n_1|}{2}} r_2^{|n_2|} = C_2$$

$\frac{N}{2}$ smallest integer $\geq \frac{N}{2}$

$n_2 \neq 0$ necessary

We may use either of these forms to discuss the behaviour of either r_1 or r_2 and having found the possible amplitude variations of one of these, the range of the other one is then easily derived by means of the quadratic invariant. [It is, however, convenient to use (48a) if $B > 0$ and (48b) if $A > 0$, because then the variables r_1 or r_2 respectively can reach zero and the picture is as in figs. 4 a, b. Otherwise the shaded area in these figures would start at some point $r \neq 0$.]

The first factor of r_1^2 and r_2^2 respectively in both formulas is $n_1 \omega_1 + n_2 \omega_2 - p = 0$. But we may treat this factor also as ^{being} slightly different from zero, thereby exploring a small neighbourhood of the resonance-line. Of course, even the slightest deviation from this resonance-line would bring us on another line. But as already pointed out (p. 25), this neighbourhood, if small enough, contains only lines of considerably higher order (apart from those crossing the considered one with a fairly large angle). We can neglect them, if the amplitudes are sufficiently small - and we shall see, that we can neglect them not only because of the smallness of higher terms, but also, because in practically all cases no resonance of order higher than four is excited. (As will be shown).

The discussion of (48a) and (48b) is as follows:

We denote

(a)
$$n_1 \omega_1 + n_2 \omega_2 - p = \epsilon \ll 1$$

Then we have in each formula at first a polynomial $\epsilon r^2 + \sum \sum \dots$ with constant coefficients and secondly, added to this, a polynomial multiplied with a cosinus which can assume any value between +1 and -1 (we do not know the phases!) We may then write

(b)
$$P_1(r) - C = P_2(r) \cdot \cos \psi, \text{ where } \psi \text{ remains undetermined.}$$

The picture looks like this (Figs. 4a, b, c, d).

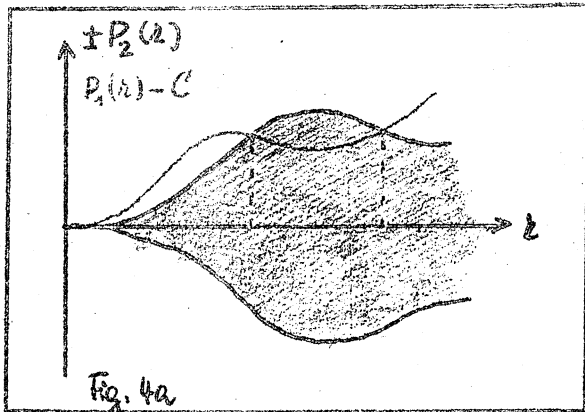


Fig. 4a. Strongly quasi-stable at the origin. Some large amplitudes possible with the same constant C ; but they are not accessible starting from zero.

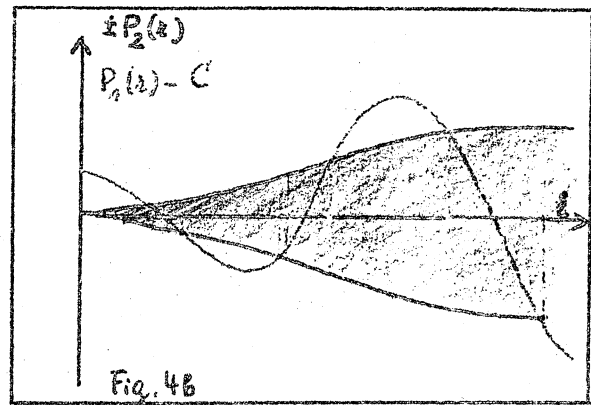


Fig. 4b. Strongly quasi-stable. For zero initial amplitudes we would have here the same picture at the origin as in fig. 4a. But this fig. 4b refers to starting with finite initial amplitudes which range of variation remains small. Larger amplitudes also possible but not directly accessible from the small one.

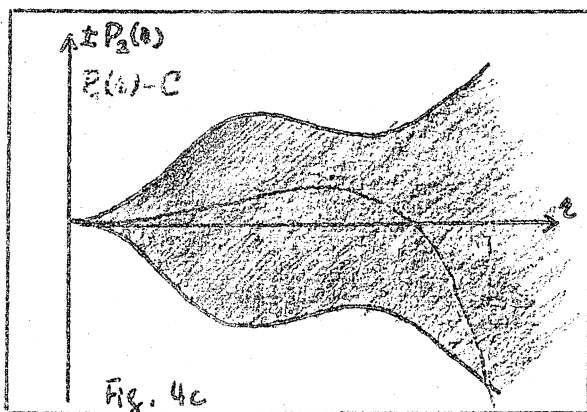


Fig. 4c. Unstable at the origin. Even vanishing initial amplitudes will grow up to considerable maximum values. If this maximum amplitude, however, is small, one has weak quasi-stability.

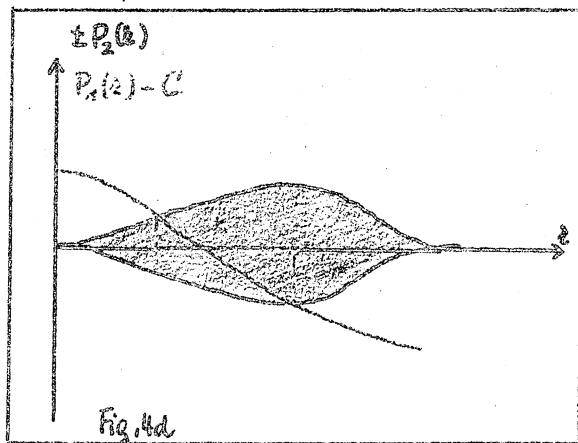


Fig. 4d. Difference resonance, elliptic quadratic invariant. Discussion of $(N+1)^{th}$ -order invariant here shows that only a part of the ellipses first quadrant is accessible. The whole quadrant corresponds to the π within the shaded area.

The polynomial $P_1(r)-C$ is bound to stay between the limits $\pm P_2(r)$. The latter region is shaded and only such values for r are possible, for which $P_1(r)-C$ lies within this area. This gives immediately the range of r -values, which however may be very large and start from the origin (instability, fig. 4c). There will be sometimes several allowed regions separated by forbidden ones. Furthermore the allowed region may cover all phases (namely if $P_1(r)-C$ crosses the shaded area Fig.4b) or some phases may be excluded (if $P_1(r)-C$ enters the shaded region and leaves it on the same side, Fig.4a). In the case of a difference-resonance (elliptic form of the quadratic invariant) $P_2(r)$ has zero's at $r=0$ and at $r = \sqrt{\left| \frac{n_2}{n_1} \right|} A$ or $\sqrt{\left| \frac{n_1}{n_2} \right|} B$ (respectively for r_2 or r_1) if n_1 and $n_2 \neq 0$

In this case the picture looks like fig 4d and generally not the whole first quadrant of the ellipse will be accessible. Therefore the difference-resonances must not lead to a total energy exchange between both spatial directions.

Now it becomes also clear what "quasi-stability" means: the "constant" C is in fact constant up to the $(N+1)^{th}$ -order only and begins to vary proportionally to terms of the order $N+2$, which contain again phase-factors $\cos(\dots)$. More exactly, the picture looks then like this (fig.5)

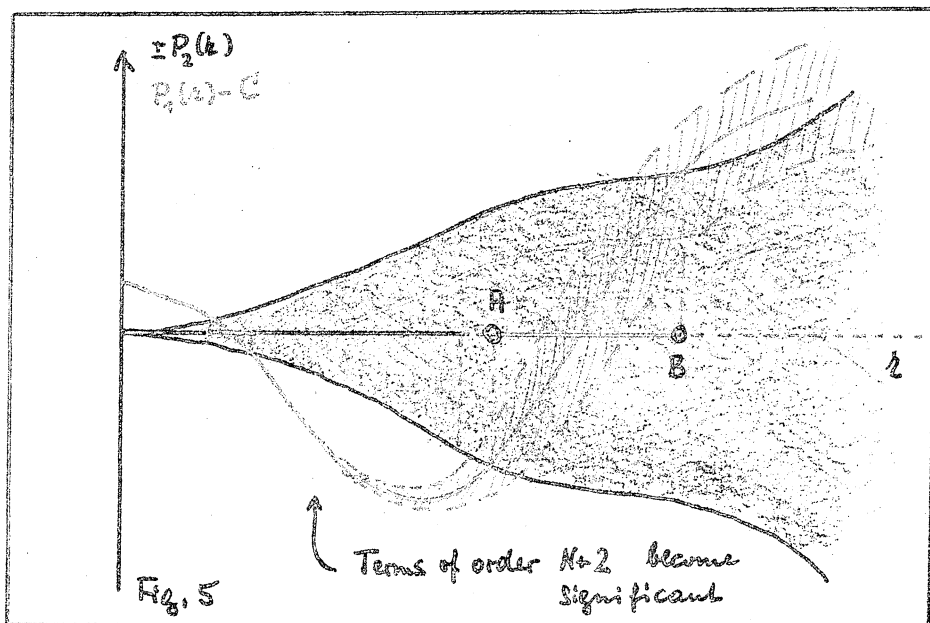


Fig.5 - Effects of neglected higher order terms limiting the validity of predictions on amplitude ranges.

One sees that near the origin, these terms are of no importance but will in fact devaluate the conclusions on the amplitude ranges for larger amplitudes. Taking the invariants literally, would lead in fig.5 to a second finite amplitude-range $A \leq r \leq B$ but considering the presence of not calculated terms of higher order shows that the upper limit B may lie far more outwards - if not in infinity. (One can, of course, also calculate higher terms. But in that case one will obtain further polynomials multiplied with $\cos(\dots)$ with another argument and the shaded area has to be taken the largest possible for all possible values of the different $\cos$ -functions. This firstly is difficult to find out and secondly may lead to too unfavourable estimates.) Finally even near to $r = 0$ such terms may become important after extremely long times because of the uncontrollable time-variation of the high transformation coefficients (see p.30-32) But after Moser's (Mo) work on the one-dimensional case one may believe without the whole estimating work, that the conclusions drawn from our "invariants" are valid for/fairly long time.

R. More explicit form of the $(N+1)^{\text{th}}$ -order invariants.

Numerical values for amplitude-ranges can be given only on the base of numerical values of the original hamiltonian-coefficients. Nevertheless, it is possible to go into more detail, even for the general formula and to give at least one general result. To obtain this, we write the $\sum \sum$ and the term combined with the $\cos(\delta - n_1 \varphi_1 - n_2 \varphi_2)$ in (48a) and (48b) more explicitly and order them with respect to increasing powers:

$r^2, r^4, r^6, \dots$. After elementary calculation we obtain in (48a) ($n_1 \neq 0!$)

$$\begin{aligned}
 (a) \quad n_1 \sum_{\lambda=2}^{\frac{N}{2}} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} r_1^{2\alpha} \left(B + \frac{n_2}{n_1} r_1^2 \right)^\beta &= n_1 \sum_{\beta=2}^{\frac{N}{2}} i g_{0\beta} B^\beta \\
 &+ r_1^2 \left[n_2 \sum_{\beta=2}^{\frac{N}{2}} i g_{0\beta} B^{\beta-1} + n_1 \sum_{\beta=1}^{\frac{N}{2}} i g_{1\beta} B^\beta \right] \\
 &+ r_1^4 \left[n_1 i g_{20} + n_2 i g_{11} + \frac{n_2^2}{n_1} i g_{02} + B(\dots) + \dots \right] r_1^6 \left[\dots \right] + \dots
 \end{aligned}$$

and in (48b) ($n_2 \neq 0!$)

(b)

$$\begin{aligned}
 n_2 \sum_{\lambda=2}^{\frac{\bar{N}}{2}} \sum_{\alpha+\beta=\lambda} i g_{\alpha\beta}^{(2\lambda)} \left(A + \frac{n_1}{n_2} r_2^2 \right) r_2^{2\beta} &= n_2 \sum_{\alpha=2}^{\frac{\bar{N}}{2}} i g_{\alpha 0} A^\alpha \\
 &+ r_2^2 \left[n_1 \sum_{\alpha=2}^{\frac{\bar{N}}{2}} i g_{\alpha 0} A^{\alpha-1} + n_2 \sum_{\alpha=1}^{\frac{\bar{N}}{2}} i g_{\alpha 1} A^\alpha \right] + \\
 &+ r_2^4 \left[n_2 i g_{02} + n_1 i g_{11} + \frac{n_1^2}{n_2} i g_{20} + A(\dots) + \dots \right] + r_2^6 [\dots] + \dots
 \end{aligned}$$

(in both formulas: $\frac{\bar{N}}{2}$ smallest integer $\geq \frac{N}{2}$)

Here the first term on the right-hand side is a constant and will be put into C_1 or C_2 respectively. The next term is r^2 with its coefficient. We observe that these coefficients (for r_1^2 or r_2^2 respectively) vanish with vanishing B or A respectively. But because

$A = r_{10}^2 - \frac{n_1}{n_2} r_{20}^2$ and $B = -\frac{n_2}{n_1} A$ they vanish simultaneously first of all for $r_{10} = r_{20} = 0$ and secondly if $r_{10}^2 = \frac{n_1}{n_2} r_{20}^2$ and $n_{1,2} \neq 0$. In these cases for $A = B = 0$ no term $\sim r^2$ is left in the $\sum \sum$. In the terms $\sim r^4$ we have picked out those parts of the coefficients, which do not contain B or A respectively. So for $A = B = 0$ a term r^4 will always remain if not accidentally

$$n_1 g_{20} + n_2 g_{11} + \frac{n_2^2}{n_1} g_{02} = 0 \quad \text{and/or} \quad n_2 g_{02} + n_1 g_{11} + \frac{n_1^2}{n_2} g_{20} = 0$$

(If $n_{1,2} \neq 0$, both vanish together or both are $\neq 0$). One sees easily that also r^6 has terms independent of A or B in its coefficients and all higher terms too.

Omitting the constant terms, we write somewhat shorter, only exhibiting the significant parameters in (48a) and (51a)

(b)

$$n_1 \sum \sum = r_1^2 \cdot B \left(n_2 \sum_2 B + n_1 \sum_1 B \right) + r_1^4 \left[n_1 i g_{20} + n_2 i g_{11} + \frac{n_2^2}{n_1} i g_{02} + B(\dots) \right] + \dots$$

and in (48b) and (52a)

$$(a) \quad n_2 \sum \sum = r_2^2 \cdot A \left(n_1 \sum_2 A + n_2 \sum_1 A \right) + r_2^4 \left[n_2^2 i g_{02} + n_1 i g_{11} + \frac{n_1^2}{n_2} i g_{20} + A(\dots) \right] + \dots$$

where the meaning of the notation is obvious from comparison with (51a), (52a).

Now we have to write the terms multiplied by $\cos(\phi - n_1 \phi_1 - n_2 \phi_2)$ also more explicitly. We get in (48a)

$$(b) \quad r_1^{|n_1|} \left(B + \frac{n_2}{n_1} r_1^2 \right)^{\frac{|n_2|}{2}} = r_1^{|n_1|} B^{\frac{|n_2|}{2}} + r_1^{|n_1|+2} \cdot \frac{|n_2|}{2} \frac{n_2}{n_1} B^{\frac{|n_2|}{2}-1} + \dots$$

$$\xrightarrow{B \rightarrow 0} \left(\frac{n_2}{n_1} \right)^{\frac{|n_2|}{2}} \cdot r_1^N$$

and in (48b)

$$(c) \quad r_2^{|n_2|} \left(A + \frac{n_1}{n_2} r_2^2 \right)^{\frac{|n_1|}{2}} = r_2^{|n_2|} A^{\frac{|n_1|}{2}} + r_2^{|n_2|+2} \frac{|n_1|}{2} \frac{n_1}{n_2} A^{\frac{|n_1|}{2}-1} + \dots$$

$$\xrightarrow{A \rightarrow 0} \left(\frac{n_1}{n_2} \right)^{\frac{|n_1|}{2}} r_2^N$$

We observe that the lowest term is $r_1^{|n_1|}$ or $r_2^{|n_2|}$ respectively if A or B is $\neq 0$, but for A or/and B equal to zero only r^N remains. Not very correctly (because $\frac{n_1}{2}$ or $\frac{n_2}{2}$ may be half-integral and the series becomes infinite) but expressing what is important here, we simply write

$$r_1^{|n_1|} \left(B + \frac{n_2}{n_1} r_1^2 \right)^{\frac{|n_2|}{2}} = r_1^{|n_1|} B^{\frac{|n_2|}{2}} + \dots + \left(\frac{n_2}{n_1} \right)^{\frac{|n_2|}{2}} r_1^N \quad \text{and}$$

$$r_2^{|n_2|} \left(A + \frac{n_1}{n_2} r_2^2 \right)^{\frac{|n_1|}{2}} = r_2^{|n_2|} A^{\frac{|n_1|}{2}} + \dots + \left(\frac{n_1}{n_2} \right)^{\frac{|n_1|}{2}} r_2^N$$

Finally with $n_1\omega_1 + n_2\omega_2 - p = \varepsilon$ we obtain for (48a)

(a)

$$r_1^2 \left[\varepsilon + B(n_2 \sum_2 B + n_1 \sum_1 B) \right] + r_1^4 \left[n_1 i g_{20} + n_2 i g_{11} + \frac{n_2^2}{n_1} i g_{02} + B(\dots) \right] + \dots = C_1 =$$

$$= 2n_1 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) \left[r_1^{\frac{|n_1|}{2}} B^{\frac{|n_2|}{2}} + \dots + \left(\frac{n_2}{n_1} \right)^{\frac{|n_2|}{2}} r_1^N \right] ; \quad n_1 \neq 0$$

and in (48b)

(b)

$$r_2^2 \left[\varepsilon + A(n_1 \sum_2 A + n_2 \sum_1 A) \right] + r_2^4 \left[n_2 i g_{02} + n_1 i g_{11} + \frac{n_1^2}{n_2} i g_{20} + A(\dots) \right] + \dots = C_2 =$$

$$= 2n_2 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) \left[r_2^{\frac{|n_2|}{2}} A^{\frac{|n_1|}{2}} + \dots + \left(\frac{n_1}{n_2} \right)^{\frac{|n_1|}{2}} r_2^N \right] ; \quad n_2 \neq 0$$

The constants A, B have to be chosen accordingly to equ. (45b) and C_1, C_2 so that (54a,b) are fulfilled for the initial-values

$r_{10}, r_{20}, \varphi_{10}, \varphi_{20}$

S. Discussion of the $(N+1)^{th}$ -order invariants

1) Stability at the origin.

We assume $r_{10} = r_{20} = 0$. Then $A=B=C_1=C_2=0$ and we obtain

$(n_1 \neq 0)$ for (54a)

(c)

$$r_1^2 \varepsilon + r_1^4 \left[n_1 i g_{20} + n_2 i g_{11} + \frac{n_2^2}{n_1} i g_{02} \right] + \dots = 2n_1 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) \times$$

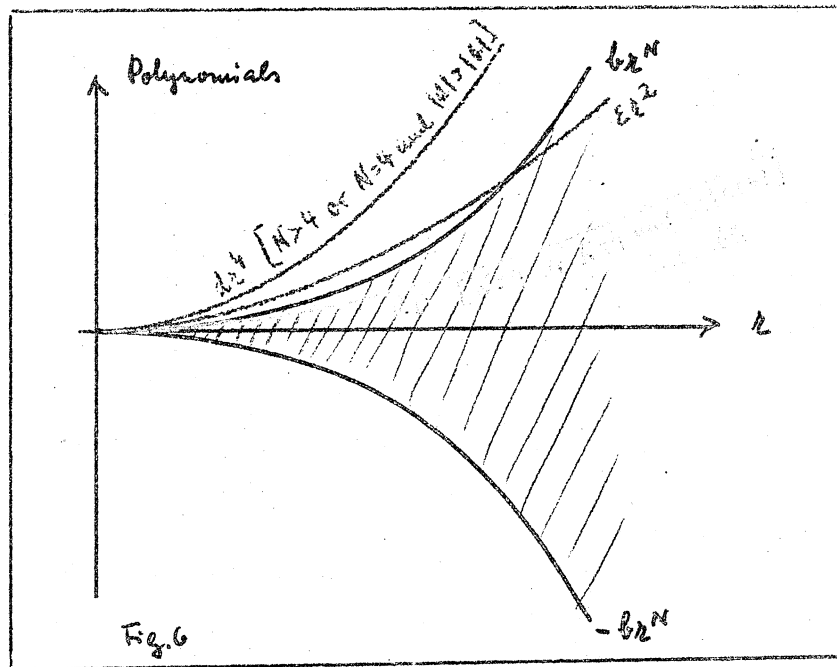
$$\times \left(\frac{n_2}{n_1} \right)^{\frac{|n_2|}{2}} r_1^N$$

and for (54b) ($n_2 \neq 0$)

$$(a) \quad r_2^2 + r_2^4 \left[n_2^2 \varepsilon_{02} + n_1^2 \varepsilon_{11} + \frac{n_1^2}{n_2} \varepsilon_{20} \right] + \dots = 2n_2 \gamma \cos(\delta - n_1 \varphi_1 - n_2 \varphi_2) \left(\frac{n_1}{n_2} \right)^{\frac{n_1}{2}} r_2^N$$

Because we do not know the phases, the cos-function lies somewhere between ± 1 . The picture is then generally of the form (fig.6)

$$(b) \quad r^2 \varepsilon + r^4 d + \dots = \pm b r^N$$



One sees immediately that this forces r to stay at zero in the following cases:

a) $\varepsilon \neq 0$; $N \geq 3$ for any b

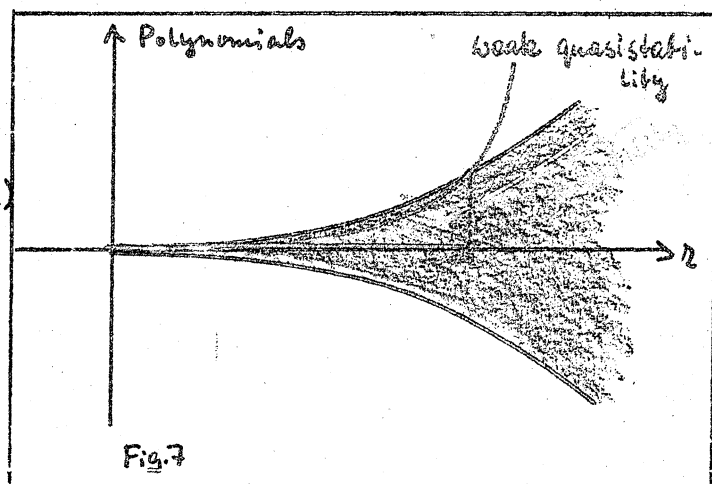
That is: very near to the resonance-line one has still strong quasi-stability.

$$\left. \begin{array}{l} \beta) \quad \varepsilon = 0 ; \quad N \geq 5 \quad \text{for any } d \neq 0 \\ \quad \quad N = 4 \quad \text{for } |b| < |d| \end{array} \right\} \text{strong quasi-stability}$$

On the other hand, if

$\gamma) \quad \varepsilon = 0$ and $N = 3$ one has no stability at the origin. But according to p. 39 the next term containing a cos-function with unknown phases in the argument is of the order $N + 2 = 5$ whereas the next term on the left-hand side is cr^4 . So it now depends on the numerical values whether the motion is bounded within a small range or not. The picture looks like this: (fig.7)

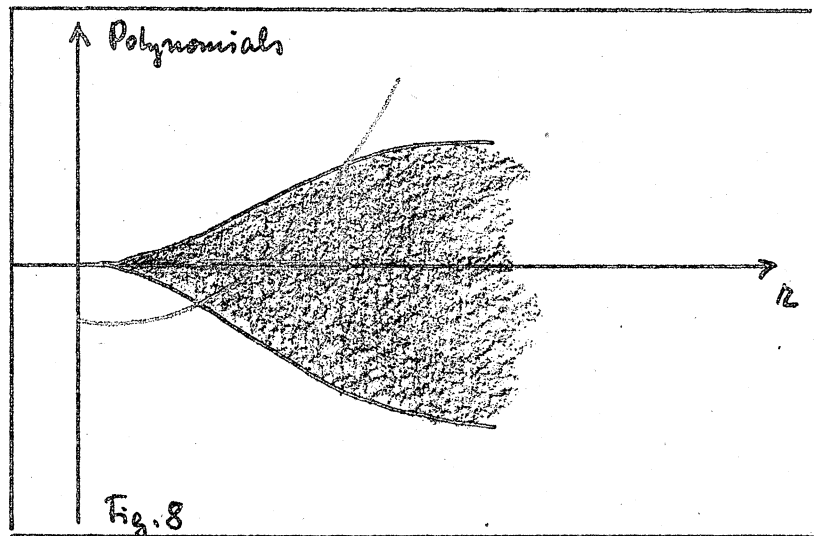
Fig.7.
Weak quasi-stability (red) and instability (green) for $N = 3$ on the resonance-line.



Surely the amplitudes will not grow to infinity but they may reach quite large values in the dotted case and quite small ones in the other one. One sees that the essential quantity which decides whether stability or not is the constant coefficient of the r^4 -term. This is $i \left[n_2 g_{02} + n_1 g_{11} + \frac{n_1^2}{n_2} g_{20} \right]$ (or the same times $\frac{n_1}{n_2}$) and one finds, by comparing it with the frequency-shifts (29a), that there the same coefficients play the determining part. This is the mathematical equivalent to the common physical argument, that no resonance would build up if the frequency-shifts corresponding to increasing amplitudes are large enough to destroy the necessary phase conditions.

2) Stability near the origin

Some of the conclusions remain valid also for small initial-values different from zero, so that A_1, B_1, C_1, C_2 are small numbers. The picture is like this (fig.8):



The criteria have to be changed somewhat; for instance, now not ϵ is significant but $\epsilon + B(\dots)$ or $\epsilon + A(\dots)$ and one has to consider all the terms in the polynomial multiplied by $\cos(\dots)$. For $N = 3$ one now has perhaps stability for $\epsilon = 0$ but no more for $\epsilon + B(\dots) = 0$. That is, the instability can, according to the initial-values, occur on a line parallel and close to the line $n_1\omega_1 + n_2\omega_2 = p$. In the r_1 - r_2 -plane, the representing point thus runs away along one of the hyperbolas (fig.2) and to each hyperbola belongs another value of ϵ (another "effective" resonance-line). Whether it is then going to small or to large amplitudes depends on the numerical values of the coefficients and this must be considered separately for each explicit case. If we collect the results, we come to the following general statements.

T. General results

- a) On and near to resonance-lines of the order $N \gg 5$ one has strong quasi-stability if

$$n_1 i g_{20} + n_2 i g_{11} + \frac{n_2^2}{n_1} i g_{02} \neq 0 \text{ (or/and) } n_2 i g_{02} + n_1 i g_{11} + \frac{n_1^2}{n_2} i g_{20} \neq 0.$$

(This will quite generally be fulfilled).

- β) On and near resonance lines of the order $N = 4$ one can have strong quasi-stability, if the just mentioned quantities are large enough to outweigh the polynomial multiplied with the $\cos(\delta - n_1 \varphi_1 - n_2 \varphi_2)$, but one can also have weak quasi-stability or even no stability.
- γ) Near resonance-lines of the order $N = 3$ one will always find a line parallel to the resonance-line on which one has no strong quasi-stability. Whether weak quasi-stability or no stability at all, depends on several numerical values.
- δ) All this concerns the sum-resonances, the difference-resonances being always stable.
- ε) All conclusions become invalid near crossing points (but presumably only near crossings of lines of low order $N < 5$).
- ξ) Estimates of the amplitude-ranges are always available if an explicit problem is given.

U. Graphical representation of resonance-lines.

In fig.9, fig.10, fig.11 we plot in a $\omega_1 \omega_2$ -plane the resonance-lines of second (linear theory), third and fourth order respectively. Dotted lines represent difference-lines which are proven to be suppressed.

(47b) In fig.12 are shown all sum-resonance-lines together up to the order four. The network of resonance lines is given for $k \leq \omega_{1,2} \leq k + 1$ and it is then continued periodically over the whole plane.

$\omega_{1,2}$ mean physically the numbers of full oscillations within one period of the hamiltonian. For an example let us consider the case that in a proton synchrotron the values for Q_x and Q_y lie between 6 and 6,5. That is with respect to the full circumference, 2π , of the machine. Thus, for the real (disturbed) machine $\omega_1 = Q_x$, $\omega_2 = Q_y$, and the pictures represent the square $6 \leq \omega_{1,2} \leq 7$.

Considering instead an ideal machine with S superperiods of length $L = \frac{2\pi}{S}$ we have $\omega_1 = \frac{Q_x}{S}$; $\omega_2 = \frac{Q_y}{S}$; [for $S = 10$ we obtain $0,6 \leq \omega_{1,2} \leq 0,65$ and the pictures represent the square $0 \leq \omega_{1,2} \leq 1$].

This picture shows that the fourth order difference-resonance would be dangerous, because it would be excited by the superperiod, which is a quite strong perturbation. Fortunately, the difference-resonance is always suppressed and no sum-resonance crosses the area in which the machine is supposed to work.

Finally for the ideal machine without superperiods and with M periods of length $l = \frac{2\pi}{M}$ we have $\omega_1 = \frac{Q_x}{M}$; $\omega_2 = \frac{Q_y}{M}$ [for $M = 50$ we have $0,12 \leq \omega_{1,2} \leq 0,13$. The pictures again represent the square $0 \leq \omega_{1,2} \leq 1$].

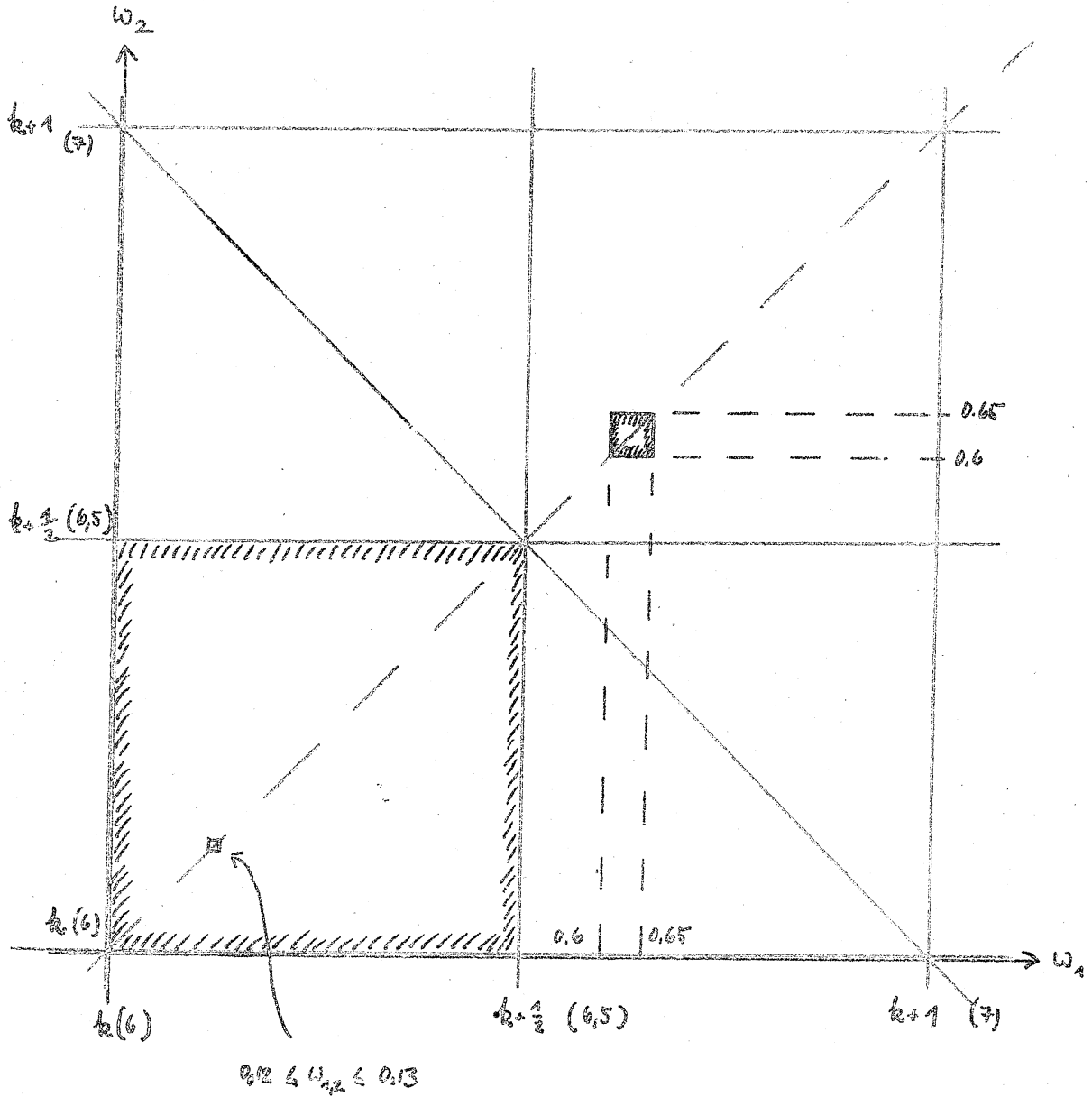


Fig.9. Resonance-lines of the order $N = 2$ ($n_1\omega_1 + n_2\omega_2 = p$; $|n_1| + |n_2| = 2$)

The figure shows the lines (red) as predicted by the linear theory (not derived in this paper) together with the working areas as provided for the CERN-proton synchrotron. The large shaded square corresponds to a perturbed machine with period 2π and Q -values between 6 and 6.5; the smaller square ($0.6 \leq \omega_{1,2} \leq 0.65$) is the same working area with respect to a machine with 10 superperiods per revolution and the smallest square ($0.12 \leq \omega_{1,1} \leq 0.13$) represents the working area with respect to the structure period $\frac{2\pi}{50}$.

Full red lines are excitable, dotted red line is suppressed and will not be excited.

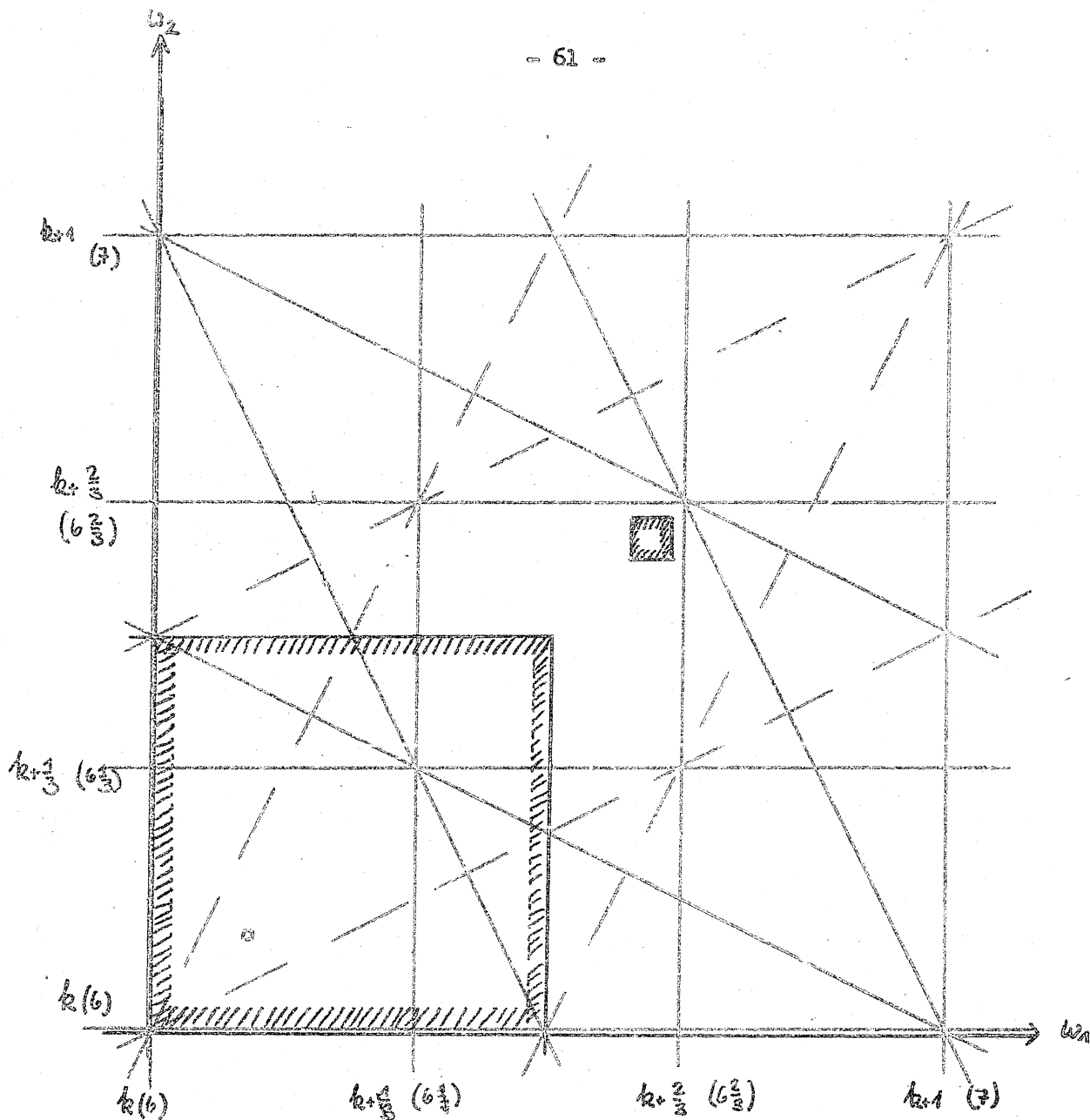


Fig.10. Resonance-lines of the order $N = 3$ ($n_1\omega_1 + n_2\omega_2 = p$; $|n_1| + |n_2| = 3$)

The figure shows the third-order resonance-lines (red), on which the solution is never strongly quasi-stable, but may reach appreciable amplitudes even if one starts from $x \equiv y \equiv 0$. The dotted lines, however, are suppressed and the solutions are there strongly quasi-stable (see 47b). The squares denote the working areas (see text below fig.9).

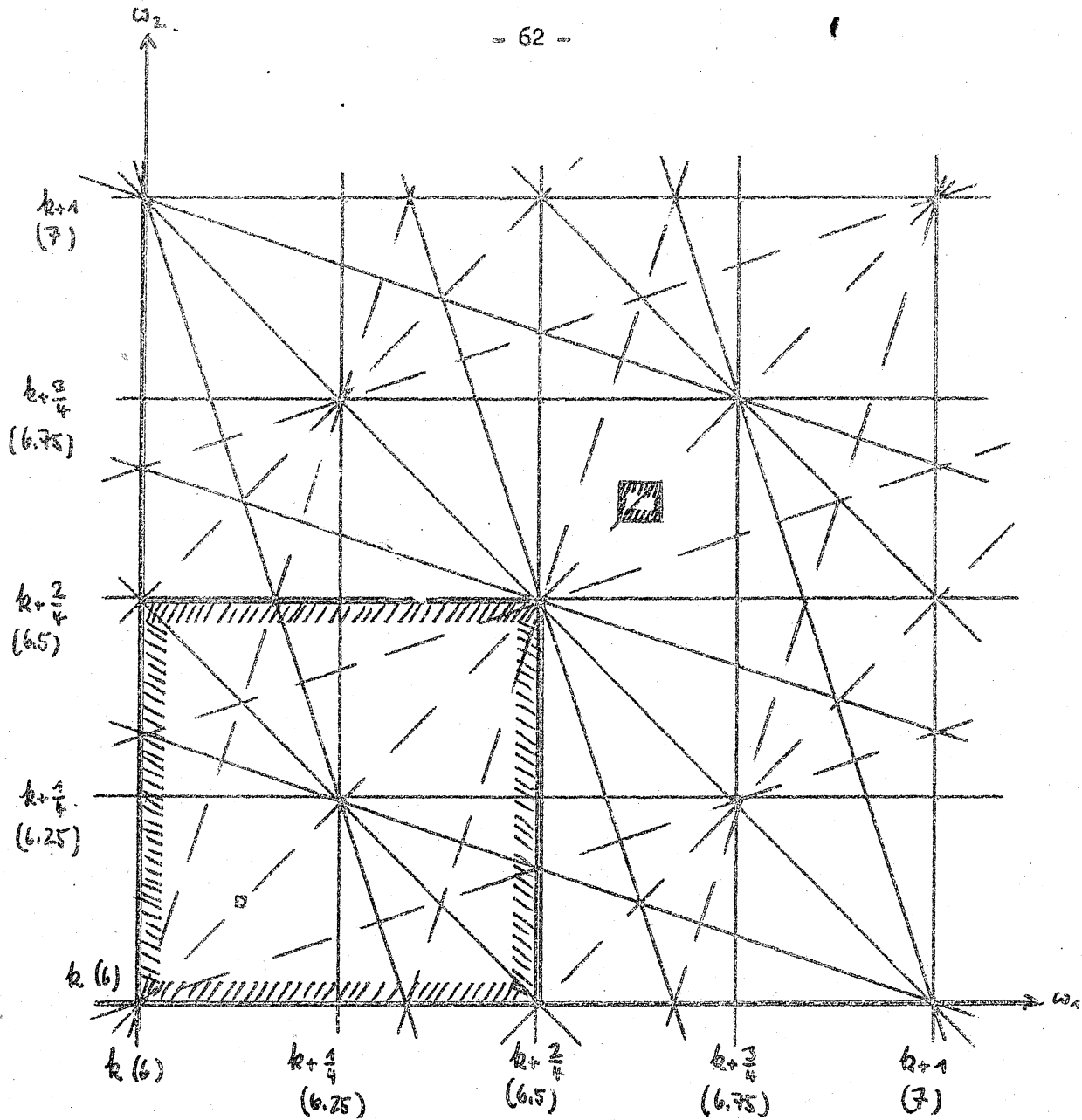


Fig.11. Resonance-lines of the order $N = 4$ ($n_1\omega_1 + n_2\omega_2 = p$; $|n_1| + |n_2| = 4$)

The figure shows the fourth-order resonance-lines. On the full red lines a solution may be either strongly quasi-stable or not, according to the criteria given on p.56. On the dotted lines the motion is always strongly quasi-stable. The squares denote the working areas (see text below fig.9).

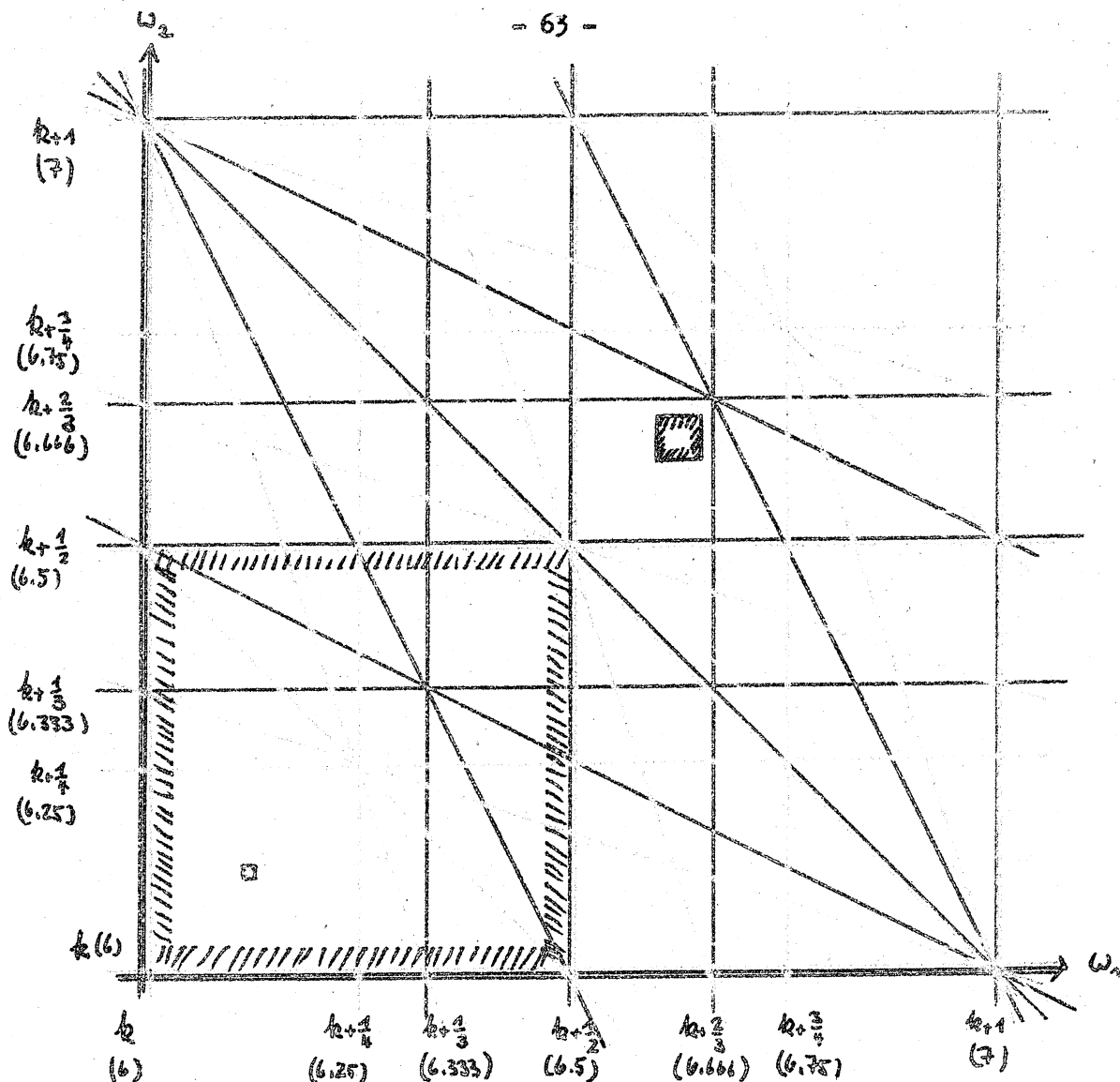


Fig.12. Excitable resonance-lines of the orders $N = 2, 3, 4$

In this figure all lines on which a resonance will surely be excited are given as full red lines. These are the sum-resonance-lines^{*} $n_1\omega_1 + n_2\omega_2 = p$ with $|n_1 + n_2| = 2$ or 3 ; the lines of the order 3 may be weakly quasi-stable and lead to only small built up, but they can become dangerous.

The green lines indicate the fourth-order sum-resonance-lines^{*} $n_1\omega_1 + n_2\omega_2 = p$; $|n_1 + n_2| = 4$. On these lines the motion will be strongly quasi-stable if some criteria (p. 56) are fulfilled.

There are no higher order resonance-lines which would become dangerous.

The squares denote the working areas (see text below fig.9).

^{*} n_1 and n_2 have equal signs.

V. Concluding remarks

We have transformed the hamiltonian with periodic coefficients into a normal form with constant coefficients by means of periodic canonical transformations. This method stems from G. Birkhoff (Bi), who treats essentially our case $n_1\omega_1 + n_2\omega_2 \neq p$ (p. 27). The idea to cancel large terms in the transformation, by adjusting some g-coefficients so that they remove the resonances, was first used by J. Moser (Mo) and this here is therefore the two-dimensional extension of the one-dimensional Birkhoff-Moser-method. On the other hand, the two- (and more) dimensional case for constant coefficients was already treated in several papers by Korteweg (Ko) and Beth (Be) at about 1900. Only the difference-resonances, which are not so interesting in our case, were treated and the essential difference between $|n_1| + |n_2| \leq 4$ and $|n_1| + |n_2| > 4$ was discovered. Of course, the case of constant coefficients in the original hamiltonian is contained in our paper also. Either the original hamiltonian is then already in the normal form and no transformation is necessary, or it is not and can still be simplified by a transformation. The equations for the coefficients $s_{jklm}^{(n)}$ are essentially the same but can all be solved by constants. For $j = k$ and $l = m$ the new coefficients of the hamiltonian are equal to the old ones and only if $n_1\omega_1 + n_2\omega_2 = 0$, one has to adjust further g-coefficients, which in this case are also constant because $p = 0$. Obviously, for $p = 0$, which is the only case of resonance character for constant coefficients, only difference lines occur for positive ω_1 and ω_2 and therefore Korteweg and Beth did not explore the sumlines.

Although the case with constant coefficients shows a similar structure on the first glance as the case with periodic coefficients, the difference is essential: Suppose a physical system with constant coefficients in cartesian coordinates. The hamiltonian can be separated uniquely into kinetic and potential energy. The kinetic energy is positively definite and the stability behaviour at the origin depends only on the curvatures of the potential energy surface at this point (minimum, maximum, saddle point) and not on any rationality relation between the basic frequencies. So one sees that the "resonance" in such cases is induced only by the perturbation treatment and expresses the fact that there is a strong energy exchange between both directions. This is easy to see on our quadratic invariant (45b): For difference-resonance it becomes an ellipse and both, $r_1^2 = y_1^2 + \dot{y}_1^2$ and $r_2^2 = y_2^2 + \dot{y}_2^2$ can alternatively become zero. That is, the whole energy is sometimes contained in one-dimensional motion only (in the new coordinates). That is not the case in sum-resonance. Here the

essential thing really is resonance with external time-functions, namely the periodic coefficients of the hamiltonian.

Independently and at the same time as the present writer, P. Sturrock attacked the two-dimensional case with periodic coefficients by another method and obtained the same general results (private communication and (Stu 2)).

One more word may be said concerning the discussion of the invariants. In explicit cases, it may happen that the hamiltonian in the normal form is such that it can be separated into two additive parts which can be interpreted (in the new coordinates) as "potential" and "kinetic" energy, the former being a function of y_1 and y_3 and the latter of y_2 and y_4 only. If the "kinetic energy" is then positively definite, the stability can be discussed from the "potential-energy-surface". This then is a function of two variables only and one can draw the equipotential lines in the y_1 - y_3 -plane and look for saddle points over which the motion can run away. In this case, however, one has to be very careful about the physical (or kinematical) meaning of the coordinates y_1 and y_3 for which one finds the "equipotential-lines" and limits of the range. They generally contain at least small contributions from the original momenta. Our $r_1^2 = y_1^2 + \dot{y}_2^2$ and $r_2^2 = y_3^2 + \dot{y}_4^2$ are somewhat simpler in this respect.

The question of the validity of calculated amplitude ranges is a very delicate one because it depends on to which order the transformation is carried out, on the initial amplitudes and on the effect of the back-transformation. In an explicit case one knows the coefficients of the higher terms in the original hamiltonian and one can then decide for which amplitudes neglected terms become really small. If then the calculated amplitude ranges lie within the limit, where neglected terms remain small, one may trust the calculation. But if the calculated ranges exceed this limit, the actual upper amplitude-limit may lie very far from the calculated one. Similar arguments apply to the question whether the back-transformation is necessary or not. From experience in handling the one-dimensional case we know that calculating higher orders in the invariant or transforming back the results make the formulas inconvenient and sometimes nearly impossible to discuss them. Fortunately the numerical values characterizing non-linearities in the CERN synchrotron turned out to be so small, that generally it does not seem necessary either to transform to high orders nor to transform back. Comparisons between predictions from considerations of invariants and numerical solutions in the one-dimensional case, showed agreeable results, so that

one can believe that amplitude ranges calculated by the methods presented here, will lead at least to very good estimates - even without back-transformations.

(See Schoch (Scho); also private communication by A.Schoch.)

Acknowledgement

The writer wishes to thank Dr.J.Moser, who visited the P.S.Group twice and who made us acquainted with his one-dimensional theory a long time before it was published. Then he is grateful for many discussions with Dr. A. Schoch (P.S.Group).

R.Hagedorn.

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W. Literature

- (Be) Beth, H.J.E. Amsterdam Proc.XII (1910) 619 and 735
- (Bi) Birkhoff, G.D. Dynamical systems; American mathematical Society Colloquium Publications, Vol. IX. New York 1927
- (Ko) Korteweg, D.J. Archives néerland. (2) (1897) 229
- (Lü) Lüders, G. in: Conference lectures, CERN PS, October 1953
- (Mo) Moser, J. Nachr. Gött. Akad. (Math.-Phys.) 1955 Nr.6
- (Scho) Schoch, A. Report CERN-PS/A.Sch.2 May 1955
- (Stu 1) Sturrock, P. Static and Dynamic Electron Optics Cambridge University Press 1955.
- (Stu 2) Sturrock, P. Non-linear effects in alternating gradient synchrotrons. A.E.R.E. X/R 1771 Harwell October 1955

X. APPENDIX 1

According to p.8 we determine the transforming matrix A_t under the following conditions

a) $\det A_t = 1$

b) $A_{t+2\pi} = A_t$

Using $\dot{\xi} = N\xi$; $\dot{\eta} = \Omega\eta$ and $\eta = A_t \xi$ we get

$$\dot{\eta} = \dot{A}\xi + AN\xi = \Omega A\xi \quad \text{or}$$

$$\dot{A} = \Omega A - AN. \quad \text{Setting } A = U_1 A_0 U_2 \text{ with } \dot{A}_0 = 0 \text{ we have}$$

$$\dot{A} = \dot{U}_1 U_1^{-1} A + A U_2^{-1} \dot{U}_2 = \Omega A - AN \quad \text{so that}$$

$$\dot{U}_1 = \Omega U_1$$

$$\dot{U}_2 = -U_2 \Omega$$

The first is easily solved remarking that the same equation holds for every columnvector of U_1 separately and that therefore four basic solutions of the equation $\dot{\eta} = \Omega\eta$ make up U_1 .

Thus

$$U_1 = U_t = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & \sin\omega_2 t \\ 0 & 0 & -\sin\omega_2 t & \cos\omega_2 t \end{pmatrix} \quad \text{with } U_0 = 1$$

Applying the same argument to the equation for U_2 we see that this does not work because of the - sign and the wrong order as compared with $\dot{\xi} = N\xi$. But let us suppose the ξ -equation as being solved. The solution can be expressed by the transfermatrix transforming from $t = 0$ to t .

$$\xi(t) = T_t \xi(0).$$

Now we have

$$\frac{d}{dt} [\xi(0)] = \frac{d}{dt} [T_t^{-1} \xi(t)] = \dot{T}_t^{-1} \xi(t) + T_t^{-1} N(t) \xi(t) = 0 \quad \text{or}$$

$\dot{T}_t^{-1} = -T_t^{-1} N$ so that U_2 is identical with the reciprocal of the transfermatrix:

$$U_2 = T_t^{-1} \quad \text{and also here} \quad U_2(0) = 1. \quad \text{Thus}$$

(a) $A_t = U_t A_0 T_t^{-1}$

A_0 now serves to fulfil the periodicity condition:

$$A_{t+2\pi} = U_{t+2\pi} A_0 T_{t+2\pi}^{-1} = U_t A_0 T_t^{-1}.$$

$$\text{Since } \dot{T}_t = N_t T_t \text{ and } N_{t+2\pi} = N_t,$$

$T_{t+2\pi}$ is again a solution and can therefore differ from T_t but only by the initial-value: $T_{t+2\pi} = T_t \cdot B$ where $B = T_{2\pi}$ because $T_0 = 1$. The same holds for $U_{t+2\pi} = U_t U_{2\pi}$.

Hence

$$A_{t+2\pi} = U_t U_{2\pi} A_0 T_{2\pi}^{-1} T_t^{-1} = U_t A_0 T_t^{-1} \quad \text{and}$$

(b)

$$U_{2\pi} A_0 T_{2\pi}^{-1} = A_0 \quad \text{or} \quad A_0^{-1} U_{2\pi} A_0 = T_{2\pi}$$

This equation must be solved for A_0 . It expresses the fact, that $T_{2\pi}$ and $U_{2\pi}$ are similar and therefore have the same diagonal representation. This will determine the so far unknown values of ω_1 and ω_2 .

There exists now a special set of four solutions $\xi^{(k)}(t)$ which are called Floquet solutions with the property that

$$\xi^{(k)}(t+2\pi) = \lambda_k \xi^{(k)}(t) \quad \text{and, with } \xi^{(k)}(0) = \xi_0^{(k)},$$

$$T_{2\pi} \xi_0^{(k)} = \lambda_k \xi_0^{(k)}.$$

If we denote the matrix composed by the columnvectors $\xi_0^{(k)}$ with X_0

$$X_{0,ik} = \xi_{0,i}^{(k)}, \quad \text{then}$$

$$T_{2\pi} X_0 = X_0 D_{2\pi} \quad \text{with } D_{2\pi} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \end{pmatrix}$$

U_t can be transformed for any time into a diagonal matrix by a constant transformation R

$$R^{-1} U_t R = D_t = \begin{pmatrix} e^{i\omega_1 t} & & & 0 \\ & e^{-i\omega_1 t} & & \\ & & e^{i\omega_2 t} & \\ & 0 & & e^{-i\omega_2 t} \end{pmatrix}$$

$$\text{That this is true and that } R = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ i\alpha & -i\beta & 0 & 0 \\ 0 & 0 & \gamma & \delta \\ 0 & 0 & i\gamma & -i\delta \end{pmatrix} \text{ with arbitrary}$$

$\alpha, \beta, \gamma, \delta$, is easily seen. We chose for simplicity's sake $\alpha = \beta = \gamma = \delta = \rho$ so that

(a)

$$R = \rho \begin{pmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix}$$

Because $U_{2\pi}$ and $T_{2\pi}$ are similar, we get

$$R^{-1} U_{2\pi} R = D_{2\pi} = X_0^{-1} T_{2\pi} X_0 \quad \text{or}$$

$$X_0 R^{-1} U_{2\pi} R X_0^{-1} = T_{2\pi} \quad \text{and comparing with (69b) we conclude}$$

(b)

$$A_0 = R X_0^{-1}$$

The elements $e^{\pm i\omega_k \cdot 2\pi}$ of $D_{2\pi}$ are the eigenvalues of $T_{2\pi}$ and the values ω_1 and ω_2 are hereby determined. If, for instance, a real machine with construction errors is considered, its period is 2π and $\omega_1 = Q_x$, $\omega_2 = Q_y$ are the common Q-values. For an ideal machine ω_1 and ω_2 denote the number of betatron-oscillations per structure period.

Finally A_0 must have a unity determinant because $\det U_t \equiv \det T_t \equiv 1$ and this can be achieved in an infinite number of ways because each columnvector of X_0 as well as of R is determined only in direction, not in length. The most convenient way is probably to normalize each columnvector of X_0 to unit-length (then $\det X_0 \neq 1$ because of lack of orthogonality) and to normalize the columnvectors of R to have equal length (as we already did) and finally to use the arbitrary constant ρ to make $\det(R X_0^{-1}) = 1$. But then $\det A_t \equiv 1$ for all the time.

Inserting (71b) into (69a) gives

(c)

$$A_t = U_t R X_0^{-1} T_t^{-1}$$

If we denote the matrix composed by the Floquet-solutions $\xi^{(k)}(t)$ as columnvectors by X_t , then

$X_t = T_t X_0$ and we can speak of X_t as "the" Floquet-solution.

Then we can write

$$(a) \quad A_t = U_t R_o X_t^{-1} = R D_t X_t^{-1}.$$

Finally we may split the Floquet-solution into two factors:

$$(b) \quad X_t = X_t \circ D_{-t} D_t = X_t D_{-t} \circ D_t = F_t \circ D_t.$$

$F_t = X_t \circ D_{-t}$ is periodic with period 2π :

$$X_{t+2\pi} \circ D_{-t-2\pi} = T_{2\pi} \circ T_{2\pi} X_t \circ D_{-t-2\pi} D_t = X_t \circ D_{-t} \text{ because}$$

$$T_{2\pi} X_o = X_o D_{2\pi}$$

This expresses the wellknown fact that the Floquet-solution can be written as a product of a 2π -periodic function F_t and another one D_t which is generally non-periodic (in the 2-dimensional case) and which we shall call the non-periodic part. If we insert $X_t^{-1} = D_t^{-1} F_t^{-1}$ into (a) we get a third representation of A_t :

$$(c) \quad A_t = R F_t^{-1} \text{ where } R \text{ is given by (71a) and } F_t \text{ is the periodic part of the Floquet-solution.}$$

So we have

$$(d) \quad A_t = U_t R X_o^{-1} T_t^{-1} = R D_t X_t^{-1} = R F_t^{-1}$$

We prove here that the system (21a) has real periodic solutions. This is not trivial because the periodic solutions are in all non-resonance cases uniquely determined and it could turn out, that they are not all real.

We prove it by concluding from n to $n+1$.

Assume the functions defined in (18b) and (19b)

$$\begin{aligned} s^{(2)}(y_1 y_2 y_3 y_4 t), \dots, s^{(n-1)}(y_1 y_2 y_3 y_4 t) \\ g^{(2)}(y_1 y_2 y_3 y_4 t), \dots, g^{(n-1)}(y_1 y_2 y_3 y_4 t) \end{aligned} \quad \text{all to be real.}$$

From this, and the fact that the old hamiltonian is real, follows that

$$f^{(n)}(y_1 y_2 y_3 y_4 t) \equiv F^{(n)}(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \equiv \sum_{jklm} f_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \quad \text{is also a real function. Then}$$

$$s^{(n)}(y_1 y_2 y_3 y_4 t) \equiv S^{(n)}(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \equiv \sum_{jklm} s_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \quad \text{and}$$

$$g^{(n)}(y_1 y_2 y_3 y_4 t) \equiv \frac{1}{2} G^{(n)}(z_1 \bar{z}_1 z_2 \bar{z}_2 t) \equiv \sum_{jklm} g_{jklm}^{(n)} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m$$

shall also be real. In the coefficients this reads:

$$\bar{f}_{jklm}^{(n)} = f_{kjml}^{(n)} \quad \text{is fulfilled}$$

$$\left. \begin{aligned} \bar{s}_{jklm}^{(n)} &= s_{kjml}^{(n)} \\ \bar{g}_{jklm}^{(n)} &= -g_{kjml}^{(n)} \end{aligned} \right\}$$

shall be fulfilled for the solutions of (23a)

If there is no resonance, $\varepsilon_{jklm}^{(n)} \equiv 0$ fulfills the condition.

In the case of resonance (23d) gives

$$\varepsilon_{jklm,-p}^{(n)} = -\frac{2}{i} f_{jklm,-p}^{(n)} \quad \text{so that on account of the } f \text{'s properties and the factor } i \text{ the condition for } g \text{ is again fulfilled.}$$

Therefore the function $f_{jklm}^{(n)} + \frac{1}{2} \varepsilon_{jklm}^{(n)} = \varphi_{jklm}^{(n)}$ in the integrand of (24b) has the property

$$\varphi_{jklm}^{(n)} = \varphi_{kjml}^{(n)} \quad \text{and therefore from (24b) and (24c)}$$

$$s_{jklm}^{(n)} = \frac{e^{-ipt}}{2\pi} \int_t^{t+2\pi} \varphi_{jklm}^{(n)}(\tau) e^{ip\tau} d\tau \quad \text{for } \Omega = \omega_1(k-j) + \omega_2(m-l) = p$$

$$s_{jklm}^{(n)} = \frac{e^{-i\Omega t}}{2\pi i \Omega} \int_t^{t+2\pi} f_{jklm}^{(n)}(\tau) e^{i\Omega \tau} d\tau \quad \text{for } \Omega = \omega_1(k-j) + \omega_2(m-l) \neq p$$

In both formulas, interchanging k with j and m with l will cause a change of sign of Ω and of the functions $f_{jklm}^{(n)}$ and $\varphi_{jklm}^{(n)}$ into the complex conjugated ones. Therefore also $s_{kjml}^{(n)} = \bar{s}_{jklm}^{(n)}$.

Finally, from the presupposition that all $s^{(m)}(y_1 \dots y_4 t)$ and $g^{(m)}(y_1 \dots y_4 t)$ up to $m = n-1$ are real, follows through the equations for $s^{(n)}(y_1 \dots y_4 t)$ and $g^{(n)}(y_1 \dots y_4 t)$ are real too. But $s^{(2)}$ and $g^{(2)}$ are real by construction [(18b) and (19b)] which completes the proof.